Physics Cup 2026 Problem 1

1 Properties of the Hodograph

Since the brick is moving relative to each plate, the kinetic friction vector on each surface must also be equal in magnitude, but in a direction given by the relative velocity of each plate with respect to the brick [1]. Since friction f in each direction is equal, the net force F lies on the angle bisector of u'_1 and u'_2.

This condition is always satisfied as the block moves. This also corresponds to

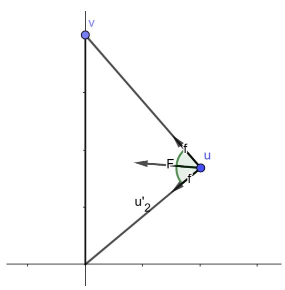

Figure 1: Hodograph of the velocity of the block

the reflection property of the hyperbola, so the trajectory of the block's velocity P will be a hyperbola with foci at the origin and $(0, v)$ in the hodograph.

2 Geometric Formalism

To formalise the diagram, let OV have length v and lie on the u_y axis, OU have length u_0 and lie on the u_x axis. A hyperbola with foci O and V passing through U be drawn. Let the the tangent of the hyperbola at point P intersect the u_y axis at F. Let α be $\angle OPF$ and β be $\angle PFO$. Let the true anomaly of the hyperbola ϕ be $\angle POU$ which takes the value of $\pi - \alpha - \beta$.

Since the friction in each direction is pointing along PO and PV, the net friction lies on PF, of value $2F\cos\alpha$ and net acceleration will be $\frac{2F\cos\alpha}{m}$. Initially, P

[1] This was directly taken from the official solutions to Problem 2 of the European Physics Olympiad 2023.

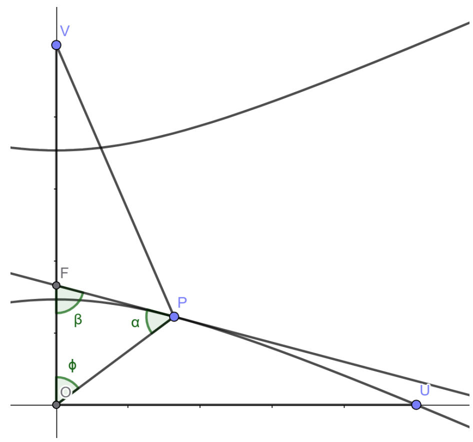

Figure 2: Diagram with the hyperbola and angles stated

coincides with U and $\alpha = \alpha_0$, so

$$a_0 = \frac{2F\cos\alpha_0}{m} \implies a = \frac{\cos\alpha}{\cos\alpha_0} a_0$$

However, we have defined the right (OU) as the positive u_x direction, so we are a minus sign off from the actual a. Therefore the x component of a will be

$$a_x = -\frac{\cos\alpha\sin\beta}{\cos\alpha_0} a_0 \tag{1}$$

2.1 Relation between α and β

In order to find the relation between α and β, we will do a series of definitions for lengths. Let $a = PV$, $b = PO$, $y = OF$, $x = FV$ and $c = PF$. Listing out 2 laws of cosines in $\triangle VPF$ gives

$$x^2 = a^2 + c^2 - 2ac\cos\alpha \tag{2}$$

$$a^2 = c^2 + x^2 - 2cx\cos(\pi - \beta) = c^2 + x^2 + 2cx\cos\beta \tag{3}$$

Repeating the same process with $\triangle OPF$,

$$y^2 = b^2 + c^2 - 2bc\cos\alpha \tag{4}$$

$$b^2 = c^2 + y^2 - 2yc\cos\beta \tag{5}$$

Subtraction Equation 4 from Equation 2, along with Equation 5 from Equation 3 gives

$$x^2 - y^2 = a^2 - b^2 - 2c(b-a)\cos\alpha \tag{6}$$

$$a^2 - b^2 = x^2 - y^2 + 2c(x+y)\cos\beta \tag{7}$$

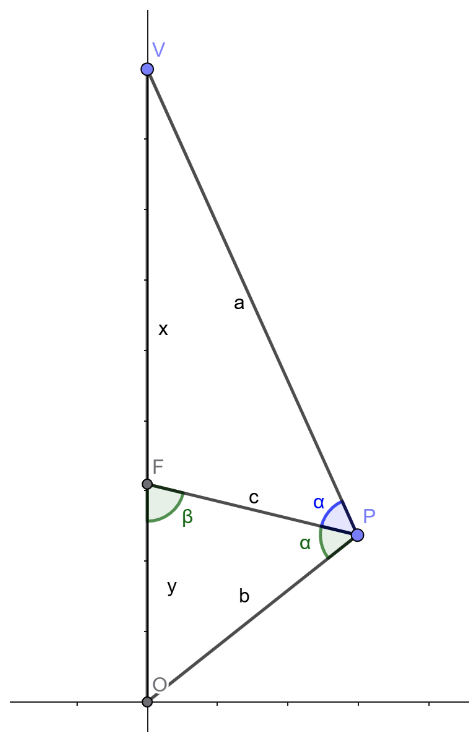

Figure 3: The trangle used for the four laws of cosines

Summing the two equations give

$$2c(b-a)cos\alpha = 2c(x+y)\cos\beta$$
$$\frac{\cos\alpha}{\cos\beta} = \frac{x+y}{b-a} \tag{8}$$

By the definition of a hyperbola, $b-a$ is the transverse axis length and $x+y$ is the focal distance. The eccentricity e is the ratio of the two, so

$$\cos\alpha = e\cos\beta \tag{9}$$

3 Integral Derivation and Evaluation

The infinitesimal displacement (in the y axis) dl_y is defined as

$$dl_y = u_y\,dt$$

This can also be represented as

$$dl_y = u_y\,dt = u_y\,du_x\frac{dt}{du_x} = \frac{u_y}{a_x}\,du_x$$

and therefore

$$l_y = \int \frac{u_y}{a_x}\,du_x \tag{10}$$

Since $u_y(u_x)$ is a hyperbola, it can be parametrized as

$$(u_x, u_y) = \left(\frac{p\cos\phi}{1+e\cos\phi}, \frac{p\sin\phi}{1+e\cos\phi}\right) \tag{11}$$

The infinitesimal element du_x can then be represented as

$$du_x = \frac{du_x}{d\phi}\,d\phi = -\frac{p\sin\phi}{(1+e\cos\phi)^2}\,d\phi \tag{12}$$

Combining Equations 1, 10, 11 and 12 gives

$$\begin{aligned} l_y &= \int_0^{\pi/2} -\frac{\cos\alpha_0}{a_0\cos\alpha\cos\beta}\frac{p\sin\phi}{1+e\cos\phi}\frac{-p\sin\phi}{(1+e\cos\phi)^2}\,d\phi \\ &= \frac{p^2\cos\alpha_0}{a_0}\int_0^{\pi/2}\frac{\sin^2\phi}{(1+e\cos\phi)^3 cos\alpha\cos\beta}\,d\phi \end{aligned} \tag{13}$$

3.1 Writing α and β in terms of ϕ

Since $\phi = \pi - \alpha - \beta$,

$$\alpha = \pi - (\phi + \beta)$$

Therefore

$$
\begin{aligned}
\cos\alpha = \cos(\pi - (\phi+\beta)) = -\cos(\phi+\beta) = -\cos\phi\cos\beta + \sin\phi\sin\beta = e\cos\beta \\
\implies (e+\cos\phi)\cos\beta = \sin\phi\sin\beta \\
\implies \tan\beta = \frac{e+\cos\phi}{\sin\phi} \\
\implies \cos^2\beta = \frac{1}{1+\tan^2\beta} = \frac{\sin^2\phi}{1+e^2+2e\cos\phi} \\
\cos\alpha\cos\beta = e\cos^2\beta
\end{aligned}
\tag{14}
$$

Substituting this into Equation 13 gives

$$
l_y = -\frac{p^2\cos\alpha_0}{a_0 e}\int_0^{\pi/2}\frac{1+2ecos\phi+e^2}{(1+e\cos\phi)^3}\,d\phi \tag{15}
$$

For convenience, let the integral (excluding the coefficients) in Equation 15 be I. Since

$$
(1+e\cos\phi)^2 + e^2\sin^2\phi = 1+2ecos\phi+e^2
$$

I can be separated into 2 separate integrals I_1 and I_2

$$
I_1 = \int_0^{\pi/2}\frac{1}{1+e\cos\phi}\,d\phi \tag{16}
$$

$$
I_2 = \int_0^{\pi/2}\frac{\sin^2\phi}{(1+e\cos\phi)^3}\,d\phi \tag{17}
$$

So

$$
I = I_1 + e^2 I_2 \tag{18}
$$

3.2 Evaluating I_1

Applying the half-angle substitution, I_1 becomes

$$
\begin{aligned}
I_1 &= \int_0^1\left(1+e\frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}\,dt \\
&= \int_0^1\frac{2}{(e+1)-(e-1)t^2}\,dt \\
&= \frac{2}{e-1}\int_0^1\frac{1}{k^2-t^2}\,dt \qquad \text{where } k^2 = \frac{e+1}{e-1}
\end{aligned}
$$

Looking up this on the table of integrals, we find that

$$
I_1 = \left[\frac{2}{e-1}\left(\frac{1}{2k}\ln\frac{k+t}{k-t}\right)\right]_0^1
$$

Evaluating at the bounds give

$$I_1 = \frac{1}{(e-1)k} \ln \frac{k+1}{k-1}$$

Rationalising the denominator of $\frac{k+1}{k-1}$ gives

$$\frac{k+1}{k-1} = \frac{\sqrt{e+1}+\sqrt{e-1}}{\sqrt{e+1}-\sqrt{e-1}} = e + \sqrt{e^2-1}$$

It turns out that

$$\ln \frac{k+1}{k-1} = \ln\left(e+\sqrt{e^2-1}\right) = \operatorname{arcosh} e$$

and

$$(e-1)k = \sqrt{e^2-1}$$

So

$$I_1 = \frac{\operatorname{arcosh} e}{\sqrt{e^2-1}} \tag{19}$$

3.3 Evaluating I_2

The half-angle substitution can be applied again to give

$$\begin{aligned}
I_2 &= \int_0^1 \frac{4t^2}{(1+t^2)^2}\left(\frac{(e+1)-(e-1)t^2}{1+t^2}\right)^{-3} \frac{2}{1+t^2}\,dt \\
&= \int_0^1 \frac{8t^2}{((e+1)-(e-1)t^2)^3} \\
&= \frac{8}{(e-1)^3}\int_0^1 \frac{t^2}{(k^2-t^2)^3}\,dt
\end{aligned}$$

Where k has the same definition as above. To evaluate this new integral, we will do the substitution $t = k\tanh x$ with $dt = k\operatorname{sech}^2 x\,dx$

$$\begin{aligned}
I_2 &= \frac{8}{(e-1)^3}\int_0^{x_0} \frac{k^2\tanh^2 x}{k^6\operatorname{sech}^6 x} k\operatorname{sech}^2 x\,dx \\
&= \frac{8}{k^3(e-1)^3}\int_0^{x_0} \sinh^2 x\cosh^2 x\,dx \\
&= \frac{2}{k^3(e-1)^3}\int_0^{x_0} \sinh^2 2x\,dx \\
&= \frac{1}{k^3(e-1)^3}\int_0^{x_0} 1-\cosh 4x\,dx \\
&= \frac{1}{k^3(e-1)^3}\left[x - \frac{1}{4}\sinh 4x\right]_0^{x_0}
\end{aligned}$$

$1 = k \tanh x_0$, so

$$x_0 = \operatorname{artanh} \frac{1}{k} = \frac{1}{2} \ln \frac{k+1}{k-1} = \frac{1}{2} \operatorname{arcosh} e$$

substituting this back into I_2 gives

$$\begin{aligned} I_2 &= \frac{1}{k^3(e-1)^3} \left(\frac{1}{2} \operatorname{arcosh} e - \frac{1}{4} \sinh(2 \operatorname{arcosh} e) \right) \\ &= \frac{1}{(e+1)^{3/2}(e-1)^{3/2}} \left(\frac{1}{2} \operatorname{arcosh} e - \frac{e}{2} \sinh(\operatorname{arcosh} e) \right) \\ &= \frac{\operatorname{arcosh} e - e\sqrt{e^2-1}}{2(e^2-1)^{3/2}} \end{aligned} \tag{20}$$

3.4 Result of the Integral

Substituting Equation 19 and 20 into Equation 18 gives

$$\begin{aligned} I &= \frac{\operatorname{arcosh} e}{\sqrt{e^2-1}} + \frac{e^2 \operatorname{arcosh} e - e^3\sqrt{e^2-1}}{2(e^2-1)^{3/2}} \\ &= \frac{e^3\sqrt{e^2-1} + (e^2-2) \operatorname{arcosh} e}{2e(e^2-1)^{3/2}} \end{aligned}$$

And therefore

$$l_y = \frac{p^2 \cos \alpha_0}{a_0 e} \frac{e^3\sqrt{e^2-1} + (e^2-2) \operatorname{arcosh} e}{2e(e^2-1)^{3/2}} \tag{21}$$

4 The Result

We have now obtained an expression for l_y in terms of p, α_0, a_0 and e, but the final answer must be only in terms of a_0, u_0 and v.
Looking back at Figure 2, we can recognise that the semi-latus rectum of the hyperbola is OU, which has length u_0 so $p = u_0$
Initially, P coincides with U so

$$\tan 2\alpha_0 = \frac{OV}{OU} \implies \alpha_0 = \frac{1}{2} \arctan \frac{v}{u_0}$$

For a hyperbola with semi-major axis a, semi minor axis b, focal distance c, semi-latus rectum p and eccentricity e,

$$p = \frac{b^2}{a}, \qquad e = \frac{c}{a}, \qquad c^2 = a^2 + b^2$$

The semi-latus rectum can also be written as

$$p = \frac{c^2 - a^2}{a} = \frac{c^2}{a} - a = c\left(\frac{c}{a} - \frac{a}{c}\right) = c\left(e - \frac{1}{e}\right)$$

So

$$ce^2 - pe - c = 0$$

And therefore

$$e = \frac{p + \sqrt{p^2 + (2c)^2}}{2c}$$

The positive root was taken as $e > 0$. Since $p = u_0$ and $2c = v$,

$$e = \frac{\sqrt{u_0^2 + v^2} + u_0}{v} \tag{22}$$

Combining everything together, the distance travelled in the y direction by the block after a long time is

$$l_y = \frac{u_0^2 \cos\alpha_0}{a_0 e} \frac{e^3\sqrt{e^2-1} + (e^2-2)\operatorname{arcosh} e}{2e(e^2-1)^{3/2}}$$

where

$$e = \frac{\sqrt{u_0^2 + v^2} + u_0}{v} \qquad \text{and} \qquad \alpha_0 = \frac{1}{2}\arctan\frac{v}{u_0}$$

The values $v = u_0 = 1\,\mathrm{ms}^{-1}$ and $a_0 = 1\,\mathrm{ms}^{-2}$ yields

$$l_y \approx 0.6631\,\mathrm{m} \tag{23}$$

to 4 decimal places.

5 Appendix

The half-angle substitution takes the form

$$\begin{aligned} t &= \tan\frac{\phi}{2} \\ \sin\phi &= \frac{2t}{1+t^2} \\ \cos\phi &= \frac{1-t^2}{1+t^2} \\ d\phi &= \frac{2}{1+t^2}\,dt \end{aligned} \tag{24}$$

The integral used in Section 3.2 was

$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\frac{a+x}{a-x} + C \tag{25}$$

The hyperbolic trigonometric identities used in Section 3.3 were

$$\operatorname{sech}^2 x = 1 - \tanh^2 x \tag{26}$$

$$\sinh 2x = 2\sinh x\cosh x \tag{27}$$

$$\cosh 4x = 2\sin^2 2x + 1 \tag{28}$$