Signals and Systems

Lecture 11: System Identification

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Outline

1. Frequency Domain System Identification
   - Introduction
   - Choice of signals inputs
2. Identifying a System based on its Impulse Response
   - Impulse response, no noise
   - Impulse response, with noise
3. Identification Using Sinusoidal Inputs
   - Experimental procedure
   - Closed-loop system identification
   - Identifying the transfer function
     - Method
     - Example
   - Weighted least squares

Introduction

In this lecture apply tools learned to identify an unknown system.

This process is known as system identification. The key idea:

1. by applying a known input to a system
2. and observing the system's output,

$\Rightarrow$ a model of the system can be estimated.

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- **Grey-box:** we have a first-principles model, but with unknown parameters. Data is used to identify these parameters. This is covered in the IDSC lecture *System Modeling*.
- **White-box:** we have a first-principles model, with no unknown parameters, or where parameters can be measured directly (mass, for example); no data is required in this case.

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1. Often first: estimate the system's frequency response $H(\Omega)$ (like Bode plots) – which can also be used for control design directly in the frequency domain (lead/lag compensation, gain and phase margin, Nyquist stability criterion, etc)
2. Next, estimate the system's transfer function of the system, based on the frequency response estimate.

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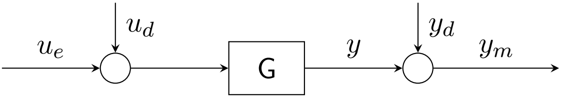
2. and then to estimate the transfer function $H(z)$ from the frequency response estimate.

⇒ We focus on the black-box identification of causal, linear time-invariant systems.

Outline

Frequency Domain System Identification

We will consider the following system identification problem:



Where G is a causal, stable LTI system, and

- u_e is a known input that we use to excite the system
- u_d is an unknown process noise, assumed to be white
- y_d is an unknown measurement noise, assumed to be white
- y_m, given by $y_m = \mathrm{G}u_e + y_d + \mathrm{G}u_d$, is a measurement of the system's output, which is corrupted by process noise and measurement noise

Outline

Input choice for frequency domain system identification

Based on the above problem structure, the task is now to design:

1. a set of inputs u_e, such that by analyzing the measured output y_m
2. and the known input u_e,

$\Rightarrow$ the frequency response of the system $H(\Omega)$ can be determined accurately, despite inherent process and measurement noise.

We will see later in the lecture how to estimate the system's transfer function based on its estimated frequency response.

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Question: What kind of input signal can be used for identification?

We now introduce two possible inputs:

- a unit impulse,
- and a sinusoid.

Outline

Identifying a System based on its Impulse Response

In Lecture 2, we saw that

an LTI system is fully characterized by its impulse response.

Conclusion

By measuring its impulse response, the system can be identified.

In the absence of process and measurement noise, we have that

$$y_m = \mathsf{G}\ u_e\ .$$

We then have a simple way of determining freq. resp. $H(\Omega)$:

1. Let $\{u_e[n]\} = \{\delta[n]\}$
2. Then $\{y_m[n]\} = \{h[n]\}$ and $H(\Omega) = \sum_{n=0}^{\infty} y_m[n] e^{-j\Omega n}$

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In practice

since most systems have an infinite impulse response

- $\Rightarrow$ impossible to measure it entirely,
- $\Rightarrow$ we collect N pieces of data, where N is large enough such that the impulse response has decayed significantly (= so that $y_m[n]$ is small enough for $n \geq N$).

We then take the DFT:

$$Y_m[k] = \sum_{n=0}^{N-1} y_m[n] e^{-j\frac{2\pi k}{N}n}.$$

At the discrete frequency $\Omega_k = 2\pi k/N$ where $k = 0, 1, \ldots, N-1$, (N DFT coefficients).

The frequency response estimate $\widehat{H}(\Omega_k)$ is defined as:

$$\widehat{H}\left(\Omega_k\right) := Y_m[k] = H\left(\Omega_k\right) - \underbrace{\sum_{n=N}^{\infty} h[n] e^{-j\Omega_k n}}_{H_N(\Omega_k)},$$

where $H_N(\Omega_k)$ is the ‘estimation” error introduced by not measuring the entire duration of the impulse response.

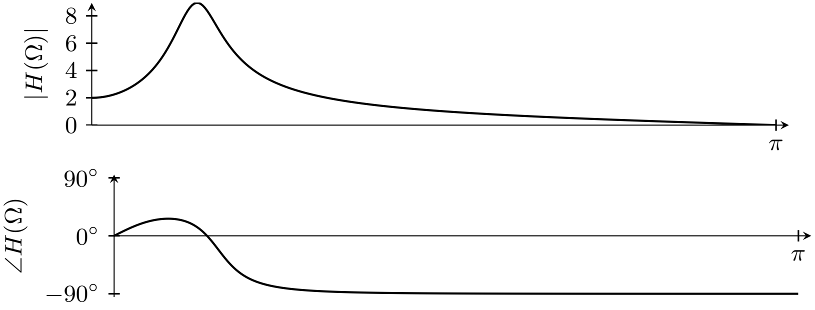
Remark: Note that $H_N(\Omega) \to 0$ as $N \to \infty$ since G is stable.

Example : Estimating frequency response, based on impulse response, without noise

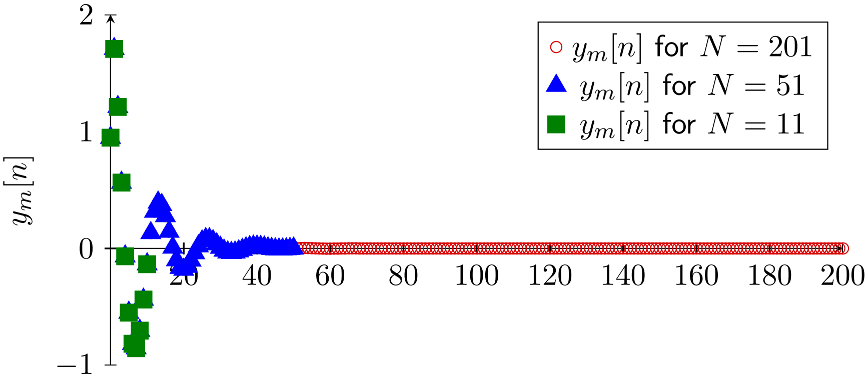
Let the system be given by the transfer function

$$H(z) = \frac{0.9474 + 0.2105z^{-1} - 0.7368z^{-2}}{1 - 1.579z^{-1} + 0.7895z^{-2}},$$

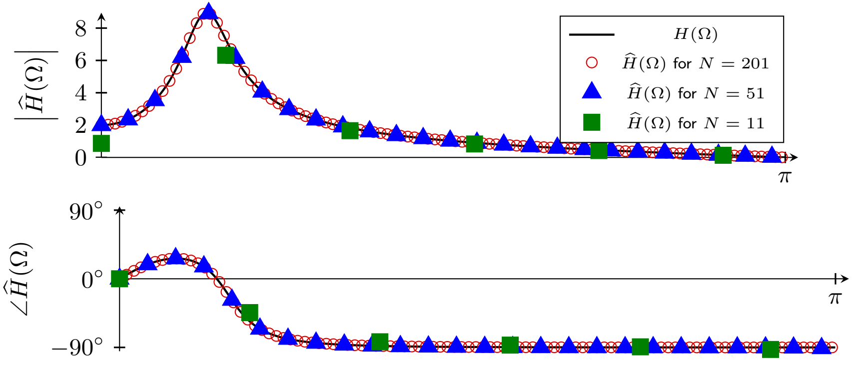
which has the following frequency response:



We apply the input $u_e[n] = \delta[n]$ in the absence of noise, and record the system's output for $N = 11$, $N = 51$ and $N = 201$ samples:



Take DFT of these measurements to estimate system's frequency response $\widehat{H}(\Omega)$:



Conclusion:

1. a larger N gives a higher frequency resolution, since we are dividing the interval $-\pi$ to π into N equally spaced frequency intervals.
2. We also note that as N increases, estimation error decreases.

Outline

With measurement noise

We study effects of measurement noise y_d on above estimation procedure.

$$y_m[n] = h[n] + y_d[n] \quad \text{for } n = 0, 1, \ldots, N-1,$$

where $E(y_d[n]) = 0$ and $E(y_d[n]y_d[m]) = \sigma_y^2 \delta[n-m]$.

It follows that $\widehat{H}(\Omega_k) = Y_m[k] = H(\Omega_k) - H_N(\Omega_k) + Y_d[k]$,
where $E(Y_d[k]) = 0$ and $E(|Y_d[k]|^2) = N\sigma_y^2$.

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- Thus, $\widehat{H}(\Omega_k) - H(\Omega_k) = -H_N(\Omega_k) + Y_d[k]$ $E\left(\widehat{H}(\Omega_k) - H(\Omega_k)\right) = -H_N(\Omega_k)$, est. error approaches zero as $N \to \infty$.

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  $E\left(\widehat{H}(\Omega_k) - H(\Omega_k)\right) = -H_N(\Omega_k)$, est. error approaches zero as $N \to \infty$.
- But, **mean-square error:** $E\left(\left|\widehat{H}(\Omega_k) - H(\Omega_k)\right|^2\right) = H_N^2(\Omega_k) + N\sigma_y^2$ approaches infinity as the length of the sample increases, because $N\sigma_y^2 \to \infty$ as $N \to \infty$.

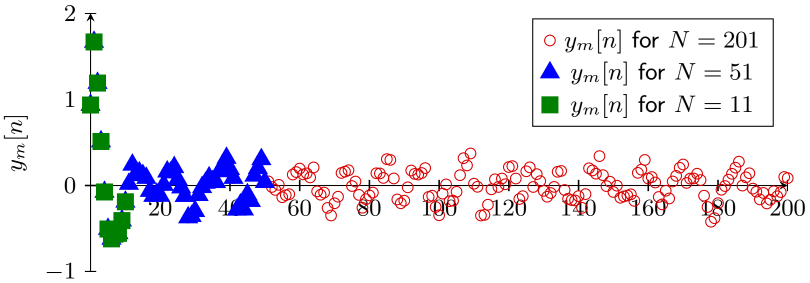
Remark : Inclusion of process noise u_d has a similar effect, but is more complex to derive analytically.

We now demonstrate the effect of zero-mean process noise u_d and measurement noise y_d with $\sigma_y^2 = \sigma_u^2 = \frac{1}{3}$.

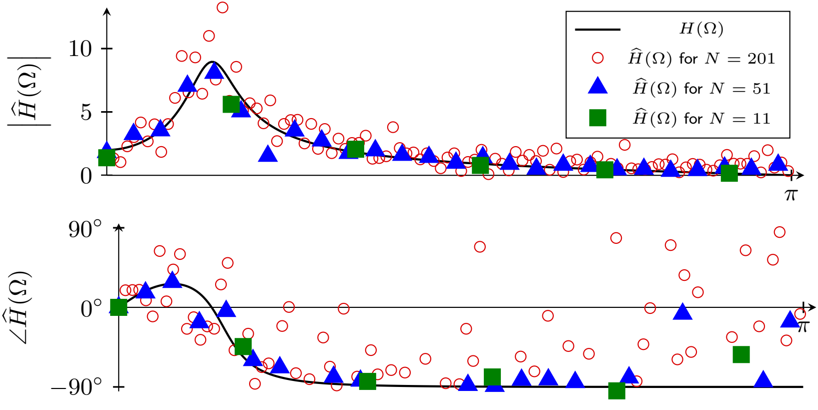
Using the identical system, given by

$$H(z) = \frac{0.9474 + 0.2105z^{-1} - 0.7368z^{-2}}{1 - 1.579z^{-1} + 0.7895z^{-2}},$$

we reapply the input $u_e[n] = \delta[n]$.
We record the system's output for $N = 11$, $N = 51$ and $N = 201$ samples:



We then take the DFT of these measurements in order to estimate the frequency response $\widehat{H}(\Omega)$ of the system:



Conclusion: Here we see that the process noise and measurement noise cause large estimation errors for large N.

This example highlights the problem of using a unit impulse input to identify a system:

- the energy of the input we apply stays constant,
- but the energy of the measurement noise increases as N increases $(E(|Y_d[k]|^2) = N\sigma_y^2.)$.

To address this problem:

- one can increase the input amplitude;
  however, beyond a certain point, we usually don't have this luxury due to saturation, non-linear effects, etc.
- An alternative solution is to select a more appropriate input signal.

Outline

Identification Using Sinusoidal Inputs

Goal: To identify the frequency response of a system from its input-output behavior.
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Advantages of this method: more robust to noise !

- Recall: the energy of the input and that of the noise both grow linearly as a function of N,
- the energy of the noise is spread out across all N frequencies (white noise),
- but a pure sinusoid has all energy at one frequency ($\Rightarrow$ better "signal to noise ratio" at every frequency).

Sinusoidal input, output with noise

Consider the case where measurement noise corrupts the output, such that

$$y_m = \mathsf{G}\, u_e + y_d.$$

Now let

- $u_e[n] = e^{j\frac{2\pi}{N}ln}$ for $n = 0, 1, \ldots, N_T + N - 1$, be a sinusoid with discrete frequency $\Omega_l = 2\pi l/N$,
- $y_e = \mathsf{G}u_e$ be the ideal output of the system.

In Lecture 6 we saw that, since $e^{j\frac{2\pi}{N}ln}$ is an eigenfunction of any LTI system,

$$y_e[n] = H(\Omega_l)\, u_e[n] - w[n], \qquad (1)$$

where $w[n]$ captures the effect of a causal input and decays to zero as $N \to \infty$ (transient).

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We now take the DFT of the sequences introduced above for $n \geq N_T$. Let

$$Y_e[l] = \sum_{n=N_T}^{N_T+N-1} y_e[n] e^{-j\frac{2\pi}{N}ln},$$

$$U_e[l] = \sum_{n=N_T}^{N_T+N-1} u_e[n] e^{-j\frac{2\pi}{N}ln} = N,$$

$$W[l] = \sum_{n=N_T}^{N_T+N-1} w[n] e^{-j\frac{2\pi}{N}ln},$$

then $\quad Y_e[l] = H(\Omega_l)\, U_e[l] - W[l]$. **Remark:** l is the index for the DT frequency

We can thus estimate the system's frequency response as follows. Let

$$Y_m[l] = \sum_{n=N_T}^{N_T+N-1} y_m[n] e^{-j\frac{2\pi}{N}ln}$$

$$Y_d[l] = \sum_{n=N_T}^{N_T+N-1} y_d[n] e^{-j\frac{2\pi}{N}ln},$$

then, since $y_m = y_e + y_d$,

$$Y_m[l] = Y_e[l] + Y_d[[l]] = H(\Omega_l)\, U_e[l] - W[l] + Y_d[[l]]$$

Frequency response estimate: $\widehat{H}(\Omega_l)$

$$\widehat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]} = H(\Omega_l) - \frac{W[l]}{N} + \frac{Y_d[l]}{N}.$$

$$\widehat{H}\left(\Omega_l\right) := \frac{Y_m[l]}{U_e[l]} = H\left(\Omega_l\right) - \frac{W[l]}{N} + \frac{Y_d[l]}{N}.$$

We therefore have

1. $$\mathsf{E}\left(\widehat{H}\left(\Omega_l\right) - H\left(\Omega_l\right)\right) = -\frac{W[l]}{N},$$

   which approaches zero as $N_T \to \infty$ or $N \to \infty$. Therefore, as $N \to \infty$, we have an unbiased estimate.

2. $$\mathsf{E}\left(\left|\widehat{H}\left(\Omega_l\right) - H\left(\Omega_l\right)\right|^2\right) = \frac{W^2[l]}{N^2} + \frac{\sigma_y^2}{N}.$$

   The mean-squared error goes to 0. A similar result holds when u_d is included.

Conclusion: A sinusoidal input at different frequencies is a very effective input for system identification.

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  - A larger N will reduce the influence of noise (as previously discussed),
  - but will increase the duration of each experiment.
- Select a frequency of interest by choosing an appropriate integer l and setting $\Omega_l = 2\pi l/N$.
  - Note that, because the system is real, we only need to identify the frequency response for $0 \leq \Omega \leq \pi$, since $H(-\Omega) = H^*(\Omega)$.
  - We will thus pick $l \in [0, \frac{N-1}{2}]$ if N is odd, or $l \in [0, \frac{N}{2}]$ if N is even.

Experimental procedure: summary

- Set $u_e[n] = A\cos(\Omega_l n)$ for $n = 0, \ldots, N_T + N - 1$, where A scales the input.
  - Choosing a large A increases the signal to noise ratio;
  - however, if A is chosen too large, nonlinearities such as saturation may be excited, thus negatively affecting the frequency response estimate.

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- Calculate

$$Y_m[l] = \sum_{n=N_T}^{N_T+N-1} y_m[n]e^{-j\Omega_l n} \quad \text{and} \quad U_e[l] = \sum_{n=N_T}^{N_T+N-1} u_e[n]e^{-j\Omega_l n}$$

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- The frequency response estimate at the frequency Ω_l is then given by

$$\widehat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]}.$$

- Repeat this for the frequencies of interest by picking the appropriate integer l.

Example:Estimating frequency response using sinusoidal inputs, with noise

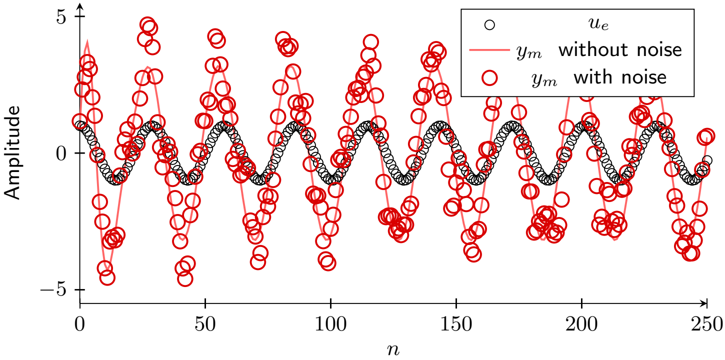
We now revisit the previous example using the identical system, given by

$$H(z) = \frac{0.9474 + 0.2105z^{-1} - 0.7368z^{-2}}{1 - 1.579z^{-1} + 0.7895z^{-2}},$$

and where the system is subject to both the process noise u_d and the measurement noise y_d.

1. Compared to the previous example, we increase the noise such that the standard deviation is increased by a factor of 5, resulting in $\sigma_y^2 = \sigma_u^2 = \frac{25}{3}$.
2. We perform system identification by applying sinusoidal inputs at N different frequencies, by choosing $l = 0, 1, \ldots, (N-1)/2$.

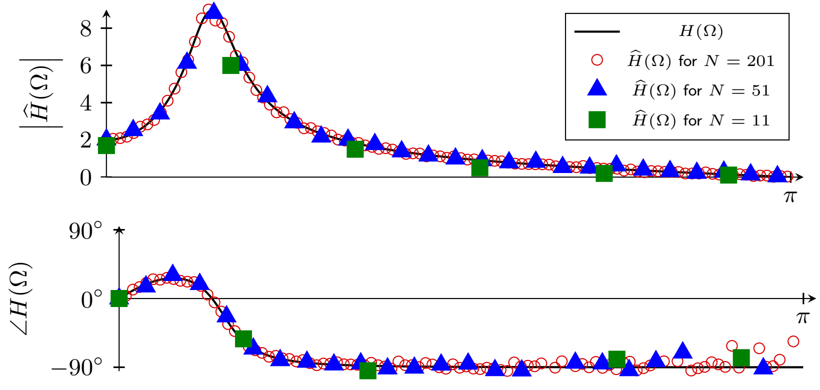
The recorded input u_e and output y_m for $N_T = 50$, $N = 201$, $A = 1$, and $l = 7$ are as follows:



Observations

1. Ideal output of system is: shifted and scaled version of input sinusoid.
2. Process noise and measurement noise corrupt the signals and their effects are noticeable.
3. By picking $N_T = 50$, we let the transient decay.

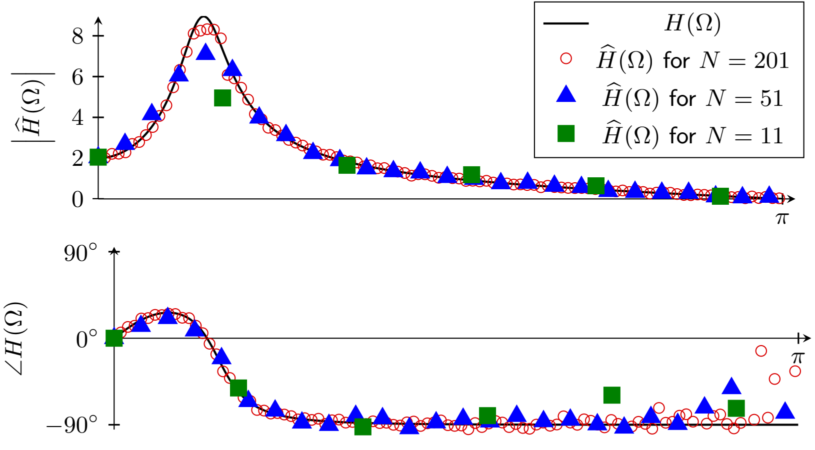
The estimate of the frequency response is as follows:



Observations: This method performs very well in the presence of noise. The above plot highlights the two main benefits of picking a larger N:

1. the effect of noise is further reduced, since more samples are averaged,
2. a larger N results in a finer frequency resolution, allowing more frequencies to be tested.

We now demonstrate what happens if we do not wait for the transient to decay, i.e. we pick $N_T = 0$.



We see that, as long as N is much larger than the transient decay time, the effect of the transient is small. However, for small N the transient significantly affects the estimate of the frequency response.

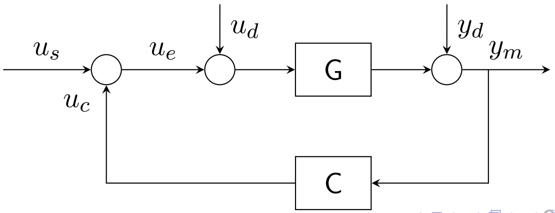
Outline

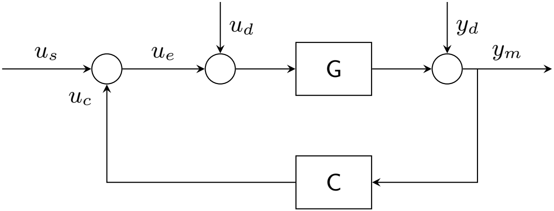
Closed-loop system identification

Often, the system to be identified is part of a closed-loop system.

This is the case, for example, if the open-loop system is unstable.
$\Rightarrow$ In this case, an appropriate, stabilizing controller C is required before performing the identification.

Consider the following stable closed-loop system (where G can be unstable)





Let $u_e = u_c + u_s$. Then, $y_m = \mathsf{G}u_e + \mathsf{G}u_d + y_d$.

Discussion :

- if u_e can be measured, the open loop method described in the previous section can be used to identify the frequency response of the plant.
- If G and C have no poles at $z = e^{j\Omega_l}$, and under some mild assumptions on u_d and y_d, it can be shown that the bias and mean-squared error of the estimate $\widehat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]}$, both approach 0 as $N \to \infty$.

Outline

Identifying the Transfer Function: Method

After estimating the frequency response at a number of distinct frequencies Ω_l, we can identify the transfer function of the system.
Given the two design parameters:

- A (number of a_k coefficients, although a_0 is fixed at 1)
- and B (number of b_k coefficients),

the goal is to identify the unknown parameters: a_k and b_k of the system's transfer function, given by

$$H(z) = \frac{\sum_{k=0}^{B-1} b_k z^{-k}}{1 + \sum_{k=1}^{A-1} a_k z^{-k}},$$

with frequency response

$$H(\Omega) = \frac{\sum_{k=0}^{B-1} b_k e^{-j\Omega k}}{1 + \sum_{k=1}^{A-1} a_k e^{-j\Omega k}}.$$

As a result of the identification experiments
we obtained estimates $\widehat{H}(\Omega_l)$ of $H(\Omega)$ at L frequencies Ω_l for $l = 0, 1, \ldots, L-1$.

Setting $\widehat{H}(\Omega_l) = H(\Omega_l)$ at all measurement frequencies, we obtain the system of equations

$$\left(1 + a_1 e^{-j\Omega_l} + \cdots + a_{A-1} e^{-j(A-1)\Omega_l}\right) \widehat{H}(\Omega_l) =$$
$$= b_0 + b_1 e^{-j\Omega_l} + \cdots + b_{B-1} e^{-j(B-1)\Omega_l} \quad \text{for } l = 0, 1, \ldots, L-1.$$

These equations must hold for both real and imaginary components.
We therefore let :

- $\widehat{H}(\Omega_l) = R_l e^{j\phi_l}$ and,
- using $e^{j\theta} = \cos\theta + j\sin(\theta)$,

rewrite the system of equations for $l = 0, 1, \ldots, L-1$ as

$$\begin{aligned} R_l\cos(\phi_l) + a_1 R_l\cos(\phi_l - \Omega_l) + \cdots + a_{A-1} R_l \cos(\phi_l - (A-1)\Omega_l) \\ = b_0 + b_1\cos(\Omega_l) + \cdots + b_{B-1}\cos((B-1)\Omega_l) \end{aligned} \qquad \textit{Real}$$

$$\begin{aligned} R_l\sin(\phi_l) + a_1 R_l\sin(\phi_l - \Omega_l) + \cdots + a_{A-1} R_l \sin(\phi_l - (A-1)\Omega_l) \\ = -b_1\sin(\Omega_l) - \cdots - b_{B-1}\sin((B-1)\Omega_l) \end{aligned} \qquad \textit{Imaginary}$$

This system of equations can be converted to the least squares problem of minimizing

$$(F\Theta - G)^{\mathsf{T}}(F\Theta - G),$$

where

$$\Theta = \begin{bmatrix} a_1 & a_2 & \dots & a_{A-1} & b_0 & b_1 & \dots & b_{B-1} \end{bmatrix}^{\mathsf{T}}$$

contains the unknown parameters and where

- Θ: size $(A + B - 1)$ vector, *unknown.*
- F: size $(2L) \times (A + B - 1)$ regression matrix, *known.*
- G: size $(2L)$ output measurement vector, *known.*

The least-squares solution

$$\Theta^* = (F^{\mathsf{T}}F)^{-1}F^{\mathsf{T}}G$$

yields the estimated transfer function coefficients Θ^*.

Note that there are $(A+B-1)$ unknown parameters in the vector Θ, and that there are $2L$ equations (L frequency measurements with both real and imaginary components). However, if any $\Omega_l = 0$ or π, an equation is lost since the equation corresponding to the imaginary component yields $0 = 0$. For the existence of a least squares solution, the matrix F must have full column rank. We therefore require

- $2L \geq A+B-1$ if frequencies $\Omega_l = 0$ and $\Omega_l = \pi$ were not tested;
- $2L \geq A+B$ if either $\Omega_l = 0$ or $\Omega_l = \pi$ were tested;
- $2L \geq A+B+1$ if both $\Omega_l = 0$ and $\Omega_l = \pi$ were tested.

In practice this is not a problem as typically $L \gg A+B$.

Identifying the transfer function

In the previous example, we applied the sinusoid method to estimate the frequency response of the system given by

$$H(z) = \frac{0.9474 + 0.2105z^{-1} - 0.7368z^{-2}}{1 - 1.579z^{-1} + 0.7895z^{-2}},$$

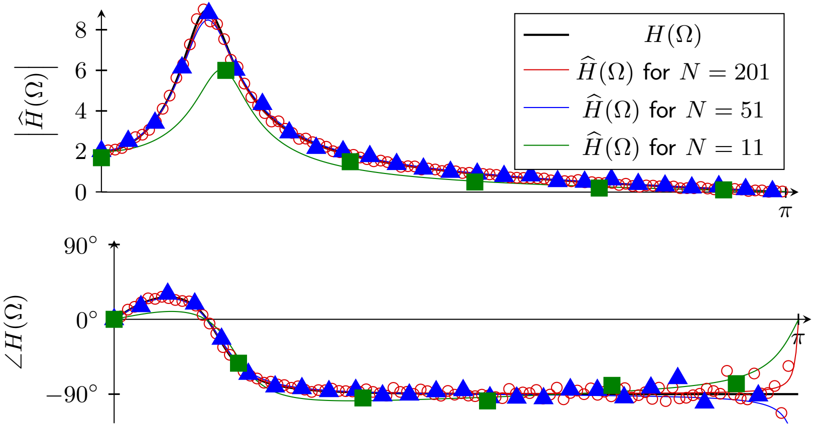
at specified frequencies Ω_l. Using the approach described above, we can now identify the transfer function. With $A = B = 3$, the transfer function is identified to be:

$$\widehat{H}(z) = \frac{0.9472 + 0.2015z^{-1} - 0.7223z^{-2}}{1 - 1.572z^{-1} + 0.7836z^{-2}}, \quad \text{for } N = 201;$$

$$\widehat{H}(z) = \frac{0.9558 + 0.2477z^{-1} - 0.7693z^{-2}}{1 - 1.559z^{-1} + 0.772z^{-2}}, \quad \text{for } N = 51;$$

$$\widehat{H}(z) = \frac{0.6371 + 0.2257z^{-1} - 0.3091z^{-2}}{1 - 1.5z^{-1} + 0.778z^{-2}}, \quad \text{for } N = 11.$$

The resulting frequency responses are the following:



Outline

A useful variant for system identification is weighted least squares: $F\Theta = G$ is replaced by $WF\Theta = WG$, where

$$W = \begin{bmatrix} w_0 & & & & & \\ & w_0 & & & & \\ & & w_1 & & & \\ & & & w_1 & & \\ & & & & \ddots & \\ & & & & & w_L \end{bmatrix}$$

is a $(2L \times 2L)$ square, diagonal matrix. Its elements w_l are weights which capture the confidence in the measurements. We obtain the solution using the substitutions $\overline{F} = WF$, $\overline{G} = WG$:

$$\begin{aligned} \Theta^* &= (\overline{F}^\mathsf{T}\overline{F})^{-1}\overline{F}^\mathsf{T}\overline{G} \\ &= (F^\mathsf{T}W^\mathsf{T}WF)^{-1}F^\mathsf{T}W^\mathsf{T}WG. \end{aligned}$$

Higher weights result in a better fit at the corresponding frequencies.
For example, it is beneficial to use low weights at frequencies where a system rolls off. There, the measured amplitudes become small compared to the noise and the measurements are less reliable.