(2) 09040 61

(d) Birth-Death process

It is a Special class of Markov chains. These may be either discrete or continuous time processes in which the defining condition is that state transitions take place between neighbouring states only.

Birth death process requires that if $X_n = i$, then $X_{n+1} = \underset{(death)}{i-1}, \underset{no\ change}{i}$, or $\underset{(birth)}{i+1}$ & no oth.

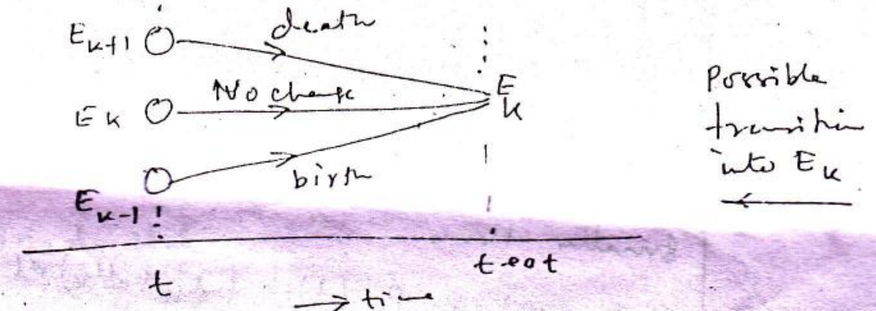
(e) ~ (g) Semi-Markov Process, Random walks & Renewal Process. (to be discussed later)

Birth-Death process: →

Let $P_k(t) \triangleq P[X(t) = k]$

Probability that the population size is k at some time t

* Consider time interval $(t, t+\Delta t)$



We will find in state E_k at $t+\Delta t$ if one of the 3 following (mutually exclusive) eventualities occurred.

1. We had k in population at t & no change occurred
2. We had k-1 at t & one birth during t & t+Δt
3. We had k+1 at t & one death during t, t+Δt

We need not concern ourselves specifically with transitions from states other than nearest neighbours to state E_k since we have assumed that such transitions in an interval of Δt are of the order of $o(\Delta t)$

Note: The Probability that, in the time interval from t to t+Δt, more than one transition

$o(\Delta t)$ denotes any function that goes to zero with Δt faster than Δt itself

i.e. $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$

Thus we may write:

$$P_k(t+\Delta t) = P_k(t)\,[1-\underbrace{\lambda_k \Delta t + o(\Delta t)}_{\text{no birth}}]\,[1-\underbrace{\mu_k \Delta t + o(\Delta t)}_{\text{no death}}]$$
$$+ P_{k-1}(t)\,[\underbrace{\lambda_{k-1}\Delta t}_{1\text{ birth}} + o(\Delta t)]$$
$$+ P_{k+1}(t)\,[\underbrace{\mu_{k+1}}_{1\text{ death}} \quad k \geq 1 \qquad -(i)$$
$$+ o(\Delta t)$$

Total Probability $= 1.0$
Probability of birth $= \lambda_k \Delta T$
$\therefore$ Probability of no birth $= 1-\lambda_k \Delta T$

$$P_0(t+\Delta t) = P_0(t)\,[1-\lambda_0 \Delta t + o(\Delta t)]$$
$$+ P_1(t)\,[\underbrace{\mu_1 \Delta t}_{1\text{ death}} + o(\Delta t)] + o(\Delta t) \qquad (ii)\quad k=0$$

(no death is possible when population size is zero)

from (i) & (ii) by expanding & neglecting smaller terms:

$$P_k(t+\Delta t) = P_k(t) - (\lambda_k + \mu_k)\Delta t\, P_k(t)$$
$$+ \lambda_{k-1}\Delta t\, P_{k-1}(t) + \mu_{k+1}\Delta t\, P_{k+1}(t)$$
$$+ o\,\Delta(t) \quad \text{------} \quad k \geq 1. \quad (iii)$$

$$P_0(t+\Delta t) = P_0(t) - \lambda_0 \Delta t\, P_0(t) + \mu_1 \Delta t\, P_1(t) + o\,\Delta(t);\; k=0 \qquad (iv)$$

we get from (iii) & (iv)

$$\frac{P_k(t+\Delta t) - P_k(t)}{\Delta t} = -(\lambda_k + \mu_k)\,P_k(t) + \lambda_{k-1}P_{k-1}(t) + \mu_{k+1}P_{k+1}(t) + \frac{o(\Delta t)}{\Delta t} \quad k \geq 1$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda_0 P_0(t) + \mu_1 P_1(t) + \frac{o(\Delta t)}{\Delta t} \quad k=0$$

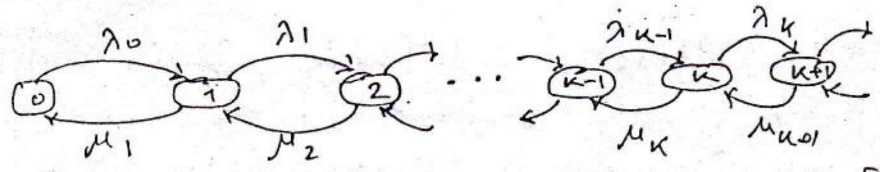
Taking limit as $\Delta t \to 0$ and also $\frac{o(\Delta t)}{\Delta t} \to 0$ we get finally

$$\frac{dP_k(t)}{dt} = -(\lambda_k + \mu_k)\,P_k(t) + \lambda_{k-1}P_{k-1}(t) + \mu_{k+1}P_{k+1}(t) \qquad (1)$$

and $\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu_1 P_1(t)$ (2)

This set of differential-difference equations are called Chapman-Kolmogorov equations.

State transition rate diagram of the birth-death process.



Flow rate into E_K (state) $= \lambda_{K-1} P_{K-1}(t) + \mu_{K+1} P_{K+1}(t)$

Flow rate out of $E_K = (\lambda_K + \mu_K) P_K(t)$

Clearly the difference between these two is the effective probability flow rate into this state that is shown in eqn (1)

at steady state

$0 = -(\lambda_K + \mu_K) P_K + \mu_{K+1} P_{K+1} + \lambda_{K-1} P_{K-1}$ (3) $K \geq 1$

$0 = -\lambda_0 P_0 + \mu_1 P_1$ $K = 0$ (4)

Pure birth process   Let $\lambda_K = \lambda$, $K = 1, 2, 3 \ldots$

from (1) $\frac{dP_K(t)}{dt} = -\lambda P_K(t) + \lambda P_{K-1}(t)$ $K \geq 1$ —(x)

$\frac{dP_0(t)}{dt} = -\lambda P_0(t)$ $K = 0$ —(y)

we assume system begins at time 0 with 0 members

$P_K(0) = \begin{cases} 1 & K = 0 \\ 0 & K \neq 0 \end{cases}$

Solving for $P_0(t) = e^{-\lambda t}$ —(z)

Inserting into (x) for $K = 1$ $\rightarrow$ soln $P_1(t) = \lambda t e^{-\lambda t}$

$\frac{dP_1(t)}{dt} = -\lambda P_1(t) + \lambda e^{-\lambda t}$

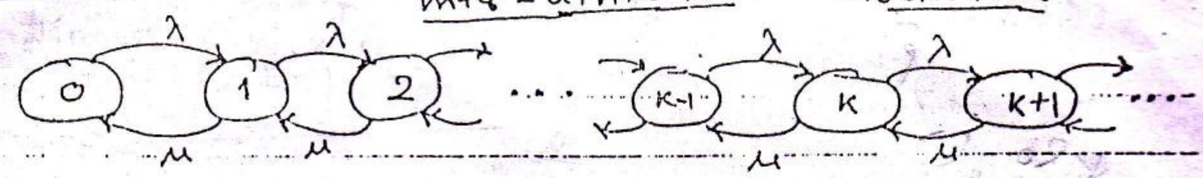
Continuing by induction, we finally get the solution of eqn (x) as

$$P_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad k \geq 0,\ t \geq 0$$

This is Poisson distribution. It is a pure birth process with constant birth rate λ and gives rise to a sequence of birth epochs which are said to constitute a "Poisson process".

A/ M/M/1 (∞) Queue : $\rightarrow a(t) = \lambda e^{-\lambda t}$, $b(t) = \mu e^{-\mu t}$ (3)

inter-arrival time, service time



state transition-rate diagram for M/M/1 (∞)

$P_k(t)$ = Probability {population is at size k at time t}

(1) ... $\dfrac{d\,p_k(t)}{dt} = -(\lambda_k + \mu_k)\,p_k(t) + \mu_{k+1}\,p_{k+1}(t) + \lambda_{k-1}(t)\,p_{k-1}(t)$ $\quad k \geq 1$

(2) ... $\dfrac{d\,p_0(t)}{dt} = -\lambda_0\, p_0(t) + \mu_1\, p_1(t)$

: Kolmogorov equations

at steady state

(3) .... $0 = -(\lambda_k + \mu_k)\,p_k + \mu_{k+1}\,p_{k+1} + \lambda_{k-1}\,p_{k-1}$ $\quad (k \geq 1)$

(4) ..... $0 = -\lambda_0 p_0 + \mu_1 p_1$ $\quad k = 0$

we consider the special case where $\lambda_k = \lambda$ & $\mu_k = \mu$
$k = 1, 2, 3, \cdots$ for μ and $k = 0, 1, 2, \cdots$ for λ we get

(5) ... $0 = -(\lambda + \mu)\,p_k + \mu\, p_{k+1} + \lambda\, p_{k-1}$ $\quad (k \geq 1)$

(6) .... $0 = -\lambda\, p_0 + \mu\, p_1$

from (5) & (6)

(7) ... $\begin{cases} p_{k+1} = \dfrac{\lambda+\mu}{\mu}\, p_k - \dfrac{\lambda}{\mu}\, p_{k-1} & (k \geq 1) \\ p_1 = \dfrac{\lambda}{\mu}\, p_0 \end{cases}$

$p_2 = \dfrac{\lambda+\mu}{\mu}\, p_1 - \dfrac{\lambda}{\mu}\, p_0 = \left(\dfrac{\lambda+\mu}{\mu}\right)\dfrac{\lambda}{\mu}\, p_0 - \dfrac{\lambda}{\mu}\, p_0$

$= \left(\dfrac{\lambda}{\mu}\right)^2 p_0$

(8) ... $p_3 = \left(\dfrac{\lambda}{\mu}\right)^3 p_0$

$\vdots \quad p_k = \left(\dfrac{\lambda}{\mu}\right)^k p_0 \qquad k \geq 0$

Boundary condition:

$$\sum_{k=0}^{\infty} P_k = 1 , \quad P_k \text{ is probability distribution of finding } k \text{ in system}$$

$$\therefore \quad 1 = \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k P_0$$

Let $\frac{\lambda}{\mu} = \rho$; ρ is called utilization factor also called traffic intensity

(9) .... $\therefore \; P_0 = \dfrac{1}{\sum_{k=0}^{\infty} \rho^k}$

$\sum_{k=0}^{\infty} \rho^k = 1 + \rho + \rho^2 + \dots$ is a geometric series

& it converges if & only if $|\rho| < 1$ $\therefore$ $\frac{\lambda}{\mu} = \rho$ must be less than 1 or λ less than μ. ie if $\lambda > \mu$, then queue will go on increasing indefinitely.

We know $\quad \sum_{k=0}^{\infty} \rho^k = \dfrac{1}{1-\rho} \quad (\rho < 1)$

$\therefore$ from (9) we get

(10) .... $\quad P_0 = 1 - \rho$

$$\therefore \quad P_k = \rho^k P_0 = \rho^k (1-\rho) , \quad \left(\rho = \frac{\lambda}{\mu} < 1\right)$$

Average number of customers in system: N

(11) ... $\quad N = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k(1-\rho)\rho^k$

$$= (1-\rho) \sum_{k=0}^{\infty} k\rho^k \quad \text{— (i)}$$

Let us consider $\quad \sum_{k=0}^{\infty} k\rho^k = \rho + 2\rho^2 + 3\rho^3 + \dots$

$$= \rho(1 + 2\rho + 3\rho^2 + \dots)$$

(12) ..... $\quad = \rho \sum_{k=1}^{\infty} k\rho^{k-1} \quad \text{— (ii)}$

$N = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k\rho^k P_0$ & $\sum_{k=0}^{\infty} k\rho^k(1-\rho)$

we can see $\sum_{k=1}^{\infty} k\rho^{k-1}$ is simply ~~that the~~ (6)

derivative of $\sum_{k=0}^{\infty} \rho^k$ with respect to ρ.

since $\sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho}$ [geometric series for $\rho < 1$

we get

$$\sum_{k=1}^{\infty} k\rho^{k-1} = \frac{d}{d\rho}\left[\frac{1}{1-\rho}\right] = \frac{1}{(1-\rho)^2}$$

Therefore from (11) & (12)

$$N = (1-\rho)\sum_{k=0}^{\infty} k\rho^k = (1-\rho)\,\rho\sum_{k=1}^{\infty} k\rho^{k-1}$$

(13) .... $$= (1-\rho)\,\rho\,\frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho}$$

$$\text{or} \quad N = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}$$

Average number of customers in queue

$$N_q = 0.P_0 + \sum_{k=1}^{\infty}(k-1)P_k$$

$$= \sum_{k=1}^{\infty} kP_k - \sum_{k=1}^{\infty} P_k$$

$\left[\sum_{k=0}^{\infty} P_k = 1 \Rightarrow P_0 + \sum_{k=1}^{\infty} P_k = 1 \right.$, $\sum_{k=1}^{\infty} P_k = 1 - P_0$

$$= N - (1-P_0)$$

using (10) & (13)

(14) ... $$= \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho} = \frac{(\lambda/\mu)^2}{1-\lambda/\mu} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

Waiting time in queue:

On an average, an arriving customer can expect to find N customers in the system. So, the mean time that a customer waits in queue

(15) ... $$W_q = \frac{N}{\mu} = \frac{\lambda}{(\mu-\lambda)\mu}$$ $\rho = \frac{\lambda}{\mu}$

using (14) $$W_q = \frac{N_q}{\lambda}$$ Little's formula

$N_q = \frac{\rho^2}{1-\rho}$

$$W_q = \frac{N}{\mu} = \frac{\rho}{\mu(1-\rho)} = \frac{N_q}{\mu\rho} = \frac{N_q}{\lambda}$$

Mean time spent in System: ω_s

waiting in queue + mean service time

$$\omega_s = \omega_q + \frac{1}{\mu} = \frac{\lambda}{(\mu-\lambda)\mu} + \frac{1}{\mu}$$

(16) . . . $$= \frac{N_q}{\lambda} + \frac{1}{\mu} = \frac{\rho^2}{\lambda(1-\rho)} + \frac{\rho}{\lambda} = \frac{\rho^2+\rho-\rho^2}{\lambda(1-\rho)} = \frac{\rho}{\lambda(1-\rho)} = \frac{N}{\lambda}$$

$$= \frac{1}{\mu-\lambda}$$

from (13)

$$N = \frac{\lambda}{\mu-\lambda}$$

$$\therefore \quad N = \lambda\cdot\omega_s$$

$W_s = \frac{N}{\lambda}$

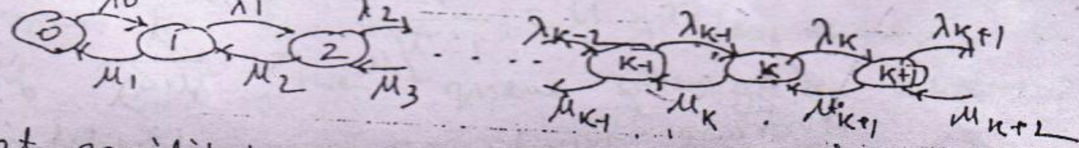
Little's formula

$N = \lambda\cdot W_s$

(B) M/M/S (∞) Queue

General state transition diagram:

(state dependent)



at equilibrium:

at 0 $\quad \lambda_0 P_0 = \mu_1 P_1 \rightarrow P_1 = \frac{\lambda_0}{\mu_1} P_0$

at state K $\quad \lambda_{K-1} P_{K-1} + \mu_{K+1} P_{K+1} = \lambda_K P_K + \mu_K P_K$

or $\quad P_{K+1} = \frac{\lambda_K + \mu_K}{\mu_{K+1}} P_K - \frac{\lambda_{K-1}}{\mu_{K+1}} P_{K-1} \; ; K \geq 1$

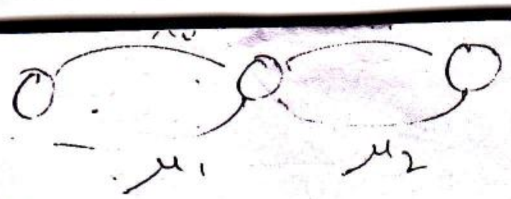
K = 1: $$P_2 = \frac{\lambda_1 + \mu_1}{\mu_2} P_1 - \frac{\lambda_0}{\mu_2} P_0$$

$$= \frac{\lambda_1+\mu_1}{\mu_2} \cdot \frac{\lambda_0}{\mu_1} P_0 - \frac{\lambda_0}{\mu_2} P_0$$

$$= \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0$$

K = 2: $$P_3 = \frac{\lambda_2 + \mu_2}{\mu_3} P_2 - \frac{\lambda_1}{\mu_3} P_1$$

Substitute P_2, P_1, we get $P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0$



(12)

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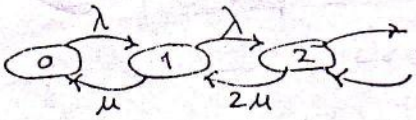
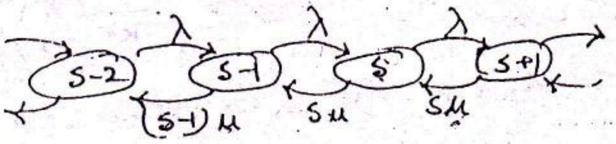
General entry

$$p_K = \frac{\lambda_{K-1}\lambda_{K-2}\cdots\lambda_0}{\mu_K\,\mu_{K-1}\cdots\mu_1}\,p_0 = p_0\prod_{i=1}^{K}\frac{\lambda_{i-1}}{\mu_i} \quad \text{--- (17)}$$

State transition rate diagram for $M|M|S(\infty)$





$\lambda_K = \lambda \qquad K = 0, 1, 2, \cdots$

$\mu_K = \min(K\mu, S\mu)$

$$= \begin{cases} K\mu, & 0 \le K \le S \\ S\mu & K \ge S \end{cases}$$

using (17) we get

$$p_K = \frac{\lambda\cdot\lambda\cdots\lambda}{\mu\cdot 2\mu\cdot 3\mu\cdots K\mu}\,p_0 = \frac{\lambda^K}{K!\,\mu^K}\,p_0$$

$$p_K = \begin{cases} \dfrac{\lambda^K}{K!\,\mu^K}\,p_0 & (1 \le K \le S) \\[2ex] \dfrac{\lambda^K}{S^{K-S}\,S!\,\mu^K}\,p_0 & (K \ge S) \end{cases} \quad \text{--- (18)}$$

Boundary condition

$$\sum_{K=0}^{\infty} p_K = 1$$

using (18)

$$p_0\left[\sum_{K=0}^{S-1}\frac{\lambda^K}{K!\,\mu^K} + \sum_{K=S}^{\infty}\frac{\lambda^K}{S^{K-S}\,S!\,\mu^K}\right] = 1$$

Let $\frac{\lambda}{\mu} = r$ and $r/S = \lambda/S\mu$, we get

$$p_0\left[\sum_{K=0}^{S-1}\frac{r^K}{K!} + \sum_{K=S}^{\infty}\frac{r^K}{S^{K-S}\,S!}\right] = 1 \quad \text{--- (19)}$$

$$\sum_{K=S}^{\infty} \frac{r^K}{S^{K-S}\, S!} = \frac{r^S}{S!} \sum_{K=S}^{\infty} \left(\frac{r}{s}\right)^{K-S}$$

$$= \frac{r^S}{S!} \sum_{m=0}^{\infty} \left(\frac{r}{s}\right)^m \qquad (K-S=m)$$

$$= \frac{r^S}{S!} \left[\frac{1}{1-r/s}\right] \qquad \{r/s = \rho < 1\}$$

$\frac{\lambda}{\mu} = r$ $\qquad \frac{r^S \cdot r^{K-S}}{S^{K-S}\, S!}$

∴ (19) becomes

$$p_0 = \left[\sum_{K=0}^{S-1} \frac{r^K}{K!} + \frac{S r^S}{S!\,(S-r)}\right]^{-1}, \qquad r = \lambda/\mu$$

$$= \left[\sum_{K=0}^{S-1} \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \left(\frac{S\mu}{S\mu-\lambda}\right)\right]^{-1} \quad \text{— (20)}$$

condition for existence of a steady state solution is $\lambda/S\mu < 1$ ie, mean arrival rate must be less than mean maximum potential service rate.

<u>expected queue size : N_q</u>

$$N_q = \sum_{K=S}^{\infty} (K-S)\, p_K$$

$$= \sum_{K=S}^{\infty} (K-S) \frac{r^K}{S^{K-S}\, S!}\, p_0 \qquad \text{(using (18))}$$

$$= \frac{r^S p_0}{S!} \sum_{K=S}^{\infty} (K-S)\, \rho^{K-S}$$

$r/S = \rho$

$$= \frac{r^S p_0}{S!} \sum_{m=1}^{\infty} m \rho^m \qquad \left[\sum_{m=0}^{\infty} \rho^m = \frac{1}{1-\rho}\right]$$

$$= \frac{r^S}{S!}\, \rho\, p_0 \sum_{m=1}^{\infty} m \rho^{m-1} \qquad = \frac{r^S}{S!}\, \rho\, p_0 \sum_{m=1}^{\infty} m\rho^{m-1}$$

$$= \frac{r^S}{S!}\, \rho\, p_0 \frac{d}{d\rho}\left[\sum_{m=1}^{\infty} \rho^m\right] \qquad = \frac{r^S}{S!}\, \rho\, p_0 \frac{1}{(1-\rho)^2}$$

$$= \frac{r^S}{S!}\, \rho\, p_0 \frac{d}{d\rho}\left[\frac{\rho}{1-\rho}\right] = \frac{r^{S+1}/S}{S!\,(1-r/S)^2}\, p_0$$

$$\sum_{m=1}^{\infty} \rho^m = \sum_{m=0}^{\infty} \rho^m - 1 = \frac{1}{1-\rho} - 1 = \frac{\rho}{1-\rho}$$

$$\therefore N_q = \left[\frac{\rho^{s+1}/s}{s!\,(1-r/s)^2}\right] p_0$$

$$= \left[\frac{(\lambda/\mu)^s \lambda\mu}{(s-1)!\,(s\mu-\lambda)^2}\right] p_0 \quad \text{--- (21)}$$

We can calculate N, mean number of customers in system by

$$N = \sum_{k=0}^{\infty} k p_k$$

$W_s = W_q + \frac{1}{\mu}$

But it will be easier to use Little's formula to find W_q and then calculate $W_s = W_q + \frac{1}{\mu}$ & finally to use Little's formula again to find N.

$\therefore$ mean waiting time in queue:

$$W_q = \frac{N_q}{\lambda} = \left[\frac{(\lambda/\mu)^s \mu}{(s-1)!\,(s\mu-\lambda)^2}\right] p_0 .$$

mean waiting time in system:

$$W_s = \frac{1}{\mu} + W_q = \frac{1}{\mu} + \left[\frac{(\lambda/\mu)^s \mu}{(s-1)!\,(s\mu-\lambda)^2}\right] p_0$$

and $N = \lambda W_s$

$$= \frac{\lambda}{\mu} + \left[\frac{(\lambda/\mu)^s \lambda\mu}{(s-1)!\,(s\mu-\lambda)^2}\right] p_0$$

$$= r + \left[\frac{r^{s+1}/s}{s!\,(1-r/s)^2}\right] p_0$$

$N_q = \frac{(\lambda/\mu)^s \lambda\mu}{(s-1)!\,(s\mu-\lambda)^2} P_0$