Computer Vision

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Textbook

Multiple View Geometry in Computer Vision, Hartley, R., and Zisserman

Richard Szeliski, **Computer Vision: Algorithms and Applications**, 1st edition, 2010

Reference books

Readings for these lecture notes:

- Hartley, R., and Zisserman, A. **Multiple View Geometry in Computer Vision**, Cambridge University Press, 2004, Chapters 1-3.

- Forsyth, D., and Ponce, J. **Computer Vision: A Modern Approach**, Prentice-Hall, 2003, Chapter 2.

These notes contain material c Hartley and Zisserman (2004) and Forsyth and Ponce (2003).

References

These notes are based on

- Dr. Matthew N. Dailey's course: AT70.20: Machine Vision for Robotics and HCI

- Dr. Sohaib Ahmad Khan CS436 / CS5310 Computer Vision Fundamentals at LUMS

2D projective geometry

A model for the projective plane

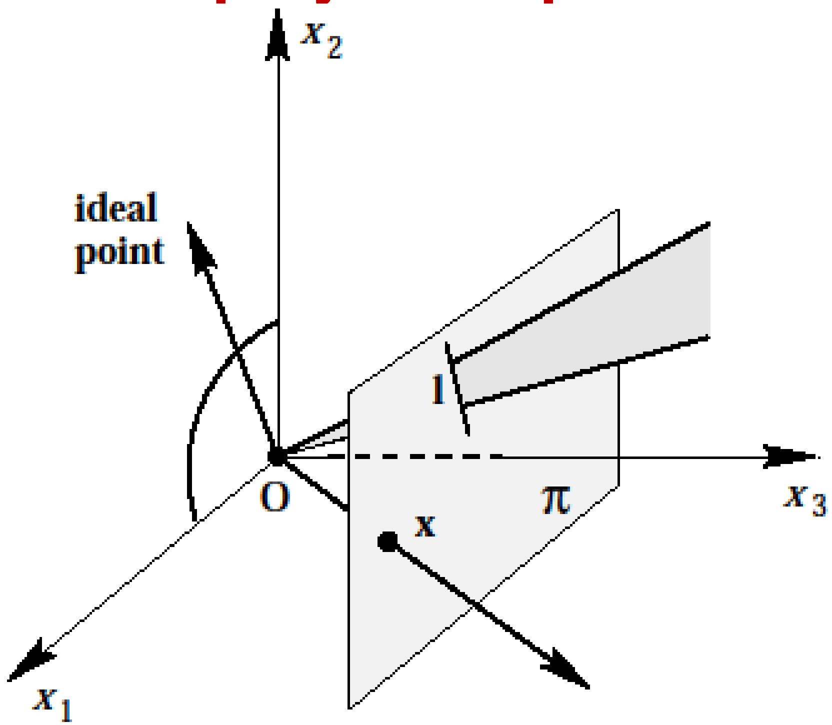


Fig 2.1 **A model of the projective plane**. **Points** and **lines** of $\mathbb{P}^2$ are represented by **rays** and **planes**, respectively, through the origin in $\mathbb{R}^3$. Lines lying in the $x_1\,x_2$ -plane represent **ideal points**, and the $x_1\,x_2$ **-plane** represents $\vec{\mathrm{l}}_\infty$ or $\mathbf{l}_\infty$.

- A **ray** has **one endpoint**, and it continues forever in **one direction** e.g.,

- A **line segment** line has **two endpoints** and continues forever in **zero directions** e.g.,

- A **line** has **zero endpoints** and continue forever in **two directions** e.g.,

- A **plane** is defined by three **non collinear points**.
- Three or more points are said to be **collinear** if they lie on a single **straight line**
- **Coplanar points** are points that all lie on the **same plane**.

Recall: Intersection of Parallel Lines

- Consider two parallel lines

$\vec{\mathrm{l}}_1$: (a, b, c)$^{\mathrm{T}}$

$\vec{\mathrm{l}}_2$: (a, b, c$'$)$^{\mathrm{T}}$

- Computing intersection (as before)

  $\vec{\mathrm{l}}_1 \times \vec{\mathrm{l}}_2 = (c' - c)(b, -a, 0)^{\mathrm{T}}$

- Thus, point of intersection is $(b, -a, 0)^T$

- Converting to inhomogeneous coordinates: $(\mathrm{b}/0, -\mathrm{a}/0)^{\mathrm{T}}$

- **Hence Parallel lines intersect at ideal points**

Recall: Ideal Points lie on a line

- Recall that all **parallel lines** intersect at an **ideal point** or point at infinity, of the form $(\mathrm{x}, \mathrm{y}, 0)^{\mathrm{T}}$
- Consider two such ideal points

$$\mathbf{x_1} = (\mathrm{x}_1, \mathrm{y}_1, 0)^{\mathrm{T}}$$
$$\mathbf{x_2} = (\mathrm{x}_2, \mathrm{y}_2, 0)^{\mathrm{T}}$$

The line joining them is given by:

$\mathbf{l} = \mathbf{x_1} \times \mathbf{x_2}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \mathrm{x}_1 & \mathrm{y}_1 & 0 \\ \mathrm{x}_2 & \mathrm{y}_2 & 0 \end{vmatrix}$$

Rule of Sarrus

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} & \hat{\imath} & \hat{\jmath} \\ \mathrm{x}_1 & \mathrm{y}_1 & 0 & \mathrm{x}_1 & \mathrm{y}_1 \\ \mathrm{x}_2 & \mathrm{y}_2 & 0 & \mathrm{x}_2 & \mathrm{y}_2 \end{matrix}$$

Ideal Points lie on a line

$$\begin{array}{ccccc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ x_1 & y_1 & 0 & x_1 & y_1 \\ x_2 & y_2 & 0 & x_2 & y_2 \end{array}$$

$= \hat{i}(y_1)(0) + \hat{j}(0)(x_2) + \hat{k}(x_1)(y_2) - \hat{k}(x_2)(y_1) - \hat{i}(y_2)(0) - \hat{j}(0)(x_1)$

$= x_1 y_2 \hat{k} - x_2 y_1 \hat{k}$

$= 0\,\hat{i} + 0\,\hat{j} + (x_1 y_2 - x_2 y_1)\,\hat{k}$ $\therefore$ after scaling by $1/(x_1 y_2 - x_2 y_1)$

$\equiv 0\,\hat{i} + 0\,\hat{j} + \hat{k}$

$= (0, 0, 1)^T$

Thus, all **points at infinity** lie on a single line, the **line at infinity**

$\mathbf{l}_\infty = (\mathbf{0}, \mathbf{0}, \mathbf{1})^T$

Line at Infinity

- Any line $\mathbf{l} = (\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$ intersects $\mathbf{l}_{\infty} = (\mathbf{0}, \mathbf{0}, \mathbf{1})^{\mathrm{T}}$ at: $(\mathbf{b}, -\mathbf{a}, \mathbf{0})^{\mathrm{T}}$

- Any line parallel to $\mathbf{l}_1 = (\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathrm{T}}$, i.e. $\mathbf{l}_2 = (\mathbf{a}, \mathbf{b}, \mathbf{c}')^{\mathrm{T}}$ will intersects $\mathbf{l}_{\infty}$ also at: $(\mathbf{b}, -\mathbf{a}, \mathbf{0})^{\mathrm{T}}$

- In inhomogeneous coordinates, $(\mathbf{b}, -\mathbf{a})^{\mathrm{T}}$ represents **line direction**

- Hence, as **line direction varies**, its intersection with $\mathbf{l}_{\infty}$ varies.

- **Line at infinity** is the **set of directions** for **lines** in a **plane**.

Examples

Image credit: https://upload.wikimedia.org/wikipedia/commons/b/b4/CTA_loop_junction.jpg

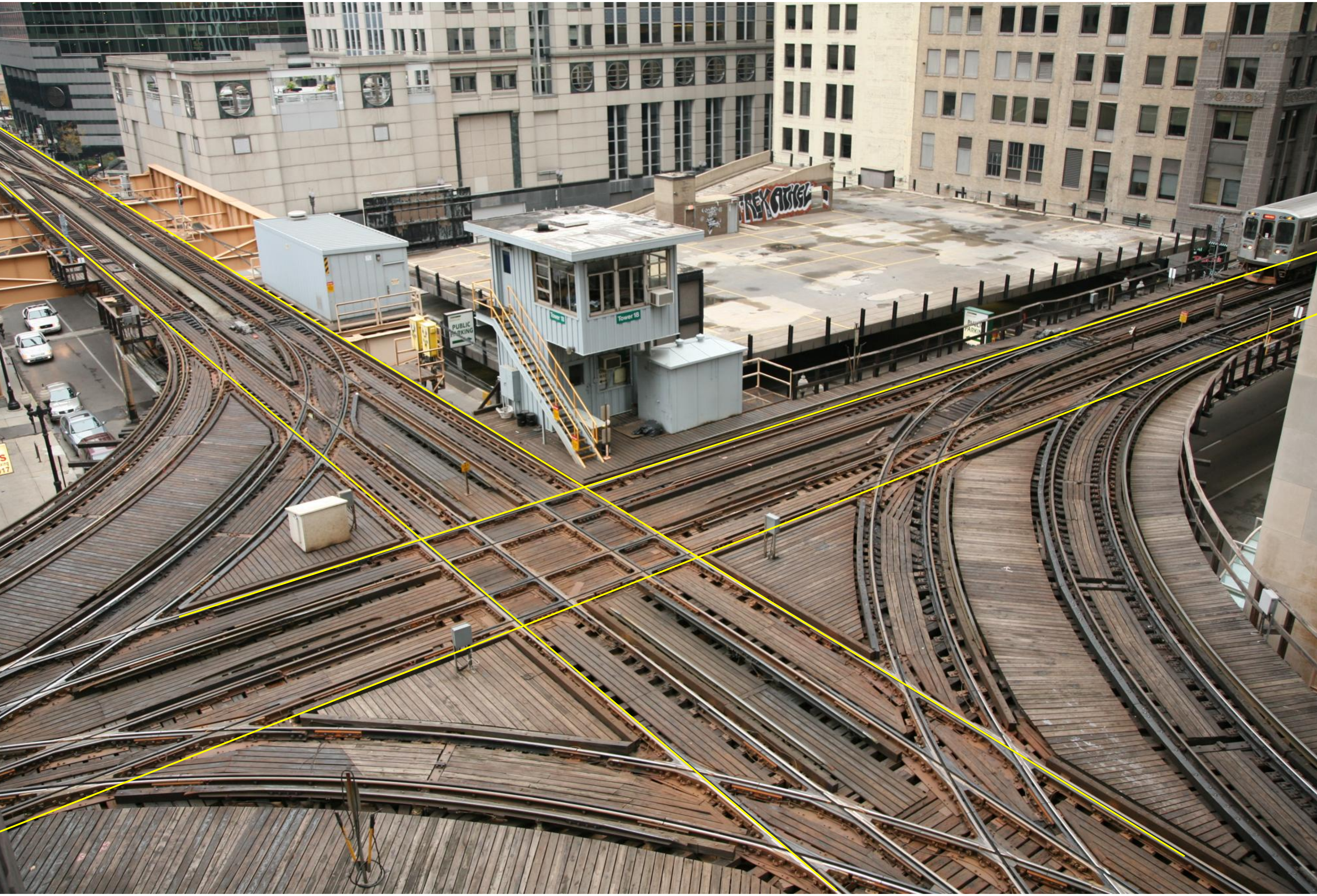

Tower 18

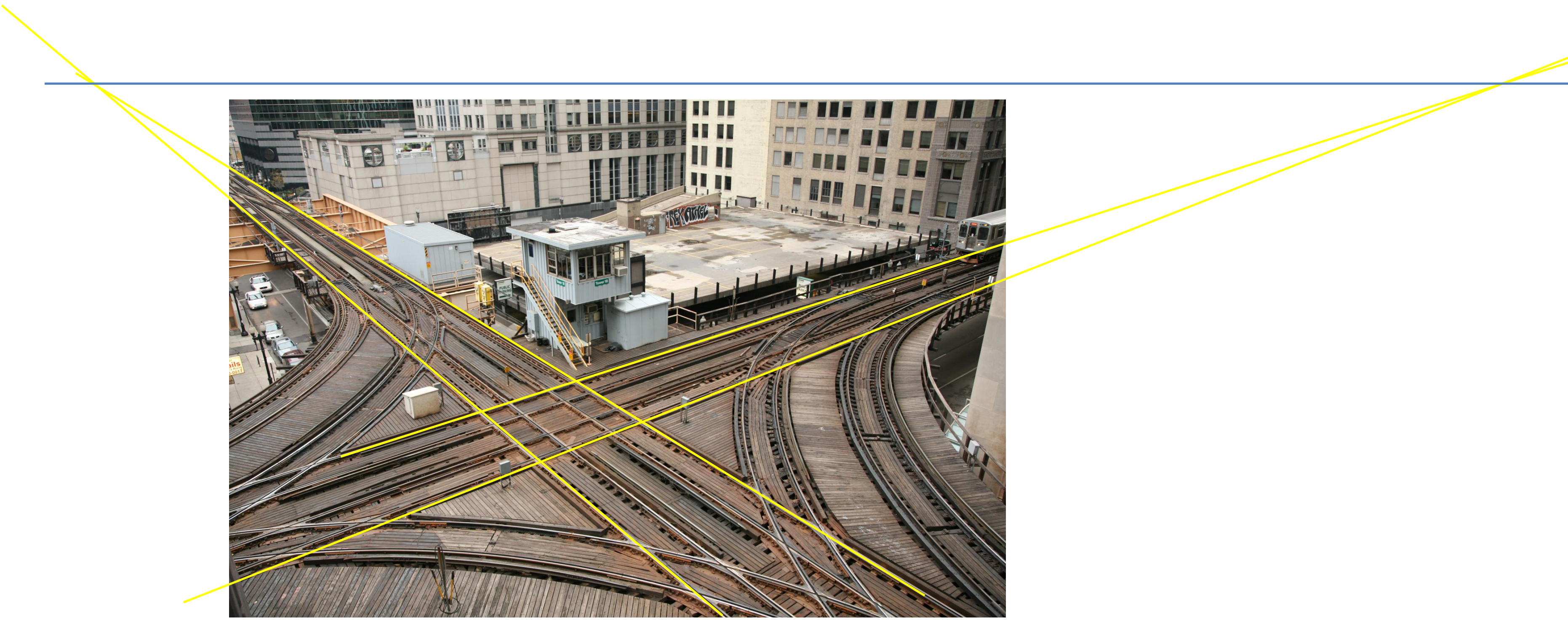

Tower 18


PUBLIC PARKING
Tower 18


Tower 18
PUBLIC PARKING


Tower 18
PUBLIC PARKING


Tower 18
PUBLIC PARKING



What is a Plane in Geometry?

A **plane** in 3D space is defined by a linear equation of the form:

$$ax + by + cz + d = 0$$

Or, rearranged:

x_3 **= constant**

- This form defines all the points in 3D space where **one coordinate is fixed** and the others **can vary freely**.

Why is x_3= 1 a Plane?

In the 3D Cartesian coordinate system (or $\mathbb{R}^3$):

- x_1, x_2 , x_3(or XYZ) represent the three spatial axes.

The equation x_3= 1 means:

- All points where the **third coordinate** is exactly 1. So, any point of the form (x_1 , x_2 , 1)lies **on this plane**.

- Even though **x_3 is fixed at 1**, and x_1, x_2 are **free to vary**.
- That makes it a **2D surface (a plane)** embedded in **3D space**

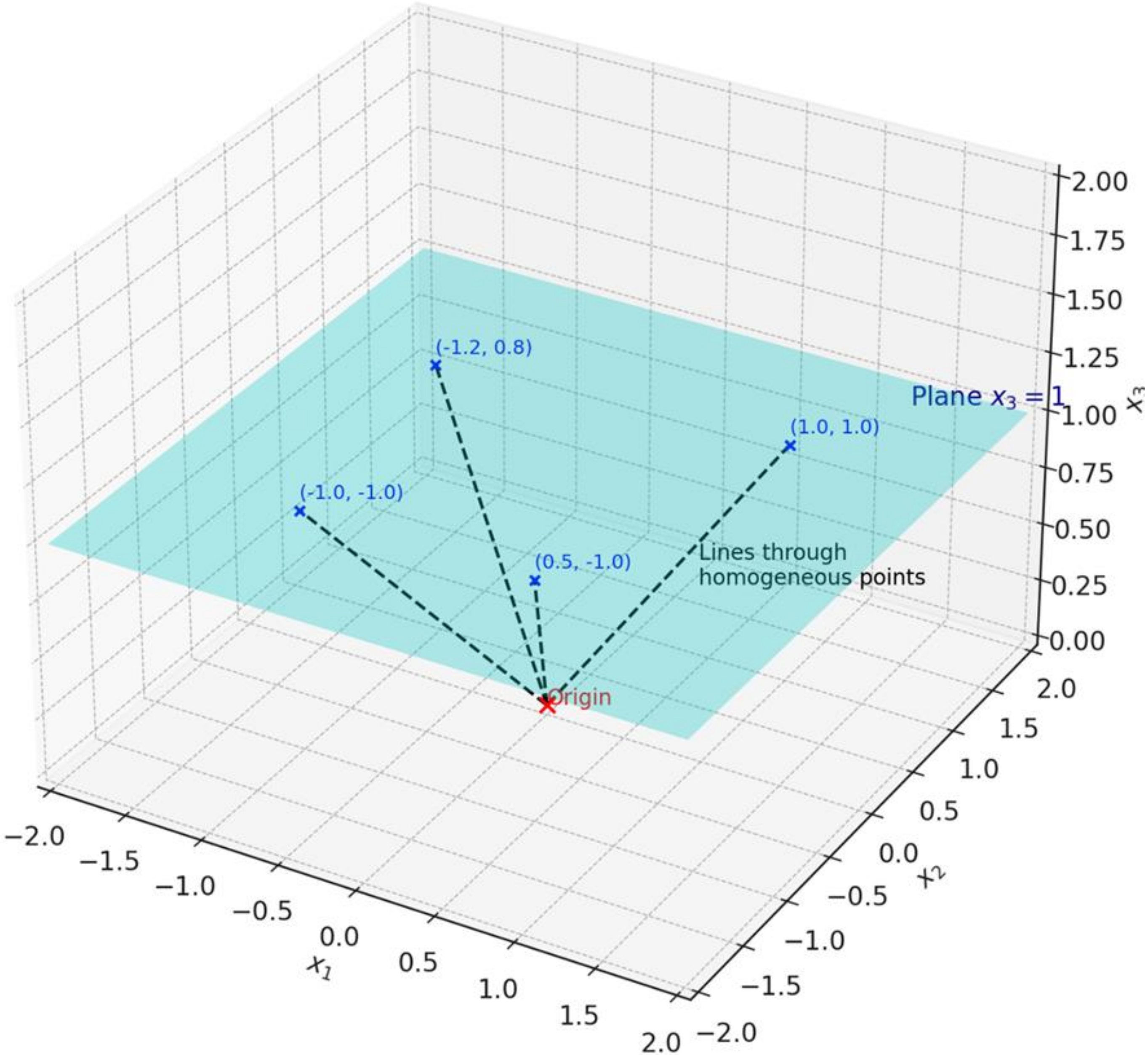
Visual Intuition of x_3= 1 a Plane

- Imagine a stack of paper sheets in 3D space:

- Each sheet is a **plane.**

- The sheet at height $\boldsymbol{x_3}$ **= 1** is one such plane parallel to the x_1x_2 (or xy) plane.

- This is like saying **"at height = 1, stretch out in x and y."**


(-1.2, 0.8)
(1.0, 1.0)
(-1.0, -1.0)
(0.5, -1.0)
Plane $x_3 = 1$
Lines through
homogeneous points
Origin
x_1
x_2
x_3
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

Why is this Plane (x_3= 1) Useful in Computer Vision?

In projective geometry:

- We often work in **homogeneous coordinates**:
  - A 3D point: (x_1 , x_2 , x_3)
  - A 2D point: (x, y, w), w $\neq$0 which becomes $(\frac{x}{w} , \frac{y}{w})$ when normalized.
- Setting $\boldsymbol{x_3}$**= 1** makes math easier:
  - It **"flattens" 3D points** onto a **2D image plane**.
  - It represents a **reference plane** where projection happens.

2D projective geometry
A model for the projective plane

- The inhomogeneous point (x_1, x_2) is represent by any vector $\begin{bmatrix} x_1 x_3 \\ x_2 x_3 \\ x_3 \end{bmatrix}$.

- **Lines** in $\mathbb{P}^2$ are planes in $\mathbb{R}^3$ intersecting the **origin**.

- The line $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ or $(\mathbf{a}, \mathbf{b}, \mathbf{c})^T$ **in** $\mathbb{R}^3$ is a vector **orthogonal** to the **plane** formed by the point on l and the **origin.**

Recall that a plane has a equation

$$\boldsymbol{ax + by + cz + d = 0}$$

- When the plane passes through the origin, **the constant term d = 0**, so the equation simplifies to:

  $$\boldsymbol{ax + by + cz = 0}$$

- It means that **any point (x, y, z)** that lies on the **plane satisfies** the equation (i.e., $\boldsymbol{ax + by + cz = 0}$**)**

- **In vector notation**, this condition can be written as:

  $$\begin{bmatrix} a \\ b \\ c \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

2D projective geometry

A model for the projective plane

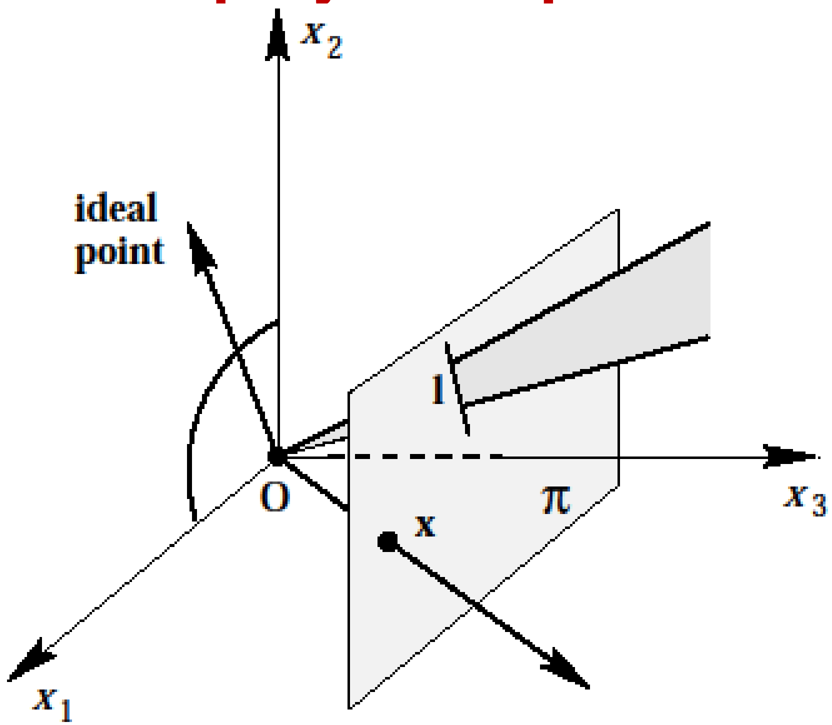


Fig 2.1 **A model of the projective plane**. **Points** and **lines** of $\mathbb{P}^2$ are represented by **rays** and **planes**, respectively, through the origin in $\mathbb{R}^3$. Lines lying in the x_1 x_2 -plane represent **ideal points**, and the x_1 x_2 **-plane** represents $\vec{l}_\infty$ or $\mathbf{l}_\infty$.

2D projective geometry

A model for the projective plane

- A fruitful way of thinking of $\mathbb{P}^2$ is as a **set of rays** in $\mathbb{R}^3$. The set of **all vectors** $\mathrm{k}\,(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3)^{\mathrm{T}}$ as *k* varies forms a **ray through the origin**.

- Each such **ray** is a **single point** in $\mathbb{P}^2$.
- In this model, the **lines** in $\mathbb{P}^2$ are **planes** passing **through the origin in** $\mathbb{R}^3$.
- One verifies that **two non identical rays** lie on exactly **one plane** and any **two planes** intersect in **one ray.**
- This is the analogue of **two distinct points** uniquely defining **a line**, and **two lines** always **intersecting in a point**.
- **Points** and **lines** may be obtained by **intersecting** this set of **rays and planes** by the **plane x_3 = 1**.

Understanding Projective Geometry: $\mathbb{P}^2$ and $\mathbb{R}^3$

- A **line** connecting any **two distinct points** in **projective space ($\mathbb{P}^2$)** corresponds to **a plane through the origin** in $\mathbb{R}^3$.
- To get the **inhomogeneous coordinates** of a **point** or a **line**, we find where its corresponding **ray** or **plane** intersects the **plane** $x_3 = 1$.
- **Remember:**
  - $\mathbb{P}^2$ is not just a flat 2D plane.
  - It represents the set of all **rays passing through the origin** in $\mathbb{R}^3$.
  - However, the **origin itself is excluded** from $\mathbb{P}^2$, because it does not define a direction (i.e., no ray).
  - One way to visualize it through **Fig 2.1** in the next slide

2D projective geometry

A model for the projective plane

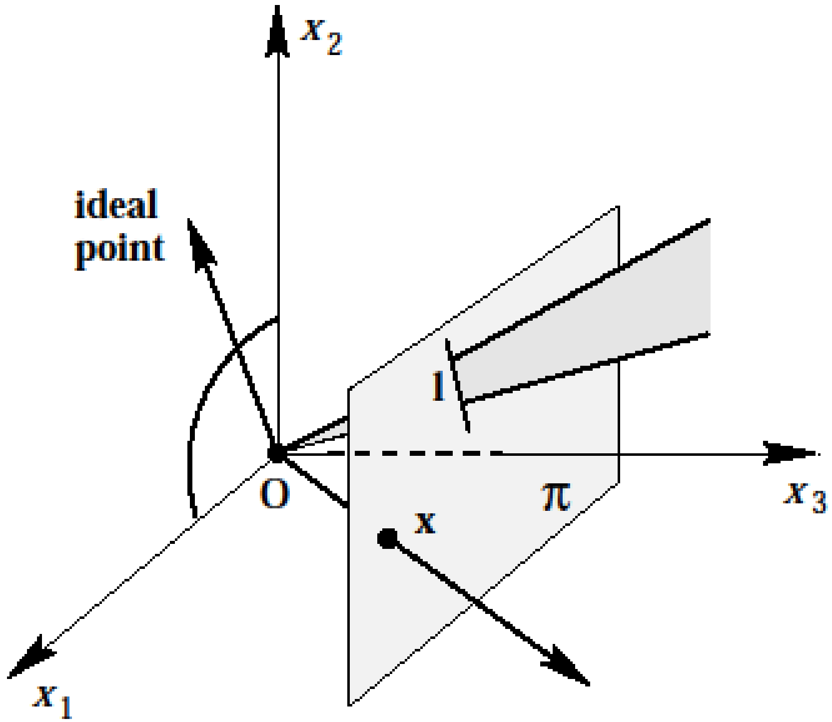


Fig 2.1 **A model of the projective plane.** Points and lines of $\mathbb{P}^2$ are represented by rays and planes, respectively, through the origin in $\mathbb{R}^3$. Lines lying in the x_1 x_2 -plane represent **ideal points**, and the x_1 x_2 -plane represents $\vec{l}_\infty$ or $\mathbf{l}_\infty$.

2D projective geometry

A model for the projective plane

- In $\mathbb{R}^3$, lines through the origin that lie in the **$x_1 x_2$ plane** represent **ideal points** in $\mathbb{P}^2$.

- All **other lines** through the **origin** represent **points** in $\mathbb{P}^2$.

- **Planes** through the **origin** in $\mathbb{R}^3$ represent **lines** in $\mathbb{P}^2$.

- The vector **(a, b, c)^T** representing a **line** in the Euclidean plane **($\mathbb{R}^3$),** when interpreted as a vector in $\mathbb{R}^3$, is **orthogonal to the $\mathbb{R}^3$ plane representing the line in $\mathbb{P}^2$** .

- We will prove it coming lecture.

2D projective geometry
A model for the projective plane

- We consider a coordinate system with axes x_1, x_2, x_3(**often called** x,y,z). The direction along the x_3 (or z-axis) is particularly important for our discussion.

Note: **2D points** are the points in the **plane x_3 = 1**.

- It means if any **homogenous vector** that does not have its **3rd component** equal to **zero** then we can divide it by the third component.

- If we look it as a **3D point**, then it is actually a point in the **plane (π) (see Fig 2.1).**

Note: If we **normalize** a **homogeneous** representation of a **point**, then we will get a **point** in the **Euclidean plane (π).**

Special Case in Projective Geometry: When the Third Component is Zero

- If the **third component** of the vector $(c' - \mathbf{c})(\mathbf{b}, -\mathbf{a}, \mathbf{0})^T = (\mathbf{b}, -\mathbf{a}, \mathbf{0})^T$ is zero, what does that imply?
  - It means $x_3 = 0$.
  - We still have a **valid vector**, because the first two components x_1 *and* x_2 may be non-zero.
- **What This Means Geometrically:**
  - The vector still passes through the **origin**, but it lies **entirely within the plane $x_3 = 0$,** this is the $\mathbf{x_1\,x_2}$ **plane**.
  - No matter how we scale this vector, it always remains in the **$x_3 = 0$ plane**, it **never intersects** the standard inhomogeneous **plane $x_3 = 1$**.

Special Case in Projective Geometry: When the Third Component is Zero

- **Why These Rays Are Special:**
  - Vectors (or rays) with $x_3 = 0$ are **not normalizable** to the standard plane $x_3 = 1$.
- That means:
  - They do **not correspond to any finite inhomogeneous point in $\mathbb{R}^3$.** These points cannot be normalized to obtain a corresponding point on the **plane (π).**
  - These represent **points at infinity** in projective geometry **($\mathbb{P}^2$).**

2D projective geometry

A model for the projective plane

- **Significance in a camera model:**

  - In the **pinhole camera model**, the interpretation shown in **Fig 2.1** is essential because the plane **π** represents the **image sensor** of the camera.

  - **Origin (i.e., O) of Fig 2.1** is the **center of the camera (or optical center)**. The point where **all projection rays originate.**

  - $\mathbb{R}^3$ going to be the **3D space** where the **camera is embedded**.

  - Using **Fig 2.1**, we aim to understand the **structure and behavior** of the **camera coordinate system** — how it captures the **3D world** and projects it onto the image plane

2D projective geometry

A model for the projective plane

What is a line in the plane? (Euclidean)

- Consider **any line** lying in the **plane π** (as shown in Fig 2.1).Now, take any **two points on that line**, and consider the span of those points starting from the origin.
- What you get is a **plane** in $\mathbb{R}^3$ — passing through both the **origin** and the **line** on **π**.

- **Key Insight:**
- **Any line** in the **plane $x_3 = 1$** corresponds to a **plane through the origin** in $\mathbb{R}^3$.
- This links **lines** in **image space** to **planes in the 3D world,** a core idea in projective geometry and camera modeling.

2D projective geometry

A model for the projective plane

Line in the plane? (Euclidean)

- How we represent a **plane** in **three-space**? It is similar as we represent a line in **2-space**
- The **general equation of the plane** is

  $\boldsymbol{ax + by + cz + d} = 0$
- But in our case, the distance from the origin is **always equal to zero** i.e., $\boldsymbol{ax + by + cz} = 0$ for any **plane through the origin**, so that three vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})^{\mathbf{T}}$ that represents **the plane** is **normal to the plane**.

The line at infinity $\vec{l}_{\infty} = (\mathbf{0}, \mathbf{0}, \mathbf{1})^{\mathbf{T}}$

"Any plane through the origin in the three-space is represented geometrically by the normal vector to the plane."

- How can we interpret the equation of a line giving us the vector which is orthogonal to the plane?

- The **line at infinity** can be represented by the **vector that is normal** to the **line at infinity**. The vector that is normal to the line at infinity is

$$\vec{\mathrm{l}}_{\infty} = (\mathbf{0}, \mathbf{0}, \mathbf{1})^{\mathbf{T}}$$

The line at infinity $\vec{l}_\infty = (\mathbf{0}, \mathbf{0}, \mathbf{1})^{\mathbf{T}}$

$\vec{l}_\infty = (\mathbf{0}, \mathbf{0}, \mathbf{1})^{\mathbf{T}}$

- So, the vector that is **normal** to the **line at infinity** which is in three-space is **orthogonal to the plane** representing the **line at infinity** is exactly the vector x_3 i.e., $(\mathbf{0}, \mathbf{0}, \mathbf{1})^{\mathbf{T}}$.

- It is a vector in the direction of x_3. So the plane that is representing **line at infinity** is just $\mathbf{x_1 x_2}$ **-plane** and the **vector orthogonal to that plane** is $(\mathbf{0}, \mathbf{0}, \mathbf{1})^{\mathbf{T}}$

- The **line at infinity** is kind of a special line. The vector that described it is **orthogonal** to the **viewing plane (π)** or the plane we are projecting to.