

# Chapter 1

## Calculus on Manifolds

### 1.1 Some proofs

**Theorem 1.** Let  $O \subseteq \mathbb{R}^{n+1}$  be open and  $x \in \mathbb{R}$ . Let's define  $O'$  as follows:

$$O' = \{(y_1, \dots, y_n) : (x, y_1, \dots, y_n) \in O\}$$

Then  $O' \subseteq \mathbb{R}^n$  is open.

*Proof.* Let  $(y_1, \dots, y_n) \in O'$  be arbitrary. By definition,  $(x, y_1, \dots, y_n) \in O$ . Since  $O$  is open there is an open rectangle  $U \subseteq O$ . Corresponding open rectangle  $U'$  also contains our  $y$ . Now suppose  $z \in U'$  is arbitrary. Then  $(x, z_1, \dots, z_n) \in U \subseteq O$ . Therefore,  $z \in O'$ . Hence,  $O'$  is open.  $\square$

**Theorem 2.** Let  $O \in \mathbb{R}^{m+n}$  be open and  $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ . Define

$$O' = \{(y_1, \dots, y_n) : (x_1, \dots, x_m, y_1, \dots, y_n) \in O\}$$

Then  $O'$  is also open in  $\mathbb{R}^n$ .

*Proof.* Previous proof works with some slight changes.  $\square$

**Theorem 3.** If  $B \subseteq \mathbb{R}^n$  is compact and  $x \in \mathbb{R}$ , then  $\{x\} \times B \subseteq \mathbb{R}^{n+1}$  is also compact.

*Proof.* Suppose  $O$  is an open cover for  $\{x\} \times B \subseteq \mathbb{R}^{n+1}$ . Then we can construct an open cover  $O'$  by having a function  $f : U \rightarrow U'$  by dropping the first coordinate. Since  $B$  is compact, there is a finite list of open sets  $U'_1, \dots, U'_2$  that covers  $B$ . From this finite set we can find a finite cover  $U_1, \dots, U_n$  by going back pre-image by pre-image so that we contain  $\alpha$ . Hence  $\{x\} \times B \in \mathbb{R}^{n+1}$  is compact.  $\square$

**Theorem 4.** If  $B \in \mathbb{R}^m$  is compact and  $x \in \mathbb{R}^n$ , then  $\{x\} \times B \in \mathbb{R}^{n+m}$  is also compact.

*Proof.* Similar as above.  $\square$

**Theorem 5.** If  $B$  is compact and  $O$  is an open cover of  $\{x\} \times B \in \mathbb{R}^{n+m}$ , then there is an open set  $U \in \mathbb{R}^n$  containing  $x$  such that  $U \times B$  is covered by a finite number of sets in  $O$ .

**Theorem 6.** If  $A \subseteq \mathbb{R}^n$  and  $B \subseteq \mathbb{R}^m$  are compact, then  $A \times B \subseteq \mathbb{R}^{m+n}$  is compact.

**Theorem 7.**  $A_1 \times \dots \times A_k$  is compact if each  $A_i$  is. In particular, a closed rectangle in  $\mathbb{R}^k$  is compact.

**Theorem 8.** A closed bounded subset of  $\mathbb{R}^n$  is compact. (The converse is also true (Problem 1-20)).

**Theorem 9.** Prove the converse of Corollary 1-7: A compact subset of  $\mathbb{R}^n$  is closed and bounded (see also Problem 1-28).

*Proof.* Let  $B \subseteq \mathbb{R}^n$  be compact. Boundedness of  $B$  is trivial. We only have to show that  $B$  is closed, we will show that  $\mathbb{R}^n \setminus B$  is open. Let  $p \in \mathbb{R}^n \setminus B$  be arbitrary. For each  $q \in B$ , we consider the open set  $W_q = B_r(q, \frac{|p-q|}{2})$ . These consists of an open cover  $O$  of  $B$ . By compactness,  $W_{q_1} \dots W_{q_m}$  covers  $B$ . Choose  $q_i$  such that  $|p - q_i|$  is smallest. Then the neighbourhood  $B(p) \subseteq \mathbb{R}^n \setminus B$ . Hence,  $\mathbb{R}^n \setminus B$  is open.  $\square$

**Exercise.** 1-21(a) If  $A$  is closed and  $x \notin A$ , prove that there is a number  $d > 0$  such that  $|y - x| \geq d$  for all  $y \in A$ .

*Proof.* Let  $x \notin A$ . Since  $\mathbb{R}^n \setminus A$  is open, there exists a neighborhood  $B(x) \subseteq \mathbb{R}^n \setminus A$ . Take  $d$  as the radius of this neighborhood  $B(x)$ .  $\square$

**Exercise.** 1-21(b) If  $A$  is closed,  $B$  is compact, and  $A \cap B = \emptyset$ , prove that there is  $d > 0$  such that  $|y - x| \geq d$  for all  $y \in A$  and  $x \in B$ . Hint: For each  $b \in B$  find an open set  $U$  containing  $b$  such that this relation holds for  $x \in U \cap B$ .

*Proof.* Since  $A$  and  $B$  are disjoint, by 1-21(a) for each  $b \in B$  we can find  $d > 0$  such that  $|y - b| \geq d$  for all  $y \in A$ . Consider the open cover  $O = \{B_{\frac{d}{2}}(b) : b \in B\}$ . Since  $B$  is compact, finitely many open set  $B_{d_1}, \dots, B_{d_k}$  covers  $B$ . By triangle inequality if  $x \in B_{d_i}$  then we have  $|y - x| \geq d_i$ . By picking the minimums of  $d_i$  we find our  $d$ .  $\square$

**Exercise.** 1-21(c) Give a counterexample in  $\mathbb{R}^2$  if  $A$  and  $B$  are closed but neither is compact.

*Proof.* Let  $A = \mathbb{N}$  and  $B = \{n + \frac{1}{2n} | n \in \mathbb{N}\}$ .  $\square$

**Exercise.** 1-22 If  $U$  is open and  $C \subseteq U$  is compact, show that there is a compact set  $D$  such that  $C \subseteq \text{interior } D$  and  $D \subseteq U$ .

*Proof.* Notice that  $U^c$  is closed and  $C \cup U^c = \emptyset$ . Therefore, by 1-21 there exists  $d > 0$  such that  $|y - x| > d$  for all  $x \in C$  and  $y \in U^c$ . Now consider the set

$$E = \bigcup_{x \in C} U_x$$

where  $U_x$  is an open neighbourhood with radius  $d/2$  and centered at  $x$ . Then we have  $C \subseteq E \subseteq U$ . Moreover since  $C$  is compact we can assume that  $E$  is an union of finitely many sets  $U_x$ . Our choice of  $d$  guarantees that if we take

$$D = \bigcup_{i=1}^k \overline{U}_{x_i}$$

then we still have  $C \subseteq D \subseteq U$ , where each  $\overline{U}_{x_i}$  is a closed neighbourhood.  $\square$

**Theorem 10.** If  $A \subseteq \mathbb{R}^n$ , a function  $f : A \rightarrow \mathbb{R}^m$  is continuous if and only if for every open set  $U \subseteq \mathbb{R}^m$  there is some open set  $V \subseteq \mathbb{R}^n$  such that  $f^{-1}(U) = V \cap A$

*Proof.* We prove the converse. Let  $a \in A$  be arbitrary, we show that  $f$  is continuous at  $a$ . Let  $\epsilon > 0$  be arbitrary. Then the neighbourhood  $B = B_\epsilon(f(a)) \subseteq \mathbb{R}^m$  is open. Therefore,  $f^{-1}(B) = V \cap A$  for some open set  $V \subseteq \mathbb{R}^n$ . But then,  $a \in V$  so there is an open set  $B_\delta(a) \subseteq V$  for some  $\delta > 0$ . Therefore, if  $x \in B_\delta(a)$  and  $x \in A$ , then  $f(x) \in B$ . Hence  $f$  is continuous at  $a$ .  $\square$

If  $A \rightarrow \mathbb{R}$  is bounded, the extent to which  $f$  fails to be continuous at  $a \in A$  can be measured in a precise way. For  $\delta > 0$  let

$$\begin{aligned} M(a, f, \delta) &= \sup\{f(x) : x \in A \text{ and } |x - a| < \delta\} \\ m(a, f, \delta) &= \inf\{f(x) : x \in A \text{ and } |x - a| < \delta\} \end{aligned}$$

The oscillation  $o(f, a)$  of  $f$  at  $a$  is defined by  $o(f, a) = \lim_{\delta \rightarrow 0^+} [M(a, f, \delta) - m(a, f, \delta)]$

**Theorem 11.** *If  $f$  is bounded, then  $o(f, a) = \lim_{\delta \rightarrow 0^+} [M(a, f, \delta) - m(a, f, \delta)]$  always exists.*

*Proof.* Consider the sequence  $M(a, f, \frac{1}{n}) - m(a, f, \frac{1}{n})$ . This is a decreasing sequence which is bounded below by 0. Therefore, there exists a limit  $a \geq 0$ . It is easy to show

$$a = \lim_{\delta \rightarrow 0^+} [M(a, f, \delta) - m(a, f, \delta)]$$

since  $1g(x) = M(a, f, x) - m(a, f, x)$  is a non-increasing function on  $x > 0$ .  $\square$

**Exercise. 1-23** If  $f : A \rightarrow \mathbb{R}^m$  and  $a \in A$ , show that  $\lim_{x \rightarrow a} f(x) = b$  if and only if  $\lim_{x \rightarrow a} f^i(x) = b^i$  for  $i = 1, \dots, m$ .

*Proof.*  $\rightarrow$  Suppose  $\lim_{x \rightarrow a} f(x) = b$ , then for each  $\epsilon > 0$  there exists  $\delta > 0$  such that if  $y \in A$  and  $0 < |x - y| < \delta$ , then  $|f(x) - b| < \epsilon$ . Same choice of  $\delta$  ensures that  $|f^i(x) - b^i| < \epsilon$ .

$\leftarrow$  Now suppose  $\lim_{x \rightarrow a} f^i(x) = b^i$  for  $i = 1, \dots, m$ . Let  $\epsilon > 0$  be arbitrary. Then we find  $\delta_1, \dots, \delta_m$  where each  $\delta_i > 0$  such that  $|f^i(x) - b^i| < \epsilon$ . Taking minimum of these  $\delta_i$  as  $\delta$  we find that  $|f(x) - b| \leq \sum |f^i(x) - b^i| \leq n\epsilon$ .  $\square$

**Exercise. 1-24.** Prove that  $f : A \rightarrow \mathbb{R}^m$  is continuous at  $a$  if and only if each  $f^i$  is.

*Proof.*  $\rightarrow$  Suppose  $f : A \rightarrow \mathbb{R}^m$  is continuous at  $a$ . Then  $\lim_{x \rightarrow a} f(x) = f(a)$  From 1-23, we see that  $\lim_{x \rightarrow a} f^i(x) = f^i(a)$ . Therefore, each  $f^i : A \rightarrow \mathbb{R}$  is continuous at  $a \in \mathbb{R}^n$ .

$\leftarrow$  Suppose each  $f^i : A \rightarrow \mathbb{R}$  is continuous at  $a \in A$ . Then we have  $\lim_{x \rightarrow a} f^i(x) = f^i(a)$ . Again, using 1-23 we conclude that  $\lim_{x \rightarrow a} f(x) = f(a)$ .  $\square$

**Exercise. 1-25** Prove that a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous. Hint: Use Problem 1-10.

*Proof.* We know from 1-10 that there is a  $M > 0$  such that  $|T(h)| \leq M|h|$  for all  $h \in \mathbb{R}^n$ . Now take  $\delta = \frac{\epsilon}{2M}$ . Then, for any  $y \in \mathbb{R}^n$  such that  $|x - y| < \delta$  we have (for  $h = y - x$ ):

$$\begin{aligned} |T(y) - T(x)| &= |T(x + h) - T(x)| \\ &= |T(x) + T(h) - T(x)| \\ &= |T(h)| \\ &\leq M|h| \\ &< M \frac{\epsilon}{2M} = \epsilon \end{aligned}$$



# Chapter 2

## Linear Algebra Done right

### 2.1 Exercise : 3.D

**Problem. 1.** Suppose  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$  are both invertible linear maps. Prove that  $ST \in \mathcal{L}(U, W)$  is invertible and that  $(ST)^{-1} = T^{-1}S^{-1}$

*Proof.* Since  $ST$  is a composition of two bijections, it is also a bijection, and hence is also a bijection. We only need to show that  $(ST)^{-1} = T^{-1}S^{-1}$ .

$$\begin{aligned}(T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}S)T \\ &= T^{-1}IT \\ &= T^{-1}T = I\end{aligned}$$

Similarly,  $(ST)(T^{-1}S^{-1}) = I$ . □

**Problem. 9.** Suppose  $V$  is finite-dimensional and  $S, T \in \mathcal{L}(V)$ . Prove that  $ST$  is invertible if and only if both  $S$  and  $T$  are invertible.

*Proof.* The reverse direction is immediate from Problem 1. Now suppose that  $ST$  is invertible. Let  $v \in V$ . Then  $STv = v$ . Hence  $S$  is surjective and therefore invertible. Suppose that  $Tu = Tv$ . Then,  $STu = STv$ . Since  $ST$  is invertible, we have  $u = v$ . Therefore,  $T$  is injective, and since  $V$  is finite dimension,  $T$  is invertible. □

**Problem. 10.** Suppose  $V$  is finite-dimensional and  $S, T \in \mathcal{L}(V)$ . Prove that  $ST = I$  if and only if  $TS = I$

*Proof.* Suppose  $ST = I$ . Then  $STv = v$ . Since  $V$  is finite dimensional,  $S$  is invertible. Now,

$$I = S^{-1}S = S^{-1}(ST)S = (S^{-1}S)TS = ITS = TS$$

□

## 2.2 Exercise : 7.C

**Problem.** 4. Suppose  $T \in \mathcal{L}(V, W)$  Prove that  $T^*T$  is a positive operator on  $V$  and  $TT^*$  is a positive operator on  $W$ .

*Proof.*

$$(T^*T)^* = T^*T$$

Therefore,  $T^*T$  is self-adjoint.

Moreover,

$$\begin{aligned}\langle T^*Tv, v \rangle &= \langle Tv, Tv \rangle \\ &= \|Tv\|^2 \geq 0\end{aligned}$$

Therefore,  $T^*T$  is positive. □