UNIT HYDROGRAPH ESTIMATION

Singh, K. P. (1976) UH – A Comparative Study, AWRA, Water Resources Bulletin, Vol. 12 No 2, pp 381-390

Mays, L.W. & Coles, L. (1980) Optimization of Unit Hydrograph Determination, ASCE, Journal of the Hydraulics Division, Vol 106, pp 85-97

HYDROGRAPH ANALYSIS
Unit Hydrograph

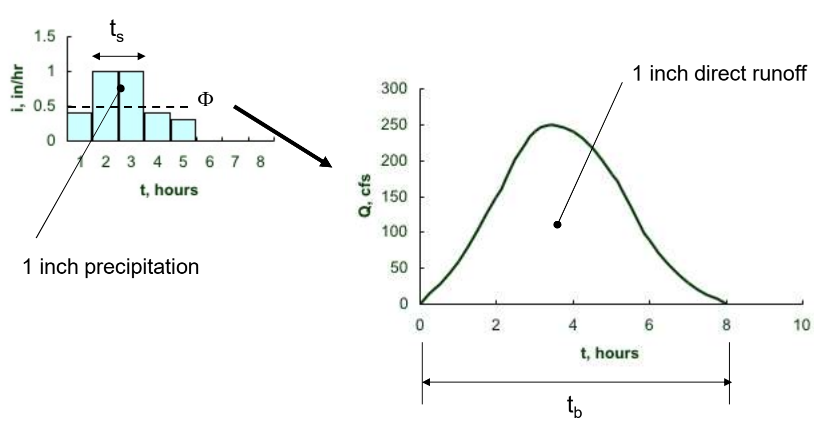
- **Unit hydrograph (UH)** of a basin: the hydrograph of surface runoff (direct runoff) resulting from 1 cm of **excess rainfall** generated uniformly over the basin area at a uniform rate during a specified period of time.
- First given by Sherman in 1932, then expanded by others.
- It is assumed that the UH is representative for the runoff process of a basin.
- Baseflow should be separated from total flow to find direct runoff, and all the losses should be subtracted from total precipitation before any analysis.

HYDROGRAPH ANALYSIS
Unit Hydrograph

There are **five assumptions** for the unit hydrograph theory.

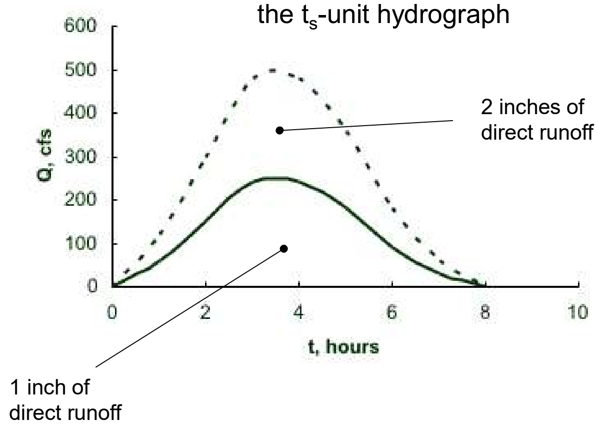
- **Excess rainfall** is uniformly distributed
  - **within a specified period of time.**
  - **within the basin area.**
- **Base time of direct runoff** is constant for a specified duration of rainfall.
- **Ordinates of the direct runoff hydrograph** of a specified duration rainfall are directly proportional to the total amount (depth) of direct runoff (= amount of excess precipitation).
- **UH** is **unique** for a basin.

Unit Hydrograph Definition



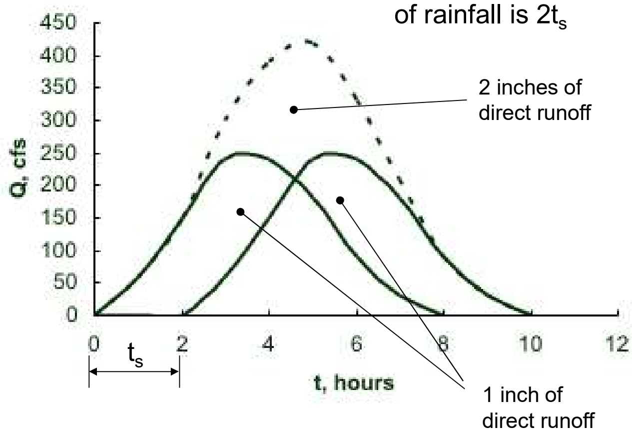
Proportional Ordinates

Ordinates are 2x those of the t_s-unit hydrograph



Superposition

Effective duration of rainfall is $2t_s$



HYDROGRAPH ANALYSIS
Unit Hydrograph Assumptions

- Ordinates of direct runoff hydrograph of a specified duration rainfall are directly proportional to the total amount of direct runoff (= net precip.).
  - principle of linearity, superposition (or proportionality) applicable
  - UHs of different durations or hydrographs of complex storms can only be obtained by applying this assumption.

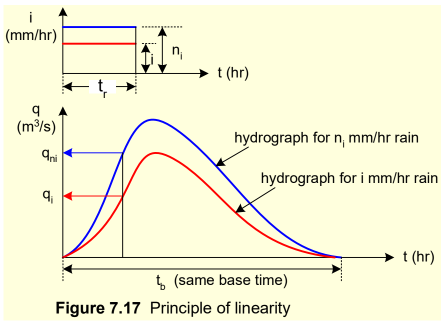
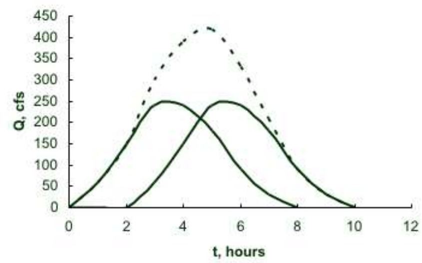
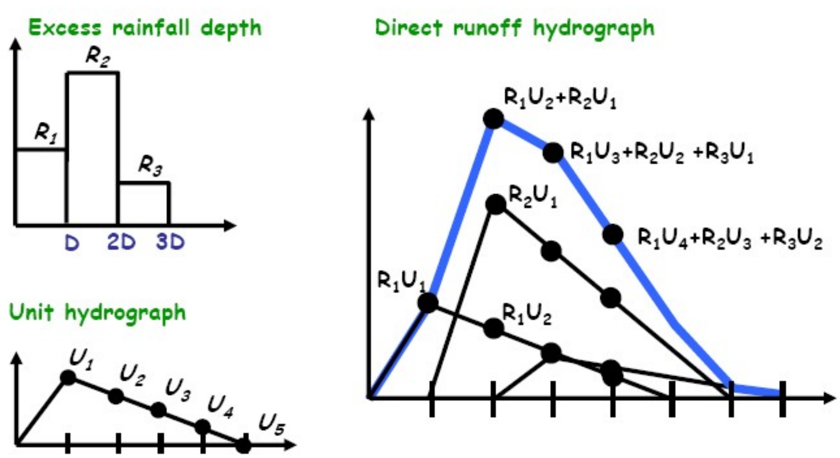


Figure 7.17 Principle of linearity

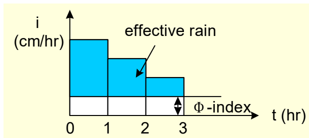


Runoff from complex storms

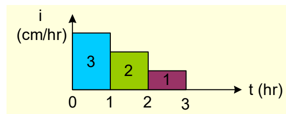


HYDROGRAPH ANALYSIS
Unit Hydrograph

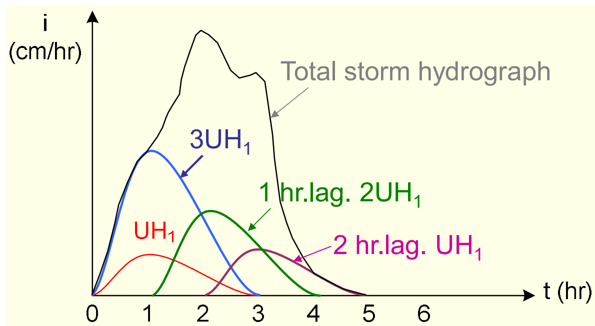
Total rainfall



Effective rainfall



Storm Hydrograph = $3UH_1$
+(1 hr lagged) $2UH_1$
+(2 hr lagged) UH_1



Example

Given a basin with the following 1-hour unit hydrograph and a Φ-index of 0.4 in/hour, determine the hydrograph corresponding to the rainfall given. Assume baseflow is 20 cfs.

t(hr)	UH(cfs)
0	0
1	80
2	240
3	200
4	80
5	20
6	0

t(hr)	i(in/hr)
1	2.4
2	3.4
3	0.3

Solution

	A	B	C
1	t(hrs)	i(in/hr)	i-Φ(in/hr)
2	1	2.4	2.0
3	2	3.4	3.0
4	3	0.3	0.0

Base Flow : 20 cfs

	A	B	C	D	E	F
1	t(hrs)	UH(cfs)	2xUH(cfs)	3xUH(cfs)	Stormflow(cfs)	Streamflow (cfs)
2	0	0	0	0	0	20
3	1	80	160	0	160	180
4	2	240	480	240	720	740
5	3	200	400	720	1120	1140
6	4	80	160	600	760	780
7	5	20	40	240	280	300
8	6	0	0	60	60	80
9	7	0	0	0	0	20
10	8	0	0	0	0	20

P_1 = 3 cm P_2 = 2 cm P_3 = 1.5cm

Base flow = 30

n	UH	3UH	2UH lag 1	1.5UH lag 2	Total	Base	Streamflow
1	40.4	121.2			121.20	30	151.20
2	107.9	323.7	80.8		404.50	30	434.50
3	234.3	702.9	215.8	60.60	979.30	30	1009.30
4	250.6	751.8	468.6	161.85	1382.25	30	1412.25
5	146.0	438.0	501.2	351.45	1290.65	30	1320.65
6	453.0	1359.0	292.0	375.90	2026.90	30	2056.90
7	381.1	1143.3	906.0	219.00	2268.30	30	2298.30
8	274.0	822.0	762.2	679.50	2263.70	30	2293.70
9	173.0	519.0	548.0	571.65	1638.65	30	1668.65
10			346.0	411.00	757.00	30	787.00
11				259.50	259.50	30	289.50

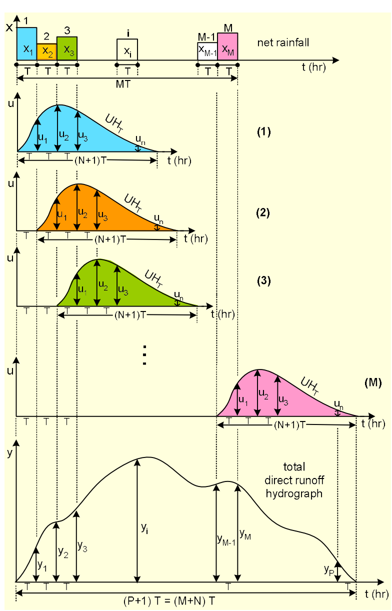
The unit hydrograph represents the time distribution of the direct stormflow resulting from 1 cm (inch) of net rainstorm of specified duration (t_s) and areal pattern.

The duration of the runoff period is termed the time base (t_b)

Let

Q_i Direct surface runoff ordinates (spaced at t hrs)
P_i The effective precipitation (spaced at t hrs)
U_i Unit Hydrograph ordinates (spaced at t hrs)

HYDROGRAPH ANALYSIS Unit Hydrograph



Direct Runoff =
$x_1\ UH_T$ +
(T hr lag) $x_2\ UH_T$ +
(2T hr lag) $x_3\ UH_T$ +
...
[(i-1) T hr lag] $x_i\ UH_T$ +
...
[(M-1) T hr lag] $x_M\ UH_T$

M hydrographs

Duration of rainfall = M T

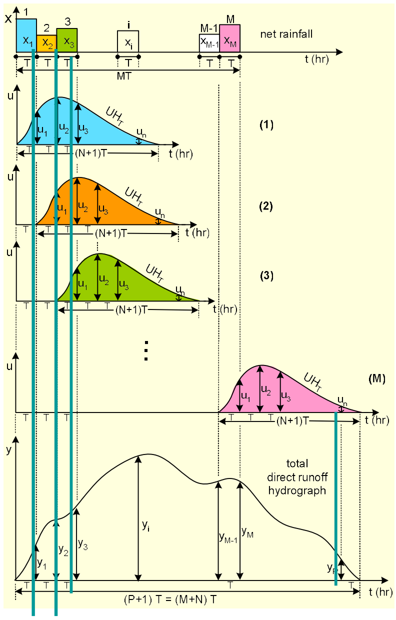
Duration of UH = (N+1) T

Duration of direct runoff = (P+1) T

= (M+N) T

(M+N) T = (P+1) T → N = P-M+1

HYDROGRAPH ANALYSIS Unit Hydrograph



ordinates of UH_T : $u_1, u_2, \dots, u_N$
ordinates of total DR : $y_1, y_2, \dots, y_P$
depth of rain blocks : $x_1, x_2, \dots, x_M$

Ordinates of total DR hydrograph:

$y_1 = x_1u_1$
$y_2 = x_2u_1 + x_1u_2$
$y_3 = x_3u_1 + x_2u_2 + x_1u_3$
.
.
$y_M = x_Mu_1 + x_{M-1}u_2 + x_{M-2}u_3 + \dots + x_{M-N+1}u_N$
.
.
$y_P = x_Mu_N$

Q_i Direct surface runoff ordinates (spaced at t hrs)
P_j The effective precipitation (spaced at t hrs)
U_m Unit hydrograph ordinates (spaced at t hrs)

$$Q_i = \sum_{m=1}^{M} (U_m * P_{i-m+1}) \quad \text{for } i = 1,2,\dots,I$$

$Q_1 = U_1 * P_1$

$Q_2 = U_2 * P_1 + U_1 * P_2$

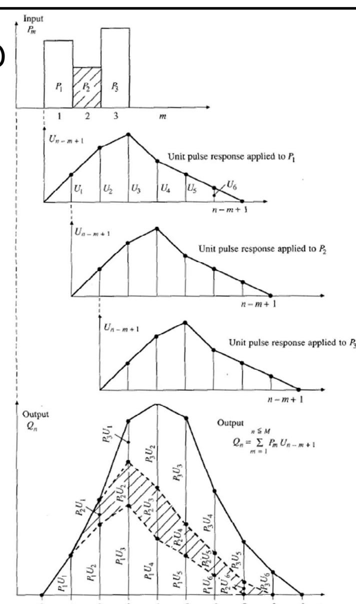
$Q_3 = U_3 * P_1 + U_2 * P_2 + U_1 * P_3$

$Q_4 = U_4 * P_1 + U_3 * P_2 + U_2 * P_3 + U_1 * P_4$

....

If # of periods of effective precipitation is J

$M = I - J + 1$



The problem is to find the "best estimates" of Unit Hydrograph oordinates for a given catchment using measured Direct Surface Runoff Oordinates and the effective precipitation data.

Q_i Direct surface runoff ordinates (spaced at t hrs)
P_j The effective precipitation (spaced at t hrs)
U_m Direct surface runoff ordinates (spaced at t hrs)

Constraints of the problem include
- Unit Hydrograph Formulation
- Unit Hydrograph Volume restriction
- Non-negativity conditions

$$\textbf{Minimize} = \sum_{i=1}^{I} (err_i)^2$$

Subject to

$$err_i = Qmeasured_i - Qest_i \quad \text{for all } i$$

$$Qest_i = \sum_{m=1}^{M} (U_m * P_{i-m+1}) \quad \text{for } i = 1,2,....,I$$

P (in)	Q (ft^3/sec)
0.7	55.1
1.7	363.4
1.2	917.3
	1198.2
Total 3.6	934.4
	539
	288.9
	143.7
	57.4
	10.3

The problem is to find best estimates of UH oordinates if the above data is measured for a catchment area of 1.94 sq. miles.

$$
\begin{aligned}
Qest_1 &= 0.7\,U_1 \\
Qest_2 &= 1.7\,U_1 + 0.7\,U_2 \\
Qest_3 &= 1.2\,U_1 + 1.7\,U_2 + 0.7\,U_3 \\
Qest_4 &= 1.2\,U_2 + 1.7\,U_3 + 0.7\,U_4 \\
Qest_5 &= 1.2\,U_3 + 1.7\,U_4 + 0.7\,U_5 \\
Qest_6 &= 1.2\,U_4 + 1.7\,U_5 + 0.7\,U_6 \\
Qest_7 &= 1.2\,U_5 + 1.7\,U_6 + 0.7\,U_7 \\
Qest_8 &= 1.2\,U_6 + 1.7\,U_7 + 0.7\,U_8 \\
Qest_9 &= 1.2\,U_7 + 1.7\,U_8 \\
Qest_{10} &= 1.2\,U_8
\end{aligned}
$$

P (in)	Q (ft^3/sec)
0.7	55.1
1.7	363.4
1.2	917.3
	1198.2
Total 3.6	934.4
	539
	288.9
	143.7
	57.4
	10.3

$Qest_1$ - 55.1 = err_1
$Qest_2$ - 363.4 = err_2
$Qest_3$ - 917.3 = err_3
$Qest_4$ - 1198.2 = err_4
$Qest_5$ - 934.4 = err_5
$Qest_6$ - 539.0 = err_6
$Qest_7$ - 288.9 = err_7
$Qest_8$ - 143.7 = err_8
$Qest_9$ - 57.4 = err_9
$Qest_{10}$ - 10.3 = err_{10}

P (in)	Q (ft^3/sec)
0.7	55.1
1.7	363.4
1.2	917.3
	1198.2
Total 3.6	934.4
	539
	288.9
	143.7
	57.4
	10.3

$$\text{Min } Z = \sum_{i=1}^{I} (err_i)^2$$

$Qest_1$ = 0.7 U_1
$Qest_2$ = 1.7 U_1 + 0.7 U_2
$Qest_3$ = 1.2 U_1 + 1.7 U_2 + 0.7 U_3
$Qest_4$ = 1.2 U_2 + 1.7 U_3 + 0.7 U_4
$Qest_5$ = 1.2 U_3 + 1.7 U_4 + 0.7 U_5
$Qest_6$ = 1.2 U_4 + 1.7 U_5 + 0.7 U_6
$Qest_7$ = 1.2 U_5 + 1.7 U_6 + 0.7 U_7
$Qest_8$ = 1.2 U_6 + 1.7 U_7 + 0.7 U_8
$Qest_9$ = 1.2 U_7 + 1.7 U_8
$Qest_{10}$ = 1.2 U_8

P (in)	Q (ft^3/sn)
0.7	55.1
1.7	363.4
1.2	917.3
Toplam 3.6	1198.2
	934.4
	539
	288.9
	143.7
	57.4
	10.3

$Qest_1$ - 55.1 = err_1
$Qest_2$ - 363.4 = err_2
$Qest_3$ - 917.3 = err_3
$Qest_4$ - 1198.2 = err_4
$Qest_5$ - 934.4 = err_5
$Qest_6$ - 539.0 = err_6
$Qest_7$ - 288.9 = err_7
$Qest_8$ - 143.7 = err_8
$Qest_9$ - 57.4 = err_9
$Qest_{10}$ - 10.3 = err_{10}

In order to find the unit hydrograph oordinates take the partial derivative of the function Z with respect to each Ui, equate to zero and solve simultaneous linear equations.

Since UH volume should be equal to 1 cm (inch)

$$k * \sum_{m=1}^{M} (U_m) = 1 \quad \text{and} \quad U_m \quad \text{non-negative}$$

time	Area	Precipitation	Runoff	UH	k
hr	-----	in	in / hr	in / hr	t
hr	sq km	cm	m3 /sec	m3 /sec	0.36t / A
hr	sq miles	in	cfs	cfs	(12*3600*t)/(5280*5280*A)

$$\text{Minimize} = \sum_{i=1}^{I} (ep_i + en_i)$$

Subject to

$$err_i = ep_i - en_i = Qmeasured_i - Qest_i \quad \text{for all } i$$

$$Qest_i = \sum_{m=1}^{M} (U_m * P_{i-m+1}) \quad \text{for } i = 1,2,....,I$$

$$k * \sum_{m=1} (U_m) = 1$$

All variables non negative

P (in)	Q (ft^3/sec)
0.7	55.1
1.7	363.4
1.2	917.3
	1198.2
Total 3.6	934.4
	539
	288.9
	143.7
	57.4
	10.3

The problem is to find best estimates of UH oordinates if the above data is measured for a catchment area of 1.94 sq. miles.

$$
\begin{aligned}
Qest_1 &= 0.7\,U_1 \\
Qest_2 &= 1.7\,U_1 + 0.7\,U_2 \\
Qest_3 &= 1.2\,U_1 + 1.7\,U_2 + 0.7\,U_3 \\
Qest_4 &= 1.2\,U_2 + 1.7\,U_3 + 0.7\,U_4 \\
Qest_5 &= 1.2\,U_3 + 1.7\,U_4 + 0.7\,U_5 \\
Qest_6 &= 1.2\,U_4 + 1.7\,U_5 + 0.7\,U_6 \\
Qest_7 &= 1.2\,U_5 + 1.7\,U_6 + 0.7\,U_7 \\
Qest_8 &= 1.2\,U_6 + 1.7\,U_7 + 0.7\,U_8 \\
Qest_9 &= 1.2\,U_7 + 1.7\,U_8 \\
Qest_{10} &= 1.2\,U_8
\end{aligned}
$$

P (in)	Q (ft^3/sec)
0.7	55.1
1.7	363.4
1.2	917.3
	1198.2
Total 3.6	934.4
	539
	288.9
	143.7
	57.4
	10.3

$Qest_1 \ - \ 55.1 = ep_1 - en_1$
$Qest_2 \ - 363.4 = ep_2 - en_2$
$Qest_3 \ - 917.3 = ep_3 - en_3$
$Qest_4 \ - 1198.2 = ep_4 - en_4$
$Qest_5 \ - 934.4 = ep_5 - en_5$
$Qest_6 \ - 539.0 = ep_6 - en_6$
$Qest_7 \ - 288.9 = ep_7 - en_7$
$Qest_8 \ - 143.7 = ep_8 - en_8$
$Qest_9 \ - 57.4 = ep_9 - en_9$
$Qest_{10} - 10.3 = ep_{10} - en_{10}$

P (in)	Q (ft^3/sec)
0.7	55.1
1.7	363.4
1.2	917.3
	1198.2
Total 3.6	934.4
	539
	288.9
	143.7
	57.4
	10.3

$$\text{Min } Z = \sum_{i=1}^{I} (ep_i + en_i)$$

SUBJECT TO

$Qest_1 = 0.7\,U_1$
$Qest_2 = 1.7\,U_1 + 0.7\,U_2$
$Qest_3 = 1.2\,U_1 + 1.7\,U_2 + 0.7\,U_3$
$Qest_4 = 1.2\,U_2 + 1.7\,U_3 + 0.7\,U_4$
$Qest_5 = 1.2\,U_3 + 1.7\,U_4 + 0.7\,U_5$
$Qest_6 = 1.2\,U_4 + 1.7\,U_5 + 0.7\,U_6$
$Qest_7 = 1.2\,U_5 + 1.7\,U_6 + 0.7\,U_7$
$Qest_8 = 1.2\,U_6 + 1.7\,U_7 + 0.7\,U_8$
$Qest_9 = 1.2\,U_7 + 1.7\,U_8$
$Qest_{10} = 1.2\,U_8$

$$\sum (U_m) = 1252.2$$

$$U_m \geq 0$$

P (in)	Q (ft^3/sn)
0.7	55.1
1.7	363.4
1.2	917.3
	1198.2
Toplam 3.6	934.4
	539
	288.9
	143.7
	57.4
	10.3

$Qest_1 \ - \ 55.1 = ep_1 - en_1$
$Qest_2 \ - 363.4 = ep_2 - en_2$
$Qest_3 \ - 917.3 = ep_3 - en_3$
$Qest_4 \ - 1198.2 = ep_4 - en_4$
$Qest_5 \ - 934.4 = ep_5 - en_5$
$Qest_6 \ - 539.0 = ep_6 - en_6$
$Qest_7 \ - 288.9 = ep_7 - en_7$
$Qest_8 \ - 143.7 = ep_8 - en_8$
$Qest_9 \ - 57.4 = ep_9 - en_9$
$Qest_{10} - 10.3 = ep_{10} - en_{10}$

$$\textbf{Minimize} = \sum_{i=1}^{I} (ep_i + en_i)$$

Subject to

$$ep_i - en_i = Qmeasured_i - Qest_i \quad \text{for all } i$$

$$Qest_i = \sum_{m=1}^{M} (U_m * P_{i-m+1}) \quad \text{for } i = 1,2,....,I$$

$$\sum (U_m) = 1252.2$$

All variables non negative

SOLUTION OF LINEAR SYSTEMS

Minimize F = DUMMY

Subject to

3X + 5Y + 4Z = 25
2X + 4Y + 6Z = 28
2X + 3Y + 3Z = 17

X , Y, Z non negative

In standart LP, all variables are non negative.

However, if X, Y, or Z can take a negative value

We have to replace these
with difference of two positive variables

Handling Unrestricted Variables

Assume X1 is unrestricted in the problem.

Let

X1 = X1P – X1N

where X1P and X1N are nonnegative.

The value of X1 is positive or negative depending on wheter

X1P > X1N or **X1P < X1N**

$X = XP - XN$
$Y = YP - YN$
$Z = ZP - ZN$

Minimize $F = DUMMY$

Subject to

$$3(XP - XN) + 5(YP - YN) + 4(ZP - ZN) = 25$$
$$2(XP - XN) + 4(YP - YN) + 6(ZP - ZN) = 28$$
$$2(XP - XN) + 3(YP - YN) + 3(ZP - ZN) = 17$$

XP, XN YP, YN ZP, ZN non-negative

IRRIGATION PLANNING

Two types of crops can be grown in a particular irrigation area each year.

	Requirements per unit of		Maximum Available
Resource	Crop A	Crop B	Resource
Water	WA	WB	W
Land	LA	LB	L
Fertilizer	FA	FB	F
Labor	HA	HB	H
Unit Benefit	**PA**	**PB**	

Find the optimum allocation of crops to maximize the benefits.

	Requirements per unit of		Maximum Available
Resource	Crop A	Crop B	Resource
Water	WA	WB	W
Land	LA	LB	L
Fertilizer	FA	FB	F
Labor	HA	HB	H
Unit Benefit	PA	PB	

Let XA and XB allocated amounts for crop A and B

Maximize Z = (PA * XA) + (PB * XB)

Subject to

$$(WA * XA) + (WB * XB) \leq W$$
$$(LA * XA) + (LB * XB) \leq L$$
$$(FA * XA) + (FB * XB) \leq F$$
$$(HA * XA) + (HB * XB) \leq H$$

XA and XB Non negative

Question An area of 300,000 dekar is to be irrigated in southern Turkey. The monthly total delivery water requirements for the considered crops are given below.

The cotton, rice, melons, and citrus are grown as single crop annually as indicated by the growing seasons on the table. On the other hand; corn, sesame, cotton, and lima beans can be grown as second crop following the harvest of wheat.

At present, the citrus (orange, lemons, etc.) covers area of 6,500 dekar. The maximum desired production level for sesame is 300 tons as the consumption is limited. Also; the yields of crops per dekar, the net relative benefit per kg of yield and available monthly flows (forecasts) which can be diverted from a river, are also given.

Write a mathematical model for this problem to determine optimal crop pattern so as to maximize relative net benefits.

Crop		Growing Season	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Production kg/dekar	Net benefits $ per kg
			Total Delivery Water Requirements (m^3 per dekar)									
Cotton		01 May - 30 Sep				147	420	400	190		275	44
Rice		01 May - 30 Sep			248	340	400	376	295		400	32
Melons		15 Apr - 31 Jul				185	330	270			3000	10
Citrus		01 Apr - 15 Nov		57	127	257	336	294	200	96	2400	12
Wheat	1st crop	30 Jan - 15 Apr	83	160							400	16
Corn	2nd crop	01 Jun - 30 Sep				150	320	445	315		3000	5
Sesame	2nd crop	01 May - 30 Sep				155	315	337	210		100	67
Cotton	2nd crop	01 Jun - 31 Oct				55	210	400	280	86	200	45
Lima Beans	2nd crop	01 Jun - 15 Sep				215	480	395	190		250	18
Available water million m^3			180	330	250	120	80	85	105	80		

BLENDING PROBLEMS

- ✓ What is the best ratios during mixing?
- ✓ **Examples :**
  - Aggregates taken from different resources
  - Chemical compounds used in plastic production
  - Products used in petroleum refinery

- ✓ **Min Z = Total cost**
- ✓ **Constraints :**

  Product min and max limits

Blending Problems : Example

- ECN Engineering firm is looking for aggregates for concrete mix. The preferred aggregate must have
  - Silt at least %20, at the most %40
  - Gravel at least %25, at the most %50
  - Sand at least %35, at the most %50
- Details of aggregates obtained from 3 different sources

LOT	Silt	Gravel	Sand	Cost
A	60%	30%	10%	250
B	25%	25%	50%	300
C	20%	50%	30%	350

- What is the best combination of aggregates?

Blending Problems : Example

- Silt min %20, at the most %40
- Gravel min %25, at the most %50
- Sand min %35, at the most %50

LOT	Silt	Gravel	Sand	Cost
A	60%	30%	10%	250
B	25%	25%	50%	300
C	20%	50%	30%	350

- **Decision variables :**

XA : A ratio in mix XB : B ratio in mix XC : C ratio in mix

- **Objective** Min Z= 250 XA + 300 XB + 350 XC
- **Subject to :** XA + XB + XC = 1

$0.20 \le 0.6\,XA + 0.25\,XB + 0.20\,XC \le 0.4$ Silt ratio in mix

$0.25 \le 0.3\,XA + 0.25\,XB + 0.50\,XC \le 0.5$ Gravel ratio in mix

$0.35 \le 0.1\,XA + 0.50\,XB + 0.30\,XC \le 0.5$ Sand ratio in mix

$XA \ge 0 \;;\; XB \ge 0 \;;\; XC \ge 0$