LINEAR MOTION

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Applied Physics

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- Derivation of velocity & acceleration in terms of integration
- 3–Equations of Motion
- Motion in 2-Dimension/3-Dimension
- Motion of Free Fall Body
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Motion in one dimension

- Here we'll discuss velocity, displacement and acceleration in terms of Cartesian components as they all are vector quantities.
- Considering motion only in the direction of a single component, for example, the x direction, that is, motion in a straight line.
- If we start timing an object moving in the x direction when it starts from $(x_0 = 0)$ point, we may write

$$\overrightarrow{v_x} = \frac{\vec{x} - \overrightarrow{x_0}}{t - t_0} \; ; (x_0 = 0)$$

$$\bar{v}_x = \frac{x - 0}{t - 0}$$

$$\boxed{x = \bar{v}_x t} \longrightarrow \textbf{Eq. 1.1}$$

- Equation 1.1 results from the definition of average velocity; **thus it holds in all cases whether or not the acceleration is constant.**

Linear Motion

Derivation of three Equations of Motion

1. $$v = v_0 + at$$

2. $$v^2 - v_0^2 = 2ax$$

3. $$x = v_0 t + \frac{1}{2} a t^2$$

1^{st} Equation of Motion

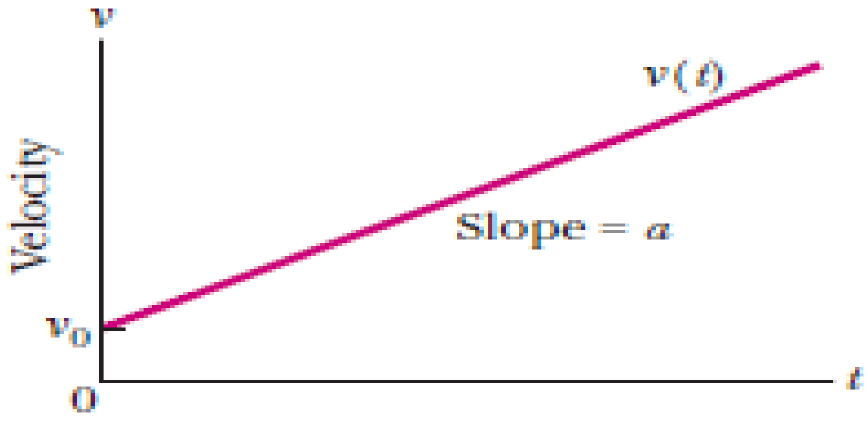
- The acceleration is defined as the rate of change of the velocity.

- If the acceleration is constant, the change in the velocity during the first, second, third, and all succeeding seconds of the motion will be the same and equal to the acceleration $\vec{a}$
- if the motion lasts t seconds, the change in the velocity $\Delta v = v - v_0 = at,$ where v is the final velocity and v_0 is the initial velocity.
- We can rewrite this result as

$$\boxed{v = v_0 + at} \longrightarrow \textbf{Eq.1.2}$$

Velocity – time Graph

- If we plot equation 1.2 on graph we will obtain a straight line, as indicated in the Figure below.
- The slope of this line is the constant acceleration *a*.



2^{nd} Equation of Motion

- Another important relation that we can have when the velocity increases at a constant rate the average velocity is one half the sum of the initial velocity v_0 and the final velocity namely, v

$$\overline{v} = \frac{v + v_0}{2} \longrightarrow \textbf{Eq. 1.3}$$

- Previously we have, $x = \overline{v}\ t \longrightarrow$ **Eq. 1.1**

- On substituting value of $\overline{v}$ in eq 1.1, we get

$$x = \frac{v + v_0}{2} t$$

2nd Equation of Motion

$$x = \frac{v + v_0}{2} t \longrightarrow \textbf{Eq. 1.4}$$

If we substitute the value of "v" from 1st equation of motion in eq.1.4 then we will have

$$x = \frac{v_0 + at + v_0}{2} t$$

Or we can have ,

$$\boxed{x = v_0 t + \frac{1}{2} a t^2}$$

3rd Equation of Motion

- Considering equation 1.4,

$$x = \frac{v + v_0}{2} t \longrightarrow \textbf{Eq. 1.4}$$

- We can find out the value of "t" from 1st equation of motion that is, $t = \frac{v - v_0}{a}$
- On substituting the above value of "t" in eq. 1.4 we will have the following results

$$x = \frac{(v + v_0)}{2} \frac{(v - v_0)}{a}$$

$$\boxed{v^2 - v_0^2 = 2ax}$$

Derivations by Integration

We may derive these equations more formally by integration.
By definition

$$a = \frac{dv}{dt}$$

Rearranging terms and integrating, we write

$$\int_{v_0}^{v} dv = \int_0^t a\,dt$$

acceleration is taken as constant, so a can be taken out of the integral and we write

Derivations by Integration

$$\int_{v_0}^{v} dv = a \int_{0}^{t} dt$$

This integrates to

$$v - v_0 = at$$

and

$$v = v_0 + at$$

1st equation of motion

Derivations by Integration

- From definition we know,

$$v = \frac{dx}{dt}$$

- Rearranging terms

$$\int_{x_0}^{x} dx = \int_{0}^{t} v\, dt$$

$$\int_{x_0}^{x} dx = \int_{0}^{t} (v_0 + at)dt$$

$$\int_{x_0}^{x} dx = v_0 \int_{0}^{t} dt + a \int_{0}^{t} t\, dt$$

Derivations by Integration

- After applying limits we will have the following result,

$$\boxed{x - x_0 = v_0 t + \frac{1}{2} a t^2}$$

- Above is the 2nd Equation of motion

Note that in this formulation we have not required that $x = 0$ at $t = 0$ as in the previous algebraic derivations.

Derivations by Integration

For 3rd equation of motion

We may use the chain rule to write

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$\int_{v_0}^{v} v\,dv = a\int_{x_0}^{x} dx$$

$$\frac{v^2}{2} - \frac{v_0^2}{2} = a(x - x_0)$$

$$v^2 - v_0^2 = 2a(x - x_0)$$

Results by Integration

- Following are the results found by integration,

$$v = v_0 + at$$
$$x - x_0 = v_0 t + \frac{1}{2}at^2$$
$$v^2 - v_0^2 = 2a(x - x_0)$$

- All the equations that we have derived for motion are in the x-direction. Similar equations can simply be written for motion in the y and z directions when the components of the acceleration in these directions are also constant.

Constant Acceleration

$$v = v_0 + at,$$

$$x - x_0 = v_0 t + \tfrac{1}{2}at^2,$$

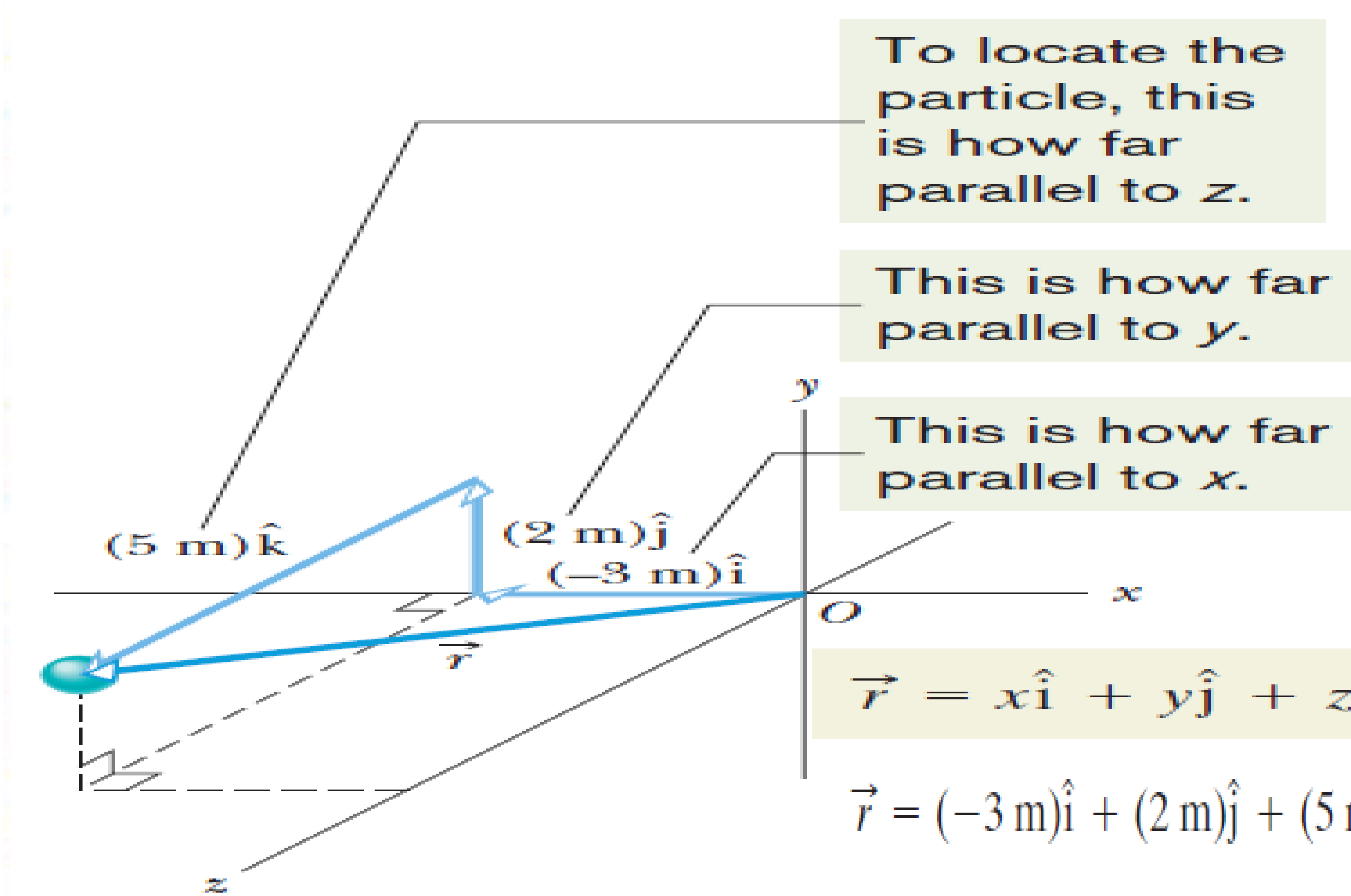
$$v^2 = v_0^2 + 2a(x - x_0),$$

$$x - x_0 = \tfrac{1}{2}(v_0 + v)t,$$

$$x - x_0 = vt - \tfrac{1}{2}at^2.$$

These are *not* valid when the acceleration is not constant.

Position of a Point in Space



$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}},$$

$$\vec{r} = (-3\,\mathrm{m})\hat{\mathrm{i}} + (2\,\mathrm{m})\hat{\mathrm{j}} + (5\,\mathrm{m})\hat{\mathrm{k}}$$

Displacement vector

As a particle moves, its position vector changes in such a way that the vector always extends to the particle from the reference point (the origin). If the position vector changes—say, from $\vec{r}_1$ to $\vec{r}_2$ during a certain time interval—then the particle's **displacement** $\Delta\vec{r}$ during that time interval is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1.$$

Using the unit-vector notation we can rewrite this displacement as

$$\Delta\vec{r} = (x_2\hat{\mathrm{i}} + y_2\hat{\mathrm{j}} + z_2\hat{\mathrm{k}}) - (x_1\hat{\mathrm{i}} + y_1\hat{\mathrm{j}} + z_1\hat{\mathrm{k}})$$

$$\Delta\vec{r} = (x_2 - x_1)\hat{\mathrm{i}} + (y_2 - y_1)\hat{\mathrm{j}} + (z_2 - z_1)\hat{\mathrm{k}},$$

$$\Delta\vec{r} = \Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}.$$

Avg. Velocity in 3-Dimensions

$$\vec{v}_{\text{avg}} = \frac{\Delta\vec{r}}{\Delta t}.$$

$$\Delta\vec{r} = \Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}.$$

Put in the formula of average velocity we'll get

$$\vec{v}_{\text{avg}} = \frac{\Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\mathrm{i}} + \frac{\Delta y}{\Delta t}\hat{\mathrm{j}} + \frac{\Delta z}{\Delta t}\hat{\mathrm{k}}.$$

Instantaneous Velocity in 3-D

When we speak of the **velocity** of a particle, we usually mean the particle's **instantaneous velocity** $\vec{v}$ at some instant. This $\vec{v}$ is the value that $\vec{v}_{avg}$ approaches in the limit as we shrink the time interval Δt to 0 about that instant. Using the language of calculus, we may write $\vec{v}$ as the derivative

$$\vec{v} = \frac{d\vec{r}}{dt}.$$

- Substitute the value of unit vector

$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}},$$

$$\vec{v} = \frac{d}{dt}(x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}) = \frac{dx}{dt}\hat{\mathrm{i}} + \frac{dy}{dt}\hat{\mathrm{j}} + \frac{dz}{dt}\hat{\mathrm{k}}.$$

Instantaneous Velocity in 3-D

- Simply we can write it as;

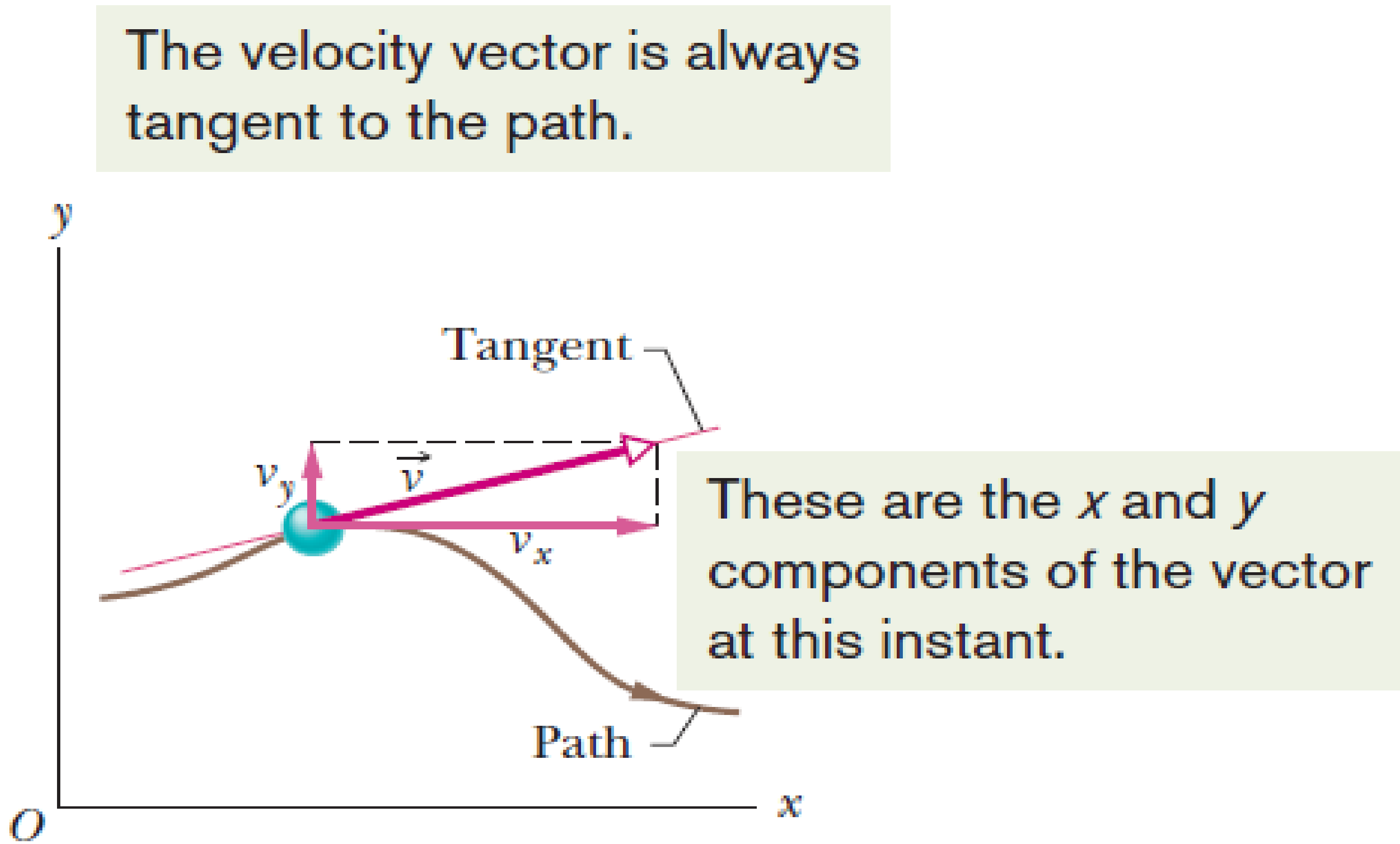
$$\vec{v} = \frac{dx}{dt}\hat{\mathrm{i}} + \frac{dy}{dt}\hat{\mathrm{j}} + \frac{dz}{dt}\hat{\mathrm{k}}.$$

$$\vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}},$$

where the scalar components of $\vec{v}$ are

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad \text{and} \quad v_z = \frac{dz}{dt}.$$

Instantaneous Velocity in 3-D



Solve for Instantaneous acceleration in 3-D

- Class task: Find out the scalar components of acceleration?

instantaneous acceleration

$$\vec{a} = \frac{d\vec{v}}{dt}.$$

$$\vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}},$$

$$\vec{a} = \frac{d}{dt}(v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}})$$

$$= \frac{dv_x}{dt}\hat{\mathrm{i}} + \frac{dv_y}{dt}\hat{\mathrm{j}} + \frac{dv_z}{dt}\hat{\mathrm{k}}$$

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}},$$

where the scalar components of $\vec{a}$ are

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad \text{and} \quad a_z = \frac{dv_z}{dt}.$$

Two Dimensional Motion

Example Problem 1

A rabbit runs across a parking lot on which a set of coordinate axes has, strangely enough, been drawn. The coordinates (meters) of the rabbit's position as functions of time t (seconds) are given by

$$x = -0.31t^2 + 7.2t + 28 \quad \text{——— (i)}$$

and

$$y = 0.22t^2 - 9.1t + 30. \quad \text{——— (ii)}$$

(a) At $t = 15$ s, what is the rabbit's position vector $\vec{r}$ in unit-vector notation and in magnitude-angle notation?

Solution:

The x and y coordinates of the rabbit's position, are the scalar components of the rabbit's position vector $\vec{r}$.

We can write

$$\vec{r}(t) = x(t)\hat{\mathrm{i}} + y(t)\hat{\mathrm{j}}.$$

(We write $\vec{r}(t)$ rather than $\vec{r}$ because the components are functions of t, and thus $\vec{r}$ is also.)

At $t = 15$ s, the scalar components are

$$x = (-0.31)(15)^2 + (7.2)(15) + 28 = 66 \text{ m}$$

and $\quad y = (0.22)(15)^2 - (9.1)(15) + 30 = -57 \text{ m},$

so $\quad \vec{r} = (66 \text{ m})\hat{\mathrm{i}} - (57 \text{ m})\hat{\mathrm{j}},$

Solution:(cont'd)

To get the magnitude and angle of $\vec{r}$, we use

$$r = \sqrt{x^2 + y^2} = \sqrt{(66\text{ m})^2 + (-57\text{ m})^2}$$

$$= 87\text{ m}.$$

and $\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-57\text{ m}}{66\text{ m}}\right) = -41°.$

Two Dimensional Motion

- Example Problem 2

For the rabbit in the preceding Sample Problem, find the velocity $\vec{v}$ at time $t = 15$ s.

- Solution:

We can find $\vec{v}$ by taking derivatives of the components of the rabbit's position vector.

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(-0.31t^2 + 7.2t + 28)$$

$$= -0.62t + 7.2.$$

Two Dimensional Motion

At $t = 15$ s, this gives $v_x = -2.1$ m/s.

Similarly, applying the v_y part

$$v_y = \frac{dy}{dt} = \frac{d}{dt}(0.22t^2 - 9.1t + 30)$$

$$= 0.44t - 9.1.$$

At $t = 15$ s, this gives $v_y = -2.5$ m/s.

$$\boxed{\vec{v} = (-2.1\ \text{m/s})\hat{\text{i}} + (-2.5\ \text{m/s})\hat{\text{j}}.}$$

Two Dimensional Motion

- Example Problem 3

For the rabbit in the preceding two Sample Problems, find the acceleration $\vec{a}$ at time $t = 15$ s.

- Solution: Class Task

Tasks for Assignment

Assignment

Here are four descriptions of the position (in meters) of a puck as it moves in an xy plane:

(1) $x = -3t^2 + 4t - 2$ and $y = 6t^2 - 4t$ (3) $\vec{r} = 2t^2\hat{\text{i}} - (4t + 3)\hat{\text{j}}$

(2) $x = -3t^3 - 4t$ and $y = -5t^2 + 6$ (4) $\vec{r} = (4t^3 - 2t)\hat{\text{i}} + 3\hat{\text{j}}$

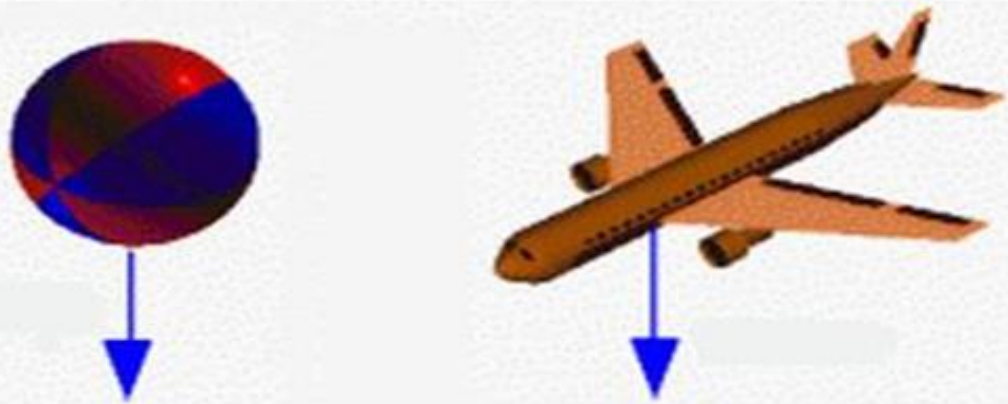
Are the x and y acceleration components constant? Is acceleration $\vec{a}$ constant?

Free Fall Motion/Acceleration

- There is one important thing to be noted here. In the solution of motion problems we must assign vector directions.
- If you tossed an object either up or down and could somehow eliminate the effects of air on its flight, you would find that the object accelerates downward at a certain constant rate.
- That rate is called the **free-fall acceleration, and its magnitude** is represented by **g**.
- The value of g varies slightly with latitude and with elevation.
- At sea level in Earth's mid latitudes the value is 9.8 m/s^2

Conceptual View

Motion of Free Falling Object
(no air resistance)



Mass and shape of object does not affect the motion.

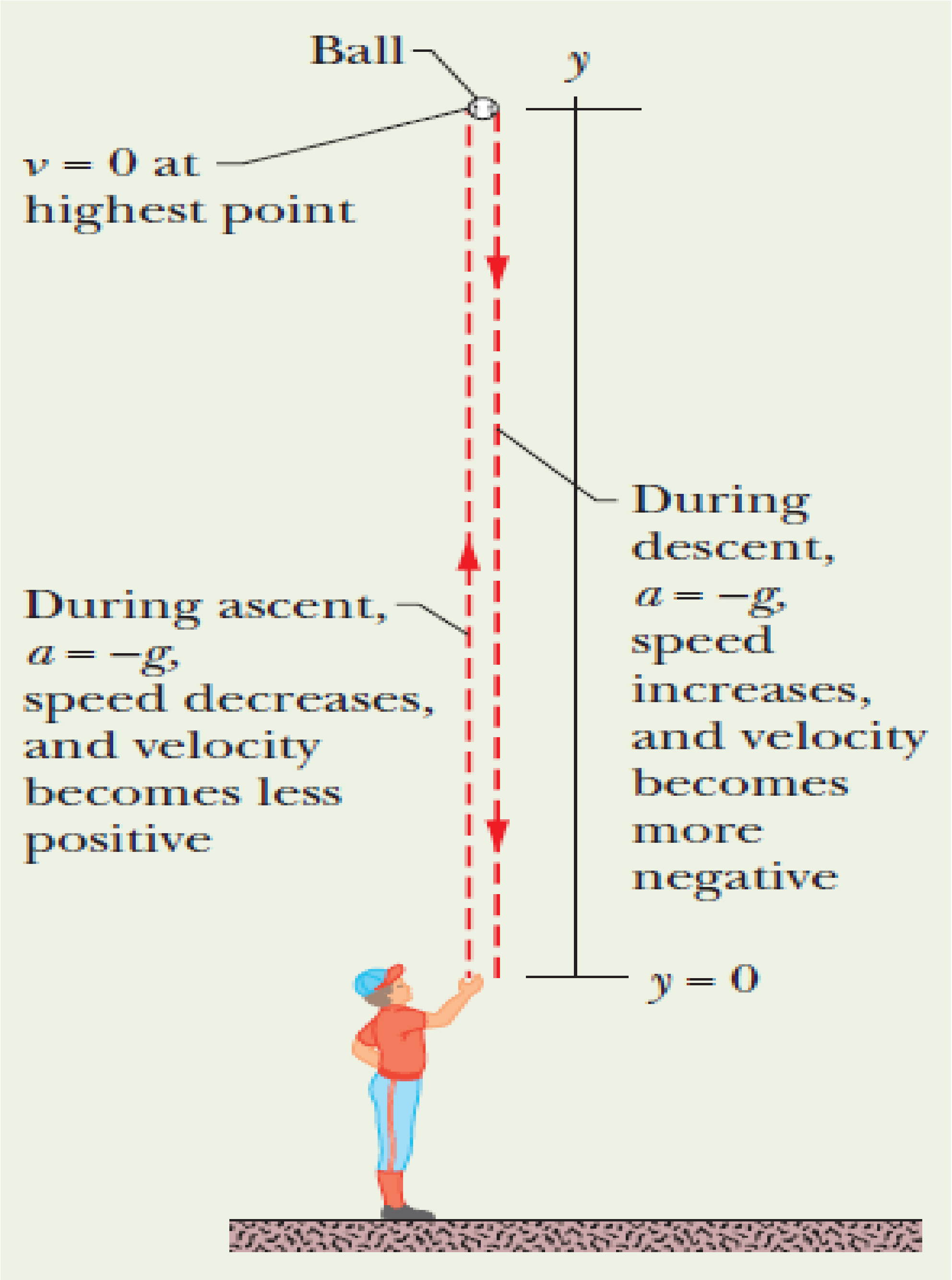
All objects fall at the same rate in a vacuum. -- Galileo.

Time – sec.	0	1	2	3	4	5	6	7	8
Accel – m/sec^2	9.8	9.8	9.8	9.8	9.8	9.8	9.8	9.8	9.8
Velocity – m/sec	0	9.8	19.6	29.4	39.2	49.0	58.8	68.6	78.4
Dist – meters	0	4.9	19.6	44.1	78.4	122.5	176.4	240.1	313.6

Direction of Velocity in free fall

- In order to work with free fall problems choosing a particular coordinate system is a matter of personal convenience
- Consider a boy throwing a ball vertically upwards then we will take the motion of ball is along "y-axis" and we will take the upward motion of ball as positive.
- It is important to note that once we choose a coordinate system, all parameters have their vector direction controlled by it.

The equations of free fall apply for rising as well as for falling objects, provided any effects from the air can be neglected



Direction of Velocity in free fall

- If we choose the positive *y* direction as up and the boy throws the ball straight up, then the vector displacement from the ground to its highest position is positive.
- During its upward travel, because velocity is the displacement divided by the scalar time, it too is positive.
- The only motion is in the “y direction”, so we therefore use y, v_y, a_y in the equations previously derived

$$\boxed{v_y = v_{0y} + a_y t}$$

Conditions of “g” when velocity (of free fall) is max or min

- Condition 1:When velocity is maximum

  We observe that in throwing the ball upward the largest value for the magnitude of the y velocity occurs as it leaves the boy's hand. For which the value of g should be positive.

- Condition 2:When velocity is minimum

  It is when the ball reaches to highest point therefore the acceleration due to gravity should be minimum or negative.

  a_y is the acceleration caused by the force of gravity acting on the ball

Solved Example 1

EXAMPLE A boy throws a ball upward with an initial velocity of 12 m/sec. How high does it go?

Solution We choose the starting point as the origin and the upward direction as positive. Because velocity is a vector displacement divided by time, upward velocity is also positive. The force of gravity is in the negative y direction, so the sign of the acceleration is therefore negative. First list what is known and what is to be found

$$v_{0y} = 12\,\text{m/sec}, \quad v_y = 0 \text{ (at its highest point)}, \quad a_y = g = -9.8\,\text{m/sec}^2$$

$$y = ?$$

We select the 3rd equation of motion because all the quantities in that equation are known except y, the quantity that we want to find

$$v_y^2 - v_{0y}^2 = 2a_y y$$

Solved Example(cont'd)

Solving for y, we write

$$y = \frac{v_y^2 - v_{0y}^2}{2a_y}$$

Substituting the numerical values for the quantities in the equation,

$$y = \frac{0 - (12\,\text{m/sec})^2}{2(-9.8\,\text{m/sec}^2)}$$

$$= 7.3\,\text{m}$$

Solved Example 2

EXAMPLE A boy throws a ball upward with an initial velocity of 12 m/sec and catches it when it returns. How long was it in the air?

Solution As in the previous example, we choose the starting point as the origin and the upward direction as positive.

$$v_{0y} = 12\,\text{m/sec}, \quad a_y = -9.8\,\text{m/sec}^2, \quad y = 0 \text{ (vector displacement is zero}$$

$$\text{because it returns to his hand)}, \; t = ?$$

Select 2nd equation of motion

$$y = v_{0y}t + \frac{1}{2}a_y t^2 \qquad \text{..........eq. 1}$$

Using the fact that $y = 0$, Eq. 1 becomes

$$0 = v_{0y}t + \frac{1}{2}a_y t^2$$

We see immediately that if we divide both sides of the equation by t, we obtain

$$0 = v_{0y} + \frac{1}{2}a_y t$$

Solved Example(cont'd)

$$t = -\frac{2\,v_{0y}}{a_y}$$

$$= -\frac{2 \times 12\,\mathrm{m/sec}}{-9.8\,\mathrm{m/sec^2}}$$

$$= 2.45\,\mathrm{sec}$$

Note: If the ball had landed on a roof, then the left side of Eq. 1 would not be zero and the equation to be solved would be quadratic.