Linear Fractional Programming

References

- A. Charnes and W.W: Cooper, *Programming with Linear Fractional Functionals,* Naval Research Log. Quarterly, Vol.9, 181-186, 1962
- Stanley Zionts, *Programming with Linear Fractional Functionals,* Naval Research Log. Quarterly, Vol.15, 449-450, 1968
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Linear Fractional Programming

Objective Function is expressed as the ratio of two linear functions.

$$\text{Max } Z = \frac{c^*x + \alpha}{d^*x + \beta}$$

α & β are known constants

c & d are given vectors

Sub to $A^*x = b$; $x \geq 0$

Although the objective function neither convex nor concave the optimal solution is again at one of the corner points!!!

Example : Benefit / Cost analysis

Cargo Loading in Marine Transportation

Definition of terms :

N : # types of cargo's available
V : Total volume
W : Total weight

For each cargo type 'i', let

Q_i : Maximum tonnage
f_i : Freight rate (\$/ton)
v_i : volume for (ft^3/ton)
x_i : amount loaded (ton)

$$\text{Max } Z = \sum (f_i * x_i)$$

Sub to
$$0 \le x_i \le Q_i$$
$$\sum (v_i * x_i) \le V$$
$$\sum (x_i) \le W$$
$$x_i \ge 0$$

This objective maximizes the income for one loading!!!

Cargo Loading in Marine Transportation

t_i : Loading / unloading time per ton for cargo 'i'
t : Trip time between ports
C_1 : Cost (\$) per unit time when ship is in port \$3,000
C_2 : Cost (\$) per unit time when ship is at sea \$5,000

Maximizing the net return for one trip

$$\text{Max } Z = \sum (f_i * x_i) - C_1 \sum (2\, x_i * t_i) - C_2 * t$$

Maximizing the net revenue per unit time

$$\text{Max } Z = \frac{\sum (f_i * x_i) - C_1 \sum (2\, x_i * t_i) - C_2 * t}{\sum (2\, x_i * t_i) + t}$$

Charnes & Cooper Algorithm

Assumptions :

- There is at least one feasible solution
- The feasible region is bounded

It can always be bounded by adding upper bounds

PROBLEM P1

$$\text{Max } Z = \frac{c^*x + \alpha}{d^*x + \beta}$$

Sub to $A^*x = b$; $x \geq 0$

Transformation of variables :

Define $y = t * x$ where "t" is a nonnegative scalar

and for every **x** in feasible region **y** satisfies

$dy + \beta t = \nu$ where $\nu > 0$ and constant.

Multiply the objective function by t/t and constraint by t :

$$\text{Max } Z = \frac{c^*y + \alpha t}{d^*y + \beta t}$$

Since the denominator is constant ($dy + \beta t = \nu$)

PROBLEM P1

$$\text{Max } Z = \frac{c^*x + \alpha}{d^*x + \beta}$$

Sub to $A^*x = b$; $x \geq 0$

The problem reduces to **new Problem P2**

Max $Z = c^*y + \alpha t$

Sub to $A^*y - b\,t = 0$ after multiplying both sides by t

$$d\,y + \beta\,t = \nu$$

$$y, t \geq 0$$

Note : Every (y,t) which satisfies the constraints of P2 always feasible to P1 Except t=0

Lemma : Every feasible solution to P2 will have **$t > 0$**.

PROBLEM P1

$$\text{Max } Z = \frac{c^* x + \alpha}{d^* x + \beta}$$

Sub to $A^* x = b$; $x \geq 0$

PROBLEM P2

Max Z = $c^* y + \alpha t$

Sub to $A^* y - b\,t = 0$
$d\,y + \beta\,t = \nu$
$y,\ t \geq 0$

- **Theorem I :**
- Assume that P1 has a finite optimum where the denominator is always positive.
- Let (y^*, t^*) be optimal for P2
- Then y^*/t^* is optimal for P1.
- For any choice of ν y/t is the same so..... Use $\nu = 1$

- Example

PROBLEM P1

$$\text{Max } Z = \frac{3x + 2}{-x + 4}$$

Sub to $0 \leq x \leq 1$

1. The feasible region is not empty.
2. Feasible region is bounded
3. Denominator is positive for all x.
4. Use $\nu = 1$

PROBLEM P2

Max Z = $3y + 2t$

Sub to $y - t \leq 0$
$-y + 4t = 1$
$y,\ t \geq 0$

Optimal solution to LP2

$y^* = 1/3$ $t^* = 1/3$

Optimal solution to LP1

$x^* = y^*/t^* = 1$

PROBLEM P1

$$\text{Max } Z = \frac{c^*x + \alpha}{d^*x + \beta}$$

Sub to $A^*x = b$; $x \geq 0$

PROBLEM P2

Max $Z = c^*y + \alpha t$

Sub to $A^*y - b\,t = 0$

$d\,y + \beta\,t = \nu$

$y,\ t \geq 0$

- Suppose that we know that the denominator is negative at the optimum point. Then one can write P1 and P2 as

PROBLEM P1

$$\text{Max } Z = \frac{-c^*x - \alpha}{-d^*x - \beta}$$

Sub to $A^*x = b$; $x \geq 0$

PROBLEM P2

Max $Z = -\,c^*y - \alpha t$

Sub to $A^*y - b\,t = 0$

$-\,d\,y - \beta\,t = 1$

$y,\ t \geq 0$

- Can denominator be zero at optimum?
  - nr. = 0 not defined nr<0 Z ===➔ - ∞ can not be optimum
  - nr > 0 Z ===➔ ∞ unbounded....not practical

Theorem II : Any LFP problem (which has a finite optimum) can always be solved by solving at most 2 LP problems

PROBLEM P2

Max $Z = c^*y + \alpha t$

Sub to $A^*y - b\,t = 0$

$d\,y + \beta\,t = 1$

$y,\ t \geq 0$

PROBLEM P1

$$\text{Max } Z = \frac{c^*x + \alpha}{d^*x + \beta}$$

Sub to

$A^*x = b$; $x \geq 0$

PROBLEM P2

Max $Z = -\,c^*y - \alpha t$

Sub to $A^*y - b\,t = 0$

$-\,d\,y - \beta\,t = 1$

$y,\ t \geq 0$

Ziont's Extension (for a practical problem)

The denominator ($d^*x + \beta$) can NOT change its sign in the feasible region.

You have to find a solution anyway (to prove that feasible set is not empty)

Find the sign of the ($d^*x + \beta$) and solve the corresponding one.

PROBLEM P2 LP1

$$\text{Max } Z = c^*y + \alpha t$$

Sub to $\quad A^*y - b\,t = 0$

$\quad d\,y + \beta\,t = \nu$

$\quad y,\ t \geq 0$

PROBLEM P1

$$\text{Max } Z = \frac{c^*x + \alpha}{d^*x + \beta}$$

Sub to

$A^*x = b\ ;\ x \geq 0$

PROBLEM P2 LP2

$$\text{Max } Z = -\,c^*y - \alpha t$$

Sub to $\quad A^*y - b\,t = 0$

$\quad -\,d\,y - \beta\,t = 1$

$\quad y,\ t \geq 0$

REMARKS

- Find a feasible solution to P1

  Check the sign of the $(d^*x + \beta)$

  If $(d^*x + \beta) > 0$ =======➔ solve **LP1**

  If $(d^*x + \beta) < 0$ =======➔ solve **LP2**

- Both LP1 & LP2 can not have optimal solutions. One will always be infeasible.

- If both LP1 & LP2 are infeasible then $(d^*x + \beta) = 0$ for all feasible x.

Example

$$\text{Max } Z = \frac{-3 + 2X_1 + 4X_2 - 5X_3}{6 + 3X_1 - X_2}$$

Subject to $\quad X_1 - X_2 \geq 0$

$\quad 7X_1 + 9X_2 + 10X_3 \leq 30$

$\quad X_2 \geq 1$ & all variables nonnegative

$X_1 = 1\ \ X_2 = 1\ X_3 = 0$ is feasible.
Feasible region is bounded (by 2nd constraint)
For the feasible solution denominator is positive.

$$\text{Max } Z = 2y_1 + 4y_2 - 5y_3 - 3t$$

Subject to $\quad y_1 - y_2 \geq 0$

$\quad 7y_1 + 9y_2 + 10y_3 - 30t \leq 0$

$\quad y_2 - t \geq 0$

$\quad 3y_1 - y_2 + 6t = 1$

$\quad y_1 \geq 0\ ;\ y_2 \geq 0\ ;\ t \geq 0$

Solution :

$y_1 = 0.1923$
$y_2 = 0.1923$
$y_3 = 0$
$t = 0.10256$

Solution :

$x_1 = 1.875$
$x_2 = 1.875$
$x_3 = 0$
$Z^* = 0.846$

Homework

$$\text{Min } Z = \frac{2X_1 + 3X_2}{-X_1 - X_2}$$

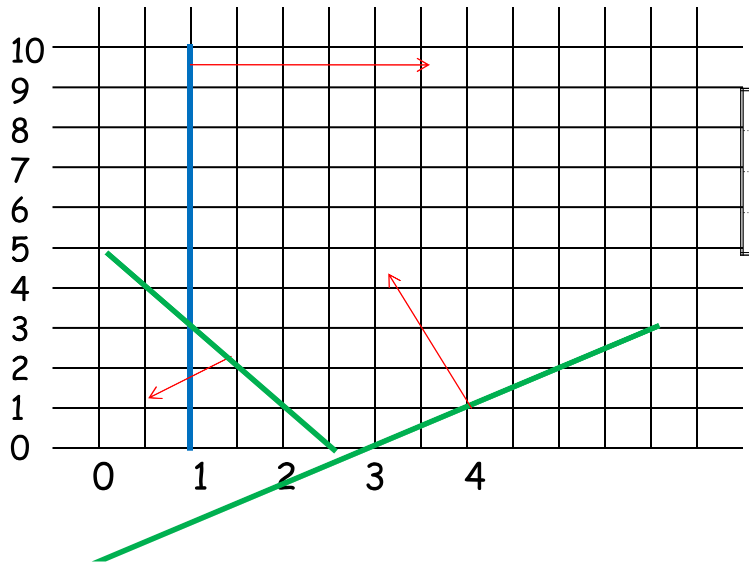
Subject to
$$2X_1 + X_2 \le 5$$
$$X_1 - X_2 \le 3$$
$$X_1 \ge 1 \quad \text{\& all variables nonnegative}$$

Convert the above LFP to an equivalent LP and solve.

$$\text{Min } Z = \frac{2X_1 + 3X_2}{-X_1 - X_2}$$

x1	x2	2x1 + X2	x1-x2	2x1+3x2	-x1-x2	Frac
1	0	2	1	2	-1	-2
1	3	5	-2	11	-4	-2.75
2.5	0	5	2.5	5	-2.5	-2

Subject to
$$2X_1 + X_2 \le 5$$
$$X_1 - X_2 \le 3$$
$$X_1 \ge 1 \quad \text{\& all variables nonnegative}$$



Nokta	X1	X2	Z
A	1	0	-2
B	1	3	-2.75
C	2.5	0	-2