Different Queueing Model Analysis:

1. $(M/M/1) : (GD/\infty/\infty)$ Model :

The parameters of this model are as follows:

1. Arrival rate follows poisson's distribution.
2. Service rate follows Poisson's distribution.
3. Number of server is one.
4. Service discipline is general discipline.
5. Maximum number of customers permitted in the system is infinity / infinite.
6. Size of celling source is infinite.

The steady state formula to obtain the probability of having 'n' customers in the system P_n and the formula for P_0, L_s, L_q, W_s & W_q are presented ~~bello~~ below.

usy idle = $P_0 = 1 - \rho$

ρ = traffic intensity / line utilizat

Probability of k population of time t.

1. $P_K = (1-\rho)\rho^K$ where $K = 0, 1, 2, 3, \ldots, \infty$ & $\rho = \frac{\lambda}{\mu} < 1$ $P_K = (1-\rho)\rho^K$

N = Number of customers in service

2. $N = L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$

Nq = Number of customers in queue

3. $N_q = L_q = L_s - \frac{\lambda}{\mu} = \frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu}$

$= \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$ $N_q = \frac{\rho^2}{1-\rho}$

Ws = ~~waiting~~ total time in the system

Wq = Waiting time in queue.

4. $W_s = \frac{L_s}{\lambda} = \frac{\lambda}{(\mu-\lambda)\lambda} = \frac{1}{\mu-\lambda} = \frac{1}{(1-\rho)\mu}$ $W_s = \frac{N}{\lambda}$

λ = arrival rate

μ = service rate

5. $W_q = \frac{L_q}{\lambda} = W_s - \frac{1}{\mu} = \frac{1}{\mu - \lambda} - \frac{1}{\mu}$

$$\frac{N}{\mu} = = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu - \lambda} = \frac{\rho}{\mu\left(1-\frac{\lambda}{\mu}\right)} = \frac{\rho}{\mu(1-\rho)}$$

Problems associated with M/M/1 model:

Example-1: The arrival rate of customers at a banking counter follows Poisson distribution with a mean of 45 persons per hours. The service rate of the counter also follows Poisson distribution, with a mean of 60 persons per hour. Find the followings:

a. Probability of having zero customer in system

b. " " 5 " " "

c. " " 10 " " "

d. Find $\frac{L_s}{N}, \frac{L_q}{N_q}, W_s$ & W_q

Soln: Given, Arrival rate, $\lambda = 45$ pr/hr

service rate, $\mu = 60$ pr/hr

$\therefore$ Utilization factor, $\rho = \frac{\lambda}{\mu} = \frac{45}{60} = 0.75$

a. $P_0 = (1-\rho)\rho^0 = 1-\rho = 1-0.75 = 0.25$

b. $P_5 = (1-\rho)\rho^5 = (1-0.75)(0.75)^5 = 0.0593$

c. $P_{10} = (1-\rho)\rho^{10} = (1-0.75)(0.75)^{10} = 0.0141$

d. $L_s = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3$ customers

$L_q = \frac{\rho^2}{1-\rho} = \frac{0.75^2}{1-0.75} = 2.25$ customers

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{60-45} = 0.067 \text{ hour}$$

$$W_q = \frac{\rho}{\mu - \lambda} = \frac{0.75}{60-45} = 0.05 \text{ hour}$$

Example 2: A supermarket has a single Cashier. During the peak hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be processed by the Cashier is 24 customers per hour. Calculate

(i) The probability that the cashier is idle

(ii) L_s (iii) L_q (iv) W_s (v) W_q

Soln: Given, $\lambda = 20$ customers per hour

$\mu = 24$ customers per hour

(i) $$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{20}{24} = 0.167$$

Here, $$\rho = \frac{\lambda}{\mu} = \frac{20}{24} = 0.833$$

(ii) $$L_s = \frac{\rho}{1-\rho} = \frac{0.833}{1-0.833} = 4.988$$

(iii) $$L_q = \frac{\rho^2}{1-\rho} = \frac{(0.833)^2}{1-0.833} = 4.155$$

(iv) $$W_s = \frac{1}{\mu - \lambda} = \cancel{\frac{1}{60-45}} = \frac{1}{24-20} = 0.25$$

(v) $$W_q = \frac{\rho}{\mu - \lambda} = \frac{0.833}{24-20} = 0.20825$$

Example - 3 : Any air lines organisation has one reservation clerk on duty in its local branch at any given time. The clerk handles information regarding passenger reservation and flight timings. Assume that the number of customers arriving during any given period is Poisson distributed and an arrival rate of eight per hour and that the reservation clerk can serve a customer in six minutes on an average with an exponential distributed service time. Calculate

(i) What is the probability that the system is busy.

(ii) W_s (iii) L_q (iv) L_s.

Soln: Given, $\lambda = 8$ customers per hour

$$\mu = \frac{60}{6} = 10 \text{ customers per hour.}$$

$$\therefore \rho = \frac{\lambda}{\mu} = \frac{8}{10} = 0.8$$

(i) The probability that the system is busy is

$$= 1 - P_0 = 1 - (1-\rho)\rho^0, = 1 - (1 - 0.8) = 0.8$$

(ii) $$W_s = \frac{1}{\mu - \lambda} = \frac{1}{10-8} = \frac{1}{2} = 0.5 \text{ hour}$$

(iii) $$L_q = \frac{\rho^2}{1-\rho} = \frac{0.8^2}{1-0.8} = 3.2 \text{ customers}$$

(iv) $$L_s = \frac{\rho}{1-\rho} = \frac{0.8}{1-0.8} = 4 \text{ customers.}$$

Example-4: The xyz company's quality control dept. is managed by a single clerk, who takes on an average 5 min in checking parts of each of the machine coming in for inspection. The machine arrive once in every 8 min on the average. One hour of the machine is valued at Rs. 15 and a clerk's time is valued at Rs. 4 per hour. What are the average hourly queueing costs associated with the quality department.

Soln: Given, Mean arrival rate, $\lambda = \frac{60}{8} = 7.5$ machine per hour

Mean service rate, $\mu = \frac{60}{5} = 12$ " "

Average time spent by a machine in the system

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{12 - 7.5} = 0.222 \text{ hour}$$

Cost for per machine $= 0.222 \times 15 = 3.33$

Cost for 7.5 machine per hour $= 3.33 \times 7.5 = 24.975$

$\cong$ Rs. 25/hour

Average hourly cost for the clerk = Rs. 4

Hence, total cost = Rs. 25 + Rs. 4 = Rs. 29 per hour.

Example: 5 A duplicating m/e maintained for office use is used and operated by people in the office who need to make copies, mostly secretaries. Since the work to be copied varies in length (number of pages of the original) and copies required, the service rate is randomly distributed, but it does approximate a poisson having a mean service rate of 10 jobs per hour. Generally, the requirements for use are random over the entire 8 hour working day but arrives at a rate of 5 per hour. Several people have noticed that a waiting line developes occasionally and have questioned the policy of maintaining only one unit. If the time of a secretary is valued at Rs. 3.50 per hour

(i) equipment utilization factor

(ii) the probability that an arrival has to wait.

(iii) the average system time &

(iv) the average cost due to waiting & operating the machine.

Soln: Here, $\lambda = 5$ jobs per hour

$\mu = 10$ " "

(i) $\rho = \frac{\lambda}{\mu} = \frac{5}{10} = 0.50$

(ii) $P_0 = 1 - \rho = 1 - \frac{1}{2} = \frac{1}{2}$

$P(busy) = 1 - P_0 = 1 - (1-\rho) = \rho = \frac{1}{2} = 0.50$

(iii) $W_s = \frac{1}{\mu - \lambda} = \frac{1}{5}$ hour

(iv) Average cost per day

= Average cost per job × No. of job proceed per day

$= 8 \times 3.5 \times L_s = 8 \times 3.5 \times \frac{\lambda}{\mu - \lambda}$ = Rs. 28 per day

where L_s = Page length / no. of unit

Birth-Death Process

* M/M/s : GD/∞/∞ Model

$$P_k = \frac{\rho^k}{k!} P_0 \quad (1 \le k \le s)$$

$$= \frac{\rho^k}{s^{k-s}\, s!} P_0 \quad (k \ge s)$$

$$P_0 = \left[\sum_{k=0}^{s-1} \frac{\rho^k}{k!} + \frac{\rho^s}{s!\left(1-\frac{\rho}{s}\right)} \right]^{-1}$$

$$L_q (N_q) = \left[\frac{\rho^s \ \lambda\mu}{(s-1)!\ (s\mu-\lambda)^2} \right] P_0 = \frac{\rho^{s+1}}{(s-1)!\ (s-\rho)^2} P_0$$

$$L_s = L_q + \rho = \lambda W_s$$

$$W_q = \frac{L_q}{\lambda}, \quad W_s = W_q + \frac{1}{\mu}$$

$$P_0 = 1 - \rho$$

$$P(k \ge s) = \frac{\rho^s}{s!\left(1-\frac{\rho}{s}\right)} P_0$$ [Prob. that a customer has to wait]

*Prob.-1: At a central warehouse, vehicles arrive at the rate of 18/hr and the arrival rate follows poisson distribution. The unloading time of the vehicles follows exponential distribution and the unloading rate is 6 vehicles/hr. There are 4 unloading crews. Find the following:

a) P_0 and P_3, b) L_q, L_s, W_q, W_s

Soln:

$\lambda = 18/hr$

$\mu = 6/hr$

$s = 4$

and $\rho = \frac{\lambda}{\mu} = \frac{18}{6} = 3$ $\quad r = \rho = \frac{\lambda}{\mu}$ $\quad r \to \rho$.

$\rho = \frac{\lambda}{\mu}$

a) $P_0 = \left[\sum_{k=0}^{s-1} \frac{\rho^k}{k!} + \frac{\rho^s}{s!\left(1-\frac{\rho}{s}\right)} \right]^{-1}$

$= \left[\sum_{k=0}^{3} \frac{3^k}{k!} + \frac{3^4}{4!\left(1-\frac{3}{4}\right)} \right]^{-1}$

$= 0.0377$

$P_3 = \frac{\rho^3}{3!} P_0$ $\quad \left[\because 1 \le k \le s, \therefore P_k = \frac{\rho^k}{k!} P_0 \right]$

$= 0.1697$

b) $L_q = \frac{\rho^{s+1}}{(s-1)!\,(s-\rho)^2} P_0 = \frac{3^5}{3!\,(4-3)^2} \times 0.0377$

$= 1.53 \approx 2$ vehicles

$L_s = L_q + \rho = 1.53 + 3 = 4.53 \approx 5$ vehicles

$W_q = \frac{L_q}{\lambda} = \frac{1.53}{18} = 0.085$ hr $= 5.1$ min.

$W_s = W_q + \frac{1}{\mu} = 0.085 + \frac{1}{6} = 0.252$ hr

$= 15.12$ min.

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* Prob.-2: A tax consulting firm has four service stations (counters) in its office to receive people who have problems and complaints about their income, wealth and sales taxes. Arrival average is 100 persons in a 10 hours service day. Each tax adviser spends an irregular amount of time servicing the arrivals which have been found to have an exponential distribution. The average service time is 20 min. Calculate

(i) L_s, (ii) L_q (iii) W_s (iv) W_q

(v) the probability that a customer has to wait before he gets service.

Sol:

Here, $\lambda = 100/10 = 10/hr$,

$\mu = 60/20 = 3/hr$

$s = 4$ and $\rho = \frac{\lambda}{\mu} = \frac{10}{3}$

Assume the model: $M/M/s : GD/\alpha/\alpha$

Now, $$P_0 = \left[\sum_{k=0}^{s-1} \frac{\rho^k}{k!} + \frac{\rho^s}{s!\left(1 - \frac{\rho}{s}\right)} \right]^{-1}$$

$$= \left[\sum_{k=0}^{3} \frac{\left(\frac{10}{3}\right)^k}{k!} + \frac{\left(\frac{10}{3}\right)^4}{4!\left(1 - \frac{10}{12}\right)} \right]^{-1}$$

$= 0.0213$

i) $$L_s = L_q + \rho = \frac{\rho^{s+1}}{(s-1)!\,(s-\rho)^2} P_0 + \rho$$

$$= \frac{\left(\frac{10}{3}\right)^5}{3!\left(4 - \frac{10}{3}\right)^2} \times 0.0213 + \frac{10}{3}$$

$= 6.567 \approx 7$

(ii) $L_q = L_s - \rho = 6.567 - \frac{10}{3} = 3.234 \approx 3$

(iii) $W_s = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{3.234}{10} + \frac{1}{3} = 0.6567h$

(iv) $W_q = \frac{L_q}{\lambda} = \frac{3.234}{10} = 0.3234\ hr$

(v) $P(k \geq 5) = \dfrac{\rho^s}{s!\left(1-\frac{\rho}{s}\right)} P_0$

$= \dfrac{\left(\frac{10}{3}\right)^4}{4!\left(1-\frac{10}{12}\right)} \times 0.0213$

$= 0.618$

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* Prob.— 3: A telephone exchange has two long distance operators. It is observed that, during the peak load, long distance calls arrive in a poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 minutes.

(i) what is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day?

(ii) If the subscribers will wait and are serviced in turn, what is the expected waiting time?

sol^n:

Here, $s = 2$, $\lambda = 15/hr$, $\mu = \frac{60}{5} = 12/hr$

$\therefore \rho = \frac{\lambda}{\mu} = \frac{15}{12} = \frac{5}{4}$

Model: M/M/S ; GD/∞/∞

Now, $P_0 = \left[\sum_{k=0}^{s-1} \frac{\rho^k}{k!} + \frac{\rho^s}{s!\left(1-\frac{\rho}{s}\right)} \right]^{-1}$

$= \left[\sum_{k=0}^{1} \frac{1}{k!}\left(\frac{5}{4}\right)^k + \frac{\left(\frac{5}{4}\right)^2}{2!\left(1-\frac{5}{8}\right)} \right]^{-1}$

$= \frac{3}{13}$

i) Probability that a subscriber will have to wait is given by

$$P(k \geq s) = \frac{\rho^s}{s!\left(1-\frac{\rho}{s}\right)} P_0 = \frac{\left(\frac{5}{4}\right)^2}{2!\left(1-\frac{5}{8}\right)} \times \frac{3}{13}$$

$= 0.48$

ii) Expected waiting time, $W_q = \frac{L_q}{\lambda}$

$$= \frac{\rho^{s+1} \cdot P_0}{(s-1)!\,(s-\rho)^2 \cdot \lambda}$$

$$= \frac{\left(\frac{5}{4}\right)^3 \times \frac{3}{13}}{1!\left(2-\frac{5}{4}\right)^2 \times \left(\frac{1}{4}\text{ min}\right)}$$

$$= \frac{\left(\frac{5}{4}\right)^3 \times \frac{3}{13}}{\left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}} = 3.2 \text{ min}$$

$= \frac{3.2}{60} = 0.0534$ hr.

[Note: $\lambda = 15/\text{hr} = \frac{1}{4}/\text{min}$]

* Prob.-4: A bank has only one typist. Since the typing work varies in length (number of pages to be typed), the typing rate is randomly distributed ~~approximately~~ approximating a poisson distribution with mean service rate of 8 letters/hour. The letters arrive at a rate of 5/hour during the entire 8-hour work day. If the waiting time cost of the letters is Tk. 15 per hr, find the average system time and the total lost time cost.

There is a possibility of either installing an additional typewritter of the same type or replacing the present one by a better and faster typewritter. The data are given below:

1. Present typewriter: service rate per hr 8 Daily rental cost 25 Tk.

2. Preposed typewriter: service rate per hr 12 Daily rental cost 45 Tk.

Suggest the better alternative.

Soln:

Here, $\lambda = 5/hr$ and $\mu = 8/hr$ $\therefore \rho = \frac{\lambda}{\mu} = \frac{5}{8}$

Average system time with present typewritter,

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{3} \text{ hr}$$

[illegible] arrival

$\therefore$ Lost time cost/day $= (8 \times 5) \times \frac{1}{3} \times 15 = $ Tk. 200

Thus total cost per day of present typewritter

= Rental cost + lost time cost

$= 25 + 200 = 225$ Tk.

For proposed typewritter:

Average system time, $W_s = \frac{1}{\mu - \lambda} = \frac{1}{12-5} = \frac{1}{7}$ hr.

Lost time cost per day $= 8 \times 5 \times \frac{1}{7} \times 15 = 85.7$ Tk.

Total cost per day $= 45 + 85.7 = 130.7$ Tk.

To calculate the waiting time in the system for two typewritter:

$$P_0 = \left[\sum_{k=0}^{s-1} \frac{\rho^k}{k!} + \frac{\rho^s}{s!\left(1 - \frac{\rho}{s}\right)} \right]^{-1}$$

$$= \left[\sum_{k=0}^{2-1} \frac{(5/8)^k}{k!} + \frac{\left(\frac{5}{8}\right)^2}{2!\left(1 - \frac{5}{16}\right)} \right]^{-1}$$

$$= \frac{11}{21}$$

Expected waiting time in the system, $W_s = W_q + \frac{1}{\mu}$

$$= \frac{\rho^{s+1}}{\lambda (s-1)! (s-\rho)^2} P_0 + \frac{1}{\mu}$$

$$= \frac{32}{231}$$

Total cost per day = rental cost of two type writter + cost of lost time $= 2 \times 25 + (8\times$

$\times \frac{32}{..} \times 15 = 133$