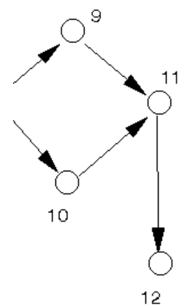
CPM and PERT

SPRING



NETWORK MODELS CPM & PERT

- Civil engineers are responsible for the planning, design and construction of the projects.
- Series of activities → in a specified sequence

- How long will the project take?
- When will we be able to start a particular task?
- If this task is not completed on time, will the entire project be delayed?
- Which tasks should we speed up (crash) in order to finish the project earlier?

Introduction

- CPM → Critical Path Method
- PERT → Program Evaluation Review Technique

- By using these tools, project manager can control the utilization of personnel, materials, and facilities and complete the project at an optimal time and cost.

- CPM was develop by DuPont de Nemours Company 1950's
- PERT was develop by U.S Navy 1958

Similarities and Differences

- They both require the network formulation to represent the activities of the project and their precedence relationships

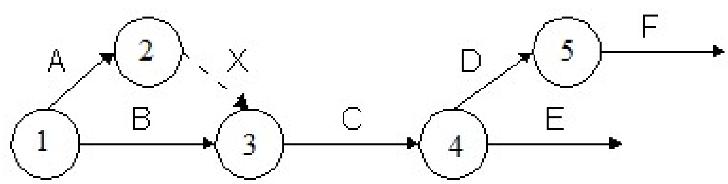
- The major difference between two methods that PERT incorporates **uncertainities** in activity times.

- CPM assumes that the activity times are proportional to the amount of resources allocated to them. Also it uses the prior experience with similar projects to obtain the relationships between resources and activity times.

PROJECT NETWORKS

Arcs in the network indicates the individual activities and the direction of the arc shows the job sequences.

- Nodes represent specific points in time and are called events. They mark the completion of activities directed to that node.
- If two or more activities need to start at event i and go to event j , a dummy is used. In order to avoid uncertainty in the job sequence and the completion time of the dummy job is always zero.



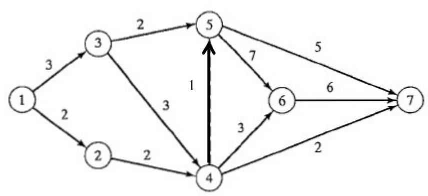
Linear Programming (CPM)

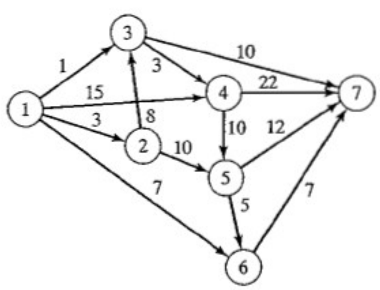
CPM requires the time estimates for each activity of the project and the precedence relations between activities to be known.

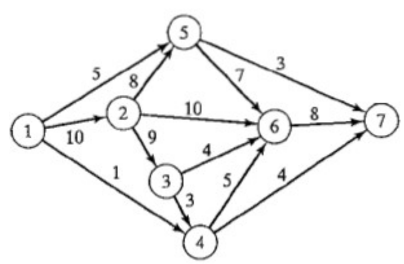
- The time required to complete each activity (T_{ij}) is estimated based on prior experience with similar projects.
- In CPM analysis the objective is to determine ***the minimum time required to complete the project and to identify the critical jobs*** whose delay can delay the entire project.

EXAMPLE

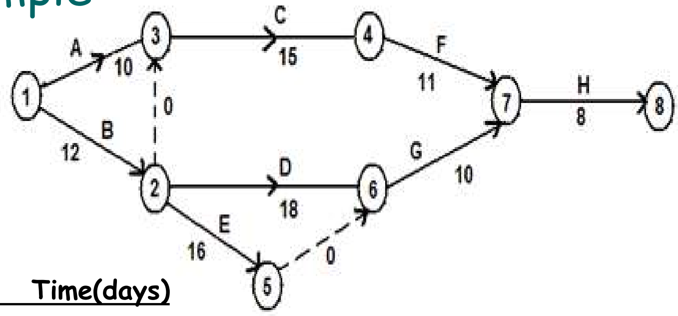
	Activity	Predecessor(s)	Duration (days)
A:	Clear site	—	1
B:	Bring utilities to site	—	2
C:	Excavate	*A*	1
D:	Pour foundation	*C*	2
E:	Outside plumbing	*B, C*	6
F:	Frame house	*D*	10
G:	Do electric wiring	*F*	3
H:	Lay floor	*G*	1
I:	Lay roof	*F*	1
J:	Inside plumbing	*E, H*	5
K:	Shingling	*I*	2
L:	Outside sheathing insulation	*F, J*	1
M:	Install windows and outside doors	*F*	2
N:	Do brick work	*L, M*	4
O:	Insulate walls and ceiling	*G, J*	2
P:	Cover walls and ceiling	*O*	2
Q:	Insulate roof	*I, P*	1
R:	Finish interior	*P*	7
S:	Finish exterior	*I, N*	7
T:	Landscape	*S*	3





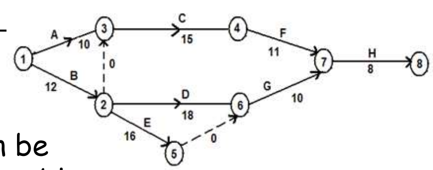


CPM Example



Job	Predecessor	Time(days)
A	-	10
B	-	12
C	A,B	15
D	B	18
E	B	16
F	C	11
G	D,E	10
H	F,G	8

Linear Programming (CPM)

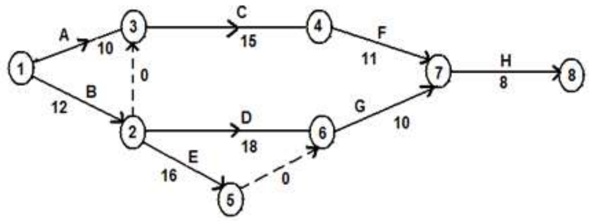


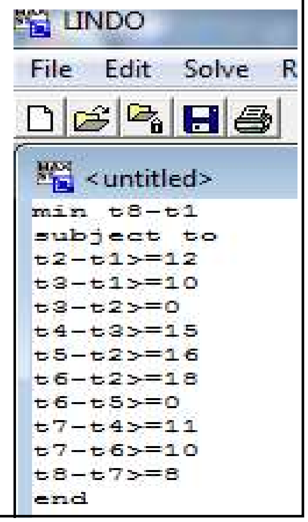
- The preliminary CPM problem can be solved by formulating it as an LP problem.
- Let "t_i" represent the time when all the activities directed toward node i are completed and any activity directed away from the node "i" can start.
- The objective of the problem is to minimize the time required to complete project.
- The constraints of the linear program ensures that the time available to complete the activity (i,j), should be greater or equal to the time required to complete the activity , T_{ij}

LP - CPM Example

General formulation for minimizing the project completion time :

- Minimize $Z = t_n - t_1$
- Subject to: $t_j - t_i \geq T_{ij}$
- $t_i \geq 0 \quad i=1,2,\ldots,n$





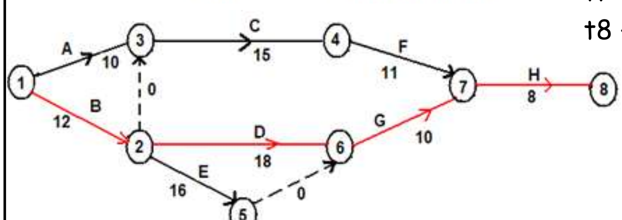
LP - CPM Example

Optimal solution obtained from the LP problem :

1) 48.00000

VARIABLE	VALUE
T8	48.000000
T1	0.000000
T2	12.000000
T3	12.000000
T4	27.000000
T5	30.000000
T6	30.000000
T7	40.000000

t2 - t1 >= 12 ; 12- 0 = 12
t3 - t2 >= 0 ; 12- 12 = 0
t3 - t1 >= 10 ; 12 - 0 = 12
t5 - t2 >= 16 ; 30 - 12 = 18
t6 - t2 >= 18 ; 30 - 12 = 18
t4 - t3 >= 15 ; 27 - 12 = 15
t7 - t4 >= 11 ; 40 - 27 = **13**
t7 - t6 >= 10 ; 40 - 30 = 10
t8 - t7 >= 8 ; 48 - 40 = 8



LP - CPM

- Erliest and latest occurance times of events, will enable managers to determine the flexibility in the non-critical jobs.

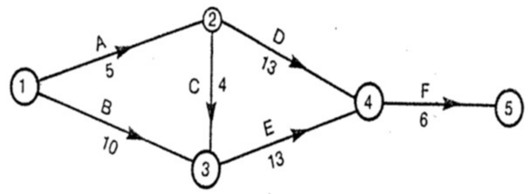
- EST Earliest Start Time
- LST (without delay) Latest Start Time
- EFT Earliest Finish Time
- LFT (without delay) Latest Finish Time
- Slack Time $(tl_j - te_i - T_{ij})$

Homework

Write a linear programming formulation to solve the following CPM problem if the numbers on the arcs indicate the project completion times.

The objective is to determine the minimum time required to complete the project.

Solve the problem using Lindo or another computer package and find the critical path.



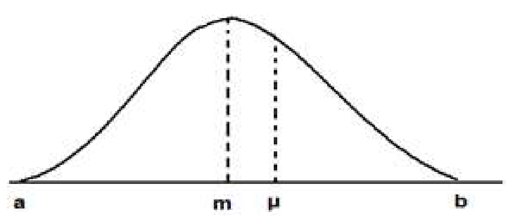
PERT (Program Evaluation Review Technique)

- Not based on prior experience,
- Uncertainities in the completion times of various activities can be included.

- PERT uses three different types of estimates of the activity times
- A most likely time (m) normal
- An optimistic time (a) min
- An pessimistic time (b) unusual circumstances

Program Evaluation Review Technique

- A Beta distribution is assumed to represent the probability distribution of activity times.



Beta distribution for the activity time

Program Evaluation Review Technique

- The expected time to complete an activity :

$$\mu = \frac{a + 4m + b}{6}$$

- The variance of the activity :

$$\sigma^2 = \left(\frac{b - a}{6}\right)^2$$

- The expected project duration is determined by summing the expected activity times along the critical path.

PERT Example

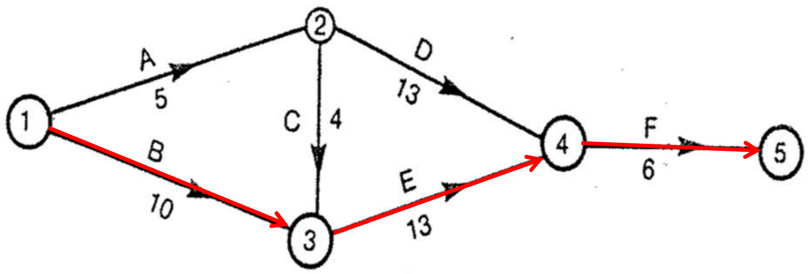
JOB	PREDECESSORS	OPTIMISTIC TIME, a	MOST LIKELY TIME, m	PESSIMISTIC TIME, b
A	-	2	5	8
B	-	7	10	13
C	A	3	4	5
D	A	9	12	21
E	B, C	5	14	17
F	D ,E	3	6	9

$$\mu = \frac{a + 4m + b}{6}$$

$$\sigma^2 = \left(\frac{b - a}{6}\right)^2$$

JOB	EXPECTED DURATION	VARIANCE
A	5	1
B	10	1
C	4	1/9
D	13	4
E	13	4
F	6	1

PERT Example



Minimize $Z = t_n - t_1$

Subject to: $t_j - t_i \geq T_{ij}$

$t_i \geq 0 \quad i=1,2,\ldots\ldots,n$

PROJECT COMPLETION TIME PROBABILITIES

Activitiy times are assumed to have Beta distribution and be independent of each other.

- The expected project duration is equal to sum of expected activity times along the critical path.
- Under these assumptions the **Project Completion Time** has a **Normal Distribution** with
- **Mean** ======➔ Expected Project Duration
- **Variance** ===➔ Sum of critical activity variances

Tables of the Normal Distribution

Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Example : Project Completion Times

Since

$E(T) = 29$ days and $\sigma = (6)^{1/2}$ days

Find the probabilities of

a) Completing the project 3 days before
b) Completing the project at most two days after

Solution

a) Completion within 26 days

$$\text{Prob } (T \leq 26) = \text{Prob } \left(S \leq \frac{26 - 29}{\sqrt{6}}\right) = 0.11$$

b) Completion at most 31 days

$$\text{Prob } (T \leq 31) = \text{Prob } \left(S \leq \frac{31 - 29}{\sqrt{6}}\right) = 0.79$$

LP - CPM Example

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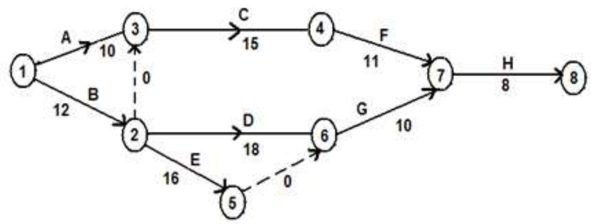

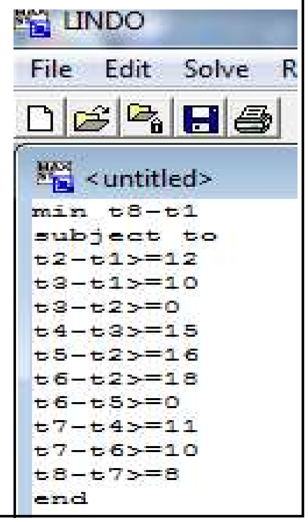

LP - CPM Example

Optimal solution obtained from the LP problem :

```
      1)        48.00000

VARIABLE         VALUE
      T8       48.000000
      T1        0.000000
      T2       12.000000
      T3       12.000000
      T4       27.000000
      T5       30.000000
      T6       30.000000
      T7       40.000000
```

t2 - t1 >= 12 ; 12- 0 = 12
t3 - t2 >= 0 ; 12- 12 = 0
t3 - t1 >= 10 ; 12 - 0 = 12
t5 - t2 >= 16 ; 30 - 12 = 18
t6 - t2 >= 18 ; 30 - 12 = 18
t4 - t3 >= 15 ; 27 - 12 = 15
t7 - t4 >= 11 ; 40 - 27 = **13**
t7 - t6 >= 10 ; 40 - 30 = 10
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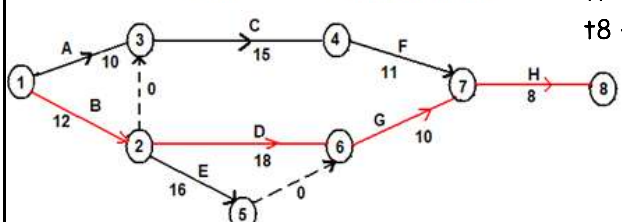


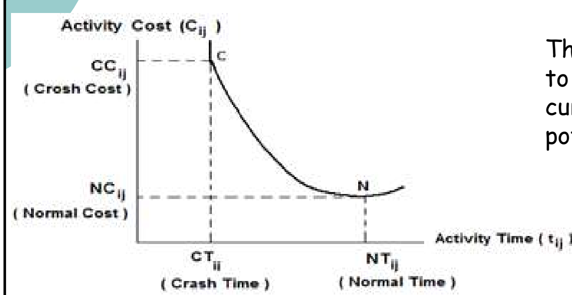
TIME-COST TRADE-OFFS FOR CPM

In preliminary CPM analysis, the objective is to determine the minimum time required to complete the project at minimum cost.

- CPM assumes that the activity times can be shortened to a certain extent by assigning additional resources (capital,labor,machines,materials,etc.) to an activity.
- Shortening the duration of an activity is known as crashing and additional cost for **crashing** is called as **crash cost**
- Crashing activities increase the total direct cost of the project, but the reduction in project completion time will result in other advantages.

TIME-COST TRADE-OFFS FOR CPM

- The basic objective of CPM is to determine how the project activities are to be expedited so that total cost of crashing is minimized and the project is completed at a required time.

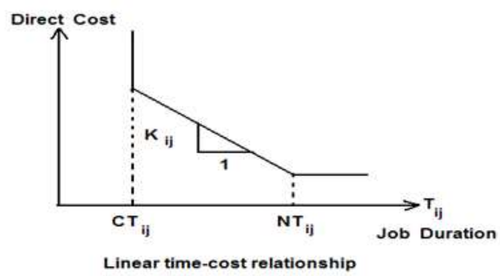


Time Cost Curve for an Activity

The first step in such analysis is to develop the activity time-cost curves for each job that has a potential of being compressed

Crashing by LP

- Assume linear time-cost relationships for every activity



- Kij denotes the unit cost of crashing the job (i,j)

Crashing by LP

The problem is to minimize the total cost of crashing and to complete the project by predetermined time TMAX.

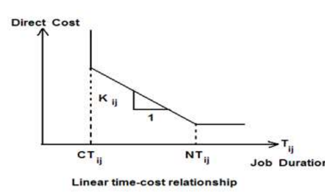
Minimize $Z = \sum K_{ij} * (NT_{ij} - T_{ij})$

Subject to: $t_j - t_i \geq T_{ij}$

$CT_{ij} \leq T_{ij} \leq NT_{ij}$

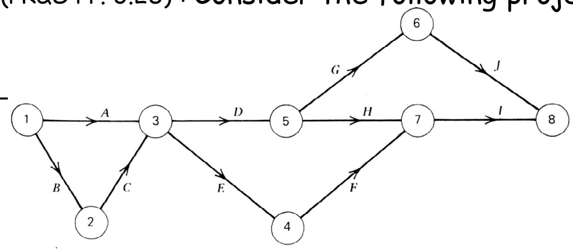
$t_n - t_1 \leq TMAX$

$t_i \geq 0 \quad i=1,2,....,n.$



- The solution of the LP yields the optimal cost of crashing "**Z**" and the best activity time Tij.

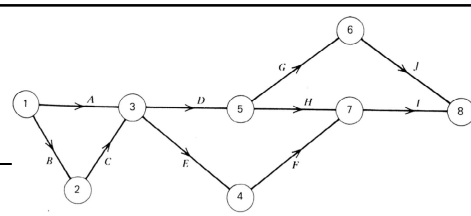
Example (PR&S Pr. 3.28) : Consider the following project network.



Activity	Normal Time	Crash Time	Unit Cost of Crashing (TL)
A	10	7	4
B	5	4	2
C	3	2	2
D	4	3	3
E	5	3	3
F	6	3	5
G	5	2	1
H	6	4	4
I	6	4	3
J	4	3	3

Find the normal and crash time of the project.

Example



Min Z = t8 – t1

Subject to $t3-t1 \geq T_A$ $\quad$ $t2-t1 \geq T_B$

$t3-t2 \geq T_C$ $\quad$ $t5-t3 \geq T_D$

$t4-t3 \geq T_E$ $\quad$ $t7-t4 \geq T_F$

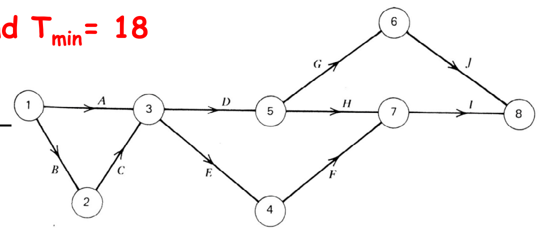
$t6-t5 \geq T_G$ $\quad$ $t7-t5 \geq T_H$

$t8-t7 \geq T_I$ $\quad$ $t8-t6 \geq T_J$

Maximum and minimum project times can be found by solving the above LP with Normal and Crash times, respectively.

T_{maks} = 27 Critical path : A, E, F, I.
T_{min} = 18 Critical path : A, D, H, I.

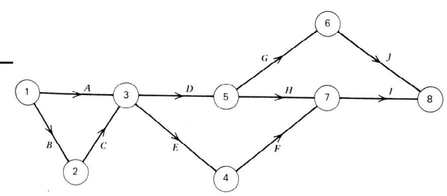
Example 1 T_{maks} = 27 and T_{min} = 18



Activity	Normal Time	Crash Time	Unit Cost of Crashing (TL)
A	10	7	4
B	5	4	2
C	3	2	2
D	4	3	3
E	5	3	3
F	6	3	5
G	5	2	1
H	6	4	4
I	6	4	3
J	4	3	3

Write an LP that would solve the problem of completing the project economically in 21 days.

Activity	Normal Time	Crash Time	Unit cost of Crashing (10^6TL)
A	10	7	4
B	5	4	2
C	3	2	2
D	4	3	3
E	5	3	3
F	6	3	5
G	5	2	1
H	6	4	4
I	6	4	3
J	4	3	3



Min Z= $4(10-t_a) + 2(5-t_b) + 2(3-t_c) + 3(4-t_d) + 3(5-t_e)$
$+ 5(6-t_f) + 1(5-t_g) + 4(6-t_h) + 3(6-t_I) + 3(4-t_j)$

Subject to $t8 - t1 \leq 21$

$t3 - t1 \geq t_a$ $t2 - t1 \geq t_b$ $t3 - t2 \geq t_c$ $t5 - t3 \geq t_d$

$t4 - t3 \geq t_e$ $t7 - t4 \geq t_f$ $t6 - t5 \geq t_g$ $t7 - t5 \geq t_h$

$t8 - t7 \geq t_I$ $t8 - t6 \geq t_j$

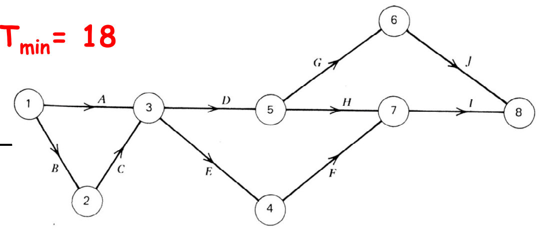
$7 \leq t_a \leq 10$

$4 \leq t_B \leq 5;$ $2 \leq t_C \leq 3;$ $3 \leq t_D \leq 4;$

$3 \leq t_E \leq 5;$ $3 \leq t_F \leq 6;$ $2 \leq t_G \leq 5;$

$4 \leq t_H \leq 6;$ $4 \leq t_I \leq 6;$ $3 \leq t_J \leq 4;$

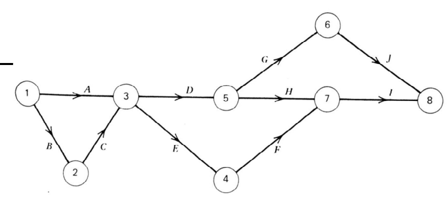
Example 2 T_{maks} = 27 and T_{min}= 18



Activity	Normal Time	Crash Time	Unit Cost of Crashing (TL)
A	10	7	4
B	5	4	2
C	3	2	2
D	4	3	3
E	5	3	3
F	6	3	5
G	5	2	1
H	6	4	4
I	6	4	3
J	4	3	3

Write an LP that would solve the problem of determining the optimal project time if a weekly bonus of 5 Million TL is given for early completion.

Activity	Normal Time	Crash Time	Unit cost of Crashing (10^6TL)
A	10	7	4
B	5	4	2
C	3	2	2
D	4	3	3
E	5	3	3
F	6	3	5
G	5	2	1
H	6	4	4
I	6	4	3
J	4	3	3



Min Z= $4(10-t_a) + 2(5-t_b) + 2(3-t_c) + 3(4-t_d) + 3(5-t_e)$
$+ 5(6-t_f) + 1(5-t_g) + 4(6-t_h) + 3(6-t_I) + 3(4-t_j)$
$- 5 * \{27 - (t8 - t1)\}$

Subject to

$t3 - t1 \geq t_a$ $t2 - t1 \geq t_b$ $t3 - t2 \geq t_c$ $t5 - t3 \geq t_d$
$t4 - t3 \geq t_e$ $t7 - t4 \geq t_f$ $t6 - t5 \geq t_g$ $t7 - t5 \geq t_h$
$t8 - t7 \geq t_I$ $t8 - t6 \geq t_j$

$7 \leq t_a \leq 10$

$4 \leq t_B \leq 5;$ $2 \leq t_C \leq 3;$ $3 \leq t_D \leq 4;$
$3 \leq t_E \leq 5;$ $3 \leq t_F \leq 6;$ $2 \leq t_G \leq 5;$
$4 \leq t_H \leq 6;$ $4 \leq t_I \leq 6;$ $3 \leq t_J \leq 4;$

Extensions

If the problem is solved by creating new variable $D_{ij} = NT_{ij} - T_{ij}$ (amount of time activity is crashed), one can obtain the amount of shortening directly.

- Suppose an additional budget of NEWB dollars is available to crash the activities.

$$\sum (K_{ij} \, . \, D_{ij}) \leq NEWB$$

- The indirect cost of the project vary linearly with the project duration.

$$\text{Min } Z = \sum K_{ij} \, . \, D_{ij} + I\,(t_n - t_1)$$

- Let B represent the additional benefits due to completion of the project before predetermined time TSET.

$$\text{Min } Z = \sum K\,ij \, . \, Dij - B\,(TSET - t_n)$$

Thank you for your attention

Do you have any questions?