Problem 1

$$a_n = 3a_{n-1} + 2^n.$$

$$a_n = -2^{n+1}, \quad a_0 = 1$$

a)

Step 1:

$$a_{(n-1)} \stackrel{?}{=} -2^{n-1+1} = -2^n$$

Step 2:

$$a_n = a_n$$

$$-2^{n+1} \stackrel{?}{=} 3 \cdot a_{n-1} + 2^n$$

$$-2^{n+1} \stackrel{?}{=} 3 \cdot (-2^n) + 2^n$$

$$-2^{n+1} \stackrel{?}{=} -2^n(3-1)$$

$$\underline{-2^{n+1}} \stackrel{?}{=} -2^n \cdot 2$$

$$-2^{n+1} = -2^{n+1} \quad \checkmark$$

Yes, It is a solution.

b)

Step 1: Homogeneous part.

$a_n(h) = 3a_{n-1}$

$a_n = r$

$a_{n-1} = 1$

$r = 3$

$a_n^{(h)} = c \cdot 3^n$

Step 2: Particular part.

$a_n^{(p)} = A \cdot 2^n$

$a_{n-1} = A \cdot 2^{n-1}$

$A.2^n = 3 . (A . 2^{n-1}) + 2^n$

$A . \cancel{2^n} = \dfrac{3A . \cancel{2^n}}{2} + 2^n$

$A = \dfrac{3A}{2} + 1$

$A - \dfrac{3A}{2} = 1$

$A = -2$

$a_n^{(p)} = -2 . 2^n$

Step 3:

$a_n = c . 3^n - 2 . 2^n$

$a_0 = 1 = c . 3^0 - 2 . 2^0$

$1 = c - 2$

$c = 3$

Result:

$a_n = 3 . 3^n - 2 . 2^n$

Problem 2

$f(n) = 4f(n-1) - 4f(n-2) + n^2$, $f(0) = 2$, $f(1) = 5$

$f(n) = f(n)^{(h)} + f(n)^{(p)}$

Step 1: Homogeneous part:

$f(n)^{(h)} = 4f(n-1) - 4f(n-2)$

$f(n) = r^2$

$f(n-1) = r$ $\Rightarrow$ $r^2 - 4r + 4 = 0 \Rightarrow r_1 = r_2 = 2$

$f(n-2) = 1$

$f(n)^{(h)} = c_1 \cdot 2^n + c_2 \cdot 2^n \cdot n$

Step 2: Particular Part

$f(n) = An^2 + Bn + C$

$f(n-1) = A(n-1)^2 + B(n-1) + C = An^2 - 2An + 1 + Bn - B + C$

$f(n-2) = A(n-2)^2 + B(n-2) + C = An^2 - 4An + 4 + Bn - 2B + C$

$An^2 + Bn + C = 4(An^2 - 2An + 1 + Bn - B + C) - 4(An^2 - 4An + 4 + Bn - 2B + C) + n^2$

$An^2 + Bn + C = 4An^2 - 8An + 4 + 4Bn - 4B \cancel{+4C} - 4An^2 + 16An - 16 - 4Bn + 8B \cancel{-4C} + n^2$

$An^2 + Bn + C = 8An + 4B - 12 + n^2$

$An^2 + Bn + C - 8An - 4B + 12 = n^2$

$An^2 + (B - 8A)n + (C - 4B + 12) = n^2$

$A = 1$

$B - 8A = 0$

$B = 8$

$C - 4B + 12 = 0$

$C = 20$

Step 3

$$f(n) = c_1 . 2^n + c_2 . 2^n . n + n^2 + 8n + 20$$

$$f(0) = 2 = c_1 . 2^0 + c_2 . 2^0 . 0 + 0^2 + 8.0 + 20$$

$$2 = c_1 + 20$$

$$c_1 = -18$$

$$f(1) = 5 = -18 . 2^1 + c_2 . 2^1 . 1 + 1^2 + 8.1 + 20$$

$$5 = 2c_2 + 11$$

$$c_2 = -3$$

Result:

$$f(n) = -18 . 2^n - 3 . 2^n . n + n^2 + 8n + 20$$

Problem 3

$a_n = 2a_{n-1} - 2a_{n-2}$ $\rightarrow$ It is homogeneous.

Step 1: $a_n = r^2$

$a_{n-1} = r$

$a_{n-2} = 1$

$\Rightarrow r^2 = 2r - 2$

$r^2 - 2r + 2 = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4.1.2}}{2.1}$$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} \Rightarrow \begin{array}{l} r_1 = 1 + i \\ r_2 = 1 - i \end{array}$$

$\rightarrow$ Characteristic Roots

$\rightarrow$ Part b, is not available.