CSE 3207 Mathematical Analysis for Computer Science

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Lesson Objectives

- Review of probability, random variable and its probability distribution

What is Probability?

- Science of Uncertainty

Probability Axioms

1. **(Nonnegativity)** $\mathbf{P}(A) \geq 0$, for every event A.
2. **(Additivity)** If A and B are two disjoint events, then the probability of their union satisfies

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B).$$

   More generally, if the sample space has an infinite number of elements and $A_1, A_2, \ldots$ is a sequence of disjoint events, then the probability of their union satisfies

$$\mathbf{P}(A_1 \cup A_2 \cup \cdots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots.$$

3. **(Normalization)** The probability of the entire sample space Ω is equal to 1, that is, $\mathbf{P}(\Omega) = 1$.

Conditional Probability

The conditional probability of an event A, given an event B with $\mathbf{P}(B) > 0$, is defined by

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)},$$

Conditional Probability

- Def. The <u>conditional</u> <u>probability</u> of *E* given *F* is the probability that an event, *E*, will occur given that another event, *F*, has occurred

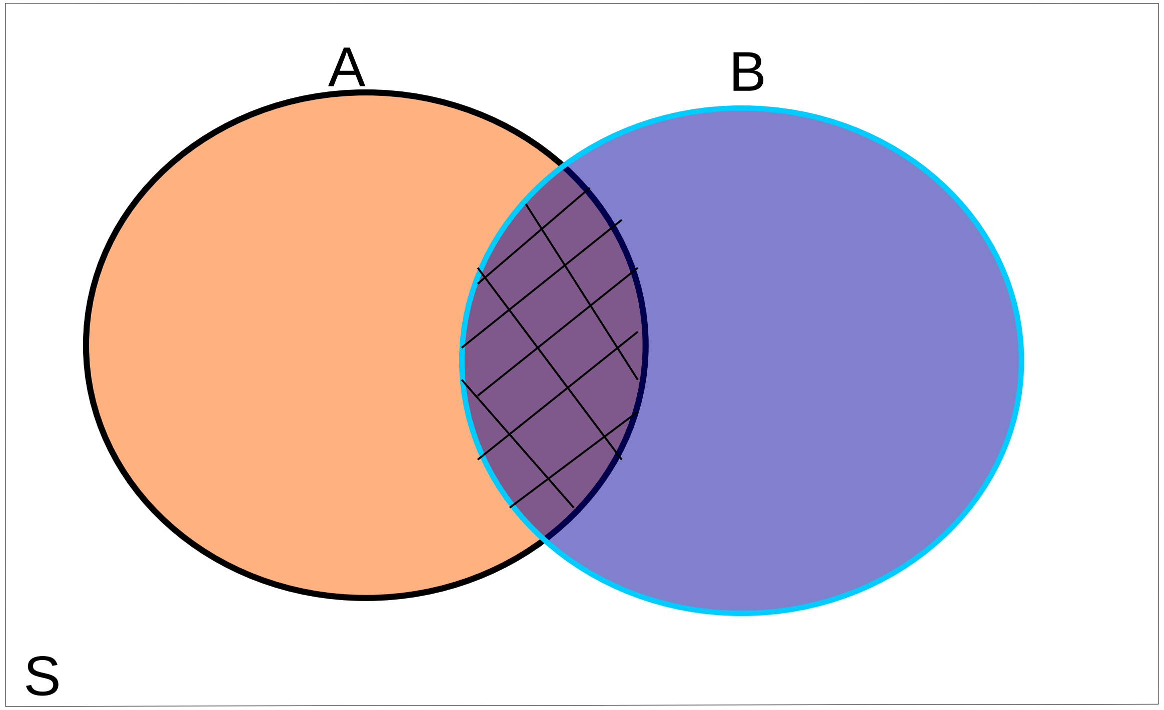
$$P(E|F)=\frac{P(E\cap F)}{P(F)} \quad \text{if} \quad P(F)\neq 0$$

- Conditional Probability can be rewritten as follows

$$P(E\cap F)=P(E|F)*P(F)$$

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Conditional Probabilitys

- Example:

Earned degrees in the United States in recent year

	B	M	P	D	Total
Female	616	194	30	16	856
Male	529	171	44	26	770
Total	1145	365	74	42	1626

$$P(Male \mid B) = \frac{529}{1145} \cong 0.4620$$

$$P(Male) = \frac{770}{1626} \cong 0.4735$$

Conditional Probability

"Probability Of" *"Given"*

$$P(\,A \text{ and } B\,) = P(\,A\,) \times P(\,B \mid A\,)$$

Event A *Event B*

"Probability of **event A and event B** equals
the probability of **event A** times the probability of **event B given event A**"

Conditional Probability

- Example: Ice Cream

70% of your friends like Chocolate, and 35% like Chocolate AND like Strawberry.

What percent of those who like Chocolate also like Strawberry?

P(Strawberry|Chocolate) = P(Chocolate and Strawberry) / P(Chocolate)

=0.35 / 0.7 = 50%

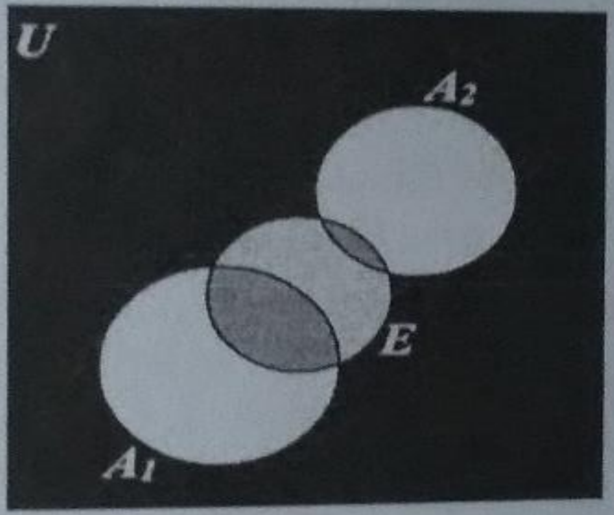
50% of your friends who like Chocolate also like Strawberry

Bayes Theorem

$$P(A_1|E) = \frac{\text{প্রকৃতই অসৎ হলে যতভাবে অ্যালার্ম বাজতে পারে}}{\text{সম্ভাব্য যতভাবে অ্যালার্ম বাজতে পারে}}$$

$$= \frac{\text{প্রকৃতই অসৎ হওয়া এবং অসৎ হওয়ার শর্তে অ্যালার্ম বাজার সম্ভাবনা}}{\text{(প্রকৃতই অসৎ হওয়া এবং অসৎ হওয়ার শর্তে) অথবা (প্রকৃতই সৎ হওয়া এবং সৎ হওয়ার শর্তে) অ্যালার্ম বাজার সম্ভাবনা}}$$

$$= \frac{P(A_1) \times P(E|A_1)}{P(A_1) \times P(E|A_1) + P(A_2) \times P(E|A_2)} \text{ [২.৫ দ্রষ্টব্য]}$$



Bayes Theorem

বেইস উপপাদ্য

$A_1, A_2, A_3, \ldots \ldots \ldots A_n$ যদি পরস্পর বিচ্ছিন্ন (disjoint) কিছু ঘটনা হয় যাদের প্রত্যেকের সাথেই E নামক একটি ঘটনার ছেদ (common অংশ) থাকে তাহলে

$$P(A_1|E) = \frac{P(A_1) \times P(E|A_1)}{P(A_1) \times P(E|A_1) + P(A_2) \times P(E|A_2) + P(A_3) \times P(E|A_3) + \ldots + P(A_n) \times P(E|A_n)}$$

অনুসিদ্ধান্ত

যদি $A_1, A_2, A_3, \ldots \ldots \ldots A_n$ ঘটনাগুলো ঘটার শর্ত ছাড়া অন্য কোনো উপায়ে E ঘটনাটি ঘটতে না পারে, অর্থাৎ যদি $P(E) = P(A_1) \times P(E|A_1) + P(A_2) \times P(E|A_2) + P(A_3) \times P(E|A_3) + \ldots + P(A_n) \times P(E|A_n)$ হয় তাহলে

$$P(A_1|E) = \frac{P(A_1) \times P(E|A_1)}{P(E)}$$

Bayes Theorem Example

In a factory, three machines I, II and III are all producing screws of the same size. Of their production, machines I, II and III produce 5%, 5% and 10% defective screws, respectively. Of the total production of screws, machine I produces 20%, machine II produces 35% and machine III produces 45%. If one screw is selected at random from the total screws produced in a day and it is found to be defective, what is the probability that it was produced by machine III ?

Solution

Let D denote the event that the selected screw is defective. Then

$$P(D) = P(I)\,P(D \mid I) + P(II)\,P(D \mid II) + P(III)\,P(D \mid III)$$

$$= \left(\frac{20}{100}\right)\left(\frac{5}{100}\right) + \left(\frac{35}{100}\right)\left(\frac{5}{100}\right) + \left(\frac{45}{100}\right)\left(\frac{10}{100}\right)$$

$$= \frac{725}{10{,}000}$$

and by Bays formula, we obtain

$$P(III \mid D) = \frac{P(III)\,P(D \mid III)}{P(D)} = \frac{\left(\frac{45}{100}\right)\left(\frac{10}{100}\right)}{\frac{725}{10000}} = \frac{450}{725},$$

where

I = { the screw was produced by machine I }

II = { the screw was produced by machine II }

III = { the screw was produced by machine III }.

Bayes Theorem Homework/Assignment

Suppose that a large number of packets of biscuits are made of two types A and B. Type A packet contains 80% lemon biscuits and 20% glucose biscuits while type B packet contains 20% lemon and 80% glucose biscuits. Further suppose that 70% of all packets are of type A and 30% are of type B. From a packet of unknown type one biscuit is tested and is found to be a lemon biscuit. Compute the probability that the biscuit came from type (i) A packet, (ii) B packet. On the basis of this test, comment on the type of packets from which the biscuit could most possibly have come ?

Assignment :

ক) একটি আলপিন কারখানায় তিনটি ভিন্ন যন্ত্র (জাপানি, জার্মান ও কোরিয়ান) থেকে মোট উৎপাদনের যথাক্রমে 30%, 40% ও 50% উৎপাদিত হয়। এটা জানা আছে যে, যন্ত্র তিনটি হতে উৎপাদিত আলপিনগুলোর যথাক্রমে 5%, 4% ও 3% ত্রুটিপূর্ণ। কিন্তু তিনটি যন্ত্র থেকে উৎপাদিত সব আলপিন একসাথে মিশে যায় বলে কোনটা কোন যন্ত্র থেকে তৈরি সেটা বোঝা মুশকিল। তারপরও কি আমরা একটা ত্রুটিপূর্ণ আলপিন যে কোরিয়ান যন্ত্র থেকে আসেনি, তার সম্ভাব্যতা নির্ণয় করতে পারবো? [সূত্র:২.৪.৩]

খ) একজন ধূমপায়ী রোগীর ঘাড়ের লসিকা গ্রন্থিতে (lymph node) ক্যান্সার ধরা পড়লো। ডাক্তাররা পরীক্ষা করে দেখলেন যে ক্যান্সারটি অন্য কোনো অঙ্গ থেকে সেখানে ছড়িয়েছে। এটা জানা ছিল যে শ্বাসনালি, খাদ্যনালি, জননাঙ্গ কিংবা মুখগহ্বরে ক্যান্সার হলে সেখান থেকে ওই ধরনের ক্যান্সার ঘাড়ের লসিকা গ্রন্থিতে ছড়াতে পারে যথাক্রমে 13%, 10%, 2% এবং 5% ক্ষেত্রে। আবার, ধূমপায়ীদের ওই চারটি অঙ্গে ওই বিশেষ ধরনের ক্যান্সার হওয়ার সম্ভাব্যতার আপেক্ষিক অনুপাত = 10:3:2:8। রোগীটির ক্যান্সারের উৎস খাদ্যনালীতে হওয়ার সম্ভাবনা কত?

Assignment

- Book: probability an introduction by a.k.m sirajul haque. Available in BAUST Library.
- Problems for conditional probability: page 89-90
- Exercise:3.1
- Question no: 3,4,5,6,12.(u may take help from example problem)

Random Variable

- A random variable is a rule/function which assigns a numerical value to each possible outcome of an experiment

Random Variable

- Discrete
  - I,e number of telephone calls in an hour
- Continuous
  - I,e Heights, Weights;
  - Lifetime of a component;
  - cpu time;

Probability Distributions Functions

- For discrete, probability mass function (pmf)
- For Continuous, probability density function (pdf)

Example

1. Toss a fair coin:

   Let $X = 1$ if head occurs, and $X = 0$ if tail occurs. Since the coin is fair, $P(X = 1) = P(X = 0) = 1/2$. The pdf is

Outcome (x)	0	1
Probability ($P(X = x)$)	.5	.5

2. Tossing two fair coins:

   Let X be the number of heads. Then the values of X are 0, 1 or 2. The sample space consists of the following equally likely outcomes. $\{(T, T), (H, T), (T, H)$ and $(H, H)\}$. So $P(X = 0) = 1/4$, $P(X = 1) = 2/4$, and $P(X = 2) = 1/4$. The pdf is

Outcome (x)	0	1	2
Probability ($P(X = x)$)	.25	.5	.25

Cumulative distribution function (CDF)

The CDF is defined as the probability that the random variable X is less than or equal to some specified x

$$F(x) = P(X \leq x)$$

Mean/Expected value of a discrete random variable

- Weighted average of all possible values .The weights are the probabilities of respective values of the random variable .

$$E(X) = \sum_x xp(x)$$

Example

- A quarter of the source programs submitted by a certain programmer compile successfully. Each day the programmer writes five programs. The compiling probabilities are:

No. that compiles	0	1	2	3	4	5
Probability	.237	.396	.264	.088	.014	.001

Calculate the expected number of programs that will compile per day?

$$E(X) = 0\times.237+1\times.396+2\times.264+3\times.088+4\times.014+5\times.001 = 1.25$$

- If X is discrete and takes integer values, the PMF and the CDF can be obtained from each other by summing or differencing:

$$F_X(k) = \sum_{i=-\infty}^{k} p_X(i),$$

$$p_X(k) = \mathbf{P}(X \le k) - \mathbf{P}(X \le k-1) = F_X(k) - F_X(k-1),$$

for all integers k.

- If X is continuous, the PDF and the CDF can be obtained from each other by integration or differentiation:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt, \qquad f_X(x) = \frac{dF_X}{dx}(x).$$

Let X be a continuous random variable with PDF f_X.

- The expectation of X is defined by

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$