# UCL Mechanical Engineering 2020/2021

ENGF0004 Coursework 2

NCWT3

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## 1 Question 1

### 1.1 a

For the line integral to be independent from the path of integration, the following conditions must be fulfilled:

$$I = \int_A^B \left( \frac{\partial u}{\partial x} \, \mathrm{d}x + \frac{\partial u}{\partial y} \, \mathrm{d}y \right) \tag{1.1}$$

$$P\left(x,\, y\right) = \frac{\partial u}{\partial x} \text{ and } Q\left(x,\, y\right) = \frac{\partial u}{\partial y} \tag{1.2}$$

$$\frac{\partial P\left(x,\, y\right)}{\partial y} = \frac{\partial Q\left(x,\, y\right)}{\partial x} \tag{1.3}$$

Considering the integral:

$$I = \int_A^B \left[ e^{-\alpha xy} \left( \frac{\alpha - 2}{x} \right) \mathrm{d}x - \frac{1}{\alpha y} \left( e^{-\alpha xy} - 1 \right) \mathrm{d}y \right] \tag{1.4}$$

$$P\left(x,\, y\right) = e^{-\alpha xy} \left( \frac{\alpha - 2}{x} \right) \text{ and } Q\left(x,\, y\right) = -\frac{1}{\alpha y} \left( e^{-\alpha xy} - 1 \right) \tag{1.5}$$

$$\frac{\partial P\left(x,\, y\right)}{\partial y} = -\alpha x \left( \frac{\alpha - 2}{x} \right) e^{-\alpha xy} = \left( 2\alpha - \alpha^2 \right) e^{-\alpha xy} \tag{1.6}$$

$$\frac{\partial Q\left(x,\, y\right)}{\partial x} = -\frac{1}{\alpha y} \left( -\alpha y \right) e^{-\alpha xy} = e^{-\alpha xy} \tag{1.7}$$

$$\therefore 2\alpha e^{-\alpha xy} - \alpha^2 e^{-\alpha xy} = e^{-\alpha xy} \tag{1.8}$$

$$e^{-\alpha xy} \left( \alpha^2 - 2\alpha + 1 \right) = 0 \tag{1.9}$$

$$e^{-\alpha xy} = 0 \rightarrow \text{no solutions} \tag{1.10}$$

$$\left(\alpha - 1\right)^2 = 0 \tag{1.11}$$

$$\alpha = 1 \tag{1.12}$$

## 1.2 b

Calculating the line integral of 1.13 from $O\,(0,\,0)$ to $A(1,\,e-1)$ along $y = e^x - 1$:

$$I = \int_O^A \left(ye^{-2x}\right)(\mathrm{d}x + \mathrm{d}y) \tag{1.13}$$

$$y = e^x - 1 \tag{1.14}$$

$$\mathrm{d}y = e^x\,\mathrm{d}x \tag{1.15}$$

$$I = \int_0^1 \left((e^x - 1)\left(e^{-2x}\right) + (e^x - 1)\left(e^{-2x}\right)(e^x)\right)\mathrm{d}x \tag{1.16}$$

$$= \int_0^1 \left(e^{-x} - e^{-x} - e^{-2x} + 1\right)\mathrm{d}x \tag{1.17}$$

$$= \int_0^1 \left(1 - e^{-2x}\right)\mathrm{d}x \tag{1.18}$$

$$= \left[x + \frac{e^{-2x}}{2}\right]_0^1 \tag{1.19}$$

$$= 1 + \frac{e^{-2}}{2} - 0 - \frac{1}{2} \tag{1.20}$$

$$I = \frac{1}{2}\left(e^{-2} + 1\right) \tag{1.21}$$

## 1.3 c

### 1.3.1 i

$$\underline{F}\,(x,\,y,\,z) = \begin{pmatrix} \frac{y}{x^2} \\ \frac{x}{y^2} \end{pmatrix} \tag{1.22}$$

$$\nabla \cdot \underline{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{y}{x^2} \\ \frac{x}{y^2} \end{pmatrix} \tag{1.23}$$

$$= \frac{\partial}{\partial x}\left(\frac{y}{x^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{y^2}\right) \tag{1.24}$$

$$= -\frac{2y}{x^3} - \frac{2x}{y^3} \tag{1.25}$$

$$= -2\left(\frac{y}{x^3} + \frac{x}{y^3}\right) \tag{1.26}$$

### 1.3.2 ii

$$I = \int_1^2 \int_1^2 \left( -2 \left( \frac{y}{x^3} + \frac{x}{y^3} \right) \right) \mathrm{d}x \, \mathrm{d}y \tag{1.27}$$

$$= \int_1^2 \left[ -2 \left( \frac{y}{-2x^2} + \frac{x^2}{2y^3} \right) \right]_1^2 \mathrm{d}y \tag{1.28}$$

$$= \int_1^2 \left[ -2 \left( -\frac{y}{8} + \frac{2}{y^3} + \frac{y}{2} - \frac{1}{2y^3} \right) \right] \mathrm{d}y \tag{1.29}$$

$$= \int_1^2 \left( -\frac{3y}{4} - \frac{3}{y^3} \right) \mathrm{d}y \tag{1.30}$$

$$= \left[ -\frac{3y^2}{8} + \frac{3}{2y^2} \right]_1^2 \tag{1.31}$$

$$= -\frac{3}{2} + \frac{3}{8} + \frac{3}{8} - \frac{3}{2} \tag{1.32}$$

$$I = -\frac{9}{4} \tag{1.33}$$

## 1.4 d

### 1.4.1 i

$$I = \int (\sin x \cos y \, \mathrm{d}y + \cos x \sin y \, \mathrm{d}y) \tag{1.34}$$

$$y = 0 \qquad \mathrm{d}y = 0 \tag{1.35}$$

$$I_{AB} = \int_{x=0}^{\pi} (\sin x) \, \mathrm{d}x = [-\cos x]_0^{\pi} = 2 \tag{1.36}$$

$$x = \pi \qquad \mathrm{d}x = 0 \tag{1.37}$$

$$I_{BC} = \int_{y=0}^{\pi} (-\sin y) \, \mathrm{d}y = [\cos y]_0^{\pi} = -2 \tag{1.38}$$

$$\therefore I = I_{AB} + I_{BC} = 2 - 2 = 0 \tag{1.39}$$

### 1.4.2 ii

$$I = \int (\sin x \cos y \, \mathrm{d}y + \cos x \sin y \, \mathrm{d}y) \tag{1.40}$$

$$y = x \qquad \mathrm{d}y = \mathrm{d}x \tag{1.41}$$

$$I_{AC} = \int_0^{\pi} (\sin x \cos x + \sin x \cos x) \, \mathrm{d}x \tag{1.42}$$

$$= \int_0^{\pi} \big( \sin(2x) \big) \, \mathrm{d}x \tag{1.43}$$

$$I_{AC} = \left[ -\frac{1}{2} \cos(2x) \right]_0^{\pi} = \frac{1}{2} - \frac{1}{2} = 0 \tag{1.44}$$

$$\tag{1.45}$$