OPTIMIZATION PROBLEMS

- The **decision variables** represent some aspect of the system that is controllable by DMs.

- **State Variables** represent some aspect of the system that is affected by a decision and hence are functions of the decision variables

- The models contain pyhsical, financial, political, economical and other restrictions that must be met in the system.

- They also contain statements that define the goals, desires or values of the decision variables with respect to the system.

- This combination of constraints and objectives which are defined in terms of decision and state variables within a mathematical model provides a powerful tool for DMs as they analyze complex systems.

- Minimize $f(x) = x^2 - 3x + 4$
- Minimize $f(X) = 3X_1 + 4X_2$
- Minimize $f(x) = \sin(X)$
- Minimize $f(x,y) = xy - 3x^2 + 4y^2$

- Subject to $g(X) = X_1 + 3X_2 = 5$
- Subject to $g(X) = X_1 + 3X_2 < 5$
- Subject to $g(X) = X_1 + 3X_2 > 5$

- Non negativity restrictions
- $X_1 \geq 0$, $X_2 \geq 0$

LINEAR PROGRAMMING

- Objective function must be linear function of variables

- Constraints must be in the form of linear function of variables

- Variables must be nonnegative

Handling Unrestricted Variables

Assume X1 is unrestricted in the problem.

Let

$$X1 = X1P - X1N$$

where X1P and X1N are nonnegative.

The value of X1 is positive or negative depending on wheter

$$X1P > X1N \quad \text{or} \quad X1P < X1N$$

EXAMPLE

A company has two grades of inspectors, 1 and 2, who are to be assigned for a quality control inspection. It is required that at least 1800 pieces be inspected per 8-hour day. Grade 1 inspectors can check pieces at the rate of 25 per hour, with an accuracy of 98 percent. Grade 2 inspectors check at the rate of 15 pieces per hour, with an accuracy of 95 percent.

The wage rate of a grade 1 inspector is $4.00 per hour, while that of grade 2 inspector is $3.00 per hour. Each time an error is made by an inspector, the cost to the company is $2.00. The company has available for the inspection job, eight grade 1 inspectors, and ten grade 2 inspectors. The company wants to determine the optimal assignment of inspectors which minimize the total cost of inspection.

STEP 1 : Let

$X1$: # grade 1 inspectors assigned for inspection.

$X2$: # grade 2 inspectors assigned for inspection.

STEP 2 : Since the # available inspectors in each grade is limited;

$$X1 \leqslant 8 \ ; \ X2 \leqslant 10$$

The company requires at least 1800 pieces to be inspected daily, thus

$$8 * 25 * X1 + 8 * 15 * X2 \geqslant 1800$$

$$200 * X1 + 120 * X2 \geqslant 1800$$

STEP 3 : i) Wages paid to the inspectors
ii) Cost of inspection errors

Grade 1 : $4 + 2 * 25 * 0.02 = \$5.0$ per hour
Grade 2 : $3 + 2 * 15 * 0.05 = \$4.5$ per hour

The objective is to minimize the daily cost of inspection

$$Z = 8 * 5\ X1 + 8 * 4.5\ X2$$

EXAMPLE

Assume that a farmer can grow different types of corps on his land. The annual water requirements for wheat and corn are 60 cm and 90 cm respectively. The yields for each corp are 9000 kg/ha for corn and 3000 kg/ha for wheat. In addition, the net profit for wheat is 60 TL/kg and for corn is 40 TL/kg.

If the farmer has plenty of land but only 50,000 m3 of water for the season, what is the best combination of corn and wheat to grow (for max profit)?

Let

XW = Land allocated for wheat (ha) XC = land allocated for corn (ha)

Constraints

(0.60* 10,000) XW + (0.90 * 10,000) XC $\leq$ 50,000

XW and XC nonnegative

Objective function

Max Z = (60 TL/kg * 3000 kg/ha) XW + (40 TL/kg * 9000 kg/ha) XC

Let X_w : Amount of water to be used for wheat (m3)
X_c : Amount of water to be used for corn (m3)

1 m3 of water will grow

3000 kg/ha / (0.6*10000) m3/ha = 0.5 kg/m3 (wheat)
9000 kg/ha / (0.9*10000) m3/ha = 1.0 kg/m3 (corn)

$$\text{Max } Z = 60 \text{ TL/kg} * 0.5 \text{ kg/m3} * X_w + 40 \text{ TL/kg} * 1.0 \text{ kg/m3} * X_c$$

$$\text{Max } Z = 30\, X_w + 40\, X_c$$

Subject to $X_w + X_c \leqslant 50{,}000$
$X_w \geqslant 0$; $X_c \geqslant 0$

Solution $X_c = 50{,}000$ m3 ; $X_w = 0$

Now, if an additional information reveals that the farmer has enough money to buy fertilizer for enough corn to utilize 30,000 m3 of water. What is the best combination in this case?

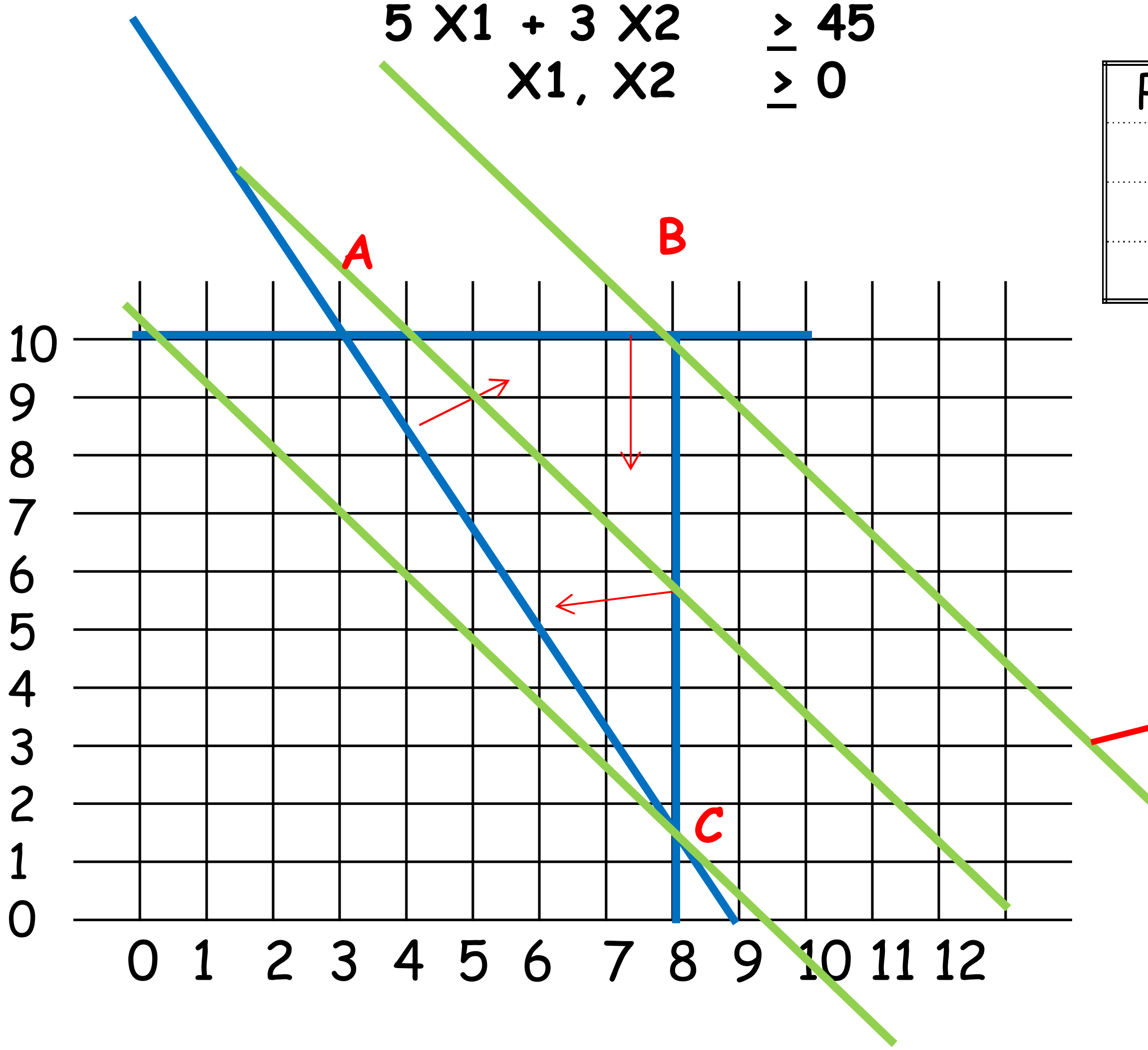
Additional constraint $X_c \leq 30,000$

Solution $X_c = 30,000$ m3 ; $X_w = 20,000$

Minimize $Z = 40 X1 + 36 X2$

Subject to

$$X1 \le 8$$
$$X2 \le 10$$
$$5 X1 + 3 X2 \ge 45$$
$$X1, X2 \ge 0$$



Point	X1	X2	Z
A	3	10	480
B	8	10	680
C	8	5/3	380

$Z^* = 380$

$Z = 40 X1 + 36 X2 = 680$

Maximize $Z = 20\,X1 + 50\,X2$

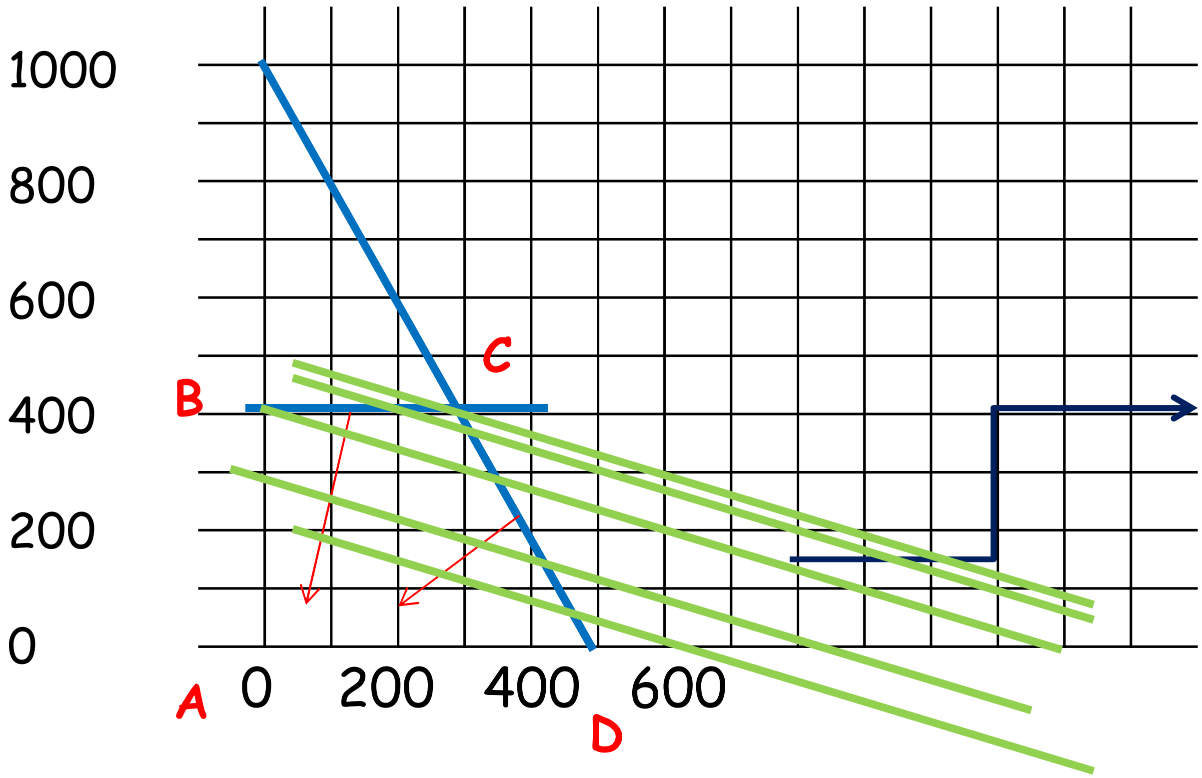
Subject to

$$2X1 + X2 \le 1000$$

$$0.1\,X2 \le 40$$

$$X1,\ X2 \ge 0$$

Point	X1	X2	Z
A	0	0	0
B	0	400	20K
C	300	400	26K
D	500	0	10K



$Z^* = 26{,}000$

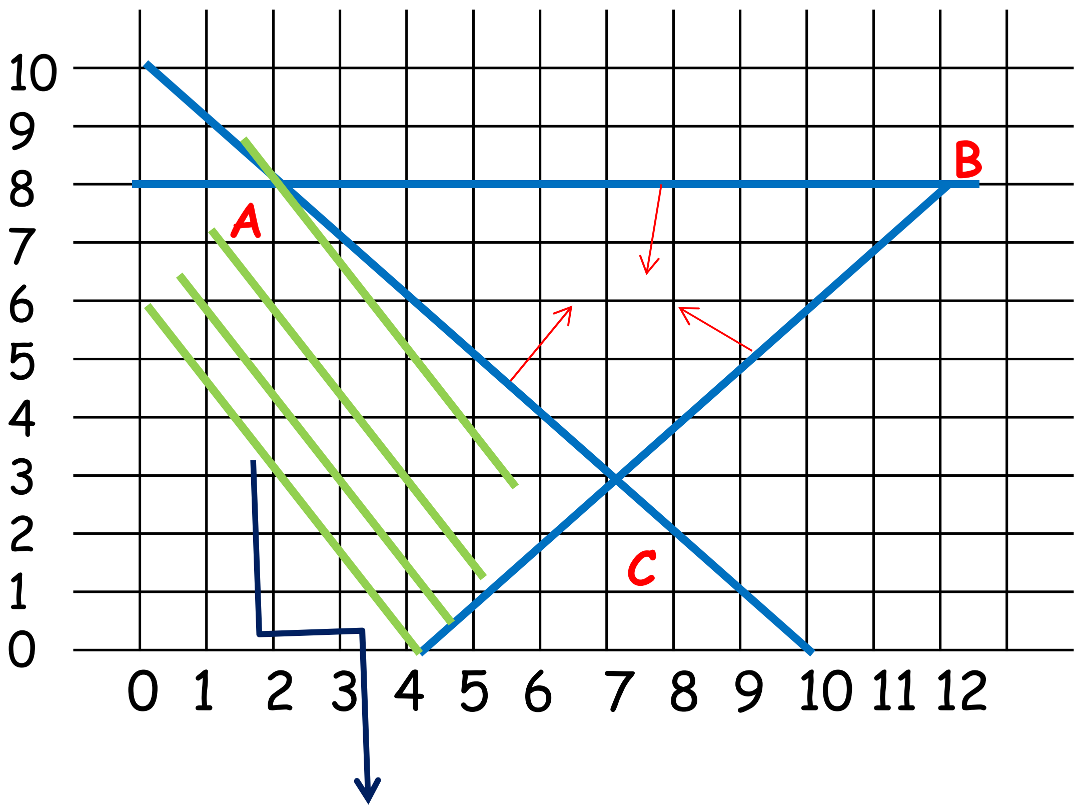
$Z = 20\,X1 + 50\,X2 = 20K$

Minimize $Z = 3X1 + 2X2$

Subject to

$$X2 \le 8$$
$$X1 + X2 \ge 10$$
$$X1 - X2 \le 4$$
$$X1, X2 \ge 0$$

Point	X1	X2	Z
A	2	8	22
B	12	8	52
C	7	3	27



$Z^* = 22$

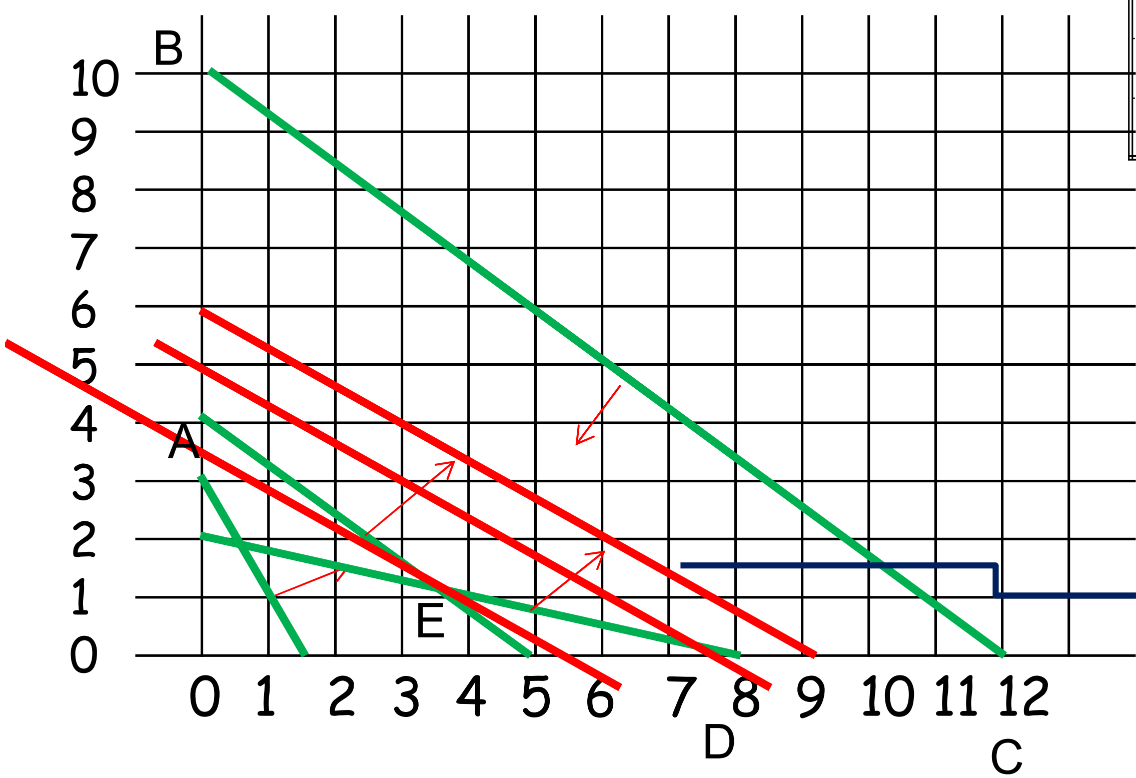
$Z = 3X1 + 2X2 = 12$

Minimize $Z = 2X1 + 3X2$

Subject to

$$2X1 + X2 \geq 3$$
$$4X1 + 5X2 \geq 20$$
$$2X1 + 8X2 \geq 16$$
$$5X1 + 6X2 \leq 60$$
$$X1, X2 \geq 0$$

Point	X1	X2	Z
A	0	4	12
B	0	10	30
C	12	0	24
D	8	0	16
E	40/11	12/11	116/11



$Z^* = 10.54$

$Z = 2X1 + 3X2 = 18$

Maximize $Z = X1 + 2\ X2$

Subject to

$$X1 + 2\ X2 \leq 10$$

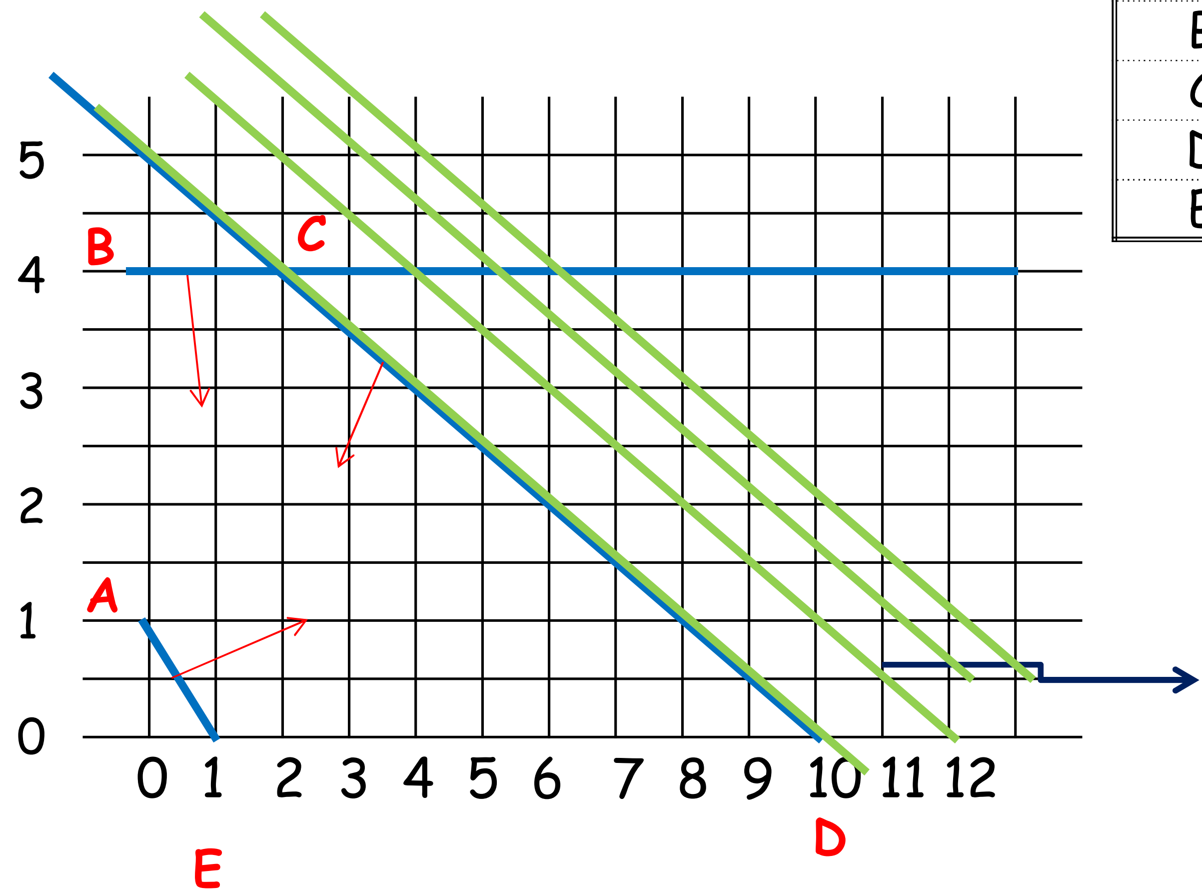
$$X1 + X2 \geq 1$$

$$X2 \leq 4$$

$$X1,\ X2 \geq 0$$

Slopes are equal!!!

Point	X1	X2	Z
A	0	1	2
B	0	4	8
C	2	4	10
D	10	0	10
E	1	0	1



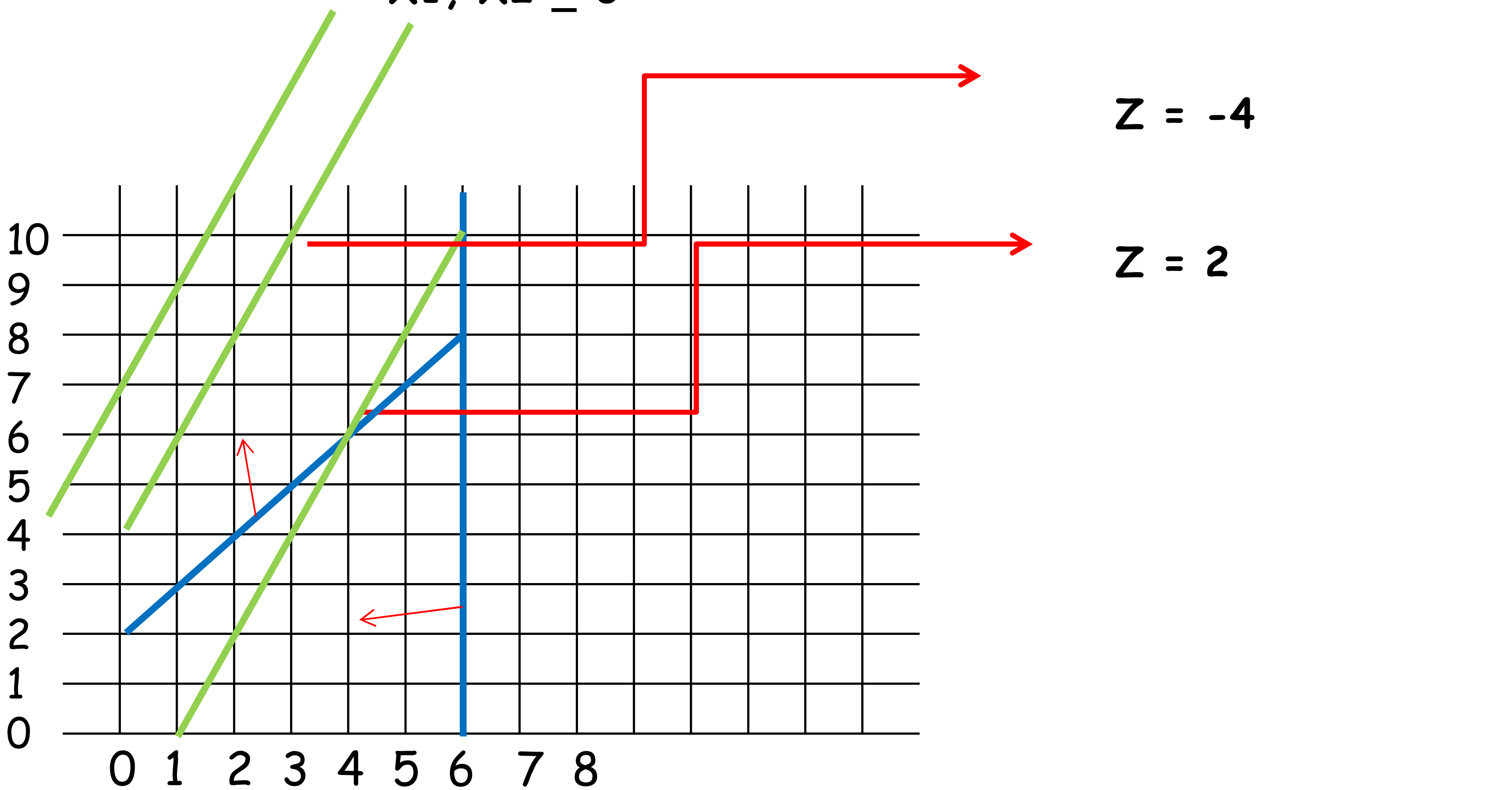
Z* = 10 (Line CD)

Z = 12

Minimize $Z = 2X1 - X2$

Subject to

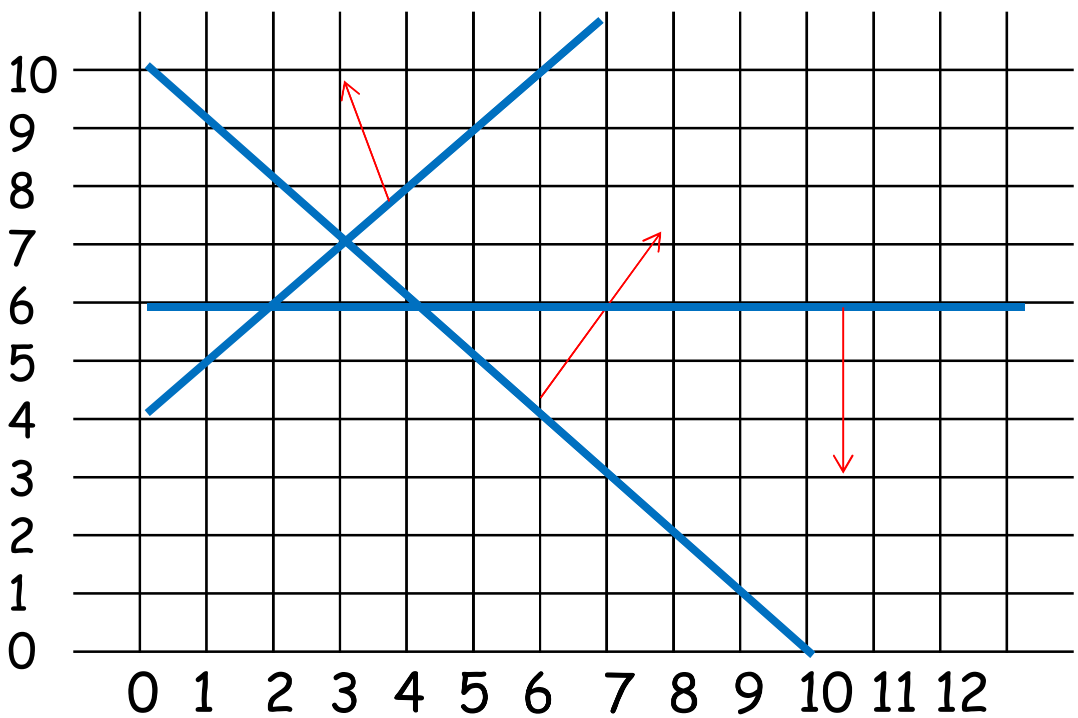
$$X1 \le 6$$
$$-X1 + X2 \ge 2$$
$$X1, X2 \ge 0$$



Minimize $Z = X1 + X2$

Subject to

$$\begin{aligned} -X1 + X2 &\geq 4 \\ X1 + X2 &\geq 10 \\ + X2 &\leq 6 \\ X1 , X2 &\geq 0 \end{aligned}$$



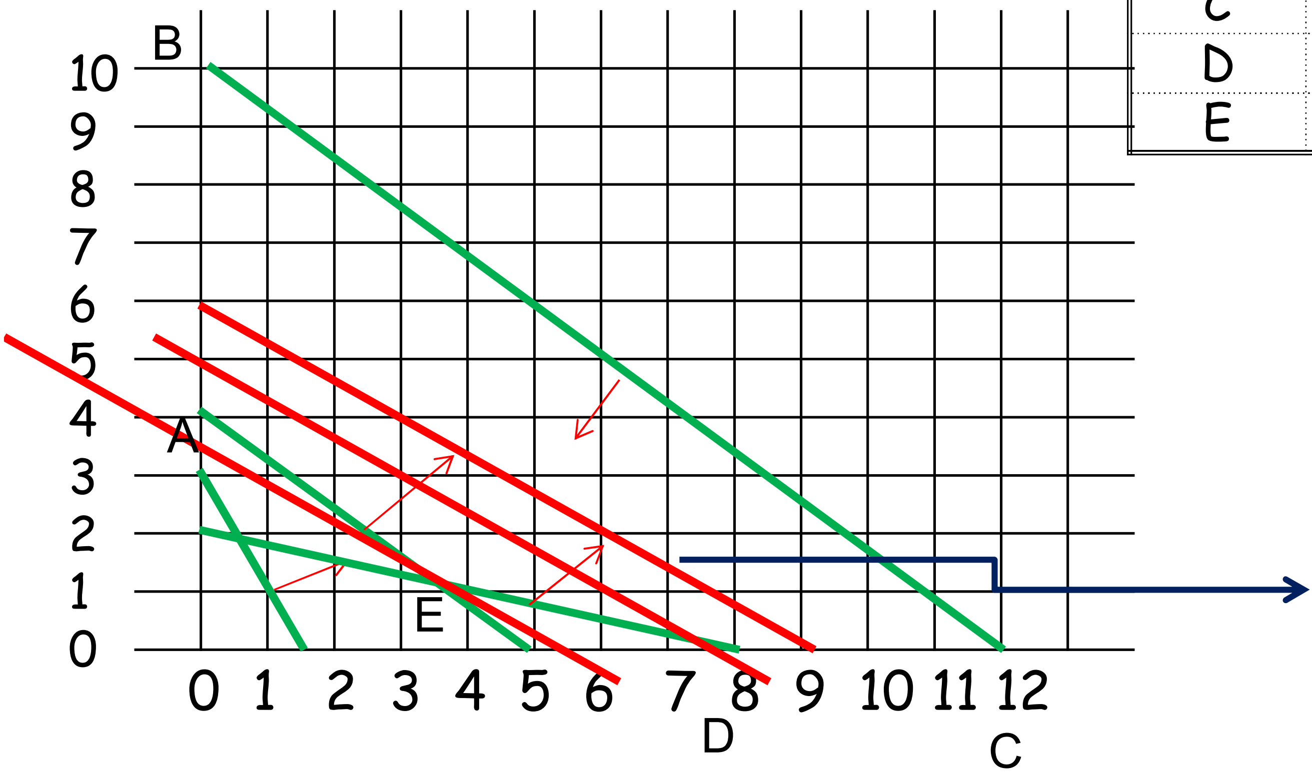
No feasible point!!

Minimize $Z = 2X1 + 3X2$

Subject to

$$2X1 + X2 \geq 3$$
$$4X1 + 5X2 \geq 20$$
$$2X1 + 8X2 \geq 16$$
$$5X1 + 6X2 \leq 60$$
$$X1, X2 \geq 0$$

Point	X1	X2	Z
A	0	4	12
B	0	10	30
C	12	0	24
D	8	0	16
E	40/11	12/11	116/11

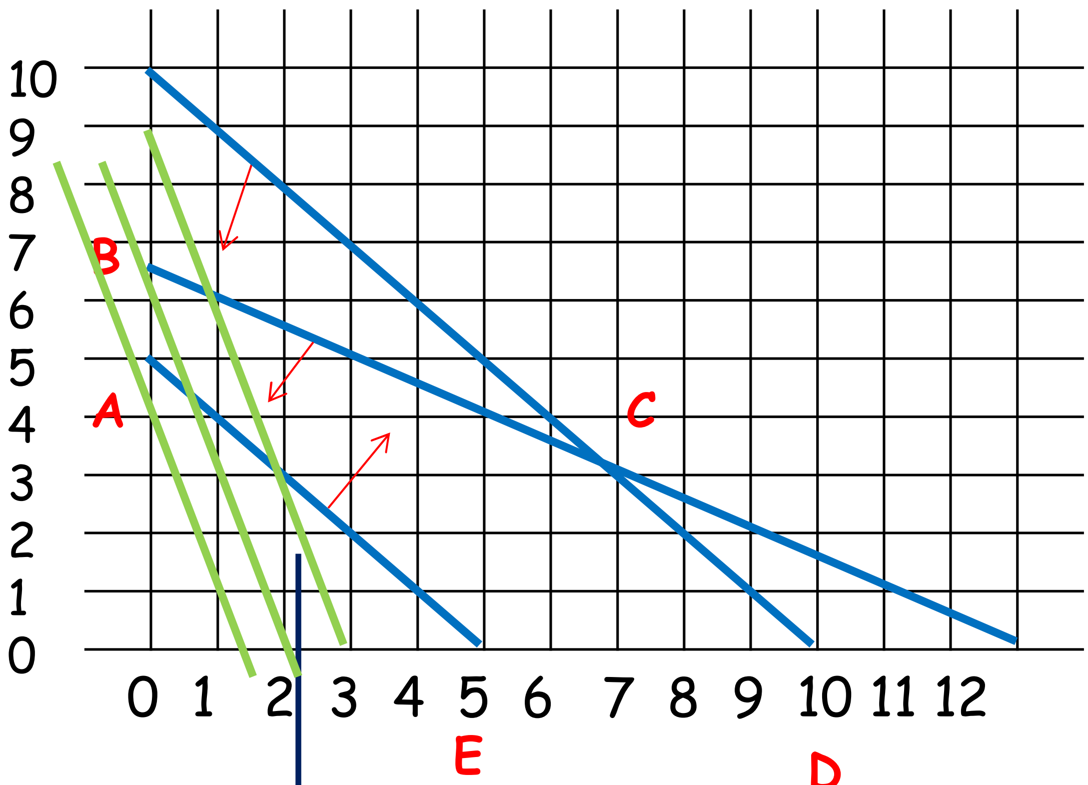


$Z^* = 10.54$

$Z = 2X1 + 3X2 = 18$

Min	$Z =$	$3X1$	$+ X2$		
Subject to	$X1 +$	$X2$	≤ 10		
	$-X1 -$	$X2$	≤ -5	$\Longrightarrow$	$X1 + X2 \geq 5$
	$X1 + 2X2$		≤ 13		
	$X1$ &	$X2$	≥ 0		

Point	X1	X2	Z
A	0	5	5
B	0	6.5	6.5
C	7	3	24
D	10	0	30
E	5	0	15



$Z = 3X1 + X2 = 9$

$Z^* = 5$

$X1^* = 0$ & $X2 = 5$

Homework Solve the following LP problems graphically.
Please note that X1 & X2 non-negative for all problems.

- Maximize $Z = 4 X1 + 2 X2$
  Subject to
  $X1 + X2 \leq 20$
  $2 X1 + 3 X2 \leq 18$
- Maximize $Z = X1 + 1.5 X2$
  Subject to
  $2 X1 + 3 X2 \leq 6$
  $X1 + 4 X2 \leq 4$
- Maximize $Z = 4 X1 + X2$
  Subject to
  $3 X1 + X2 \leq 9$
  $0.5 X1 + X2 \leq 4$
- Maximize $Z = 2 X1 + X2$
  Subject to
  $X1 + X2 \geq 2$
  $X1 + 2 X2 \geq 7$
- Maximize $Z = 4 X1 + X2$
  Subject to
  $12 X1 + 3 X2 \leq 36$
  $X1 + X2 \geq 4$
  $X1 \geq 2$