# Computer Vision

**Dr. Syed Faisal Bukhari**
**Associate Professor**
**Department of Data Science**
**Faculty of Computing and Information Technology**
**University of the Punjab**

# Textbook

- Linear Algebra and its application
by David C Lay

# Reference books

- Elementary Linear Algebra
 by Howard Anton and Chris Rorres

# References

Readings for these lecture notes:

**Linear Algebra and its application**

**by David C Lay**

These notes contain material from the above recourses.

# The Invertible Matrix Theorem (Continued)

**Theorem:** Let $A$ be an **$n \times n$ matrix**. Then ***A*** is **invertible if and only if**:

**s.** The number **0** is ***not*** an **eigenvalue** of ***A***.

**t.** The **determinant** of ***A*** is ***not*** **zero**.

- In linear algebra, **eigenvalues** and **eigenvectors** are used to understand the behavior of a linear transformation represented by a square matrix.
- **Eigenvector**: A **non-zero vector** $\vec{v}$ such that when a matrix **A** is multiplied by $\vec{v}$, the result is a **scalar multiple** of $\vec{v}$

$$\mathbf{A}\vec{v} = \alpha\vec{v}$$

**Eigenvalue λ:** The scalar in the equation above. It tells us how the direction of the vector is scaled.

- To find **eigenvalues**: Solve **det(A−λI) = 0**
- To find **eigenvectors**: Solve **(A−λI)v = 0**

**Example**

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Find eigenvalues and eigenvectors.

**Solution:**

**Eigenvalues**: Solve **det(A–λI) = 0**

**det(A–λI) =** $\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$$= \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right)$$

$$= (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda^2 - 3\lambda - \lambda + 3 = 0$$

$$\lambda = 3 \text{ and } \lambda = 1$$

**Eigenvectors**: Solve **(A−λI)v = 0**

For λ = 1

**(A − $\boldsymbol{I}$ )$\boldsymbol{v}$ = 0**

A - $\boldsymbol{I}$ = $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$-$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A - $\boldsymbol{I}$ = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Row reduce the augmented matrix for (A - $\boldsymbol{I})\boldsymbol{v} = \mathbf{0}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

- 1 × $R_1$ + $R_2$ → $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $\boldsymbol{x_1 + x_2}$ **= 0**

$\Rightarrow$ $\boldsymbol{x_1}$**= -** $\boldsymbol{x_2}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

**Eigenvector** $= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

**Example 4:** Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An **eigenvalue** of ***A*** is **2**. Find a **basis** for the corresponding **eigenspace**.

**Solution:**

$$\mathbf{A - 2}\,\boldsymbol{I} = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $\mathbf{(A -}\ \boldsymbol{I})\boldsymbol{x} = \mathbf{0}$

$$[A - 2I \quad 0] = \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix}$$

$$-1 \times R_1 + R_2, -1 \times R_1 + R_3 \longrightarrow \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 - x_2 + 6x_3 = 0$$

The general solution is $x_1 = \frac{1}{2}x_2 - 3x_3$ with $x_2$ and $x_3$ are **free variables**.

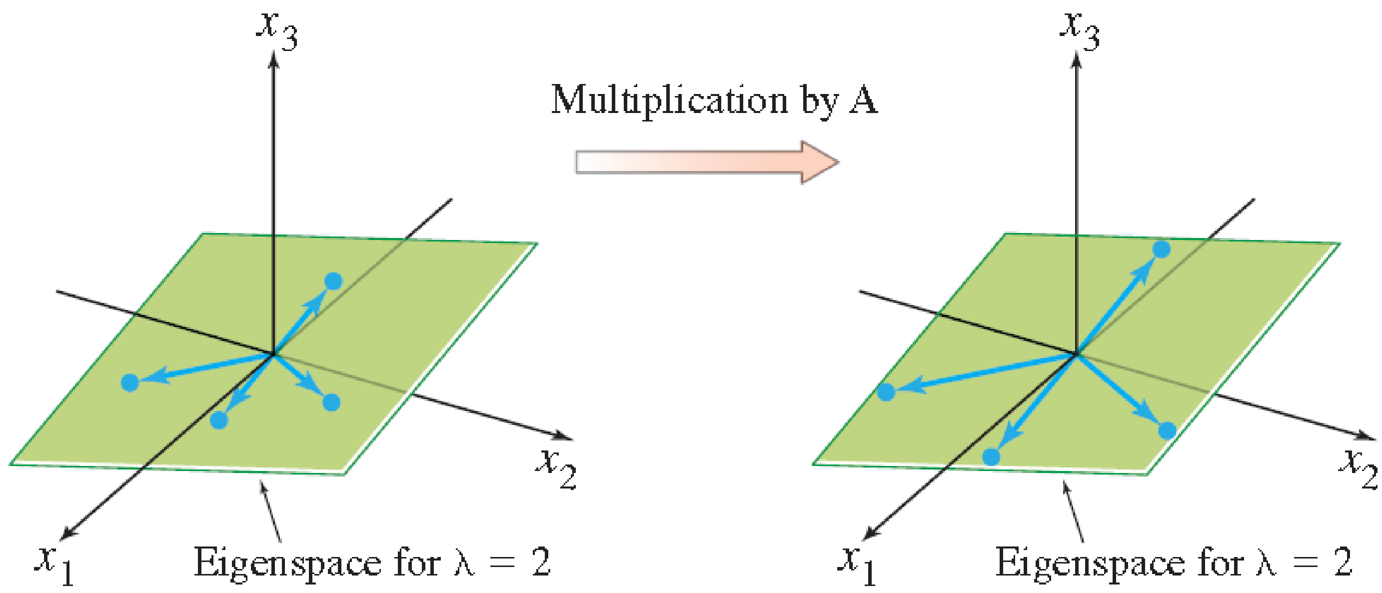
$\boldsymbol{x_1 = \frac{1}{2}x_2 - 3x_3}$ with $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$ are **free variables**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\, x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The **eigenspace**, shown in the following figure, is a two-dimensional subspace of $\mathbb{R}^3$. A **basis** is

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The **eigenspace**, shown in the figure below, is a two-dimensional subspace of $\mathbb{R}^3$




**A acts as a dilation on the eigenspace**.

# Eigenvectors And Eigenvalues

**Theorem 1:** The **eigenvalues** of a **triangular matrix** are the **entries** on its **main diagonal**.

**Proof:** For simplicity, consider the **3 × 3** case.

If ***A*** is **upper triangular**, the $A - \lambda I$ has the form

$$A - \lambda \boldsymbol{I} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{a_{11} - \lambda} & a_{12} & a_{13} \\ 0 & \boldsymbol{a_{22} - \lambda} & 0 \\ 0 & 0 & \boldsymbol{a_{33} - \lambda} \end{bmatrix}$$

- The scalar **λ** is an eigenvalue of ***A*** if and only if the equation **$(A - \lambda I)x = 0$** has a **nontrivial solution**, that is, if and only if the equation has a **free variable**.

- Because of the **zero entries** in $A - \lambda I$, it is easy to see that **$(A - \lambda I)x = 0$** has a **free variable** if and only if **at least one of the entries** on the **diagonal** of $A - \lambda I$ is zero.

- This happens if and only if **λ** equals one of the entries $a_{11}$, $a_{22}$, $a_{33}$ in $A$.

**Theorem 2:** If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are **eigenvectors** that correspond to **distinct eigenvalues** $\lambda_1, \ldots, \lambda_r$ of an matrix $A$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is **linearly independent**.

**Example 5**

Let A = $\begin{bmatrix} \mathbf{3} & 6 & -8 \\ 0 & \mathbf{0} & 6 \\ 0 & 0 & \mathbf{2} \end{bmatrix}$ and B = $\begin{bmatrix} \mathbf{4} & 0 & 0 \\ -2 & \mathbf{1} & 0 \\ 5 & 3 & \mathbf{4} \end{bmatrix}$.

The **eigenvalues** of **A** are **3**, **0**, and **2**.

The **eigenvalues** of **B** are **4** and **1**.

What does it mean for a matrix A to have an **eigenvalue** of **0**, such as in Example 5?

**Solution**

This happens if and only if the equation

$A\mathbf{x} = 0\mathbf{x}$ **------------------------(1)**

has a **nontrivial solution**. But (1) is equivalent to

$A\mathbf{x} = \mathbf{0}$, which has a **nontrivial solution** if and only if A is **not invertible**.

- Thus **0** *is an* ***eigenvalue*** *of* A ***if and only if*** **A *is not invertible***.

**Note:** This fact will be added to the **Invertible Matrix Theorem in Section**

# Eigenvectors and Eigenvalues

**Theorem 2:** If *A* is a **triangular matrix**, then **det *A*** is the **product of the entries** on the **main diagonal** of ***A***.

# Determinants

- Let ***A*** be an **n × n** matrix, let ***U*** be any echelon form obtained from *A* by **row replacements** and **row interchanges (without scaling)**, and let *r* be the number of such **row interchanges**.

- Then the **determinant** of *A*, written as **det *A***, is **$(-1)^r$ times** the **product** of the **diagonal entries** $u_{11}, \ldots, u_{nn}$ in *U*.

- If *A* is **invertible**, then $u_{11}, \ldots, u_{nn}$ are **all pivots** (because $A \sim I_n$ and the $u_{ii}$ have not been **scaled** to 1's).

Otherwise, at least $u_{nn}$ is **zero**, and the product $u_{11} \ldots u_{nn}$ is zero.

**If there are r interchanges**

$$\begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in U} \end{pmatrix} & \text{when A is invertible} \\ 0 & \text{when A is not invertible} \end{cases} \quad \text{---(1)}$$

**Example 1:** Compute det $A$ for

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

**Solution:** The following row reduction uses one row interchange

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$-2 \times R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$R_2 \longleftrightarrow R_3 \rightarrow \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & -1 \end{bmatrix} \qquad (r = 1)$$

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & -1 \end{bmatrix}$$

$$-3 \times R_2 + R_3 \rightarrow \begin{bmatrix} \mathbf{1} & 5 & 0 \\ 0 & \mathbf{-2} & 0 \\ 0 & 0 & \mathbf{-1} \end{bmatrix} = U_1$$

$$\det A = (-1)^1(1)(-2)(-1)$$

$$= -2$$

# Alternative Solution

- The following alternative row reduction avoids the **row interchange** and produces a **different echelon form**.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$-2 \times R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$-\frac{1}{3} \times R_2 + R_3 \rightarrow \begin{bmatrix} \mathbf{1} & 5 & 0 \\ 0 & \mathbf{-6} & -1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = U_2$$

$$\det A = (-1)^0(1)(-6)\left(\frac{1}{3}\right)$$

$= -2.$ **The answer is same as before.**

# The Invertible Matrix Theorem (Continued)

**Theorem:** Let $A$ be an **$n \times n$ matrix**. Then ***A*** is **invertible if and only if**:

**s.** The number **0** is ***not*** an **eigenvalue** of ***A***.

**t.** The **determinant** of ***A*** is ***not*** **zero**.

# Properties of Determinants

**Theorem 3:** Let $A$ and $B$ be **n × n** matrices.

*a.* $A$ is invertible if and only if **det ≠ 0**

b. $\det AB = (\det A)(\det B)$

c. $\det A^T = \det A$

# Properties of Determinants

d. If $A$ is **triangular**, then det $A$ is the **product of the entries** on the **main diagonal** of $A$.

e. A **row replacement operation** on $A$ does not change the **determinant**. A **row interchange changes** the **sign** of the determinant. A **row scaling** also **scales** the determinant by the **same scalar factor**.

# The Characteristic Equation

**Theorem 3(a)** shows how to determine when a matrix of the form $(A - \lambda I)x$ is ***not* invertible**.

- The **scalar equation** det $(A - \lambda I)x = 0$ is called the **characteristic equation** of *A*.
- A scalar **λ** is an **eigenvalue** of an **n × n matrix *A*** if and only if **λ** satisfies the characteristic equation

  $\det(A - \lambda I)x = 0$

**Example 3:** Find the **characteristic equation** of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Solution:** Form $A - \lambda \boldsymbol{I}$, and use Theorem 3(d):

det $(A - \lambda \boldsymbol{I})$

$$= \det \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$= (5 - \lambda)(3 - \lambda)(5 - \lambda)(1 - \lambda)$

The **characteristic equation** is

$$(5 - \lambda)^2 (3 - \lambda)(1 - \lambda) = 0$$

$$\Rightarrow (5 - \lambda)^2 (\lambda - 3)(\lambda - 1) = 0$$

Expanding the product, we can also write

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

If $A$ is an **n × n matrix**, then det $(A - \lambda I)$ is a polynomial of degree $n$ called the **characteristic polynomial** of $A$.

- The **eigenvalue 5** in Example 3 is said to have ***multiplicity* 2** because **$(\lambda - 5)$** occurs **two times** as a factor of the **characteristic polynomial**.
- In general, the **(algebraic) multiplicity** of an **eigenvalue λ** is its multiplicity as a **root** of the **characteristic equation**.

**Example 4** The characteristic polynomial of a $6 \times 6$ matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the **eigenvalues** and their **multiplicities**.

*Factor the polynomial*

$$\lambda^6 - 4\lambda^5 - 12\lambda^4 = \lambda^4(\lambda^2 - 4\lambda - 12) = \lambda^4(\lambda^2 - 6\lambda + 2\lambda - 12)$$
$$= \lambda^4(\lambda - 6)(\lambda + 2)$$

- The **eigenvalues** are **0 (multiplicity 4)**, **6 (multiplicity 1)**, and **-2 (multiplicity 1)**. We could also list the **eigenvalues** in Example 4 as **0, 0, 0, 0, 6**, and **-2**, so that the **eigenvalues** are **repeated** according to their **multiplicities**.

# Orthogonal Matrix

**Orthogonal Matrix:** A **square invertible matrix U** such that $\mathbf{U^{-1} = U^T}$.

**Theorem 6** An $\mathbf{m \times n}$ **matrix U** has **orthonormal columns** if and only if $\mathbf{U^T U} = \boldsymbol{I}$.

# The Singular Value Decomposition

The **decomposition** of **A** involves an **m × n "diagonal"** matrix $\Sigma$ of the form

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \quad \text{---------------(1)}$$

**m – r rows**

**n – r columns**

$$A_{m\times n} = U_{m\times m}\, \Sigma_{m\times n}\, V^{T}{}_{n\times n}$$

- where **D** is an **r × r diagonal matrix** for **some r not exceeding** the **smaller of m and n**. (If r equals m or n or both, some or all of the zero matrices do not appear.)

# The Singular Value Decomposition

**Theorem 10: The Singular Value Decomposition** Let $A$ be an $\boldsymbol{m \times n}$ **matrix** with rank r. Then there exists an $\boldsymbol{m \times n}$ **matrix** $\boldsymbol{\Sigma}$ as in (1) for which the **diagonal entries** in ***D*** are the first ***r* singular values** of $A$, $\boldsymbol{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0}$, and there exist an $\boldsymbol{m \times m}$ **orthogonal matrix *U*** and an $\boldsymbol{n \times n}$ **orthogonal matrix *V*** such that

$$\boldsymbol{A = U\Sigma V^T}$$

$$A_{m\times n} = \boldsymbol{U_{m\times m}\, \Sigma_{m\times n}\, V^T{}_{n\times n}}$$

# The Singular Value Decomposition

- Any factorization $A = U\Sigma V^T$, with $U$ and $V$ orthogonal, $\boldsymbol{\Sigma}$ as in **(1),** and positive diagonal entries in $D$, is called a **singular value decomposition** (or **SVD**) of $A$.

- The columns of $U$ in such a decomposition are called **left singular vectors** of $A$, and the columns of $V$ are called **right singular vectors** of $A$.