IISc Lecture Notes Series

Y Narahari

Game Theory and Mechanism Design

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Game Theory and Mechanism Design

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Y Narahari
Indian Institute of Science, India

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Series Preface

World Scientific Publishing Company - Indian Institute of Science Collaboration

IISc Press and WSPC are co-publishing books authored by world renowned scientists and engineers. This collaboration, started in 2008 during IISc's centenary year under a Memorandum of Understanding between IISc and WSPC, has resulted in the establishment of three Series: IISc Centenary Lectures Series (ICLS), IISc Research Monographs Series (IRMS), and IISc Lecture Notes Series (ILNS).

This pioneering collaboration will contribute significantly in disseminating current Indian scientific advancement worldwide.

The **"IISc Centenary Lectures Series"** will comprise lectures by designated Centenary Lecturers - eminent teachers and researchers from all over the world.

The **"IISc Research Monographs Series"** will comprise state-of-the-art monographs written by experts in specific areas. They will include, but not limited to, the authors' own research work.

The **"IISc Lecture Notes Series"** will consist of books that are reasonably self-contained and can be used either as textbooks or for self-study at the postgraduate level in science and engineering. The books will be based on material that has been class-tested for most part.

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Dedication

To My Beloved Parents
for giving me this wonderful life,
for teaching me the fundamentals of the game of life,
and for continuously inspiring me in this life
through their exemplary mechanisms,

and

To the 2007 Economic Sciences Nobel Laureates
Leonid Hurwicz, Eric S. Maskin, and Roger B. Myerson
for creating the wonderful edifice of mechanism design
using game theory building blocks.

Foreword

Game theory and mechanism design have come a long way. Thirty-five years ago, they were fringe subjects, taught – if at all – in specialty courses. Today they are at the center of economic theory and have become an important part of engineering disciplines such as computer science and electronic commerce.

I am very pleased that Y. Narahari has written this lovely text, which presents the fundamentals of game theory and mechanism design clearly and concisely. In doing so, Dr. Narahari has performed a great service to students and researchers interested in the lively interface between engineering sciences and economics.

Eric Maskin
Nobel Laureate in Economic Sciences - 2007
Adams University Professor
Department of Economics
Faculty of Arts and Sciences
Harvard University
Cambridge, MA, USA

16 July 2013

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Opinions on the Book

The theory of *Games and Mechanism Design* find today wide applications in Economics, Engineering, and Operations Research. This is one of the few books which present a detailed account of both Non-Cooperative and Cooperative Games as well as Mechanism Design, all under one cover. Proofs of important theorems are given in a clear and succinct manner and the bibliographical and biographical references are particularly valuable. The book can serve both as a graduate text as well as a reference volume. I highly recommend it.

– **Sanjoy K. Mitter**, Massachusetts Institute of Technology, Cambridge, MA, USA

This is a splendid book for engineers by an engineer. It has the ideal choice of topics and emphasis that reflects the driving themes in game theory, such as mechanism design, that have lead the revival of game theory in recent times and its multifarious applications in cybercommerce and allied areas. The lucidly written byte-sized chapters rich with examples and historical details make it an exciting read. This is the *right book at the right time*.

– **Vivek Borkar**, Indian Institute of Technology-Bombay, Mumbai, India

This book covers a subject which now straddles at least three subjects – Economics, Mathematics and Computer Science. It is a comprehensive presentation for a wide range of readers from the novice to experts in related areas who want to inform themselves of Game Theory and Mechanism Design. The book has a very readable from-first-principles approach to topics which commendably illuminates while not sacrificing rigor.

– **Ravi Kannan**, Microsoft Research and Indian Institute of Science, Bangalore, India

Narahari's book is a beautifully written text that handles both introductory material and advanced topics well.

– **Preston McAfee**, Google, Mountain View, CA, USA

This marvelous book on *Game Theory and Mechanism Design* is an essential reference for beginners and practitioners alike. The book covers the basic concepts needed to understand game theory and powerful practical implications of the theory embodied in mechanism design. Narahari excels at elucidating the essentials of game theory, while motivating the reader with a number of illustrative examples and real-world applications from engineering, economics and networks. It is fun to read and should be on the shelf of any student or practitioner interested in the practical applications of game theory.

– **Krishna Pattipati**, University of Connecticut, Storrs, CT, USA

Game Theory is the formal analysis of strategic behavior. It originated with the classic book of von Neumann and Morgenstern in the 1940's and over the last 70 years, has become a vital ingredient in both the social and engineering sciences. Professor Narahari is a leading expert in the burgeoning area of game theoretic applications to computer science. His lucid and elegant book, packed with examples and historical background, is a wonderful introduction to modern Game Theory. It clearly lays out the central concepts and results of the theory while conveying its potential for providing insights to a range of interesting practical problems. The book will be invaluable to students from diverse backgrounds such as economics, mathematics, and engineering.

– **Arunava Sen**, Indian Statistical Institute, New Delhi, India

Game Theory and Mechanism Design is impressive in its broad coverage of cooperative games, non-cooperative games and mechanism design from an engineering perspective. The book is rich in examples and exercises, and couples historical appraisals of the evolution of the field with careful mathematical proofs. It should be valuable both as a graduate text and for reference.

– **Chris Dance**, Xerox Research Centre Europe, Grenoble, France

About the Author

Professor Y. Narahari is currently teaching at the Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India. The focus of his research in the last decade has been to explore problems at the interface of computer science and microeconomics. In particular, he is interested in applications of game theory and mechanism design to design of auctions and electronic markets, multiagent systems, and social network research. He has coauthored a large number of influential research papers in these and other areas. Many of his doctoral and master's students have bagged best thesis prizes for their dissertations.

He is the lead author of a research monograph *Game Theoretic Problems in Network Economics and Mechanism Design Solutions* published by Springer, London, in 2009. He coauthored an acclaimed book earlier, *Performance Modeling of Automated Manufacturing Systems* (Prentice Hall, USA, 1992). He has also created a web-based teaching resource on *Data Structures and Algorithms*.

His work has been recognized through many fellowships and awards. He is an elected Fellow of the following Institutions and Academies: IEEE, New York; Indian National Science Academy; Indian Academy of Sciences; Indian National Academy of Engineering; and the National Academy of Sciences. He has been a Senior Editor of the IEEE Transactions on Automation Science and Engineering and an Associate Editor of several reputed journals. He is currently a J.C. Bose National Fellow, a recognition awarded to distinguished scientists by the Department of Science and Technology, Government of India. In 2010, he received the Institute Award for Research Excellence in Engineering at the Indian Institute of Science.

During the past 15 years, he has been an active scientific collaborator with a host of global R & D companies and research labs including General Motors R & D, IBM Research, Infosys Technologies, Intel, and Xerox Research.

The current book represents a culmination of his teaching and research efforts in game theory and mechanism design during the past decade.

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Preface

The project of writing this book was conceived and conceptualized in December 2008 during the *Centenary Conference* of the *Indian Institute of Science*, my Alma mater that has shaped my career, and life as well, for the past three and half decades. On December 16, 2008, Professor Eric Maskin who had received the 2007 Sveriges Riksbank prize (aka Nobel Prize in Economic Sciences) (jointly with Professors Leonid Hurwicz and Roger Myerson) gave a lively, lucid, enthralling, and inspirational talk entitled *Mechanism Design: How to Implement Social Goals* to an audience comprising more than 1500 scientists, engineers, and graduate students. Soon after this talk, it occurred to me that a book on game theory emphasizing not only non-cooperative games and cooperative games but also mechanism design would be valuable for engineering audience (in general) and computer science audience (in particular). I had been teaching a game theory and mechanism design course to our master's and doctoral students in computer science since 2004. This coupled with the brief but breathtakingly stimulating interaction with Professor Maskin sowed the seeds for undertaking this ambitious project of writing the book. It is therefore befitting that the book is dedicated to Professor Eric Maskin and his co-laureates Professors Leonid Hurwicz and Roger Myerson. This triumvirate, through their path-breaking work on mechanism design, have opened up this discipline to numerous powerful applications cutting across boundaries of disciplines.

Studying the rational behavior of entities interacting with each other in the context of a variety of contemporary applications such as Internet advertising, electronic marketplaces, social network monetization, crowdsourcing, and even carbon footprint optimization, has been the bread and butter of our research group here at the Game Theory Lab at the Department of Computer Science and Automation, Indian Institute of Science. Specifically, the application of game theoretic modeling and mechanism design principles to the area of Internet and network economics has been an area of special interest to the group for a decade now.

More than eight decades ago, the legendary John von Neumann played a significant role in the creation of two different exciting disciplines: *Game Theory* and *Computer Science*. Astonishingly, in the past fifteen years (1998-2013), There has been a spectacular convergence of the above two intellectual currents. The applications of game theory and mechanism design to problem solving in engineering and computer science applications have exploded in these fifteen years. This phenomenon certainly spurred us to dive into this area in the last decade.

Further, during this period, there were other developments that made sure we got locked into this area. Intel India, Bangalore, funded a collaborative project in 2000 that required the development of a multi-attribute combinatorial procurement auction for their indirect materials procurement. General Motors R & D, Warren, Michigan, next collaborated with our group to develop procurement auction mechanisms during 2002-2007. Meanwhile, Infosys Technologies, Bangalore, collaborated with us in 2006-07 on applying game theory and mechanism design to an interesting web services composition problem. The current collaboration with the Infosys team is focused on using game theory and mechanism design techniques to carbon footprint optimization. IBM India and IBM India Research Labs provided us with funding and a faculty award to make further explorations into this area. All this work culminated in a 2009 research monograph entitled *Game Theoretic Problems in Network Economics and Mechanism Design Solutions* (co-authored with my graduate students Dinesh Garg, Ramasuri Narayanam, and Hastagiri Prakash) and a string of research papers. We are also currently engaged with Xerox Research on fusing mechanism design with machine learning to extract superior performance from service markets. These projects have helped us to investigate deep practical problems, providing a perfect complement to our theoretical work in the area.

We have also been fortunate to be working in this area during an eventful period when game theorists and mechanism designers have been awarded the Nobel Prize in Economic Sciences. We were excited when Professors Robert Aumann and Thomas Schelling were awarded the Prize 2005. In fact, we had an illuminating visit by Robert Aumann in January 2007 to the Indian Institute of Science. We were delighted when, just two years later, Professors Leonid Hurwicz, Eric Maskin, and Roger Myerson were awarded the Prize in 2007 for their fundamental contributions to mechanism design theory. Finally, our excitement knew no bounds in October 2012 when Professors Lloyd Shapley and Professor Al Roth were announced as the winners of the prize for 2012.

Objectives of the Book

Set in the above backdrop, this book strives to distill the key results in game theory and mechanism design and present them in a way that can be appreciated by students at senior undergraduate engineering level and above. The book includes a number of illustrative examples, carefully chosen from different domains including computer science, networks, engineering, and microeconomics; however they are fairly generic.

There are numerous excellent textbooks and monographs available on game theory. This book has drawn inspiration from the following reference texts: Mas-Colell, Whinston, and Green [1]; Myerson [2]; Nisan, Roughgarden, Tardos, and Vazirani [3]; Shoham and Leyton Brown [4]; Straffin [5]; Osborne [6]; and the very recent book by Maschler, Solan, and Zamir [7]. The dominating theme in many of the above texts is social sciences, particularly microeconomics. Our book is different in two ways. First, it has the primary objective of presenting the essentials of game theory and mechanism design to an engineering audience. Since I happen to be from a computer science department, there is also an inevitable emphasis on computer science based applications. Second, the book has a detailed coverage of mechanism design unlike most books on game theory. A precursor to this current book is an earlier monograph by Narahari, Garg, Narayanam, and Prakash [8].

Outline and Organization of the Book

The book is structured into three parts: *Non-cooperative game theory* (Chapters 2 to 13); *Mechanism design* (Chapters 14 to 24); and *Cooperative game theory* (Chapters 25 to 31). Chapter 1 is an introduction to the book and Chapter 32 is an epilogue while Chapter 33 attempts to provide a succinct discussion of mathematical preliminaries required for understanding the contents of the book.

Each chapter commences with a motivation and central purpose of the chapter, and concludes with a crisp summary of key concepts and results in the chapter and a set of references to probe further. At the end of each chapter, a set of exercise problems is also included. In relevant chapters, programming assignments are also suggested. The book has a table of acronyms and notations at the beginning of the book. The book further contains, at relevant places, informative biographical sketches of legendary researchers in game theory and mechanism design. We now present a chapter-by-chapter outline of the book.

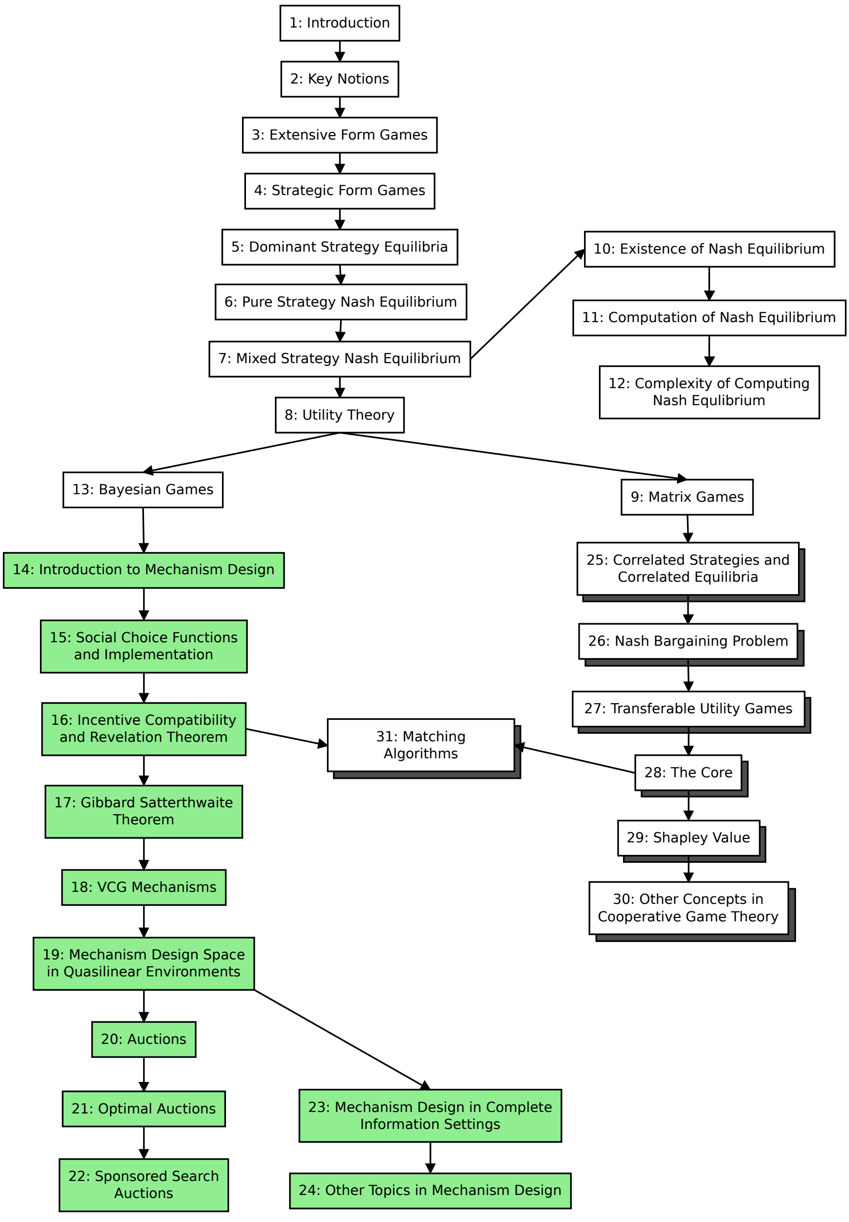
Chapter Reading Sequence

The picture appearing overleaf depicts the sequential dependency among the main chapters of the book. The rectangles corresponding to Part 1, Part 2, and Part 3 are shaded differently in the picture. The diagram is self-explanatory. Since Chapter 32 (Epilogue) and Chapter 33 (Mathematical Preliminaries) have a special purpose, they are not depicted in the diagram.

Part 1: Non-cooperative Game Theory

We first introduce, in Chapter 2, key notions in game theory such as *preferences*, *utilities*, *rationality*, *intelligence*, and *common knowledge*. We then study two representations for non-cooperative games: *extensive form representation* (Chapter 3) and *strategic form representation* (Chapter 4).

In Chapters 5,6, and 7, we describe different solution concepts which are fundamental to the analysis of strategic form games: dominant strategies and *dominant*



Reading sequence of chapters in the book

strategy equilibria (Chapter 5); *pure strategy Nash equilibrium* (Chapter 6); and *mixed strategy Nash equilibrium* (Chapter 7). In Chapter 8, we introduce the *utility theory* of von Neumann and Morgenstern which forms the foundation for game theory.

Chapters 9, 10, 11, and 12 are devoted to studies on existence and computation of Nash equilibria. In Chapter 9, we focus on two player zero-sum games. In

Chapter 10, we provide a detailed treatment of the Nash theorem that establishes the existence of a mixed strategy Nash equilibrium in finite strategic form games. Chapter 11 is concerned with algorithmic computation of Nash equilibria while Chapter 12 deals with computational complexity of finding Nash equilibria.

In Chapter 13, we introduce *Bayesian games* which are games with *incomplete information*. These games play a central role in mechanism design which is the subject of Part 2 of the book.

Part 2: Mechanism Design

Mechanism design is the art of designing games so that they exhibit desirable equilibrium behavior. In this part (Chapters 14-24), we study fundamental principles and key issues in mechanism design.

In Chapter 14, we introduce mechanisms with simple, illustrative examples and discuss the key notions of *social choice functions*, *direct mechanisms*, and *indirect mechanisms*. In Chapter 15, we bring out the principles underlying implementation of social choice functions by mechanisms. In Chapter 16, we define the important notion of incentive compatibility and bring out the difference between dominant strategy incentive compatibility (DSIC) and Bayesian incentive compatibility (BIC). We prove the *revelation theorem*, an important fundamental result. Chapter 17 is devoted to two key impossibility results: the Gibbard-Satterwaite theorem and the Arrow theorem.

Chapters 18-22 are devoted to different classes of quasilinear mechanisms which are either DSIC or BIC. In Chapter 18, we study VCG (Vickrey-Clarke-Groves) mechanisms, by far the most extensively investigated class of mechanisms. Chapter 19 is devoted to an exploration of mechanism design space in quasilinear environment, including Bayesian mechanisms. In Chapter 20, we discuss auctions which are a popular example of mechanisms. In Chapter 21, we study optimal mechanisms, in particular the Myerson auction. In Chapter 22, we study the sponsored search auction problem in detail to illustrate a compelling application of mechanism design.

In Chapter 23, we discuss *implementation in Nash equilibrium* which assumes a complete information setting. Finally, Chapter 24 provides a brief description of important advanced topics in mechanism design.

Part 3: Cooperative Game Theory

We commence our study of cooperative game theory in Chapter 25 with a discussion on *correlated strategies* and *correlated equilibrium*. The *Nash bargaining problem* represents one of the earliest and most influential results in cooperative game theory. Chapter 26 describes the problem and proves the Nash bargaining result. We introduce in Chapter 27, *multiplayer coalitional games* or *characteristic form games*. In particular, we introduce *transferable utility games* (TU games) with several illustrative examples.

Chapters 28-30 are devoted to solution concepts in cooperative game theory. In Chapter 28, we study *the core*, a central notion in cooperative game theory. The *Shapley value* is a popular solution concept that provides a unique allocation to a set of players in a cooperative game. In Chapter 29, we present the Shapley axioms and prove the existence and uniqueness of the Shapley value. In Chapter 30, we briefly study five other important solution concepts in cooperative game theory: *Stable sets*, *Bargaining sets*, *Kernel*, *Nucleolus*, and *Gately point*. Chapter 31 is devoted to the interesting topic of matching algorithms.

We conclude the book in Chapter 32 with some thoughts on how best to utilize the insights from the book. We have included, in Chapter 33, an appendix that contains key notions and results from probability theory, linear algebra, linear programming, mathematical analysis, and computational complexity, which are used in a crucial way at various points in this textbook.

Intended Audience

The primary audience for the book include: senior undergraduate, first year master's, and first year research students studying computer science, networks, communications, electrical engineering, industrial engineering and operations research, microeconomics, and management science. Researchers and industry professionals who wish to explore game theory and mechanism design in Internet and network economics applications will find the book useful. After a thorough reading of this book, we expect that readers would be able to apply game theory and mechanism design in a principled and mature way to solve relevant problems. It is our sincere hope that the book will whet the appetite of the intended audience and arouse curiosity in this exciting subject. To provide an idea of how different types of audience could potentially benefit from this book, here are several examples:

- Computer science students will be able to make forays into topical areas such as algorithmic game theory, algorithmic mechanism design, computational social choice, auctions and market design, electronic commerce, Internet monetization, social network research, and mechanism design for multiagent systems.
- Computer science, electronics, and electrical engineering students would be able to explore research areas like network protocol design, dynamic resource allocation in networked systems, design of multiagent smart grid networks, and network science.
- Industrial engineering or management science students would be in a position to undertake research in supply chain network design, logistics engineering, dynamic pricing in e-business, etc.
- Researchers on inter-disciplinary topics such as cyberphysical systems, intelligent transportation, service science, green supply chains, and human

computation systems (such as crowdsourcing networks) would be able to formulate and solve topical problems using the tools covered in this book.

Possible Course Offerings

I have taught for several years a course on game theory to master's and doctoral students at the Indian Institute of Science, Bangalore. About 80 percent of the students have been from a computer science background with the rest of the students drawn from communications, electrical engineering, and management. In fact the book can be considered as a culmination of my lovely experience with this course spread over a number of years. The lecture notes of the course have survived the scrutiny of the talented students and in fact many of the students have contributed to this book by providing critical comments and suggestions. The course taught by me typically covers about 60 percent each of the contents in Part 1 (Non-cooperative game theory); Part 2 (Mechanism design); and Part 3 (Cooperative game theory).

With a judicious selection of topics, it is possible to design several courses based on this book. We provide three possibilities below.

Undergraduate Level Course on Game Theory

To an audience consisting of third year or fourth year undergraduate students, the following collection of topics would make an interesting course.

- *Non-cooperative game theory*: Chapter 1, Chapter 2, Chapter 3, Chapter 4, Chapter 5, Chapter 6, Chapter 7, Chapter 9.
- *Cooperative game theory*: Parts of Chapter 25, Chapter 26, Chapter 27, Chapter 28, Chapter 29, Chapter 31.
- *Mechanism design* (optional): Parts of Chapter 13, Chapter 14, Chapter 15, Chapter 16, Chapter 20

Master's Level Course on Game Theory

To an audience consisting of final year undergraduate students, master's students, and first year graduate students, the entire book would be relevant. To cover the entire book as a one semester course would be challenging, so a judicious choice of topics will be the key.

Graduate Level Course on Game Theory

About 40 percent material of a graduate level course could be covered from this book. If the students have already gone through an undergraduate level course on game theory (as explained above), then the remaining chapters of this book (especially Chapter 10, Chapter 11, Chapter 12, Chapter 13, all chapters in mechanism design (Chapters 14-24), and all chapters in cooperative game theory (Chapters 25-31) would provide the initial content. Appropriate material from advanced books and from the current literature should complement and complete such a course offering.

Convention in Usage of Certain Common Words and Phrases

We wish to draw the attention of the readers regarding use of certain words and phrases. We use the words *players* and *agents* interchangeably throughout the text. The words *bidders*, *buyers*, and *sellers* are often used to refer to players in an auction or a market. The words *he* and *his* are used symbolically to refer to both the genders. This is not to be mistaken as gender bias. Occasionally we have also used the words *she* and *her*. We have also sporadically used the words *it* and *its* while referring to players or agents.

Supplementary Resources

The URL http://lcm.csa.iisc.ernet.in/hari/book.html will take the interested readers to supplementary material which would be continuously updated. The material includes additional references, additional problems, solutions to selected exercises, and viewgraphs for selected chapters.

Feedback is Welcome!

No book is flawless. We invite you to report any flaws and provide your valuable comments and suggestions by sending email to me at hari@csa.iisc.ernet.in. We would be delighted to post the clarifications on the website at the URL mentioned above.

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It is my pleasant duty to recall the exemplary support I have received from numerous individuals and organizations. It is a true privilege and pleasure to be associated with the Indian Institute of Science, Bangalore. I salute this magnificent temple of learning with all devotion and humility. I would like to thank the Institute Director Professor P. Balaram and the Associate Director Professor N. Balakrishnan for their fabulous support and encouragement. Professor Balaram encouraged me to bring out a special section on game theory in *Current Science*, a journal that he edited with great distinction for more than a decade. The special section appeared in November 2012 and had a foreword by Professor Eric Maskin. Professor Balakrishnan has been wonderfully supportive all these years.

Similarly, the Department of Computer Science and Automation (CSA) has been a paradise for me. Professor Viswanadham (currently a senior distinguished professor at the department) who was my master's and doctoral adviser during 1983-88 in CSA has been my friend, philosopher, and guide at all times ever since 1983. He has provided rock solid support to me in all my academic endeavors and struggles. He is directly responsible for imbibing in me the culture of writing books with contemporary content and his positive influence can be seen in many parts of this book.

I would like to remember the support and encouragement of all colleagues and staff at the Department of Computer Science and Automation. I like to specially mention Professors V.V.S. Sarma, V. Rajaraman, U.R. Prasad, C.E. Veni Madhavan, M. Narasimha Murty, and Y.N. Srikant who all provided encouragement to me whenever I needed it most.

My forays into and explorations in game theory and mechanism design have been largely due to a string of collaborative projects starting with Intel India in 2000. I thank General Motors R & D, Warren, Michigan, and the General Motors India Science Lab, Bangalore for their wonderful support during the past eight years. I must thank, for their splendid support, Intel India, Bangalore, during 2000-2003; Infosys Technologies, Bangalore, during 2007-13; the Office of Naval Research, Arlington, Virginia, during 2007-08; IBM India and IBM India Research Labs during 2009-11; and Xerox Research (2009-2013). I also would like to thank the Homi Bhabha Fellowships Council, Mumbai, for awarding me a fellowship during 2006-07. My special thanks to the Department of Science and Technology for awarding me me the prestigious J.C. Bose Fellowship for the duration 2010-15. Such fellowships go a long way in lifting the spirits of academics.

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Acronyms

AE	Allocatively efficient (or allocative efficiency)
BB	Budget balance
dAGVA	d'Aspremont, Gérard-Varet, and Arrow (mechanism)
DSE	Dominant strategy equilibrium
DSIC	Dominant strategy incentive compatible
EFG	Extensive form game
EPE	Ex-Post efficient
EPIC	Ex-Post incentive compatible
EPIR	Ex-Post individually rational
GSP	Generalized second price (mechanism)
GST	Gibbard - Satterthwaite Theorem
GVA	Generalized Vickrey auction
IC	Incentive compatible
IIR	Interim individually rational
IR	Individually rational
LP	Linear program
MSNE	Mixed strategy Nash equilibrium
NBS	Nash bargaining solution
ND	Non-dictatorial (social choice function)
NE	Nash equilibrium
NTU	Non-transferable utility (game)
PSNE	Pure strategy Nash equilibrium
SBB	Strong budget balance
SCF	Social choice function
SDSE	Strongly dominant strategy equilibrium
SFG	Strategic form game
SGPE	Subgame perfect equilibrium
SSA	Sponsored search auction
TU	Transferable utility (game)
VWDSE	Very weakly dominant strategy equilibrium
WBB	Weak budget balance
WDSE	Weakly dominant strategy equilibrium
VCG	Vickrey-Clarke-Groves (mechanism)

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Symbols and Notations

General Notation

$\mathbb{R}$	Set of all real numbers
$\mathbb{R}_+$	Set of all non-negative real numbers
$\mathbb{N}$	Set of all non-negative integers
$\emptyset$	Empty set
$\lvert A \rvert$	Cardinality of set A
2^A	Power set of a set A
$d(x, y)$	Euclidean distance between vectors x and y

Strategic Form Games

Γ (strategic form game)	$\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ or $\langle N, (S_i), (u_i) \rangle$
n	Number of players in the game
N	A set of players, $\{1, 2, \ldots, n\}$
S_i	Set of actions or pure strategies of player i
$(S_i)_{i \in N}$	Short form for $(S_1, \ldots, S_n)$
(S_i)	Short form for $(S_i)_{i \in N}$ (when context is clear)
S	Set of all pure strategy profiles $= S_1 \times \ldots \times S_n$
s	A strategy profile, $s = (s_1, \ldots, s_n) \in S$
$-i$	All players other than i
s_{-i}	A profile of strategies of agents except i
(s_i, s_{-i})	Another representation for profile $(s_1, \ldots, s_n)$
S_{-i}	Set of all strategy profiles of all agents except i
u_i (strategic form game)	Utility function of player i; $u_i : S \to \mathbb{R}$
$(u_i)_{i \in N}$	Short form for $(u_1, \ldots, u_n)$
(u_i)	Short form for $(u_i)_{i \in N}$ (when context is clear)
s_i^*	An equilibrium strategy of player i
$s^* = (s_1^*, \ldots, s_n^*)$	An equilibrium strategy profile
b_i (pure strategies)	Best response correspondence of player i $b_i : S_{-i} \to 2^{S_i}$

Extensive Form Games

Γ (extensive form game)	$\langle N, (A_i)_{i\in N}, \mathbb{H}, P, (\mathbb{I}_i)_{i\in N}, (u_i)_{i\in N}\rangle$
A_i (extensive form game)	Set of actions of players i
$\mathbb{H}$	Set of all terminal histories
$S_{\mathbb{H}}$	Set of proper subhistories of terminal histories
$P : S_{\mathbb{H}} \to N$	Mapping of proper subhistories to players
$\mathbb{I}_i$	Set of all information sets of player i
u_i (extensive form game)	Utility function of player i; $u_i : \mathbb{H} \to \mathbb{R}$
$s_i(\cdot)$ (Extensive form game)	A strategy of player i; $\quad s_i : \mathbb{I}_i \to A_i$

Mixed Strategy Games

$\Delta(S_i)$	Set of all probability distributions on the set S_i
σ_i	A mixed strategy of player i; $\sigma_i \in \Delta(S_i)$
$\sigma = (\sigma_1, \ldots, \sigma_n)$	A mixed strategy profile
σ_{-i}	A mixed strategy profile of agents except i
(σ_i, σ_{-i})	Another representation for profile $(\sigma_1, \ldots, \sigma_n)$
σ_i^*	An equilibrium mixed strategy of player i
$\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$	A mixed strategy Nash equilibrium
u_i (mixed strategies)	Utility of player i; $u_i : \Delta(S_1) \times \ldots \times \Delta(S_n) \to \mathbb{R}$
b_i (mixed strategies)	Best response correspondence of player i
$\delta(\sigma_i)$	Support of mixed strategy σ_i
$\delta(\sigma)$; $\ \sigma = (\sigma_1, \ldots, \sigma_n)$	$\delta(\sigma_1) \times \ldots \times \delta(\sigma_n)$; $\quad$ Support of strategy profile σ

Bayesian Games

Γ (Bayesian Game)	$\langle N, (\Theta_i), (S_i), (p_i), (u_i)\rangle$
Θ_i	Set of types of player i
Θ	Set of all type profiles $= \Theta_1 \times \Theta_2 \times \ldots \times \Theta_n$
S	Set of all action profiles $= S_1 \times S_2 \times \ldots \times S_n$
θ	$\theta = (\theta_1, \ldots, \theta_n) \in \Theta$; a type profile
Θ_{-i}	Set of type profiles of agents except i $= \Theta_1 \times \ldots \Theta_{i-1} \times \Theta_{i+1} \times \ldots \times \Theta_n$
θ_{-i}	$\theta_{-i} \in \Theta_{-i}$; type profile of agents except i
p_i	Belief function of player i; $\quad p_i : \Theta_i \to \Delta(\Theta_{-i})$
$\mathbb{P}$ (Bayesian game)	A common prior; $\mathbb{P} \in \Delta(\Theta)$
u_i (Bayesian game)	Utility function of player i; $u_i : \Theta \times S \to \mathbb{R}$
$s_i(\cdot)$ (Bayesian game)	A strategy of player i; $s_i : \Theta_i \to S_i$
$s(\cdot) = (s_1(\cdot), \ldots, s_n(\cdot))$	A strategy profile in a Bayesian game
$s^*(\cdot) = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$	A pure strategy Bayesian Nash equilibrium

Matrix Games

Γ	$\langle\{1,2\}, S_1, S_2, u_1, -u_1\rangle$
1	Player 1 (row player)
2	Player 2 (column player)
S_1	$\{s_{11}, s_{12}, \ldots, s_{1m}\} = \{1, 2, \ldots, m\}$
S_2	$\{s_{21}, s_{22}, \ldots, s_{2n}\} = \{1, 2, \ldots, n\}$
a_{ij}	$u_1(i,j)$ where $i \in S_1$; $j \in S_2$
A	Matrix of payoff values of player 1; $A = [a_{ij}]$
$x = (x_1, \ldots, x_m)$	A mixed strategy of row player (player 1)
$y = (y_1, \ldots, y_n)$	A mixed strategy of column player (player 2)
$\underline{v}$	Maxmin value or lower value
$\overline{v}$	Minmax value or upper value
v	Value of a matrix game when $\underline{v} = \overline{v} = v$

Mechanism Design

X	A set of alternatives or outcomes
θ_i	A preference or type of player i
$\hat{\theta}_i$	Type announced by player i
Θ_i	Set of all types of player i
Θ	$\Theta_1 \times \ \ldots \ \times \Theta_n$ (set of all type profiles)
$\Delta(\Theta)$	Set of all probability distributions on Θ
$\mathbb{P} \in \Delta(\Theta)$	A common prior distribution on type profiles
$p_i : \Theta_i \to \Delta(\Theta_{-i})$	Belief distribution of player i
$u_i : X \times \Theta_i \to \mathbb{R}$	Utility function of player i
$f : \Theta_1 \times \ldots \times \Theta_n \to X$	A social choice function
$\mathscr{D} = (\Theta_1, \Theta_2, \ldots, \Theta_n, f(.))$	A direct mechanism
$g : S_1 \times S_2 \times \ldots \times S_n \to X$	A mapping from action profiles to outcomes
$\mathscr{M} = (S_1, S_2, \ldots, S_n, g(.))$	An indirect mechanism
$\succsim$	A preference relation on a set of outcomes
$\mathscr{R}$	Set of all rational preference relations on X
$\mathscr{P}$	Set of all strict rational preference relations on X
$\succsim_i (\theta_i)$	A rational preference relation induced by u_i and θ_i
$\mathscr{R}_i$	Set of all $\succsim_i (\theta_i)$ where $\theta_i \in \Theta_i$
K	A set of project allocations (or project choices)
$k \in K$	A project choice
$k(\theta)$	Project choice when type profile is θ
t_i	Monetary transfer to player i
$t_i(\theta)$	Monetary transfer to player i when type profile is θ
$x = (k, t_1, \ldots, t_n)$	A typical outcome in quasilinear environment
$x(\theta) = (k(\theta), t_1(\theta), \ldots, t_n(\theta))$	Outcome in quasilinear setting with type profile θ
$v_i(k, \theta_i)$	Value of allocation k for player i when type is θ_i
$h_i : \Theta_{-i} \to \mathbb{R}$	Mapping used in Groves payment rule
b_i	Bid of player i in an auction

Cooperative Game Theory

$C \subseteq N$	A coalition (subset) of players
$S_C = \times_{i \in C} \; S_i$	Set of strategy profiles of players in coalition C
$S = S_N = \times_{i \in N} \; S_i$	Set of strategy profiles of all players in N
$\Delta(S_C)$	Set of all probability distributions on S_C
$\tau_C \in \Delta(S_C)$	A correlated strategy of players in coalition C
$\tau = (\tau_C)_{C \subseteq N}$	A contract signed by all players in N
$\alpha \in \Delta(S)$	A correlated equilibrium
$F \subseteq \mathbb{R}^2$	Set of feasible allocations in bargaining problem
$v = (v_1, v_2) \in \mathbb{R}^2$	Disagreement point (default point) (de facto point)
(F, v) (Nash bargaining)	Instance with feasible set F; default point v
$f(F, v)$	Nash bargaining solution $= (f_1(F, v), f_2(F, v))$
$v : 2^N \to \mathbb{R}$	Value function in a transferable utility game
(N, v)	A transferable utility game
$\Pi(N)$	Set of all permutations of $N = \{1, 2, \ldots, n\}$
$\pi \in \Pi(N)$	A typical permutation of $N = \{1, 2, \ldots, n\}$
$P(\pi, i)$	Set of all predecessors of player i in permutation π
$S(\pi, i)$	Set of all successors of player i in permutation π
$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$	A payoff allocation
$\mathbb{C}(N, v)$	The core of the transferable utility game (N, v)
$(\phi_1(N, v), \ldots, \phi_n(N, v))$	Shapley value of a transferable utility game (N, v)
$(\phi_1(v), \ldots, \phi_n(v))$	Another notation for Shapley value (N implied)
$e(C, x)$	Excess of coalition C with respect to allocation x

Contents

Chapter 1

Introduction and Overview

In this chapter, we bring out the importance and current relevance of game theory and mechanism design. The modern era, marked by magnificent advances in information and communication technologies, has created possibilities for fascinating new applications. In many of these applications, the research challenges can be effectively addressed using game theory and mechanism design. In this chapter, we describe a few motivational examples and present several modern research trends that have brought game theory and mechanism design to the forefront.

Game theory and mechanism design deal with interactions among *strategic agents*. While game theory is concerned with *analysis of games*, mechanism design involves *designing games* with desirable outcomes. Currently these are lively and active areas of research for inter-disciplinary problem solving. The central objective of this book is to gain a sound understanding of the science behind the use of game theory and mechanism design in solving modern problems in the Internet era. This book deals with three broad areas: *non-cooperative game theory*, *cooperative game theory*, and *mechanism design*.

Disciplines where game theory and mechanism design have traditionally been used include economics, business science, sociology, political science, biology, philosophy, and engineering. In engineering, it has been most widely used in industrial engineering, inventory management, supply chain management, electronic commerce, and multiagent systems. More recently, game theory has been embraced by computer science and electrical engineering disciplines in the context of many emerging applications.

1.1 Game Theory: The Science of Strategic Interactions

The term *game* used in the phrase *game theory* corresponds to an interaction involving decision makers or players who are rational and intelligent. Informally, *rationality* of a player implies that the player chooses his strategies so as to maximize a well defined individualistic payoff while *intelligence* means that players are capable enough to compute their best strategies. Game theory is a tool for logical

1	2	
	IISc	MG Road
IISc	$100, 100$	$0, 0$
MG Road	$0, 0$	$10, 10$

Table 1.1: Payoffs for the students in different situations

and mathematical analysis that models conflict as well as cooperation between the decision makers and provides a principled way of predicting the result of the interactions among the players using equilibrium analysis. Traditional games such as chess and bridge represent games of a fairly straightforward nature. Games that game theory deals with are much more general and could be viewed as abstractions and extensions of the traditional games. The abstractions and extensions are powerful enough to include all complexities and characteristics of social interactions. For this reason, game theory has proved to be an extremely valuable tool in social sciences in general and economics in particular. While *game theory* focuses on analysis of games, *mechanism design* is concerned with design of games to obtain desirable outcomes - mechanism design could be described as reverse engineering of games. In the sequel, whenever there is no need for emphasis, we use the single phrase *game theory* instead of the phrases *game theory* and *mechanism design*.

Value of Game Theory and Mechanism Design

We provide four simple, stylized examples which bring out the value of game theory and mechanism design in modeling situations of conflict and cooperation among strategic agents. These examples are abstractions of representative real-world situations and applications.

Student Coordination Problem

Imagine two typical students (call them 1 and 2), say belonging to the Indian Institute of Science (IISc), Bangalore, who are close friends. The students derive utility by spending time together either studying (in IISc) or going to the MG Road (Mahatma Gandhi Road, a location in Bangalore, frequented by young students seeking entertainment). Thus to spend time together, they have two options (or strategies): IISc and MG Road. If both of them are in IISc, each one gets a payoff of 100. If both of them go to MG Road, each gets a payoff of only 10. If one of them remains in IISc and the other goes to MG Road, the payoff is 0 for each. The payoffs are shown in Table 1.1 and are self-explanatory. Suppose the two friends have to choose their strategies simultaneously and independently of each other. Being rational and intelligent, each one would like to select the best possible strategy. It is clear that both opting for IISc is the best possible outcome and both opting for MG Road is also fine though clearly worse than both opting for IISc. The worst happens when they choose different options since each ends up with zero utility.

Game theory helps us with a principled way of predicting the options that would be chosen by the two students. In this case, the outcome of both opting for IISc and the outcome of both opting for MG Road can be shown to be what are called *Nash equilibria* which are strategy profiles in which no player is better off by unilaterally deviating from her equilibrium strategy. Game theory also provides one more prediction for this game which on the face of it is counter-intuitive but represents an equilibrium outcome that the students will not be averse to playing. This outcome which is technically called a *mixed strategy Nash equilibrium* corresponds to the situation where each student chooses IISc with probability $\frac{1}{11}$ and MG Road with probability $\frac{10}{11}$. This perhaps explains why some students are found mostly in MG Road and rarely in IISc!

The above game which is often called the *coordination game* is an abstraction of many social and technical situations in the real world. We will not get into details here but only leave the comment that game theory enables a scientific way of predicting the outcome of such interactions among decision makers.

Braess Paradox

We now illustrate the Braess paradox which is named after the German mathematician Dietrich Braess. This paradox is usually associated with transportation networks and brings out the counter-intuitive fact that a transportation network with extra capacity added may actually perform worse for commuters (in terms of time delays) than without the extra capacity. The game that we describe here is developed on the lines presented in the book by Easley and Kleinberg [1].

Figure 1.1 shows a network that consists of a source S and a destination T, and two intermediate hubs A and B. All vehicles traveling from S can go via hub A or hub B. Suppose, regardless of the number of vehicles on the route, it takes 25 minutes to travel from S to B or from A to T. On the other hand, the travel time from S to A is $\frac{m}{50}$ minutes where m is the number of vehicles traveling on that link. Similarly, the travel time from B to T is $\frac{m}{50}$ minutes where m is the number of vehicles on that link.

Suppose we now introduce an additional fast link from A to B to ease the congestion in the network (as a degenerate case, we will assume the travel time from A to B to be zero). Figure 1.2 depicts this new network with an extra link added from A to B. Now a vehicle can go from S to T in three different ways: (1) S to A to T; (2) S to B to T; and (3) S to A to B to T. The users of this network would be happier if the time to travel from S to T is lower. Intuition tells us that the second configuration where we have an additional link should make the users happier. However, game theoretic analysis proves, using equilibrium analysis, that the first configuration is in fact better for the users.

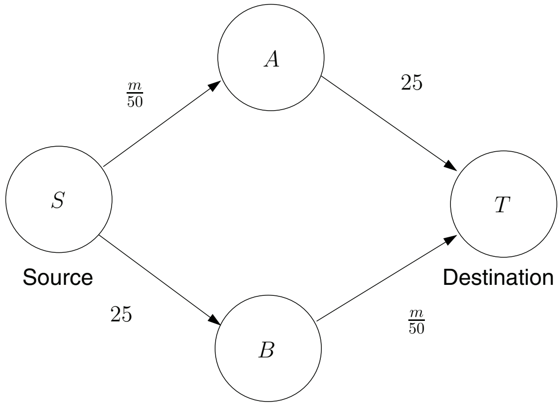

Fig. 1.1: A transportation network with four nodes

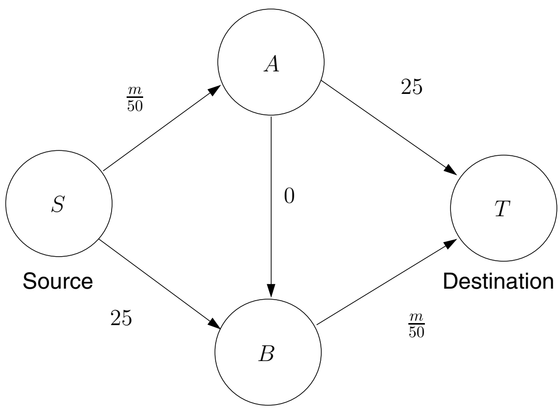

Fig. 1.2: Transportation network with an additional high speed link from A to B

There is considerable evidence for the Braess paradox. For example, in Seoul, South Korea, traffic congestion around the city dramatically reduced when a particular high speed arterial link was closed for traffic as a part of the Cheonggyecheon restoration project. In Stuttgart, Germany, a huge investment was made in decongesting the traffic on the roads by building additional roads but the traffic situation improved only when some of the newly-built roads were closed for traffic. Game theory could be used to obtain scientific predictions of what is likely to happen, by modeling the situation as a game involving the users of the transportation network and capturing their interactions. In chapters 4, 5, and 6, we study this example in some detail.

Vickrey Auction

Consider a seller who wishes to allocate an indivisible item to one of n prospective buyers in exchange for a payment. An example would be the sale of a spectrum license by the Government to one of several telecom service providers seeking to buy the license (See Figure 1.3). Each player has a certain valuation for the item on sale. For example, in the spectrum license case, imagine that there are four service providers 1, 2, 3, 4 who value the license at Rs. 400 million, Rs. 500 million, Rs. 700 million, and Rs. 1000 million. In a spectrum auction, the Government invites bids from prospective buyers and allocates the license based on an auction protocol. Two simple and common auction methods are *first price sealed bid auction* and *second price sealed bid auction*. In the first price auction, the one who bids highest will be allocated the item and the winning bidder will pay an amount equal to the bid. In the second price auction, the one who bids highest will be allocated the item but the winning bidder will pay an amount equal to the second highest bid.

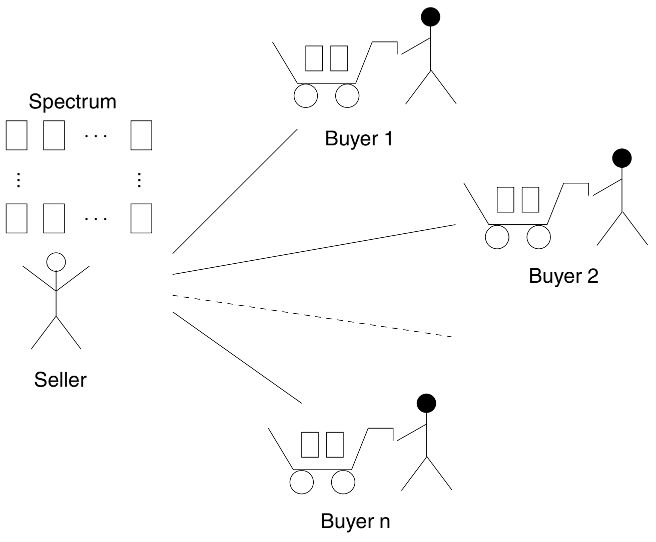

Fig. 1.3: A spectrum auction

Each auction above can be modeled as a game involving the seller and the buyers. In the first price auction, the bidders will bid amounts which are less than their valuations. In the second price auction, the bidding will be more aggressive since the bidders know that they would be paying less than what they bid in case they win. William Vickrey, in his Nobel prize winning work, proved the remarkable result that the bids in the second price auction will be exactly equal to the respective valuations. In fact, Vickrey showed that it is best for every bidder to bid her true valuation irrespective of whatever is bid by the other players. In the example above, if second price auction is employed, then the players will bid their valuations and the license will be awarded to player 4. This player will pay an amount equal to Rs. 700 million which is the second highest bid. Thus the seller who does not know the valuations of the bidders is able to extract these valuations in the form of their bids. Game theory and mechanism design constitute the science behind the design of a whole gamut of auction protocols which are ubiquitous and extensively used these days.

Divide the Dollar Game

Suppose there are three individuals who wish to divide a total wealth of 300 among themselves. Each player can propose an allocation such that no player's payoff is negative and the sum of all the payoffs does not exceed 300. Assume that if two or more players propose the same allocation, then that allocation will be implemented. For example, if players 1 and 2 propose an allocation $(150, 150, 0)$ and player 3 proposes $(100, 100, 100)$, the allocation $(150, 150, 0)$ will be implemented. However, player 3 may tempt player 2 with the allocation $(0, 225, 75)$ and if players 2 and 3 propose this, the original allocation $(150, 150, 0)$ gets overturned. Note that this allocation is strictly better for both 2 and 3. Player 1 may now entice player 3 and jointly propose with player 3 an allocation $(200, 0, 100)$ which is better for both 1 and 3. Bargaining of this kind can be never ending leading to the perpetual breaking and making of coalitions. This is a situation that is common in the real world (for example in politics and business).

Predicting the final outcome in such situations is hard using conventional techniques. Cooperative game theory helps us analyze such situations in a systematic and scientific way. For example, by modeling the above as a cooperative game, one can show that the *core* of this game is empty implying that none of the allocations is stable and can always by derailed by a pair of players coming together. One can also show that the *Shapley value* of this game is $(100, 100, 100)$ which provides a fair way of allocating the wealth among the three players in this case.

Game Theory: A Rich History

Game theory, as a mathematical discipline and modeling tool, has a rich history and its foundations and advances have been the contributions of some of the most brilliant minds of the twentieth century. Figure 1.4 shows the legends who have made path breaking contributions to game theory and mechanism design. John von Neumann and Oskar Morgenstern were the principal architects of game theory in the late 1920s, 1930s, and early 1940s. Their marvelous collaboration built the foundations of game theory and yielded a monumental book entitled *The Theory of Games and Economic Behavior* [2]. This book continues to be an authentic source of early pioneering results in game theory. Following their work, several celebrated game theorists have contributed to developing game theory as the science of economics. The importance of the discipline of game theory and their contributions have been recognized through a number of Sveriges Riksbank prizes (Nobel Prize in Economic Sciences) being awarded to game theorists, including the 1994, 1996, 2005, 2007, and 2012 prizes. In fact, between 1994 and 2012, as many as 11 game theorists have been awarded the prize.

Fig. 1.4: Legends of game theory and mechanism design

John Nash, John Harsanyi, and Reinhard Selten received the prize in 1994 for their path breaking work in equilibrium analysis of games. William Vickrey won the prize in 1996 for his influential work in auction theory. In 2005, Robert Aumann and Thomas Schelling received the prize for having enhanced the understanding of conflict and cooperation through game theory analysis. In 2007, the prize was awarded to Leonid Hurwicz, Eric Maskin, and Roger Myerson for their fundamental contributions to mechanism design theory. More recently, in 2012, Lloyd Shapley and Alvin Roth have been awarded the prize for advancing the theory of stable allocations and the practice of market design. Before all these contributions, Kenneth Arrow had been awarded the prize in 1972 for his masterly work on social choice theory which had been carried out as early as 1950s. Clearly, game theory and mechanism design have held the center-stage for several decades now in the area of social sciences. The development of game theory can be truly described as one of the most significant achievements of the twentieth century since it has shown that mathematical reasoning can be applied to studying complex human interactions.

1.2 Current Trends and Modern Applications

Since the 1990s, two related threads have catapulted game theory to the centerstage of problem solving in modern times. The first thread is the emergence of theoretical research areas at the interface of game theory and varied subjects like computer science, network science, and other engineering sciences. The second thread is the natural and often compelling use of game theory in breathtaking new applications in the Internet era. In the modern era, game theory has become a key ingredient for solving problems in areas as diverse as electronic commerce and business, Internet advertising, social network analysis and monetization, wireless networks, intelligent transportation, smart grids, and carbon footprint optimization. We touch upon a few relevant current trends and modern applications.

Current Trends

To illustrate the first thread above, we allude to a lively new theoretical research area, algorithmic game theory, at the interface of game theory and computer science. The importance and limelight can be appreciated by the fact that the 2012 Gödel Prize which recognizes outstanding papers in theoretical computer science has been awarded to six researchers (Elias Koutsoupias, Christos Papadimitriou, Tim Roughgarden, Eva Tardos, Noam Nisan, and Amir Ronen) in algorithmic game theory. The award has cited three papers [3, 4, 5] which have laid the foundations in this area. Here is a brief overview of the three papers to get a quick idea of the central themes in this area.

Koutsoupias and Papadimitriou [3] introduced the key notion of *price of anarchy* in their paper entitled *Worst-case Equilibria*. The price of anarchy measures the extent to which selfish behavior by decentralized agents affects the achievement of a social optimum. In particular, the paper quantifies how much efficiency is lost due to selfish behavior on the Internet which does not have a central monitor or authority to coordinate or control the actions of its users. Their study is based on a game theoretic model of the Internet and they use the notion of Nash equilibrium to formalize the concept of price of anarchy.

The concept of price of anarchy is used by Roughgarden and Tardos [4] to study the specific problem of routing traffic in large scale transportation networks or communication networks. Their beautiful analysis explains the well known Braess's paradox (see Chapters 4 and 5) in transportation science using a game theoretic model and establishes the relationship between centrally optimized routing and selfish routing in congested networks. Through such studies, game theory becomes a valuable tool for design of routing policies and traffic networks.

The third Gödel prize winning paper by Nisan and Ronen [5] proposes a fascinating new problem domain which they call *algorithmic mechanism design*. In this paper, the authors show how game theory and mechanism design could be used to solve algorithmic problems where the inputs to the problem constitute the private information of rational and intelligent agents. Traditional computer science assumes that algorithms once designed will work as per design when executed. The computing systems that execute the algorithms will follow the rules written for them faithfully. However if self-interested participants are required to provide inputs to the computing system during the execution of the algorithm, the inputs provided to the algorithm may or may not be truthful. Making algorithms robust to manipulation by strategic agents is the central theme of algorithmic mechanism design. Algorithmic game theory is now an active research area in many leading computer science departments in the world. It represents one of many such recent research trends in which game theory is a key ingredient.

We now take a look at the second thread which has pushed game theory to the forefront of problem solving. This thread is inspired by a natural relevance of game theory to many emerging applications in the Internet era.

Some Modern Applications

Modern applications often involve the Internet which often encourages strategic behavior by the users due to its decentralized nature. Also, modern applications in the social, economic, or business domain invariably involve individuals and organizations which have their own self-interests and act strategically. To make these modern applications perform as intended in spite of the presence of strategic users in the system, one could use creative techniques offered by game theory and mechanism design as a part of system design. This explains the second trend that has pushed game theory and mechanism design to the forefront. To drive home the point that game theory has proved crucial for advancing the current art in modern day problem solving, we provide four examples below.

Matching Markets

This is a traditional problem setting that continues to throw up exciting new applications in modern times as well. Matching is the process of allocating one set of resources or individuals to another set of resources or individuals. Examples include matching buyers to sellers in a market; matching resources to tasks; matching new doctors to hospitals; matching job-seeking engineers to companies; and matching students to schools (see Figure 1.5). There are also examples with deep societal impact such as matching kidneys to patients (or in general organ donors to organ recipients). Such matching problems are broadly categorized into marriage problems and house allocation problems. In a marriage problem, the resources on each side of the market have preferences over the resources on the other side. In house allocation, only resources on one of the sides have preferences over the resources on the other side. In either case, the matching has to be accomplished so that the individual preferences are honored and performance is optimized.

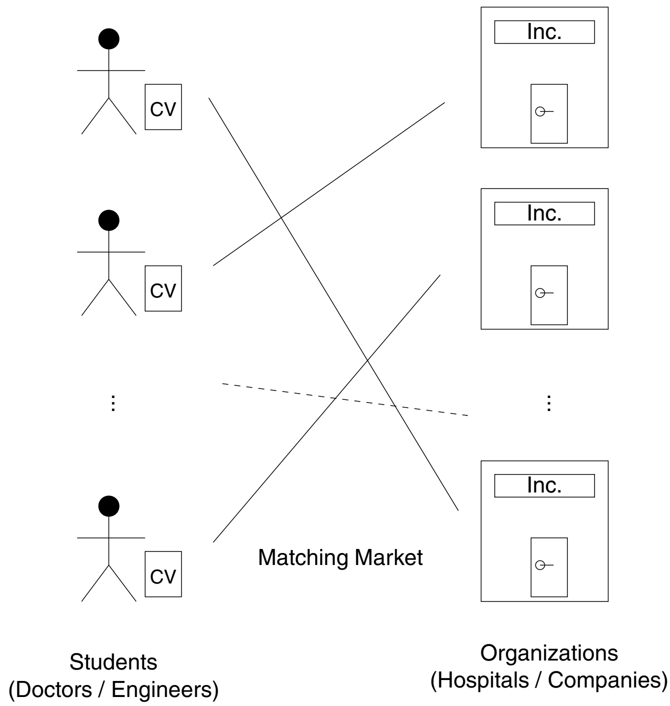

Fig. 1.5: A matching market

Two key requirements of any solution to the matching problem are *stability* and *incentive compatibility*. Informally, a solution is said to be stable if the solution cannot become strictly better through a reallocation. A solution is called incentive compatible if the preferences are reported truthfully by all the agents. Game theory has been used to analyze in a rigorous manner both stability and incentive compatibility. Since the 1960s, game theory and game theorists have contributed immensely to the development of a comprehensive theory of matching markets. The existence of a large number of successful matching markets in real world applications is one of the significant successes of game theory. In fact, the Nobel Prize in Economic Sciences for the year 2012 has been awarded to Lloyd Shapley and Alvin Roth for their pioneering work on matching theory and matching markets [6].

Matching markets have many socially important applications such as competitive matching of colleges with students and hospitals with interns, leading to maximization of social welfare. They have also saved precious human lives through better and faster matching of kidneys and human organs. Game theory and mechanism design have played a significant role in ensuring the success of these markets.

Sponsored Search Auctions

Sponsored search is by now a well known example of an extremely successful business model in Internet advertising. When a user searches a keyword, the search engine delivers a page with numerous results containing the links that are relevant to the keyword and also sponsored links that correspond to advertisements of selected advertisers. Figure 1.6 depicts a typical scenario.

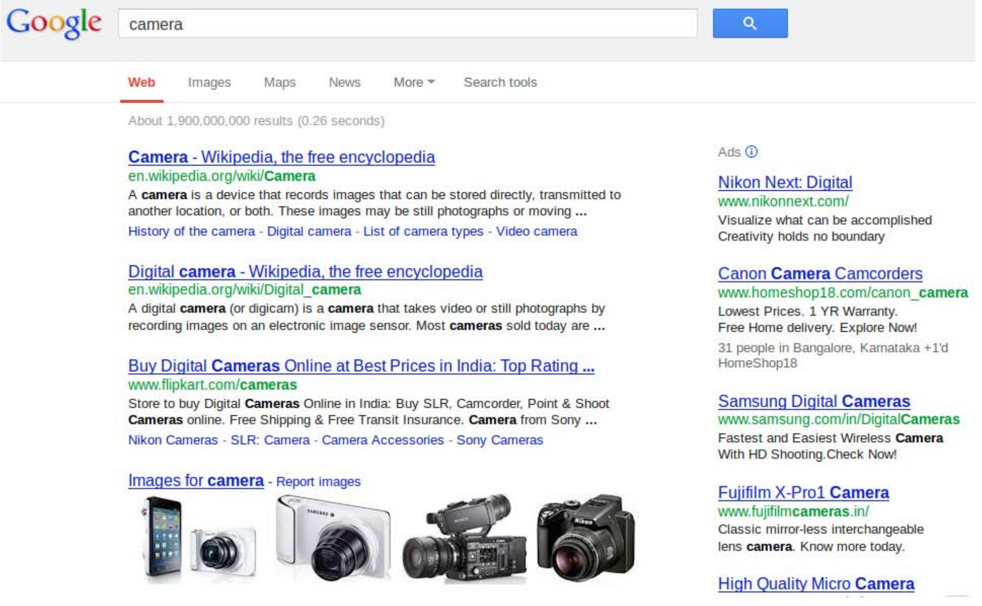

Fig. 1.6: Keyword auction on a search engine

When a sponsored link is clicked, the user is directed to the corresponding advertiser's web page. In the commonly used pay-per-click model, the advertiser makes a certain payment to the search engine for directing the user to its web page. Against every search performed by any user on any keyword, the search engine faces the problem of matching a set of advertisers to the (limited number of) sponsored slots. In addition, the search engine also needs to decide on a payment to be made by the advertiser against each click. Most search engines currently use an auction mechanism for this purpose, known as sponsored search auction. A significant percentage of the revenue of Internet giants such as Google, Microsoft, Yahoo!, etc., accrues from sponsored search auctions. In a typical sponsored search auction, advertisers are invited to specify their willingness to pay for their preferred keywords, that is, the maximum amount they would be willing to pay when an Internet user clicks on the respective sponsored slots. This willingness to pay is typically referred to as *cost-per-click*. Based on the bids submitted by the advertisers for a particular keyword, the search engine determines (1) a subset of advertisements to display; (2) the order in which the selected advertisements are displayed; and (3) the payments to be made by the selected advertisers when their respective slots are clicked by a user. The actual payment to be made depends on the bids submitted by the advertisers. The decisions (1), (2), and (3) constitute the sponsored search auction mechanism.

The search engine would typically like to maximize its revenue whereas the advertisers would wish to achieve maximum payoffs within a given budget. This leads to a classic game situation where the search engine and the advertisers are the players. The players are rational in the sense of trying to maximize their payoffs and this induces the advertisers to bid strategically after computing their best possible bids. The problem of designing a sponsored search auction mechanism becomes a problem of designing a game involving the search engine and the advertisers. The rules of the game have to be designed in a way that a well defined set of criteria would be realized by an equilibrium solution for the game.

Crowdsourcing Mechanisms

In the recent years, crowdsourcing has emerged as a major paradigm for getting work done through a large group of human resources. It can be described as distribution of work to a possibly unknown group of human resources in the form of an open call. There is a proliferation of crowdsourcing platforms in the past few years. Some of the prominent ones are Amazon Mechanical Turk, CrowdCloud, CrowdFlower, Elance, Innocentive, Taskcn, Topcoder, etc. Examples of tasks typically performed using crowdsourcing include: labeling of images, graphical design of logos, preparation of marketing plans, design of websites, developing efficient code for algorithmic business problems, classification of documents (legal documents, patents, etc.), translation services from one language to another, eliciting answers for questions, search and rescue missions in a wide geographical area, etc.

A well known crowdsourcing experiment in the recent times is the DARPA red balloon challenge which involved discovering, in as short a time as possible, 10 red balloons that were launched at ten undisclosed locations in the United States (locations shown in Figure 1.7). The total prize money was US$ 40000. The winning team from the Massachusetts Institute of Technology (MIT) employed the following mechanism. First a team of volunteers was recruited (first level volunteers) and each member of this team recruited second level volunteers. The second level volunteers recruited third level volunteers, and so on. The volunteer (say X) who first discovers a red balloon and reports it will get an incentive of US$ 2000 while the volunteer (say Y) who recruited X will get an incentive of US$ 1000, the volunteer who recruited Y will get US$ 500, and so on. The above mechanism proved highly successful and the MIT team was able to discover all ten red balloons in less than 10 hours time. The winning mechanism is an excellent example of application of game theory and mechanism design to this fascinating challenge.

In general, there are many research questions involved in deriving success out of

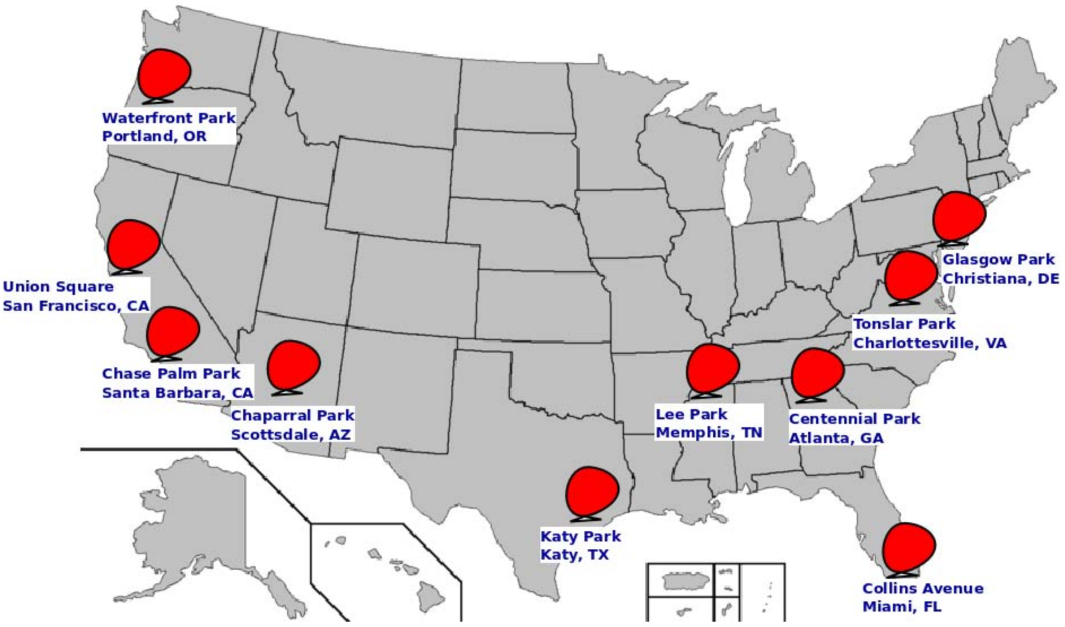

Fig. 1.7: Location of ten red balloons in the DARPA challenge

crowdsourcing. These issues include: attracting participation in required numbers, deciding on the nature and extent of incentives (cash or kind), eliciting truthful reports from the participants, and ensuring quality, timeliness, and cost-effectiveness of task execution. Game theory and mechanism design prove to be critical ingredients in designing such crowdsourcing campaigns.

Social Network Analysis

Social networks are now ubiquitous and are useful for many applications including information diffusion, electronic business, and search. Social network analysis is central to numerous Internet-based applications, for example, viral marketing, influence maximization, and influence limitation, that are based on social networks. Existing methods and tools for social network analysis have a lacuna: they do not capture the behavior (such as rationality and intelligence) of individual nodes nor do they model the strategic interactions that occur among these nodes. Game theory is a natural tool to overcome this inadequacy since it provides rigorous mathematical models of strategic interaction among autonomous, intelligent, and rational agents which form the nodes of a social network. The books by Jackson [7] and Easley and Kleinberg [1] emphasize the use of game theory in studying several social network analysis problems such as predicting topologies of social networks, modeling information diffusion, etc. For example, Figure 1.8 shows a social network in which the four most influential nodes have been identified using Shapley value, a solution concept in cooperative game theory [8]. Game theoretic approaches provide a suitable approach to designing scalable algorithms for social network analysis. Mechanism design has proved valuable in the area of social network monetization. Numerous applications using social networks have emerged in the recent times which have been enabled by the use of game theory and mechanism design.

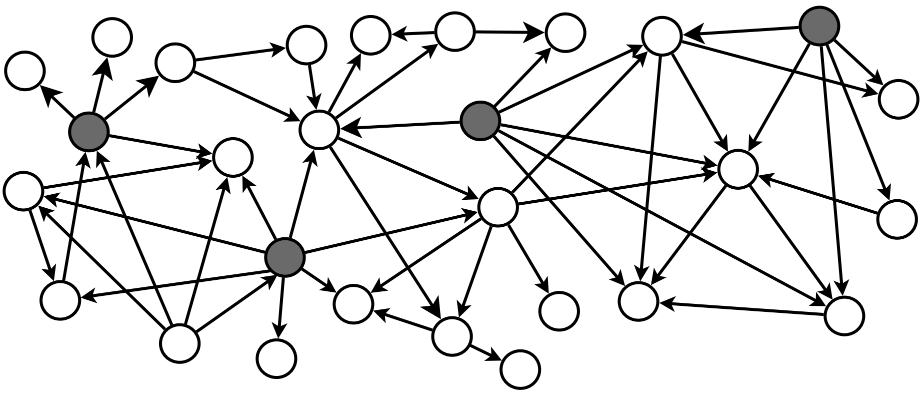

Fig. 1.8: Influential nodes in a social network

1.3 Outline of this Book

In the foregoing discussion, we have seen the increasingly influential and useful role game theory and mechanism design have come to play in inter-disciplinary research and modern applications. There is thus a heightened need to digest the foundations of game theory and mechanism design to gain a deeper understanding and appreciation of the value of game theory in the emerging applications. This textbook strives to fulfill this need.

After a thorough reading of the book, we expect that the reader will be able to use game theory and mechanism design to model, analyze, and solve centralized as well as decentralized design problems involving multiple autonomous agents that interact strategically in a rational and intelligent way. The book only assumes familiarity with an elementary course on calculus and probability. Familiarity with foundational aspects of linear algebra, real analysis, and optimization will be useful. The mathematical appendix included in Chapter 33 presents the key mathematical concepts and results that are used in the book.

There are numerous excellent textbooks and monographs available on game theory. Many of these textbooks are inspired by social sciences in general and microeconomics in particular. Our book has the primary objective of presenting the essentials of game theory and mechanism design to senior undergraduate students and above from various branches of engineering.

The book is structured into three parts:

(1) Non-cooperative game theory (Chapters 2 to 13)
(2) Mechanism design (Chapters 14 to 24)
(3) Cooperative game theory (Chapters 25 to 31)

In Part 1 (non-cooperative game theory), the chapters are devoted to key notions (such as utilities, rationality, intelligence, and common knowledge); extensive form games; strategic form games; dominant strategy equilibria; pure strategy Nash equilibria; mixed strategy Nash equilibria; utility theory; two person zero-sum games; existence theorems for Nash equilibrium (including the Nash theorem); computation of Nash equilibria; complexity of computing Nash equilibria; and Bayesian games,

Part 2 (mechanism design) is concerned with design of games. The chapters cover the following topics: building blocks of mechanisms; social choice functions and their implementation using mechanisms; notion of incentive compatibility and the equivalence of direct mechanisms and indirect mechanisms; the Gibbard-Satterthwaite theorem and the Arrow impossibility theorem; Vickrey-Clarke-Groves mechanisms; possibility and impossibility results in mechanism design; auctions and revenue equivalence theorem; optimal auctions; case study of sponsored search auctions; and mechanism implementation in ex-post Nash equilibrium.

Cooperative game theory is covered in Part 3. The chapters are devoted to correlated strategies and correlated equilibrium; Nash bargaining theory; coalitional games in characteristic form; the core of coalitional games; Shapley value; other solution concepts; and matching algorithms.

Chapter 32 (*Epilogue*) brings out the value game theory and mechanism design provide to a researcher in engineering sciences. Chapter 33 consists of a mathematical appendix that includes key concepts and results in probability, linear algebra, linear programming, mathematical analysis, and computational complexity which are often used in the textbook.

Each of the chapters commences with a motivating introduction to the chapter and concludes with a crisp summary of the chapter and a list of references to probe further. A set of problems is included in every chapter. Concepts and results are illustrated using a number of examples. These examples are carefully chosen from different domains including computer science, networks, and microeconomics; however they are fairly generic. The chapters also contain, at relevant places, informative biographical sketches of game theorists and mechanism designers who have made

We need to emphasize that our book is inspired by, and, indeed, has immensely benefited from the superb expositions available in the following books and monographs: Mas-Colell, Whinston, and Green [9]; Myerson [10]; Maschler, Solan, and Zamir [11]; Nisan, Roughgarden, Tardos, and Vazirani [12]; Shoham and Leyton-Brown [13]; Straffin [14]; and Osborne [15]. The monograph by Narahari, Garg, Narayanam, and Prakash [16] can be considered as a precursor to the current effort.

A superb collection of classic papers in game theory brought out in 1997 [17] is a must read for passionate students and researchers. We also refer the readers to a recent, very comprehensive book by Maschler, Solan, and Zamir [11].

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PART 1

NON-COOPERATIVE GAME THEORY

Informally, non-cooperative games are those in which the actions of individual players form the primitives while in cooperative games, joint actions of groups of players form the primitives. In this part, we study non-cooperative games spread over 12 chapters (Chapters 2-13).

- We first introduce, in Chapter 2, key notions in game theory such as *preferences*, *utilities*, *rationality*, *intelligence*, and *common knowledge*. We then study two representations for non-cooperative games: *extensive form representation* (Chapter 3) and *strategic form representation* (Chapter 4).
- In Chapters 5, 6, and 7, we describe different solution concepts which are fundamental to the analysis of strategic form games: dominant strategies and *dominant strategy equilibria* (Chapter 5); *pure strategy Nash equilibrium* (Chapter 6); and *mixed strategy Nash equilibrium* (Chapter 7). In Chapter 8, we introduce the *utility theory* of von Neumann and Morgenstern which forms the foundation for game theory.
- Chapters 9, 10, 11, and 12 are devoted to studies on existence and computation of Nash equilibria. In Chapter 9, we focus on two player zero-sum games. In Chapter 10, we provide a detailed treatment of the Nash theorem that establishes the existence of a mixed strategy Nash equilibrium in finite strategic form games. Chapter 11 is concerned with algorithmic computation of Nash equilibria while Chapter 12 deals with computational complexity of finding Nash equilibria.
- In Chapter 13, we introduce *Bayesian games* which are games with *incomplete information*. These games play a central role in mechanism design which is the subject of Part 2 of the book.

Chapter 2

Key Notions in Game Theory

> We commence this chapter with a discussion of a simple coordination game to illustrate the building blocks of a strategic form game. Next we introduce the readers to key notions which are fundamental to game theory. These notions include *preferences*, *utilities* or *payoffs*, *rationality*, *intelligence*, and *common knowledge*. We also include a brief discussion on different types of games.

Game theory may be defined as the study of mathematical models of interaction between rational, intelligent decision makers [1]. The decision makers are usually referred to as players or agents. The interaction may involve *conflict* as well as *cooperation*. Game theory provides general mathematical techniques for analyzing situations in which two or more players make decisions that influence one another's welfare. A game could be considered as a mathematical model of a situation where every player strives to obtain her best possible outcome, knowing fully well that all other players are also striving to obtain their respective best possible outcomes [2].

2.1 Strategic Form Games

Normal Form

Before we describe key notions in game theory, we first introduce a representation of games called *strategic form games* or *normal form games*, a very commonly used representation for games. In fact, this book mostly deals with this representation of games.

Example 2.1. Consider the example of the student coordination problem discussed in Section 1.1 of the previous chapter. For the sake of convenience, let us rename IISc as A and MG Road as B. We have two players, namely, students 1 and 2. Each of them can choose any action or strategy from the set $\{A, B\}$. They choose their individual actions simultaneously, independent of each other. Depending on the strategies chosen, the two players obtain payoffs as shown in Table 2.1. This situation motivates the following definition. □

Definition 2.1. (Strategic Form Game). A strategic form game Γ is a tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where

- $N = \{1, 2, \ldots, n\}$ is a set of players;

set of players

strategy profiles

utility / payoff.

	2	
1	A	B
A	$10, 10$	$0, 0$
B	$0, 0$	$1, 1$

Table 2.1: Payoffs for the students in different outcomes

- $S_1, S_2, \ldots, S_n$ are sets called the strategy sets of the players $1, \ldots, n$, respectively; and
- $u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$ for $i = 1, 2, \ldots, n$ are mappings called the utility functions or payoff functions.

Example 2.2. For the example being discussed, it is clear that

$$N = \{1, 2\};\ S_1 = S_2 = \{A, B\};$$

$$u_1(A, A) = 10;\ u_1(A, B) = 0; u_1(B, A) = 0;\ u_1(B, B) = 1;$$
$$u_2(A, A) = 10;\ u_2(A, B) = 0; u_2(B, A) = 0;\ u_2(B, B) = 1.$$

Note that the utilities of a player depend not only on his strategies but also on the strategies played by the other players. □

The strategies are also called *actions* or more specifically *pure strategies*. We denote by S, the Cartesian product $S_1 \times S_2 \times \cdots \times S_n$. The set S is the collection of all strategy profiles or strategy vectors (we use the phrase strategy profiles in the rest of the book) of the players. Every profile of strategies corresponds to an *outcome* in the game. We use the phrases *strategy profile* and *outcome* synonymously. Also, we use the terms *players*, *individuals*, *persons*, *decision makers*, and *agents* synonymously.

A strategic form game is a simultaneous move game that captures each agent's decision problem of choosing a strategy that will counter the strategies adopted by the other agents. Each player is faced with this problem and therefore the players can be thought of as simultaneously choosing their strategies from the respective sets $S_1, S_2, \ldots, S_n$. We can view the play of a strategic form game as follows: each player simultaneously selects a strategy and informs this to a neutral observer who then computes the outcome and the utilities. We will be presenting several examples of strategic form games in Chapter 4.

There are certain key notions which are fundamental to game theory. We now discuss these notions and a few related issues.

2.2 Preferences

The student coordination game has four outcomes, namely (A, A), (A, B), (B, A), and (B, B) which are also the four strategy profiles. Each student has certain preferences over these outcomes. Clearly, in this case, each student prefers outcome (A, A) over (B, B); prefers outcome (B, B) over outcomes (A, B) and (B, A); and has no preference between (A, B) and (B, A). The preferences that a player has over outcomes can be formalized as a *preference relation* over the set of outcomes S. We will be studying this relation formally in Chapter 8. In the current context, it is useful to know that the preference relation of each player will be reflexive, transitive, and complete (that is, every pair of outcomes is covered by the relation). Obviously, in a general situation, the preference relations of different players will be different (though in the current example, both players have the same preference relation).

preference relation → reflexive, transitive & complete.

2.3 Utilities

The utility function or payoff function of a player is a real valued function defined on the set of all outcomes or strategy profiles. The utility function of each player maps multi-dimensional information (strategy profiles) into real numbers to capture preferences. It is important to note that the utility of a player in an outcome depends not only on his own strategy but also on the strategies of the rest of the players. One can ask the question whether it is possible at all to map preference profiles to real numbers without losing any preference information. The utility theory of von Neumann and Morgenstern [3] deals with this problem in a systematic and scientific way. In fact, von Neumann and Morgenstern stated and proved in [3] a significant result that establishes that there must exist a way of assigning real numbers to different strategy profiles in a way that the decision maker would always choose the option that maximizes her expected utility. This theorem holds under quite weak assumptions about how a rational decision maker behaves. We will defer a detailed discussion of this topic to Chapter 8.

2.4 Rationality

One of the key assumptions in game theory is that the players are rational. An agent is said to be rational if the agent always makes decisions in pursuit of her own objectives. In particular, it is assumed that each agent's objective is to maximize the expected value of her own payoff measured in some utility scale. The above notion of rationality (maximization of expected utility) was initially proposed by Bernoulli (1738) and later formalized by von Neumann and Morgenstern (1944) [3].

A key observation would be that rationality implies selfishness of the agent if her utility function captures her *self-interest*. It is important to note that self-interest does not mean that each player wants to harm the other players. It also does not necessarily mean that the players only care about themselves. Self-interest only means that each player has certain individual preferences over the outcomes and the player consistently seeks to obtain these preferred outcomes. A player's preferred outcomes could include outcomes which are preferred by some other players as well. For example, if the utility function of a player captures the altruistic nature of that player, then rationality would imply altruism.

John von Neumann (1903 - 1957) is respected as one of the foremost mathematicians of the 20th century. He is regarded as the founding father of game theory. He was born in Budapest, Hungary on December 28, 1903. He was a mathematical genius from early childhood. Ironically and interestingly, however, his first major degree was in chemical engineering from the Swiss Federal Institute of Technology in Zurich. In 1926, he earned a Doctorate in Mathematics from the University of Budapest, working with Professor Leopold Fezer.

During 1926 to 1930, he taught in Berlin and Hamburg, and from 1930 to 1933, he taught at the Princeton University. In 1933, he was appointed as one of the six professors of the School of Mathematics at the Institute for Advanced Study in Princeton and he was the youngest among them. Albert Einstein and Kurt Godël were two of his distinguished colleagues at the center. During his glittering scientific career, von Neumann created several intellectual currents, two of the major ones being game theory and computer science. The fact that these two disciplines have converged during the 1990s and 2000s, almost sixty years after von Neumann brilliantly created them, is a true example of his visionary genius. In addition to game theory and computer science, he made stunning contributions to a wide array of disciplines including set theory, functional analysis, quantum mechanics, ergodic theory, continuous geometry, numerical analysis, hydrodynamics, and statistics. He is best known for his minimax theorem, utility theory, von Neumann algebras, von Neumann architecture, and cellular automata. In game theory, von Neumann's first significant contribution was the minimax theorem, which proves the existence of a *randomized saddle point* in two player zero sum games. His collaboration with Oskar Morgenstern at the Institute for Advanced Study resulted in the classic book *The Theory of Games and Economic Behavior*, which to this day continues to be an authentic source of early game theory results. This book contains a deep discussion of many fundamental notions of game theory such as utilities, saddle points, coalitional games, bargaining sets, etc. von Neumann was associated with the development of the first electronic computer in the 1940s. He wrote a widely circulated paper entitled the *First Draft of a Report on the EDVAC* in which he described a computer architecture (which is now famously called the von Neumann architecture). He is also credited with the development of the notions of a computer algorithm and algorithm complexity.

Maximizing expected utility is not necessarily the same as maximizing expected monetary returns. In general, utility and money are nonlinearly related. For example, a certain amount of money may provide different utilities to different players depending on how endowed or desperate they are.

When there are two or more players, it would be the case that the solution to each player's decision problem depends on the others' individual problems and vice-versa. When such rational decision makers interact, their decision problems have to be analyzed together, like a system of simultaneous equations [1]. Game theory provides an apt and natural mathematical framework to deal with such analysis.

Oskar Morgenstern (1902-1977) is widely known for his famous collaboration with John von Neumann which led to the celebrated book *The Theory of Games and Economic Behavior* in 1944. Their collaboration at the Institute for Advanced Study is quite legendary and was spread over 1928 - 44. The utility theory which is fundamental to game theory is rightly named after von Neumann and Morgenstern. Prior to this book, Morgenstern had authored another pioneering book *Economic Prediction*.

He also wrote a scholarly book in 1950, *On the Accuracy of Economic Observations*. In this book, he came down heavily on what he described as unscientific use of data on national income to deduce far-reaching conclusions about the state of the economy and to formulate major government policies. He is well known for applying game theory to business problems. Morgenstern was born in Germany in 1902 and studied economics in Vienna. When Adolf Hitler invaded Vienna, he was fortunately at Princeton where he continued to work until retirement. He was initially in the Princeton University and later moved to the Institute for Advanced Study at Princeton to collaborate with von Neumann. Morgenstern passed away in 1977.

2.5 Intelligence

→ can determine best response strategy

Another key notion in game theory is that of intelligence of the players. This notion means that each player in the game knows everything about the game that a game theorist knows, and the player is competent enough to make any inferences about the game that a game theorist can make. In particular, an intelligent player is *strategic*, that is, would fully take into account his knowledge or expectation of behavior of other agents in determining what his optimal response should be. We call such a strategy a *best response strategy*. Each player is assumed to have enough resources to carry out the required computations involved in determining a best response strategy.

Myerson [1] provides a convincing explanation to show that the two assumptions of rationality and intelligence are indeed logical and reasonable. The assumption that all individuals are rational and intelligent may not exactly be satisfied in a typical real-world situation. However, any theory that is not consistent with the assumptions of rationality and intelligence loses credibility on the following count:

"If a theory predicts that some individuals will be systematically fooled into making mistakes, then this theory will lose validity when individuals learn to better understand the situations." On the other hand, a theory based on rationality and intelligence assumptions would be sustainable.

resources to get best response
strategy to get max payoff

maximizing payoff

Robert Aumann is a celebrated game theorist who has made path-breaking contributions to a wide spectrum of topics in game theory such as repeated games, correlated equilibria, bargaining theory, cooperative game theory, etc. Aumann provided in 1976 [4] a convincing explanation of the notion of common knowledge in game theory, in a classic paper entitled *Agreeing to disagree* (which appeared in the Annals of Statistics). Aumann's work in the 1960s on repeated games clarified the difference between infinitely and finitely repeated games.

With Bezalel Peleg in 1960, Aumann formalized the notion of a coalitional game with non-transferable utility (NTU), a significant advance in cooperative game theory. With Michael Maschler (1963), he introduced the concept of a *bargaining set*, an important solution concept in cooperative game theory. In 1974, Aumann went on to define and formalize the notion of *correlated equilibrium* in Bayesian games. In 1975, he proved a convergence theorem for the Shapley value. In 1976, in an unpublished paper with Lloyd Shapley, Aumann provided the perfect folk theorem using the limit of means criterion. All of these contributions have advanced game theory in significant ways. His book *Values of Non-Atomic Games* (1984) co-authored with Lloyd Shapley and the book *Repeated Games with Incomplete Information* (1995) co-authored with Michael Maschler are widely regarded as game theory classics.

Aumann was born in Frankfurt am Main, Germany on June 8, 1930. He earned an M.Sc. Degree in Mathematics in 1952 from the Massachusetts Institute of Technology where he also received his Ph.D. Degree in 1955. His doctoral adviser at MIT was Professor George Whitehead Jr. and his doctoral thesis was on knot theory. He has been a professor at the Center for Rationality in the Hebrew University of Jerusalem, Israel, since 1956 and he also holds a visiting appointment with Stonybrook University, USA. Robert Aumann and Thomas Schelling received the 2005 Nobel Prize in Economic Sciences for their contributions toward a clear understanding of conflict and cooperation through game theory analysis.

Common Knowledge

A fact that is known to everyone and everyone knows that everyone knows it

The notion of common knowledge is an important implication of *intelligence*. Aumann [4] defines *common knowledge* as follows: A fact is common knowledge among the players if every player knows it, every player knows that every player knows it, and so on. That is, every statement of the form "every player knows that every player knows that ⋯ every player knows it" is true forever. If it happens that a fact is known to all the players, without the requirement of all players knowing that all players know it, etc., then such a fact is called *mutual knowledge*. In game

Common K   Mutual K.

* Model is a common knowledge if the assumption of intelligence is given.

theory, analysis often requires the assumption of common knowledge to be true; however, sometimes, the assumption of mutual knowledge suffices for the analysis. A player's *private information* is any information that the player has that is not common knowledge or mutual knowledge among any of the players.

The intelligence assumption means that whatever a game theorist knows about the game must be known to or understood by the players of the game. Thus the model of the game is also known to the players. Since all the players know the model and they are intelligent, they also know that they all know the model; they all know that they all know that they all know the model, etc. Thus the model is common knowledge.

In a strategic form game with complete information, $\langle N, (S_i), (u_i) \rangle$, the set N, the strategy sets $S_1, \ldots, S_n$, and the utility functions $u_1, \ldots, u_n$ are common knowledge, that is every player knows them, every player knows that every player knows them, and so on. We will be studying strategic form games with complete information in this and the next few chapters. We will study games with incomplete information in Chapter 13.

Example 2.3 (Common Knowledge). This example is a variant of the one presented by Myerson [1]. Assume that there are five rational and intelligent mothers A, B, C, D, and E and let a, b, c, d, and e be their daughters (or sons), respectively. The kids go to the school every day, escorted by their respective mothers and the mothers get an opportunity everyday to indulge in some conservation. The conversation invariably centers around the performance and behavior of the kids. Everyday when the five mothers meet, the conversation protocol is the following. If a mother thinks her kid is *well behaved*, she will praise the virtues of her kid. On the other hand, if a mother knows that her kid is *not well behaved*, she will cry. All mothers follow this protocol.

The fact is that none of the kids is well behaved but their behaviors are unknown to their respective mothers. However, whenever a mother finds that the kid of another mother is not well behaved, she would immediately report it to all mothers except the kid's mother. For example, if A finds b was not well behaved, then A would report it to C, D, and E, but not to B. This protocol is also known to all the mothers. Let us therefore take as fact that the knowledge that kid a is not well behaved is known to all the mothers except A (who believes that a is well behaved). Similar is the knowledge and belief about other kids' well behavedness or lack thereof.

Since each mother does not know that her kid is not well behaved, it turns out that every mother keeps praising her kid everyday. On a fine day, the class teacher meets all the mothers and makes the following statement: "at least one of the kids is not well behaved." Thus the fact that one of the kids is not well behaved is now common knowledge among all the mothers. Subsequently, when the five mothers meet the next day, all of them praise their respective kids; the same happens on the 2^{nd} day, 3^{rd} day, and the 4^{th} day. On the 5^{th} day, however, all the mothers cry together because all of them realize that their respective kids are not well behaved. The readers are urged to convince themselves why the above two statements are true.

Note that the announcement made by the class teacher is common knowledge and that is what makes all the mothers cry on the fifth day. □

2.6 Classification of Games

Any well developed subject like game theory which has been extensively explored for more than eight decades will abound in numerous kinds of games being defined and studied. We only provide a listing of some of the well known ones here.

Non-cooperative Games and Cooperative Games decision of group of individuals

primitive is individual decision

Non-cooperative games are those in which the actions of individual players are the primitives; in cooperative games, joint actions of groups of players are the primitives. John Harsanyi (1966) [5] explained that a game is cooperative if commitments (agreements, promises, threats) among players are enforceable and that a game becomes non-cooperative if the commitments are not enforceable.

It would be false to say that non-cooperative game theory applies only to situations in which there is a conflict or non-cooperation among the players. It is just that each individual player and the preferences of the player provide the basic modeling unit. In contrast, in cooperative games, the basic modeling unit is a group of players. If all groups are singletons, then we have a non-cooperative game.

Static Games and Dynamic Games extensive form

strategic form repre

In static games, players choose their actions simultaneously and no information is received during the play. An immediate example is the situation in Example 2.1 where two students simultaneously decide their strategies and receive a certain amount of reward based on the outcomes obtained. These are often called single-stage games. In a dynamic game which is often called a multi-stage game, there is a temporal order in which actions are played by the players. Typically, in a multi-stage game, a certain player chooses an action before other players do and the player knows that the choice of actions by other players will be influenced by her action. Players who choose their actions subsequently make their choices dependent on their knowledge of the actions that others have chosen. An immediate example of a dynamic game is Chess. In dynamic games, information is received and could be used by the players to plan their actions during the play of the game.

Different Representational Forms

A strategic form game (also called simultaneous move game or normal form game), which was introduced in this chapter, is a model or a situation where each player chooses the plan of action once and for all, and all players exercise their decisions simultaneously. Strategic form representation does not capture sequence of moves by the players and does not capture any information accrual to the players during the play of a game. This representation is therefore very convenient for static games. If used for dynamic games, it is to be noted that the strategic form representation could suppress the dynamics of the game.

An extensive form game specifies a possible order of events and each player can consider his plan of action whenever a decision has to be made by him. The extensive form representation can capture the sequence of moves by the players and can also capture the information accrual to the players during the play of a game. It is therefore a suitable form of representation for dynamic games. Strategic form can be considered as a static equivalent of extensive form.

A *coalitional form game* or *characteristic form game* is one where every subset of players is represented with an associated value. This form is appropriate for cooperative games.

history

Games with Perfect Information and Games with Imperfect Information

When the players are fully informed about the entire past history (each player, before making a move, knows the past moves of all other players as well as his own past moves), the game is said to be of perfect information. Otherwise it is called a game with imperfect information.

common knowlege.

Complete Information and Incomplete Information Games

A game with incomplete information is one in which, at the first point in time when the players can begin to plan their moves, some players have private information about the game that other players do not know. In a game with complete information, every aspect of the game is common knowledge.

Other Categories

There are many other categories of games, such as repeated games, evolutionary games, stochastic games, multi-level games (Stackelberg games), differential games, etc. We do not get into the details of these games in this book. We refer the reader to the books by Osborne [6] and by Maschler, Solan, and Zamir [2] for a discussion of other categories of games.

2.7 Summary and References

In this chapter, we have introduced several fundamental notions and assumptions which are key to game theory. These include: preferences, utilities or payoffs, rationality, intelligence, and common knowledge.

- Preferences of a player specify qualitatively the player's ranking of the different outcomes of the game.
- Utilities are real valued payoffs that players receive when they play different actions. The utility of a player depends not only on the action played by that player but also on the actions played by the rest of the players.
- Rationality intuitively means that players always choose their actions so as to

maximize their expected utilities. Depending on how the utility function is defined, rationality could mean self-interest, altruism, indifference, etc.

- Intelligence means that the players are as knowledgeable as game theorists and have enough computational power to compute their best response actions.
- Common knowledge is an implication of intelligence and means that all players know the entire structure of the game, all players know that all players know the game, all players know that all players know that all players know the game, etc.

Game theory is founded on the above notions. Some of the above assumptions may or may not be valid in real world situations, however the abstractions provided by game theory under the above assumptions will provide a perfect starting point for a scientific investigation into strategic situations.

The material discussed in this chapter draws mainly upon the following sources, namely the books by Myerson [1], Mas-Colell, Whinston, and Green [7], Osborne [6], Osborne and Rubinstein [8], and Maschler, Solan, and Zamir [2].

A detailed discussion of the notion of common knowledge can be found in the original paper by Aumann [4]. The book by Maschler, Solan, and Zamir [2] discusses this notion extensively with several illustrative examples.

For an undergraduate level treatment of game theory, we recommend the books by Osborne [6], Straffin [9], and Binmore [10]. For a graduate level treatment, we recommend the books by Myerson [1], Maschler, Solan, and Zamir [2], and Osborne and Rubinstein [8].

We also refer the readers to books by Rasmussen [11], Gibbons [12], Basar and Olsder [13], and Fudenberg and Tirole [14] for a scholarly treatment of game theory.

The classic treatise by John von Neumann and Oskar Morgenstern [3], published in 1944, provides a comprehensive foundation for game theory. To this day, even after many decades of its first appearance, the book continues to be a valuable reference.

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Chapter 3

Extensive Form Games

ekatrit hona

In this chapter, we study *extensive form representation* of a game, which provides a detailed representation of a game, including the sequence of moves and information accrual to players. We explain the important notions underlying extensive form representation. In this book, we mostly deal with strategic form representation; in this chapter, we bring out the connection between extensive form and strategic form representations. We show how any extensive form game can be transformed into a strategic form game.

The extensive form representation of a game provides a detailed and richly structured way to describe a game. This form was first proposed by von Neumann and Morgenstern [1] and was later refined by Kuhn [2]. The extensive form captures complete sequential play of a game. Specifically it captures (1) who makes a move at any given time (2) what actions each player may play (3) what the players know before playing at each stage (4) what the outcomes are as a function of the actions, and (5) payoffs that players obtain from each outcome. Extensive form games with a finite number of players and with a finite number of actions available to each player are depicted graphically using game trees.

3.1 Illustrative Examples

We first present several examples before we formally define an extensive form game.

Example 3.1 (Matching Pennies with Observation). In the matching pennies game, there are two players, 1 and 2, each of whom has a rupee coin. One of the players puts down his rupee coin heads or tails up. The other player sees the outcome and puts down her coin heads up or tails up. If both the coins show heads or both the coins show tails, player 2 gives one rupee to player 1 who thus becomes richer by one rupee. If one of the coins shows heads and the other coin shows tails, then player 1 pays one rupee to player 2 who becomes richer by one rupee. Depending on whether player 1 or player 2 moves first, there are two versions of this game. Figure 3.1 shows the game tree when player 1 moves first while Figure 3.2 shows the game tree when player 2 moves first. In the game tree representation, the nodes are of three types: (1) *root node* (initial decision node); (2)

internal nodes (which are decision nodes); and (3) *leaf nodes* or *terminal nodes* (which are outcome nodes). Each possible sequence of events that could occur in the game is captured by a path of links from the root node to one of the terminal nodes. When the game is played, the path that represents the sequence of events is called the *path of play*. Each decision node is labeled with the player who takes a decision at that node. Also note that each decision node can be uniquely identified by a sequence of actions leading to that decision node from the root node. The links that are outgoing at the decision node are labeled with the actions the player may select at that node. Note that each node represents not only the current position in the game but also how it was reached. The terminal nodes are labeled with the payoffs that the players would get in the outcomes corresponding to those nodes. □

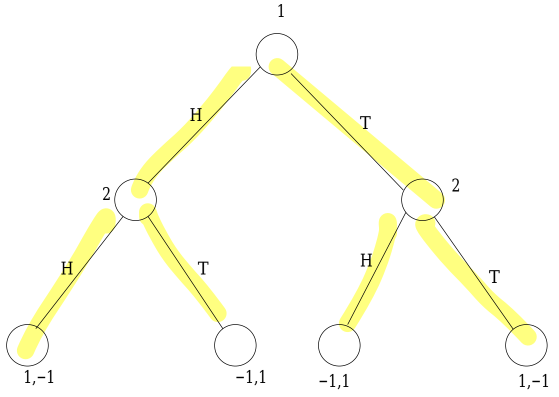

Fig. 3.1: Matching pennies game with observation when player 1 moves first

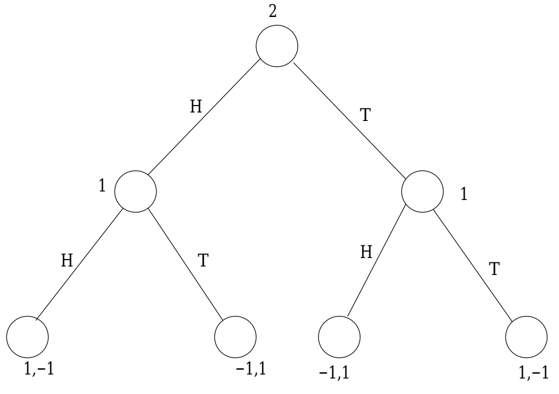

Fig. 3.2: Matching pennies game with observation when player 2 moves first

Example 3.2 (Matching Pennies without Observation). In this case, one of the players places his rupee coin heads up or tails up. The other player *does not observe the outcome* and only puts down her rupee coin heads up or tails up. Depending on whether player 1 moves first or player 2 moves first, we obtain the game tree of Figure 3.3 or Figure 3.4, respectively. Note that the game trees of Figures 3.1 and 3.3 are virtually the same except that the two decision nodes corresponding to player 2 in Figure 3.3 are connected with dotted lines. Similarly the game trees of Figures 3.2 and 3.4 are the same except that the two decision nodes corresponding to player 1 in Figure 3.4 are connected with dotted lines. A set of nodes that are connected with dotted lines is called an *information set*. When the game reaches a decision node in an information set, the player involved at that node does not know the node in the information set she is in. The reason for this is that the player cannot observe the previous moves in the game. □

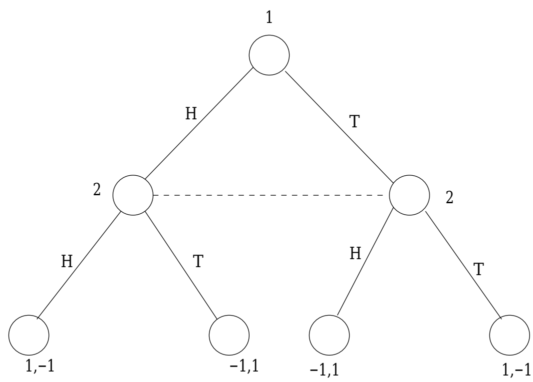

Fig. 3.3: Matching pennies game without observation when player 1 moves first

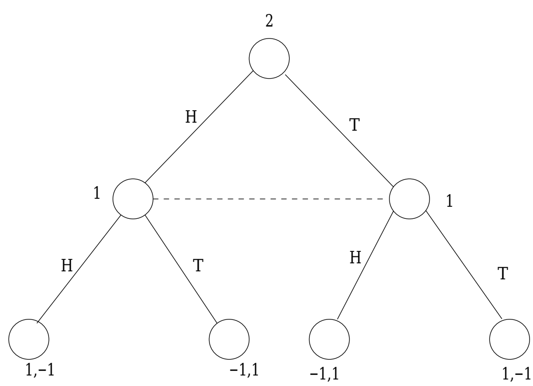

Fig. 3.4: Matching pennies game without observation when player 2 moves first

Example 3.3 (Matching Pennies with Simultaneous Play). In this version of the game, the two players put down their rupee coins simultaneously. Clearly, each player has no opportunity to observe the outcome of the move of the other player. The order of play is obviously irrelevant here. Thus both the game trees depicted in Figure 3.3 and Figure 3.4 provide a valid representation of this version of the game. □

3.2 Extensive Form Games: Definitions

We now formally define an extensive form game. This definition follows closely the one given by Osborne [3]. First we define an information set.

Definition 3.1 (Information Set). *An information set of a player is a set of that player's decision nodes that are indistinguishable to her.*

An information set of a player describes a collection of all possible distinguishable circumstances in which the player is called upon to make a move. Since each decision node corresponds uniquely to a sequence of actions from the root node to the decision node, each information set of a player consists of all proper subhistories relevant to that player which are indistinguishable to that player. Clearly, in every node within a given information set, the corresponding player must have the same set of possible actions.

Example 3.4. In the matching pennies game shown in Figure 3.3, the only information set of a player 1 is the singleton $\{\varepsilon\}$ consisting of the empty history. The information set of player 2 is the set $\{H, T\}$ that consists of the proper histories H and T which are indistinguishable to player 2. On the other hand, in the game shown in Figure 3.1, player 1 has only one information set namely $\{\varepsilon\}$ whereas player 2 has two information sets $\{H\}$ and $\{T\}$ because these two proper subhistories are distinguishable to player 2. □

Definition 3.2 (Extensive Form Game). *An extensive form game* Γ *consists of a tuple* $\Gamma = \langle N, (A_i)_{i\in N}, \mathbb{H}, P, (\mathbb{I}_i)_{i\in N}, (u_i)_{i\in N}\rangle$ *where*

- $N = \{1, 2, \ldots, n\}$ *is a finite set of players*
- A_i *for* $i = 1, 2, \ldots, n$ *is the set of actions available to player* i *(action set of player* i*)*
- $\mathbb{H}$ *is the set of all* terminal histories *where a terminal history is a path of actions from the root to a terminal node such that it is not a proper subhistory of any other terminal history. Denote by* $S_{\mathbb{H}}$ *the set of all proper subhistories (including the empty history* ε*) of all terminal histories.*
- $P : S_{\mathbb{H}} \to N$ *is a player function that associates each proper subhistory to a certain player*
- $\mathbb{I}_i$ *for* $i = 1, 2, \ldots, n$ *is the set of all information sets of player* i
- $u_i : \mathbb{H} \to \mathbb{R}$ *for* $i = 1, 2, \ldots, n$ *gives the utility of player* i *corresponding to each terminal history.*

Example 3.5. We illustrate the above definition for the matching pennies game shown in Figure 3.1.

$$N = \{1, 2\}$$

$$A_1 = A_2 = \{H, T\}$$

$$\mathbb{H} = \{(H, H), (H, T), (T, H), (T, T)\}$$

$$S_{\mathbb{H}} = \{\varepsilon, H, T\}$$ → Can be used to know the player after some actions.

$$P(\varepsilon) = 1;\ P(H) = 2;\ P(T) = 2$$

$$\mathbb{I}_1 = \{\{\varepsilon\}\};\ \mathbb{I}_2 = \{\{H\}, \{T\}\}$$

$$u_1(HH) = 1;\ u_1(HT) = -1;\ u_1(TH) = -1;\ u_1(TT) = 1$$

$$u_2(HH) = -1;\ u_2(HT) = 1;\ u_2(TH) = 1;\ u_2(TT) = -1$$

It is clear that action sets of different players can be deduced from the terminal histories and the player function. □

Note. Though the action sets of players can be deduced from terminal histories and the player function, we explicitly include action sets as a part of definition of an extensive form game for ease of understanding.

Example 3.6 (Entry Game). In this game, there are two players, 1 and 2. Player 1 is called *challenger* and player 2 is called *incumbent*. Figure 3.5 shows the game tree. Player

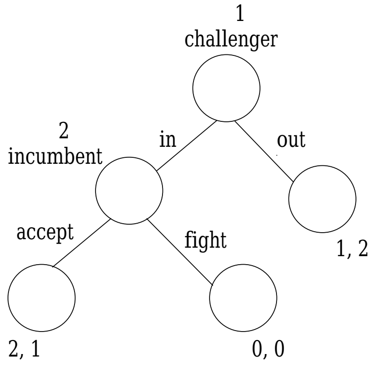

Fig. 3.5: Game tree for entry game

1 (challenger) either decides to challenge the incumbent (action: in) or drops out (action: out). Player 2 (incumbent) either decides to fight or accommodate the challenger in case the challenger decides to confront the incumbent. The respective payoffs are shown in the

game tree. For this game, we have

$$N = \{1,2\}; \quad A_1 = \{\text{in, out}\}; \quad A_2 = \{\text{accept, fight}\}$$
$$\mathbb{H} = \{(\text{in, accept}), (\text{in, fight}), (\text{out})\}$$
$$S_{\mathbb{H}} = \{\epsilon, (\text{in})\}$$
$$P(\epsilon) = 1$$
$$P(\text{in}) = 2$$
$$\mathbb{I}_1 = \{\{\varepsilon\}\}; \ \ \mathbb{I}_2 = \{\{\text{in}\}\}$$
$$u_1(\text{in, accept}) = 2; \quad u_1(\text{in, fight}) = 0; \quad u_1(\text{out}) = 1$$
$$u_2(\text{in, accept}) = 1; \quad u_2(\text{in, fight}) = 0; \quad u_2(\text{out}) = 2$$

Note again that the action sets A_1 and A_2 can be deduced from the terminal histories and the player function. □

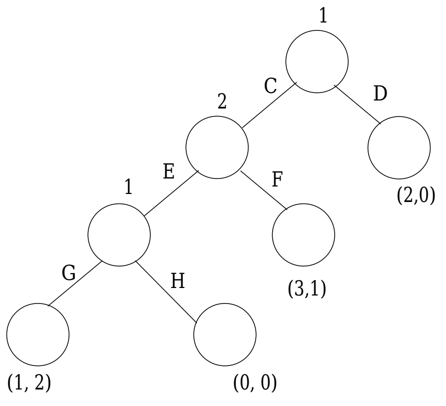

Fig. 3.6: Another game tree

Example 3.7. Consider the game tree shown in Figure 3.6. For this game, it is easy to see that

$$N = \{1,2\}; \quad A_1 = \{C, D, G, H\}; \quad A_2 = \{E, F\}$$

The terminal histories are given by

$$\mathbb{H} = \{(C, E, G), (C, E, H), (C, F), (D)\}$$

The proper subhistories of terminal histories are given by

$$S_{\mathbb{H}} = \{\epsilon, (C), (C, E)\}$$

The player function is given by

$$P(\epsilon) = 1; \quad P(C) = 2; \quad P(C, E) = 1$$

The information sets are given by

$$\mathbb{I}_1 = \{\{\varepsilon\}, \{(C, E)\}\}; \;\; \mathbb{I}_2 = \{\{(C)\}\}$$

The utility functions are given by

$$\begin{aligned} &u_1(C, E, G) = 1; \quad u_1(C, E, H) = 0; \\ &u_1(C, F) = 3; \quad u_1(D) = 2; \\ &u_2(C, E, G) = 2; \quad u_2(C, E, H) = 0; \\ &u_2(C, F) = 1; \quad u_2(D) = 0. \end{aligned}$$

This completes the description of the game shown in Figure 3.6. □

Definition 3.3 (Perfect Information and Imperfect Information Games). *An extensive form game with perfect information is one in which all the information sets are singletons. If at least one information set of at least one player has two or more elements, the game is said to be of imperfect information.*

In a game with perfect information, each player is able to observe all previous moves or the entire history thus far. Each player knows precisely where she is currently and also knows precisely how she has reached that node.

Example 3.8. As immediate examples, the games depicted in Figures 3.1, 3.2, 3.5, and 3.6 are games with perfect information while the games shown in Figures 3.3 and 3.4 are games with imperfect information. The matching pennies game with simultaneous play is obviously a game with imperfect information. □

Strategic Form N, S, u_i Extensive Form $N, A, H, P, u, \mathcal{I}$

3.3 Transforming Extensive Form to Strategic Form

The notion of a strategy is one of the most important notions in game theory. A *strategy* can be described as a complete action plan that specifies what a player will do at each of the information sets where he is called upon to play.

Recall that $\mathbb{I}_i$ denotes the set of all information sets of player i in the given game. Let A_i as usual denote the actions available to player i. Given an information set $J \in \mathbb{I}_i$, let $C(J) \subseteq A_i$ be the set of all actions possible to player i in the information set J. Then we define a strategy of a player formally as follows.

Definition 3.4 (Strategy). *A strategy s_i of player i is a mapping $s_i : \mathbb{I}_i \to A_i$ such that $s_i(J) \in C(J) \; \forall J \in \mathbb{I}_i$.*

The strategy s_i for player i is a complete contingent plan that specifies an action for every information set of the player. A strategy thus determines the action the player will choose in every stage or history of the game the player is called upon to play. In fact, the player can prepare a look-up table with two columns, one for her information sets and the other for corresponding actions; the player or a representative of the player can then take over and play the game using table look-up. Different strategies of the player correspond to different contingent plans of actions. We illustrate the notion of strategy through an example.

Example 3.9 (Strategies in Matching Pennies with Observation). Consider the game shown in Figure 3.1. We have $\mathbb{I}_1 = \{\{\varepsilon\}\}$; $\mathbb{I}_2 = \{\{H\}, \{T\}\}$. Player 1 has two strategies:

$$s_{11} : \{\varepsilon\} \to H$$

$$s_{12} : \{\varepsilon\} \to T$$

Player 2 has the following four strategies:

$$s_{21} : \{H\} \to H; \quad \{T\} \to H$$

$$s_{22} : \{H\} \to H; \quad \{T\} \to T$$

$$s_{23} : \{H\} \to T; \quad \{T\} \to H$$

$$s_{24} : \{H\} \to T; \quad \{T\} \to T$$

The payoffs obtained by the players 1 and 2 can now be described by Table 3.1. Note, for example, that when the strategy of player 1 is s_{11}, the player plays H and when the strategy of player 2 is s_{21}, the player 2 plays H, leading to the payoffs 1, -1.

1 \ 2	s_{21}	s_{22}	s_{23}	s_{24}
s_{11}	$1, -1$	$1, -1$	$-1, 1$	$-1, 1$
s_{12}	$-1, 1$	$1, -1$	$-1, 1$	$1, -1$

Table 3.1: Payoffs obtained in matching pennies with observation

The above game is a *strategic form game* equivalent of the original extensive form game. For the game shown in Figure 3.2, player 2 will have two strategies and player 1 will have four strategies and a payoff matrix such as above can be easily derived. □

Example 3.10 (Strategies in Matching Pennies without Observation). Consider the game shown in Figure 3.3. It is easy to see that $\mathbb{I}_1 = \{\{\varepsilon\}\}$ and $\mathbb{I}_2 = \{\{H, T\}\}$. Here player 1 has two strategies and player 2 has two strategies as shown below.

$$s_{11} : \{\varepsilon\} \to H$$

$$s_{12} : \{\varepsilon\} \to T$$

$$s_{21} : \{H, T\} \to H$$

$$s_{22} : \{H, T\} \to T$$

The payoff matrix corresponding to all possible strategies that can be played by the players is shown in Table 3.2.

1	2: s_{21}	s_{22}
s_{11}	$1, -1$	$-1, 1$
s_{12}	$-1, 1$	$1, -1$

Table 3.2: Payoffs obtained in matching pennies without observation

Clearly, the matching pennies game with simultaneous moves also will have the same strategies and payoff matrix as above. □

Example 3.11. Consider the game in Figure 3.6. Player 1 has four strategies given by

$$s_{11} : \{\varepsilon\} \to C; \quad \{(C, E)\} \to G$$

$$s_{12} : \{\varepsilon\} \to C; \quad \{(C, E)\} \to H$$

$$s_{13} : \{\varepsilon\} \to D; \quad \{(C, E)\} \to G$$

$$s_{14} : \{\varepsilon\} \to D; \quad \{(C, E)\} \to H$$

For the sake of convenience, let us denote the above strategies by CG, CH, DG, and DH, respectively. Player 2 has two strategies given by

$$s_{21} : \{C\} \to E$$

$$s_{21} : \{C\} \to F$$

For the sake of convenience, let us denote the above strategies by E and F, respectively. If S_1 and S_2 are the sets of strategies of players 1 and 2 respectively, it can be seen that

$$S_1 = \{CG, CH, DG, DH\}$$
$$S_2 = \{E, F\}$$

The set of strategy profiles, $S_1 \times S_2$, is given by

$$S_1 \times S_2 = \{(CG, E), (CG, F), (CH, E), (CH, F), (DG, E), (DG, F), (DH, E), (DH, F)\}$$

Note that a strategy profile uniquely determines a terminal history. For example, the profile (CG, E) corresponds to the terminal history (C, E, G); the profile (CG, F) corresponds to the terminal history (C, F); the profiles (DH, E) as well as (DH, F) correspond to the terminal history (D), etc. This example motivates the following definition. □

Definition 3.5 (Outcome). *Given an extensive form game Γ and a strategy profile $s = (s_1, \ldots, s_n)$ in the game, the outcome resulting out of the terminal history corresponding to the strategy profile s is called the outcome of s and is denoted by $O(s)$.*

Note. It is to be noted that every extensive form game has a unique strategic form representation. The uniqueness is up to renaming or renumbering of strategies. We can also immediately observe that a given strategic form game may correspond to multiple extensive form games.

Note. We have seen that any given extensive form game has an equivalent strategic form game. However, the equivalent strategic form representation may or may not contain all of the strategically relevant information present in the extensive form representation. In fact, the strategic form representation suppresses the dynamics of the game because of simultaneous play. In this book, we mostly focus on the strategic form representation. It is to be noted that dynamic games where there is sequential play as well as information accrual to the players during the play of the game warrant extensive form representation.

3.4 Summary and References

Following is a summary of salient points that we have covered in this chapter.

- Extensive form games provide a detailed representation of sequence of play and information accrual by players in a game. Finite extensive form games can be represented using game trees which consist of decision nodes and terminal nodes. Each decision node corresponds to a certain player and the player is required to choose an action in the decision node.
- An important notion in an extensive form games is that of an *information set* of a player. An information set of a player is a set of decision nodes of the player that are indistinguishable to the player (the player does not know in which of these decision nodes she is in).
- An extensive form game with perfect information is one in which all information sets of all players are singletons. This implies that at every decision node, the corresponding player has knowledge of the entire history until reaching that decision node. An extensive form game with imperfect information is one where at least one information set of at least one player is not a singleton.
- An extensive form game can be transformed into an equivalent strategic form game using the notion of a strategy. A strategy of a player is a complete action plan that specifies which action the player will choose in each of her information sets.
- A strategic form game often suppresses the dynamics of the game. However, it simplifies the analysis of games and it suffices to work with the strategic form representation for finding answers to many useful analysis questions.
- A given strategic form game could correspond to multiple extensive form games while a given extensive form game when transformed into strategic form yields a representation that is unique in structure.

Much of the material in this chapter is based on relevant discussions in the books by Osborne [3] and by Mas-Colell, Whinston, and Green [4]. In this book, we will be dealing mostly with strategic form games. In Chapter 6, we briefly return to extensive form games to introduce the notion of subgame perfect equilibrium. For a detailed treatment of extensive form games, we refer the reader to the books by Osborne [3], Myerson [5], and Maschler, Solan, and Zamir [6].

References

[1] John von Neumann and Oskar Morgenstern. *Theory of Games and Economic Behavior.* Princeton University Press, 1944.
[2] H.W. Kuhn. "Extensive form games and the problem of information". In: *Contributions to the Theory of Games II.* Princeton University Press, 1953, pp. 193–216.
[3] Martin J. Osborne. *An Introduction to Game Theory.* The MIT Press, 2003.
[4] Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green. *Microeconomic Theory.* Oxford University Press, 1995.
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3.5 Exercises

(1) You might know the tic-tac-toe game. Sketch a game tree for this game.
(2) In a game, a certain player has m information sets indexed by $j = 1, 2, \ldots, m$. There are k_j possible actions for information set j. How many strategies does the player have?
(3) For game shown in Figure 3.7, write down the terminal histories, proper subhistories, information sets, and the strategic form representation.

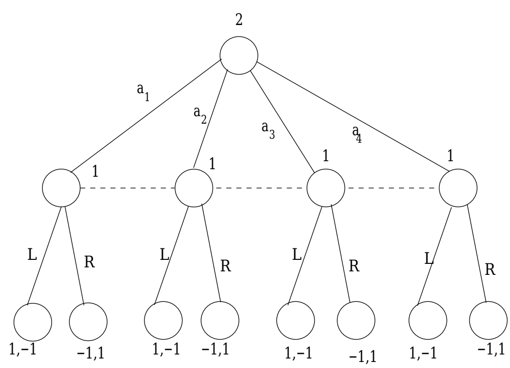

Fig. 3.7: An extensive form game

Chapter 4

Strategic Form Games

Strategic form representation is the most extensively discussed representation for games in this book. In this chapter, we provide a number of illustrative examples to help gain an intuitive understanding of strategic form games. The examples involve finite games as well as infinite games. The games discussed include matching pennies, rock-paper-scissors, Bach or Stravinsky, student's co-ordination, prisoner's dilemma, company's dilemma, duopoly pricing, tragedy of the commons, bandwidth sharing, sealed bid auction, Pigou network routing game, and Braess paradox. The examples are fairly representative of the wide canvas of applications where game theoretic modeling is relevant.

4.1 Preliminaries

We have seen in Chapter 2 (Definition 2.1) that a strategic form game Γ is a tuple $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where $N = \{1, 2, \ldots, n\}$ is a finite set of players; $S_1, S_2, \ldots, S_n$ are the strategy sets of the players; and $u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$ for $i = 1, 2, \ldots, n$ are utility functions. We have seen in Chapter 3 that games in extensive form can be transformed into strategic form games by mapping contingent action plans into strategies. The phrases *strategic form games*, *strategic games*, and *normal form games* are synonymous. When there is no confusion, we use the notation $\Gamma = \langle N, (S_i), (u_i) \rangle$ for a strategic form game.

We denote by S, the set of all strategy profiles or strategy vectors, which is the Cartesian product $S_1 \times \cdots \times S_n$. A typical strategy profile is represented by $(s_1, \ldots, s_n)$ where s_i is the strategy of player i $(i = 1, \ldots, n)$. We denote by S_{-i} the Cartesian product $S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots S_n$ of strategy sets of all players other than player i. We denote by s_{-i} a typical strategy profile in S_{-i}. When we are focusing on a particular player i, a convenient way of representing a strategy profile is (s_i, s_{-i}) where $s_i \in S_i$ and $s_{-i} \in S_{-i}$.

The idea behind the strategic form representation is that a player's decision problem is to essentially choose a strategy that will counter most effectively the strategies adopted by the other players. Such a strategy is called a *best response strategy* which is formally defined as follows.

Definition 4.1 (Best Response Strategy). *Given a strategic form game* $\Gamma = \langle N, (S_i), (u_i) \rangle$ *and a strategy profile* $s_{-i} \in S_{-i}$, *we say* $s_i \in S_i$ *is a best response strategy of player* i *with respect to* s_{-i} *if* $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \; \forall s'_i \in S_i$.

Given a strategy profile $s_{-i} \in S_{-i}$ of all players except player i, there could exist multiple best response strategies for player i.

In a strategic form game, each player is faced with the problem of choosing his best response strategy and the players can be thought of as simultaneously choosing their strategies from the respective sets $S_1, \ldots, S_n$.

Many interpretations have been provided in the game theory literature for strategic form games. We present here two interpretations provided by Osborne and Rubinstein [1].

In the first interpretation, a strategic form game is a model of an event that occurs only once. Each player knows the details of the game and the fact that all players are rational. The players choose their strategies simultaneously and independently. Each player is unaware of the choices being made by the other players.

In the second interpretation, a player is assumed to form an expectation of the other players' behavior on the basis of information about the way the game or a similar game was played in the past. A strategic form game models a sequence of plays of the game under the condition that there is no strategic link between the plays of the game. That is, a player who plays the game many times should only worry about his own instantaneous payoff and ignore the effects of his current action on the future behavior of the other players. A class of games called *repeated games* is relevant if there is a strategic link between plays of a game.

The extensive form representation, discussed in Chapter 3, is a more detailed representation than the strategic form representation. A given strategic form game could correspond to multiple extensive form games. The strategic form game suppresses the dynamics of the game but is more convenient for certain kinds of analysis of games. It is enough to work with strategic form for finding answers to many useful analysis questions.

4.2 Matching Pennies with Simultaneous Moves

We have already studied this game in Chapter 3. Recall that in this game, two players 1 and 2 put down their respective rupee coins, heads or tails up. If both the coins match (both heads or both tails), then player 2 pays one rupee to player 1. Otherwise, player 1 pays one rupee to player 2. Let us say A denotes *heads* and B denotes *tails*. It is easy to see that:

$$N = \{1, 2\}$$
$$S_1 = S_2 = \{A, B\}$$
$$S = S_1 \times S_2 = \{(A, A), (A, B), (B, A), (B, B)\}$$

The payoff matrix is given by

	2	
1	A	B
A	$1, -1$	$-1, 1$
B	$-1, 1$	$1, -1$

This is perhaps the simplest example of a two player game. This belongs to a class of games called two player zero-sum games (so called because the sum of utilities in every outcome is equal to zero). In the above game, it is easy to see that the best response strategy of player 1 is A (B) when player 2 plays A (B). On the other hand, the best response strategy of player 2 is A (B) when player 1 plays B (A).

To understand why a simple game such as the above is important to study, let us consider the following situation. There are two companies, call them 1 and 2. Each company is capable of producing two products A and B, but at any given time, a company can only produce one product, due to high setup and switchover costs. Company 1 is known to produce superior quality products but company 2 scores over company 1 in terms of marketing and advertising. If both the companies produce the same product (A or B), it turns out that company 1 makes all the profits and company 2 loses out, because of the superior quality of products produced by company 1. This is reflected in our model with a payoff of $+1$ for company 1 and a payoff of -1 for company 2, corresponding to the strategy profiles (A, A) and (B, B).

On the other hand, if one company produces product A and the other produces product B, it may turn out that because of aggressive marketing by company 2 in differentiating the product offerings A and B, company 2 captures all the market, resulting in a payoff of $+1$ for company 2 and a payoff of -1 for company 1.

The two companies have to simultaneously decide (each one does not know the decision of the other) which product to produce. This is the strategic decision facing the two companies. This situation is captured by a strategic form game $\Gamma = \langle N, S_1, S_2, u_1, u_2 \rangle$, where $N = \{1, 2\}$; $S_1 = S_2 = \{A, B\}$, and the utility functions are as described in the above table.

4.3 Rock-Paper-Scissors Game

This is an example of another two player zero-sum game where each player has three strategies, called *rock*, *paper*, and *scissors*. Actually, this is a popular hand game played by two persons. Another name for this game is *roshambo*. Two players simultaneously display one of three symbols: a *rock*, a *paper*, or *scissors*. The *rock* symbol beats *scissors* symbol; *scissors* symbol beats *paper* symbol; *paper* symbol beats *rock* symbol (symbolically, rock can break scissors; scissors can cut paper; and paper can cover rock). The payoff matrix for this game is given as follows.

1	2		
	Rock	Paper	Scissors
Rock	$0,0$	$-1,1$	$1,-1$
Paper	$1,-1$	$0,0$	$-1,1$
Scissors	$-1,1$	$1,-1$	$0,0$

4.4 BOS (Bach or Stravinsky) Game

This game is named after the famous musicians Bach and Stravinsky. This is also called the *Battle of Sexes* game. This is a game where the players want to coordinate with each other; however, they have a disagreement about which of the outcomes is better. Let us say two players 1 and 2 wish to go out together to an event A or to an alternative event B. Player 1 prefers to go to event A and player 2 prefers to go to event B. The payoff matrix is as shown.

1	2	
	A	B
A	$2,1$	$0,0$
B	$0,0$	$1,2$

Clearly, this game captures a situation where the players want to coordinate but they have conflicting interests. The outcomes (A, B) and (B, A) are unfavorable to either player. The choice is essentially between (A, A) and (B, B). Recalling the company analogy, suppose we have two companies, 1 and 2. Each company can produce only one of two competing products A and B, but at any given time, a company can only produce one type of product. Assume product A is a niche product of company 1 while product B is a niche product of company 2. If both the companies produce product A, the consumers are compelled to buy product A and would naturally prefer to buy it from company 1 rather than from 2. Assume that company 1 captures two thirds of the market. We can reflect this fact by making the payoff to company 1 twice that of company 2. If both the companies produce product B, the reverse situation will prevail and company 2 will make twice as much payoff as company 1.

On the other hand, if the two companies decide to produce different products, then the market gets segmented and each company tries to outwit the other through increased spending on advertising. In fact, their competition may actually benefit a third company and, effectively, neither company 1 nor company 2 makes any payoff. The above table depicts the payoff structure for this game.

4.5 A Coordination Game

We have already presented this game in Chapter 1 (student's coordination game). This game is similar to the BOS game but the two players now have a preference for the same option, namely event A. The payoff matrix is as shown below; note that the outcomes (A, A) and (B, B) in that order are preferred.

	2	
1	A	B
A	$10, 10$	$0, 0$
B	$0, 0$	$1, 1$

Continuing our analogy of companies, the above game corresponds to a situation wherein the two companies produce the same product, and they have equal market share. This market share is ten times as much for product A as for product B. On the other hand, if the two companies produce different products, a third company may capture all the market share leaving nothing for companies 1 and 2.

Since the payoffs are the same for both the players in all outcomes, such games are also called *common payoff games*.

4.6 Prisoner's Dilemma Game

This is one of the most extensively studied problems in game theory, with many interesting interpretations in a wide variety of situations. Two individuals are arrested for allegedly committing a crime and are lodged in separate cells. The interrogator questions them separately. The interrogator privately tells each prisoner that if he is the only one to confess, he will get a light sentence of 1 year in jail while the other would be sentenced to 10 years in jail. If both players confess, they would get 5 years each in jail. If neither confesses, then each would get 2 years in jail. The interrogator also informs each prisoner what has been told to the other prisoner. Thus the payoff matrix is common knowledge.

	2	
1	NC	C
NC	$-2, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

How would the prisoners behave in such a situation? They would like to play a strategy that offers a best response to a best response strategy that the other player may adopt, the latter player also would like to play a strategy that offers a best response to the other player's best response strategy, and so on. First observe that

C is each player's best response strategy regardless of what the other player plays:

$$u_1(C, C) = -5 > u_1(NC, C) = -10; \quad u_1(C, NC) = -1 > u_1(NC, NC) = -2.$$
$$u_2(C, C) = -5 > u_2(C, NC) = -10; \quad u_2(NC, C) = -1 > u_2(NC, NC) = -2.$$

Thus (C, C) is a natural prediction for this game. However, the outcome (NC, NC) is the best outcome jointly for the players. Prisoner's dilemma is a classic example of a game where rational, intelligent behavior does not lead to an outcome where the sum of utilities of the players is maximal. Also, each prisoner has a negative effect or externality on the other. When a prisoner moves away from (NC, NC) to reduce his jail term by 1 year, the jail term of the other prisoner increases by 8 years.

An alternate way of interpreting the strategies of the prisoners is also popular. In this interpretation, each prisoner has two strategies, *cooperate* and *defect*. The strategy *cooperate* corresponds to cooperating with the other player by not confessing and therefore is equivalent to the strategy NC. The strategy *defect* corresponds to betraying the other player by confessing to the crime and therefore is equivalent to the strategy C. In the rest of the book, we will consistently use the C and NC strategies and not the *defect* and *cooperate* nomenclature.

4.7 Company's Dilemma Game

On the lines of the prisoner's dilemma problem, we present an analogous game involving two companies. Consider two companies 1 and 2, each of which can produce two competing products A and B, but only one at a time. The companies are known better for product A than for product B. However, environmentalists have launched a negative campaign on product A branding it as eco-unfriendly.

If both the companies produce product A, then, in spite of the negative campaign, their payoff is quite high since product A happens to be a niche product of both the companies. On the other hand, if both the companies produce product B, they still make some profit, but not as much as they would if they both produced product A.

On the other hand, if one company produces product A and the other company produces product B, then because of the negative campaign about product A, the company producing product A makes zero payoff while the other company captures all the market and makes a high payoff. The table below depicts the payoff structure for this game.

	2	
1	A	B
A	$6, 6$	$0, 8$
B	$8, 0$	$3, 3$

4.8 A Non-Symmetric Company's Dilemma Game

The examples we have provided so far, namely, matching pennies; rock-paper-scissors; BOS; coordination; prisoner's dilemma; and company's dilemma are instances of *symmetric games*. A two player strategic form game is called symmetric if $S_1 = S_2$ and $u_1(s_1, s_2) = u_2(s_2, s_1) \quad \forall s_1 \in S_1, \ \forall s_2 \in S_2$. We now provide an example of a non-symmetric game involving two competing companies 1 and 2. In this game also, each company has to simultaneously decide which of the two products A, B, it will produce. Company 1 is better known for product A and if it happens that both companies produce A, company 1 prospers. If both companies produce B, then they share the profits equally. If one of them produces A and the other produces B, then company 2 prospers (perhaps due to its more aggressive marketing). The following payoff matrix captures the strategic situation facing the two companies.

	2	
1	A	B
A	$4, 1$	$0, 4$
B	$1, 5$	$1, 1$

4.9 A Duopoly Pricing Game

This is due to Bertrand (1883). There are two companies 1 and 2 which wish to maximize their profits. The demand as a function of a price p is given by a continuous and strictly decreasing function $x(p)$. The cost for producing each unit of product is c where $c > 0$. The companies simultaneously choose their prices p_1 and p_2. The amount of sales for each company is given by:

$$
\begin{aligned}
x_1(p_1, p_2) &= x(p_1) && \text{if} \quad p_1 < p_2 \\
&= \tfrac{x(p_1)}{2} && \text{if} \quad p_1 = p_2 \\
&= 0 && \text{if} \quad p_1 > p_2
\end{aligned}
$$

$$
\begin{aligned}
x_2(p_1, p_2) &= x(p_2) && \text{if} \quad p_2 < p_1 \\
&= \tfrac{x(p_2)}{2} && \text{if} \quad p_1 = p_2 \\
&= 0 && \text{if} \quad p_2 > p_1
\end{aligned}
$$

It is assumed that the firms incur production costs only for an output level equal to their actual sales. Given prices p_1 and p_2, the utilities of the two companies are:

$$
u_1(p_1, p_2) = (p_1 - c)\, x_1(p_1, p_2)
$$
$$
u_2(p_1, p_2) = (p_2 - c)\, x_2(p_1, p_2)
$$

Note that for this game, $N = \{1, 2\}$ and $S_1 = S_2 = [0, \infty)$. This is an infinite game since the strategy sets are infinite.

4.10 Tragedy of the Commons

The *Tragedy of the Commons* represents a type of social paradox or social tragedy. The problem involves a conflict over use of resources between individual interests and social interests. A village has n farmers represented by the set $N = \{1, 2, \ldots, n\}$. Each farmer has the option of keeping a sheep or not. If 1 corresponds to keeping a sheep and 0 corresponds to not keeping a sheep, the strategy sets are given by

$$S_1 = S_2 = \cdots = S_n = \{0, 1\}.$$

The utility from keeping a sheep (that arises because of milk, wool, etc.) is equal to 1 unit. The village has a limited stretch of grassland and when a sheep grazes on this, there is a damage to the environment, equal to 5 units. This damage to the environment is to be shared equally by the farmers.

Let s_i be the strategy of each farmer. Then $s_i \in \{0, 1\}$. The payoff to farmer i is given by:

$$u_i(s_1, \ldots, s_i, \ldots, s_n) = s_i - \left[\frac{5(s_1 + \cdots + s_n)}{n}\right]$$

For the case $n = 2$, the payoff matrix would be:

	2	
1	0	1
0	$0, 0$	$-2.5, -1.5$
1	$-1.5, -2.5$	$-4, -4$

If $n > 5$, a farmer gains more utility by keeping a sheep rather than not having one. If $n < 5$, then the farmer gets less utility by keeping a sheep than not having one. If $n = 5$, the farmer can be indifferent between keeping a sheep and not keeping a sheep.

If the Government now imposes a pollution tax of 5 units to every farmer keeping a sheep, the payoff becomes:

$$u_i(s_1, \ldots, s_i, \ldots, s_n) = s_i - 5s_i - \frac{5(s_1 + \cdots + s_n)}{n}$$

Now every farmer will prefer not to keep a sheep. We will be analyzing this game in Chapters 5 and 6 to gain insights into this social situation.

4.11 Bandwidth Sharing Game

This problem is based on an example presented by Tardos and Vazirani [2]. There is a shared communication channel of maximum capacity 1. There are n users of this channel, and user i wishes to send x_i units of flow, where $x_i \in [0, 1]$. We have

$$N = \{1, 2, \ldots, n\}$$
$$S_1 = S_2 = \ldots = S_n = [0, 1].$$

If $\sum_{i \in N} x_i \geq 1$, then the transmission cannot happen since the capacity is exceeded, and the payoff to each player is zero. If $\sum_{i \in N} x_i < 1$, then assume that the following is the payoff to user i:

$$u_i(x_1, \ldots, x_n) = x_i\big(1 - \sum_{j \in N} x_j\big) \;\; \forall i \in N.$$

The above expression models the fact that the payoff to a player is proportional to the flow sent by the player but is negatively impacted by the total flow. The second term captures the fact that the quality of transmission deteriorates with the total bandwidth used. Note that the above is an infinite game (since the strategy sets are real intervals). We will show in Chapter 6 that an equilibrium outcome here is not socially optimal.

4.12 A Sealed Bid Auction

There is a seller who wishes to allocate an indivisible item to one of n prospective buyers in exchange for a payment. Here, $N = \{1, 2, \ldots, n\}$ represents the set of buying agents. Let $v_1, v_2, \ldots, v_n$ be the valuations of the players for the object. The n buying agents submit sealed bids and these bids need not be equal to the valuations. Assume that the sealed bid from player i $(i = 1, \ldots, n)$ could be any real number greater than 0. Then the strategy sets of the players are: $S_i = (0, \infty)$ for $i = 1, \ldots, n$. Assume that the object is awarded to the agent with the lowest index among those who bid the highest. Let $b_1, \ldots, b_n$ be the bids from the n players. Then the allocation function will be:

$$y_i(b_1, \ldots, b_n) = \begin{cases} 1 & \text{if } b_i > b_j \text{ for } j = 1, \ldots, i-1; \quad b_i \geq b_j \text{ for } j = i+1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

In the first price sealed bid auction, the winner pays an amount equal to his bid, and the losers do not pay anything. In the second price sealed bid auction, the winner pays an amount equal to the highest bid among the players who do not win, and as usual the losers do not pay anything. The payoffs or utilities to the bidders in these two auctions are of the form:

$$u_i(b_1, \ldots, b_n) = y_i(b_1, \ldots, b_n)(v_i - t_i(b_1, \ldots, b_n))$$

where $t_i(b_1, \ldots, b_n)$ is the amount to be paid by bidder i in the auction when player i bids b_i $(i = 1, \ldots, n)$. Assume that $n = 4$, and suppose the valuations are $v_1 = 20;\ v_2 = 20;\ v_3 = 16;\ v_4 = 16$, and the bids are $b_1 = 10;\ b_2 = 12;\ b_3 = 8;\ b_4 = 14$. Then for both first price and second price auctions, we have the allocation $y_1(b) = 0;\ y_2(b) = 0;\ y_3(b) = 0;\ y_4(b) = 1$, where $b = (b_1, b_2, b_3, b_4)$. The payments for the first price auction are $t_1(b) = 0;\ t_2(b) = 0;\ t_3(b) = 0;\ t_4(b) = 14$ whereas the payments for the second price auction would be: $t_1(b) = 0;\ t_2(b) = 0;\ t_3(b) = 0;\ t_4(b) = 12$. The utilities can be easily computed from the valuations and the payments.

4.13 Pigou's Network Game

This example captures the effect of selfish routing when strategic players act independently of one another. A directed graph consists of two nodes S and T and there are two disjoint edges A and B connecting S to T (see Figure 4.1). A certain amount of traffic has to move from S to T. Each edge is associated with a cost function $c(\cdot)$ which describes the cost (for example travel time) from S to T, incurred by the users of that edge, as a function of the fraction of total traffic that is routed on that edge. Suppose $x \in [0, 1]$ denotes the fraction of traffic routed. On the edge A, the cost function is $c(x) = x\ \forall x \in [0, 1]$. On the edge B, the cost function is constant and equal to unity, that is, $c(x) = 1\ \forall x \in [0, 1]$. A routing network of the above type is called a Pigou network [3].

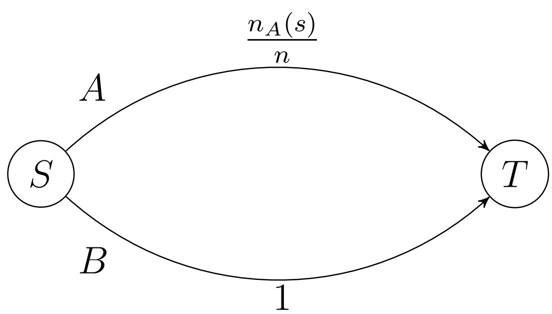

Fig. 4.1: A Pigou network

We shall consider a simple, stylized, discrete version of the above network with just two users, that is, $N = \{1, 2\}$. Each user has two strategies A and B corresponding to the two routes she may select. Thus we have $S_1 = S_2 = \{A, B\}$. Each user selects a route to be followed simultaneously and independent of the other user. A natural way of defining the payoff of a user for a strategy profile here would be negative of the cost of travel of the user. For example, suppose the strategy profile is (A, B) (player 1 selecting route A and player 2 selecting route B). The cost of travel for player 1 would be $\frac{1}{2}$ since the fraction of traffic routed through edge A is $\frac{1}{2}$. The payoff for player 1 becomes $-\frac{1}{2}$. The cost of travel for player 2 is 1 because the edge B is selected and the payoff becomes -1. The payoff matrix for this simple routing game is therefore given by

	2	
1	A	B
A	$-1, -1$	$-\frac{1}{2}, -1$
B	$-1, -\frac{1}{2}$	$-1, -1$

If we have n users, we have $N = \{1, \ldots, n\}$ and $S_1 = \ldots = S_n = \{A, B\}$. Let $s = (s_1, \ldots, s_n)$ be a strategy profile chosen by all the users. To define the payoff function, we shall first define $n_A(s)$ as the number of players who have chosen the route A in the strategy profile s. Similarly, $n_B(s)$ denotes the number of users who have chosen route B in the strategy profile s. Clearly, $n_A(s) + n_B(s) = n$ for all strategy profiles s. The payoff function is given by

$$\begin{aligned} u_i(s) &= -\frac{n_A(s)}{n} \quad \text{if } s_i = A \\ &= -1 \qquad\quad \text{if } s_i = B \end{aligned}$$

We will analyze the above game in Chapter 6.

4.14 Braess Paradox Game

This game is developed on the lines of the game presented in the book by Easley and Kleinberg [4]. This game illustrates the Braess paradox which is named after the German mathematician Dietrich Braess. This paradox is usually associated with transportation networks and brings out the counter-intuitive fact that a transportation network with extra capacity added may actually perform worse (in terms of time delays) than when the extra capacity did not exist. Figure 4.2 shows a network that consists of a source S and a destination T, and two intermediate hubs A and B. It is required to travel from S to T. One route is via the hub A and the other route proceeds via the hub B.

Regardless of the number of vehicles on the route, it takes 25 minutes to travel from S to B or from A to T. On the other hand, the travel time from S to A takes time $\frac{m}{50}$ minutes where m is the number of vehicles on that link. Similarly, the travel time from B to T takes time $\frac{m}{50}$ minutes where m is the number of vehicles on that link. Assume that there are $n = 1000$ vehicles that wish to move from S to T. This means $N = \{1, 2, \ldots, 1000\}$. The strategy sets are $S_1 = \ldots = S_n = \{A, B\}$. Given a strategy profile $(s_1, \ldots, s_n)$, let $n_A(s_1, \ldots, s_n)$ $(n_B(s_1, \ldots, s_n))$ denote the number of vehicles taking the route via A (B). It is easy to note that

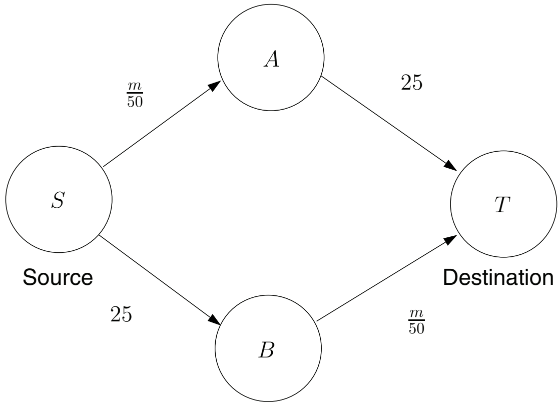

Fig. 4.2: A transportation network with four nodes

$n_A(s_1, \ldots, s_n) + n_B(s_1, \ldots, s_n) = n$. Since we wish to minimize travel time from S to T, it is convenient to define the utility of a player (in this case vehicle) as the negative of the travel time for that player from S to T. It is easy to see that

$$\begin{aligned} u_i(s_1, \ldots, s_n) &= -25 - \frac{n_A(s_1, \ldots, s_n)}{50} \quad \text{if } s_i = A \\ &= -25 - \frac{n_B(s_1, \ldots, s_n)}{50} \quad \text{if } s_i = B \end{aligned}$$

This defines a strategic form game. Note that

$$u_i(A, A, \ldots, A) = u_i(B, B, \ldots, B) = -45$$

$$u_i(s_1, s_2, \ldots, s_n) = -35 \quad \text{whenever } n_A(s_1, \ldots, s_n) = n_B(s_1, \ldots, s_n) = 500$$

Let us say we now introduce a fast link from A to B to ease the congestion in the network (as a degenerate case, we will assume the travel time from A to B to be zero). Figure 4.3 depicts this new network with extra capacity added from A to B. Now the strategies available to each vehicle are to go from S to A to T (call this strategy A); S to B to T (call this strategy B); and S to A to B to T (call this strategy AB). So we have $S_1 = \ldots = S_n = \{A, B, AB\}$. Defining $n_A(s_1, \ldots, s_n), n_B(s_1, \ldots, s_n), n_{AB}(s_1, \ldots, s_n)$ on the same lines as before, we get

$$\begin{aligned} u_i(s_1, \ldots, s_n) &= -25 - \frac{n_A(s_1, \ldots, s_n) + n_{AB}(s_1, \ldots, s_n)}{50} \quad \text{if } s_i = A \\ &= -25 - \frac{n_B(s_1, \ldots, s_n) + n_{AB}(s_1, \ldots, s_n)}{50} \quad \text{if } s_i = B \\ &= -\frac{n_A(s_1, \ldots, s_n) + n_{AB}(s_1, \ldots, s_n)}{50} \\ &\quad -\frac{n_B(s_1, \ldots, s_n) + n_{AB}(s_1, \ldots, s_n)}{50} \quad \text{if } s_i = AB \end{aligned}$$

We will analyze the above two games in Chapters 5 and 6 and illustrate the Braess paradox.

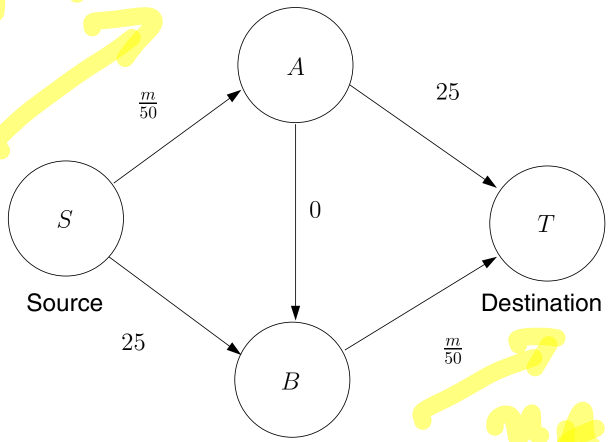

Fig. 4.3: Transportation network with an extra link from A to B

4.15 Summary and References

In this chapter, we presented many examples of strategic form games (also called normal form games) to gain an appreciation of how real world strategic situations could be abstracted as games. The examples we presented include:

- *Matching Pennies*, a two person zero-sum game and *Rock-Paper-Scissors* game, another two person zero-sum game. The games are called zero-sum because the sum of the utilities of the two players is zero in every outcome of the game. These are also called *strictly competitive games*.
- *BOS (Bach or Stravinsky) game*, also called the battle of the sexes game, captures a two player situation where the two players wish to coordinate with each other but they have different preferences for outcomes.
- *Coordination game* or the student's coordination game where two players derive a positive payoff only when they are together, however the payoff they derive when they are together depends on which actions they select.
- *Prisoner's Dilemma*, a classic two player game which illustrates many nuances of strategic conflict and cooperation. This is a game where rational and intelligent behavior does not lead to a socially optimal outcome.
- *Company's Dilemma* which shows how the prisoner's dilemma can be used to model a strategic situation facing two competing companies trying to outwit each other.
- *Non-Symmetric Company's Dilemma* which captures the strategic situation facing two companies which do not have a symmetric payoff structure.
- *Duopoly Pricing Game* which models the strategic situation facing two competing companies in deciding the price at which to sell a particular product.

- *Tragedy of the Commons*, a well studied problem that illustrates a common social paradox involving a conflict over resource sharing. This is a multi-person game.
- *Bandwidth Sharing* game, a multi-person game with infinite strategy sets which illustrates the conflict that arises due to sharing of a common resource.
- *Sealed Bid Auction*, which introduces the strategic form games underlying the well known first price auction and second price auction.
- *Pigou's Network* game which illustrates the notion of selfish routing by strategic agents acting independently of one another
- *Braess Paradox* game which faithfully captures the strategic conflict leading to the famous paradox in routing games.

The above examples are fairly representative of typical situations discussed in this book. We state cautiously that there are numerous other interesting and popular examples that we have left out in our discussion – in fact, we have covered only a minuscule subset of popular examples here. The following books also contain many illustrative examples of strategic form games: Osborne [1], Straffin [5], and Binmore [6], and Maschler, Solan, and Zamir [7].

The material discussed in this chapter draws upon mainly from three sources, namely the books by Myerson [8], Mascolell, Whinston, and Green [9], and Osborne and Rubinstein [1]. The paper by Tardos and Vazirani [2] is a fine introduction to concepts in game theory; we have taken many examples from their paper.

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[1] Martin J. Osborne and Ariel Rubinstein. *A Course in Game Theory.* Oxford University Press, 1994.

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[9] Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green. *Microeconomic Theory.* Oxford University Press, 1995.

[10] Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay Vazirani (Editors). *Algorithmic Game Theory.* Cambridge University Press, 2007.

[11] Martin J. Osborne. *An Introduction to Game Theory.* The MIT Press, 2003.

4.16 Exercises

(1) There are many interesting games that we have not discussed in this chapter. Explore other examples such as hawk and dove (also called chicken), cold war, pollution control, Cournot pricing game, ISP routing game, available from the literature, for example, the following books [10, 11, 5, 6, 7].

(2) There are n players. Each player announces a number in the set $\{1, ..., K\}$, where K is a fixed positive integer. A prize of \$1 is split equally between all the people whose number is closest to $\frac{2}{3}$ of the average number. Formulate this as a strategic form game.

(3) Develop the strategic form game for the Pigou network game for $n = 3$ and $n = 4$.

(4) Consider the following strategic form game (network formation game). The nodes of the network are the players: $N = \{1, 2, \ldots, n\}$. The strategy set S_i of player i is the set of all subsets of $N \setminus \{i\}$. A strategy of a node is to decide on with which other nodes it would like to have links. A strategy profile corresponds to a particular network or graph. Assume that δ where $0 < \delta < 1$ is the benefit that accrues to each node of a link while $c > 0$ is the cost to each node of maintaining the link. Further, assume that δ^k is the benefit that accrues from a k-hop relationship, where, k is the length of a shortest path between the two involved nodes. A link is formed under mutual consent while it can be broken unilaterally. Given a graph g formed out of a strategy profile, let the utility $u_i(g)$ of node i be given by

$$u_i(g) = \sum_{j \neq i} \delta^{l_{ij}(g)} - c.d_i(g)$$

where $l_{ij}(g)$ is the number of links in a shortest path between i and j and $d_i(g)$ is the degree of node i. Write down the strategy profiles corresponding to following structures of graphs assuming n nodes and compute the utilities of all individual nodes.

- Complete graph
- Straight line graph
- Star graph

Chapter 5

Dominant Strategy Equilibria

In the previous chapter, we presented several examples of strategic form games but we stopped short of analyzing the games. Commencing from this chapter, we start analyzing strategic form games. We define the notions of dominated strategies, dominating strategies, dominant strategies, and dominant strategy equilibria. We explore three categories of dominance: strong, weak, and very weak. We illustrate these notions through the examples of prisoner's dilemma, Braess paradox, and second price sealed bid auction.

We start the chapter with the notion of strong dominance. Subsequently, we introduce the notions of weak dominance and very weak dominance.

5.1 Strong Dominance

Definition 5.1 (Strongly Dominated Strategy). *Given a strategic form game $\Gamma = \langle N, (S_i), (u_i) \rangle$, a strategy $s_i \in S_i$ of player i is said to be strongly dominated by another strategy $s_i' \in S_i$ if*

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \ \ \forall s_{-i} \in S_{-i}$$

We also say strategy s_i' strongly dominates strategy s_i.

It is easy to note that player i will always prefer to play strategy s_i' over strategy s_i.

Definition 5.2 (Strongly Dominant Strategy). *A strategy $s_i^* \in S_i$ is said to be a strongly dominant strategy for player i if it strongly dominates every other strategy $s_i \in S_i$. That is, $\forall s_i \neq s_i^*$,*

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}) \ \ \forall s_{-i} \in S_{-i}$$

Definition 5.3 (Strongly Dominant Strategy Equilibrium). *A strategy profile $(s_1^*, \ldots, s_n^*)$ is called a strongly dominant strategy equilibrium of the game $\Gamma = \langle N, (S_i), (u_i) \rangle$ if, $\forall i = 1, 2, \ldots, n$, the strategy s_i^* is a strongly dominant strategy for player i.*

Example 5.1. Recall the prisoner's dilemma problem where $N = \{1, 2\}$ and $S_1 = S_2 = \{C, NC\}$ and the payoff matrix is given by: