INTEGER PROGRAMMING

Some or all decision variables are limited to be **INTEGER**

Examples : Number of workers
Pipe diameters (limited to commercial pipe sizes)

PURE	All variables are integer
MIXED	Some variables are integer
ZERO – ONE	Integer variables are limited to 0 – 1

One way is to solve the problem without integer restrictions and truncate or round-off the result.

Maximize Z = 10 X1

Subject to 5 X1 + 3X2 $\leq$ 24

X1 integer and non negative
X2 non - negative

Solution without IP restriction :

X1 = 4.8 X2 = 0 Z = 48

Rounding of X1 = 5.0

Is it feasible????

Maximize $Z = X1 + X2$

Subject to
$-8\,X1 + 10\,X2 \leq 13$
$-2\,X1 + 2\,X2 \geq 1$
X1 & X2 non-negative integers

Solution — Ignoring integer restrictions

	X1 = 4.0	X2 = 4.5	Z= 8.5
Truncating	X1 = 4.0	X2 = 4	Not feasible
or	X1 = 4.0	X2 = 5	Not feasible
IP solution	X1 = 1	X2 = 2	Z= 3

Maximize $Z = X1 + 1.9\,X2$

Subject to $1.3\,X1 + 2.2\,X2 \leq 7.15$

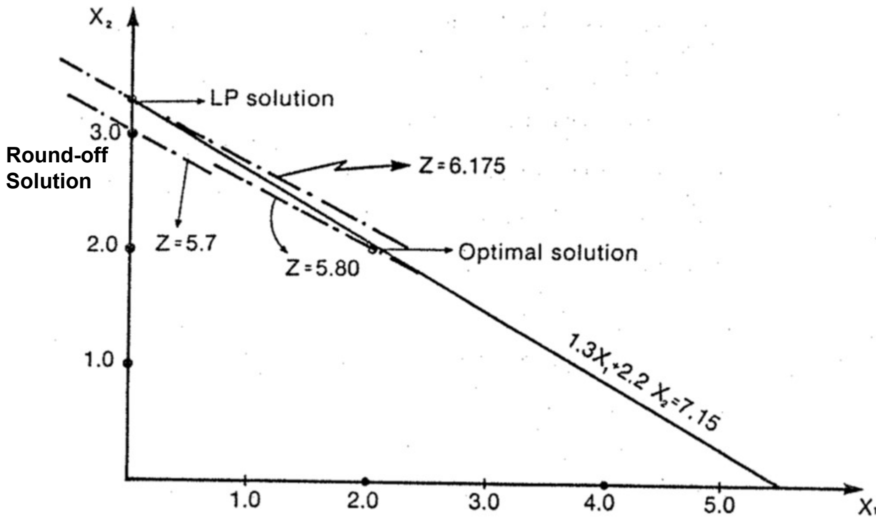
X1 & X2 non-negative integers

Solution — Ignoring integer restrictions

	X1 = 0.0	X2 = 3.25	Z= 6.175
Truncating	X1 = 0	X2 = 3	Z = 5.7

Feasibility satisfied!!

IP solution	X1 = 2	X2 = 2	Z= 5.8



Another disadvantage of rounding of or truncating will be the computational effort

Assume a solution

X1 = 2.5
X2 = 3.5
X3 = 6.5
X4 = 7.5

One should check the feasibility for

these 16 combinations

and compare the results

X1	X2	X3	X4
2	3	6	7
3	3	6	7
2	4	6	7
3	4	6	7
2	3	7	7
3	3	7	7
2	4	7	7
3	4	7	7
2	3	6	8
3	3	6	8
2	4	6	8
3	4	6	8
2	3	7	8
3	3	7	8
2	4	7	8
3	4	7	8

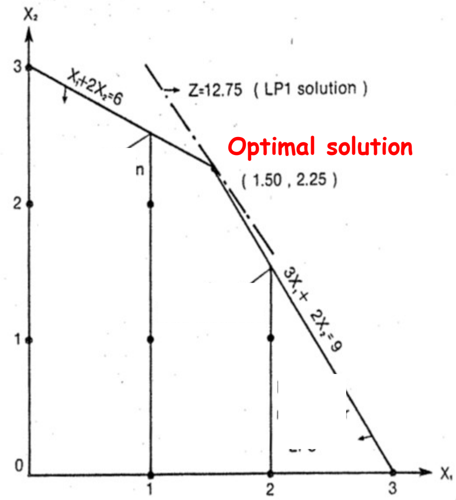
BRANCH & BOUND ALGORITHM

Maximize Z = 4 X1 + 3 X2

Sub to X1 + 2 X2 $\leq$ 6
3 X1 + 2 X2 $\leq$ 9
X1 , X2 $\geq$ 0
and integers

STEP 1

Solve LP 1 ignoring integer restrictions



Maximize Z = 4 X1 + 3 X2

Sub to X1 + 2 X2 $\leq$ 6
3 X1 + 2 X2 $\leq$ 9
X1 , X2 $\geq$ 0

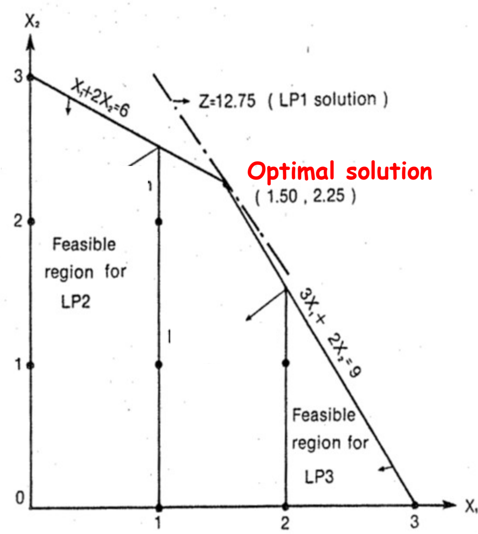
STEP 2

Since solution is fractional

Form two new LP's by adding constraints that eliminates current solution but includes all integer points

LP 2 ……… X1 $\leq$ 1

LP 3 ………. X1 $\geq$ 2



LP 2 :
Maximize Z = 4 X1 + 3 X2

Sub to X1 + 2 X2 $\leq$ 6
3 X1 + 2 X2 $\leq$ 9
X1 $\leq$ 1
X1 & X2 $\geq$ 0

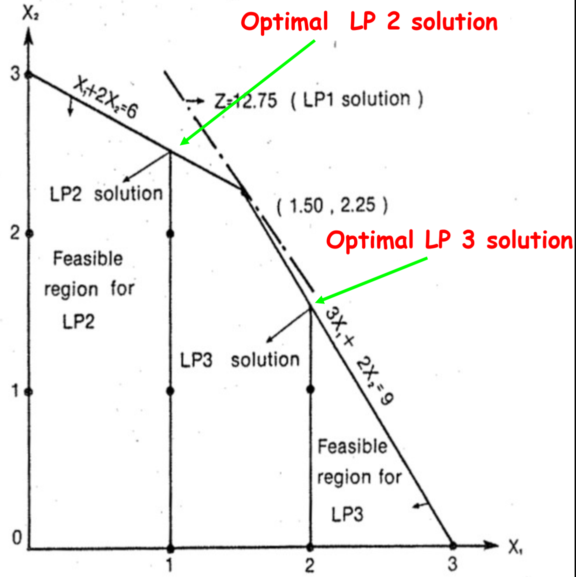
Solution Z2 = 11.5

X1 = 1 X2 = 2.5

LP 3 :
Maximize Z = 4 X1 + 3 X2

Sub to X1 + 2 X2 $\leq$ 6
3 X1 + 2 X2 $\leq$ 9
X1 $\geq$ 2
X1 & X2 $\geq$ 0

Solution Z3 = 12.5 (2, 1.5)



LP 2 :

Maximize Z = 4 X1 + 3 X2

Sub to X1 + 2 X2 $\leq$ 6
3 X1 + 2 X2 $\leq$ 9
X1 $\leq$ 1
X1 & X2 $\geq$ 0

Solution Z2 = 11.5 (1,2.5)

LP 3 :

Maximize Z = 4 X1 + 3 X2

Sub to X1 + 2 X2 $\leq$ 6
3 X1 + 2 X2 $\leq$ 9
X1 $\geq$ 2
X1 & X2 $\geq$ 0

Solution Z3 = 12.5 (2, 1.5)

Both solutions are fractional. Thus, one should branch from both sides.

However, since

Z3 > Z2

branching should start from LP3 side

Since it may be possible to find

an integer solution with optimal value

greater than Z2.

Z3 = 12.5 (2, 1.5)

LP 4 :

Maximize Z = 4 X1 + 3 X2

Sub to
$$\begin{aligned} X1 + 2\,X2 &\le 6 \\ 3\,X1 + 2\,X2 &\le 9 \\ X1 &\ge 2 \\ X2 &\le 1 \\ X1\ \&\ X2 &\ge 0 \end{aligned}$$

Solution Z4 = 37/3 (7/3, 1)

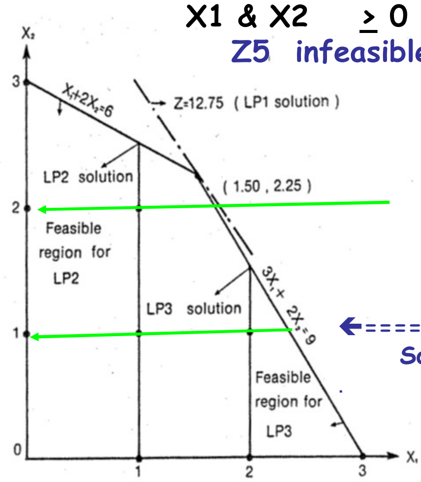
Since Z4 is fractional and the value is greater than Z2 one should continue branching from this side.

LP 5 :

Maximize Z = 4 X1 + 3 X2

Sub to
$$\begin{aligned} X1 + 2\,X2 &\le 6 \\ 3\,X1 + 2\,X2 &\le 9 \\ X1 &\ge 2 \\ X2 &\ge 2 \\ X1\ \&\ X2 &\ge 0 \end{aligned}$$

Z5 infeasible



<==== LP 4 Solution

Z4 = 37/3 (7/3, 1)

LP 6 :

Maximize Z = 4 X1 + 3 X2

Sub to
$$\begin{aligned} X1 + 2\,X2 &\le 6 \\ 3\,X1 + 2\,X2 &\le 9 \\ X1 &\ge 2 \\ X2 &\le 1 \\ X1 &\le 2 \\ X1\ \&\ X2 &\ge 0 \end{aligned}$$

Solution Z6 = 11

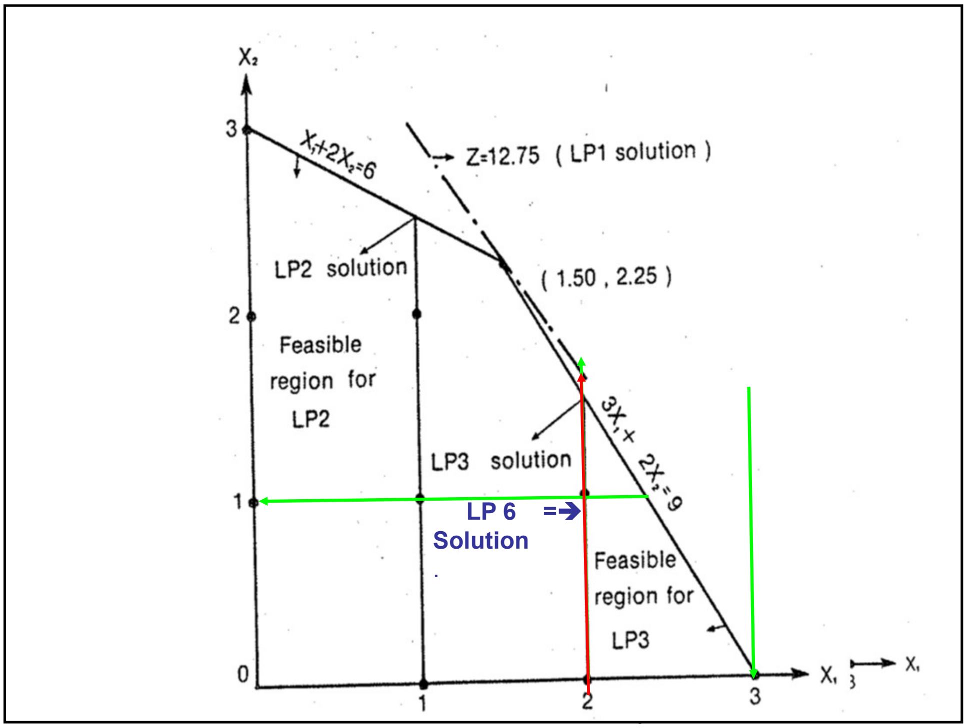
X1= 2   X2 = 1

Integer solution found. Lower bound for the problem.

LP 7 :

Maximize Z = 4 X1 + 3 X2

Sub to
$$\begin{aligned} X1 + 2\,X2 &\le 6 \\ 3\,X1 + 2\,X2 &\le 9 \\ X1 &\ge 2 \\ X2 &\le 1 \\ X1 &\ge 3 \\ X1\ \&\ X2 &\ge 0 \end{aligned}$$



Z4 = 37/3 (7/3, 1)

LP 6 :

Maximize Z = 4 X1 + 3 X2

Sub to
X1 + 2 X2 $\leq$ 6
3 X1 + 2 X2 $\leq$ 9
X1 $\geq$ 2
X2 $\leq$ 1
X1 $\leq$ 2
X1 & X2 $\leq$ 0

Solution Z6 = 11

X1= 2 X2 = 1

LP 7 :

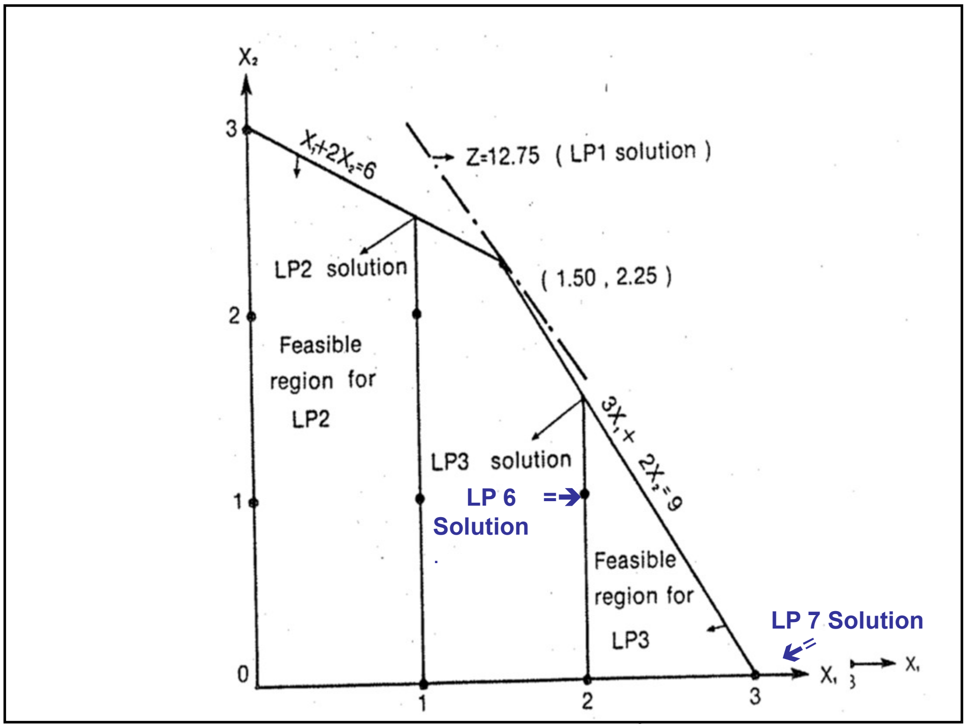
Maximize Z = 4 X1 + 3 X2

Sub to
X1 + 2 X2 $\leq$ 6
3 X1 + 2 X2 $\leq$ 9
X1 $\geq$ 2
X2 $\leq$ 1
X1 $\geq$ 3
X1 & X2 $\geq$ 0

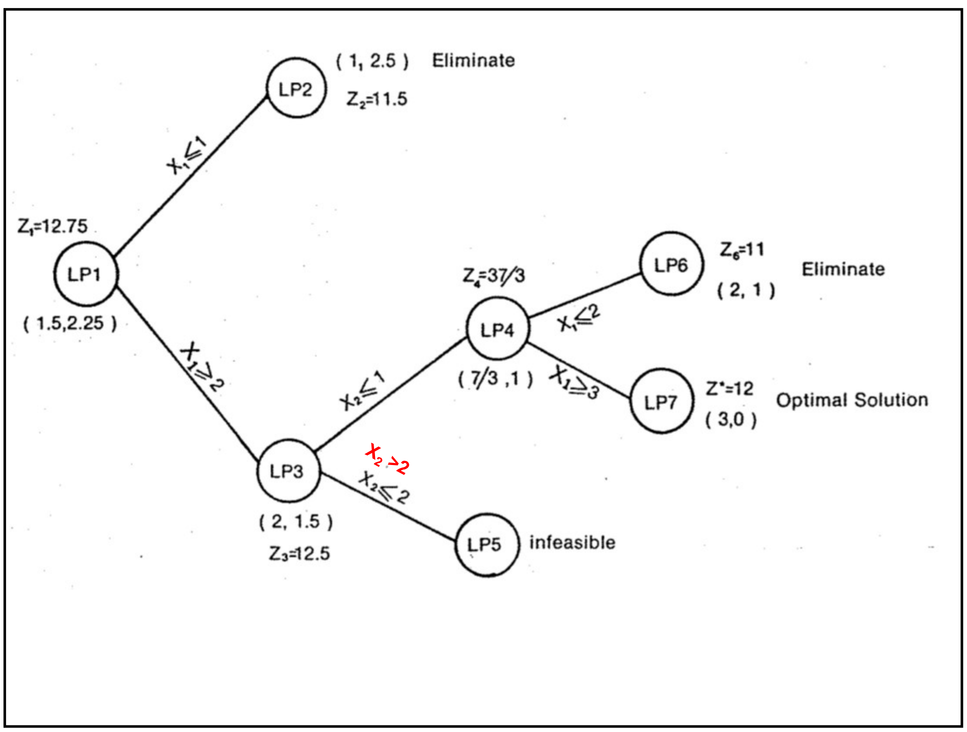
Solution Z7 = 12

X1 = 3 X2 = 0

New integer solution. Lower bound revised.



X_2
3
$X_1+2X_2=6$
Z=12.75 (LP1 solution)
LP2 solution
(1.50 , 2.25)
2
Feasible region for LP2
$3X_1+2X_2=9$
LP3 solution
1
LP 6 Solution
Feasible region for LP3
LP 7 Solution
0
1
2
3
X_1



LP2
(1, 2.5) Eliminate
Z_2=11.5
$X_1 \leq 1$
Z_1=12.75
LP1
(1.5,2.25)
$X_1 \geq 2$
LP3
(2, 1.5)
Z_3=12.5
$X_2 \leq 1$
$X_2 > 2$
$X_2 \leq 2$
LP5
infeasible
Z_4=37/3
LP4
(7/3 ,1)
$X_1 \leq 2$
$X_1 \geq 3$
LP6
Z_6=11
(2, 1)
Eliminate
LP7
Z^*=12
(3,0)
Optimal Solution

Homework

- Maximize $Z = 21\ X1 + 11\ X2$

  Subject to $7\ X1 + 4\ X2 + X3 = 13$

  X1 , X2 & X3 non-negative integers

- Minimize $Z = 10\ X1 + 9\ X2$

Subject to

$$X1 \le 8$$
$$X2 \le 10$$
$$5\ X1 + 3\ X2 \ge 45$$
$$X1 \ge 0$$
$$X2 \ge 0 \text{ \& integer}$$

LINDO Integer Options

- **INT X** Variable X is limited to [0-1]
- **INT n** First n-variables are limited to [0-1]
- **INT ALL** All variables are limited to [0-1]

- How can you handle $X \le 4$ & integer
- Let $X = y_1 + y_2 + y_3 + y_4$ where y_i [0-1]
- But what if the upper limit is too high?
- $X = y_1 + 2\ y_2 + 4\ y_3 + 8\ y_4 + 16\ y_5$ ……
- In addition
- **GIN Y** **Makes Y a general integer variable.**

0-1 Integer Variables

Consider design of pipe distribution systems.

X3 = {0, 50, 100, 200}

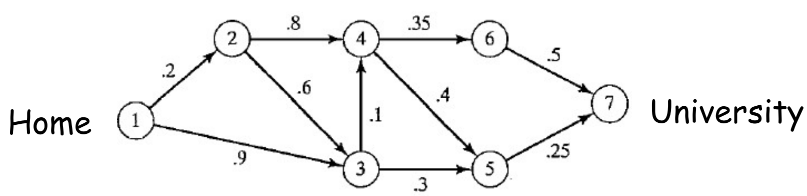
X3 = 50 Y1 + 100 Y2 + 200 Y3

Y1 + Y2 + Y3 $\leq$ 1

Y1, Y2, Y3 {0-1}

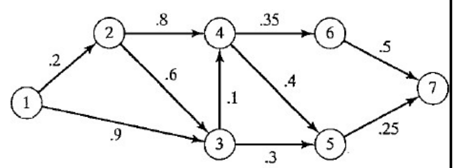
SHORTEST ROUTE PROBLEMS

Determine the shortest time it will take the driver to reach the destination if the numbers on the arcs indicate the driving times between nodes.



One way to solve this problem is to reinterpret the situation as you would like to send one unit of flow from node 1 to node 7 at "minimum cost".

SHORTEST ROUTE PROBLEMS



If T_{ij} is the travel time between i &j,

Let X_{ij} [0-1] variables for every link

X_{ij} = 1 if the link is used X_{ij} = 0 if the link is not used

Then LP can be formulated as to

$$MIN\ Z = \Sigma \Sigma\ T_{ij} * X_{ij}$$

Subject to Continuities at nodes

& nonnegativities

ASSIGNMENT MODELS

Optimal assignment of resources (man, machine, vehicle) to tasks,

Let a machine shop has N machines **M1, M2,…,MN**.

A group of N different jobs **J1, J2,…., JN** are to be assigned to these machines.

For each job (i) assigned to machine (j), the machining cost is Cij

The problem is to assign the jobs to the machines which will minimize the total cost of machining.

$$MIN\quad Z = \Sigma \Sigma\ C_{ij} * X_{ij}$$

Subject to $\Sigma X_{ij} \leq 1$ i = 1,2,….,I

$\Sigma X_{ij} \geq 1$ j = 1,2,….,J

$X_{ij} \geq 0$ for all links

0-1 Integer Variables

Maximize $Z = X1^2 + (X2 * X3) - X3^2$

Subject to $-2X1 + 3X2 + X3 \leq 3$

$X1, X2, X3 \quad \{0\text{-}1\}$

$Xi^2 = Xi$ .... $Z = X1 + (X2 * X3) - X3$

X2 * X3 is either 0 or 1

Introduce new {0-1} variable Y

$$X2 + X3 - Y \leq 1$$
$$-X2 - X3 + 2Y \leq 0$$

X2	X3	X2*X3
0	0	0
1	0	0
0	1	0
1	1	1

0-1 Integer Variables

$$X2 + X3 - Y \leq 1$$
$$-X2 - X3 + 2Y \leq 0$$

X2	X3	X2*X3	$X2 + X3 - Y \leq 1$	$-X2 - X3 + 2Y \leq 0$	Y
0	0	0	$-Y \leq 1$	$2Y \leq 0$	0
1	0	0			
0	1	0			
1	1	1			

0-1 Integer Variables

$X2 + X3 - Y \leq 1$
$- X2 - X3 + 2Y \leq 0$

X2	X3	X2*X3	$X2 + X3 - Y \leq 1$	$- X2 - X3 + 2Y \leq 0$	Y
0	0	0	$- Y \leq 1$	$2Y \leq 0$	0
1	0	0	$- Y \leq 0$	$2Y \leq 1$	0
0	1	0			
1	1	1			

0-1 Integer Variables

$X2 + X3 - Y \leq 1$
$- X2 - X3 + 2Y \leq 0$

X2	X3	X2*X3	$X2 + X3 - Y \leq 1$	$- X2 - X3 + 2Y \leq 0$	Y
0	0	0	$- Y \leq 1$	$2Y \leq 0$	0
1	0	0	$- Y \leq 0$	$2Y \leq 1$	0
0	1	0	$- Y \leq 0$	$2Y \leq 1$	0
1	1	1			

0-1 Integer Variables

$X2 + X3 - Y \leq 1$
$- X2 - X3 + 2Y \leq 0$

X2	X3	X2*X3	$X2 + X3 - Y \leq 1$	$- X2 - X3 + 2Y \leq 0$	Y
0	0	0	$- Y \leq 1$	$2Y \leq 0$	0
1	0	0	$- Y \leq 0$	$2Y \leq 1$	0
0	1	0	$- Y \leq 0$	$2Y \leq 1$	0
1	1	1	$-Y \leq -1$	$2Y \leq 2$	1

0-1 Integer Variables

Maximize $Z = X1^2 + (X2 * X3) - X3^2$

Subject to $-2\ X1 + 3\ X2 + X3 \leq 3$
$X1, X2, X3 \quad \{0\text{-}1\}$

$Xi^2 = Xi$ .... $Z = X1 + (X2 * X3) - X3$

Introduce new {0-1} variable Y

$X2 + X3 - Y \leq 1$
$- X2 - X3 + 2Y \leq 0$

Maximize Z = X1 + Y – X3
with additional constraints

$X2 + X3 - Y \leq 1$

$Y \leq X2$ No integer restriction for Y!

$Y \leq X3$

Multiple products : $X_1 * X_2 * X_3 * \ldots X_k$

$$\Sigma (x_j) - Y \leq k-1$$

$$-\Sigma (x_j) + k * Y \leq 0$$

For a continuous variable Y

$$\Sigma (x_j) - Y \leq k-1$$

$$Y \leq X_j \text{ for all } j$$

Either – or statements

Consider a problem where

Either $X1 + X2 \leq 2$........................ $X1 + X2 - 2 \leq 0$

or $2 X1 + 3 X2 \geq 8$.................. $-2 X1 - 3 X2 + 8 \leq 0$

Let XNEW {0-1} and M large positive number

$X1 + X2 - 2 \leq M * XNEW$

$- 2 X1 - 3 X2 + 8 \leq M * (1-XNEW)$

If ==➔ Then statements

If $X4 - 4 \leq 0$ then $6 - X5 \leq 0$
Otherwise $4.000001 - X4 \leq 0$ $X5 - 3 \leq 0$

Introduce new {0-1} variable Xnew and
let M be large positive number

$$X4 - 4 \leq M * Xnew$$
$$6 - X5 \leq M * Xnew$$
$$4.000001 - X4 \leq M * (1 - Xnew)$$
$$X5 - 3 \leq M * (1 - Xnew)$$

Question : How can you handle

At least two out of 4 constraints must be satisfied?