CENG 3549 – Functional Programming

Mathematical Induction & Induction over Lists & Structural Induction & Formal Verification of Functional Programs

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Outline

When to use Mathematical Induction?

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assume $P(k)$ (induction hypothesis), show $P(k+1)$

Why does this Work?

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  2. for arbitrary k, if P true for k then P true for $k+1$
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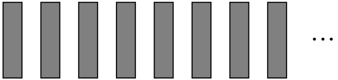

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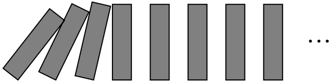

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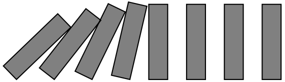

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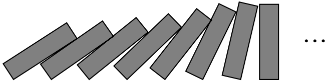

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Induction Principle

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1. Mathematical Induction
2. Structural Induction
3. The Coq Proof Assistant

Structural Induction

- proof principle similar to mathematical induction
- works in the domain of recursively defined structures (not only in $\mathbb{N}$)

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data List a where
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Remark (notes)

- lists are recursive structures
- non-recursive constructor (base case): []
- recursive constructor (step case): x:xs

Induction Principle for Lists – Informally

- to show $P(l)$ for all lists l
- show base case: $P([])$
- show step case: $P(xs) \implies P(x:xs)$ for arbitrary x and xs

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Remark

- for lists, P can be seen as function `p :: [a] -> Bool`

Definition (Append)

$$
\begin{aligned}
[\,] \quad &+\!\!+\ l_2 = l_2 \\
(x{:}xs)\ &+\!\!+\ l_2 = x : (xs +\!\!+ l_2)
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Lemma (Append is associative)

$\mathbin{+\!\!+}$ is associative, that is,

$$(l_1 \mathbin{+\!\!+} l_2) \mathbin{+\!\!+} l_3 \;=\; l_1 \mathbin{+\!\!+} (l_2 \mathbin{+\!\!+} l_3)$$

Definition (Length)

```
length []      =  0
length (_:xs)  =  1 + length xs
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Lemma (Length and append)

length of combined list is sum of lengths, that is,

$$\texttt{length}\,(l_1 +\!\!+\; l_2) \;=\; \texttt{length}\; l_1 + \texttt{length}\; l_2$$

Example (Binary Trees)

```
data BTree a where
       Empty :: BTree a
       Node  :: a -> BTree a -> BTree a -> BTree a
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Induction Principle for Binary Trees

$$(P(\mathsf{Empty}) \land \forall x.\, \forall l.\, \forall r.\, ((P(l) \land P(r)) \implies P(\mathsf{Node}\ x\ l\ r))) \implies \forall t.\, P(t)$$

Example (Perfect Binary Trees)

- a binary tree is perfect if all leaf nodes have same depth

```
perfect Empty        = True
perfect (Node _ l r) =
  height l == height r && perfect l && perfect r

height Empty        = 0
height (Node _ l r) =
  max (height l) (height r) + 1

size Empty        = 0
size (Node _ l r) = size l + size r + 1
```

Lemma

a perfect binary tree t of height n has exactly $2^n - 1$ nodes, that is,

$$P(t) = \left(\texttt{perfect}\ t \implies \texttt{size}\ t = 2^{\texttt{height}\ t} - 1\right)$$

Example

```
data Term where
        Var    :: String -> Term
        Lambda :: String -> Term -> Term
        App    :: Term   -> Term -> Term
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General Structures – Induction Principle

- for every non-recursive constructor, show base case
  - base case: $P(\texttt{Var x})$
- for every recursive constructor, show step case
  - step case 1: $(P(\texttt{s}) \wedge P(\texttt{t})) \implies P(\texttt{App s t})$
  - step case 2: $P(\texttt{t}) \implies P(\texttt{Lambda x t})$

Outline

Curry-Howard Isomorphism

Logic	∼	Type Theory
Proposition		Type
Proof		Program
⋮		⋮

The Coq Proof Assistant

- by Thierry Coquand (1985)
  - implements Curry-Howard Isomorphism
  - provides recursors + inductors

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- In CIC, proofs are programs and formulae are types
- CIC is higher-order: quantifiers over predicates (propositional valued function) allowed

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1. Coq allows for
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impredicativity + LEM ~~+ computation~~	Prop : Type
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Coq Features (Stratification of Type)

$$Type_0 : Type_1 : Type_2 : Type_3 \cdots$$

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- terms can also be definitionaly equal (next slide, ι-reduction)

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- runs termination checker over every single recursive definition

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  - Program Fixpoint: takes a measure as an argument and generates a proof obligation that the measure decreases in each recursive call

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`split`	replaces a goal consisting of a conjunction $P \wedge Q$ with two sub-goals P and Q

Coq Features (Proofs: Basic Logical Reasoning)

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Lemma ex1_v1: ∀ (A B: Prop), A ∨ B → B ∨ A.
Proof. intros A B H.
       destruct H as [ H | H ].
       - apply or_intror.
         exact H.
       - apply or_introl.
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Qed.
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         exact H.
       - apply or_introl.
         exact H.
Qed.
```

```
Lemma ex1_v2: ∀ (A B: Prop), A ∨ B → B ∨ A.
Proof. intros A B H.
       destruct H as [ H | H ].
       - right.
         exact H.
       - left.
         exact H.
Qed.
```

```
Lemma ex2_v1: ∀ (A B: Prop), A ∧ B → B ∨ A.
Proof. intros A B H.
       destruct H as (H1, H2).
       left.
       exact H2.
Qed.
```

```
Lemma ex2_v2: ∀ (A B: Prop), A ∧ B → B ∨ A.
Proof. intros A B H.
       destruct H as (H1, H2).
       right.
       exact H1.
Qed.
```

Coq Features (Proofs: Basic Logical Reasoning (cont'd))

```
Theorem ex3: ∀ (X: Type) (P: X → Prop),
  ~ (∃ (x: X), P x) → (∀ (x: X), ~ P x).
Proof. intros X P H x.
       unfold not.
       unfold not in H.
       intro px.
       apply H.
       ∃ x.
       exact px.
Qed.
```

Coq Features (Proofs: Basic Logical Reasoning (cont'd))

```
Theorem ex3: ∀ (X: Type) (P: X → Prop),
  ~ (∃ (x: X), P x) → (∀ (x: X), ~ P x).
Proof. intros X P H x.
       unfold not.
       unfold not in H.
       intro px.
       apply H.
       ∃ x.
       exact px.
Qed.
```

```
Theorem ex4: ∀ (X: Type) (P: X → Prop),
  (∀ (x: X), ~ P x) → ~ (∃ (x: X), P x).
Proof. intros X P H.
       unfold not.
       intro He.
       unfold not in H.
       destruct He as (x, He).
       specialize (H x He).
       exact H.
Qed.
```

Coq Features (Proofs: Basic Classical Logical Reasoning)

```
Axiom LEM: ∀ P: Prop, P ∨ ~P.
```

Coq Features (Proofs: Basic Classical Logical Reasoning)

```
Axiom LEM: ∀ P: Prop, P ∨ ~P.
```

```
Theorem dne: ∀ P, ~~P → P.
Proof. intros P H.
       unfold not in H.
       specialize (LEM P); intro HL.
       destruct HL as [ HL | HL ].
       - exact HL.
       - unfold not in HL.
         specialize (H HL).
         contradiction.
Qed.
```

Coq Features (Proofs: Basic Classical Logical Reasoning)

```
Axiom LEM: ∀ P: Prop, P ∨ ~P.
```

```
Theorem dne: ∀ P, ~~P → P.
Proof. intros P H.
       unfold not in H.
       specialize (LEM P); intro HL.
       destruct HL as [ HL | HL ].
       - exact HL.
       - unfold not in HL.
         specialize (H HL).
         contradiction.
Qed.
```

```
Theorem ex5: ∀ (X: Type) (P: X → Prop),
  ~ (∀ (x: X), ~ P x) → (∃ (x: X), P x).
Proof. intros X P H.
       specialize (LEM (∃ (x: X), P x)); intros HL.
       destruct HL as [ HL | HL ].
       - exact HL.
       - unfold not in *.
         destruct H.
         intros x px.
         apply HL.
         ∃ x.
         exact px.
Qed.
```

Coq Features (Proofs: Basic Equational Reasoning)

```
Lemma eq_trans_v1:
 ∀ (A: Type) (a b c: A), a = b → b = c → a = c.
Proof. intros A a b c Ha Hb.
       specialize (@eq_ind A b (fun x ⇒ a = x) Ha c Hb); intro H.
       simpl in H.
       exact H.
Qed.
```

Coq Features (Proofs: Basic Equational Reasoning)

```
Lemma eq_trans_v1:
 ∀ (A: Type) (a b c: A), a = b → b = c → a = c.
Proof. intros A a b c Ha Hb.
       specialize (@eq_ind A b (fun x ⇒ a = x) Ha c Hb); intro H.
       simpl in H.
       exact H.
Qed.
```

```
Lemma eq_trans_v2:
 ∀ (A: Type) (a b c: A), a = b → b = c → a = c.
Proof. intros A a b c Ha Hb.
       induction Ha.
       exact Hb.
Qed.
```

Coq Features (Proofs: Basic Equational Reasoning)

```
Lemma eq_trans_v1:
 ∀ (A: Type) (a b c: A), a = b → b = c → a = c.
Proof. intros A a b c Ha Hb.
       specialize (@eq_ind A b (fun x ⇒ a = x) Ha c Hb); intro H.
       simpl in H.
       exact H.
Qed.
```

```
Lemma eq_trans_v2:
 ∀ (A: Type) (a b c: A), a = b → b = c → a = c.
Proof. intros A a b c Ha Hb.
       induction Ha.
       exact Hb.
Qed.
```

```
Lemma eq_trans_v3:
 ∀ (A: Type) (a b c: A), a = b → b = c → a = c.
Proof. intros A a b c Ha Hb.
       rewrite Ha.
       exact Hb.
Qed.
```

Coq Features (Proofs: Basic Equational Reasoning (cont'd))

```
Lemma add_comm: ∀ (a b: nat), a + b = b + a.
```

Thanks! & Questions?