UCL Mechanical Engineering 2020/2021

ENGF0004 Open Book Exam 1

NCWT3

December 10, 2020

1 Question 1

a

Boundary conditions

$$\left.\frac{dx}{dt}\right|_{t=0} = v,\ x(0) = k \tag{1.1}$$

Laplace transform:

$$\frac{d^2x}{dt^2} + 2p\omega_0\frac{dx}{dt} + \omega_0^2 = 0 \tag{1.2}$$

$$s^2X(s) - sX(0) - X'(0) + 2p\omega_0 sX(s) - 2p\omega_0 X(0) + \omega_0^2 X(s) = 0 \tag{1.3}$$

$$X(s)\left(s^2 - 2p\omega_0 + \omega_0^2\right) - sX(0) - X'(0) - 2p\omega_0 X(0) = 0 \tag{1.4}$$

$$X(s)\left(s^2 - 2p\omega_0 + \omega_0^2\right) = sk + v + 2p\omega_0 k \tag{1.5}$$

$$X(s) = \frac{sk + v + 2p\omega_0 k}{s^2 - 2p\omega_0 + \omega_0^2} \tag{1.6}$$

b

Completing the square

$$s^2 + 2p\omega_0 + \omega_0^2 \tag{1.7}$$

$$(s + p\omega_0)^2 + \omega_0^2 - p^2\omega_0^2 \tag{1.8}$$

Substituting:

$$X(s) = \frac{ks + v + sp\omega_0 k}{(s + p\omega_0)^2 + \omega_0^2(1 - p^2)} \tag{1.9}$$

$$X(s) = \frac{-\frac{sv}{p\omega_0} + v - \frac{2p\omega_0 v}{p\omega_0}}{(s + p\omega_0)^2 + \omega_0^2(1 - p^2)} \tag{1.10}$$

$$X(s) = \frac{-\frac{sv}{p\omega_0} - v}{(s + p\omega_0)^2 + \omega_0^2(1 - p^2)} \tag{1.11}$$

$$X(s) = -\frac{v}{p\omega_0} \cdot \frac{s + p\omega_0}{(s + p\omega_0)^2 + \omega_0^2(1 - p^2)} \tag{1.12}$$

$$\tag{1.13}$$

From Laplace table:

$$x(t) = -\frac{v}{p\omega_0} e^{-p\omega_0 t} \cos\left(\omega_0 \sqrt{1 - p^2} t\right) \tag{1.14}$$

Constants:

$$a = -p\omega_0 \tag{1.15}$$

$$b = \omega_0 \sqrt{1 - p^2} \tag{1.16}$$

$$C = -\frac{v}{p\omega_0} \tag{1.17}$$

c

The magnitude of the b term has the term $\sqrt{1 - p^2}$. Hence, when $0 < p < 1$, we see that the cosine term has a magnitude and $x(t)$ is sinusoidal. For values of p greater than 1, we get a complex input into our cosine function and get a complex $x(t)$. We also see that:

$$\lim_{p \to 0} \left(-\frac{v}{p\omega_0}\right) \to \infty, \ p \neq 0 \tag{1.18}$$

d

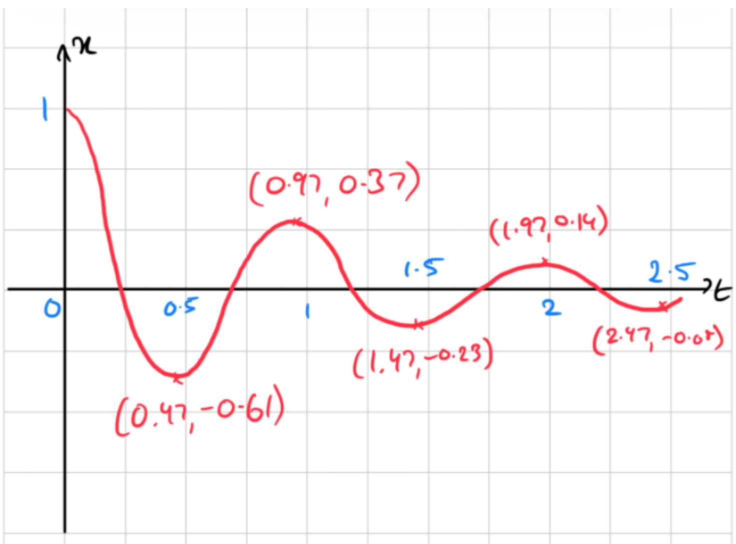

Figure 1: Graph to show the solution of $x(t)$, when $a = -1$, $b = 2\pi$, $C = 1$.

e

Constants:

$$a = -1 \tag{1.19}$$
$$b = 2\pi \tag{1.20}$$
$$C = 1 \tag{1.21}$$

Derivatives:

$$x(t) = Ce^{at}\cos(bt) \tag{1.22}$$
$$x(0.5) = -e^{0.5} \tag{1.23}$$
$$x'(t) = aCe^{at}\cos(bt) - bCe^{at}\sin(bt) \tag{1.24}$$
$$x'(0.5) = e^{-0.5} \tag{1.25}$$
$$x''(t) = a^2Ce^{at}\cos(bt) - abCe^{at}\sin(bt) - \left[abCe^{at} + b^2Ce^{at}\cos(bt)\right] \tag{1.26}$$
$$x''(0.5) = -e^{0.5} + 4\pi^2e^{-0.5} \tag{1.27}$$

Final equation:

$$x(t) \approx -e^{-0.5} + e^{-0.5}(t-0.5) + \frac{-e^{-0.5} + 4\pi^2e^{-0.5}}{2}(t-0.5)^2 + ... \tag{1.28}$$

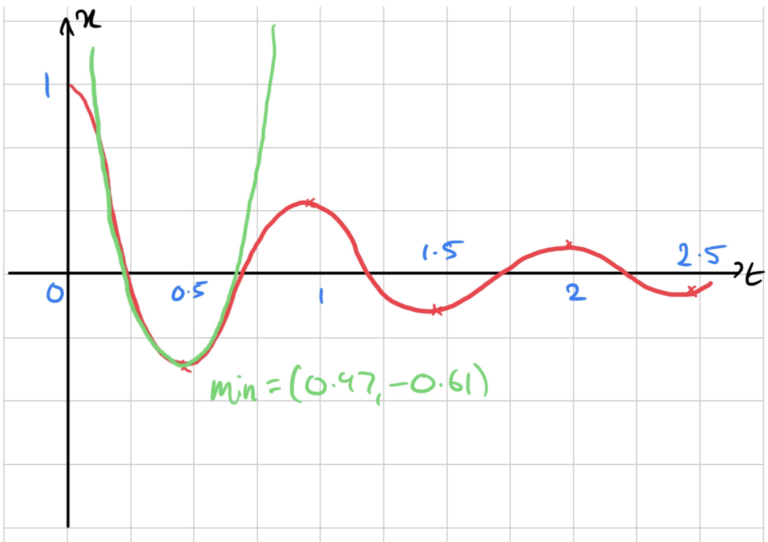

Figure 2: Graph to show the solution of $x(t)$, when $a = -1$, $b = 2\pi$, $C = 1$ (red), alongside Taylor series approximation (green).

2 Question 2

Boundary conditions: Assuming a solution such that $u(x, y) = X(x)Y(y)$, $X(x) = X$ and $Y(y) = y$:

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0 \tag{2.1}$$
$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = k \tag{2.2}$$

For a positive constant

Solving $X(x)$:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k \tag{2.3}$$
$$\frac{d^2X}{dx^2} + kX = 0 \tag{2.4}$$
$$m^2 + k = 0 \tag{2.5}$$
$$m = \pm i\sqrt{k} \tag{2.6}$$
$$\text{Let } k = k_1^2 \tag{2.7}$$
$$X(x) = A\cos(k_1 x) + B\sin(k_1 x) \tag{2.8}$$
$$\tag{2.9}$$

Solving $Y(y)$:

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = k \tag{2.10}$$
$$\frac{d^2Y}{dy^2} - kY = 0 \tag{2.11}$$
$$m^2 - k_1^2 = 0 \tag{2.12}$$
$$m = \pm\sqrt{k_1^2} \tag{2.13}$$
$$Y(y) = C'e^{k_1 y} + D'e^{-k_1 y} \tag{2.14}$$
$$Y(y) = C\cosh(k_1 y) + D\sinh(k_1 y) \tag{2.15}$$

Hence for positive constant k:

$$u_1(x,y) = \big[A\cos(k_1 x) + B\sin(k_1 x)\big] \cdot \big[C\cosh(k_1 y) + D\sinh(k_1 y)\big] \tag{2.16}$$

At $x = 0$	$\frac{\partial u}{\partial x} = 0$
At $x = a$	$\frac{\partial u}{\partial x} = 0$
At $x = 0$	$u = 0$
At $y = b$	$u = f(x)$

Table 1: Boundary conditions

Hence, for $y = 0$, $u = 0$ leading to $X(x)Y(y) = 0$. When $y = 0$, $\cosh 0 = 1$, $\sinh(0) = 0$, hence:

$$u_1(x,0) = \big[A\cos(k_1 x) + B\sin(k_1 x)\big] \cdot [C] = 0 \tag{2.17}$$

We see here that C must be 0. At $x = 0$, $\frac{du}{dx} = 0$:

$$\left.\frac{du}{dx}\right|_{x=0} = \big[-Ak_1\sin(k_1 x) + Bk_1\cos(k_1 x)\big] \cdot [D\sinh k_1 y] = 0 \tag{2.18}$$
$$-Ak_1\sin(k_1 x) = 0 \tag{2.19}$$
$$\cos(k_1 x) = 0 \tag{2.20}$$

We see here that B must be 0. At $x = a$, $\frac{du}{dx} = 0$:

$$\left.\frac{du}{dx}\right|_{x=a} = \big[-Ak_1\sin(k_1 x)\big] \cdot [D\sinh k_1 y] = 0 \tag{2.21}$$

If A or $D = 0$, we get a trivial solution. Hence:

$$\sin(k_1 a) = 0 \tag{2.22}$$

Hence:

$$u_1(x, y) = \left[A\cos(k_1 x)\right] \cdot \left[D\sinh k_1 y\right] \tag{2.23}$$