Chapter 1

Calculas on Manifolds

1.1 Some proofs

Theorem 1. *Let $O \subseteq \mathbb{R}^{n+1}$ be open and $x \in \mathbb{R}$. Let's define O' as follows:*

$$O' = \{(y_1, \ldots, y_n) : (x, y_1, \ldots, y_n) \in O\}$$

Then $O' \subseteq \mathbb{R}^n$ is open.

Proof. Let $(y_1, \ldots, y_n) \in O'$ be arbitrary. By definition, $(x, y_1, \ldots, y_n) \in O$. Since O is open there is an open rectangle $U \subseteq O$. Corrsponding open rectangle U' also contains our y. Now suppose $z \in U'$ is arbitrary. Then $(x, z_1, \ldots, z_n) \in U \subseteq O$. Therefore, $z \in O'$. Hence, O' is open. $\square$

Theorem 2. *Let $O \in \mathbb{R}^{m+n}$ be open and $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$. Define*

$$O' = \{(y_1, \ldots, y_n) : (x_1, \ldots, x_m, y_1, \ldots, y_n) \in O\}$$

Then O' is also open in $\mathbb{R}^n$.

Proof. Previous proof works with some slight changes. $\square$

Theorem 3. *If $B \subseteq \mathbb{R}^n$ is compact and $x \in \mathbb{R}$, then $\{x\} \times B \subseteq \mathbb{R}^{n+1}$ is also compact.*

Proof. Suppose O is an open cover for $\{x\} \times B \subseteq \mathbb{R}^{n+1}$. Then we can construct an open cover O' by having a function $f : U \to U'$ by dropping the first coordinate. Since B is compact, there is a finite list of open sets $U_1', \ldots, U_2'$ that covers B. From this finite set we can find a finite cover $U_1, \ldots, U_n$ by going back pre-image by pre-image so that we contain α. Hence $\{x\} \times B \in \mathbb{R}^{n+1}$ is compact. $\square$

Theorem 4. *If $B \in \mathbb{R}^m$ is compact and $x \in \mathbb{R}^n$, then $\{x\} \times B \in \mathbb{R}^{n+m}$ is also compact.*

Proof. Similar as above. $\square$

Theorem 5. *If B is compact and O is an open cover of $\{x\} \times B \in \mathbb{R}^{n+m}$, then there is an open set $U \in \mathbb{R}^n$ containing x such that $U \times B$ is covered by a finite number of sets in O.*

Theorem 6. *If $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are compact, then $A \times B \subseteq \mathbb{R}^{m+n}$ is compact.*

Theorem 7. *$A_1 \times \cdots \times A_k$ is compact if each A_i is. In particular, a closed rectangle in $\mathbb{R}^k$ is compact.*

Theorem 8. *A closed bounded subset of $\mathbb{R}^n$ is compact. (The converse is also true (Problem 1-20)).*

Theorem 9. *Prove the converse of Corollary 1-7: A compcat subset of $\mathbb{R}^n$ is closed and bounded (see also Problem 1-28).*

Proof. Let $B \subseteq \mathbb{R}^n$ be compact. Boundededness of B is trivial. We only have to show that B is closed, we will show that $\mathbb{R}^n \setminus B$ is open. Let $p \in \mathbb{R}^n \setminus B$ be arbitrary. For each $q \in B$, we consider the open set $W_q = B_r(q, \frac{|p-q|}{2})$. These consiststs of an open cover O of B. By compactness, $W_{q_1} \ldots W_{q_m}$ covers B. Choose q_i such that $|p - q_i|$ is smallest. Then the neighbourhood $B_(p) \subseteq \mathbb{R}^n \setminus B$. Hence, $\mathbb{R}^n \setminus B$ is open. ☐

Exercise. *1-21(a) If A is closed and $x \notin A$, prove that there is a number $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.*

Proof. Let $x \notin A$. Since $\mathbb{R}^n \setminus A$ is open, there exists a neighborhood $B(x) \subseteq \mathbb{R}^n \setminus A$. Take d as the radius of this neighborhood $B(x)$. ☐

Exercise. *1-21(b) If A is closed, B is compact, and $A \cap B = \emptyset$, prove that there is $d > 0$ such that $|y - x| \geq d$ for all $y \in A$ and $x \in B$. Hint: For each $b \in B$ find an open set U containing b such that this relation holds for $x \in U \cap B$.*

Proof. Since A and B are disjoint, by 1-21(a) for each $b \in B$ we can find $d > 0$ such that $|y - b| \geq d$ for all $y \in A$. Consider the open cover $O = \{B_{\frac{d}{2}}(b) : b \in B\}$. Since B is compact, finitely many open set $B_{d_1}, \ldots B_{d_k}$ covers B. By triangale inequality if $x \in B_{d_i}$ then we have $|y - x| \geq d_i$. By picking the minimums of d_i we find our d. ☐

Exercise. *1-21(c) Give a counterexample in $\mathbb{R}^2$ if A and B are closed but neither is compact.*

Proof. Let $A = \mathbb{N}$ and $B = \{n + \frac{1}{2n} | n \in \mathbb{N}\}$. ☐

Exercise. *1-22 If U is open and $C \subseteq U$ is compact, show that there is a compact set D such that $C \subseteq interior D$ and $D \subseteq U$.*

Proof. Notice that U^c is closed and $C \cup U^c = \emptyset$. Therefore, by 1-21 there exists $d > 0$ such that $|y - x| > d$ for all $x \in C$ and $y \in U^c$. Now consider the set

$$E = \bigcup_{x \in C} U_x$$

where U_x is an open neighbourhood with radius $d/2$ and centered at x. Then we have $C \subseteq E \subseteq U$. Moreover since C is compact we can assume that E is an union of finitely many sets U_x. Our choice of d guaruntees that if we take

$$D = \bigcup_{i=1}^{k} \overline{U}_{x_i}$$

then we still have $C \subseteq D \subseteq U$, where each $\overline{U}_{x_i}$ is a closed neighbourhood. ☐

Theorem 10. *If $A \subseteq \mathbb{R}^n$, a function $f : A \to \mathbb{R}^m$ is continuous if and only if for every open set $U \subseteq \mathbb{R}^m$ there is some open set $V \subseteq \mathbb{R}^n$ such that $f^{-1}(U) = V \cap A$*

Proof. We prove the converse. Let $a \in A$ be arbitrary, we show that f is continuos at a. Let $\epsilon > 0$ be arbitrary. Then the neighbourhood $B = B_\epsilon(f(a)) \subseteq \mathbb{R}^m$ is open. Therefore, $f^{-1}(B) = V \cap A$ for some open set $V \subseteq \mathbb{R}^n$. But then, $a \in V$ so there is an open set $B_\delta(a) \subseteq V$ for some $\delta > 0$. Therefore, if $x \in B_\delta(a)$ and $x \in A$, then $f(x) \subseteq B$. Hence f is continuous at a. □

If $A \to \mathbb{R}$ is bounded, the extent to which f fails to be continuous at $a \in A$ can be measured in a precise way. For $\delta > 0$ let

$$M(a, f, \delta) = \sup\{f(x) : x \in A \text{ and } |x - a| < \delta\}$$
$$m(a, f, \delta) = \inf\{f(x) : x \in A \text{ and } |x - a| < \delta\}$$

The oscillation $o(f, a)$ of f at a is defined by $o(f, a) = \lim_{\delta \to 0^+} [M(a, f, \delta) - m(a, f, \delta)]$

Theorem 11. *If f is bounded, then $o(f, a) = \lim_{\delta \to 0^+} [M(a, f, \delta) - m(a, f, \delta)]$ always exists.*

Proof. Consider the sequence $M(a, f, \frac{1}{n}) - m(a, f, \frac{1}{n})$. This is a decreasing sequence which is bounded below by 0. Therefore, there exists a limit $a \geq 0$. It is easy to show

$$a = \lim_{\delta \to 0^+} [M(a, f, \delta) - m(a, f, \delta)]$$

since $1g(x) = M(a, f, x) - m(a, f, x)$ is a non-increasing function on $x > 0$. □

Exercise. *1-23 If $f : A \to \mathbb{R}^m$ and $a \in A$, show that $\lim_{x \to a} f(x) = b$ if and only if $\lim_{x \to a} f^i(x) = b^i$ for $i = 1, \ldots, m$.*

Proof. $\to$ Suppose $\lim_{x \to a} f(x) = b$, then for each $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in A$ and $0 < |x - y| < \delta$, then $|f(x) - b| < \epsilon$. Same choice of δ ensures that $|f^i(x) - b^i| < \epsilon$.

$\leftarrow$ Now suppose $\lim_{x \to a} f^i(x) = b^i$ for $i = 1, \ldots, m$. Let $\epsilon > 0$ be arbitrary. Then we find $\delta_1, \ldots, \delta_m$ where each $\delta_i > 0$ such that $|f^i(x) - b^i| < \epsilon$. Taking minimum of these δ_i as δ we find that $|f(x) - b| \leq \sum |f^i(x) - b^i| \leq n\epsilon$. □

Exercise. *1-24. Prove that $f : A \to \mathbb{R}^m$ is continuous at a if and only if each f^i is.*

Proof. $\to$ Suppose $f : A \to \mathbb{R}^m$ is continuous at a. Then $\lim_{x \to a} f(x) = f(a)$ From 1-23, we see that $\lim_{x \to a} f^i(x) = f^i(a)$. Therefore, each $f^i : A \to \mathbb{R}$ is continuous at $a \in \mathbb{R}^n$.

$\leftarrow$ Suppose each $f^i : A \to \mathbb{R}$ is continuous at $a \in A$. Then we have $\lim_{x \to a} f^i(x) = f^i(a)$. Again, using 1-23 we conclude that $\lim_{x \to a} f(x) = f(a)$. □

Exercise. *1-25 Prove that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is continous. Hint: Use Problem 1-10.*

Proof. We know from 1-10 that there is a $M > 0$ such that $|T(h)| \leq M|h|$ for all $h \in \mathbb{R}^n$. Now take $\delta = \frac{\epsilon}{2M}$. Then, for any $y \in \mathbb{R}^n$ such that $|x - y| < \delta$ we have (for $h = y - x$):

$$\begin{aligned} |T(y) - T(x)| &= |T(x + h) - T(x)| \\ &= |T(x) + T(h) - T(x)| \\ &= |T(h)| \\ &\leq M|h| \\ &< M\frac{\epsilon}{2M} = \epsilon \end{aligned}$$

□

1.2 Differentiation

Problem. *2-1. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then it is continuous at a. Hint: User Problem 1-10.*

Proof. Since f is differentiable at a, it is easy to show the weaker statement

$$\lim_{h \to 0} |f(a+h) - f(a) - L(h)| = 0$$

.
Now,

$$0 \le |f(a+h) - f(a)| \le |f(a+h) - f(a) - L(h)| + |L(h)|$$

Since both of the terms on the right side goes to 0 as $h \to 0$, it shows that f is continuous at a. □

Problem. *2-2. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is independent of the second variable if for each $x \in \mathbb{R}$ we have $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbb{R}$. Show that f is independent of the second variable if and only if there is a funciton $g : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x)$. What is $f'(a, b$ in terms of g'?*

Proof. Suppose f is independent of the second variable. Define

$$g(x) = f(x,y)$$

Then g is well defined.
On the other hand, suppose there exists such function $g : \mathbb{R} \to \mathbb{R}$. Then f is independent of the second variable. □

Problem. *2-12: A function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is bilinear if for $x, x_1, x_2 \in \mathbb{R}^n$, $y, y_1, y_2 \in \mathbb{R}^m$, and $a \in \mathbb{R}$ we have*

$$\begin{aligned} f(ax, y) &= af(x,y) = f(x, ay) \\ f(x_1 + x_2, y) &= f(x_1, y) + f(x_2, y) \\ f(x, y_1 + y_2) &= f(x, y_1) + f(x, y_2) \end{aligned}$$

(a) Prove that if f is bilinear, then

$$\lim_{(h,k) \to 0} = \frac{|f(h,k)|}{|(h,k)|} = 0$$

(b) Prove that $Df(a,b)(x,y) = f(a,y) + f(x,b)$.
(c) Show that the formula for $Dp(a,b)$ in Theorem 2-3 is a special case of (b).

Proof. (a) We will be a little bit more verbose in our proof than the problem statement, and will use *norm* and *absolute* value sign properly. Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$, and similarly $f_1, \ldots, f_m$ ben an orthonormal basis of $\mathbb{R}^m$. Moreover, let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ such that :

$$\begin{aligned} x &= x_1 e_1 + \cdots + x_n e_n \\ y &= y_1 f_1 + \cdots + y_m e_m \end{aligned}$$

.
Then we have

$$f(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j f(e_i, f_j)$$

Hence,

$$\begin{aligned} \|f(x,y)\| &\leq \sum_{i=1}^{n} \sum_{j=1}^{m} |x_i||y_j| \|f(e_i, f_j)\| \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{m} \|x\|\|y\|\|f(e_i, f_j)\| \\ &= \|x\|\|y\| \sum_{i=1}^{n} \sum_{j=1}^{m} \|f(e_i, f_j)\| \\ &= C\|x\|\|y\| \end{aligned}$$

Therefore,

$$\frac{\|f(h,k)\|}{\|(h,k)\|} \leq \frac{C\|h\|\|k\|}{\|(h,k)\|}$$

But $\|h\|\|k\| \leq \|h\|^2 + \|k\|^2 = \|(h,k)\|^2$. Hence.

$$\frac{\|f(h,k)\|}{\|(h,k)\|} \leq \frac{C\|h\|\|k\|}{\|(h,k)\|} \leq \frac{C\|(h,k)\|^2}{\|(h,k)\|} \leq C\|(h,k)\|$$

Therefore,

$$\lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} = 0$$

□

Chapter 2

Linear Algebra Done right

2.1 Excercise : 3.D

Problem. *1. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$*

Proof. Since ST is a composition of two bijections, it is also a bijection, and hence is also a bijection. We only need to show that $(ST)^{-1} = T^{-1}S^{-1}$.

$$\begin{aligned}(T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}S)T \\ &= T^{-1}IT \\ &= T^{-1}T = I\end{aligned}$$

Similarly, $(ST)(T^{-1}S^{-1}) = I$. □

Problem. *9. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.*

Proof. The reverse direction is immediate from Problem 1. Now suppose that ST is invertible. Let $v \in V$. Then $STv = v$. Hence S is surjective and therefore invertible. Suppose that $Tu = Tv$. Then, $STu = STv$. Since ST is invertible, we have $u = v$. Therefore, T is injective, and since V is finite dimension, T is invertible. □

Problem. *10. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$*

Proof. Suppose $ST = I$. Then $STv = v$. Since V is finite dimensional, S is invertible. Now,

$$I = S^{-1}S = S^{-1}(ST)S = (S^{-1}S)TS = ITS = TS$$

□

2.2 Exercise: 6.A

Problem. *(29) Suppose $V_1, \ldots, V_m$ are inner product spaces. Show that the equation*

$$\langle (u_1, \ldots, u_m), (v_1, \ldots, v_m) \rangle = \langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle$$

defines an inner product on $V_1 \times \cdot \times V_m$.

2.3 Excercise : 7.C

Problem. *4. Suppose* $T \in \mathcal{L}(V, W)$ *Prove that* T^*T *is a positive operator on* V *and* TT^* *is a positive operator on* W*.*

Proof.

$$(T^*T)^* = T^*T$$

Therefore, T^*T is self-adjoint.
Moreover,

$$\begin{aligned} \langle T^*Tv, v\rangle &= \langle Tv, Tv\rangle \\ &= ||Tv||^2 \geq 0 \end{aligned}$$

Therefore, T^*T is positive. □