Goal is finding a transformation for each random variable to get a uniform random variable.

Way is:

- preface
  - definition of PDF and CDF
  - concept of differentiation of CDF and the PDF
  - CDF of a random variable and its map under a monotonically increasing transformation
- proposing a special transformation

1 Random variable, CDF and PDF

Random Variable is a function like f which:

$$f : S \to \mathbb{R}$$

Probability function for random variable $Pr(B)$: Sum of probabilities for each member of B. For specific case $Pr(X = x_0) = Pr(x_0)$.

Cumulative distribution function (CDF)

- for both discrete and continuous
- its nature is Probability

Probability density function (PDF)

- only for continuous random variables
- because Pr for continuous random variables is 0
- help us to calculate the Pr
- nature of PDF is **not** probability
  - $pdf\ dx$ is of type probability
  - there may be times that $PDF > 1$; but the role of dx will cause that not problematic

$$\begin{aligned} CDF(a) &= Pr(X < a) \\ &= \int_{-\infty}^{a} pdf\ dx \end{aligned}$$

$$pdf = \frac{d(CDF)}{dx}$$

$$Pr(a < X < b) = \int_{a}^{b} pdf\ dx$$

2 Continuous functions and differentiation

If f is continuous then:

- $$\exists x_0 \in [a, b] : \int_a^b f(x)\ dx = f(x_0)(b-a) \tag{1}$$

- $$\lim_{x \to x_0} (f(x) - f(x_0)) = 0 \tag{2}$$

- Concept of '$f(x)\ dx$' when $x = x_0$ denotes

  $$\lim_{\Delta x \to 0} f(x_0)\ \Delta x \tag{3}$$

  It means the idea is about limits and no one of Δx and dx do not depend to the x_0 and nor on the $f(x)$. So we do not write $\cancel{d(x)}$.

Note For a continuous random variable, the CDF also is continuous.

3 Limit of continuous random variable

For an arbitrary random variable S we can say:

$$Pr(s_0 < S < s_1) = CDF_s(s_1) - CDF_s(s_0)$$
$$= \int_{s_0}^{s_1} PDF_s(s)\ ds \tag{4}$$
$$(1),(4) \Rightarrow \exists s_2 \in [s_0, s_1] : Pr(s_0 < S < s_1) = PDF_s(s_2)\ (s_1 - s_0) \tag{5}$$
$$(2),(5) \Rightarrow \lim_{s_1 \to s_0} \Big(PDF_s(s_3)\ (s_1 - s_0) \Big) = PDF_s(s_0)\ (s_1 - s_0) \tag{6}$$
$$(3),(6) \Rightarrow \lim_{s_1 \to s_0} \Big(PDF_s(s_3)\ (s_1 - s_0) \Big) = PDF_s(s_0)\ ds$$

So we can say:

$$\lim_{s_1 \to s_0} \Big(CDF_s(s_1) - CDF_s(s_0) \Big) = PDF_s(s_0)\ ds \tag{7}$$

4 Monotonically increasing transformation

Suppose

- $s = T(r)$ is a monotonically increasing transformation from a random variable r to another one s, so $T(r)$ is $1-1$.
- $s_0 = T(r_0), s_1 = T(r_1)$.

We know:

$$\begin{aligned} CDF_s(s_1) - CDF_s(s_0) &= Pr(s_0 < S < s_1) \\ &= Pr\Big(T^{-1}(s_0) < T^{-1}(S) < T^{-1}(s_1)\Big) \qquad (8) \\ &= Pr(r_0 < R < r_1) \end{aligned}$$

$$= CDF_r(r_1) - CDF_r(r_0) \qquad (9)$$

Note The eq.(8) is proved in *measure theory.*

$$(7),(9) \Rightarrow PDF_s(s_0)\ ds = PDF_r(r_0)\ dr \qquad (10)$$

Also you can see:

- https://dsp.stackexchange.com/a/61112/11856
- https://dsp.stackexchange.com/q/30502/11856

Let

- r is a *continuous* random variable
- p_r is PDF_r
- s is another random variable

$$s = T(r) = (L-1)\int_0^r p_r(w)\ dw \tag{11}$$

- p_s is PDF_s

Now we want to show that

$$p_s = \frac{1}{L-1}$$

Really $T(r)$ is $CDF_r(r) - CDF_r(0)$ [1] , so it is a continuous and monotonically increasing function. So we can use eq.(10):

$$p_s = p_r \frac{dr}{ds}$$

Proof. To use eq.(10), we only need to get the ds:

$$\begin{aligned}
&(11) \Rightarrow \\
&\quad ds = d(T(r))\ dr = (L-1)p_r\ dr \\
&\quad p_s = p_r \frac{dr}{ds} && \text{replacing } ds \text{ in eq.(10)} \\
&\qquad = p_r \frac{dr}{(L-1)p_r\ dr} \\
&\qquad = \frac{1}{L-1}
\end{aligned}$$

□

[1] This is true for continuous random variables. For a discrete random variable, $T(r)$ will be $CDF_r(r)$.

5 Summary

- preface
  - relation between CDF and PDF
  - CDF and PDF d
  - measure theory and monotonically transformation $\Rightarrow$ equal CDF s
  - $p_s = \frac{dr}{ds} p_r$
- $$T(r) = (L-1)\int_0^r p_r(w)\ dw$$

For a continuous random variable, r, above $T(r)$ will get a uniform random variable s which p_s is $\frac{1}{L-1}$.

Note for histogram equalization, $L-1$ is max possible intensity level of the range of image data type, or equivalently you can think L is the number of possible intensity levels in the image:

- double $\rightarrow$ 1
- uint8 $\rightarrow$ 255
- uint16 $\rightarrow$ $2^{16}-1$