UCL Mechanical Engineering 2020/2021

ENGF0004 Coursework 1

NCWT3

January 1, 2021

1 Question One

a

b

We are given:

$$f(x) = \frac{x}{\sqrt{1-x}} \tag{1.1}$$

$$f(x) = x\,(1-x)^{-\frac{1}{2}} \tag{1.2}$$

Differentiating three times yields:

$$f'(x) = (1-x)^{-\frac{1}{2}} + \frac{x}{2}\,(1-x)^{-\frac{3}{2}} \tag{1.3}$$

$$f''(x) = \frac{1}{2}\,(1-x)^{-\frac{3}{2}} + \frac{1}{2}\,(1-x)^{-\frac{3}{2}} + \frac{3x}{4}\,(1-x)^{-\frac{5}{2}} \tag{1.4}$$

$$= (1-x)^{-\frac{3}{2}} + \frac{3x}{4}\,(1-x)^{-\frac{5}{2}} \tag{1.5}$$

$$f'''(x) = \frac{3}{2}\,(1-x)^{-\frac{5}{2}} + \frac{3}{4}\,(1-x)^{-\frac{5}{2}} + \frac{15x}{8}\,(1-x)^{-\frac{7}{2}} \tag{1.6}$$

$$= \frac{9}{4}\,(1-x)^{-\frac{5}{2}} + \frac{15x}{8}\,(1-x)^{-\frac{7}{2}} \tag{1.7}$$

Inputting $x = 0$:

$$f(0) = 0 \cdot (1-0)^{-\frac{1}{2}} \tag{1.8}$$

$$= 0 \tag{1.9}$$

$$f'(0) = (1-0)^{-\frac{1}{2}} + \frac{0}{2}\,(1-0)^{-\frac{3}{2}} \tag{1.10}$$

$$= 1 \tag{1.11}$$

$$f''(0) = (1-0)^{-\frac{3}{2}} + \frac{3 \cdot 0}{4}\,(1-0)^{-\frac{5}{2}} \tag{1.12}$$

$$= 1 \tag{1.13}$$

$$f'''(0) = \frac{9}{4}\,(1-0)^{-\frac{5}{2}} + \frac{15 \cdot 0}{8}\,(1-0)^{-\frac{7}{2}} \tag{1.14}$$

$$= \frac{9}{4} \tag{1.15}$$

General form of Maclaurin series:

$$f(x) \approx f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... \tag{1.16}$$

Inputting the above variables into Eq.1.16:

$$f(x) \approx x + \frac{x^2}{2} + \frac{3x^3}{8} \tag{1.17}$$

c

i

We are given:

$$E = \frac{kq}{x^2} \tag{1.18}$$

Sum of electric fields due to both charged particles is:

$$E = \frac{ke}{(x-r)^2} - \frac{ke}{(x+r)^2} \tag{1.19}$$

$$= ke\left[\frac{1}{x^2\left(1-\frac{r}{x}\right)^2} - \frac{1}{x^2\left(1+\frac{r}{x}\right)^2}\right] \tag{1.20}$$

$$E = \frac{ke}{x^2}\left[(1-y)^{-2} - (1+y)^{-2}\right] \tag{1.21}$$

Where $y = \frac{r}{x}$.

ii

Calculation of constants to be used in Maclaurin series expansion:

$$\begin{aligned} f(y) &= (1-y)^{-2} & f(0) &= 1 \\ f'(y) &= 2(1-y)^{-3} & f'(0) &= 2 \\ f''(y) &= 6(1-y)^{-4} & f''(0) &= 6 \\ f'''(y) &= 24(1-y)^{-5} & f'''(0) &= 24 \end{aligned}$$

$$\begin{aligned} g(y) &= (1+y)^{-2} & g(0) &= 1 \\ g'(y) &= -2(1+y)^{-3} & g'(0) &= -2 \\ g''(y) &= 6(1+y)^{-4} & g''(0) &= 6 \\ g'''(y) &= -24(1+y)^{-5} & g'''(0) &= -24 \end{aligned}$$

Inputting the above variables into Eq.1.16:

$$f(y) \approx 1 + \frac{2y}{1!} + \frac{6y^2}{2!} + \frac{24y^3}{3!} + ... \tag{1.22}$$

$$f(y) \approx 1 + 2y + 3y^2 + 4y^3 \tag{1.23}$$

$$g(y) \approx 1 - \frac{2y}{1!} + \frac{6y^2}{2!} - \frac{24y^3}{3!} + ... \tag{1.24}$$

$$g(y) \approx 1 - 2y + 3y^2 - 4y^2 \tag{1.25}$$

Substitution:

$$E \approx \frac{ke}{x^2}\left[f(y) - g(y)\right] \tag{1.26}$$

$$\approx \frac{ke}{x^2}\left[1 + 2y + 3y^2 + 4y^3 - 1 + 2y - 3y^2 + 4y^3\right] \tag{1.27}$$

$$\approx \frac{ke}{x^2}\left[4y + 8y^3\right] \tag{1.28}$$

$$E \approx \frac{4ke}{x^2}\left[y + 2y^3\right] \tag{1.29}$$

iii

$y = 0.01$. Exact:

$$E_E = \frac{ke}{x^2}\left[(1-0.01)^{-2} - (1+0.01)^{-2}\right] \tag{1.30}$$

$$E_E = \frac{ke}{x^2}\left[0.0400080012\right] \tag{1.31}$$

$$\tag{1.32}$$

Approximation:

$$E_A = \frac{ke}{x^2}\left[0.01 + 2\left(0.01\right)^3\right] \tag{1.33}$$

$$E_A = \frac{ke}{x^2}\left[0.010002\right] \tag{1.34}$$

Percentage error:

$$\frac{E_A}{E_E} \cdot 100 = \frac{0.010002}{0.0400080012} \cdot 100 = 25\% \text{ error (2sf)} \tag{1.35}$$

d

We are given:

$$y'' - 2y' + y = te^t \tag{1.36}$$

$$y(0) = 0,\ y'(0) = 1 \tag{1.37}$$

Laplace transformation (from tables):

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{te^t\} \tag{1.38}$$

$$s^2Y(s) - sy(0) - y'(0) - 2\left(sY(s) - y(0)\right) + Y(s) = \frac{1!}{(s-1)^2} \tag{1.39}$$

$$s^2Y(s) - 1 - 2(sY(s) - 1) + Y(s) = \frac{1}{(s-1)^2} \tag{1.40}$$

$$Y(s)\left[s^2 - 2s + 1\right] = \frac{1}{(s-1)^2} \tag{1.41}$$

$$Y(s) = \frac{1}{(s-1)^2\left(s^2 - 2s + 1\right)} \tag{1.42}$$

$$= \frac{1}{(s-1)^2\,(s-1)^2} \tag{1.43}$$

$$Y(s) = \frac{1}{(s-1)^4} \tag{1.44}$$

e

i

$a = 1 \therefore -3 \leq t \leq 3$. Sketch:

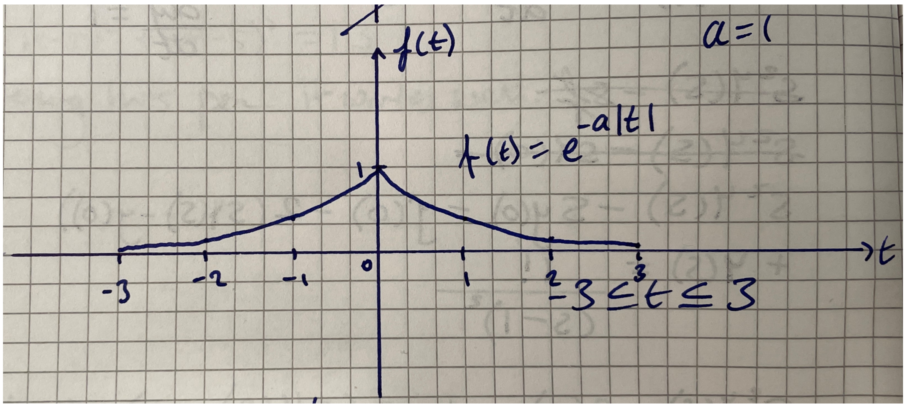

Figure 1:

ii

$$
\begin{aligned}
F(u) &= \int_{t=-\infty}^{0} e^{at} e^{-j2\pi ut}\ \mathrm{dt} + \int_{t=0}^{\infty} e^{-at} e^{-j2\pi ut}\ \mathrm{dt} && (1.45)\\
&= \int_{t=-\infty}^{0} e^{t(a-j2\pi u)}\ \mathrm{dt} + \int_{t=0}^{\infty} e^{-t(a+j2\pi u)}\ \mathrm{dt} && (1.46)\\
&= \frac{1}{(a-j2\pi u)} e^{-t(a-j2\pi u)} \Big|_{t=-\infty}^{0} + \frac{1}{-(a+j2\pi u)} e^{-t(a+j2\pi u)} \Big|_{t=0}^{\infty} && (1.47)\\
&= \frac{1}{a-j2\pi u} + \frac{1}{a+j2\pi u} && (1.48)\\
&= \frac{a+j2\pi u + a - j2\pi u}{a^2+4\pi^2u^2} && (1.49)\\
F(u) &= \frac{2a}{a^2+4\pi^2u^2} && (1.50)
\end{aligned}
$$

iii

Substituting $\omega = 2\pi u$, $\omega^2 = 4\pi^2u^2$:

$$
F(\omega) = \frac{2a}{a^2+\omega^2} \tag{1.51}
$$

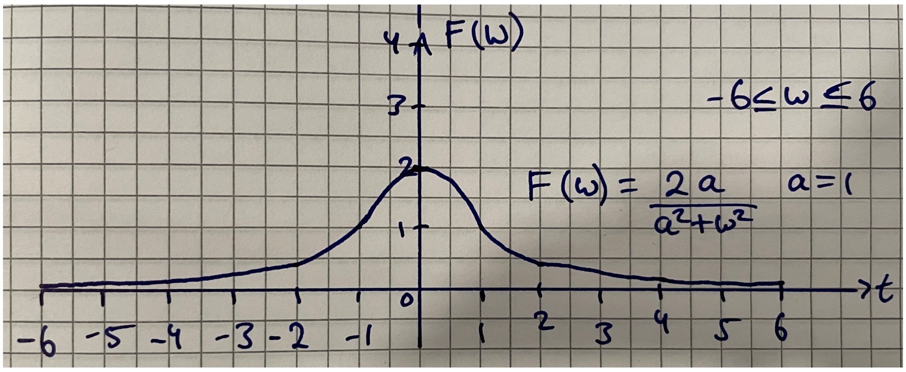

Figure 2:

iv

v

2 Question Two

a

i

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{2.1}$$

We know that $u(x,t) = X(x)T(t)$. Substituting:

$$\frac{\partial XT}{\partial t} = k\frac{\partial^2 XT}{\partial x^2} \tag{2.2}$$

$$X\frac{\partial T}{\partial t} = kT\frac{\partial^2 X}{\partial x^2} \tag{2.3}$$

Divide by XTk:

$$\frac{1}{kT}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\mu \tag{2.4}$$

$$\frac{\partial T}{\partial t} + k\mu T = 0 \tag{2.5}$$

$$T'(t) = -\mu k T(t) \tag{2.6}$$

$$\frac{\partial^2 X}{\partial x^2} + \mu X = 0 \tag{2.7}$$

$$-X''(x) = \mu X(x) \tag{2.8}$$

ii

$$
\begin{aligned}
X''(x) + \mu X(x) &= 0 && (2.9)\\
\text{Let } X(x) &= e^{mx} && (2.10)\\
X'(x) &= me^{mx} && (2.11)\\
X''(x) &= m^2 e^{mx} && (2.12)\\
m^2 + \mu &= 0 && (2.13)\\
m &= \pm j\sqrt{\mu} && (2.14)\\
& && (2.15)
\end{aligned}
$$

General solution:

$$X(x) = A\cos\left(\sqrt{\mu}x\right) + B\sin\left(\sqrt{\mu}x\right) \tag{2.16}$$

Boundary conditions:

$$
\begin{aligned}
u\,(0, t) &= 0 && (2.17)\\
X(0)T(t) &= 0 && (2.18)\\
& && (2.19)
\end{aligned}
$$

We require that $X(0) = 0$. Hence:

$$
\begin{aligned}
X(0) &= A\cos(0) + B\sin(0) && (2.20)\\
\therefore A &= 0 && (2.21)\\
X(x) &= B\sin\left(\sqrt{\mu}x\right) && (2.22)\\
u\,(l, t) &= 0 && (2.23)\\
X(l) &= B\sin\left(\sqrt{\mu}x\right) = 0 && (2.24)\\
\sqrt{\mu}x &= n\pi \text{ where } n = 1,\ 2,\ 3,\ ... && (2.25)\\
\mu &= \frac{n^2\pi^2}{x^2} && (2.26)
\end{aligned}
$$

iii

b

i

ii

iii

iv

v