Chapter 1

Linear Algebra Done right

1.1 Excercise : 3.D

Problem. *1. Suppose $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U,W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$*

Proof. Since ST is a composition of two bijections, it is also a bijection, and hence is also a bijection. We only need to show that $(ST)^{-1} = T^{-1}S^{-1}$.

$$\begin{aligned}(T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}S)T \\ &= T^{-1}IT \\ &= T^{-1}T = I\end{aligned}$$

Similarly, $(ST)(T^{-1}S^{-1}) = I$. □

Problem. *9. Suppose V is finite-dimensional and $S,T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.*

Proof. The reverse direction is immediate from Problem 1. Now suppose that ST is invertible. Let $v \in V$. Then $STv = v$. Hence S is surjective and therefore invertible. Suppose that $Tu = Tv$. Then, $STu = STv$. Since ST is invertible, we have $u = v$. Therefore, T is injective, and since V is finite dimension, T is invertible. □

Problem. *10. Suppose V is finite-dimensional and $S,T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$*

Proof. Suppose $ST = I$. Then $STv = v$. Since V is finite dimensional, S is invertible. Now,

$$I = S^{-1}S = S^{-1}(ST)S = (S^{-1}S)TS = ITS = TS$$

□

1.2 Excercise : 7.C

Problem. *4. Suppose $T \in \mathcal{L}(V, W)$ Prove that T^*T is a positive operator on V and TT^* is a positive operator on W.*

Proof.

$$(T^*T)^* = T^*T$$

Therefore, T^*T is self-adjoint.
Moreover,

$$\begin{aligned}\langle T^*Tv, v\rangle &= \langle Tv, Tv\rangle \\ &= ||Tv||^2 \geq 0\end{aligned}$$

Therefore, T^*T is positive. □