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Analytical and Numerical Solution of Ground Water Flow Equation

By

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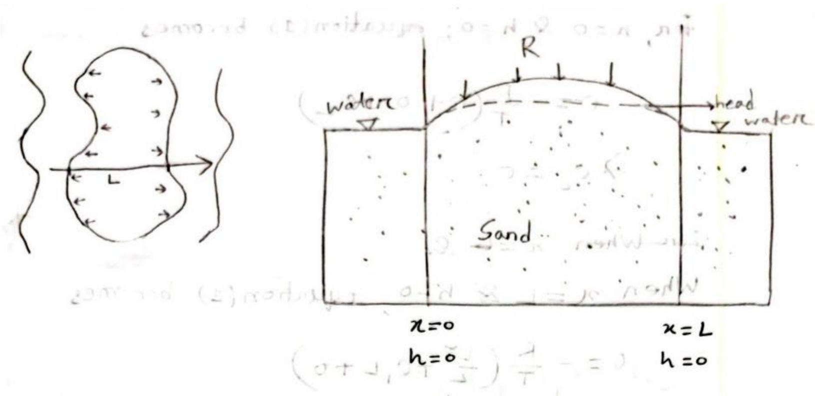


Introduction

This study investigates the behavior of groundwater flow within a confined island aquifer. By analytically solving the island problem, we aim to understand how aquifer properties and recharge rates influence hydraulic head distribution. Our focus is on the scenario where head reaches its maximum at the island's center and gradually decreases towards the periphery.

To model this system, we employ Darcy's Law, a fundamental principle governing groundwater flow through porous media. By making simplifying assumptions about the aquifer, we derive an analytical solution that provides insights into the system's behavior under various conditions.

Assumptions

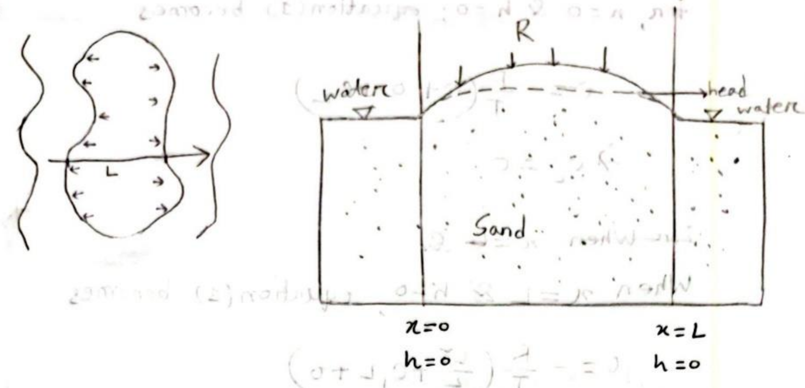
To simplify our analysis, we make the following assumptions:

1. **Confined Aquifer:** The aquifer is completely enclosed by impermeable layers, preventing water from entering or leaving at the top or bottom. This allows us to assume linear flow and apply Darcy's Law directly.
2. **Steady-State Conditions:** The groundwater flow has reached a stable state, meaning that the water levels are not changing over time. This simplifies our calculations by eliminating time-dependent factors.
3. **Radial Symmetry:** The island is perfectly circular, and groundwater flows outward from the center in a radial pattern. This simplifies the problem by allowing us to focus on the radial distance from the center.

4. **Uniform Recharge:** Rainwater seeps into the aquifer at a constant rate across the entire island. This ensures a consistent and predictable water supply.
5. **Homogeneous Aquifer:** The aquifer has the same properties (like permeability) throughout, meaning that water flows equally well in all directions.
6. **Horizontal Flow:** We assume that groundwater flows primarily horizontally, neglecting any vertical movement. This is reasonable for a relatively thin aquifer.

1) Solve the island problem analytically and plot the calculated head over distance for the given base case parameter value of T = 2000 m2/d, L=5km, R= 0.002 m/d.

1)



This is one dimensional steady-state condition, where

$$\frac{dh}{dt} = 0$$

So, the equation can be written like following

$$T\frac{d^2h}{dx^2} + R = 0$$

$$\Rightarrow T\frac{d^2h}{dx^2} = -R$$

$$\Rightarrow d^2h = -\frac{R}{T}dx^2$$

$$\Rightarrow \int d\int dh = -\frac{R}{T}\int dx \int dx$$

$$\Rightarrow \int dh = -\frac{R}{T}\int (x+c_1)\,dx$$

$$\Rightarrow h = -\frac{R}{T}\left(\frac{x^2}{2} + c_1x + c_2\right) \quad \text{——} (1)$$

For, $x=0$ & $h=0$; equation(1) becomes

$$0=-\frac{R}{T}(0+0+c_2)$$

$$\Rightarrow c_2=0$$

~~For When $x=L$ &~~

When $x=L$ & $h=0$, equation(1) becomes

$$0=-\frac{R}{T}\left(\frac{L^2}{2}+c_1L+0\right)$$

$$\Rightarrow \frac{L^2}{2}+c_1L=0$$

$$\Rightarrow \frac{L^2}{2}=-c_1L$$

$$\Rightarrow c_1=\frac{L}{2}$$

Using the value of c_1 & c_2, equation 1 could be written as;

$$h=-\frac{R}{T}\left(\frac{x^2}{2}-\frac{Lx}{2}\right)$$

$$\Rightarrow h=-\frac{R}{T}\left(\frac{x^2-Lx}{2}\right)$$

$$\Rightarrow h=-\frac{R}{2T}(x^2-Lx) \quad \text{--- (11)}$$

1. Base Case Analysis

For the given parameters:

- T (Transmissivity) = 2000 m^2/d
- L (Distance from center to island boundary) = 5 km
- R (Recharge rate) = 0.002 m/d

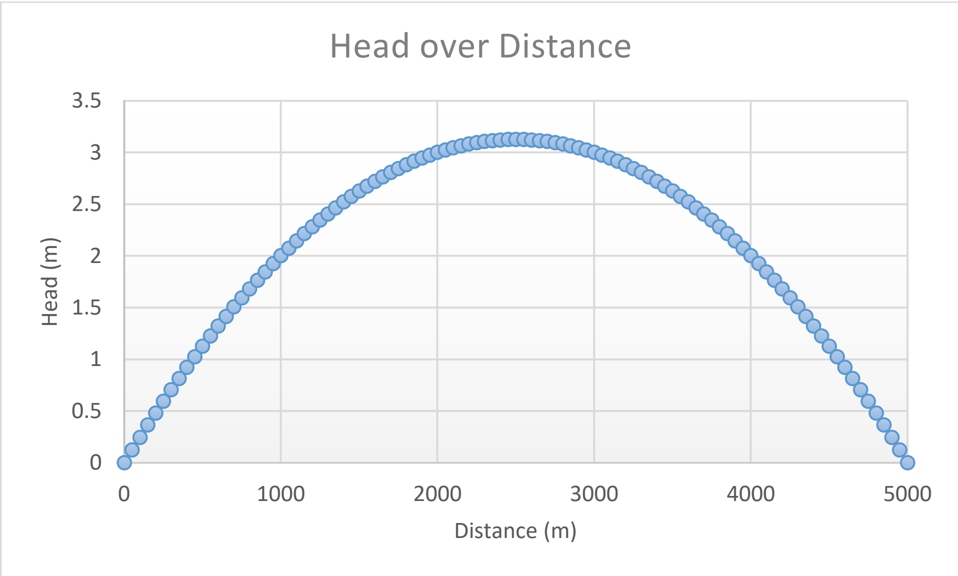


Figure 1: Head over Distance

As depicted in the graph, the hydraulic head reaches its maximum near the center of the island (around 2500 meters) before gradually decreasing towards the edges. This observed pattern aligns with our theoretical expectations, confirming that the uniform recharge leads to a central peak in head distribution.

2. Sensitivity Analysis

Transmissivity	T	2000	m^2/d
Radial Distance	L	5000	m
Discharge	R	0.002	m/d

For Sensitivity Analysis				
Transmissivity	T	200	20000	m^2/d
Radial Distance	L	500	50000	m

Discharge	R	0.02	0.0002	m/d

2.1 Sensitivity analysis based on Transmissivity

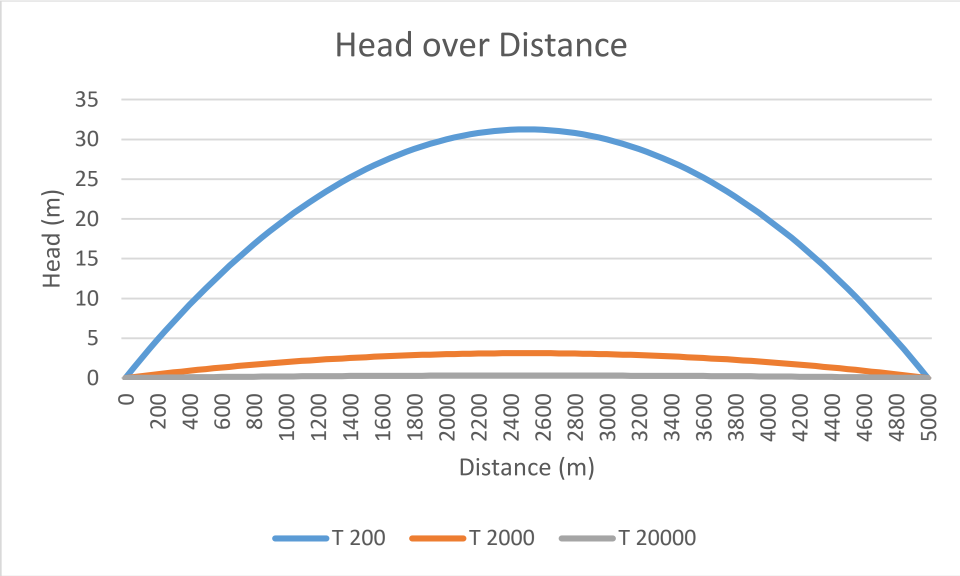


Figure 2: Sensitivity analysis of the head whenever the transmissivity (T) is varied by an order of magnitude

The plot shows three different curves for different values of T (Transmissivity):

The blue curve for T = 200 m²/day, the orange curve for T = 2000 m²/day, the gray curve for T = 20,000 m²/day.

Transmissivity significantly influences groundwater head distribution. Aquifers with high transmissivity allow water to move rapidly, resulting in lower head levels. Conversely, aquifers with low transmissivity hinder water movement, leading to higher head buildup. This relationship was demonstrated in a sensitivity analysis where transmissivity was varied by an order of magnitude. The findings revealed that lower transmissivity leads to increased head accumulation and steeper gradients, while higher transmissivity results in flatter gradients and minimal head buildup. Understanding the impact of transmissivity is crucial for effective groundwater management and prediction of aquifer behavior.

2.2 Sensitivity analysis based on Discharge

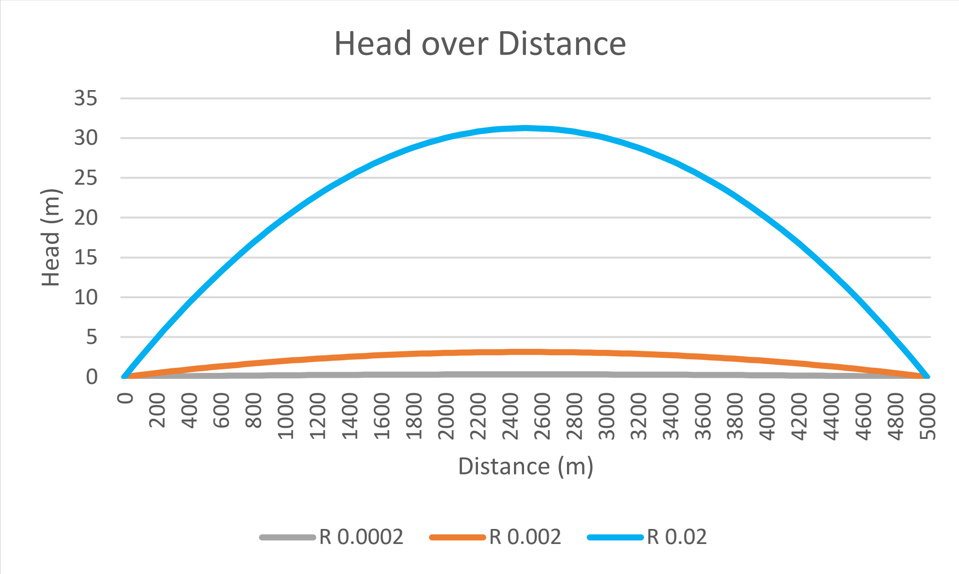


Figure 3 Sensitivity analysis of the head whenever the Discharge (R) is varied by an order of magnitude

The plot contains three curves for different values of R (Recharge rate):

The blue curve for R = 0.02 m/day, the orange curve for R = 0.002 m/day, the gray curve for R = 0.0002 m/day.

Recharge rate plays a pivotal role in determining hydraulic head distribution within an island aquifer. By varying the recharge rate by an order of magnitude, we observed significant changes in head levels. Higher recharge rates led to increased water accumulation, resulting in higher peak heads and steeper gradients. Conversely, lower recharge rates resulted in lower head levels and flatter gradients. These findings highlight the importance of understanding recharge patterns for effective groundwater management and prediction of aquifer behavior in response to varying hydrological conditions.

2.3 Sensitivity analysis based on Length

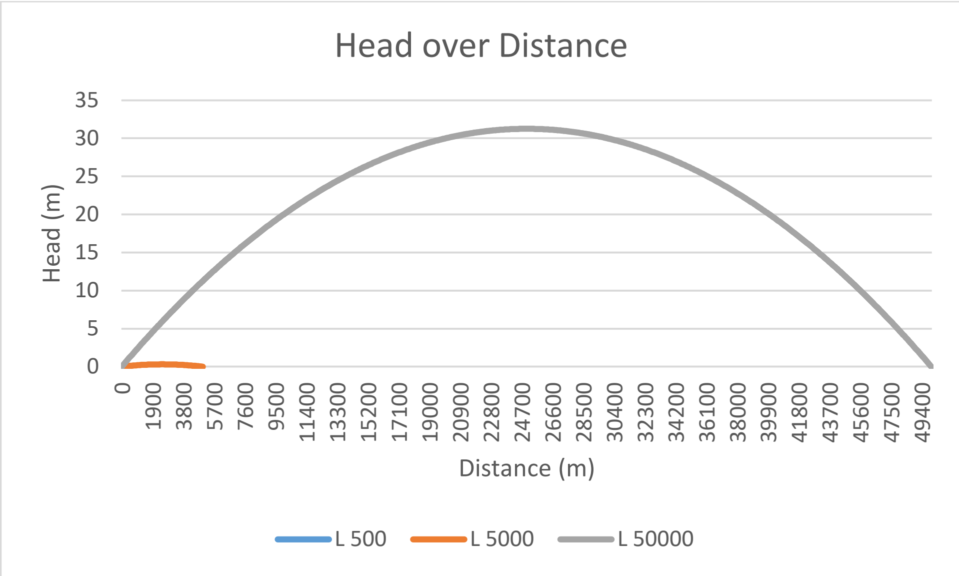


Figure 4: Changes of Head with respect to Length of the Island

The graph contains three curves, each representing different values of L (Length of the island): The blue curve for L = 500 meters, the orange curve for L = 5000 meters, and the gray curve for L

= 50,000 meters.

Island length plays a crucial role in determining hydraulic head distribution within a confined aquifer. As the island size increases, the peak head levels rise significantly, while the gradients become broader and flatter. This is because larger islands provide more space for water to accumulate, leading to higher head buildup. Conversely, smaller islands exhibit lower peak heads and steeper gradients due to limited space for water accumulation. Understanding the relationship between island length and hydraulic head is essential for effective groundwater management and prediction of aquifer behavior in various island environments.

3. Analysis using excel grid spacing of 500 m

3. From 1-D steady-state ~~eqn~~ equation,

We know; $\dfrac{d^2h}{dx^2} = -\dfrac{R}{T}$

$$\Rightarrow \frac{h_{i+1} - 2h_i + h_{i-1}}{dx^2} = -\frac{R}{T}$$

$$\Rightarrow h_{i+1} - 2h_i + h_{i-1} = -\frac{R}{T}dx^2 \quad \text{(iii)}$$

h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}
0										0

We have to determine the head values from h_2 to h_9.

From, equ^n(3); For h_1,

$$h_2 - 2h_1 + 0 = -\frac{R}{T}dx^2$$

$$\Rightarrow -2h_1 + h_2 = -\frac{Rdx^2}{T}$$

For h_2, $\quad h_3 - 2h_2 + h_1 = -\dfrac{Rdx^2}{T}$

$$\Rightarrow h - 2h_2 + h_3 = -\frac{Rdx^2}{T}$$

For, h_3, $\quad h_4 - 2h_3 + h_2 = -\dfrac{Rdx^2}{T}$

$$\Rightarrow h_2 - 2h_3 + h_4 = -\frac{Rdx^2}{T}$$

For h_4; $h_3 - 2h_4 + h_5 = -\frac{R\,dx^2}{T}$

h_5; $h_4 - 2h_5 + h_6 = -\frac{R\,dx^2}{T}$

h_6; $h_5 - 2h_6 + h_7 = -\frac{R\,dx^2}{T}$

h_7; $h_6 - 2h_7 + h_8 = -\frac{R\,dx^2}{T}$

h_8; $h_7 - 2h_8 + h_9 = -\frac{R\,dx^2}{T}$

h_9; $h_8 - 2h_9 = -\frac{R\,dx^2}{T}$

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \times \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = \begin{bmatrix} -\frac{R\,dx^2}{T} \\ -\frac{R\,dx^2}{T} \\ -\frac{R\,dx^2}{T} \\ -\frac{R\,dx^2}{T} \\ -\frac{R\,dx^2}{T} \\ -\frac{R\,dx^2}{T} \\ -\frac{R\,dx^2}{T} \\ -\frac{R\,dx^2}{T} \\ -\frac{R\,dx^2}{T} \end{bmatrix}$$

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}^{-1} \times \begin{bmatrix} -\frac{Rdx^2}{T} \\ -\frac{Rdx^2}{T} \\ -\frac{Rdx^2}{T} \\ -\frac{Rdx^2}{T} \\ -\frac{Rdx^2}{T} \\ -\frac{Rdx^2}{T} \\ -\frac{Rdx^2}{T} \\ -\frac{Rdx^2}{T} \\ -\frac{Rdx^2}{T} \end{bmatrix}$$

Inverse Matrix								
-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
-0.8	-1.6	-1.4	-1.2	-1	-0.8	-0.6	-0.4	-0.2
-0.7	-1.4	-2.1	-1.8	-1.5	-1.2	-0.9	-0.6	-0.3
-0.6	-1.2	-1.8	-2.4	-2	-1.6	-1.2	-0.8	-0.4
-0.5	-1	-1.5	-2	-2.5	-2	-1.5	-1	-0.5
-0.4	-0.8	-1.2	-1.6	-2	-2.4	-1.8	-1.2	-0.6
-0.3	-0.6	-0.9	-1.2	-1.5	-1.8	-2.1	-1.4	-0.7
-0.2	-0.4	-0.6	-0.8	-1	-1.2	-1.4	-1.6	-0.8
-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9

Right side	Distance	Head
-0.25	500	1.125
-0.25	1000	2
-0.25	1500	2.625
-0.25	2000	3
-0.25	2500	3.125
-0.25	3000	3
-0.25	3500	2.625
-0.25	4000	2
-0.25	4500	1.125

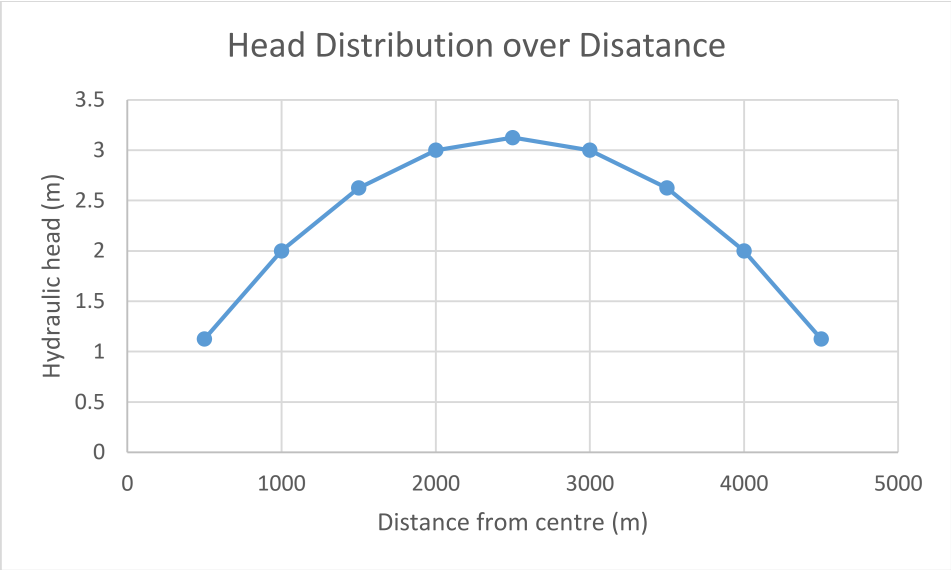


Figure 5: Head Distribution with Grid Spacing of 500 m

4. Analysis using R grid spacing 50 m

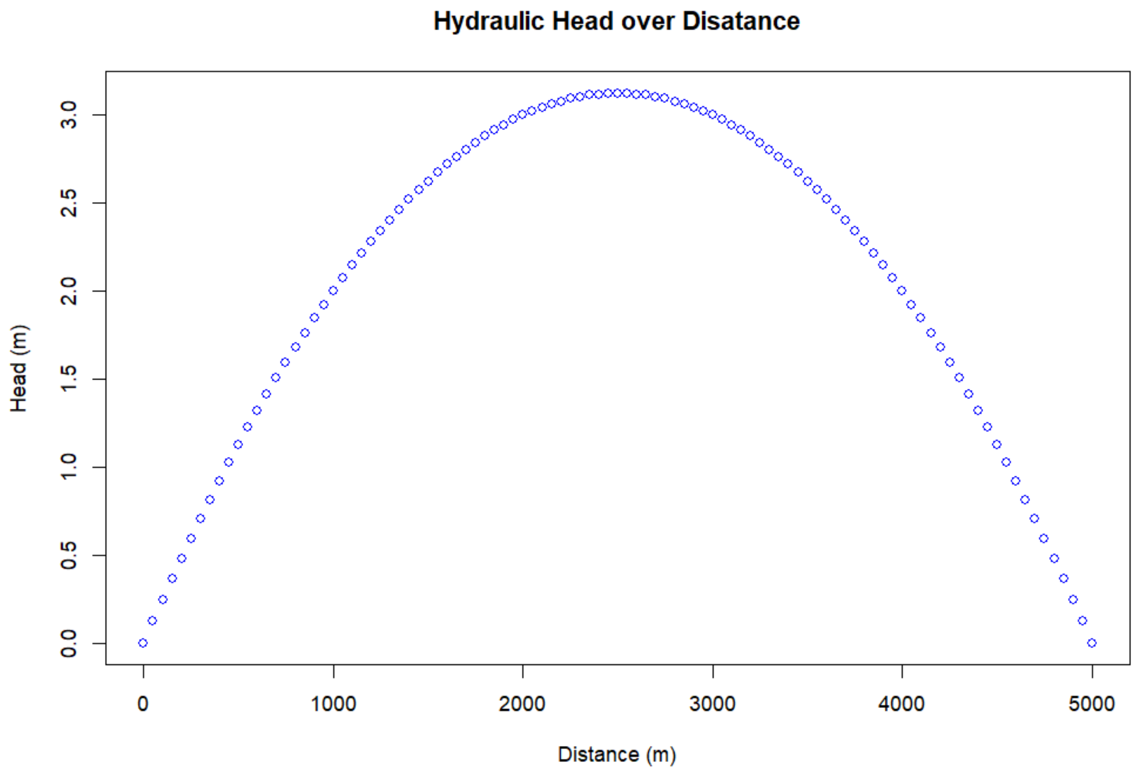


Figure 6: Hydraulic Head with Grid Spacing 50 m

By reducing the grid spacing from 500 meters to 50 meters, we significantly improved the accuracy and smoothness of our groundwater flow simulations. This finer grid allows for a more detailed representation of hydraulic head variations, leading to more reliable predictions of groundwater behavior. While requiring more computational resources, the increased accuracy justifies the additional effort.

Conclusion

The parabolic water table profile, characteristic of symmetric islands with uniform rainfall, is influenced by transmissivity and recharge rate. Understanding this relationship is essential for sustainable groundwater management on islands, guiding water extraction, contamination assessment, climate change adaptation, and artificial recharge planning.