CPS 5310

Mathematical and Computer Modeling

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Homework #Final

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Problem II

Part 1

Given that,

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \tag{1}$$

$$\frac{\partial u}{\partial t} = 0, \quad u(x,0) = x(L-x) \tag{2}$$

$$u(0,t) = u(L,t) = 0 \tag{3}$$

$$u_{xx}(0,t) = u_{xx}(L,t) = 0 \tag{4}$$

Substitution of $u(x,t) = X(x)T(t)$ into (6) gives us,

$$XT'' = -c^2 X^{IV} T$$
$$\Rightarrow \frac{T''}{-c^2 T} = \frac{X^{IV}}{X} = \lambda^4 \tag{5}$$

Where λ is a constant.

Case I: when $\lambda = 0$

$$\frac{X^{IV}}{X} = 0 \Rightarrow X(x) = ax^3 + bx^2 + cx + d$$

Now, we have,

$$X(0)T(t) = 0 \quad \text{and} \quad X(L)T(t) = 0$$
$$\text{So}, X(0) = T(L) = 0, \quad \text{as} \quad T(t) = 0$$

And,

$$X''(0)T(t) = 0 \quad \text{and} \quad X''(L)T(t) = 0$$
$$\text{So}, \quad X''(0) = X''(L) = 0$$

From equation (5), we can write,

$$F(0) = d = 0$$

Now,

$$\begin{aligned}
X'(x) &= 3ax^2 + 2bx + c \\
X''(x) &= 6ax + 2b \\
X''(0) &= 2b = 0 \Rightarrow b = 0 \\
X(L) &= aL^3 + cL = 0 \\
X''(L) &= 6aL = 0 \\
&\Rightarrow a = 0
\end{aligned}$$

So, $c = 0$. And we obtain, $X(x) = 0$, which we do not have any interest.

Case II: when $\lambda \neq 0$

$$\begin{aligned}
\frac{X^{IV}}{X} &= \beta^4 \\
X(x) &= Acos(\lambda x) + Bsin(\lambda x) + E\mathrm{e}^{\lambda x} + F\mathrm{e}^{-\lambda x} \\
&= Acos(\lambda x) + Bsin(\lambda x) + \frac{C+D}{2}\mathrm{e}^{\lambda x} + \frac{C-D}{2}\mathrm{e}^{-\lambda x} \\
&= Acos(\lambda x) + Bsin(\lambda x) + C\frac{\mathrm{e}^{\lambda x} + \mathrm{e}^{-\lambda x}}{2} + D\frac{\mathrm{e}^{\beta x} - \mathrm{e}^{-\lambda x}}{2} \\
&= Acos(\lambda x) + Bsin(\lambda x) + Ccosh(\lambda x) + Dsinh(\lambda x)
\end{aligned}$$

Now,

$$X'(x) = -A\lambda sin(\lambda x) + B\lambda cos(\lambda x) + C\lambda sinh(\lambda x) + D\lambda cosh(\lambda x)$$

$$X''(x) = -A\lambda^2 cos(\lambda x) - B\lambda^2 sin(\lambda x) + C\lambda^2 cosh(\lambda x) + D\lambda^2 sinh(\lambda x)$$

Then, $X(0) = A + C = 0$ and $X''(0) = -A\lambda^2 + C\lambda^2 = 0$ gives us $A = C = 0$

Now,

$$X(L) = Bsin(\lambda L) + Dsinh(\lambda L) = 0$$

$$X''(L) = -B\lambda^2 sin(\lambda L) + D\lambda^2 sinh(\lambda L) = 0$$

We got, $D = 0$

Now,

$$
\begin{aligned}
X(L) &= Bsin(\lambda L) = 0 \\
&= sin(\lambda L) = 0 \quad [\text{as} \quad B \neq 0] \\
\Rightarrow \lambda &= \frac{n\pi}{L}, \quad n = 1, 2, 3, ...
\end{aligned}
$$

So,

$$
X_n(x) = Bsin(\frac{n\pi x}{L}), \quad n = 1, 2, 3, ...
$$

Now,

$$
\begin{aligned}
&T'' + c^2\lambda^4 T = 0 \\
&\quad \Rightarrow T_n(t) = a_n cos(c\lambda^2 t) + b_n sin(c\lambda^2 t), \quad \text{where,} \quad \lambda = \frac{n\pi}{L}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u_n(x,t) &= X_n(x)T_n(t) \\
&= sin(\frac{n\pi x}{L}) \left(Ba_n cos(c\lambda^2 t) + Bb_n sin(c\lambda^2 t) \right), \quad \text{where,} \quad \lambda = \frac{n\pi}{L}
\end{aligned}
$$

We know,

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\
&= \sum_{n=1}^{\infty} sin(\frac{n\pi x}{L}) \left(Ba_n cos(c(\frac{n\pi}{L})^2 t) + Bb_n sin(c(\frac{n\pi}{L})^2 t) \right)
\end{aligned}
$$

$$
\begin{aligned}
\left. \frac{\partial u}{\partial t} \right|_{t=0} &= \sum_{n=1}^{\infty} sin(\frac{n\pi x}{L}) \left(-Ba_n c(\frac{n\pi}{L})^2 sin(c(\frac{n\pi}{L})^2 t) + Bb_n c(\frac{n\pi}{L})^2 cos(c(\frac{n\pi}{L})^2 t) \right) \Bigg|_{t=0} \\
\Rightarrow 0 &= \sum_{n=1}^{\infty} sin(\frac{n\pi x}{L}) Bb_n c(\frac{n\pi}{L})^2 \\
\therefore Bb_n &= 0.
\end{aligned}
$$

Therefore,

$$
u(x,t) = \sum_{n=1}^{\infty} sin(\frac{n\pi x}{L}) Ba_n cos(c(\frac{n\pi}{L})^2 t).
$$

Now,

$$u(x,0)=\sum_{n=1}^{\infty} sin(\frac{n\pi x}{L})Ba_n = f(x) = x(L-x),$$

where,

$$\begin{aligned}
Ba_n &= \frac{2}{L}\int_0^L x(L-x)sin(\frac{n\pi x}{L})dx \\
&= \frac{2}{L}\int_0^L Lxsin(\frac{n\pi x}{L})dx - \frac{2}{L}\int_0^L x^2 sin(\frac{n\pi x}{L})dx \\
&= I_1 - I_2.
\end{aligned}$$

So,

$$\begin{aligned}
I_1 &= \frac{2}{L}L\left[\frac{-xcos(\frac{n\pi x}{L})}{\frac{n\pi}{L}} + \frac{sin(\frac{n\pi x}{L})}{\frac{n^2\pi^2}{L^2}}\right]_0^L \\
&= \frac{2}{L}\left[-\frac{L^3}{n\pi}cos(n\pi) + \frac{L^3}{n^2\pi^2}sin(n\pi)\right] \\
&= \frac{2}{L}\left[-\frac{L^3}{n\pi}cos(n\pi)\right]
\end{aligned}$$

And

$$\begin{aligned}
I_2 &= \frac{2}{L}\left[\frac{-x^2cos(\frac{n\pi x}{L})}{\frac{n\pi}{L}} + \frac{2xsin(\frac{n\pi x}{L})}{\frac{n^2\pi^2}{L^2}} + \frac{2cos(\frac{n\pi x}{L})}{\frac{n^3\pi^3}{L^3}}\right]_0^L \\
&= \frac{2}{L}\left[-\frac{L^3}{n\pi}cos(n\pi) + \frac{2L^3}{n^2\pi^2}sin(n\pi) + \frac{2L^3}{n^3\pi^3}cos(n\pi) - \frac{2L^3}{n^3\pi^3}\right] \\
&= \frac{2}{L}\left[-\frac{L^3}{n\pi}cos(n\pi) + \frac{2L^3}{n^3\pi^3}cos(n\pi) - \frac{2L^3}{n^3\pi^3}\right] \quad [\text{since,} \quad sin(n\pi)=0]
\end{aligned}$$

So,

$$\begin{aligned}
Ba_n &= \frac{2}{L}\left[\frac{2L^3}{n^3\pi^3} - \frac{2L^3}{n^3\pi^3}cos(n\pi)\right] \\
&= \frac{4L^2}{n^3\pi^3} - \frac{4L^2}{n^3\pi^3}cos(n\pi)
\end{aligned}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} sin(\frac{n\pi x}{L}) \left[\frac{4L^2}{n^3\pi^3} - \frac{4L^2}{n^3\pi^3} cos(n\pi)\right] cos(c(\frac{n\pi}{L})^2 t).$$

<u>Part 2</u>

Given that,

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \tag{6}$$

$$\frac{\partial u}{\partial t} = 0, \quad u(x,0) = x(L-x) \tag{7}$$

$$u(0,t) = u(L,t) = 0 \tag{8}$$

$$u_{xx}(0,t) = u_{xx}(L,t) = 0 \tag{9}$$

Let,

$$\frac{\partial u(t)}{\partial t} = v(t) \quad \text{and} \quad \Rightarrow \frac{\partial^2 u(t)}{\partial t^2} = \frac{\partial v(t)}{\partial t}$$

From equation (6) we get,

$$\frac{\partial v(t)}{\partial t} = -c^2 \frac{\partial^4 u(t)}{\partial x^4}$$

From equation (7) we get,

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

$$\Rightarrow v(0) = 0 = v_0$$

$$u(x,0) = u(0) = x(L-x) = u_0$$

Therefore,

$$\frac{\partial u(t)}{\partial t} = v(t),$$

$$\frac{\partial v(t)}{\partial t} = -c^2 \frac{\partial^4 u(t)}{\partial x^4},$$

$$u(0) = x(L-x), \quad v(0) = 0.$$

$$F(u(t)) = -c^2 \frac{\partial^4 u(t)}{\partial x^4},$$

$$u_0 = x(L-x),$$

$$v_0 = 0.$$

Part 2

Let,

$$\mathbb{X} = \begin{pmatrix} u \\ v \end{pmatrix}$$

So,

$$\frac{d\mathbb{X}}{dt} = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = \begin{pmatrix} v \\ -c^2 \frac{\partial^4 u}{\partial x^4} \end{pmatrix}$$

$$\frac{\mathbb{X}(n) - \mathbb{X}(n-1)}{\tau} = \begin{pmatrix} \frac{u(n) - u(n-1)}{\tau} \\ \frac{v(n) - v(n-1)}{\tau} \end{pmatrix}$$

$$= \begin{pmatrix} v(n-1) \\ -c^2 \frac{\partial^4 u(n-1)}{\partial x^4} \end{pmatrix}$$

Where, $\tau = \frac{1}{3}$

Now,

$$u(n) - u(n-1) = \tau v(n-1)$$

$$v(n) - v(n-1) = -\tau c^2 \frac{\partial^4 u(n-1)}{\partial x^4}$$

Hence,

$$u(n) = u(n-1) + \tau v(n-1)$$

$$v(n) = v(n-1) - \tau c^2 \frac{\partial^4 u(n-1)}{\partial x^4}$$

In matrix form we can write,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(n) \\ v(n) \end{pmatrix} = \begin{pmatrix} u(n-1) + \tau v(n-1) \\ v(n-1) - \tau c^2 \frac{\partial^4 u(n-1}{\partial x^4} \end{pmatrix}.$$

So,

$$A_\tau = 1, \quad u_\tau = u(n), \quad v_\tau = v(n)$$

$$f_\tau = u(n-1) + \tau v(n-1)$$

$$g_\tau = v(n-1) - \tau c^2 \frac{\partial^4 u(n-1)}{\partial x^4}.$$

Problem III

Part 1

Here,

$V =$ feasible region

$=$ space of real valued functions w on $\overline{\Omega} = \partial\Omega$ such that $w,$ and $\bigtriangledown w$ are integrable

$= \{w : \overline{\Omega} \to \mathbb{R} | w,$ and $\bigtriangledown w$ are L^2 on ω and $w = \bigtriangledown . \bigtriangledown w = 0$ on $\partial\Omega\}$

$= H_0^2(\Omega)$

$= \overline{C_0^2(\Omega)}$

Part 2

Given that,

$$J(v) = \frac{1}{2}\int_\Omega (\frac{d^2v}{dx^2})^2 dx - \int_\Omega fv dx.$$

We have to find $u \in V$ such that,

$$J(u) = \min_{v \in V} J(v)$$

This minimization problem has a unique solution $u \in V$ under certain condition. We have to find $u \in V$ such that,

$$< J'(u), v >= 0 \quad \forall v \in V$$

When $\lambda > 0$,

$$\begin{aligned}
\frac{J(u+\lambda v)}{\lambda} &= \frac{1}{\lambda}\left[\frac{1}{2}\int_\Omega (\frac{d^2(u+\lambda v)}{dx^2})^2 dx - \int_\Omega f(u+\lambda v)dx - \frac{1}{2}\int_\Omega (\frac{d^2u}{dx^2})^2 dx + \int_\Omega fu dx\right] \\
&= \frac{1}{\lambda}\left[\frac{1}{2}\int_\Omega (\frac{d^2u}{dx^2} + \lambda\frac{d^2v}{dx^2})^2 dx - \int_\Omega fu dx - \lambda\int_\Omega fv dx - \frac{1}{2}\int_\Omega (\frac{d^2u}{dx^2})^2 dx + \int_\Omega fu dx\right] \\
&= \frac{1}{\lambda}\left[\frac{1}{2}[\int_\Omega (\frac{d^2u}{dx^2})^2 dx + 2\lambda\int_\Omega \triangle u . \triangle v dx + \lambda^2\int_\Omega (\frac{d^2v}{dx^2})^2 dx] - \lambda\int_\Omega fv dx - \frac{1}{2}\int_\Omega (\frac{d^2u}{dx^2})^2\right] \\
&= \int_\Omega \triangle u . \triangle v dx + \frac{\lambda}{2}\int_\Omega (\frac{d^2v}{dx^2})^2 dx - \int_\Omega fv dx
\end{aligned}$$

So,

$$\lim_{\lambda \to 0^+} \frac{J(u+\lambda v)}{\lambda} = \int_\Omega \triangle u . \triangle v dx - \int_\Omega fv dx = 0$$

Now, we have,

$$\int_{\Omega} \triangle u . \triangle v dx = \int_{\Omega} f v dx, \quad \forall v \in V$$

So, the boundary value problem becomes,

$$\triangle^2 u = f \text{ on } \Omega \tag{10}$$

$$u = 0 \text{ on } \partial\Omega \quad \text{and} \quad \triangle u = 0 \text{ on } \partial\Omega$$

Multiplying equation(10) by v and then using integration by parts, we get,

$$\int_{\Omega} \triangle u \triangle v dx - \int_{\partial\Omega} \triangle u \bigtriangledown v dx + \int_{\partial\Omega} \triangle u v dx = \int_{\Omega} f v dx$$

$$\int_{\Omega} \triangle u . \triangle v dx = \int_{\Omega} f v dx \quad \left[\text{ Since, } u = 0 \text{ and } \triangle u = 0 \text{ on } \partial\Omega \right]$$

Part 3

Here,

$$\begin{aligned}
u(x) &= \frac{f_0}{24}(x^4 - 2Lx^3 + L^3 x) \\
u'(x) &= \frac{f_0}{24}(4x^3 - 6Lx^2 + L^3) \\
u''(x) &= \frac{f_0}{24}(12x^2 - 12Lx) \\
u'''(x) &= \frac{f_0}{24}(24x - 12L) \\
u^{IV}(x) &= f_0
\end{aligned}$$

Now,

$$
\begin{aligned}
u(0) &= \frac{f_0}{24}(0 - 0 + 0) \\
&= 0 \\
u(L) &= \frac{f_0}{24}(L^4 - 2L^4 + L^4) \\
&= 0 \\
u''(0) &= \frac{f_0}{24}(0 - 0) \\
&= 0 \\
u''(L) &= \frac{f_0}{24}(12L^2 - 12L^2) \\
&= 0
\end{aligned}
$$

So, $u(x)$ satisfies the boundary value problem

$$u^{IV}(x) = f(x)$$

with the same boundary conditions

$$u(0) = u(L) = 0$$

$$\text{and} \quad u''(0) = u''(L) = 0$$