BAB III
Rantai Markov

Keadaan hari ini hanya ditentukan oleh keadaan kemarin, atau keadaan besok hanya dipengaruhi oleh keadaan hari ini

$$P(X_{n+1}|X_0, X_1, \ldots X_n) = P(X_{n+1}|X_n)$$

Probabilitas Transisi

- $P(X_{n+1}=x_{n+1}|Xo=x0, X_1=x_1, \ldots, X_n=x_n) = P(Xn+1 = xn+1|Xn=xn)$
- Ini berarti bahwa kondisi $Xo=x_0$, $X_1=x_1$, …, $X_{n-1}=x_{n-1}$ tidak mempunyai pengaruh terhadap keadaan tersebut, yang mempengaruhi probabilitas X_{n+1} hanya $X_n=x_n$. Jadi keadaan (state) sebelumnya tidak berpengaruh terhadap keadaan besok (X_{n+1}) , yang mempengaruhi hanya keadaan sekarang($X_n=x_n$).

- Secara formula ditulis :

  $P_{ij}=P[X_{n+1}=j|X_n=i], \ \forall \ i,j$

  dengan $P_{ij} \geq 0$ dan $\sum_j P_{ij} = 1, \forall i$

Catatan :

Jika$P[X_{n+1}=j|X_n=i]=P[X_1=j|X_0=i], \ \forall \ n=0,1,2,\ldots$ maka probabilitas transisinya bersifat stasioner.

Probabilitas Transisi m langkah :

$P_{ij}^{(m)} = P[X_{n+m}=j|X_n=i], \ \forall \ i,j$

$P_{ij}^{(m)} \geq 0, \ \forall \ i,j \in S$ dan $m = 0,1,2,\ldots$

$\sum_j P_{ij}^{(m)} = 1, \ \forall \ i \in S$ dan $m = 0,1,2,\ldots$

Secara matriks P – P$^{(m)}$

$$
\mathrm{P} = \begin{array}{c c}
 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} & \left[\begin{array}{ccccc}
p_{00} & p_{01} & p_{02} & p_{03} & \cdots \\
p_{10} & p_{11} & p_{12} & p_{13} & \cdots \\
p_{20} & p_{21} & p_{22} & p_{23} & \cdots \\
p_{30} & p_{31} & p_{32} & p_{33} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]
\end{array}
$$

$$
\mathrm{P}^{(\mathrm{m})} = \begin{array}{c c}
 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} & \left[\begin{array}{ccccc}
p^{(m)}_{00} & p^{(m)}_{01} & p^{(m)}_{02} & p^{(m)}_{03} & \cdots \\
p^{(m)}_{10} & p^{(m)}_{11} & p^{(m)}_{12} & p^{(m)}_{13} & \cdots \\
p^{(m)}_{20} & p^{(m)}_{21} & p^{(m)}_{22} & p^{(m)}_{23} & \cdots \\
p^{(m)}_{30} & p^{(m)}_{31} & p^{(m)}_{32} & p^{(m)}_{33} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]
\end{array}
$$

- Contoh :

  Pemakaian sebuah mesin pada saat tertentu mesin dalam kondisi baik atau rusak. Misal state 0 : mesin rusak dan state 1 : mesin baik

State space : S={0,1}

$P_{01}=P[X_{n+1}=1|X_n=0]=p$

$P_{00}=P[X_{n+1}=0|X_n=0]=1-p$

$P_{10}=P[X_{n+1}=0|X_n=1]=q$

$P_{11}=P[X_{n+1}=1|X_n=1]=1-q$

Matriks probabilitas transisinya:

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} \end{array} = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \end{array}$$

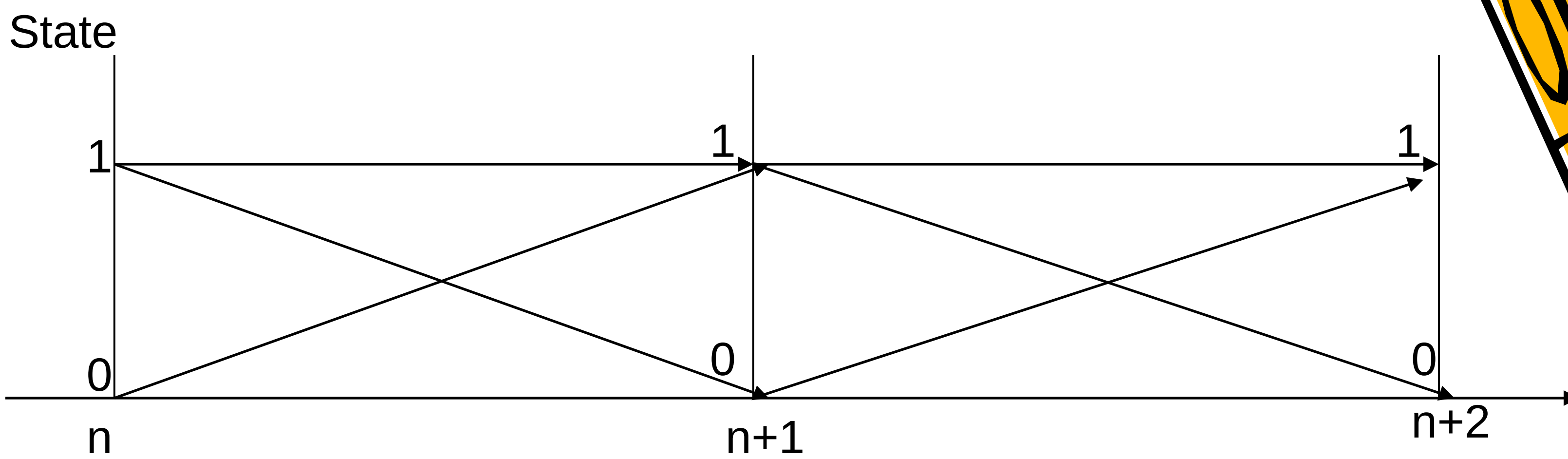
Dari contoh tersebut ingin dicari probabilitas transisi dua langkah :

$P^{(2)}_{00} = P[X_{n+2}=0|X_n=0]$

$P^{(2)}_{01} = P[X_{n+2}=1|X_n=0]$

$P^{(2)}_{10} = P[X_{n+2}=0|X_n=1$

$P^{(2)}_{11} = P[X_{n+2}=1|X_n=1]$



Dari grafik terlihat :

$P^{(2)}_{00} = P_{01}P_{10} + P_{00}P_{00} = pq + (1-p)^2$

$P^{(2)}_{01} = P_{01}P_{11} + P_{00}P_{01} = p(1-q) + (1-p)p$

$P^{(2)}_{10} = P_{11}P_{10} + P_{10}P_{00} = (1-q)q + q(1-p)$

$P^{(2)}_{11} = P_{11}P_{11} + P_{10}P_{01} = (1-q)^2 + qp$

Elemen-elemen tersebut merupakan elemen-elemen dari matriks:

$$P^2 = PP = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$$

$$= \begin{bmatrix} p^2{}_{00} + p_{01}p_{10} & p_{00}p_{01} + p_{01}p_{11} \\ p_{10}p_{00} + p_{11}p_{10} & p_{10}p_{01} + p^2{}_{11} \end{bmatrix}$$

- Sehingga dapat disimpulkan bahwa untuk mendapatkan probabilitas transisi 2 langkah diperoleh dengan mengambil elemen-elemen dari P^2 yang sesuai. Demikian seterusnya untuk probabilitas transisi 3 langkah, dihitung P^3 dan seterusnya :

  $P^2 = PP$

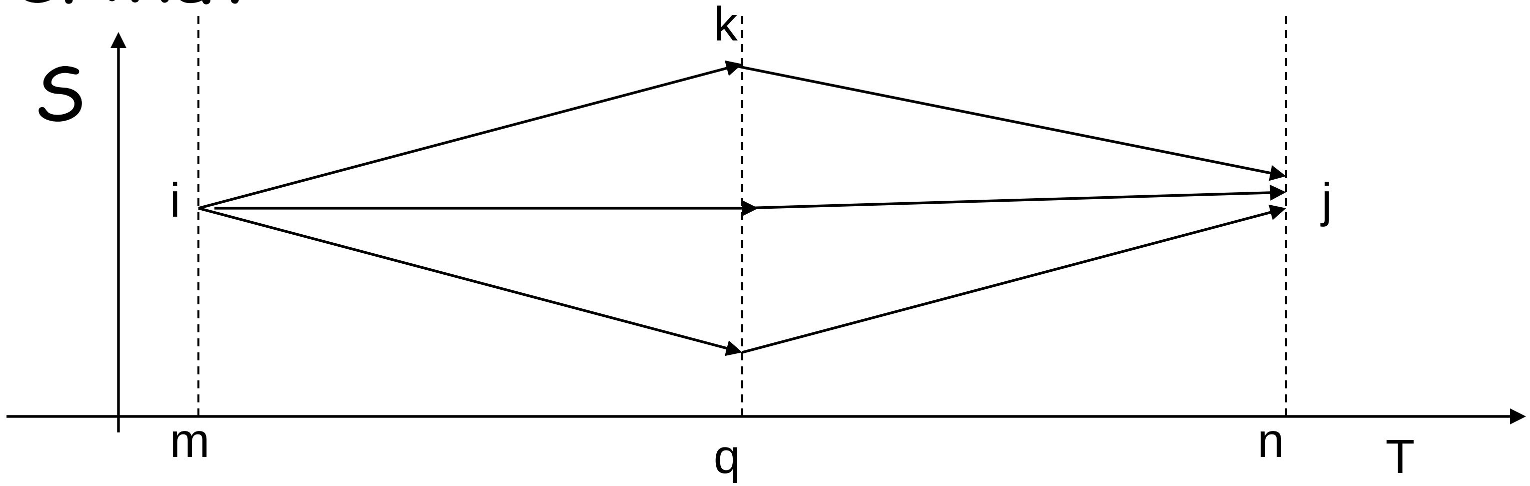
  $P^3 = P^2P$

  ....

  $P^n = P^{n-1}P$

Generalisasi oleh persamaan Chapman-Kolmogorov sebagai berikut:

- S



- $P_{ij}(m,n)=P[X_n=j|X_m=i]=\Sigma_k P[X_n=j,X_q=k|X_m=i]$

$$= \Sigma_k P[X_n=j,X_q=k]\ P[X_q=k|X_m=i]$$

$$= \Sigma_k\ P[X_q=k|X_m=i]\ P[X_n=j,X_q=k]$$

$$= \Sigma_k p_{ik}(m,q)\ p_{kq}(q,n)$$

Secara matriks : $P(m,n)=P(m,q)\ P(q,n)$

- Keadaan khusus:

a. $m=0,\ q=1$

$$P^{(n)}_{ij} = \Sigma_k p_{ik} p_{kq}(n-1)$$

b. $m=0,\ q=n-1$

$$P_{ij}(0,n) = P_{ij}(n) = \Sigma_k p_{ik}(0,n-1) p_{kq}(n-1,n)$$
$$= \Sigma_k p_{ik}^{(n-1)} p_{kj}$$

Kesimpulan :

Probabilitas transisi n langkah dapat diperoleh dari probabilitas transisi 1 langkah secara berulang (rekursif)

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \;\;0\;\; & \;\;\;\;1\;\; \end{array} \\ \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

$$P^2 = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}\begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix}$$

$$P^2 = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \;\;0\;\; & \;\;\;\;1\;\; \end{array} \\ \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix} \end{array}$$

Dari contoh matriks transisi di atas

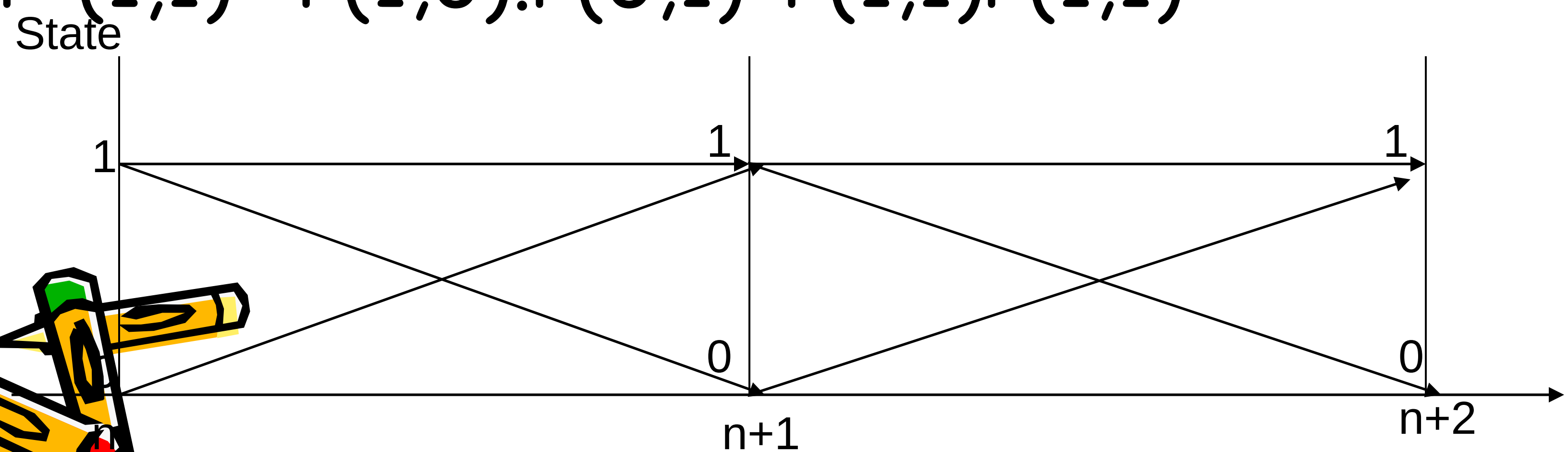
Rantai markov dengan $S=\{0,1\}$ terlihat bahwa:

$P^2(0,0) = P(0,0).P(0,0)+P(0,1)P(1,0)$

$P^2(0,1) = P(0,0).P(0,1)+P(0,1)P(1,1)$

$P^2(1,0) = P(1,0).P(0,0)+P(1,1)P(1,0)$

$P^2(1,1) = P(1,0).P(0,1)+P(1,1)P(1,1)$



- Sehingga untuk menentukan peluang dua langkah $P^2(i,j)$ untuk i, j bernilai 0 atau 1 dapat dilakukan dengan dua cara, yaitu :

(1) Menentukan matriks transisi dua langkah P^2

(2) Berdasarkan matriks transisi satu langkah P dengan penjumlahan aljabar kombinasi langkah yang mungkin

- Misalkan untuk contoh di atas :

Nilai $P(X_{n+2}=0|X_n=0)$ diperoleh dari :

(1) Matriks $P^2(0,0) = 0.61$

(2) Berdasarkan matriks P dengan kombinasi langkah yang mungkin yaitu

$$P^2(0,0) = P(0,0).P(0,0)+P(0,1)P(1,0)$$
$$=(0.7)(0.7) + (0.3)(0.4)$$
$$= 0.49 + 0.12$$
$$= 0.61$$

Contoh:

Suatu proses Markov $X_0, X_1, X_2, \ldots$ dengan matriks peluang transisi sebagai berikut :

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0,5 & 0,5 & 0 \\ 0,5 & 0 & 0,5 \\ 0 & 0,5 & 0,5 \end{Vmatrix} \end{array}$$

Tentukan peluang untuk :

a. $P(X_2 = 1, X_1 = 1 \mid X_0 = 0)$

b. $P(X_3 = 1, X_2 = 1 \mid X_0 = 0)$

c. Jika diketahui proses berawal dari $X_0 = 0$, tentukan

   (i) $P(X_2 = 0)$

   (ii) $P(X_2 = 0, X_1 = 0)$

Penyelesaian:

a. $P(X_2 = 1, X_1 = 1 \mid X_0 = 0) = \frac{P(X_2=1, X_1=1, X_0=0)}{P(X_0=0)} = \frac{P(X_2=1 \mid X_1=1, X_0=0) P(X_1=1, X_0=0)}{P(X_0=0)}$

$$= \frac{P(X_2 = 1 | X_1 = 1) P(X_1 = 1 | X_0 = 0) P(X_0 = 0)}{P(X_0 = 0)}$$

$$= P(X_2 = 1 | X_1 = 1) P(X_1 = 1 | X_0 = 0) = P_{01} P_{11} = 0{,}5 \times 0 = 0$$

b. $P(X_3 = 1, X_2 = 1 \mid X_0 = 0) = \frac{P(X_3=1, X_2,=1 X_1=*, X_0=0)}{P(X_0=0)}$

$$= \frac{P(X_3 = 1, X_2, = 1, X_1 = 0, X_0 = 0) + P(X_3 = 1, X_2, = 1, X_1 = 1, X_0 = 0) + P(X_3 = 1, X_2, = 1, X_1 = 2, X_0 = 0)}{P(X_0 = 0)}$$

$$P(X_3 = 1, X_2, = 1, X_1 = 0, X_0 = 0) =$$

$$P(X_3 = 1 | X_2 = 1) P(X_2 = 1 | X_1 = 0) P(X_1 = 0 | X_0 = 0) P(X_0 = 0)$$

$$= \ P_{00} P_{01} P_{11} P(X_0 = 0) = 0{,}5 \times 0{,}5 \times 0 \times P(X_0 = 0) = 0$$

$$P(X_3 = 1, X_2, = 1, X_1 = 1, X_0 = 0)$$

$$= P(X_3 = 1 | X_2 = 1) P(X_2 = 1 | X_1 = 1) P(X_1 = 1 | X_0 = 0) P(X_0 = 0)$$

$$= P_{01} P_{11} P_{11} P(X_0 = 0) = 0{,}5 \times 0 \times 0 \times P(X_0 = 0) = 0$$

$$P(X_3 = 1, X_2, = 1, X_1 = 2, X_0 = 0)$$

$$= P(X_3 = 1 | X_2 = 1) P(X_2 = 1 | X_1 = 2) P(X_1 = 2 | X_0 = 0) P(X_0 = 0)$$

$$= P_{02} P_{21} P_{11} P(X_0 = 0) = 0 \times 0{,}5 \times 0 \times P(X_0 = 0) = 0$$

$$P(X_3 = 1, X_2, = 1 | X_0 = 0) = \frac{P(X_0=0)(P_{00}P_{01}P_{11} + P_{01}P_{11}P_{11} + P_{02}P_{21}P_{11})}{P(X_0=0)} = 0$$

Jadi $P(X_3 = 1, X_2 = 1 \mid X_0 = 0) = 0$

Atau juga dapat diselesaikan dengan :

$P(X_3 = 1, X_2 = 1 \mid X_0 = 0) = P^2_{01}\, P_{11} = 0{,}25 \,.\, 0 = 0$

c. $$P^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0{,}5 & 0{,}5 & 0 \\ 0{,}5 & 0 & 0{,}5 \\ 0 & 0{,}5 & 0{,}5 \end{Vmatrix} \end{array} \begin{Vmatrix} 0{,}5 & 0{,}5 & 0 \\ 0{,}5 & 0 & 0{,}5 \\ 0 & 0{,}5 & 0{,}5 \end{Vmatrix} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0{,}50 & 0{,}25 & 0{,}25 \\ 0{,}25 & 0{,}50 & 0{,}25 \\ 0{,}25 & 0{,}25 & 0{,}50 \end{Vmatrix} \end{array}$$

i) $P(X_2 = 0) = P^2_{00} p_0 + P^2_{10} p_1 + P^2_{20} p_2 = 0{,}50 \times 1 + 0{,}25 \times 0 + 0{,}25 \times 0 = 0{,}5$

ii) $P(X_2 = 0, X_1 = 0) = P(X_2 = 0, X_1 = 0, X_0 = 0) + P(X_2 = 0, X_1 = 0, X_0 = 1) +$

$P(X_2 = 0, X_1 = 0, X_0 = 2)$

$= P_{00} P_{00} p_0 + P_{10} P_{00} p_1 + P_{20} P_{00} p_2 = P_{00} P_{00} 1 + P_{10} P_{00} 0 + P_{20} P_{00} 0 = 0{,}5 \times 0{,}5 =$

$0{,}25$

Soal latihan

1. Suatu rantai Markov $\{X_n\}$ dengan state 0,1,2 memiliki matriks probabilitas transisi

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{c}
\begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\left\| \begin{array}{ccc} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{array} \right\|
\end{array}
$$

a. Tentukan matriks transisi dua langkah P^2

b. $P(X_3=1|X_1=0)=$ ...?

c. $P(X_3=1|X_0=0)=$...?

Penyelesaian :

a) $P^2 = P.P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & 0.1 & 0.2 & 0.7 \\ 1 & 0.2 & 0.2 & 0.6 \\ 2 & 0.6 & 0.1 & 0.3 \end{array} \begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & 0.1 & 0.2 & 0.7 \\ 1 & 0.2 & 0.2 & 0.6 \\ 2 & 0.6 & 0.1 & 0.3 \end{array} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & 0.47 & 0.13 & 0.40 \\ 1 & 0.42 & 0.14 & 0.44 \\ 2 & 0.26 & 0.17 & 0.57 \end{array}$

b) $P(X_3 = 1|X_1 = 0) = P_{01}^2 = P^2(0,1) = 0.13$ atau secara aljabar :

$$P(X_3 = 1|X_1 = 0) = \frac{P(X_3=1,X_2=\cdot,X_1=0)}{P(X_1=0)}$$

$$\begin{aligned} P(X_3 = 1, X_2 = \cdot, X_1 = 0) \\ &= P(X_3 = 1, X_2 = 0, X_1 = 0) + P(X_3 = 1, X_2 = 1, X_1 = 0) \\ &+ P(X_3 = 1, X_2 = 2, X_1 = 0) \end{aligned}$$

$$\begin{aligned} P(X_3 = 1, X_2 = 0, X_1 = 0) &= P(X_3 = 1|X_2 = 0, X_1 = 0)P(X_2 = 0, X_1 = 0) \\ &= P(X_3 = 1|X_2 = 0)P(X_2 = 0|\ X_1 = 0)P(\ X_1 = 0) \\ &= P(X_2 = 0|\ X_1 = 0)P(X_3 = 1|X_2 = 0)P(\ X_1 = 0) \\ &= P(0,0)P(0,1)P(\ X_1 = 0) = 0.1 \times 0.2\ P(\ X_1 = 0) = 0.02\ P(\ X_1 = 0) \end{aligned}$$

$$
\begin{aligned}
P(X_3=1,X_2=1,X_1=0) &= P(X_3=1|X_2=1,X_1=0)P(X_2=1,X_1=0)\\
&= P(X_3=1|X_2=1)P(X_2=1|\,X_1=0)P(\,X_1=0)\\
&= P(X_2=1|\,X_1=0)P(X_3=1|X_2=1)P(\,X_1=0)\\
&= P(0{,}1)P(1{,}1)P(\,X_1=0) = 0.2\ \times 0.2\,P(\,X_1=0) = 0.04\,P(\,X_1=0)
\end{aligned}
$$

$$
\begin{aligned}
P(X_3=1,X_2=2,X_1=0) &= P(X_3=1|X_2=2,X_1=0)P(X_2=2,X_1=0)\\
&= P(X_3=1|X_2=2)P(X_2=2|\,X_1=0)P(\,X_1=0)\\
&= P(X_2=2|\,X_1=0)P(X_3=1|X_2=2)P(\,X_1=0)\\
&= P(0{,}2)P(2{,}1)P(\,X_1=0) = 0.7\ \times 0.1\,P(\,X_1=0) = 0.07\,P(\,X_1=0)
\end{aligned}
$$

$$
P(X_3=1,X_2=\cdot\,,X_1=0) = (\,0.02 +\ 0.04 +\ 0.07)P(\,X_1=0) = 0.13\,P(\,X_1=0)
$$

Jadi $P(X_3=1|X_1=0) = \frac{P(X_3=1,X_2=\cdot\,,X_1=0)}{P(X_1=0)} = \frac{0.13\,P(\,X_1=0)}{P(\,X_1=0)} = 0.13$

c. $P(X_3=1|X_0=0)=\ldots$

c) $P(X_3 = 1|X_0 = 0) = P^3(0,1) = 0.160$

$$P^3 = P^2.P = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & 0.47 & 0.13 & 0.40 \\ 1 & 0.42 & 0.14 & 0.44 \\ 2 & 0.26 & 0.17 & 0.57 \end{array} \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & 0.1 & 0.2 & 0.7 \\ 1 & 0.2 & 0.2 & 0.6 \\ 2 & 0.6 & 0.1 & 0.3 \end{array}$$

$$= \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & 0,313 & 0.160 & 0.527 \\ 1 & \square & \square & \square \\ 2 & \square & \square & \square \end{array}$$

Cara aljabar …… (coba sendiri)

2. Suatu partikel bergerak dalam state 0,1,2 mengikuti proses markov dengan matriks transisi sebagai berikut :

$$
\begin{array}{c}
\\
P =
\end{array}
\begin{array}{c|ccc|}
 & 0 & 1 & 2 \\
0 & 0 & 0.5 & 0.5 \\
1 & 0.5 & 0 & 0.5 \\
2 & 0.5 & 0.5 & 0
\end{array}
$$

Bila X_n menyatakan posisi partikel pada langkah ke-n, hitunglah $P(X_n=0|X_0=0)$ untuk n=0,1,2,3,4

$$P = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{array} \right\| \end{array}$$

$$P^2 = P.P = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{array} \right\| \end{array} \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{array} \right\| \end{array} = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{array} \right\| \end{array}$$

$$P^3 = P^2.P = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{array} \right\| \end{array} \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{array} \right\| \end{array} = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} 0.250 & 0.375 & 0.375 \\ 0.375 & 0.250 & 0.375 \\ 0.375 & 0.375 & 0.250 \end{array} \right\| \end{array}$$

$$P^4 = P^3 . P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{Vmatrix} 0.250 & 0.375 & 0.375 \\ 0.375 & 0.250 & 0.375 \\ 0.375 & 0.375 & 0.250 \end{Vmatrix} \end{array} \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{Vmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{Vmatrix} \end{array}$$

$$= \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{Vmatrix} 0.3750 & 0.3125 & 0.3125 \\ 0.3125 & 0.3750 & 0.3125 \\ 0.3125 & 0.3125 & 0.3750 \end{Vmatrix} \end{array}$$

Jadi :

Untuk n = 0, $P(X_0 = 0 | X_0 = 0) = 1$

Untuk n = 1, $P(X_1 = 0 | X_0 = 0) = P(0,0) = 0$

Untuk n = 2, $P(X_2 = 0 | X_0 = 0) = P^2(0,0) = 0.5$

Untuk n = 3, $P(X_3 = 0 | X_0 = 0) = P^3(0,0) = 0.25$

Untuk n = 4, $P(X_4 = 0 | X_0 = 0) = P^4(0,0) = 0.375$

Quis

- Mata Kuliah A menjadi prasyarat mata kuliah B dengan ketentuan bahwa mahasiswa boleh mengambil mata kuliah B apabila sudah pernah mengambil mata kuliah A. Jika diketahui peluang kelulusan mahasiswa dalam mengambil mata kuliah A adalah a dan peluang kelulusan mata kuliah B = b.

- Juga diketahui bahwa jika seorang mahasiswa lulus mata kuliah A maka ia akan lulus mata kuliah B adalah p, dan peluang mahasiswa tidak lulus mata kuliah B jika diketahui tidak lulus mata kuliah A adalah q maka tentukan peluang mahasiswa :
  - Lulus kedua mata kuliah tersebut!
  - Tidak lulus kedua mata kuliah tersebut!
  - Lulus salah satu mata kuliah tersebut!

Penyelesaian :

Misalkan A_0 : tidak lulus mata kuliah A ; $P(A_0) = 1 - a$

A_1 : lulus mata kuliah A : $P(A_1) = a$

B_1 : lulus mata kuliah B ; $P(B_1) = b$

B_0 : tidak lulus mata kuliah B ; $P(B_0) = 1 - b$

$P(B_1 \mid A_1) = p$

$P(B_0 \mid A_0) = q$

Maka dapat disusun matriks peluang transisi sbb :

$$P = \begin{array}{c} \\ A_0 \\ A_1 \end{array} \begin{array}{c} \begin{array}{cc} B_0 & B_1 \end{array} \\ \begin{Vmatrix} q & 1-q \\ 1-p & p \end{Vmatrix} \end{array}$$

a. $P(A_1, B_1) = P(B_1 \mid A_1) \cdot P(A_1) = p.a$

b. $P(A_0, B_0) = P(B_0 \mid A_0) \cdot P(A_0) = q.(1-a)$

c. $P(A_1, B_0) + (A_0, B_1) = P(B_0 \mid A_1) \cdot P(A_1) + P(B_1 \mid A_0) \cdot P(A_0)$

$= (1-p)\, a + (1-q)\,(1 - a)$

$= a - ap + 1 - q - a + aq$

$= 1 - ap + aq - q$

3. Rantai Markov $X_0, X_1, X_2, \ldots$ dengan matriks transisi sebagai berikut :

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{Vmatrix} \end{array}$$

Tentukan probabilitas bersyarat :

a. $P(X_3=1|X_0=0)$

b. $P(X_4=1|X_0=0)$

$$P^2 = P.P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{Vmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{Vmatrix} \end{array} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.54 & 0.26 & 0.20 \\ 0.20 & 0.36 & 0.44 \\ 0.60 & 0.10 & 0.30 \end{Vmatrix} \end{array}$$

$$P^3 = P^2.P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.54 & 0.26 & 0.20 \\ 0.20 & 0.36 & 0.44 \\ 0.60 & 0.10 & 0.30 \end{Vmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{Vmatrix} \end{array} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.478 & 0.264 & 0.258 \\ 0.360 & 0.256 & 0.384 \\ 0.570 & 0.180 & 0.250 \end{Vmatrix} \end{array}$$

$$P^4 = P^3.P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.478 & 0.264 & 0.258 \\ 0.360 & 0.256 & 0.384 \\ 0.570 & 0.180 & 0.250 \end{Vmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{Vmatrix} \end{array} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.4636 & 0.2540 & 0.2824 \\ 0.4440 & 0.2256 & 0.3304 \\ 0.5240 & 0.2220 & 0.2540 \end{Vmatrix} \end{array}$$

a. $P(X_3=1|X_0=0)=0.264$

b. $P(X_4=1|X_0=0)=0.2540$

4. Rantai Markov X_0, X_1,X_2, ... dengan matriks transisi sebagai berikut :

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{Vmatrix} \end{array}$$

Diketahui proses berawal dari state X_0=1, tentukan $P(X_2=2)$

Diketahui proses berawal dari state $X_0=1$, berarti $P(X_0=1)=1$ atau vektor distribusi awal

$$\pi_0 = \left(\pi_0(0), \pi_0(1), \pi_0(2)\right) = (0,1,0)$$

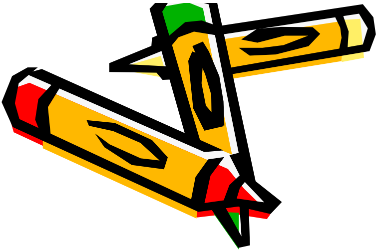
$$P^2 = P.P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array}\right\| \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array}\right\| \end{array} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0.49 & 0.28 & 0.23 \\ 0.43 & 0.22 & 0.35 \\ 0.47 & 0.20 & 0.33 \end{array}\right\| \end{array}$$

Maka:

$$P(X_2=2) = P(X_2=2|X_0=0)\pi_0(0) + P(X_2=2|X_0=1)\pi_0(1) + P(X_2=2|X_0=2)\pi_0(2)$$

$$= P_{02}^2\pi_0(0) + P_{12}^2\pi_0(1) + P_{22}^2\pi_0(2)$$

$$= 0.23 \times 0 + 0.35 \times 1 + 0.33 \times 0 = 0.35$$



- Bila diselesaikan secara aljabar:

$$P(X_2 = 2) = P(X_2 = 2, X_1 = i, X_0 = j)\ ; i, j = 0,1,2$$

Penjabaran ini seharusnya terurai menjadi 9 suku kombinasi unsur-unsur matriks transisi satu langkah dengan distribusi awal $\pi_0 = \left(\pi_0(0), \pi_0(1), \pi_0(2)\right) = (0, 1, 0)$. Karena nilai distribusi awal tersebut hanya $\pi_0(1) = 1$ yang akan memberikan nilai bukan nol, maka hanya 6 kombinasi pasti memberikan nilai nol, sehingga dapat dinyatakan sebagai:

$$P(X_2 = 2) = P(X_2 = 2, X_1 = i, X_0 = 1)\ ; i = 0,1,2$$

$$= P(X_2 = 2, X_1 = 0, X_0 = 1) + P(X_2 = 2, X_1 = 1, X_0 = 1) + P(X_2 = 2, X_1 = 2, X_0 = 1)$$
$$= \{P(X_2 = 2|X_1 = 0)P(X_1 = 0|\ X_0 = 1) + P(X_2 = 2|X_1 = 1)P(X_1 = 1|\ X_0 = 1) +$$
$$P(X_2 = 2|X_1 = 2)P(X_1 = 2|\ X_0 = 1)\}P(\ X_0 = 1)$$

$$=\{P_{10}P_{02} + P_{11}P_{12} + P_{12}P_{22}\}\pi_0(1)$$

$$=0.3 \times 0.1 + 0.3 \times 0.4 + 0.4 \times 0.5$$

$$=0.03 + 0.12 + 0.20 = 0.35$$

5. Rantai Markov X_0, X_1,X_2, … dengan matriks transisi sebagai berikut :

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{array} \right\| \end{array}$$

Tentukan probabilitas bersyarat :

a. $P(X_3=1|X_1=0)$

b. $P(X_2=1|X_0=0)$

6. Rantai Markov X_0, X_1,X_2, ... dengan matriks transisi sebagai berikut :

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{c}
\begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array} \right\|
\end{array}
$$

diketahui distribusi awal p_0=0.5 dan p_1=0.5

Tentukan

a. $P(X_2=0)=$ ...?

b. $P(X_3=0)=$...?