CENG 3549 – Functional Programming

Formal Verification of Functional Programs with Coq

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December 15, 2022

Outline

Definition (lists)

```
Inductive list (A : Type) : Type ≜
    | nil : list A
    | cons : A → list A → list A.

(** induction principle *)
list_ind =
fun (A : Type) (P : list A → Prop) (f : P [])
  (f0 : ∀ (a : A) (l : list A), P l → P (a :: l)) ⇒
fix F (l : list A) : P l ≜
  match l as l0 return (P l0) with
  | [] ⇒ f
  | y :: l0 ⇒ f0 y l0 (F l0)
  end
     : ∀ (A : Type) (P : list A → Prop),
       P [] → (∀ (a : A) (l : list A), P l → P (a :: l)) → ∀ l : list A, P l
```

Definition (list append)

```
Fixpoint append {A: Type} (l1 l2: list A): list A ≜
  match l1 with
    | []    ⇒ l2
    | x::xs ⇒ x::append xs l2
  end.
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nil is right identity of append, that is

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Proof.

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Lemma app_nil: ∀ {A: Type} (l: list A), append l [] = l.
Proof. intros A l.
       induction l; intros.
       - simpl. easy.
       - simpl. rewrite IHl. easy.
Qed.
```

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append is associative, that is

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Lemma app_assoc: ∀ {A: Type} (l1 l2 l3: list A), append (append l1 l2) l3 = append l1 (append l2 l3).
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       - simpl. rewrite IHl1. easy.
Qed.
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Definition (list length)

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Fixpoint length {A: Type} (l: list A): nat ≜
  match l with
    | []    ⇒ 0
    | x::xs ⇒ S (length xs)
  end.
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Qed.
```

Definition (binary trees)

```
Inductive BTree: Set → Type ≜
  | Empty A: BTree A
  | Node A : BTree A → BTree A → BTree A.

(** induction principle *)
BTree_ind =
fun (P : ∀ S : Set, BTree S → Prop) (f : ∀ A : Set, P A (Empty A))
  (f0 : ∀ (A : Set) (b : BTree A),
        P A b → ∀ b0 : BTree A, P A b0 → P A (Node A b b0)) ⇒
fix F (S : Set) (b : BTree S) {struct b} : P S b ≜
  match b as b0 in (BTree S0) return (P S0 b0) with
  | Empty A ⇒ f A
  | Node A b0 b1 ⇒ f0 A b0 (F A b0) b1 (F A b1)
  end
     : ∀ P : ∀ S : Set, BTree S → Prop,
       (∀ A : Set, P A (Empty A)) →
       (∀ (A : Set) (b : BTree A),
        P A b → ∀ b0 : BTree A, P A b0 → P A (Node A b b0)) →
       ∀ (S : Set) (b : BTree S), P S b
```

Definition (hight, perfectness and size of a binary tree)

```
Fixpoint height {A: Set} (t: BTree A): nat ≜
  match t with
    | Empty A    ⇒ 0
    | Node A l r ⇒ max (@height A l) (@height A r) + 1
  end.

Fixpoint perfect {A: Set} (t: BTree A): bool ≜
  match t with
    | Empty A    ⇒ true
    | Node A l r ⇒ Nat.eqb (height l) (height r) && perfect l && perfect r
  end.

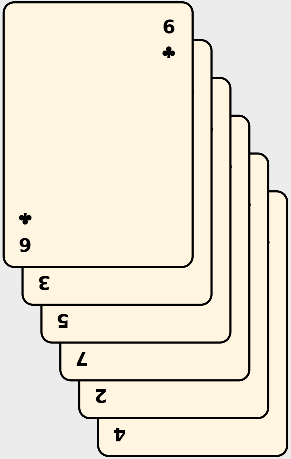
Fixpoint size {A: Set} (t: BTree A): nat ≜
  match t with
    | Empty A    ⇒ 0
    | Node A l r ⇒ size l + size r + 1
  end.
```

Lemma

a perfect binary tree t of height n has exactly $2^n - 1$ nodes, that is,

```
Lemma perfectness: ∀ {A: Set} (t: BTree A), perfect t = true → size t = Nat.pow 2 (height t) - 1.
```

Example (Insertion Sort)



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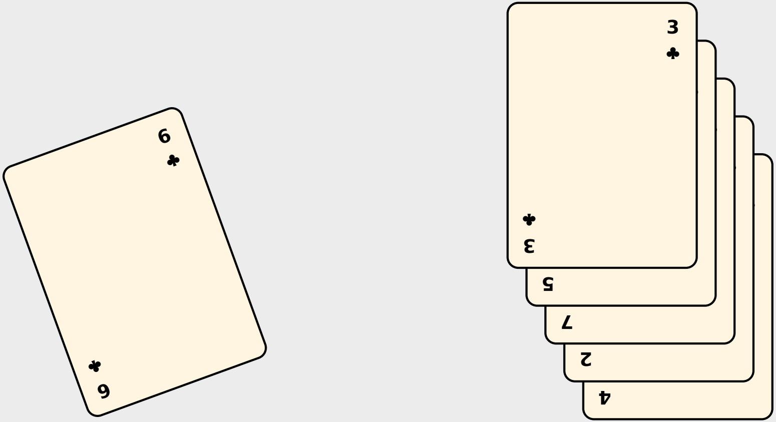

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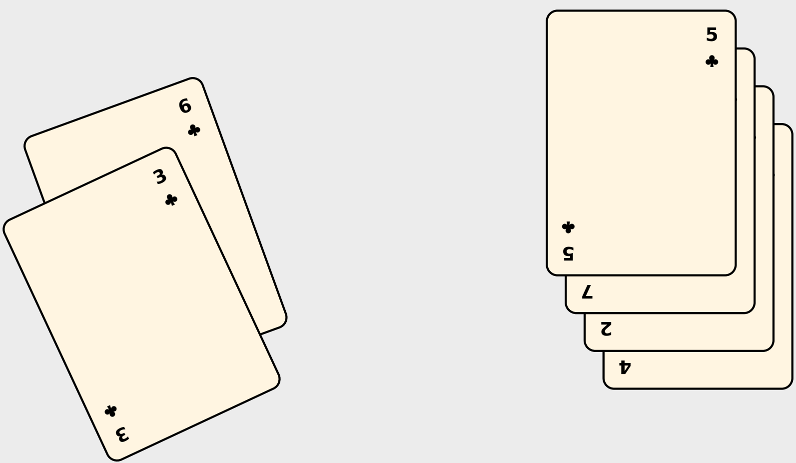

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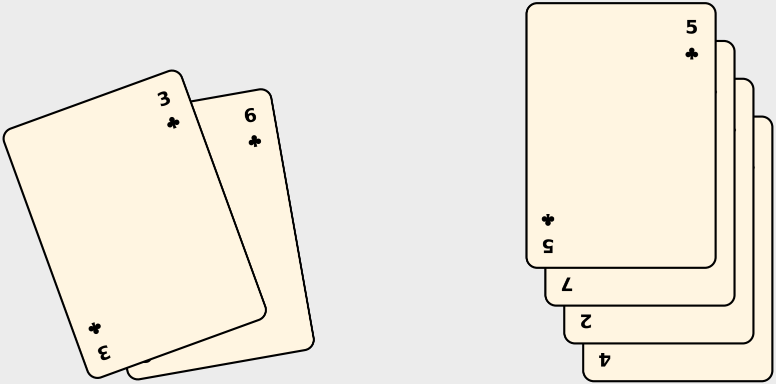

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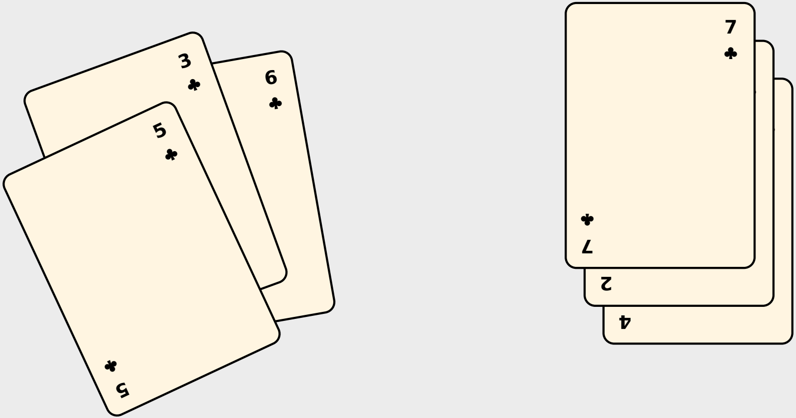

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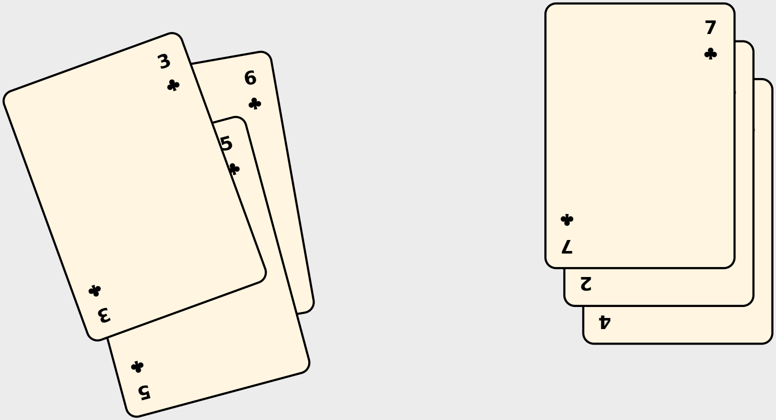

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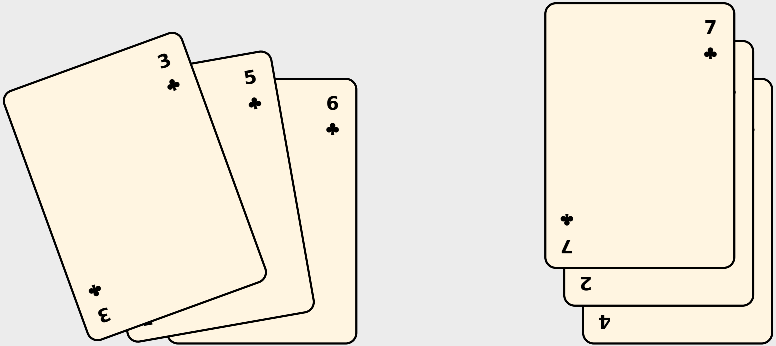

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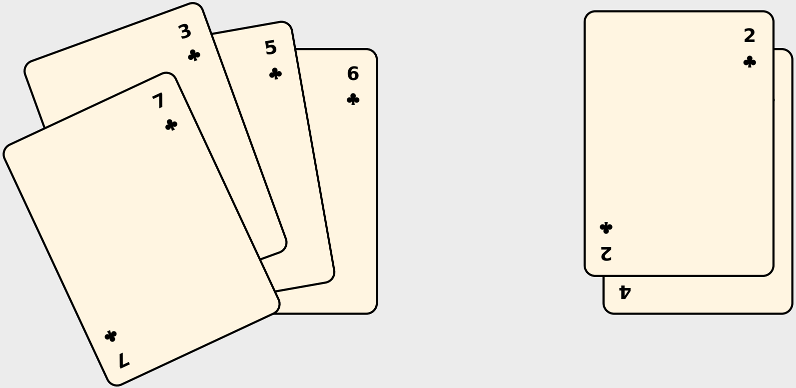

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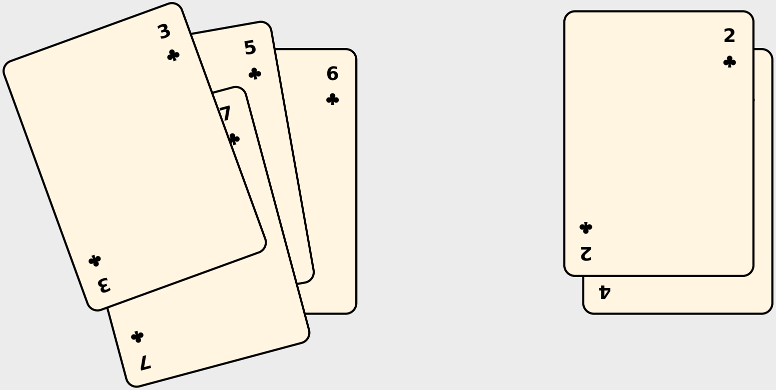

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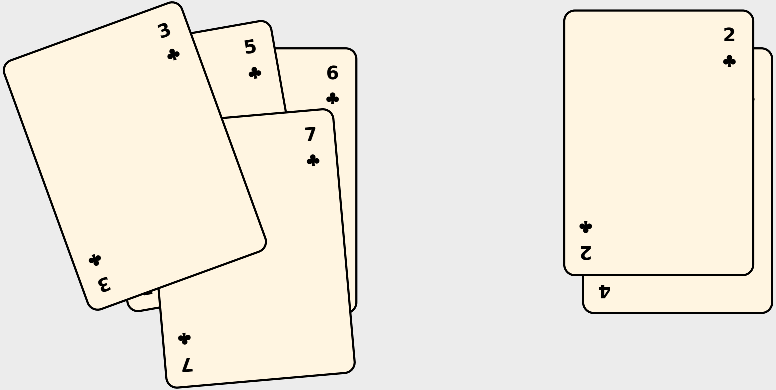

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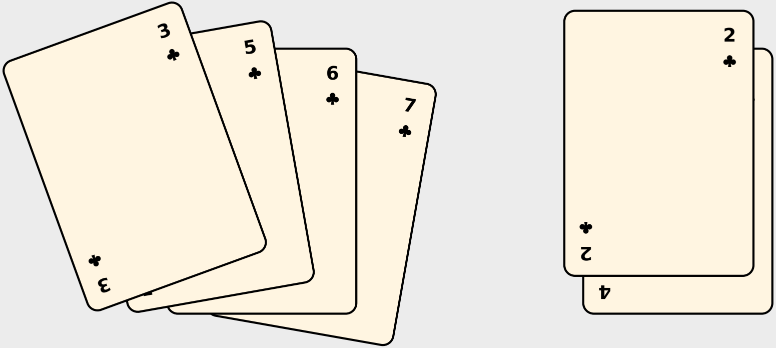

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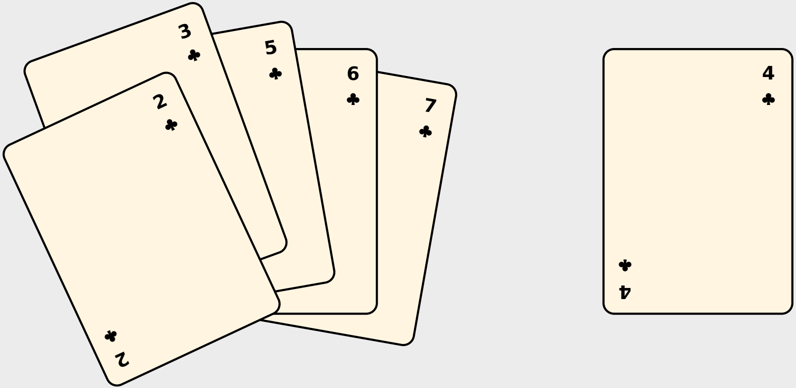

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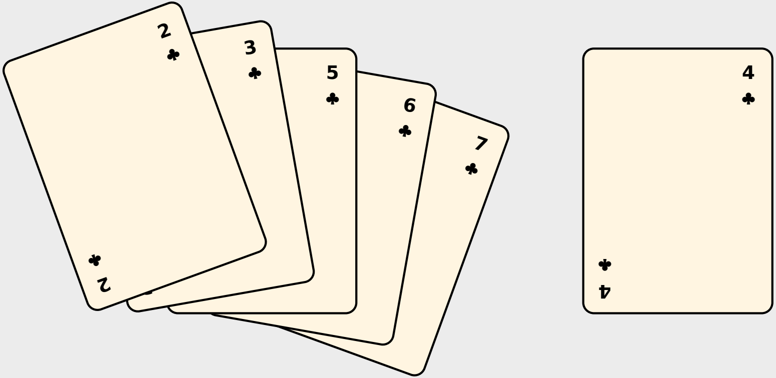

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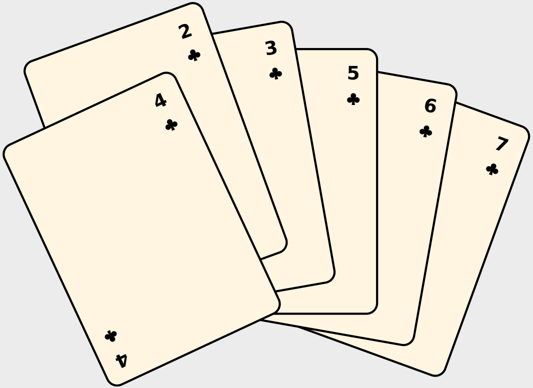

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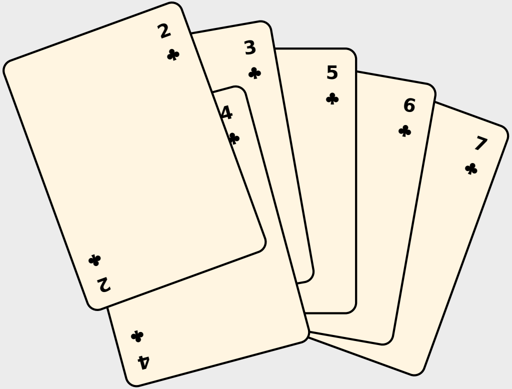

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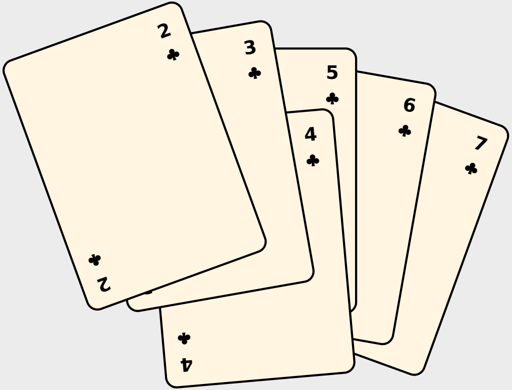

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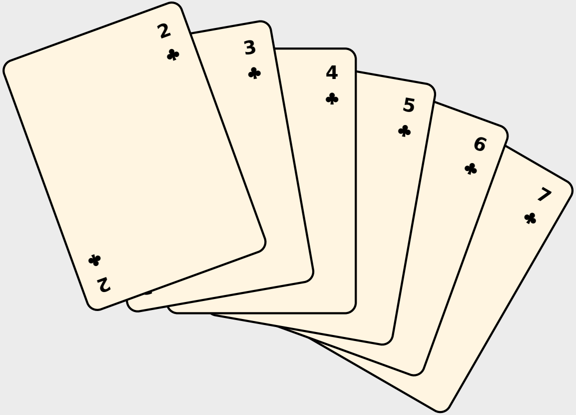

Definition (inserting an element into a sorted list)

```
Fixpoint insert (a: nat) (l: list nat): list nat ≜
  match l with
    | []    ⇒ [a]
    | x::xs ⇒ if Nat.leb a x then a::x::xs
              else x::(insert a xs)
  end.
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```

Definition (sorting by repeatedly inserting elements into the empty list)

```
Definition insertionsort ≜ foldr insert [].
```

Theorem

Insertion sort is a valid sorting algorithm

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Theorem

```
Theorem insertionsort_sorts l: is_sorted (insertionsort l) = true.
```

where

Definition (is sorted predicate)

```
Fixpoint is_sorted (l:list nat) : bool ≜
  match l with
    | nil ⇒ true
    | x::xs ⇒
      match xs with
        | nil ⇒ true
        | y::_ ⇒ (Nat.leb x y) && (is_sorted xs)
      end
  end.
```

Thanks! & Questions?