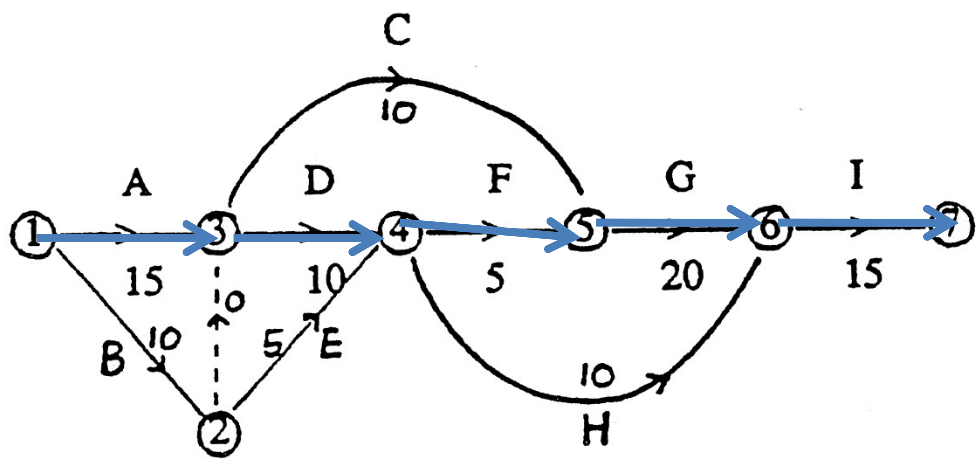
Example (PR&S Pr. 3.26) : .

Activity	Preceeding Activity	Duration (day)
A	—	15
B	—	10
C	A, B	10
D	A, B	10
E	B	5
F	D, E	5
G	C, F	20
H	D, E	10
I	G, H	15



Min Z = t7

Sub to $t_3 - t_1 \geq 15$ $t_2 - t_1 \geq 10$ $t_3 - t_2 \geq 0$

$t_5 - t_3 \geq 10$ $t_4 - t_3 \geq 10$ $t_4 - t_2 \geq 5$

$t_5 - t_4 \geq 5$ $t_6 - t_5 \geq 20$ $t_6 - t_4 \geq 10$ $t_7 - t_6 \geq 15$

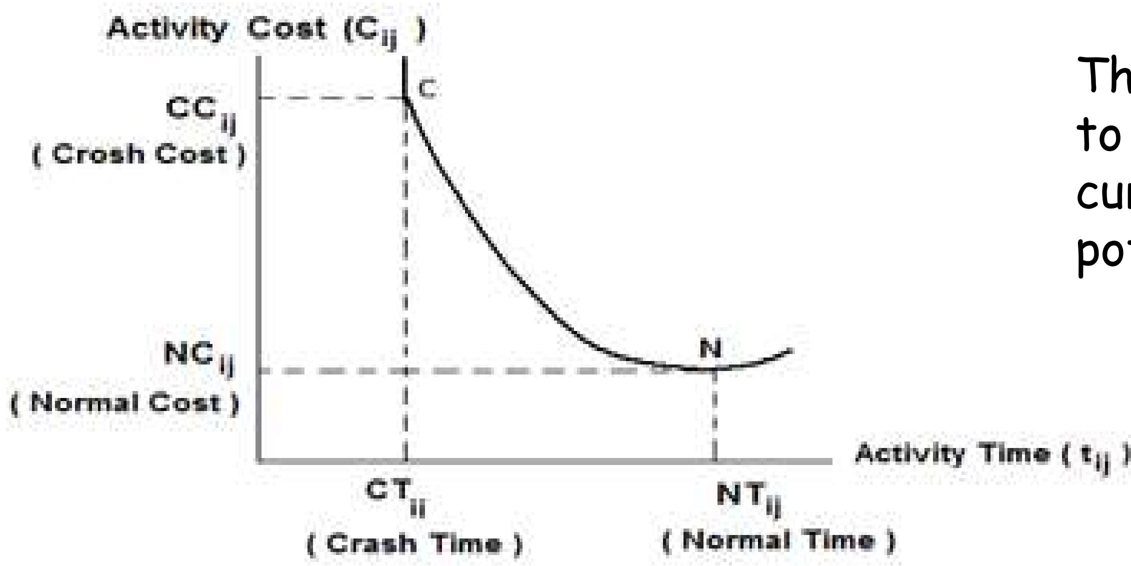
t_7^* = 65 days Critical path : A-D-F-G-I

TIME-COST TRADE-OFFS FOR CPM

- In preliminary CPM analysis, the objective is to determine the minimum time required to complete the project at minimum cost.
- CPM assumes that the activity times can be shortened to a certain extent by assigning additional resources (capital,labor,machines,materials,etc.) to an activity.
- Shortening the duration of an activity is known as crashing and additional cost for **crashing** is called as **crash cost**
- Crashing activities increase the total direct cost of the project, but the reduction in project completion time will result in other advantages.

TIME-COST TRADE-OFFS FOR CPM

- The basic objective of CPM is to determine how the project activities are to be expedited so that total cost of crashing is minimized and the project is completed at a required time.

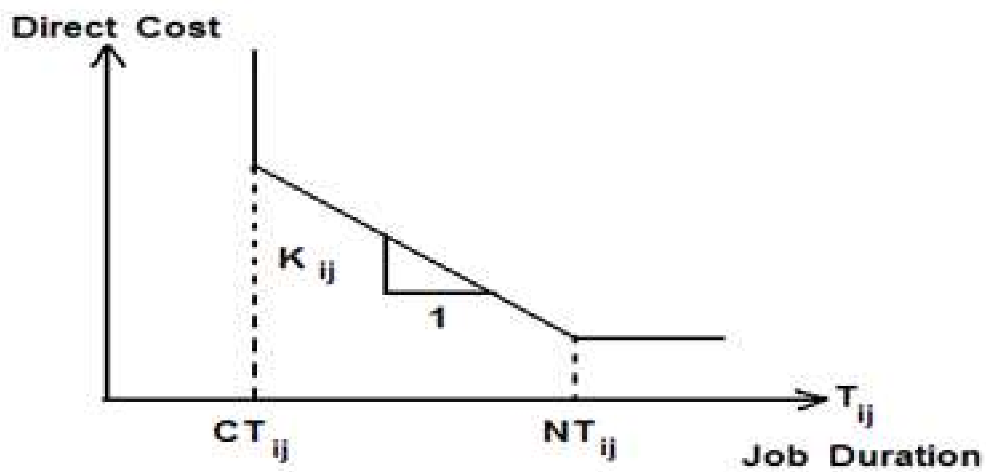


Time Cost Curve for an Activity

The first step in such analysis is to develop the activity time-cost curves for each job that has a potential of being compressed

Crashing by LP

- Assume linear time-cost relationships for every activity



Linear time-cost relationship

- Kij denotes the unit cost of crashing the job (i,j)

Crashing by LP

- The problem is to minimize the total cost of crashing and to complete the project by predetermined time TMAX.

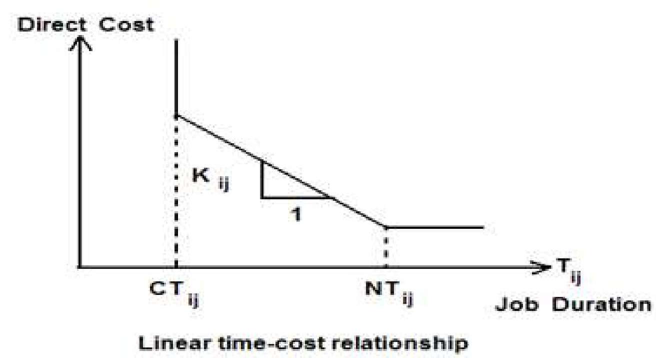
Minimize $Z = \sum K_{ij} * (NT_{ij} - T_{ij})$

Subject to: $t_j - t_i \geq T_{ij}$

$CT_{ij} \leq T_{ij} \leq NT_{ij}$

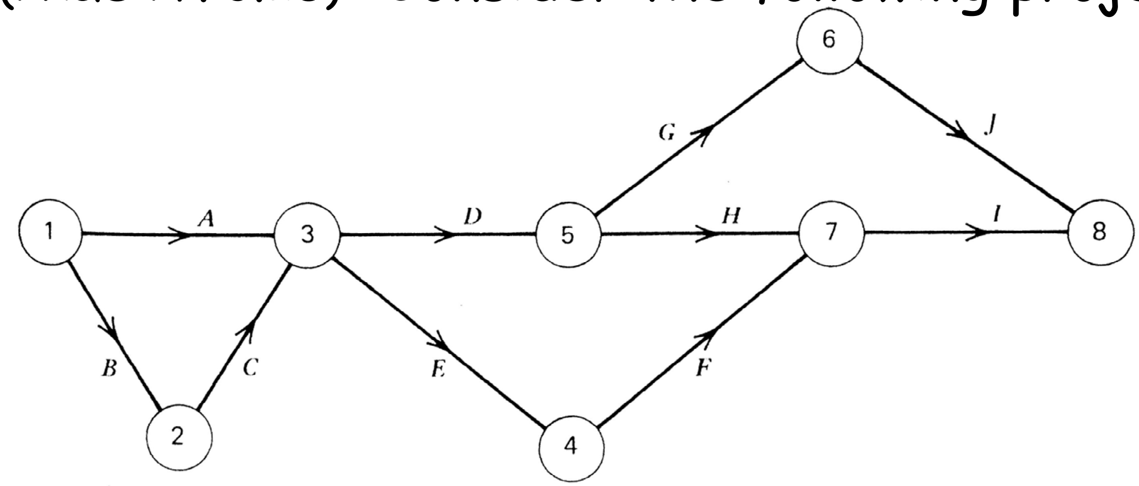
$t_n - t_1 \leq TMAX$

$t_i \geq 0 \qquad i=1,2,....,n.$



- The solution of the LP yields the optimal cost of crashing "**Z**" and the best activity time Tij.

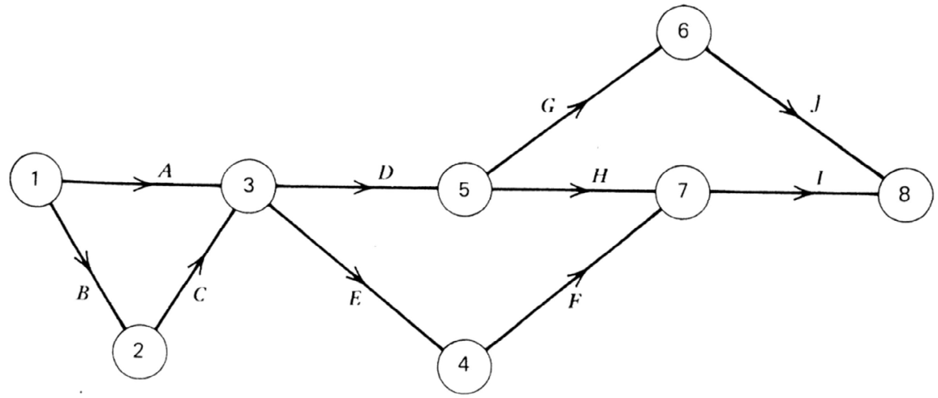
Example (PR&S Pr. 3.28) : Consider the following project network.



Activity	Normal Time	Crash Time	Unit Cost of Crashing (TL)
A	10	7	4
B	5	4	2
C	3	2	2
D	4	3	3
E	5	3	3
F	6	3	5
G	5	2	1
H	6	4	4
I	6	4	3
J	4	3	3

Find the normal and crash time of the project.

Example



Min Z = t8 – t1

Subject to $t3 - t1 \geq T_A$ $\quad$ $t2 - t1 \geq T_B$

$t3 - t2 \geq T_C$ $\quad$ $t5 - t3 \geq T_D$

$t4 - t3 \geq T_E$ $\quad$ $t7 - t4 \geq T_F$

$t6 - t5 \geq T_G$ $\quad$ $t7 - t5 \geq T_H$

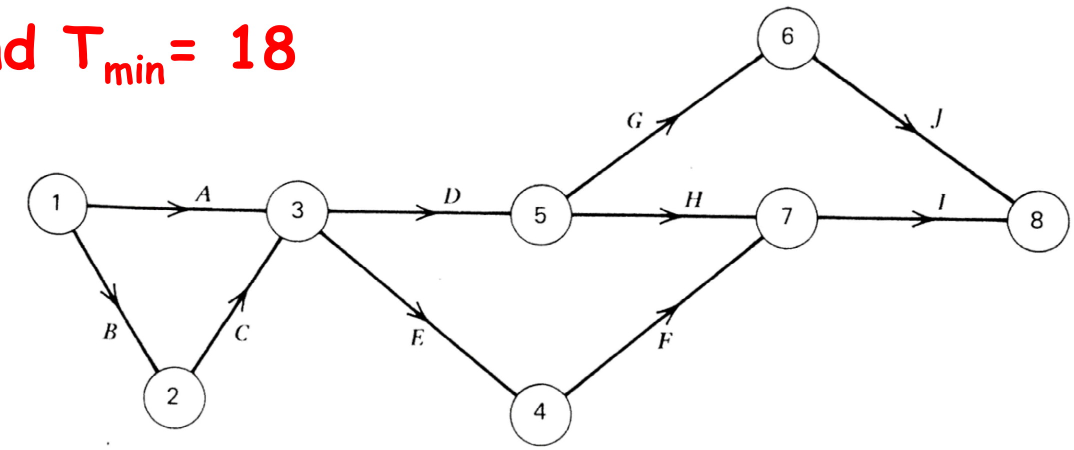
$t8 - t7 \geq T_I$ $\quad$ $t8 - t6 \geq T_J$

Maximum and minimum project times can be found by solving the above LP with Normal and Crash times, respectively.

T_{maks} = 27 Critical path : A, E, F, I.

T_{min} = 18 Critical path : A, D, H, I.

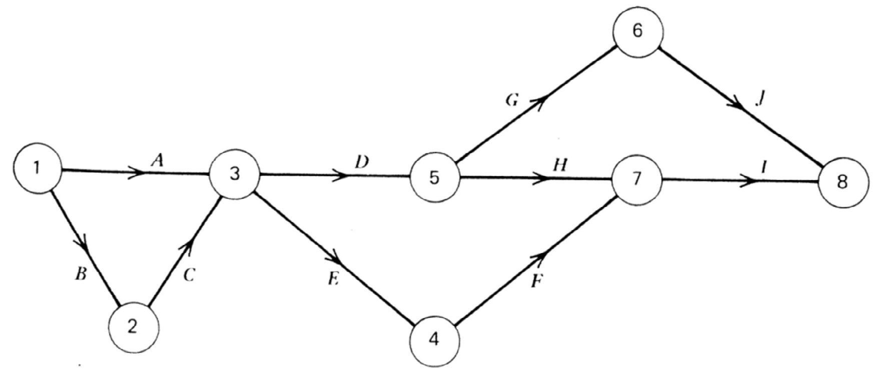
Example 1 T_{maks} = 27 and T_{min}= 18



Activity	Normal Time	Crash Time	Unit Cost of Crashing (TL)
A	10	7	4
B	5	4	2
C	3	2	2
D	4	3	3
E	5	3	3
F	6	3	5
G	5	2	1
H	6	4	4
I	6	4	3
J	4	3	3

Write an LP that would solve the problem of completing the project economically in 21 days.

Activity	Normal Time	Crash Time	Unit cost of Crashing (10^6TL)
A	10	7	4
B	5	4	2
C	3	2	2
D	4	3	3
E	5	3	3
F	6	3	5
G	5	2	1
H	6	4	4
I	6	4	3
J	4	3	3



Min Z= $4(10-t_a) + 2(5-t_b) + 2(3-t_c) + 3(4-t_d) + 3(5-t_e) + 5(6-t_f) + 1(5-t_g) + 4(6-t_h) + 3(6-t_I) + 3(4-t_j)$

Subject to $t8 - t1 \leq 21$

$t3 - t1 \geq t_a$ $\quad t2 - t1 \geq t_b$ $\quad t3 - t2 \geq t_c$ $\quad t5 - t3 \geq t_d$

$t4 - t3 \geq t_e$ $\quad t7 - t4 \geq t_f$ $\quad t6 - t5 \geq t_g$ $\quad t7 - t5 \geq t_h$

$t8 - t7 \geq t_I$ $\quad t8 - t6 \geq t_j$

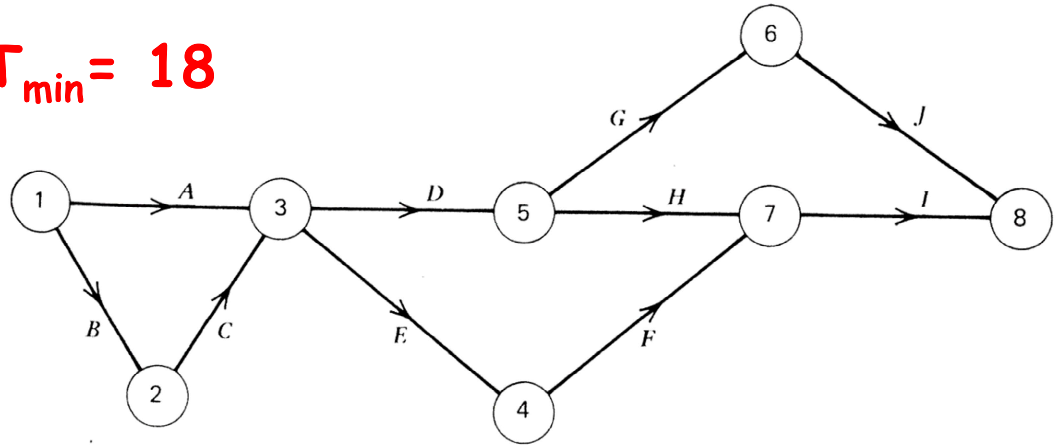
$7 \leq t_a \leq 10$

$4 \leq t_B \leq 5;\quad 2 \leq t_C \leq 3;\quad 3 \leq t_D \leq 4;$

$3 \leq t_E \leq 5;\quad 3 \leq t_F \leq 6;\quad 2 \leq t_G \leq 5;$

$4 \leq t_H \leq 6;\quad 4 \leq t_I \leq 6;\quad 3 \leq t_J \leq 4;$

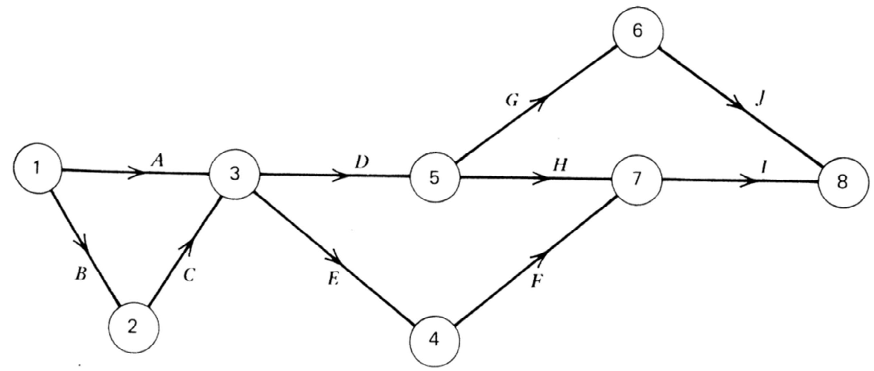
Example 2 $T_{maks} = 27$ and $T_{min} = 18$



Activity	Normal Time	Crash Time	Unit Cost of Crashing (TL)
A	10	7	4
B	5	4	2
C	3	2	2
D	4	3	3
E	5	3	3
F	6	3	5
G	5	2	1
H	6	4	4
I	6	4	3
J	4	3	3

Write an LP that would solve the problem of determining the optimal project time if a weekly bonus of 5 Million TL is given for early completion.

Activity	Normal Time	Crash Time	Unit cost of Crashing (10^6TL)
A	10	7	4
B	5	4	2
C	3	2	2
D	4	3	3
E	5	3	3
F	6	3	5
G	5	2	1
H	6	4	4
I	6	4	3
J	4	3	3



Min Z= $4\,(10-t_a) + 2\,(5-t_b) + 2\,(3-t_c) + 3\,(4-t_d) + 3\,(5-t_e)$
$+ 5\,(6-t_f) + 1\,(5-t_g) + 4\,(6-t_h) + 3\,(6-t_I) + 3\,(4-t_j)$
$- 5 * \{27 - (t8 - t1)\}$

Subject to

$t3 - t1 \geq t_a$ $\quad t2 - t1 \geq t_b$ $\quad t3 - t2 \geq t_c$ $\quad t5 - t3 \geq t_d$

$t4 - t3 \geq t_e$ $\quad t7 - t4 \geq t_f$ $\quad t6 - t5 \geq t_g$ $\quad t7 - t5 \geq t_h$

$t8 - t7 \geq t_I$ $\quad t8 - t6 \geq t_j$

$7 \leq t_a \leq 10$

$4 \leq t_B \leq 5;\quad 2 \leq t_C \leq 3;\quad 3 \leq t_D \leq 4;$

$3 \leq t_E \leq 5;\quad 3 \leq t_F \leq 6;\quad 2 \leq t_G \leq 5;$

$4 \leq t_H \leq 6;\quad 4 \leq t_I \leq 6;\quad 3 \leq t_J \leq 4;$

Extensions

- If the problem is solved by creating new variable $D_{ij} = NT_{ij} - T_{ij}$ (amount of time activity is crashed), one can obtain the amount of shortening directly.
- Suppose an additional budget of NEWB dollars is available to crash the activities.

$$\sum (K_{ij} \cdot D_{ij}) \leq NEWB$$

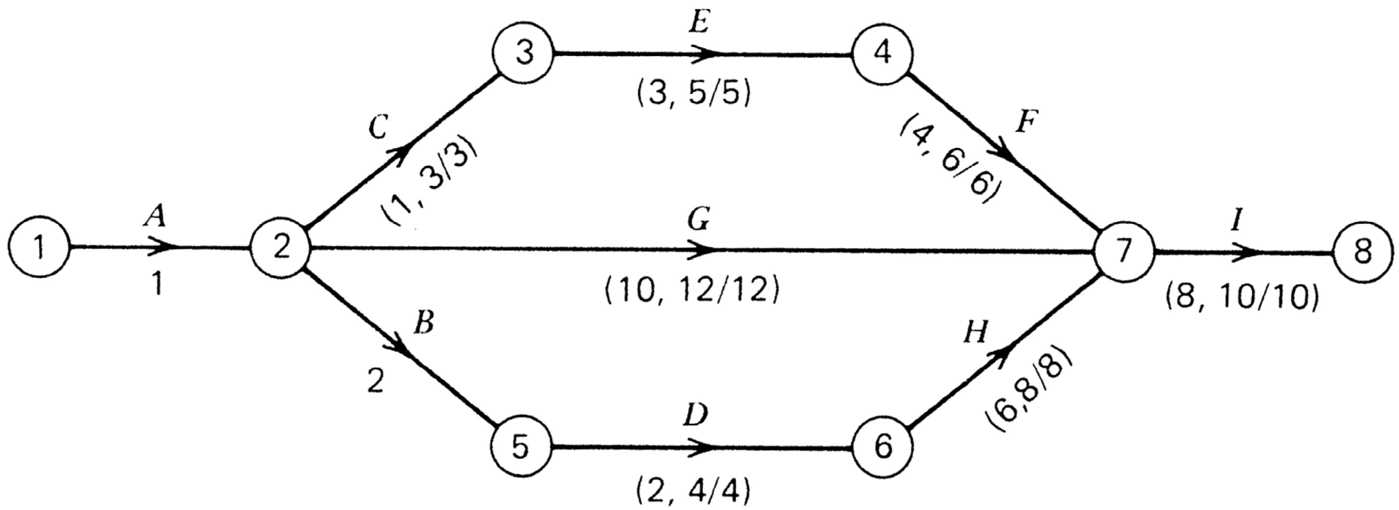
- The indirect cost of the project vary linearly with the project duration.

$$\text{Min } Z = \sum K_{ij} \cdot D_{ij} + I\,(t_n - t_1)$$

- Let B represent the additional benefits due to completion of the project before predetermined time TSET.

$$\text{Min } Z = \sum K\,ij \cdot Dij - B\,(TSET - t_n)$$

(PR&S Pr. 3.29) : The numbers in the network indicates the crash time (weeks), normal time (weeks) and unit crash costs (1000TL / week), respectively.

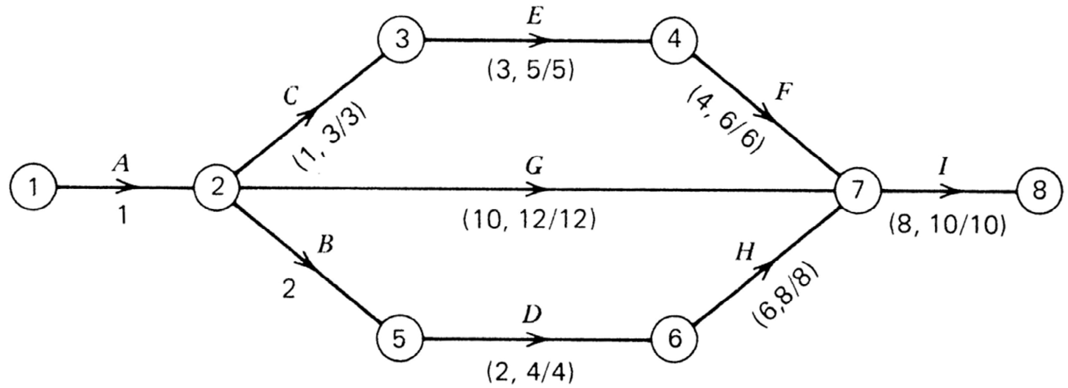


Note : It is not possible to crash activities A & B.

Write an LP program to determine the project completion time with minimum cost.

Variables :

$T_A, \ldots\ldots, T_I$ activity times
$t_1, t_2, \ldots\ldots, t_8$ time of nodes



Constraints :

Job A	$t_2 - t_1 \geq t_A$
Job B	$t_5 - t_2 \geq t_B$
Job C	$t_3 - t_2 \geq t_C$
Job D	$t_6 - t_5 \geq t_D$
Job E	$t_4 - t_3 \geq t_E$
Job F	$t_7 - t_4 \geq t_F$
Job G	$t_7 - t_2 \geq t_G$
Job H	$t_7 - t_6 \geq t_H$
Job I	$t_8 - t_7 \geq t_I$

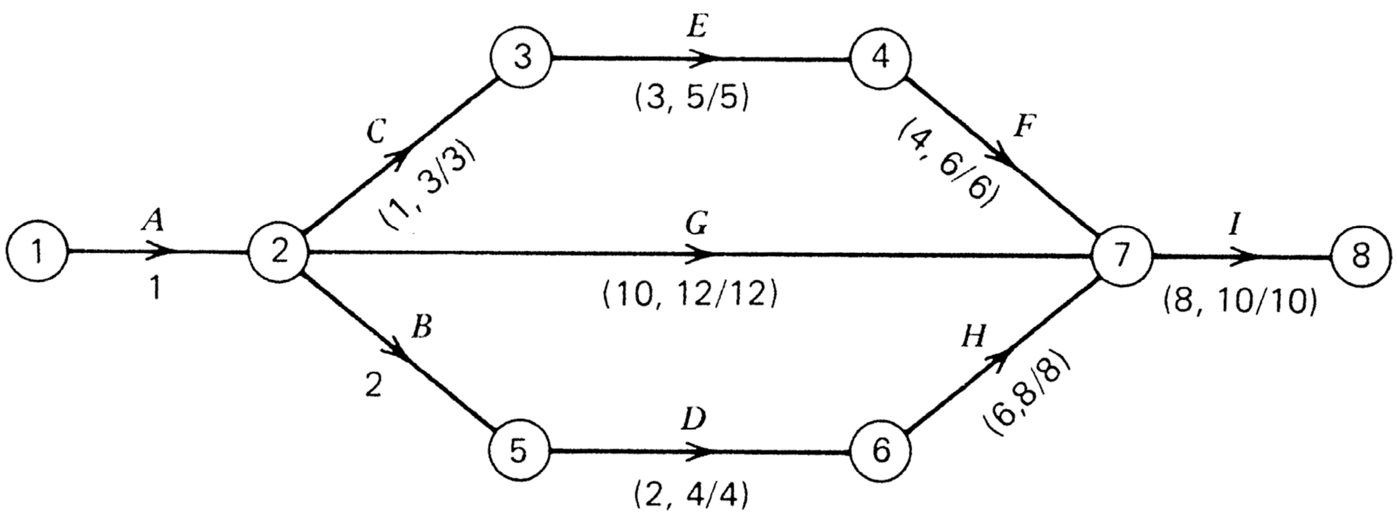
All variables non negative

Normal project time : 25 weeks
Critical paths : A-B-D-H-I and
A-C-E-F-I

Objective function

Min $Z = t_8 - t_1$

(PR&S Pr. 3.29) : The numbers in the network indicates the crash time (weeks), normal time (weeks) and unit crash costs (1000TL / week), respectively.



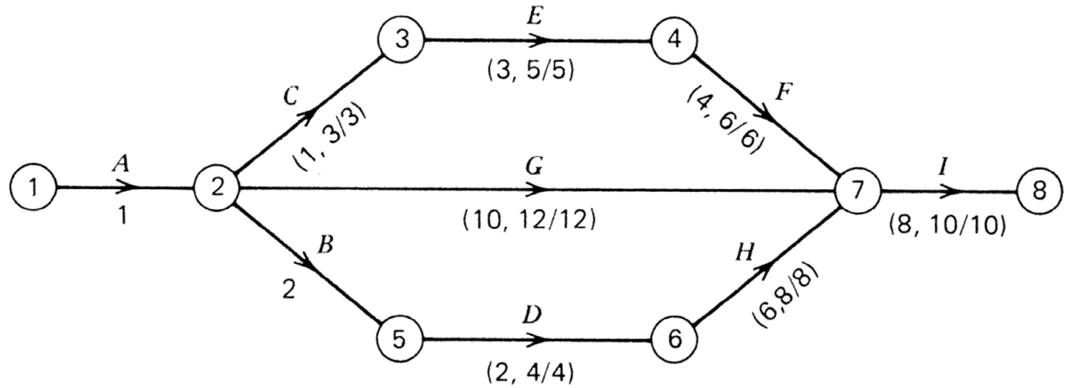
Note : It is not possible to crash activities A & B.

Normal project time : 25 weeks
Critical paths : A-B-D-H-I & A-C-E-F-I

Write an LP program that determines the most economical crashing to complete the project in 20 weeks.

Variables :

$T_A, \ldots, T_I$ activity times
$t_1, t_2, \ldots, t_8$ time of nodes



Constraints :

- Project activitiy time limits

$t_A = 1$; $t_B = 2$; $1 \le t_C \le 3$;
$2 \le t_D \le 4$; $3 \le t_E \le 5$; $4 \le t_F \le 6$;
$10 \le t_G \le 12$; $6 \le t_H \le 8$; $8 \le t_I \le 10$;

- Time limits

Job A	$t_2 - t_1 \ge t_A$
Job B	$t_5 - t_2 \ge t_B$
Job C	$t_3 - t_2 \ge t_C$
Job D	$t_6 - t_5 \ge t_D$
Job E	$t_4 - t_3 \ge t_E$
Job F	$t_7 - t_4 \ge t_F$
Job G	$t_7 - t_2 \ge t_G$
Job H	$t_7 - t_6 \ge t_H$
Job I	$t_8 - t_7 \ge t_I$

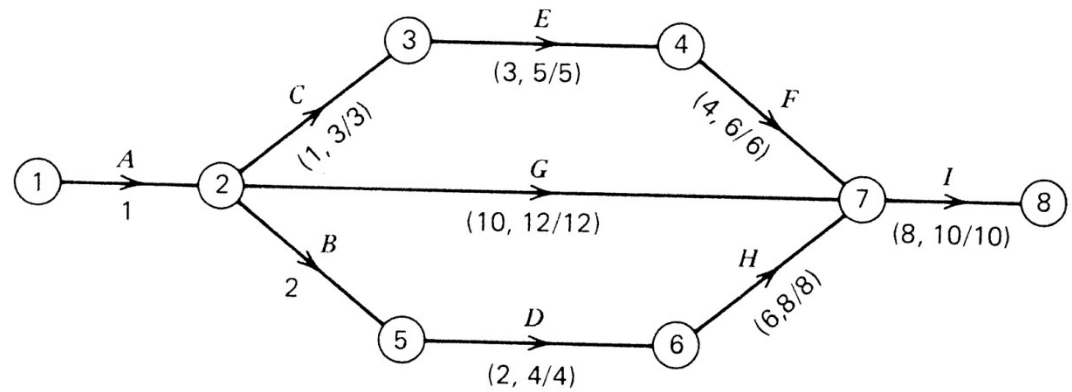
- Project completion time limit

$t8 - t1 \le 20$

Objective function

$$\text{Min } Z = 3(3-t_C) + 4(4-t_D) + 5(5-t_E) + 6(6-t_F) + 12(12-t_G) + 8(8-t_H) + 10(10-t_I)$$

Normal project time : 25 days
Critical paths : A-B-D-H-I & A-C-E-F-I



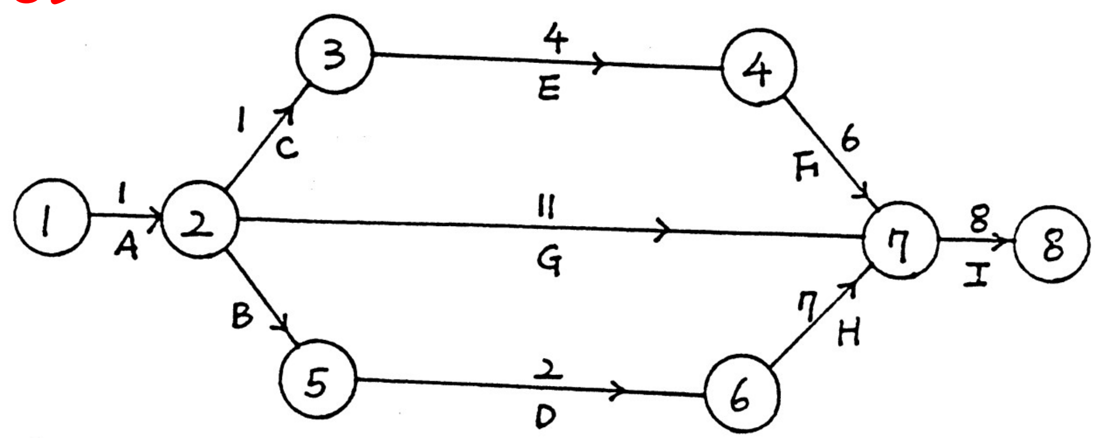
Total Crashing Cost Z = $59

C ===➔ 2 days $6
E ===➔ 1 days $5

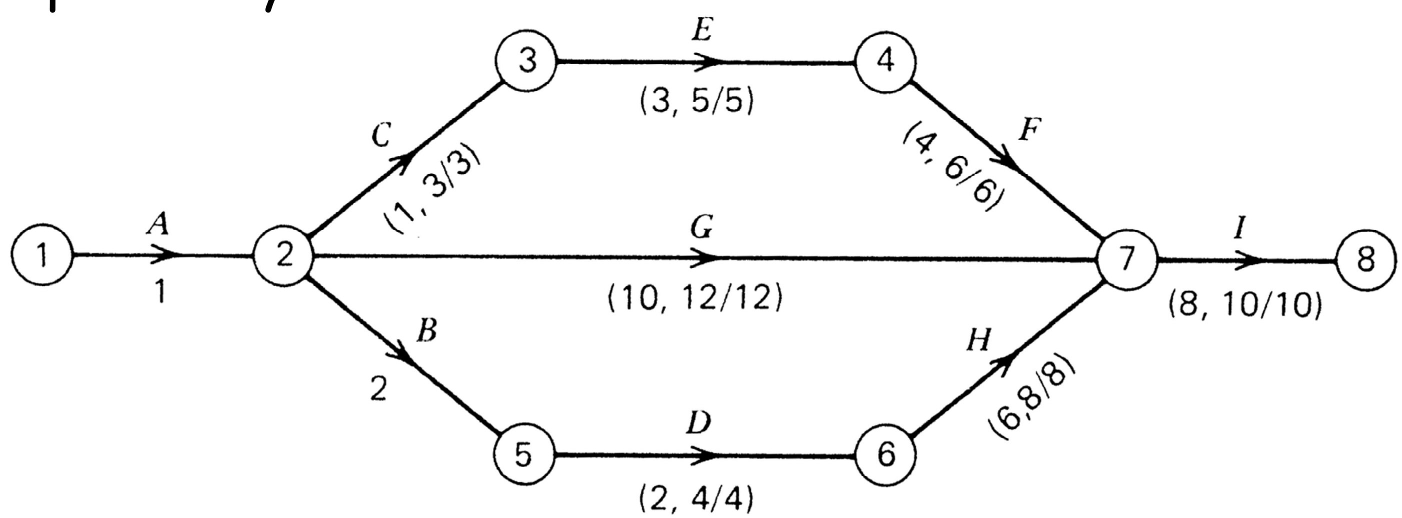
D ===➔ 2 days $8
G ===➔ 1 day $12

H ===➔ 1 day $8

I ===➔ 2 days $20



(PR&S Pr. 3.29) : The numbers in the network indicates the crash time (weeks), normal time (weeks) and unit crash costs (1000TL / week), respectively.



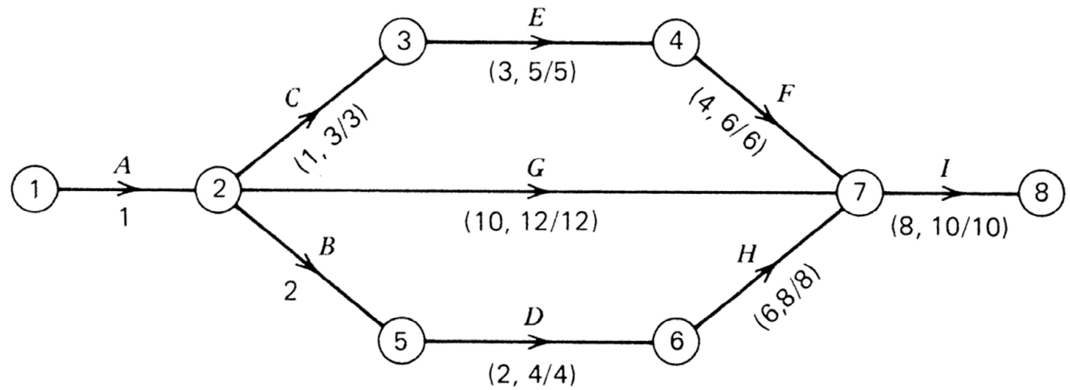
Assume that there is a 30,000 TL/week income for completion before 25 weeks.

Formulate an LP problem that will estimate the most economical solution and project completion duration.

Variables :

$t_A, \ldots\ldots, t_I$ activity times

$t_1, t_2, \ldots\ldots, t_8$ time of nodes



Constraints :

- Project activitiy time limits

$t_A = 1;\ \ t_B = 2;\ \ 1 \le t_C \le 3;$

$2 \le t_D \le 4;\ \ 3 \le t_E \le 5;\ \ 4 \le t_F \le 6;$

$10 \le t_G \le 12;\ \ 6 \le t_H \le 8;\ \ 8 \le t_I \le 10;$

- Project completion time limit

  $t8 - t1 \le 20$ Not required

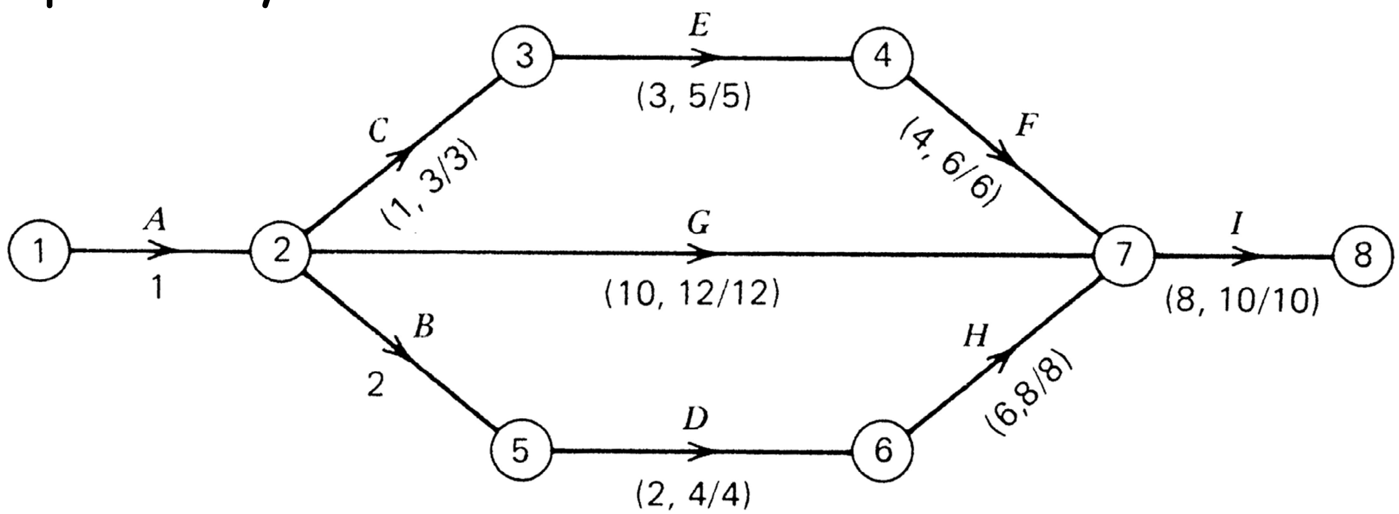
- Time limits

Job A	$t_2 - t_1 \ge t_A$
Job B	$t_5 - t_2 \ge t_B$
Job C	$t_3 - t_2 \ge t_C$
Job D	$t_6 - t_5 \ge t_D$
Job E	$t_4 - t_3 \ge t_E$
Job F	$t_7 - t_4 \ge t_F$
Job G	$t_7 - t_2 \ge t_G$
Job H	$t_7 - t_6 \ge t_H$
Job I	$t_8 - t_7 \ge t_I$

Objective function

$$\text{Min } Z = 3\,(3-t_C) + 4\,(4-t_D) + 5\,(5-t_E) + 6\,(6-t_F) + 12\,(12-t_G) + 8\,(8-t_H) + 10\,(10-t_I) - 30 * [25 - (t_8 - t_1)]$$

(PR&S Pr. 3.29) : The numbers in the network indicates the crash time (weeks), normal time (weeks) and unit crash costs (1000TL / week), respectively.



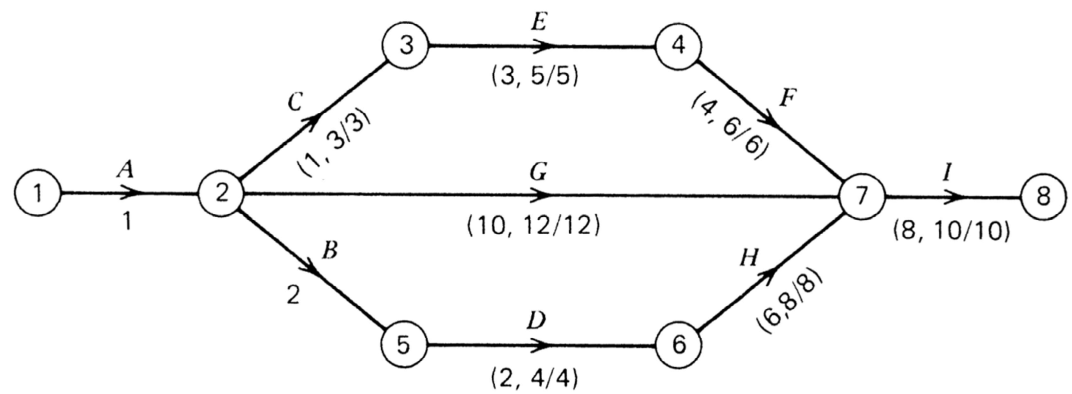
Assume that there is a 20,000 TL available for crashing..

Formulate an LP problem that will estimate the project completion duration with this additional budget.

Variables :

$t_A, \ldots, t_I$ activity times

$t_1, t_2, \ldots, t_8$ time of nodes



Constraints :

- Project activitiy time limits

$t_A = 1;\ t_B = 2;\ 1 \le t_C \le 3;$

$2 \le t_D \le 4;\ 3 \le t_E \le 5;\ 4 \le t_F \le 6;$

$10 \le t_G \le 12;\ 6 \le t_H \le 8;\ 8 \le t_I \le 10;$

- Time limits

Job	A	$t_2 - t_1 \ge t_A$
Job	B	$t_5 - t_2 \ge t_B$
Job	C	$t_3 - t_2 \ge t_C$
Job	D	$t_6 - t_5 \ge t_D$
Job	E	$t_4 - t_3 \ge t_E$
Job	F	$t_7 - t_4 \ge t_F$
Job	G	$t_7 - t_2 \ge t_G$
Job	H	$t_7 - t_6 \ge t_H$
Job	I	$t_8 - t_7 \ge t_I$

COST < 20,000

$$COST = 3(3-t_C) + 4(4-t_D) + 5(5-t_E) + 6(6-t_F) + 12(12-t_G) + 8(8-t_H) + 10(10-t_I)$$

Objective function **Minimize** **$Z = t8 - t1$**

		Normal		Crash	
Activity	Predecessors	Time	Cost ($)	Time	Cost ($)
A	-	3	5,000	2	6,500
B	A	4	12,000	3	15,000
C	B	6	6,000	4	7,000
D	B	10	20,000	6	30,000
E	C	12	12,000	8	15,000
F	C,D	4	11,000	3	12,000
G	D	6	3,500	4	7,500
H	F,G	8	5,000	5	8,000
I	E,H	5	8,000	3	14,000

HOMEWORK For the project shown above

a) Plot the network
b) Find the minimum project completion time under Normal conditions
c) Find the most economical crashing if the project must be completed within 30 time units.
d) Find the minimum completion time if the total budget is limited to $100,000.

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