SVD

Singular Value Decomposition is very popular algorithm for Recommendation Technique. For Example, Netflix movie recommendation.

Mathematical Formula:

Given Matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A = U \sum V^t$$

Where,

$$V^t = \text{eigenvector}(A^T A^T)$$

Example,

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Finding eigenvector :

$$AA^T = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} . \begin{bmatrix} 2 & 4 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 14 & 0 & 0 \\ 14 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = W$$

Now that we have a n x n matrix we can determine the eigenvalues of the matrix W.

Since $W \mathbf{x} = \lambda \mathbf{x}$ then $(W - \lambda I) \mathbf{x} = 0$

$$\begin{bmatrix} 20-\lambda & 14 & 0 & 0 \\ 14 & 10-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} \mathbf{x} = (W - \lambda I)\mathbf{x} = 0$$

For a unique set of eigenvalues to determinant of the matrix $(W-\lambda I)$ must be equal to zero. Thus from the solution of the characteristic equation, $|W-\lambda I|=0$ we obtain:

$\lambda=0$, $\lambda=0$; λ = 15+Ö221.5 ~ 29.883; λ = 15-Ö221.5 ~ 0.117 (four eigenvalues since it is a fourth degree polynomial). This value can be used to determine the eigenvector that can be placed in the columns of U. Thus we obtain the following equations:

19.883 x1 + 14 x2 = 0

14 x1 + 9.883 x2 = 0

x3 = 0

x4 = 0

Upon simplifying the first two equations we obtain a ratio which relates the value of x1 to x2. The values of x1 and x2 are chosen such that the elements of the S are the square roots of the eigenvalues. Thus a solution that satisfies the above equation x1 = -0.58 and x2 = 0.82 and x3 = x4 = 0 (this is the second column of the U matrix).

Substituting the other eigenvalue we obtain:

-9.883 x1 + 14 x2 = 0

14 x1 - 19.883 x2 = 0

x3 = 0

x4 = 0

Thus a solution that satisfies this set of equations is x1 = 0.82 and x2 = -0.58 and x3 = x4 = 0 (this is the first column of the U matrix). Combining these we obtain:

$$U = \begin{bmatrix} 0.82 & -0.58 & 0 & 0 \\ 0.58 & 0.82 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Together everything,

$$A^T.A = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.40 & -0.91 \\ 0.91 & 0.40 \end{bmatrix}$$

Finally as mentioned previously the S is the square root of the eigenvalues from AA^T or A^TA. and can be obtained directly giving us:

$$S = \begin{bmatrix} 5.47 & 0 \\ 0 & 0.37 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$