CENG 223

Discrete Computational Structures

Fall 2022-2023
Take Home Exam 1

Solution

Question 1 (15 pts)

a) Show that whether the following statement is a tautology or a contradiction by using a **truth table**.

$$(p \land q) \leftrightarrow (\neg p \lor \neg q)$$

(5/15 pts)

p	q	$\neg p$	$\neg q$	$p \land q$	$\neg p \lor \neg q$	$(p \land q) \leftrightarrow (\neg p \lor \neg q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

It is a contradiction.

b) Show that $p \to ((q \lor \neg q) \to (p \land q))$ and $(\neg p \lor q)$ are logically equivalent. Use tables 6, 7 and 8 given under the section "*Propositional Equivalences*" in the course textbook and give the reference to the table and the law in each step. REMARK: You can use $\neg T \equiv F$ directly.

(10/15 pts)

$$
\begin{aligned}
p \to ((q \lor \neg q) \to (p \land q)) &\equiv p \to (T \to (p \land q)) && \textit{Negation laws} \\
&\equiv p \to (\neg T \lor (p \land q) && \textit{Table 7, line 2} \\
&\equiv p \to (F \lor (p \land q) && \textit{Remark} \\
&\equiv p \to (p \land q) && \textit{Identity laws} \\
&\equiv \neg p \lor (p \land q) && \textit{Table 7, line 2} \\
&\equiv (\neg p \lor p) \land (\neg p \lor q) && \textit{Distributive laws} \\
&\equiv T \land (\neg p \lor q) && \textit{Negation laws} \\
&\equiv \neg p \lor q && \textit{Identity laws}
\end{aligned}
$$

Question 2 (30 pts)

Assume the following:

$W(x, y)$: x works on project y.
$S(x, y)$: x supervises project y.
$A(x, y)$: x is the advisor of y.
$F(x, y)$: x funds project y.

Translate the following sentences into predicate logic using $\vee, \wedge, \rightarrow, \neg, \forall, \exists$. You are not allowed to use any other predicate symbols. **f** and **g** are 5 points, while all others are 4 points each. (Note: You can use constants to denote individuals or specific projects like Ali or P).

a) Every student works on a project.

b) Not all projects are funded.

c) Instructor Ali is the advisor of all the students that work on project P.

d) Büşra works on a project that is funded by TUBITAK.

e) Some supervisors supervise more than one project.

f) No two students work on the same project.

g) Exactly two students work on some project.

a)

$$\forall x \exists y (W(x, y))$$

b)

$$\neg(\forall x \exists y (F(y, x)))$$

c)

$$\forall x (W(x, P) \rightarrow A(Ali, x))$$

d)

$$\exists x (F(TUBITAK, x) \wedge (W(Busra, x)))$$

e)

$$\exists x \exists y \exists z (S(x, y) \wedge S(x, z) \wedge y \neq z)$$

f)

$$\forall x \forall y \forall z (W(x, z) \wedge W(y, z) \rightarrow x = y)$$

g)

$$\exists x \exists y \exists z ((W(x, z) \wedge W(y, z) \wedge (x \neq z)) \wedge \forall t (W(t, z)) \rightarrow (t = x \vee t = y))$$

Question 3 (15 pts)

Prove the following claims by natural deduction. Use only the natural deduction rules $\vee$, $\wedge$, $\rightarrow$, $\neg$, introduction and elimination. If you attempt to make use of a lemma or equivalence, you need to prove it by natural deduction too.

$$p \rightarrow q, (q \wedge \neg r) \rightarrow s, \neg s \vdash p \rightarrow r$$

1.	$p \rightarrow q$	*premise*
2.	$(q \wedge \neg r) \rightarrow s$	*premise*
3.	$\neg s$	*premise*
4.	$(q \wedge \neg r)$	*assumption*
5.	s	$\rightarrow e$ $2, 4$
6.	$\perp$	$\neg e$ $3, 5$
7.	$\neg(q \wedge \neg r)$	$\neg i$ $4, 6$
8.	p	*assumption*
9.	q	$\rightarrow e$ $1, 8$
10.	$\neg r$	*assumption*
11.	$q \wedge \neg r$	$\wedge i$ $9, 10$
12.	$\perp$	$\neg e$ $7, 11$
13.	r	$\neg i$ $10, 12$
14.	$p \rightarrow r$	$\rightarrow i$ 8-13

Question 4 (20 pts)

Ekin has four children: Ayşe, Barış, Can and Duygu. They make the following claims:

Ayşe: We went to park.
Barış: If we played hide and seek, then we did not eat candy.
Can: If we went to park, then we both ate candy and played games.
Duygu: If we played games, then we played hide and seek.

Ekin knows that three of them are telling truth but Barış is lying. Help her to prove this by using natural deduction. Use the following clauses;

p: We went to park.
q: We ate candy.
r: We played games.
s: We played hide and seek.

Barış's comment is $s \to \neg q$ which is equivalent to $\neg s \vee \neg q \equiv \neg(s \wedge q)$ as we know from De Morgan's laws. Then the question becomes;

$$p, p \to (q \wedge r), r \to s \vdash \neg\neg(s \wedge q)$$

1.	p	*premise*
2.	$p \to (q \wedge r)$	*premise*
3.	$r \to s$	*premise*
4.	$q \wedge r$	$\to e$ $1, 2$
5.	q	$\wedge e_1$ 4
6.	r	$\wedge e_2$ 4
7.	s	$\to e$ $3, 6$
8.	$s \wedge q$	$\wedge i$ $5, 7$

Question 5 (20 pts)

Prove the following claim by natural deduction. Use only the natural deduction rules $\vee$, $\wedge$, $\to$, $\neg$ introduction and elimination. If you attempt to make use of a lemma or equivalence, you need to prove it by natural deduction too.

$$\forall x(P(x) \to (Q(x) \to R(x))), \exists x(P(x)), \forall x(\neg R(x)) \vdash \exists x(\neg Q(x))$$

1.	$\forall x(P(x) \to (Q(x) \to R(x)))$	*premise*
2.	$\exists x(P(x))$	*premise*
3.	$\forall x(\neg R(x))$	*premise*
4.	$x_0 P(x_0)$	*assumption*
5.	$P(x_0) \to (Q(x_0) \to R(x_0))$	$\forall x$ e 1
6.	$Q(x_0) \to R(x_0)$	$\to$ e $4, 5$
7.	$Q(x_0)$	*assumption*
8.	$R(x_0)$	$\to$ e $6, 7$
9.	$\neg R(x_0)$	$\forall x$ e 3
10.	$\perp$	$\neg$ e $8, 9$
11.	$\neg Q(x_0)$	$\neg i$ $7 - 10$
12.	$\exists x(\neg Q(x))$	$\exists i$ 11
13.	$\exists x(\neg Q(x))$	$\exists x$ e $2, 4 - 12$