CSE'08

Solutions of Queueing Problems

1. Exact solution — previously discussed.
2. Approximation Techniques —
   a) Fluid Approximation
   b) Diffusion Approximation
   c) Simulation
   d) Other Approx. Techniques
      — Numerical Approx. etc.

* Why approximate solution is necessary?

— Because for many queueing problems exact model cannot be made and exact equation cannot be found. So exact solution cannot be obtained and it is very limited. Hence approximation is taken.

* Fluid Approximation:

— considers mean only

XX * Diffusion Processes:

— Considers mean and variance.

* Classification of stochastic process:

Simulation

* What is simulation?
* Why simulation is done?
* What should be considired (requirements) for doing simulation?
* Generation of random deviates: (Page-377)
* Generation of random numbers: (Page-378)
* Techniques for generating random deviates: (Page-388)
* Simulation Process:

Fig.- 9.1, 9.2, 9.3 only

* Examples of simulation Modeling:

– Sales of Life Insurance (self)

1.2 FLUID APPROXIMATION METHOD [4]

In trying to solve real world problems, engineers first try to solve the problems by crude estimations and try to get some idea about the type of expected solutions. Then they consider about the improvement and try to find out the exact solution if it is possible to obtain. Fluid approximation is a *first order* approximation method, where stochastic process is replaced by its simple average, and which considers discrete customer flow into the system as continuous fluid flow. This assumption can be made if the queue is very large , and many customers must arrive and depart before the queue can change very much. In a period of time sufficiently long for many arrivals, the uncertainity due to random fluctuations in the number of arrivals can be taken as small compared to the observed number of arrivals. That is, let $A(t)$ be the total number of arrivals in $(0,t)$ time interval, and when $A(t)$ gets large compared to unity then we expect only small percentage of deviation from its average value $E\{A(t)\} \triangleq \overline{A(t)}$, that is by the *law of large numbers*

$$\lim_{t \to \infty} \frac{A(t) - \overline{A(t)}}{A(t)} = 0 \qquad (1..$$

with probability one. This suggests that a first order approximation to the stochastic process is to replace it by its average value as a function of time. This is the fluid approximation for queues, in which the discontinuous stochastic process $A(t)$ is replaced by the continuous deterministic process $\overline{A(t)}$. Similarly we allow the discontinuous stochastic departure process $B(t)$ (=total number of departures in interval $(0,t)$) be replaced by its mean value $\overline{B(t)}$. SO, if we assume $L(0) = 0$ i.e., number of customers in the system is 0 initially, then the fluid approximation predicts the number in the system at time t as

$$L(t) = \overline{A(t)} - \overline{B(t)} \tag{1.2}$$

which also is a deterministic continuous function of time.

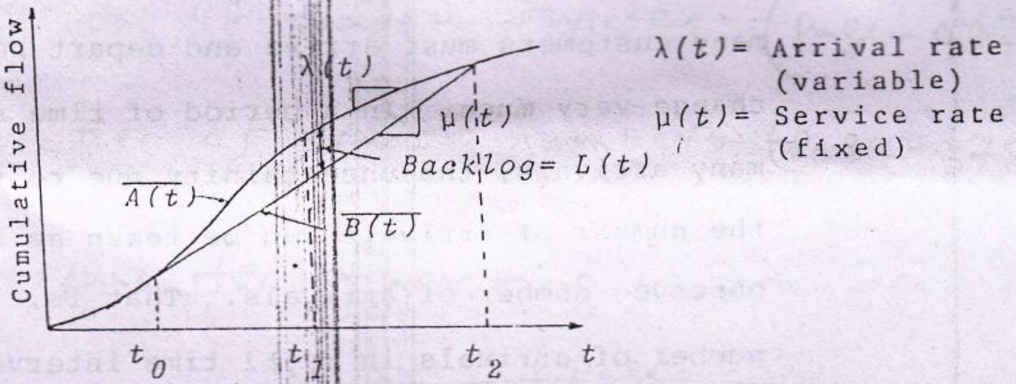

Fig. 1.1

This simple model has many draw backs, i.e., (it says no queue forms as the system approaches saturation from left at t_0 *(Fig.1.1)*. But we know the fact that size of backlog at this time is strongly dependent on the manner in which rate of arrival approaches rate of departure in the interval prior to t_0. As a result of this the queue length is badly under estimated. We can get a gross picture through fluid approximation method for rush hour queues. The fluid approximation is actually a $D/D/1(\infty)$ continuous approximation.)

I.3 DIFFUSION PROCESSES [4]

In the previous section first order approximation for queues was discussed in which (the arrival and the departure

In case of fluid App.

processes were replaced by their mean values. Since , these processes are random in nature, the first order approximation method can be improved by permitting $A(t)$ and $B(t)$ to have variations about the mean. This can be done by considering variances for the arrival and the departure processes. So, if these processes are represented by *Normal Stochastic* processes, then the variations are included. This can be justified by *Central Limit Theorem* which is as follows : observing $A(t)$ represents the total number of arrivals up to time t, then the probability that $A(t) \geq n$, is the same as the probability that the nth customer arrives at a time τ_n that occurs at or before t i.e.,

$$P[A(t) \geq n] = P[\tau_n \leq t]$$

where
$A(t)$ be the total no of arrivals up to time t
τ_n be the arrival time of nth customer that occurs at or before

The arrival time of nth customer is merely the sum of n interarrival times , i.e.,

$$\tau_n = t_1 + t_2 + t_3 + \ldots + t_n$$

where we assume $\tau_0 = 0$. For general independent input and output system we assume that the set t_i is a set of independent and identically distributed random variables. When the time t and therefore the number n gets large then τ_n is the sum of a large number of independent and identically distributed random variables. Thus we expect that the central limit theorem should apply and permits us to describe the random variable τ_n and therefore also the random process $A(t)$ as *Gaussian function*. This assumption of normality for $A(t)$ (and for $B(t)$) is the basic requirement for our diffusion approximation. Further, for the diffusion approximation we need to consider the departure process as an independent process, which we can have if the system is considered as a non-empty one i.e., when the number of customers in the system is large. If we have two independent normally distributed random processes, say $A(t)$ and $B(t)$, then any linear combination of these two processes is also a normally distributed process with some appropriate mean and variance. Here we are interested in $A(t)-B(t)$, which represents the number of the customers in the system. Thus we conclude that the number of customers in the system at time t given by $x(t)=A(t)-B(t)$ is also a normal random process, whose mean is $\overline{x}(t)=\overline{A}(t)-\overline{B}(t)$ and with variance $\sigma_x^2=\sigma_A^2+\sigma_B^2$, since variances of the independent processes must add.

So, we consider the arrival and the departure processes as continuous quantities, and the number of the customers in the system varies as *normal stochastic process*, then for large queue, for the probability density function $(p.d.f)$ $f(x,t)$ of the customer number x in the system, the following diffusion equation (*Forward Kolmogorov eqn. or Focker - Plank eqn.*) exsits.

$$\frac{\partial}{\partial t} f(x,t) = -\frac{\partial}{\partial x}\{\eta(x,t)f(x,t)\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\{\sigma^2(x,t)f(x,t)\} \qquad (1.3)$$

, where $\eta(x,t)$ is the infinitesimal mean and $\sigma^2(x,t)$ is the infinitesimal variance, which means the rate of change of the above mentioned quantities with respect to time.

For stationary arrival and departure processes mean and variance can be considered as

$$\eta(x,t) = \eta(x) \quad \text{and} \quad \sigma^2(x,t) = \sigma^2(x)$$

and postulate that *eqn.(1.3)* has a solution with $f(x,t) = f(x)$, independent of time. So the *eqn.(1.3)* becomes

$$0 = -\frac{d}{dx}\{\eta(x)f(x)\} + \frac{1}{2}\frac{d^2}{dx^2}\{\sigma^2(x)f(x)\} . \tag{1.4}$$

If this equation is integrated with respect to x we get

$$C = -\eta(x)f(x) + \frac{1}{2}\frac{d}{dx}\{\sigma^2(x)f(x)\} \tag{1.5}$$

, for some constant C. But from the boundary conditions of queues i.e., queue must be finite in finite time with probability one and queue is restricted to remain non-negative i.e., as $x \to \infty$ or as $x \to 0$, the right hand side of *eqn.(1.5)* vanishes. Therefore we get, from *eqn.(1.5)*

[illegible] $x \to \alpha$ or $x \to 0$

$$0 = -\eta(x)f(x) + \frac{1}{2}\frac{d}{dx}\{\sigma^2(x)f(x)\} \tag{1.6}$$

Eqn.(1.6) is the basic equation, based on which we shall be deriving approximation formulas for different type of queueing models in the following chapters. We shall be defining $\eta(x)$ and $\sigma^2(x)$ in each case, considerations about which remain as one of the central point in the proposed approximation method.

* Why approx. soln is necessary?
Because for many queueing problems, exact model can not be made and eqn can not be found. So exact soln can not be obtained.