Risk and Return on Financial Securities

A description on risk preference of investors and volatility of financial securities

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Introduction

Measurement and management of risk and uncertainty is very important in modern finance. Different individuals have different appetites for risk. Some investors prefer riskier strategies with more upside while others want safer strategies. It is very important to assess risk correctly and invest accordingly. The concept of expected utility is used to make decisions under uncertainty. The volatility of investment can be decreased by making different uncorrelated bets

Expected Utility

If we have three payoffs X,Y and Z such that:

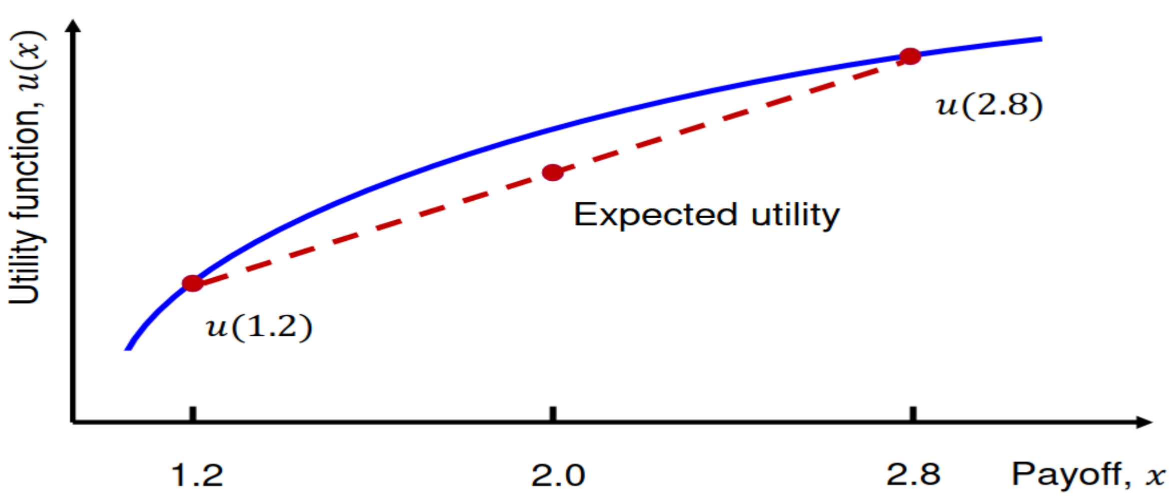
X = $1000 or $0 (50/50 odds)
Y= $700 or $300 (50/50 odds)
Z= $500 (100% certain)

All three payoffs have an expected value of $500 but they have different volatilities. Are they all equally valuable? Which one is better or preferable? To answer these questions we use the concept of expected utility.

Expected utility is the most used model to make decisions under uncertainty. Investors value each bet by its expected utility and not by its expected value. Expected utility is the nonlinear transformation of the payoff and it is mostly a concave function.

Illustration

Suppose that the payoff is represented by X and its utility is represented by a function u(X) then the graph can be of the form:



Description

In the graph, it can be seen that the utility of a certain payoff of $2 is greater than the expected utility of a payoff of $1.2 or $2.8 with 50/50 odds.
It can be written mathematically as:

$$u(\mathrm{E}[x]) \geq \mathrm{E}[u(x)] \qquad (\mathrm{E}[x]=2.0)$$

This shows that if there are two payoffs X and Y and expected value of X is equal to expected value of Y (E[x]=E[y]) then the payoff with low volatility will have higher utility. This is due to the concave shape of the utility function.

Risk Premium

Risk premium is excess return that the investors require for holding a risky asset. It is directly proportional to the variance of the returns.

Mathematical Description

Consider an investment with random return x. Investor starts with W and ends with $W(1+x)$. Expected return of x is zero ($\mathrm{E}[x]=0$). Variance of x is $\sigma_x^2 = \mathrm{E}[x^2]$.

Due to concave utility function, in this case Risk-averse investor would prefer zero riskless return, which means

$$\mathrm{E}[u(W(1+x))] \leq u(W).$$

We define the Risk Premium.π, such that investor is indifferent between a random return x and losing a fraction π of wealth for sure:

$$\mathrm{E}[u(W(1+x))] = u(W(1-\pi))$$

Assume that x is close to 0 then the risk premium is small. We can use Taylor expansion on both sides.

For right hand side:

$$\mathrm{E}[u(W(1+x))] = \mathrm{E}[u(W)+u'(W)Wx+0.5u''(W)W^2x^2+\cdots]$$

By simplification we get

$$\mathrm{E}[u(W(1+x))] = u(W)+0.5u''(W)W^2\sigma_x^2$$

Similarly for Left hand side

$u(W(1-\pi)) = u(W) - u'(W)W\pi$

By expanding and simplifying we get

$u(W) - u'(W)W\pi = u(W)+0.5u''(W)W^2\sigma^2_x$

Solving for risk premium π

$$\pi = \frac{-Wu''(W)\sigma^2_x}{2u'(W)}$$

In this equation, it can be seen that the risk premium is directly proportional to variance σ^2_x of the random return x. This means that if the variance of the return is high, investors should demand a higher risk premium.

Reducing risk through diversification

Volatility of the investment can be decreased by making many uncorrelated bets, this is called diversification. We can diversify by constructing a portfolio of investments. Portfolio is a collection of assets.

Mathematical description of portfolio

Suppose that we have a collection of n assets. We have N_i shares of each asset i and the price is P_i. Then the portfolio value V will be

V=$N_1P_1 + N_2P_2 + N_3P_3 +.... N_nP_n$= $\sum_{i=1}^{n} N_iP_i$

Weight of each asset can be found as

$$w_i = \frac{N_i P_i}{V}$$

Expected return and Variance of portfolio

Expected return on portfolio is the weighted average of expected returns on individual assets.

$$E[r_p] = \overline{r_p} = w_1 r_1 + w_2 r_2 + \ldots.. w_n r_n = \sum_{i=1}^{n} w_i r_i$$

To find variance, we can take variance on both sides of the equation.

$$A)\ Var[\overline{r_p}] = Var\left[\sum_{i=1}^{n} w_i r_i\right]$$

We know that for all random variables x_i

$$Var[X_1 + X_2 \cdots + X_N] = \sum_{i=1}^{N} Var[X_i] + \sum_{i=1}^{N} \sum_{j \neq i}^{N} Cov[X_i, X_j]$$

We also know that if x is random variable and a is a constant

$$Var[ax] = a^2 Var[x]$$

Solving for eq A, we get

$$Var[\overline{r_p}] = \sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}, \quad \sigma_{ii} = \sigma_i^2$$

We see that the variance of the portfolio is directly proportional to the individual variance and pairwise covariance.

Simplifying the Variance

We know that

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

Suppose we have an equally weighted portfolio of n assets. In this case:

$w_i = w_j = \frac{1}{n}$

Then the variance will be

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2 \sigma_{ii} + \sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(\frac{1}{n}\right)^2 \sigma_{ij}$$

We can see that we will have n variance terms and $n^2 - n$ covariance terms

$$\sigma_p^2 = \left(\frac{1}{n}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right) + \left(\frac{n^2-n}{n^2}\right)\left(\frac{1}{n^2-n}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sigma_{ij}\right)$$

So we have

$$\sigma_p^2 = \left(\frac{1}{n}\right)(average\ variance) + \left(1 - \frac{1}{n}\right)(average\ covariance)$$

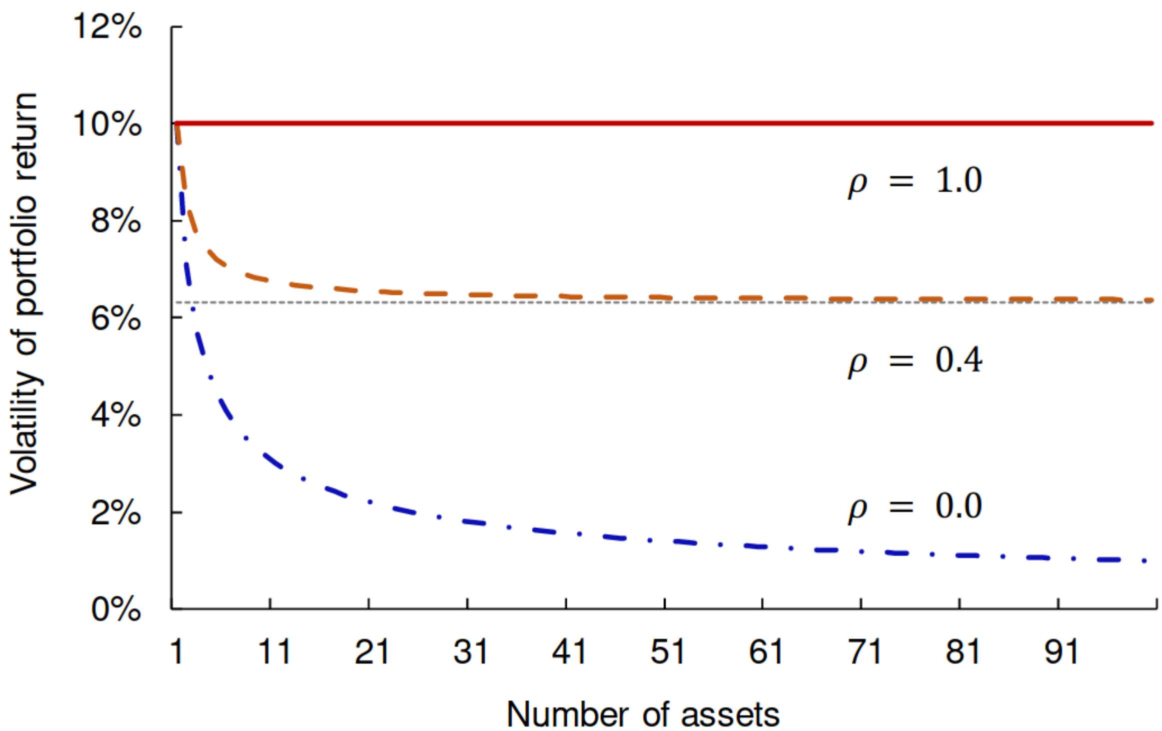
Result

We can see that as n gets large, the contribution of variance term goes to zero and the contribution of covariance goes to average covariance.

This means that if we have a large number of assets, the variance of the portfolio will depend entirely on average covariance. So we can reduce the volatility of our investment by making many uncorrelated bets.

Return correlation and limits of diversification

The result can be visualized by the graph



We can see that initially, volatility of portfolio return decreases sharply as the number of assets increase but gradually it stops decreasing. This is due to the presence of systematic risk. By diversification, we can eliminate individual risk but we cannot eliminate systematic risk.