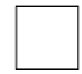
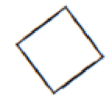
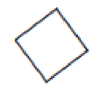
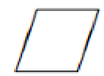
Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} I \mid t \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} R \mid t \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} sR \mid t \end{bmatrix}_{2\times 3}$	4	angles Length Ratios	
affine	$\begin{bmatrix} A \end{bmatrix}_{2\times 3}$	6	parallelism Length Ratios along a line	
projective	$\begin{bmatrix} \tilde{H} \end{bmatrix}_{3\times 3}$	8	straight lines Length Cross-Ratios along a line	

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Transformations	Operations that move or change shapes. Types: translation, rotation, scaling, etc.
GL(3)	Set of all invertible 3×3 matrices. Allows scale, rotation, skew, etc.
PL(3)	Projective Linear Group; GL(3) modulo scale (i.e., ignores scalar multiples).
Affine Transformation	Preserves parallel lines but can skew or scale.
Similarity Transformation	Preserves shapes and angles (no skew).
Euclidean Transformation	Rigid body motion—only rotation and translation.

Property	GL(3)	PL(3)
Type	Linear	Projective
Includes	All invertible 3×3 matrices	Equivalence classes of GL(3)
Scale	Matters	Ignored
Use Case	Classical linear algebra	Projective geometry (e.g., homographies)

> 🧠 In simple terms: If A and 2A are both invertible matrices, they are different in GL(3) but **equivalent in PL(3)**.

PL(3) is useful because **camera transformations** (homographies) often involve scaling — but **we only care about the mapping**, not the scale itself. So, PL(3) is perfect for modeling **real-world image transformations**, like mapping a plane in one view to another.

Subgroup	Condition on Matrix
Affine	Bottom row = (0, 0, 1)
Similarity	2x2 submatrix is **orthogonal**
Euclidean	2x2 submatrix is **orthonormal**

ORTHOGONAL:

- A matrix is **orthogonal** if: (Transpose × Matrix = Identity)
- **Interpretation**: Rows and columns are **perpendicular** (dot product = 0)

Check:

- Take two rows or two columns of matrix A
- Do the dot product → It should be **0**
- Their lengths can be **any non-zero number**

ORTHONORMAL:

A matrix is **orthonormal** if:

- It is orthogonal **AND**
- Each row and column has length = **1**

Check:

- Step 1: Check if A is orthogonal (like above)
- Step 2: Find norm (length) of each row/column = √(sum of square
- Must be exactly 1

- **Affine transformations** preserve the **ideal nature** of points.
- **Projective transformations** can **move ideal points into the finite plane**, making them visible or measurable in projective space.

Affinities map **ideal points** to **ideal points**:

$$\begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ 0 \end{bmatrix}$$

but **general projectivities** can map ideal points to **finite points**:

$$\begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{V}^T & v \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ v_1 x_1 + v_2 x_2 \end{bmatrix}$$

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- **Point at infinity in homogenous representation of a 2D point:**

  The third component of your vector representing the point is zero. If the **bottom row is 0 0 1** then **points remain ideal**.

- **Point at finite in homogeneous representation of a 2D point:**

  If you have **arbitrary bottom row** of your homography the that means you have arbitrary homography. In this case **ideal points are not preserved.**

Date

Conic: the curve obtained by intersecting plane with cone

↳ circle, hyperbola, parabola.

$$ax^2+bxy+cy^2+dx+ey+f=0.$$

$$x=\frac{x_1}{x_3},\ y=\frac{x_2}{x_3} \text{ as } (x,y)=(x_1,x_2,x_3)$$

$$\frac{ax_1^2}{x_3^2}+\frac{bx_1x_2}{x_3^2}+\frac{cx_2^2}{x_3^2}+\frac{dx_1}{x_3}+\frac{ex_2}{x_3}+f=0.$$

multiply by x_3^2

$$ax_1^2+bx_1x_2+cx_2^2\cancel{x_3}+dx_1x_3+ex_2x_3+fx_3^2=0.$$

$$\boxed{\vec{x}^T C\vec{x}=0}$$

$$x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \quad C=\begin{bmatrix}a & b/2 & d/2\\ b/2 & c & e/2\\ d/2 & e/2 & f\end{bmatrix}$$

6 unique elements

dof = 5

why 5 dof?

↳ conic is symmetric.

↳ defined by 6 values, scale doesn't matter

↳ 5 points need to define conic

Five points define a conic:

$(x_1, y_1), (x_2 y_2) \ldots\ldots (x_5, y_5)$

↳ Each point must satisfy:

$ax_i^2 + bxy + cy^2 + dx + ey + f = 0$

$$[x_i^2 \cdot x_i y_i \quad y_i^2 \quad x_i \quad y_i \quad 1] \cdot c = 0$$

$$c = [a, b \ c \ d \ e \ f]^T$$

$$\boxed{\underset{(5\times6)}{A} \circ \underset{(6\times1)}{C} = \underset{(5\times1)}{0}}$$

homogenous equation

$$\underset{(5\times6)}{\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}} \circ \underset{(6\times1)}{C} = [0]_{5\times1}$$

Date:

$$\vec{x}^T C \vec{x} = 0 \quad \text{—— A}$$

$\because$ $\vec{x}' = H\vec{x}$

$$\vec{x} = H^{-1}\vec{x}' \quad \text{—— B}$$

Put B in A.

$$(H^{-1}x')^T C x = 0$$

$$H^{-1T} x'^T C x = 0 \quad \text{—— C}$$

$\because$ $\vec{x}' = H\vec{x} \rightarrow x = H^{-1}\vec{x}'$

$$H^{-1T} x'^T C H^{-1} x' = 0$$