Chapter 1

Calculas on Manifolds

1.1 Some proofs

Theorem 1. *Let $O \subseteq \mathbb{R}^{n+1}$ be open and $x \in \mathbb{R}$. Let's define O' as follows:*

$$O' = \{(y_1, \ldots, y_n) : (x, y_1, \ldots, y_n) \in O\}$$

Then $O' \subseteq \mathbb{R}^n$ is open.

Proof. Let $(y_1, \ldots, y_n) \in O'$ be arbitrary. By definition, $(x, y_1, \ldots, y_n) \in O$. Since O is open there is an open rectangle $U \subseteq O$. Corrsponding open rectangle U' also contains our y. Now suppose $z \in U'$ is arbitrary. Then $(x, z_1, \ldots, z_n) \in U \subseteq O$. Therefore, $z \in O'$. Hence, O' is open. □

Theorem 2. *Let $O \in \mathbb{R}^{m+n}$ be open and $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$. Define*

$$O' = \{(y_1, \ldots, y_n) : (x_1, \ldots, x_m, y_1, \ldots, y_n) \in O\}$$

Then O' is also open in $\mathbb{R}^n$.

Proof. Previous proof works with some slight changes. □

Theorem 3. *If $B \subseteq \mathbb{R}^n$ is compact and $x \in \mathbb{R}$, then $\{x\} \times B \subseteq \mathbb{R}^{n+1}$ is also compact.*

Proof. Suppose O is an open cover for $\{x\} \times B \subseteq \mathbb{R}^{n+1}$. Then we can construct an open cover O' by having a function $f : U \to U'$ by dropping the first coordinate. Since B is compact, there is a finite list of open sets $U_1', \ldots, U_2'$ that covers B. From this finite set we can find a finite cover $U_1, \ldots, U_n$ by going back pre-image by pre-image so that we contain α. Hence $\{x\} \times B \in \mathbb{R}^{n+1}$ is compact. □

Theorem 4. *If $B \in \mathbb{R}^m$ is compact and $x \in \mathbb{R}^n$, then $\{x\} \times B \in \mathbb{R}^{n+m}$ is also compact.*

Proof. Similar as above. □

Theorem 5. *If B is compact and O is an open cover of $\{x\} \times B \in \mathbb{R}^{n+m}$, then there is an open set $U \in \mathbb{R}^n$ containing x such that $U \times B$ is covered by a finite number of sets in O.*

Theorem 6. *If $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are compact, then $A \times B \subseteq \mathbb{R}^{m+n}$ is compact.*

Theorem 7. *$A_1 \times \cdots \times A_k$ is compact if each A_i is. In particular, a closed rectangle in $\mathbb{R}^k$ is compact.*

Theorem 8. *A closed bounded subset of $\mathbb{R}^n$ is compact. (The converse is also true (Problem 1-20)).*

Theorem 9. *Prove the converse of Corollary 1-7: A compcat subset of $\mathbb{R}^n$ is closed and bounded (see also Problem 1-28).*

Proof. Let $B \subseteq \mathbb{R}^n$ be compact. Boundededness of B is trivial. We only have to show that B is closed, we will show that $\mathbb{R}^n \setminus B$ is open. Let $p \in \mathbb{R}^n \setminus B$ be arbitrary. For each $q \in B$, we consider the open set $W_q = B_r(q, \frac{|p-q|}{2})$. These consiststs of an open cover O of B. By compactness, $W_{q_1} \dots W_{q_m}$ covers B. Choose q_i such that $|p - q_i|$ is smallest. Then the neighbourhood $B_(p) \subseteq \mathbb{R}^n \setminus B$. Hence, $\mathbb{R}^n \setminus B$ is open. $\square$

Theorem 10. *1-21(a) If A is closed and $x \notin A$, prove that there is a number $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.*

Proof. Let $x \notin A$. Since $\mathbb{R}^n \setminus A$ is open, there exists a neighborhood $B(x) \subseteq \mathbb{R}^n \setminus A$. Take d as the radius of this neighborhood $B(x)$. $\square$

Theorem 11. *1-21(b) If A is closed, B is compact, and $A \cap B = \emptyset$, prove that there is $d > 0$ such that $|y - x| \geq d$ for all $y \in A$ and $x \in B$. Hint: For each $b \in B$ find an open set U containing b such that this relation holds for $x \in U \cap B$.*

Chapter 2

Linear Algebra Done right

2.1 Excercise : 3.D

Problem. *1. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$*

Proof. Since ST is a composition of two bijections, it is also a bijection, and hence is also a bijection. We only need to show that $(ST)^{-1} = T^{-1}S^{-1}$.

$$\begin{aligned}(T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}S)T \\ &= T^{-1}IT \\ &= T^{-1}T = I\end{aligned}$$

Similarly, $(ST)(T^{-1}S^{-1}) = I$. □

Problem. *9. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.*

Proof. The reverse direction is immediate from Problem 1. Now suppose that ST is invertible. Let $v \in V$. Then $STv = v$. Hence S is surjective and therefore invertible. Suppose that $Tu = Tv$. Then, $STu = STv$. Since ST is invertible, we have $u = v$. Therefore, T is injective, and since V is finite dimension, T is invertible. □

Problem. *10. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$*

Proof. Suppose $ST = I$. Then $STv = v$. Since V is finite dimensional, S is invertible. Now,

$$I = S^{-1}S = S^{-1}(ST)S = (S^{-1}S)TS = ITS = TS$$

□

2.2 Excercise : 7.C

Problem. *4. Suppose $T \in \mathcal{L}(V, W)$ Prove that T^*T is a positive operator on V and TT^* is a positive operator on W.*

Proof.

$$(T^*T)^* = T^*T$$

Therefore, T^*T is self-adjoint.
Moreover,

$$\begin{aligned} \langle T^*Tv, v\rangle &= \langle Tv, Tv\rangle \\ &= ||Tv||^2 \geq 0 \end{aligned}$$

Therefore, T^*T is positive. □