Domino Tiling problem

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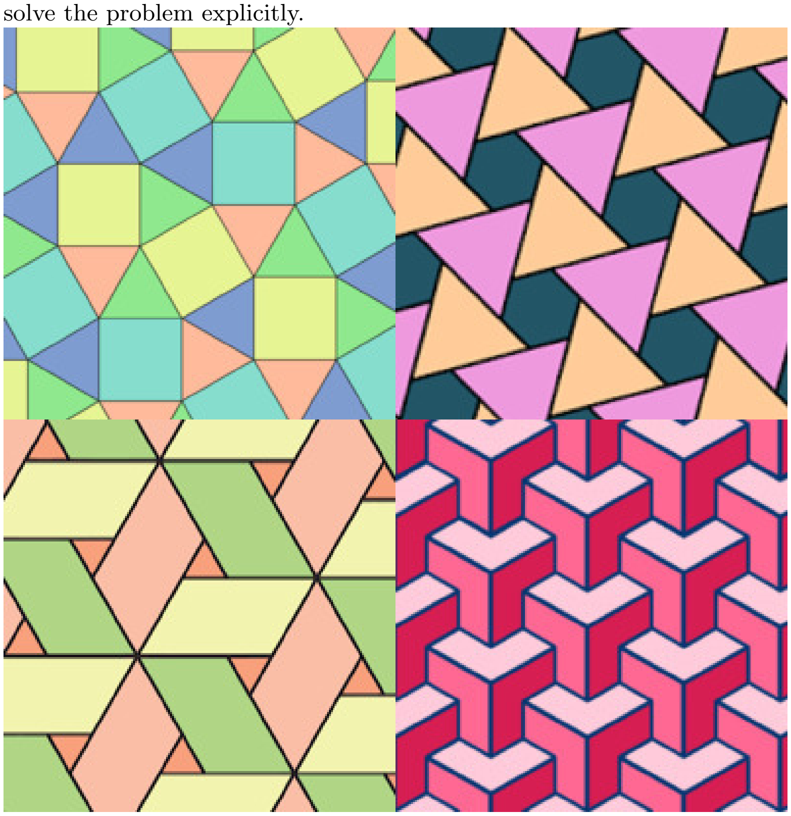
Abstract

Consider than an 8×8 chessboard has 64 squares. Suppose we have 32 dominos, each of which is large enough to cover a 2×1 region of our chessboard. How many different ways are there of tiling the chessboard with these dominos? More generally, How many different ways are there of tiling an $m \times n$ chessboard? In this report we study some basic bounds on the problem, and then introduce a method called Kasteleyn theory to solve the problem explicitly.



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1 Introduction and overview

1.1 History of the problem

For thousands of years, humans have been interested in tesselating space using shapes. As early as 4000 BC, the Sumerians were using tesselating shapes for wall decorations . Decorative mosaic tilings made of small squared blocks called tesserae were widely employed in classical antiquity, sometimes displaying geometric patterns.

Figure 1: A temple mosaic from the ancient Sumerian city of Uruk [1]

Later on it would most commonly associated with Muslim civilisations, this was in part due to the preference to not use people or animals in art.

Figure 2: Zellige terracotta tiles in Marrakech [2]

In the report at hand we will focus on a simple problem that nonetheless has a rich mathematical structure, namely, that of domino tilings. As in the abstract, consider that an 8×8 chessboard has 64 squares. Suppose we have 32 dominos, each of which is large enough to cover a 2×1 region of our chessboard. How many different ways are there of tiling the chessboard with these dominos? More generally, how many different ways are there of tiling an $m \times n$ chessboard? The topic of this report is to tackle this difficult combinatorial problem.

1.2 Further Definitions

Consider the following problem. Take an $m \times n$ chessboard and consider tiling the chessboard with 2×1 dominos. How many different ways are there of doing this? We consider all of the dominos all of the dominos to be indistinguishable from one another.

1. Dominos must be placed either horizontally or vertically.
2. Dominos must cover the whole grid to be a valid tiling pattern.
3. Dominos must fit on the board.
4. Dominos are fungible.

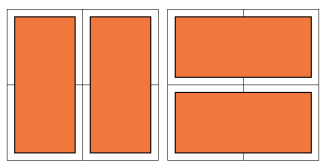


Figure 3: There are 2 valid domino tilings for a 2 by 2 grid

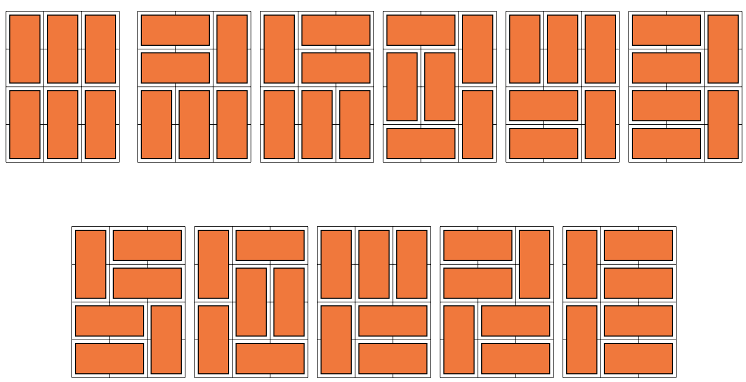


Figure 4: There are 11 valid domino tilings for a 3 by 4 grid

1.3 Our success Criteria

We will also be interested in taking a computational perspective on the domino tiling problem. This means, we are interested in efficient algorithms for computing the number of domino tilings. Given an $n \times n$ chessboard, we will show that the number of tilings grows exponentially in n^2, the number of squares. We will also see later that there is an explicit formula to compute the number of tilings. For any method of counting domino tilings to be it needs to fulfil two criteria.

- It should work for any m by n grid size regardless of shape, eg(It can't just work for $2 \times m$ grids).
- It should be efficient and work for large grids in a reasonable amount of time.

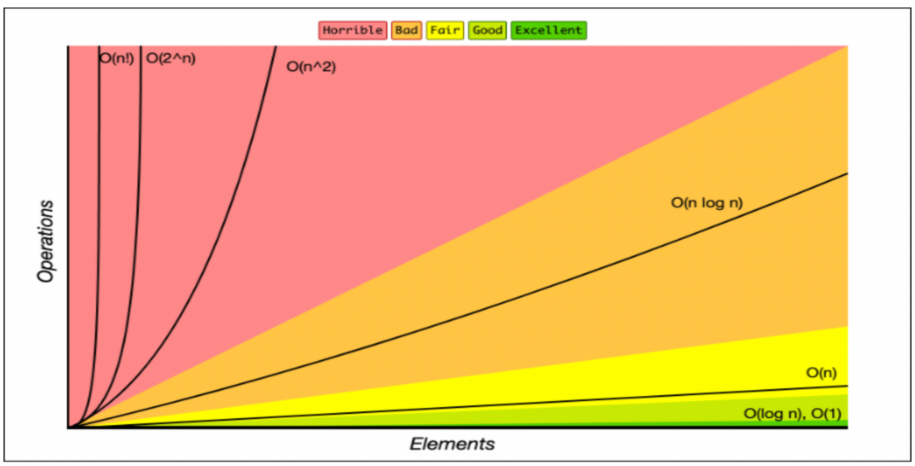


Figure 5: Shows the computational efficiency of algorithms [3]

1.4 Overview of the remainder of the article

The remainder of the article is structured as follows. In Section 2 we make some basic observations and prove some upper and lower bounds for the number of tilings. We also show that the number of domino tilings of a $2 \times n$ chessboard is intimately related to the Fibonacci numbers. In Section 3 we find out how to calculate the number of domino tilings explicitly. Finally in Section 4 we discuss problems we could try solve e.g.(Tetris, or even 3D Tetris) and possible real life applications of our findings.

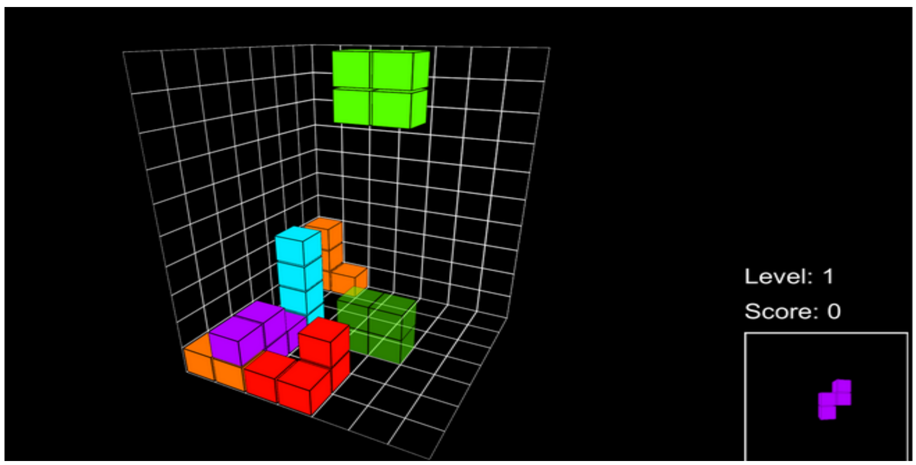


Figure 6: Shows possible further use in 3D Tetris[4]

2 Basic Observations and bounds

2.1 Basic Observations

Let $H_{m,n}$ = Number of valid domino tilings for a m by n grid where $m, n \in \mathbb{N}$.

Each domino is a 2 by 1 meaning it covers an area of 2, and the area of the grid is m*n. Since Rule 2 tells us that the grid must be entirely covered to be a valid domino tiling. It therefore follows that any m by n grid requires $\frac{m*n}{2}$ dominos to be valid. Since Rule 3 tells us that dominos must fit on the board it means that you can only have a whole number of dominos on a board.

Meaning $H_{m,n} = 1$ if both m and n are odd.

There is a bijection with any valid domino tiling pattern in a $H_{m,n}$ and $H_{n,m}$.
You can simply rotate any valid domino tiling from a $H_{m,n}$ to produce a valid one for a $H_{n,m}$ and vice versa.

Meaning $H_{m,n} = H_{n,m}$ For all values of n and m.

Proof. **Lemma 2.1.** Thus $H_{m,n} \geq H_{m_1,n_1} * H_{m_2,n_2} * H_{m_3,n_3} ... H_{m_K,n_K}$.

Where H_{m_i,n_j} are sub grids that together can form $H_{m,n}$.

This is because each of the smaller grids is self contained which means that the way one sub grid is tiled does not effect the others.
The large grid can have tiling patterns that don't follow these self contained rules, meaning it can have more tiling patterns.

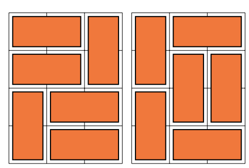


Figure 7: Shows two valid 3 by 4 domino tiling grids

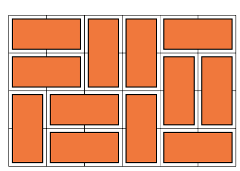


Figure 8: Shows that they can form a 6 by 4 grid

□

2.2 Finding the number of tilings for 2 by n

In this section we explore the problem of counting the number of domino tilings of a 2×n chessboard, and see that these are intimately connected with the Fibonacci numbers.

Proof.

Lemma 2.2. For a grid in the form $H_{2,n} = F_{n+1}$.

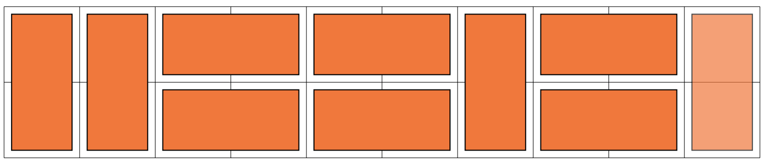


Figure 9: Here we can see that for any valid tiling for a $H_{2,n-1}$ we can create a valid tiling for a $H_{2,n}$ simply by a adding a vertical domino.

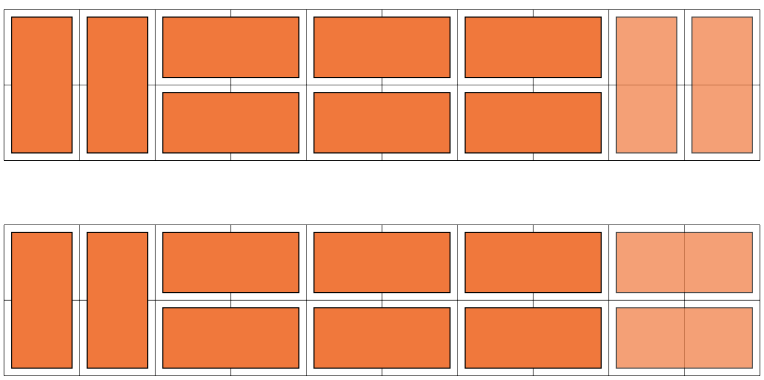


Figure 10: Here we can see that for any valid tiling for a $H_{2,n-2}$ we can create a valid tiling for a $H_{2,n}$ in two ways

However the first one ,also forms a valid tiling for a $H_{2,n-1}$ which means we would be double counting by including it.

$$\text{This gives us } H_{2,n} = H_{2,n-1} + H_{2,n-2}$$

Which is equivalent to the Fibonacci sequence. □

2.3 Brute force method of counting tilings

We can display every domino tiling pattern as a series of 1's, 2's and 0s each corresponding to an individual grid square, which we will refer to as a grid value. A 0 means that no domino that starts there for it to be occupied a domino from an adjacent tile would need to enter it.
A 1 means a vertical domino facing downwards starts in that square.
A 2 means a horizontal domino facing leftwards starts in that square.
This is because a vertical domino facing downwards and a vertical domino facing upwards covering the same spaces are interchangeable, with it also being true for leftward and rightward domino's.

If we go through every possible grid value we would have 3^{m*n} combinations for a $H_{m,n}$.
We can verify if each grid value forms a valid domino tiling, by checking if it passes the three requirements, discussed in section 1.2.

1	2	0	1
0	1	1	0
1	0	0	1
0	2	0	0

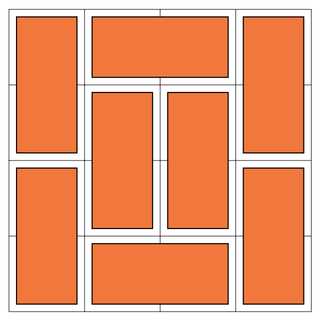


Figure 11: Shows how a 4 by 4 domino tiling would get represented

1	2	0	1
0	1	1	0
1	0	0	1
2	0	0	0

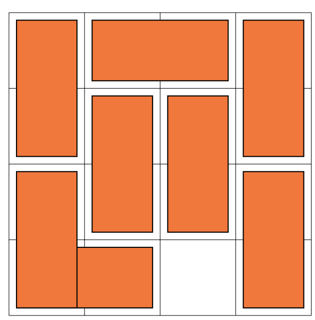


Figure 12: Shows that not every grid tiling forms a valid domino tiling

Every domino tiling has a unique grid tiling associated with it,but not every grid tiling creates a valid domino tiling.

2.4 What can the brute force method shows us

Proof.

Lemma 2.4 For a grid $H_{m,n} \leq 3^{(n-1)*(m-1)-1} * 2^{n+m}$.

Since there is only 3^{m*n} combinations for a $H_{m,n}$, and every grid tiling can form at most one valid domino tiling pattern, if we assume every single grid value formed a valid domino tiling.

$$\text{That would mean } H_{m,n} \leq 3^{n*m}.$$

We can do better since the value in the bottom corner must have the value of 0 for it to form a domino tiling.
A value of 1 means there is a vertical domino facing downwards but since it is on the last row of the grid there is no space for it.
Likewise a value of 2 indicates there is a horizontal domino facing leftward but since its the last column it is impossible for it to exist without being off the grid.

$$\text{Thus } \ H_{m,n} \leq 3^{(n*m)-1}.$$

Next if we consider that values on the bottom row can't have the value 1 since there would be no space for the domino, but can have 2 or 0.
The same applies to the values on the most leftward column can only have values 1 or 0.

$$\text{Therefore } H_{m,n} \leq 3^{(n-1)*(m-1)} * 2^{n+m-1}.$$

Lastly if we consider the value in the top corner it can't have the value 0 , as there are only downwards dominoes and leftward dominoes.

□

2.5 Limitations of the brute force method

The number of combinations triple every time you add a square, for smaller grids this isn't too much of a problem.
But for larger grid such as a 8 by 8 chess board it is too inefficient to solve in a reasonable amount of time.
It would have to go throught $3^{8*8} \approx 3.4 * 10^{30}$ combinations.

Figure 13: Shows the supercomputer New Exascale Supercomputer [5]

New Exascale Supercomputer Can Do a Quintillion Calculations a Second, that is $1 * 10^{18}$ calculations a secound. Which would mean it would take $3 * 10^{12}$ secounds. One day has about $1 * 10^{5}$ secounds, one year has roughly $1 * 10^{4}$ days. Meaning it would take a minimum of $3 * 10^{3}$=3000 years to calculate.
This ignores any calculations we need to go through to check if it forms a valid grid, meaning the true answer will be magnitudes longer, then what this estimate is.

2.6 The need for explicit method to count

Even if we designed the brute force method to be 100% efficient at creating domino tilings that are valid while still creating every valid domino tiling. It would still be inefficient for large grid sizes.

This is due to the fact that $H_{m,n}$ increases exponentially as m or n increases. We can prove this by referring to lemma 2.1. It states that $H_{m,n} \geq H_{m_1,n_1} * H_{m_2,n_2} * H_{m_3,n_3} ... H_{m_K,n_K}$.Where H_{m_i,n_j} are sub grids that together can form $H_{m,n}$.

If we assume that both m and n are even, then $H_{m,n}$ can be formed by $\frac{m*n}{4}$ 2 by 2 grids.
Meaning $H_{m,n} \geq 2^{\frac{m*n}{4}}$ since each 2 by 2 sub grids can only form 2 valid domino tilings.

If we now take the case where either m or n is odd, lets take m to be odd as the same logic applies if n is odd. Then $H_{m,n}$ can form $\frac{(m-1)*n}{4}$ 2 by 2 grids. And it will have $\frac{n}{2}$ 1 by 2 grids required to form the grid $H_{m,n}$. Meaning $H_{m,n} \geq 2^{\frac{(m-1)*n}{4}} * 1^{\frac{n}{2}}$ since each 2 by 1 sub grids can only form 1 valid domino tilings.

This number of domino tilings increasing exponentially, means that any algorithm that doesn't count domino tilings explicitly will be exponentially efficient. For an 8 by 8 chessboard there is a minimum of $2^{\frac{8*8}{4}}$ domino tilings. While this might be possible to calculate in a reasonable amount of time, if we increase the grid size even a little it quickly requires an unreasonable amount of computational power to compute.

3 Kasteleyn theory

3.1 Representing tilings with arrows

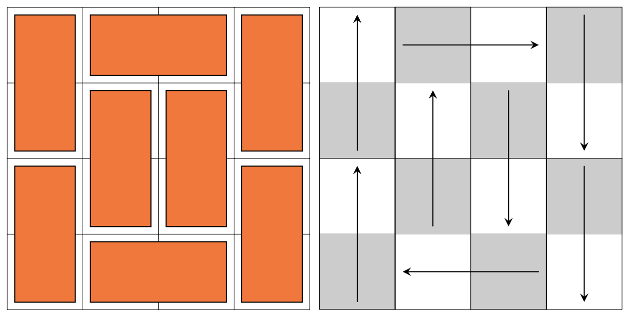


Figure 14: Shows a how a domino tiling pattern represented with cycles

If we draw a check board pattern on the grid, then created arrows to correspond to each domino, such that a vertical domino will have a vertical arrow and horizontal domino will have a horizontal arrow.The arrows will always start in a dark square and end in a light square.

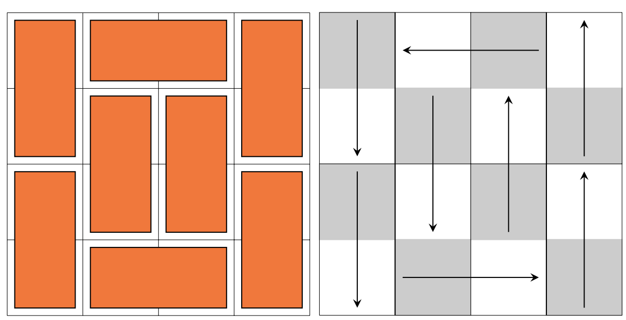


Figure 15: Shows a how a different arrow configuration can represent a domino tiling

If we decide to change the checkerboard pattern we can get an entirely different arrow tiling to represent the same domino tiling.

3.2 Cycles

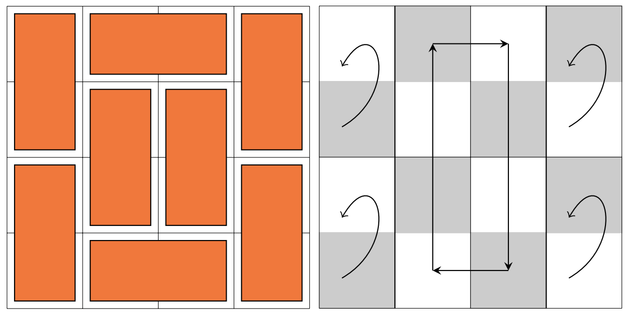


Figure 16: Shows a how a domino tiling pattern would be represent by the arrows

If we divide an m by n grid into 2 by 1 regions, with arrows being able to leave or enter 2 by 1 regions. Some arrows wont leave the 2 by 1 region and will form a self contained cycle.
In every odd column it impossible to draw an up arrow since it would form a self cycle arrow, likewise you can't draw a down arrow in a even column.
Arrows that leave the region will eventually return to the region and form a cycle

In each cycle there is an equal number of up and down arrows and left and right arrow, since every cycle returns to the original position.

Number of arrows = 2*(Number of up arrows + Number of left arrows)

Every cycle that isn't self contained has an even number of arrows.

3.3 Determinant of a matrix

"We define the determinant of a matrix A to be the quantity

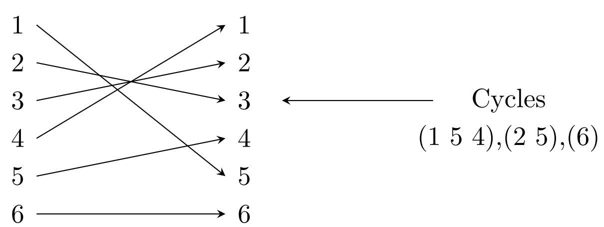
$$\det(A) = \sum_{\sigma \in T} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i)i}$$

Using Leibniz formula for determinants we can calculate the number of domino tilings. T is the set of bijections from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$. Where n is the total number of tiles. [6]

3.4 Cycles in the matrix

A permutation of a set is simply a bijection on this set. We write $\mathcal{S}_n$ for the set of permutations on $\{1, \ldots, n\}$. A cycle in a permutation is a collection of points $i_1, \ldots, i_r$ such that $\sigma(i_1) = i_2, \ldots, \sigma(i_{r-1}) = i_r, \sigma(i_r) = i_1$. Every permutation can be decomposed into cycles.

Let σ:$\{1, \ldots, n\} \ \rightarrow \{1, \ldots, n\}$ Be a bijection.



The sign of a permutation Sgn(σ)= $(-1)^{\sum_{j=1}^{r}(C_j-1)} \in \{\ 1, -1.\}$

Where σ has r cycle of lengths $c_1, \ldots, c_r$.

Since cycles are all of even length its follows that

Lemma 3.4. Sgn(σ)= $(-1)^{\text{Number of cycles}}$.

3.5 Domino tilings and arrow configurations

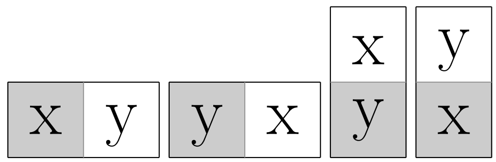
Each arrow configuration gives rise to two domino configurations.

Lemma 3.5. $Z_{m,n}$ =Number of arrow configeration= (Number of domino tilings)^2.

3.6 Creating a matrix to find the number of arrow configurations

$$\text{Let D=}\{1,\ldots,n\}\times\{1,\ldots,\frac{n}{2}\}$$

$$\text{Define } A_{x,y} :\in D.$$



$$A_{x,y} = 1 \text{ if any of these conditions are met or if x=y.}$$

$$\text{Otherwise } A_{x,y} = 0.$$

For example:

$$x = (1,1), y = (2,1) \implies A_{x,y} = 1$$
$$x = (2,1), y = (1,1) \implies A_{x,y} = 1$$
$$x = (1,1), y = (1,2) \implies A_{x,y} = 1$$
$$x = (1,2), y = (1,1) \implies A_{x,y} = 1$$
$$x = (1,1), y = (1,1) \implies A_{x,y} = 1$$
$$x = (7,4), y = (3,6) \implies A_{x,y} = 0$$

This matrix $A_{x,y}$ represents every arrow configuration.

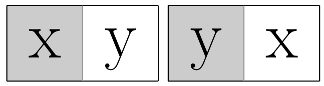
Under these rules $det(A_{x,y})$= N.Arrow config with even cycles - N.Arrow config with odd cycles.

This is due to $Sgn(\sigma) = -1^{\text{N.Cycles}}$ meaning that when the number cycles in an arrow configuration are odd it times it by -1.

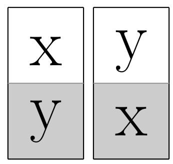
3.7 Improving the matrix

Let D={ 1,2,...,n } x { 1,2,...,$\frac{n}{2}$ }

Define $B_{x,y} :\in D$.



$B_{x,y} = 1$ if any of these conditions are met.



$B_{x,y} = i$ if any of these conditions are met or if x=y.

Else $B_{x,y} = 0$.

Under these rules $det(B_{x,y})$= N.Arrow config with even cycles + N.Arrow config with odd cycles.

N.Cycles+N.Rights=Even.

N.Cycles+N.Lefts=Even.

N.Cycles+$\frac{1}{2}$(N.Rights+N.Lefts)=Even.

$$\prod_{i=1}^{n} B_{\sigma(i)i} = i^{\text{N.Rights+N.Lefts}} \text{or 0 if an invalid arrow configuration.}$$

$$= -1^{\text{N.Rights}} \text{due to N.Rights=N.Lefts.}$$

This is because every cycle must return to same point.

$= -1^{\text{N.Rights+N.Cycles}}$ Which we have established is even meaning it always take the value 1.

3.8 Eigenfunctions of the Kasteleyn operator

The eigenvalue λ_{j_1,j_2} associated with ϕ_{j_1,j_2} is

$$\lambda_{j_1,j_2} = 2\pi \cos(\frac{j_1}{m+1}) + 2\pi i \cos(\frac{j_2}{n+1}).$$

$$Z_{m,n} = det_{x,y \in T_{m,n}}(B_{x,y}) = \text{Product of eigenvalues of B}$$

$$= \prod_{1 \leq j_1 \leq m, 1 \leq j_2 \leq n} \lambda_{j_1,j_2}.$$

We are going to show that for each

$$1 \leq j_1 \leq m, 1 \leq j_2 \leq n.$$

The function $\Phi_{j,k} : T_{m,n} \to T_{m,n}$

$$\Phi_{j_1,j_2}(x_1, x_2) = sin(\frac{j_1 x_1}{m+1}) sin(\frac{j_2 x_2}{n+1})$$

is an "eigenfunction" of the matrix $B_{x,y \in T_{m,n}}$.

Show that $\forall (x_1, x_2) \in \{1, ..., m\}\text{x}\{1, ..., n\} = T_{m,n}$ we have

$$(B\phi)(X) = \sum_{y \in T_{m,n}} B_{x,y} \phi_{j_1,j_2}(y) = \lambda_{j_1,j_2} \phi_{j_1,j_2}(x).$$

$B_{x,y} \neq 0$ only when y is a neighbour of x.

$$= 1 * \phi_{j_1,j_2}(x_1, x_2 + 1) + 1 * \phi_{j_1,j_2}(x_1, x_2 - 1)$$

$$+ i * \phi_{j_1,j_2}(x_1 + 1, x_2) + i * \phi_{j_1,j_2}(x_1 - 1, x_2).$$

$$\lambda\phi = B\phi.$$

Theorem 3.8

The number of domino tilings is given by

$$H_{m,n} = \sqrt{\prod_{1 \leq j_1 \leq m, 1 \leq j_2 \leq n} 2\cos(\frac{\pi j_2}{n+1}) + 2i \cos(\frac{\pi j_1}{m+1})}$$

For n,m=4,6,8 we get the number of domino tilings is 36, 6728, 12988816.

4 Further topics

4.1 Closing Remarks

We found how many 2 by 1 domino's combinations we can have on a m by n grid. We did not care about the order of the domino's, we restricted the domino's to being placed either horizontally or vertically. Additionally the whole grid had to be covered and no domino could have parts of it not on the grid. We can solve any n by m grids with an efficiency of $O(\text{nm})$ making it efficient at solving even large grids such as an 8 by 8 trivial.
However for extremely large grids such as a 100 by 100 it still takes a lot of computations, additionally the values must be precise enough to give the correct answer.

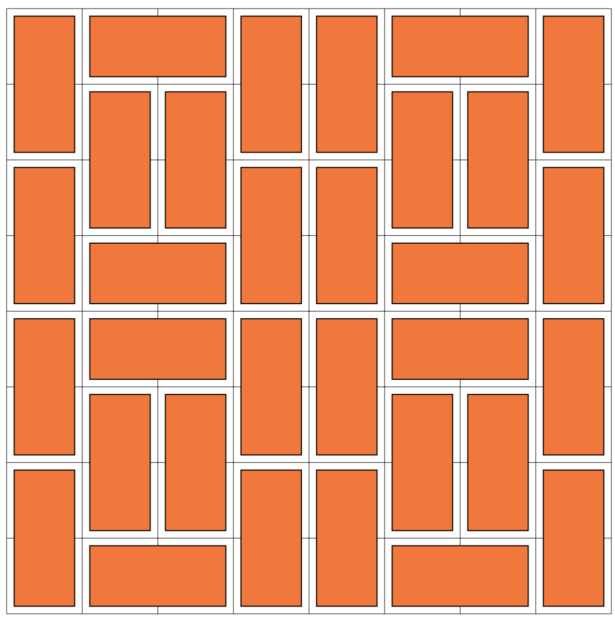


Figure 17: Shows how a 8 by 8 domino tiling on a chessboard can be represented

For extremely large grids, such as 1 billion by 1 billion, it can't be solved explicitly in a reasonable amount of time. But the upper and lower bounds proven can still be calculated in a reasonable amount of time, giving us information about $H_{m,n}$ we couldn't calculate explicitly.

4.2 Additionally problems

We can try solving n by m grids with holes in them, the brute force method can be used on grids with holes in them, if the grid is small enough.

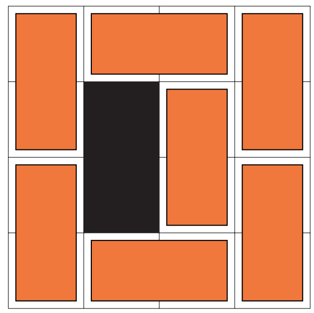


Figure 18: Shows how a 4 by 4 domino tiling with a hole

We can also try solving n by m grids with multiple different shapes to tessellate, for example the Tetris shapes. This could have real life applications, as a way to for an AI to use to make decisions while playing tetris, for example a block placement that results in an empty shape that has more combinations that can fill it may be preferable to one that has fewer ways to fill.

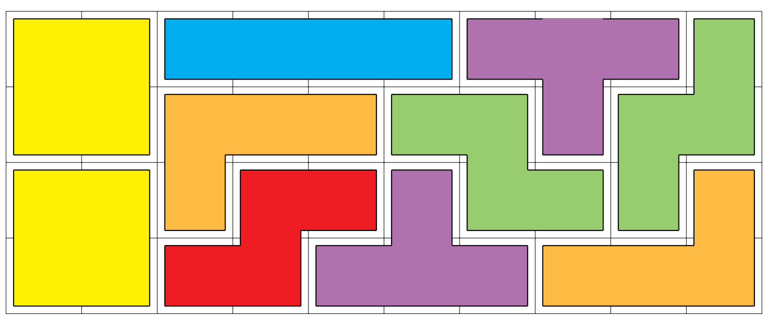


Figure 19: A 4 by 10 grid being filled by tetris shapes

5 References

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