Physics Cup 2026 Problem 3

1 Notations and Simplifications

Firstly, we propose a naming system for each node in the circuit: they will be written in the form (m, n) where m denotes the layer its on (starting on 0) and $n \in \{0, 1, ..., 7\}$ denotes that it is the n^{th} node in its octagonal ring, going clockwise. For example, A will be $(0, 0)$, the point directly above A will be $(1, 0)$ and the point left of A will be $(0, 1)$
The circuit's symmetry can be made more apparent by rotating layer m by $\frac{m\pi}{4}$, so $(1, 1)$ will become $(1, 0)$ as it is directly on top of A, $(2, 2)$ will become $(2, 0)$ and so on. This way, as shown in Figure 1, all the capacitors and inductors line up, while the inter-layer wires will become diagonal. It is now apparent that the circuit is actually made up of 1-layer blocks rather than 2-layer blocks.
Since the problem seeks a time dependent solution $I(t)$, a differential equation

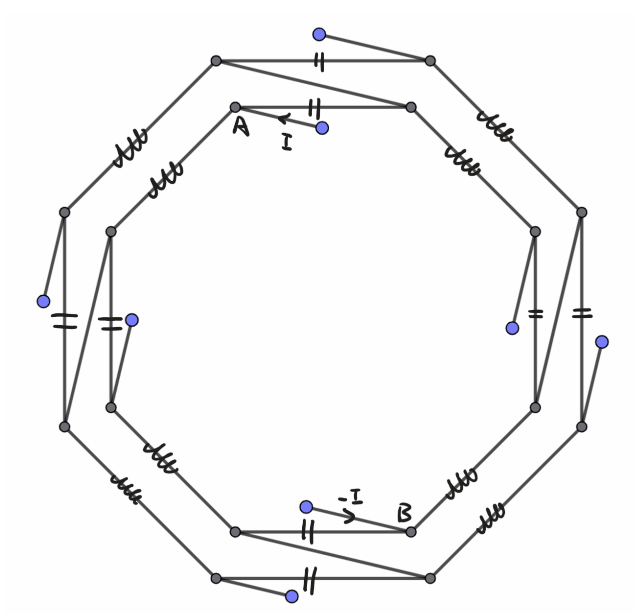

Figure 1: Diagram of the circuit after twisting

can be turned into an algebraic one by using the complex frequency domain (s-domain) instead through the laplace transform. The impedances of capacitors and inductors will be modified by the substitution $i\omega \to s$, so

$$Z_L = sL_0, \qquad Z_C = \frac{1}{sC_0}$$

For notational simplicity (although it might introduce some ambiguities), we make the following definitions:

$$L = Z_L, \qquad C = Z_C \tag{1}$$

Since we are now working in s-domain, the input voltage will go from V_i to V_i/s.[1] The last tweak is to write the potential difference V_i/s as $+V_i/2s$ at A and $-V_i/2s$ at B since this also abides by the C_4 symmetry in the circuit.

2 Utilising Symmetry

Since the circuit has C_4 rotational symmetry, its voltages and currents can be written as a superposition of the 4 Fourier modes:

$$\begin{aligned} V_{2n} &= V_0\lambda^n \\ V_{2n+1} &= V_1\lambda^n \\ I_{2n} &= I_0\lambda^n \\ I_{2n+1} &= I_1\lambda^n \end{aligned} \tag{2}$$

where $\lambda \in \{1, i, -1, -i\}$ as applying the rotation operator 4 times returns to the same position so $\lambda^4 = 1$. The input current on even nodes take the form $[I, 0, -I, 0]$ which lies entirely on the subspace of $\lambda = \pm i$.

3 The Infinite Part

In order to find the total admittance of the infinite circuit, the admittance of one layer must be found first, which we have chosen to be layer 0. Taking KCL around $(0, 0)$ gives

$$\frac{V_0 - V_1}{C} + \frac{V_0 - V_7}{L} = I_0 \tag{3}$$

where we have defined current going into the circuit for both currents to be positive Taking KCL around $(0, 1)$ gives

$$\frac{V_1 - V_0}{C} + \frac{V_1 - V_2}{L} = I_1$$

[1] We have decided to replace the potential differene V_0 defined in the problem with V_i due to clashes in notation

This simplifies to

$$I_0 LC = V_0(L+C) - V_1(L+\lambda^3 C)$$
$$I_1 LC = V_1(L+C) - V_0(L+\lambda C)$$

Since the admittance matrix takes the form

$$\begin{pmatrix} I_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} Y_{00} & Y_{01} \\ Y_{10} & Y_{11} \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \tag{4}$$

The 4 elements of Y can be found by setting V_0 and V_1 to 0 and 1 respectively and reading the value of I_0 and I_1.
When $V_0 = 1$, $V_1 = 0$,

$$Y_{00} = I_0 = \frac{L+C}{LC}$$
$$Y_{01} = I_1 = -\frac{L+\lambda C}{LC}$$

When $V_0 = 0$, $V_1 = 1$,

$$Y_{10} = I_0 = -\frac{L+\lambda^3 C}{LC}$$
$$Y_{11} = I_1 = \frac{L+C}{LC}$$

Eliminating port 1 via Kron Reduction gives

$$Y = Y_{00} - \frac{Y_{01}Y_{10}}{Y_{11}+Y}$$

where Y is the admittance of the self-similar tower. Substituting the values of Y in gives

$$Y = \pm\sqrt{\frac{2}{LC}}$$

for both $\lambda = i$ and $\lambda = -i$. The passive branch $\Re(Y) \geq 0$ is chosen, which is the positive root and gives

$$Y = \sqrt{\frac{2}{LC}} = \sqrt{\frac{2C_0}{L_0}} \tag{5}$$

Here we notice that there is actually no dependence on s in the circuit!

4 Final Input Current

The voltage at node A in s-domain is $V_0 = V_i/2s$, so by Ohm's Law, the input current is

$$I = YV_0 = \frac{V_i}{s}\sqrt{\frac{C_0}{2L_0}}$$

Taking the inverse laplace transform of $I(s)$ gives

$$I(t) = V_i\sqrt{\frac{C_0}{2L_0}}u(t) \tag{6}$$

Where $u(t)$ is the Heaviside Step Function.