Notes on bounded-distance decoding for stabilizer states

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Recall the setting of agnostic tomography. There is a class $\mathcal{C}$ of states of interest, e.g., stabilizer states, and we no longer assume that our unknown state $\rho = |\psi\rangle\langle\psi|$ comes from that class, but we want to output $|\phi\rangle\langle\phi|$ for which

$$|\langle\psi|\,|\phi\rangle|^2 \geq \sup_{|\phi\rangle^* \in \mathcal{C}} |\langle\psi|\,|\phi^*\rangle|^2 - \epsilon\,. \tag{1}$$

Throughout, we will denote the supremum on the right-hand side by the variable τ. Let $|\phi^*\rangle \in \mathcal{C}$ be a stabilizer state satisfying $|\langle\psi|\,|\phi^*\rangle|^2 = \tau$, and let $W \subset \mathbb{F}_2^{2n}$ denote the isotropic subspace corresponding to the stabilizer group of $|\phi^*\rangle$.

Below, we sketch a result of [GIKL24] for agnostic tomography of stabilizer states when τ is at least some absolute constant.

The main structural result they show, which we provide notes for in the full lecture notes but will not cover in this lecture is the following:

Theorem 1 (Anti-concentration theorem)**.** *Let $\mathcal{B}_\rho$ denote the distribution over strings $\vec{s}$ obtained by Bell difference sampling.*

For any proper subspace $V \subsetneq W$,

$$\Pr[\vec{s} \sim \mathcal{B}_\rho]\vec{s} \in W \backslash V \geq \Omega(\tau^4)\,. \tag{2}$$

This ensures that if we run Bell difference sampling $\Omega(n/\tau^4)$ times, the set of strings obtained contains a basis for the stabilizer subspace W. The key challenge in the agnostic setting however is that the set of strings obtained will *also* contain strings that have nothing to do with the true subspace. How do we filter these out and only retain the strings in W? It turns out that if τ is sufficiently large, we can do this just by thresholding on the correlation between ρ and the associated Pauli operators.

Lemma 1 (Good Paulis have high correlation)**.** *If $\tau > 1/2$, then if $\vec{s} \in W$, we have*

$$\mathrm{tr}(P_{\vec{s}}\rho)^2 \geq (2\tau - 1)^2\,. \tag{3}$$

Proof. Write $|\psi\rangle = \sqrt{\tau}\,|\phi^*\rangle + \sqrt{1-\tau}\,|\phi^\perp\rangle$, where $|\phi^\perp\rangle$ is orthogonal to $|\phi^*\rangle$. Then

$$\begin{aligned}
\mathrm{tr}(P_{\vec{s}}\rho) &= (\sqrt{\tau}\,\langle\phi^*| + \sqrt{1-\tau}\,\langle\phi^\perp|)P_{\vec{s}}(\sqrt{\tau}\,|\phi^*\rangle + \sqrt{1-\tau}\,|\phi^\perp\rangle) \\
&= \tau\,\langle\phi^*|\,P_{\vec{s}}\,|\phi^*\rangle + (1-\tau)\,\langle\phi^\perp|\,P_{\vec{s}}\,|\phi^\perp\rangle + \sqrt{\tau(1-\tau)}\,\langle\phi^*|\,P_{\vec{s}}\,|\phi^\perp\rangle + \sqrt{\tau(1-\tau)}\,\langle\phi^\perp|\,P_{\vec{s}}\,|\phi^*\rangle \\
&= \tau\,\langle\phi^*|\,P_{\vec{s}}\,|\phi^*\rangle + (1-\tau)\,\langle\phi^\perp|\,P_{\vec{s}}\,|\phi^\perp\rangle\,,
\end{aligned}$$

where in the third step we used that $P_{\vec{s}}\,|\phi^*\rangle = \pm\,|\phi^*\rangle$ is orthogonal to $|\phi^\perp\rangle$ to get rid of the cross terms.

Note that $|\,\langle\phi^\perp|\,P_{\vec{s}}\,|\phi^\perp\rangle\,| \leq 1$, whereas $|\,\langle\phi^*|\,P_{\vec{s}}\,|\phi^*\rangle\,| = 1$. So because $\tau > 1/2$, $|\,\mathrm{tr}(P_{\vec{s}}\rho)| > 2\tau - 1$, from which the claim follows. □

Lemma 2 (Bad Paulis have low correlation)**.** *If $\tau > 1/2$, then if $\vec{s} \notin W$, we have*

$$\mathrm{tr}(P_{\vec{s}}\rho)^2 \leq 4\tau - 4\tau^2 \,. \tag{4}$$

Proof. By the uncertainty principle (Lemma 3), for any $\vec{t} \in W$, we have

$$\mathrm{tr}(P_{\vec{s}})^2 \leq 1 - \mathrm{tr}(P_{\vec{t}})^2 \leq 1 - (2\tau - 1)^2 = 4\tau - 4\tau^2 \tag{5}$$

as claimed. □

Lemma 3 (Uncertainty principle)**.** *If Pauli operators P, Q anti-commute, then*

$$\mathrm{tr}(P\rho)^2 + \mathrm{tr}(Q\rho)^2 \leq 1 \,. \tag{6}$$

Proof. Let $O = \mathrm{tr}(P\rho)P + \mathrm{tr}(Q\rho)Q$.

Claim 1. *The variance of the observable O, namely*

$$\mathrm{tr}(O^2\rho) - \mathrm{tr}(O\rho)^2 \tag{7}$$

is nonnegative.

Proof. Let $O = UDU^\dagger$ for diagonal matrix D, so $O^2 = UD^2U^\dagger$. Then

$$\mathrm{tr}(O^2\rho) = \mathrm{tr}(D^2U^\dagger\rho U) = \mathrm{tr}(D^2\mathrm{diag}(U^\dagger\rho U)) \tag{8}$$

and similarly $\mathrm{tr}(O\rho) = \mathrm{tr}(D\mathrm{diag}(U^\dagger\rho U))$. Note that the diagonal entries of $\mathrm{diag}(U^\dagger\rho U)$ are nonnegative and sum to 1, so $\mathrm{tr}(O^2\rho)$ is the second moment of a random variable on d elements, and $\mathrm{tr}(O\rho)^2$ is the square of the first moment. $\mathrm{tr}(O^2\rho) - \mathrm{tr}(O\rho)^2$ is thus the variance of a *bona fide* classical random variable, implying nonnegativity. □

The claim immediately implies the uncertainty principle:

$$\mathrm{tr}(O\rho) = \mathrm{tr}(P\rho)^2 + \mathrm{tr}(Q\rho)^2 \,, \tag{9}$$

while

$$\begin{aligned}\mathrm{tr}(O^2\rho) &= \mathrm{tr}(P\rho)^2\,\mathrm{tr}(P^2\rho) + \mathrm{tr}(Q\rho)^2\,\mathrm{tr}(Q^2\rho) + \mathrm{tr}(P\rho)\,\mathrm{tr}(Q\rho)(\mathrm{tr}(PQ\rho) + \mathrm{tr}(QP\rho)) &(10)\\ &= \mathrm{tr}(P\rho)^2 + \mathrm{tr}(Q\rho)^2 \,, &(11)\end{aligned}$$

where we used that $P^2 = Q^2 = I$ and $PQ + QP = 0$ by assumption.

Nonnegativity of the variance of O then implies that $(\mathrm{tr}(P\rho)^2 + \mathrm{tr}(Q\rho)^2)^2 \leq \mathrm{tr}(P\rho)^2 + \mathrm{tr}(Q\rho)^2$, which yields the lemma. □

With Lemmas 1 and 2 in place, the learning result for agnostic tomography of stabilizer results follows immediately for sufficiently large τ, namely provided the lower bound for good Paulis is at least the upper bound for bad Paulis:

$$(2\tau - 1)^2 \geq 4\tau - 4\tau^2 \Leftrightarrow \tau \geq \cos^2(\pi/8) \,. \tag{12}$$

References

[GIKL24] Sabee Grewal, Vishnu Iyer, William Kretschmer, and Daniel Liang. Improved stabilizer estimation via bell difference sampling. In *Proceedings of the 56th Annual ACM Symposium on Theory of Computing*, pages 1352–1363, 2024.