# Chapter 6

# Stochastic Processes

## 6.1 INTRODUCTION

In the previous chapters we have seen the need to consider a collection or a family of random variables instead of a single random variable. A family of random variables that is indexed by a parameter such as time is known as a **stochastic process** (or **chance** or **random process**).

***Definition (Stochastic Process).*** A stochastic process is a family of random variables $\{X(t) \mid t \in T\}$, defined on a given probability space, indexed by the parameter $t$, where $t$ varies over an index set $T$.

The values assumed by the random variable $X(t)$ are called **states**, and the set of all possible values forms the **state space** of the process. The state space will be denoted by $I$.

Recall that a random variable is a function defined on the sample space $S$ of the underlying experiment. Thus the above family of random variables is a family of functions $\{X(t, s) \mid s \in S, t \in T\}$. For a fixed $t = t_1$, $X_{t_1}(s) = X(t_1, s)$ is a random variable [denoted by $X(t_1)$] as $s$ varies over the sample space $S$. At some other fixed instant of time $t_2$, we have another random variable $X_{t_2}(s) = X(t_2, s)$. For a fixed sample point $s_1 \in S$, $X_{s_1}(t) = X(t, s_1)$ is a single function of time $t$, called a **sample function** or a **realization** of the process. When both $s$ and $t$ are varied, we have the family of random variables constituting a stochastic process.

**Example 6.1 [STAR 1979]**

Consider the experiment of randomly choosing a resistor $s$ from a set $S$ of thermally agitated resistors and measuring the noise voltage $X(t, s)$ across the resistor at time $t$. Sample functions for two different resistors are shown in Figure 6.1.





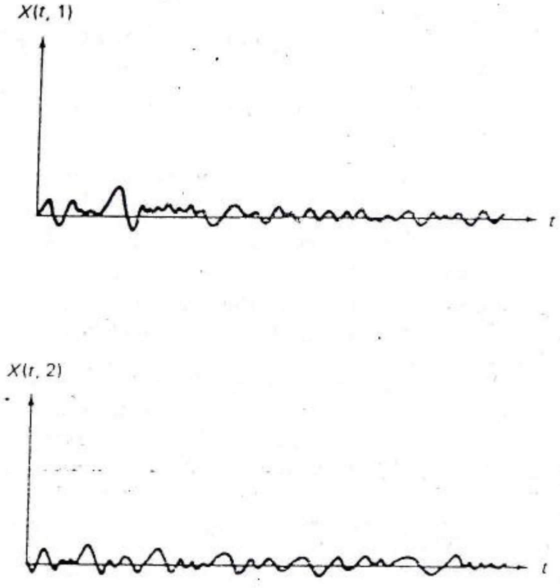



**Figure 6.1** Noise voltages across two resistors

At a fixed time $t = t_1$, suppose we measure the voltages across all the resistors in the set $S$, count the number of resistors with a voltage level less than or equal to $x_1$ and divide this count by the total number of resistors in $S$. Using the frequency interpretation of probability, this will give the distribution function, $F_{X(t_1)}(x_1) = P[X(t_1) \leqslant x_1]$, of the random variable $X(t_1)$. This calculation can be repeated at other instants of time $t_2, t_3, \ldots$, to obtain the distribution functions of $X(t_2)$, $X(t_3)$, .... The joint distribution function of $X(t_1)$ and $X(t_2)$ can similarly be obtained by computing the relative frequency of the event $[X(t_1) \leqslant x_1$ and $X(t_2) \leqslant x_2]$. Continuing in this fashion, we can compute the joint distribution function of $X(t_1)$, $X(t_2)$, $\ldots$, $X(t_n)$. #

If the state space of a stochastic process is discrete, then it is called a **discrete-state process**, often referred to as a **chain**. In this case, the state space is often assumed to be $\{0, 1, 2, \ldots\}$. Alternatively, if the state space is continuous, then we have a **continuous-state process**. Similarly, if the index set $T$ is discrete, then we have a **discrete (time)-parameter process**; otherwise we have a **continuous parameter process**. A discrete-parameter process is also called a stochastic sequence and is denoted by $\{X_n \mid n \in T\}$. This gives us four different types of stochastic processes, as shown in Table 6.1.

The theory of queues (or waiting lines) provides many examples of stochastic processes. Before introducing these processes, we present a notation to describe the queues. A queue may be generated when customers (jobs) arrive at a station (computing center) to receive service (see Figure 6.2). Assume that successive interarrival times $Y_1$, $Y_2$, ..., between jobs are independent identically distributed random variables having a distribution

Table 6.1 A Classification of Stochastic Processes

| | | Index Set *T* | |
|---|---|---|---|
| | | *Discrete* | *Continuous* |
| State Space *I* | Discrete | Discrete-parameter stochastic chain | Continuous-parameter stochastic chain |
| | Continuous | Discrete-parameter continuous-state process | Continuous-parameter continuous-state process |

$F_Y$. Similarly, the service times $S_1$, $S_2$, . . . , are assumed to be independent identically distributed random variables having a distribution $F_S$. Let $m$ denote the number of servers (computer systems) in the station (computing center). We use the notation $F_Y/F_S/m$ to describe the queuing system. To denote the specific types of interarrival-time and service-time distributions, we use the following symbols:

- $M$ (for memoryless) for the exponential distribution
- $D$ for a deterministic or constant interarrival or service time
- $E_k$ for a $k$-stage Erlang distribution
- $H_k$ for a $k$-stage hyperexponential distribution
- $G$ for a general distribution
- $GI$ for general independent interarrival times

Thus $M/G/1$ denotes a single-server queue with exponential interarrival times and an arbitrary service-time distribution. The most frequent example of a queue that we will use is $M/M/1$. Besides the nature of the interarrival-time and service-time distributions, we also need to specify a scheduling discipline that decides how the server is to be allocated to the jobs waiting for service. Unless otherwise specified, we will assume that jobs are selected for

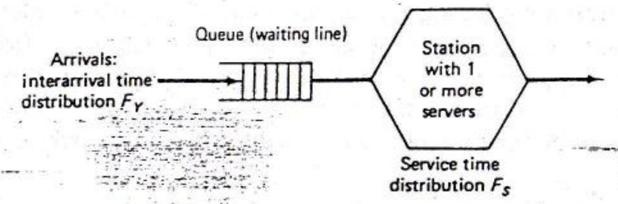



**Figure 6.2** A queuing system



service in the order of their arrivals; that is, we will assume FCFS (first-come, first-served) scheduling discipline. Now we will describe various stochastic processes associated with a queue.

**Example 6.2**

Consider a computer system with jobs arriving at random points in time, queuing for service, and departing from the system after service completion.

Let $N_k$ be the number of jobs in the system at the time of the departure of the $k$th customer (after service completion). The stochastic process $\{N_k \mid k = 1, 2, \ldots\}$ is a discrete-parameter, discrete-state process with the state space $I = \{0, 1, 2, \ldots\}$ and the index set $T = \{1, 2, 3, \ldots\}$. A realization of this process is shown in Figure 6.3.

Next let $X(t)$ be the number of jobs in the system at time $t$. Then $\{X(t) \mid t \in T\}$ is a continuous-parameter, discrete-state process with $I = \{0, 1, 2, \ldots\}$ and $T = \{t \mid 0 \leq t < \infty\}$. A realization of this process is shown in Figure 6.4.

Let $W_k$ be the time that the $k$th customer has to wait in the system before receiving service. Then $\{W_k \mid k \in T\}$, with $I = \{x \mid 0 \leq x < \infty\}$ and $T = \{1, 2, 3, \ldots\}$, is a discrete-parameter, continuous-state process. A realization of this process is shown in Figure 6.5.

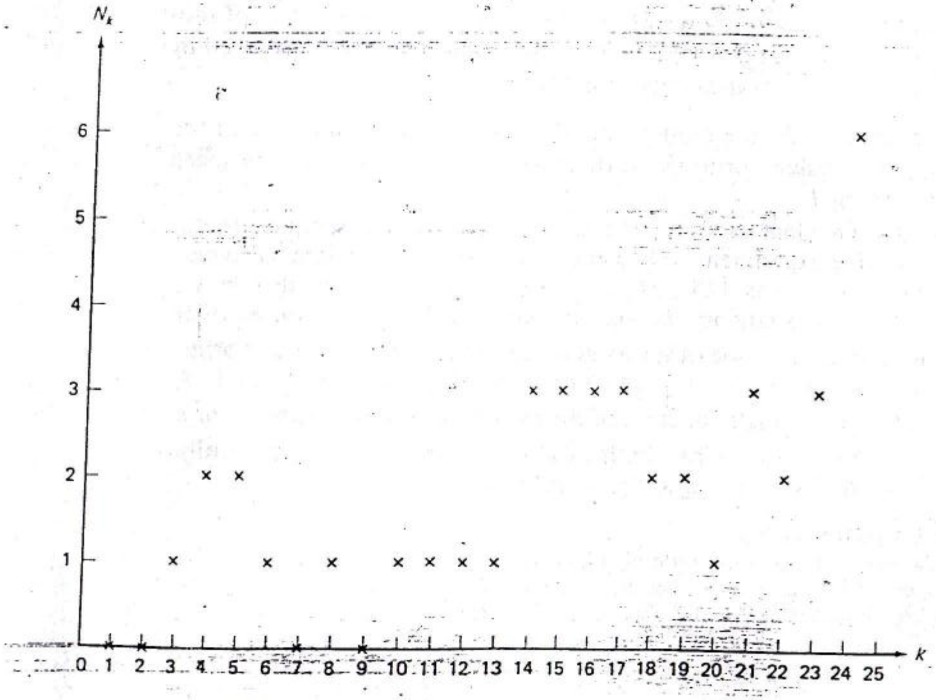



**Figure 6.3** Typical sample function of a discrete-parameter, discrete-state process

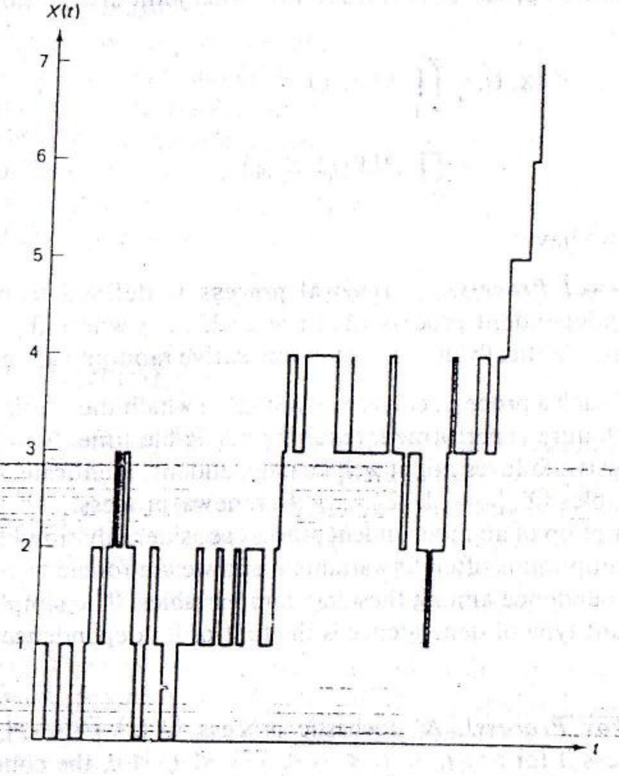



**Figure 6.4** Typical sample function of a continuous-parameter, discrete-state process

Finally, let $Y(t)$ denote the cumulative service requirement of all jobs in the system at time $t$. Then $\{Y(t) \mid 0 \leqslant t < \infty\}$ is a continuous-parameter, continuous-state process with $I = [0, \infty)$. A realization of this process is shown in Figure 6.6. #

### *Problems*

1. Write and run a program to simulate an $M/E_2/1$ queue and obtain realizations of the four stochastic processes defined in Example 6.2. Plot these realizations. You may use a simulation language such as SIMULA or GPSS or you may use one of the standard high-level languages. You will have to generate random deviates of the interarrival-time distribution (assume arrival rate $\lambda = 1$) and the service-time distribution (assume mean service time 0.8) using methods of Chapter 3.

   Study the process $\{N_k \mid k = 1, 2, \ldots\}$ in detail as follows:

   By varying the seeds for generating random numbers you get different realizations. For a fixed $k$, different observed values of $N_k$ for these distinct realizations can be used to estimate the mean and variance of $N_k$. Using a sample size of 30, estimate $E[N_k]$, $\text{Var}[N_k]$ for $k = 1, 5, 10, 100, 200, 1000$. What can you conclude from this experiment?



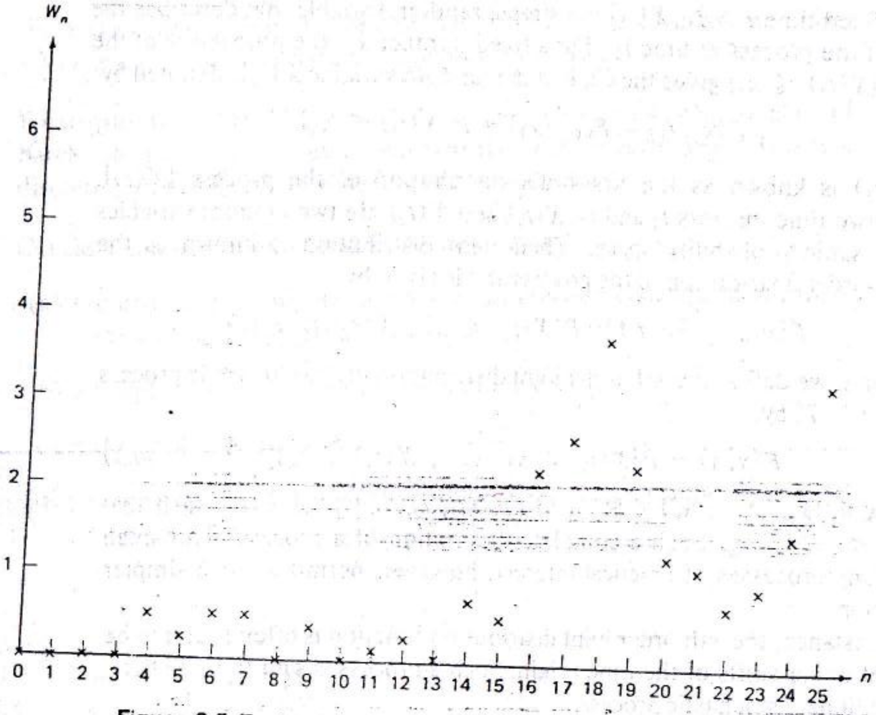



**Figure 6.5** Typical sample function of a discrete-parameter, continuous-state process

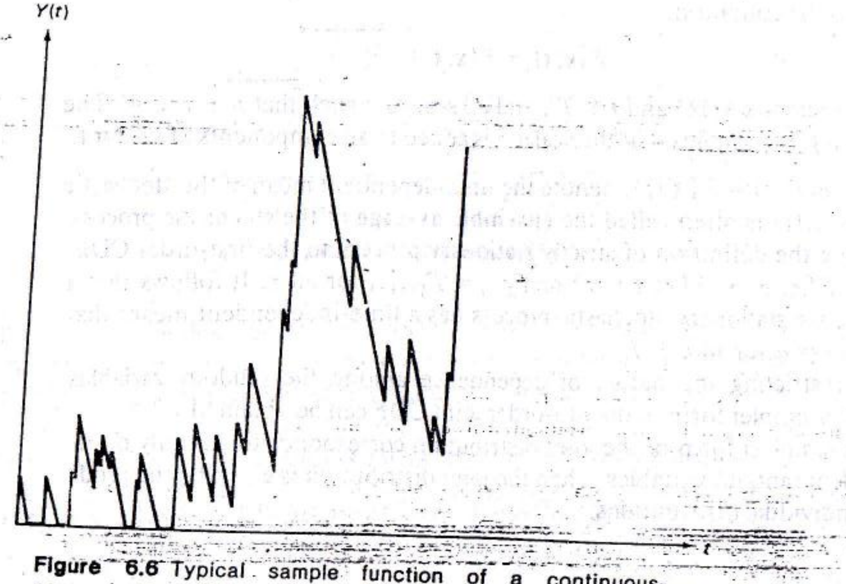



**Figure 6.6** Typical sample function of a continuous-parameter, continuous-state process

## 6.2 CLASSIFICATION OF STOCHASTIC PROCESSES

For a fixed time $t = t_1$, $X(t_1)$ is a simple random variable that describes the state of the process at time $t_1$. For a fixed number $x_1$, the probability of the event $[X(t_1) \leqslant x_1]$ gives the CDF of the random variable $X(t_1)$, denoted by:

$$F(x_1; t_1) = F_{X(t_1)}(x_1) = P[X(t_1) \leqslant x_1].$$

$F(x_1; t_1)$ is known as the first-order distribution of the process $\{X(t)\}$. Given two time instants $t_1$ and $t_2$, $X(t_1)$ and $X(t_2)$ are two random variables on the same probability space. Their joint distribution is known as the second-order distribution of the process and is given by:

$$F(x_1, x_2; t_1, t_2) = P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2).$$

In general, we define the $n$th order joint distribution of the stochastic process $\{X(t), t \in T\}$ by:

$$F(\mathbf{x}; \mathbf{t}) = P[X(t_1) \leqslant x_1, \ldots, X(t_n) \leqslant x_n] \tag{6.1}$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in T^n$ such that $t_1 < t_2 < \cdots < t_n$. Such a complete description of a process is no small task. Many processes of practical interest, however, permit a much simpler description.

For instance, the $n$th order joint distribution function is often found to be invariant under shifts of the time origin. Such a process is said to be a strict-sense stationary stochastic process.

***Definition (Strictly Stationary Process).*** A stochastic process $\{X(t)\}$ is said to be stationary in the strict sense if for $n \geqslant 1$, its $n$th-order joint CDF satisfies the condition:

$$F(\mathbf{x}; \mathbf{t}) = F(\mathbf{x}; \mathbf{t} + \tau)$$

for all vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{t} \in T^n$, and all scalars $\tau$ such that $t_i + \tau \in T$. The notation $\mathbf{t} + \tau$ implies that the scalar $\tau$ is added to all components of vector $\mathbf{t}$.

We let $\mu(t) = E[X(t)]$ denote the time-dependent mean of the stochastic process. $\mu(t)$ is often called the **ensemble average** of the stochastic process. Applying the definition of strictly stationary process to the first-order CDF, we get $F(x; t) = F(x; t + \tau)$ or $F_{X(t)} = F_{X(t+\tau)}$ for all $\tau$. It follows that a strict-sense stationary stochastic process has a time-independent mean; that is, $\mu(t) = \mu$ for all $t \in T$.

By restricting the nature of dependence among the random variables $\{X(t)\}$, a simpler form of the $n$th-order joint CDF can be obtained.

The simplest form of the joint distribution corresponds to a family of independent random variables. Then the joint distribution is given by the product of individual distributions.





***Definition (Independent Process).*** A stochastic process $\{X(t) \mid t \in T\}$ is said to be an **independent process** provided its $n$th-order joint distribution satisfies the condition:

$$F(\mathbf{x}; \mathbf{t}) = \prod_{i=1}^{n} F(x_i; t_i)$$
$$= \prod_{i=1}^{n} P[X(t_i) \leqslant x_i]. \tag{6.2}$$

As a special case we have:

***Definition (Renewal Process).*** A **renewal process** is defined to be a discrete-parameter independent process $\{X_n \mid n = 1, 2, \ldots\}$ where $X_1, X_2, \ldots$, are independent, identically distributed, nonnegative random variables.

As an example of such a process, consider a system in which the repair (or replacement) after a failure is performed, requiring negligible time. Now the times between successive failures might well be independent, identically distributed random variables $\{X_n \mid n = 1, 2, \ldots\}$ of a renewal process.

Though the assumption of an independent process considerably simplifies analysis, such an assumption is often unwarranted, and we are forced to consider some sort of dependence among these random variables. The simplest and the most important type of dependence is the first-order dependence or **Markov dependence.**

***Definition (Markov Process).*** A stochastic process $\{X(t) \mid t \in T\}$ is called a **Markov process** if for any $t_0 < t_1 < t_2 < \cdots < t_n < t$, the conditional distribution of $X(t)$ for given values of $X(t_0), X(t_1), \ldots, X(t_n)$ depends only on $X(t_n)$; that is:

$$P[X(t) \leqslant x \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_0) = x_0]$$
$$= P[X(t) \leqslant x \mid X(t_n) = x_n]. \tag{6.3}$$

Although this definition applies to Markov processes with continuous state space, we will be mostly concerned with discrete-state Markov processes—that is, Markov chains. We will study both discrete-parameter and continuous-parameter Markov chains.

In many problems of interest, the conditional distribution function (6.3) has the property of invariance with respect to the time origin $t_n$; that is:

$$P[X(t) \leqslant x \mid X(t_n) = x_n] = P[X(t - t_n) \leqslant x \mid X(0) = x_n].$$

In this case the Markov chain is said to be **(time)-homogeneous.** Note that the stationarity of the conditional distribution function (6.3) does not imply the stationarity of the joint distribution function (6.1). Thus, a homogeneous Markov process need not be a stationary stochastic process.

A.O. Allen

*We are the music-makers,*
*And we are the dreamers of dreams,*
*Wandering by lone sea-breakers,*
*And sitting by desolate streams;*
*World-losers and world-forsakers,*
*On whom the pale moon gleams:*
*Yet we are the movers and shakers*
*Of the world for ever, it seems.*

Arthur O'Shaughnessy

## 4.1 Definitions

A family of random variables $\{X(t), t \in T\}$ is called a *stochastic process*. Thus, for each $t \in T$, where $T$ is the *index set* of the process, $X(t)$ is a random variable. An element of $T$ is usually referred to as a time parameter and we often refer to $t$ as time, although this is not part of the definition. The *state space* of the process is the set of all possible values that the random variables $X(t)$ can assume. Each of these values is called a *state* of the process.

Stochastic processes are classified in a number of ways, such as by the index set and by the state space. If $T = \{0, 1, 2, \cdots\}$ or $T = \{0, \pm 1, \pm 2, \cdots\}$, the stochastic process is said to be a *discrete parameter process* and we will usually indicate the process by $\{X_n\}$. If $T = \{t : -\infty < t < \infty\}$ or $T = \{t : t \geq 0\}$, the stochastic process is said to be a *continuous parameter process* and will be indicated by $\{X(t), -\infty < t < \infty\}$ or $\{X(t), t \geq 0\}$. The state space is classified as *discrete* if it is finite or countable; it is *continuous* if it consists of an interval (finite or infinite) of the real line. For a stochastic process $\{X(t)\}$, for each $t$, $X(t)$ is a random variable and thus a function from the underlying sample space, $\Omega$, into the state space. For any $\omega \in \Omega$, there is a corresponding collection $\{X(t)(\omega), t \in T\}$ called a *realization* or *sample path* of $X$ at $\omega$ (usually the $\omega$ is elided.)

**Example 4.1.1** The waiting time of an arriving inquiry message until processing is begun, is $\{W(t), t \geq 0\}$. The arrival time, $t$, of the message is the continuous parameter. The state space is also continuous. □

**Example 4.1.2** The number of messages that arrive in the time period from 0 to $t$ is $\{N(t), t \geq 0\}$. This is a continuous parameter, discrete state space process. □

**Example 4.1.3** Let $\{X_n\}, n = 1, 2, 3, 4, 5, 6, 7\}$ denote the average time to run a batch job at the computer center on the $n$th day of the week. Thus, $X_1$ is the average job execution time on Sunday, $X_2$ on Monday, etc. Then $\{X_n\}$ is a discrete parameter, continuous state space process. □



**Example 4.1.4** Let $\{X_n, n = 1, 2, \ldots, 365(366)\}$ denote the number of batch jobs run at a computer center on the $n$th day of the year. This is a discrete parameter, discrete state space process. □

Consider random (unpredictable) events such as

(a) the arrival of an inquiry at the central processing system of an interactive computer system,

(b) a telephone call to an airline reservation center,

(c) and end-of-file interrupt, or

(d) the occurrence of a hardware or software failure in a computer system.

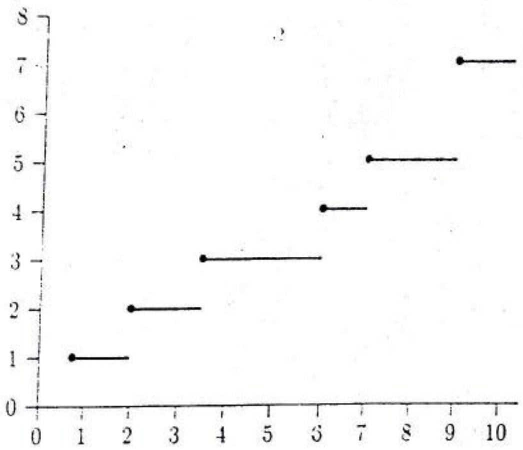



Figure 4.1.1. Realization of counting process $N(t)$.

Such events can be described by a *counting process* $\{N(t), t \geq 0\}$, where $N(t)$ is the number of events that have occurred after time 0 but not later than time $t$. (The realization of a typical counting process is shown in Figure 4.1.1.)

The idea of a counting process is formalized in the following definition.

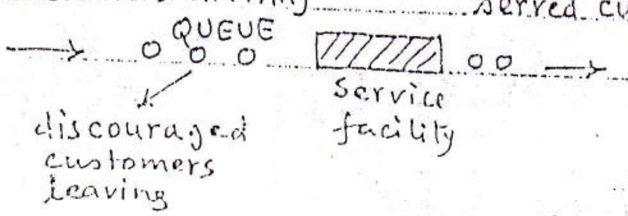
**Definition 4.1.1** $\{N(t), t \geq 0\}$ constitutes a *counting process* provided that

CSE-08 (4A) Queueing Systems



Types –

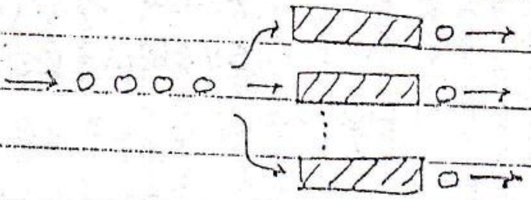
a) customers arriving ........ served customers leaving.




single server queueing system

e.g., one printer

b)



multi-server queueing system

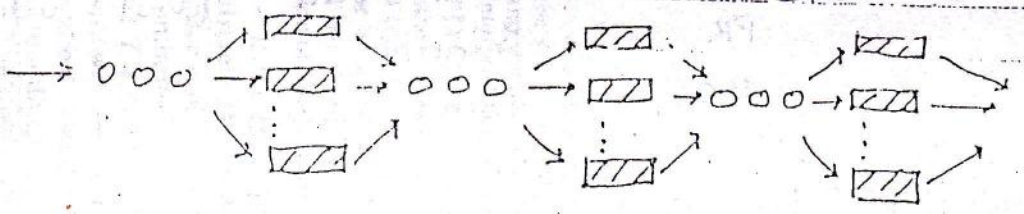
e.g. Internet access through several servers

c)

Single server multi-stage queueing system
(Series Q. System)

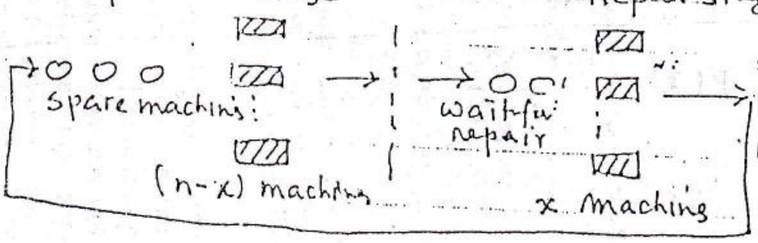
e.g., In a company some one connecting mouse, some one speaker etc.

d)



series queue : multi-server - multi-stage
(Series - Parallel Q. System)

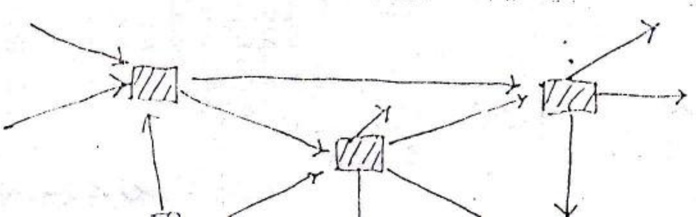
e)




e.g. There are three system analyst in a lab, when a machine needs to repair, then it will send to one of the ser

cyclic system (machine repair model)

f)



## Queueing Systems

Notation for describing a queuing system: ②

Kendall's notation

$$A / B / X / Y / Z$$

A : inter-arrival time distribution
B : distribution of service time
X : number of parallel server
Y : restriction on system capacity
Z : queue discipline

types: (Symbols)

A : M – Exponential, D – Deterministic.
$E_K$ – Erlangian ($K = 1, 2, \ldots$), G – General,

B : —do— GI – General Independ

X : 1, 2, ...

Y : 1, 2, ... ∞.

Z : FCFS First come, first served
LCFS Last come, first served
PR Priority
RSS Random selection for service

Probability distribution: examples:

| Name | probability function P(x) : discrete f(x) : continuous | | Parameter | Mean E(X) | Variance $E\{X-E(X)\}^2$ |
|---|---|---|---|---|---|
| poisson | $P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ | $(x = 0, 1, 2 \ldots)$ | $\lambda > 0$ | $\lambda$ | $\lambda$ |
| Exponential | $f(x) = \theta e^{-\theta x}$ | $(x > 0)$ | $\theta > 0$ | $\frac{1}{\theta}$ | $\frac{1}{\theta^2}$ |
| Erlang-k | $f(x) = \frac{(\theta k)^k}{(k-1)!} x^{k-1} e^{-k\theta x}$ | $x > 0$ | $\theta > 0$, $k = 1, 2, \ldots$ | $\frac{1}{\theta}$ | $\frac{1}{k\theta^2}$ |
| Normal | $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ | $-\infty < x < \infty$ | $\mu$ | $\mu$ | $\sigma^2$ |