# Chapter 1

# Calculas on Manifolds

## 1.1 Some proofs

**Theorem 1.** *Let $O \subseteq \mathbb{R}^{n+1}$ be open and $x \in \mathbb{R}$. Let's define $O'$ as follows:*

$$O' = \{(y_1, \ldots, y_n) : (x, y_1, \ldots, y_n) \in O\}$$

*Then $O' \subseteq \mathbb{R}^n$ is open.*

*Proof.* Let $(y_1, \ldots, y_n) \in O'$ be arbitrary. By definition, $(x, y_1, \ldots, y_n) \in O$. Since $O$ is open there is an open rectangle $U \subseteq O$. Corrsponding open rectangle $U'$ also contains our $y$. Now suppose $z \in U'$ is arbitrary. Then $(x, z_1, \ldots, z_n) \in U \subseteq O$. Therefore, $z \in O'$. Hence, $O'$ is open. $\square$

**Theorem 2.** *Let $O \in \mathbb{R}^{m+n}$ be open and $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$. Define*

$$O' = \{(y_1, \ldots, y_n) : (x_1, \ldots, x_m, y_1, \ldots, y_n) \in O\}$$

*Then $O'$ is also open in $\mathbb{R}^n$.*

*Proof.* Previous proof works with some slight changes. $\square$

**Theorem 3.** *If $B \subseteq \mathbb{R}^n$ is compact and $x \in \mathbb{R}$, then $\{x\} \times B \subseteq \mathbb{R}^{n+1}$ is also compact.*

*Proof.* Suppose $O$ is an open cover for $\{x\} \times B \subseteq \mathbb{R}^{n+1}$. Then we can construct an open cover $O'$ by having a function $f : U \to U'$ by dropping the first coordinate. Since $B$ is compact, there is a finite list of open sets $U_1', \ldots, U_2'$ that covers $B$. From this finite set we can find a finite cover $U_1, \ldots, U_n$ by going back pre-image by pre-image so that we contain $\alpha$. Hence $\{x\} \times B \in \mathbb{R}^{n+1}$ is compact. $\square$

**Theorem 4.** *If $B \in \mathbb{R}^m$ is compact and $x \in \mathbb{R}^n$, then $\{x\} \times B \in \mathbb{R}^{n+m}$ is also compact.*

*Proof.* Similar as above. $\square$

**Theorem 5.** *If $B$ is compact and $O$ is an open cover of $\{x\} \times B \in \mathbb{R}^{n+m}$, then there is an open set $U \in \mathbb{R}^n$ containing $x$ such that $U \times B$ is covered by a finite number of sets in $O$.*

**Theorem 6.** *If $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are compact, then $A \times B \subseteq \mathbb{R}^{m+n}$ is compact.*

**Theorem 7.** *$A_1 \times \cdots \times A_k$ is compact if each $A_i$ is. In particular, a closed rectangle in $\mathbb{R}^k$ is compact.*

**Theorem 8.** *A closed bounded subset of $\mathbb{R}^n$ is compact. (The converse is also true (Problem 1-20)).*

**Theorem 9.** *Prove the converse of Corollary 1-7: A compcat subset of $\mathbb{R}^n$ is closed and bounded (see also Problem 1-28).*

*Proof.* Let $B \subseteq \mathbb{R}^n$ be compact. Boundededness of $B$ is trivial. We only have to show that $B$ is closed, we will show that $\mathbb{R}^n \setminus B$ is open. Let $p \in \mathbb{R}^n \setminus B$ be arbitrary. For each $q \in B$, we consider the open set $W_q = B_r(q, \frac{|p-q|}{2})$. These consiststs of an open cover $O$ of $B$. By compactness, $W_{q_1} \dots W_{q_m}$ covers $B$. Choose $q_i$ such that $|p - q_i|$ is smallest. Then the neighbourhood $B_(p) \subseteq \mathbb{R}^n \setminus B$. Hence, $\mathbb{R}^n \setminus B$ is open. □

**Exercise.** *1-21(a) If $A$ is closed and $x \notin A$, prove that there is a number $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.*

*Proof.* Let $x \notin A$. Since $\mathbb{R}^n \setminus A$ is open, there exists a neighborhood $B(x) \subseteq \mathbb{R}^n \setminus A$. Take $d$ as the radius of this neighborhood $B(x)$. □

**Exercise.** *1-21(b) If $A$ is closed, $B$ is compact, and $A \cap B = \emptyset$, prove that there is $d > 0$ such that $|y - x| \geq d$ for all $y \in A$ and $x \in B$. Hint: For each $b \in B$ find an open set $U$ containing $b$ such that this relation holds for $x \in U \cap B$.*

*Proof.* Since $A$ and $B$ are disjoint, by 1-21(a) for each $b \in B$ we can find $d > 0$ such that $|y - b| \geq d$ for all $y \in A$. Consider the open cover $O = \{B_{\frac{d}{2}}(b) : b \in B\}$. Since $B$ is compact, finitely many open set $B_{d_1}, \dots B_{d_k}$ covers $B$. By triangale inequality if $x \in B_{d_i}$ then we have $|y - x| \geq d_i$. By picking the minimums of $d_i$ we find our $d$. □

**Exercise.** *1-21(c) Give a counterexample in $\mathbb{R}^2$ if $A$ and $B$ are closed but neither is compact.*

*Proof.* Let $A = \mathbb{N}$ and $B = \{n + \frac{1}{2n} | n \in \mathbb{N}\}$. □

**Exercise.** *1-22 If $U$ is open and $C \subseteq U$ is compact, show that there is a compact set $D$ such that $C \subseteq interior D$ and $D \subseteq U$.*

*Proof.* Notice that $U^c$ is closed and $C \cup U^c = \emptyset$. Therefore, by 1-21 there exists $d > 0$ such that $|y - x| > d$ for all $x \in C$ and $y \in U^c$. Now consider the set

$$E = \bigcup_{x \in C} U_x$$

where $U_x$ is an open neighbourhood with radius $d/2$ and centered at $x$. Then we have $C \subseteq E \subseteq U$. Moreover since $C$ is compact we can assume that $E$ is an union of finitely many sets $U_x$. Our choice of $d$ guaruntees that if we take

$$D = \bigcup_{i=1}^{k} \overline{U}_{x_i}$$

then we still have $C \subseteq D \subseteq U$, where each $\overline{U}_{x_i}$ is a closed neighbourhood. □

**Theorem 10.** *If $A \subseteq \mathbb{R}^n$, a function $f : A \to \mathbb{R}^m$ is continuous if and only if for every open set $U \subseteq \mathbb{R}^m$ there is some open set $V \subseteq \mathbb{R}^n$ such that $f^{-1}(U) = V \cap A$*

*Proof.* We prove the converse. Let $a \in A$ be arbitrary, we show that $f$ is continuos at $a$. Let $\epsilon > 0$ be arbitrary. Then the neighbourhood $B = B_\epsilon(f(a)) \subseteq \mathbb{R}^m$ is open. Therefore, $f^{-1}(B) = V \cap A$ for some open set $V \subseteq \mathbb{R}^n$. But then, $a \in V$ so there is an open set $B_\delta(a) \subseteq V$ for some $\delta > 0$. Therefore, if $x \in B_\delta(a)$ and $x \in A$, then $f(x) \subseteq B$. Hence $f$ is continuous at $a$. □

If $A \to \mathbb{R}$ is bounded, the extent to which $f$ fails to be continuous at $a \in A$ can be measured in a precise way. For $\delta > 0$ let

$$M(a, f, \delta) = \sup\{f(x) : x \in A \text{ and } |x - a| < \delta\}$$
$$m(a, f, \delta) = \inf\{f(x) : x \in A \text{ and } |x - a| < \delta\}$$

The oscillation $o(f, a)$ of $f$ at $a$ is defined by $o(f, a) = \lim_{\delta \to 0^+} [M(a, f, \delta) - m(a, f, \delta)]$

**Theorem 11.** *If $f$ is bounded, then $o(f, a) = \lim_{\delta \to 0^+} [M(a, f, \delta) - m(a, f, \delta)]$ always exists.*

*Proof.* Consider the sequence $M(a, f, \frac{1}{n}) - m(a, f, \frac{1}{n})$. This is a decreasing sequence which is bounded below by $0$. Therefore, there exists a limit $a \geq 0$. It is easy to show

$$a = \lim_{\delta \to 0^+} [M(a, f, \delta) - m(a, f, \delta)]$$

since $1g(x) = M(a, f, x) - m(a, f, x)$ is a non-increasing function on $x > 0$. □

**Exercise.** *1-23 If $f : A \to \mathbb{R}^m$ and $a \in A$, show that $\lim_{x \to a} f(x) = b$ if and only if $\lim_{x \to a} f^i(x) = b^i$ for $i = 1, \ldots, m$.*

*Proof.* $\to$ Suppose $\lim_{x \to a} f(x) = b$, then for each $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in A$ and $0 < |x - y| < \delta$, then $|f(x) - b| < \epsilon$. Same choice of $\delta$ ensures that $|f^i(x) - b^i| < \epsilon$.

$\leftarrow$ Now suppose $\lim_{x \to a} f^i(x) = b^i$ for $i = 1, \ldots, m$. Let $\epsilon > 0$ be arbitrary. Then we find $\delta_1, \ldots, \delta_m$ where each $\delta_i > 0$ such that $|f^i(x) - b^i| < \epsilon$. Taking minimum of these $\delta_i$ as $\delta$ we find that $|f(x) - b| \leq \sum |f^i(x) - b^i| \leq n\epsilon$. □

**Exercise.** *1-24. Prove that $f : A \to \mathbb{R}^m$ is continuous at $a$ if and only if each $f^i$ is.*

*Proof.* $\to$ Suppose $f : A \to \mathbb{R}^m$ is continuous at $a$. Then $\lim_{x \to a} f(x) = f(a)$ From 1-23, we see that $\lim_{x \to a} f^i(x) = f^i(a)$. Therefore, each $f^i : A \to \mathbb{R}$ is continuous at $a \in \mathbb{R}^n$.

$\leftarrow$ Suppose each $f^i : A \to \mathbb{R}$ is continuous at $a \in A$. Then we have $\lim_{x \to a} f^i(x) = f^i(a)$. Again, using 1-23 we conclude that $\lim_{x \to a} f(x) = f(a)$. □

**Exercise.** *1-25 Prove that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is continous. Hint: Use Problem 1-10.*

*Proof.* We know from 1-10 that there is a $M > 0$ such that $|T(h)| \leq M|h|$ for all $h \in \mathbb{R}^n$. Now take $\delta = \frac{\epsilon}{2M}$. Then, for any $y \in \mathbb{R}^n$ such that $|x - y| < \delta$ we have (for $h = y - x$):

$$\begin{aligned}
|T(y) - T(x)| &= |T(x + h) - T(x)| \\
&= |T(x) + T(h) - T(x)| \\
&= |T(h)| \\
&\leq M|h| \\
&< M\frac{\epsilon}{2M} = \epsilon
\end{aligned}$$

□

## 1.2 Differentiation

**Problem.** *2-1. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then it is continuous at $a$. Hint: User Problem 1-10.*

*Proof.* Since $f$ is differentiable at $a$, it is easy to show the weaker statement

$$\lim_{h \to 0} |f(a+h) - f(a) - L(h)| = 0$$

.
Now,

$$0 \leq |f(a+h) - f(a)| \leq |f(a+h) - f(a) - L(h)| + |L(h)|$$

Since both of the terms on the right side goes to $0$ as $h \to 0$, it shows that $f$ is continuous at $a$. □

**Problem.** *2-2. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is independent of the second variable if for each $x \in \mathbb{R}$ we have $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbb{R}$. Show that $f$ is independent of the second variable if and only if there is a funciton $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x)$. What is $f'(a, b$ in terms of $g'$?*

*Proof.* Suppose $f$ is independent of the second variable. Define

$$g(x) = f(x, y)$$

Then $g$ is well defined.
On the other hand, suppose there exists such function $g : \mathbb{R} \to \mathbb{R}$. Then $f$ is independent of the second variable. □

**Problem.** *2-12: A function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is bilinear if for $x, x_1, x_2 \in \mathbb{R}^n$, $y, y_1, y_2 \in \mathbb{R}^m$, and $a \in \mathbb{R}$ we have*

$$\begin{aligned} f(ax, y) &= af(x, y) = f(x, ay) \\ f(x_1 + x_2, y) &= f(x_1, y) + f(x_2, y) \\ f(x, y_1 + y_2) &= f(x, y_1) + f(x, y_2) \end{aligned}$$

*(a) Prove that if $f$ is bilinear, then*

$$\lim_{(h,k) \to 0} = \frac{|f(h, k)|}{|(h, k)|} = 0$$

*(b) Prove that $Df(a, b)(x, y) = f(a, y) + f(x, b)$.*
*(c) Show that the formula for $Dp(a, b)$ in Theorem 2-3 is a special case of (b).*

*Proof.* (a) We will be a little bit more verbose in our proof than the problem statement, and will use *norm* and *absolute* value sign properly. Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$, and similarly $f_1, \ldots, f_m$ ben an orthonormal basis of $\mathbb{R}^m$. Moreover, let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ such that :

$$\begin{aligned} x &= x_1 e_1 + \cdots + x_n e_n \\ y &= y_1 f_1 + \cdots + y_m e_m \end{aligned}$$

.
Then we have

$$f(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j f(e_i, f_j)$$

Hence,

$$\begin{aligned} \|f(x,y)\| &\leq \sum_{i=1}^{n} \sum_{j=1}^{m} |x_i||y_j| \|f(e_i, f_j)\| \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{m} \|x\|\|y\| \|f(e_i, f_j)\| \\ &= \|x\|\|y\| \sum_{i=1}^{n} \sum_{j=1}^{m} \|f(e_i, f_j)\| \\ &= C\|x\|\|y\| \end{aligned}$$

Therefore,

$$\frac{\|f(h,k)\|}{\|(h,k)\|} \leq \frac{C\|h\|\|k\|}{\|(h,k)\|}$$

But $\|h\|\|k\| \leq \|h\|^2 + \|k\|^2 = \|(h,k)\|^2$. Hence.

$$\frac{\|f(h,k)\|}{\|(h,k)\|} \leq \frac{C\|h\|\|k\|}{\|(h,k)\|} \leq \frac{C\|(h,k)\|^2}{\|(h,k)\|} \leq C\|(h,k)\|$$

Therefore,

$$\lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} = 0$$

□

*Proof.* (b)

$$\begin{aligned} &\lim_{(x,y)\to 0} \frac{\|f(a+x, y+b) - f(a,b) - f(a,y) - f(x,b)\|}{\|(x,y)\|} \\ &= \lim_{(x,y)\to 0} \frac{\|f(a,b) + f(a,y) + f(x,b) + f(x,y) - f(a,b) - f(a,y) - f(x,b)\|}{\|(x,y)\|} \\ &= \lim_{(x,y)\to 0} \frac{\|f(x,y)\|}{\|(x,y)\|} = 0 \end{aligned}$$

We also need to show that $Dp(a,b)(x,y) = f(a,y) + f(x,b) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is linear, but that is easy to show. □

*Proof.* For $n = m = p = 1$, the product is bilinear, and follows from there. $\square$

**Problem.** *2-13 Define* $IP : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *by* $IP(x, y) = \langle x, y \rangle$.
*(a) Find* $D(IP)(a, b)$ *and* $(IP)'(a, b)$.

*(b) If* $f, g : \mathbb{R} \to \mathbb{R}^n$ *are differentiable and* $h : \mathbb{R} \to \mathbb{R}$ *is defined by* $h(t) = \langle f(t), g(t) \rangle$*, show that*

$$h'(a) = \langle f'(a), g(a) \rangle + \langle f(a), g'(a) \rangle$$

*(c) If* $f : \mathbb{R} \to \mathbb{R}^n$ *is differentiable and* $|f(t)| = 1$ *for all* $t$*, show that*

$$\langle f'(t), f(t) \rangle = 0$$

*(d) Exhibit a differentiable functoin* $f : \mathbb{R} \to \mathbb{R}$ *such that the function* $|f|$ *defined by* $|f|(t) = |f(t)|$ *is not differentiable.*

# Chapter 2

# Linear Algebra Done right

## 2.1 Excercise : 3.D

**Problem.** *1. Suppose $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U,W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$*

*Proof.* Since $ST$ is a composition of two bijections, it is also a bijection, and hence is also a bijection. We only need to show that $(ST)^{-1} = T^{-1}S^{-1}$.

$$\begin{aligned}(T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}S)T \\ &= T^{-1}IT \\ &= T^{-1}T = I\end{aligned}$$

Similarly, $(ST)(T^{-1}S^{-1}) = I$. □

**Problem.** *9. Suppose $V$ is finite-dimensional and $S,T \in \mathcal{L}(V)$. Prove that $ST$ is invertible if and only if both $S$ and $T$ are invertible.*

*Proof.* The reverse direction is immediate from Problem 1. Now suppose that $ST$ is invertible. Let $v \in V$. Then $STv = v$. Hence $S$ is surjective and therefore invertible. Suppose that $Tu = Tv$. Then, $STu = STv$. Since $ST$ is invertible, we have $u = v$. Therefore, $T$ is injective, and since $V$ is finite dimension, $T$ is invertible. □

**Problem.** *10. Suppose $V$ is finite-dimensional and $S,T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$*

*Proof.* Suppose $ST = I$. Then $STv = v$. Since $V$ is finite dimensional, $S$ is invertible. Now,

$$I = S^{-1}S = S^{-1}(ST)S = (S^{-1}S)TS = ITS = TS$$

□

## 2.2 Exercise: 6.A

**Problem.** *(29) Suppose $V_1, \ldots, V_m$ are inner product spaces. Show that the equation*

$$\langle (u_1, \ldots, u_m), (v_1, \ldots, v_m) \rangle = \langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle$$

*defines an inner product on $V_1 \times \cdot \times V_m$.*

## 2.3 Excercise : 7.C

**Problem.** *4. Suppose* $T \in \mathcal{L}(V, W)$ *Prove that* $T^*T$ *is a positive operator on* $V$ *and* $TT^*$ *is a positive operator on* $W$.

*Proof.*

$$(T^*T)^* = T^*T$$

Therefore, $T^*T$ is self-adjoint.
Moreover,

$$\begin{aligned} \langle T^*Tv, v\rangle &= \langle Tv, Tv\rangle \\ &= ||Tv||^2 \geq 0 \end{aligned}$$

Therefore, $T^*T$ is positive. □