0.1 Derivation of Bending Equations

0.1.1 Equilibrium – Beam Bending Equations

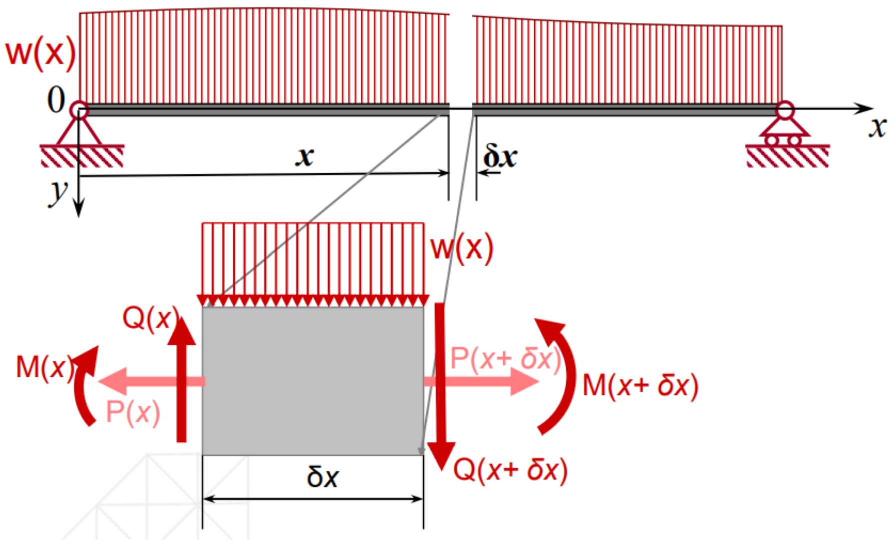


Figure 1: Horizontal beam with vertical distributed load. We focus a section of the beam with the very small length δx.

y-direction

$$Q(x+\delta x) - Q(x) + w(x)\delta x = 0 \tag{1}$$

x-direction

$$M(x+\delta x) - M(x) - Q(x)\delta x + w(x) \cdot \delta x \cdot \frac{\delta x}{2} = 0 \tag{2}$$

We assume that δx is so small that no matter the distribution of the load on the beam, $w(x)$ is constant along the investigated beam section.

Rearranging the terms in the y direction yields:

$$\frac{Q(x+\delta x) - Q(x)}{\delta x} = -w(x) \tag{3}$$

Taking δx to the smallest limit (0), the above equation can be written as:

$$\delta x = 0 \tag{4}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -w(x) \tag{5}$$

$$w(x) = -\frac{\mathrm{d}Q}{\mathrm{d}x} \tag{6}$$

The load w in the y direction is the derivative of the shear force Q. Integrating equation (6):

$$Q = -\int w \, \mathrm{d}x \tag{7}$$

We consider the bending moment at the distance δx (right-side for the example). Rearranging the terms in the x direction yields:

$$\frac{M(x+\delta x) - M(x)}{\delta x} = Q(x) - \frac{1}{2} w(x) \delta x \tag{8}$$

Taking δx to the smallest limit (0), the above equation can be written as:

$$\delta x = 0 \tag{9}$$

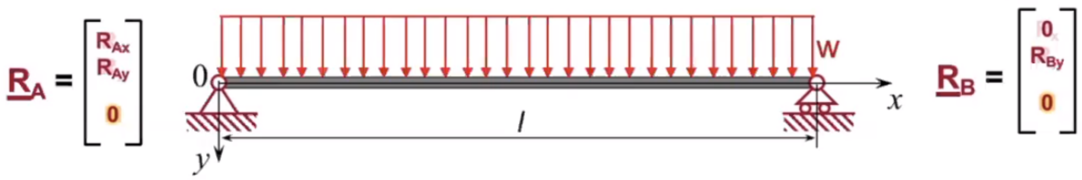
$$Q(x) = \frac{\mathrm{d}M}{\mathrm{d}x} \tag{10}$$

$$\tag{11}$$

The shear force Q is the derivative of the bending Moment M. Integrating equation (11):

$$M = \int Q \, \mathrm{d}x \tag{12}$$

Example:



$$Q = -\int w \, \mathrm{d}x \tag{13}$$

$$= -wx + Q_0 \tag{14}$$

$$\tag{15}$$

$$M = \int Q \, \mathrm{d}x \tag{16}$$

$$= \int (-wx + Q_0) \, \mathrm{d}x \tag{17}$$

$$= -\frac{1}{2} wx^2 + Q_0 \cdot x + M_0 \tag{18}$$

Applying the boundary conditions:

$$M(0) = 0 \rightarrow -\frac{1}{2} w \cdot 0^2 + Q_0 \cdot 0 + M_0 = 0 \tag{19}$$

$$M_0 = 0 \tag{20}$$

$$\tag{21}$$

$$M(l) = 0 \rightarrow -\frac{1}{2} wl^2 + Q_0 \cdot l + 0 = 0 \tag{22}$$

$$Q_0 = \frac{1}{2} wl \tag{23}$$

Overall:

$$Q = -wx + \frac{1}{2}wl \tag{24}$$

$$M = -\frac{1}{2}wx^2 + \frac{1}{2}wlx \tag{25}$$

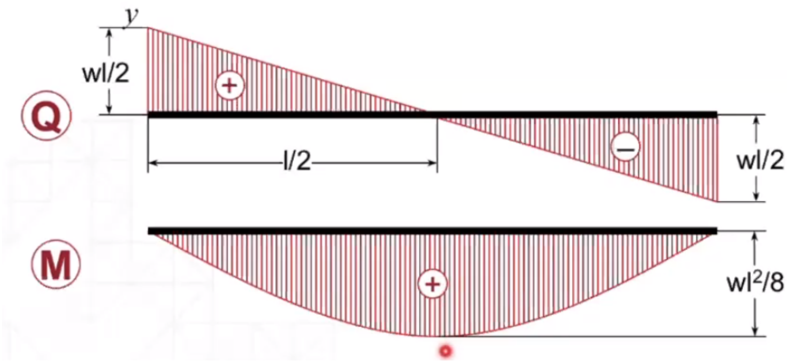


Figure 2: The shear force and bending moment varying along x

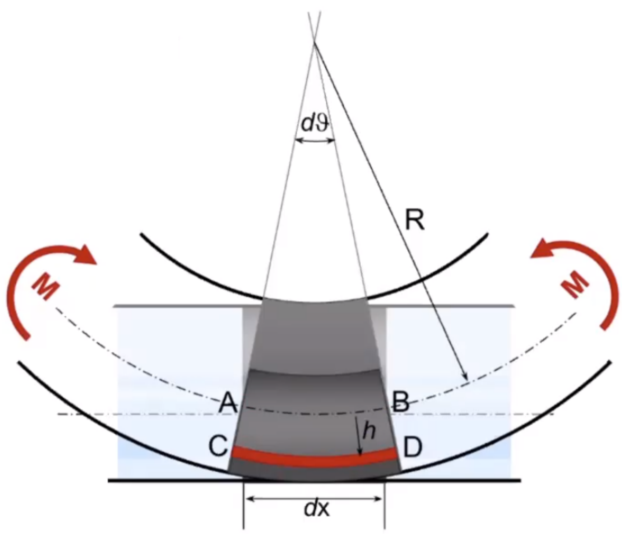
0.2 Differential Equations for Deflection

0.2.1 Theory of Pure Bending

- The beam is initially straight and unstressed;
- The beam material is perfectly homogeneous and isotropic;
- Plane cross-sections remain plane before and after bending;
- Every cross-section in the beam is symmetrical about the plane of bending;
- There is no resultant force perpendicular to any cross-section.
- The elastic limit is nowhere exceeded;
- Young's Modulus for the material is the same in tension and compression;

0.2.2 Compatibility - Strains in Pure Bending

Consider a portion of the length dx from the beam subject to uniform bending.



Lower fibres stretch and upper fibres shorten. Hence, since cross-sections remain plane, there must be a plane where the fibre elongation is zero. This is called **neutral plane** (it is a plane because section is symmetrical) and the intersection with the plane of bending is called **neutral axis**.

If we indicate with R the radius of curvature of the neutral axis, along the neutral plane:

$$\overline{AB} = \widehat{AB} \tag{26}$$

$$\widehat{AB} = \mathrm{d}x = R\,\mathrm{d}\theta \tag{27}$$

For a generic plane CD, distant h from N.A.:

$$\widehat{CD} = (R+h)\,\mathrm{d}\theta \tag{28}$$

The longitudinal strain of CD is given by:

$$\epsilon(h) = \frac{elongation}{initial\ length} = \frac{\widehat{CD} - \overline{CD}}{\overline{CD}} \tag{29}$$

$$\tag{30}$$

$$= \frac{(R+h)\cdot \mathrm{d}\theta - R\cdot \mathrm{d}\theta}{R\cdot \mathrm{d}\theta} = \frac{h}{R} \tag{31}$$

$$\tag{32}$$

$$\epsilon(h) = \frac{h}{R} \tag{33}$$

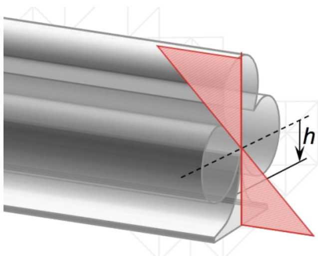


Figure 3: Strains are distributed linearly across the section

Stresses and strains can be associated with each other. For most materials, under small deformation:

$$\sigma = E \cdot \epsilon \tag{34}$$

Where:

- σ is the stress
- E is the Young Modulus
- ϵ is the strain

Therefore, from equations (33) and (34):

$$\sigma_x(h) = E\frac{h}{R} \tag{35}$$

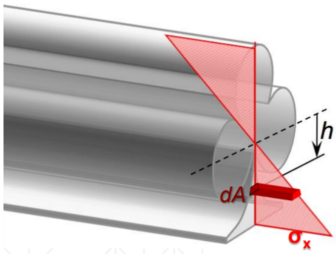
0.2.3 Constitutive - Stress-Curvature Relation

Stress is also distributed linearly across the section, being 0 at the neutral plane and maximum (in tension and compression) at the outer surfaces, where the distance from the neutral plane is maximum.

$$\sigma_{min} = E\frac{h_{min}}{R} \tag{36}$$
$$\sigma_{max} = E\frac{h_{max}}{R} \tag{37}$$

The minimum value is at the compression and the maximum value is at the tension. This just follows the convention; the tension is taken as positive and compression is taken as negative, in terms of the stress direction.

0.2.4 Equilibrium - Force-Stress Relation



Considering an elemental area dA, the force associated with bending stress is:

$$\mathrm{d}F_x = \sigma_x \cdot \mathrm{d}A \tag{38}$$
$$= E\frac{h}{R} \cdot \mathrm{d}A \tag{39}$$

For the force equilibrium of the entire section:

$$\int_A \mathrm{d}F_x = \int_A E\frac{h}{R} \cdot \mathrm{d}A = 0 \tag{40}$$
$$\tag{41}$$

E and R are constants, so they don't affect the integral and can be taken out. Hence the first moment of area is:

$$\int_A h \cdot \mathrm{d}A = 0 \tag{42}$$

The first moment of area of a section is zero if it is calculated about the centroid. **The neutral axis corresponds to the centroid of the section.**

0.2.5 Equilibrium - Bending-Stress Relation

Considering an elemental area dA, the internal moment produced by the bending stress is:

$$\mathrm{d}M = \sigma_x \cdot h \cdot \mathrm{d}A \tag{43}$$
$$= \mathrm{d}F_x \cdot h \tag{44}$$
$$= E\frac{h}{R} \cdot h \cdot \mathrm{d}A \tag{45}$$

Moment of the entire section:

$$M = \int_A \mathrm{d}M = \int_A E\frac{h^2}{R} \cdot \mathrm{d}A = \frac{E}{R}\int_A h^2 \cdot \mathrm{d}A \tag{46}$$

The second moment of area is:

$$\int_A h^2 \cdot \mathrm{d}A = I \tag{47}$$

The second moment of area represents how easy or difficult it is to bend a beam, depending on the shape of the cross section. The overall relation is:

$$M = \frac{EI}{R} \tag{48}$$

This defines how the cross section, elastic modulus, curvature, and bending moment are related to each other.

0.2.6 Solid Mechanics Equations

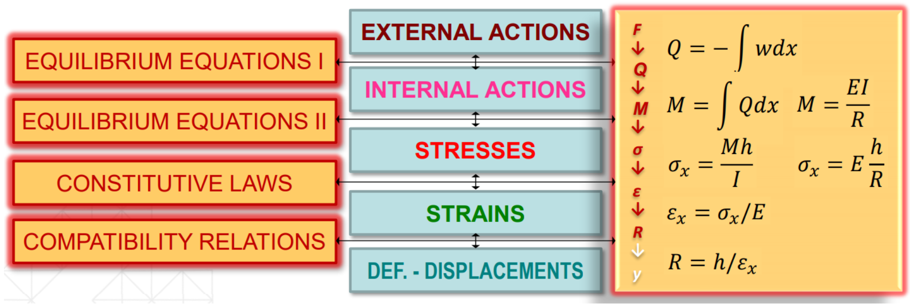


Figure 4: The relationships between Force, Shear Force, Bending Moment, Stress, Strain, Curvature

0.2.7 Geometric - Slope-Deflection Relation

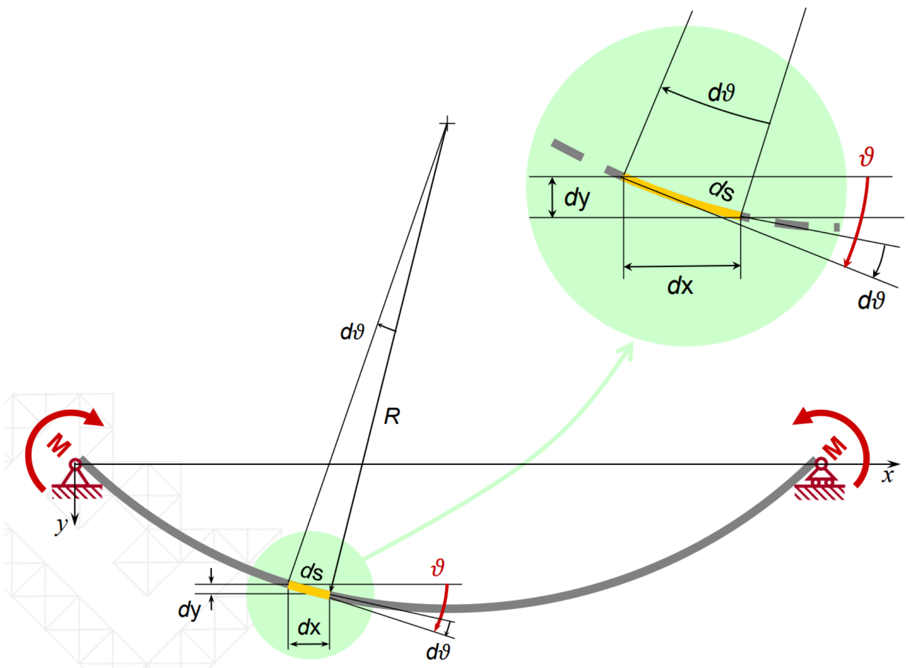


Figure 5: The geometry of the beam under bending. The direction of the angle θ is based on the "right-hand rule".

For infinitesimal deformations:

$$\mathrm{d}x \approx \mathrm{d}s = -R \cdot \mathrm{d}\theta \tag{49}$$

$$\frac{1}{R} = -\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{50}$$

The negative sign comes from the situation that the x axis direction is towards right while the angle (θ) direction is to the left. Hence, they are opposite. Since:

$$M = \frac{EI}{R} \tag{51}$$

$$\rightarrow M = -EI\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{52}$$

Assuming anlge θ is small:

$$\theta \approx \tan(\theta) = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{53}$$

- The load w in the y direction is the derivative of the shear force Q.
- The shear force Q is the derivative of the bending moment M.
- The bending moment M is proportional to the derivative of the slope θ.
- The slope θ is the derivative of the deflection $y : \theta = \frac{\mathrm{d}y}{\mathrm{d}x}$

0.2.8 Stresses Due to Shear Force

The shear force also produces stresses into the section (shear stresses). However:

- They are much lower than bending stresses
- They are zero at surfaces, where bending stresses are maximum
- Their effect on the deformation is negligible compared to bending stresses

In general, neglecting the shear stresses due to the shear force is a good approximation for both the calculation of failure and deflections.

0.2.9 Summary of Beam Bending Equations

Deflection:

$$y \tag{54}$$

Slope:

$$\theta = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{55}$$

Bending Moment:

$$M = -EI\frac{\mathrm{d}\theta}{\mathrm{d}x} = -EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \tag{56}$$

Shear Force:

$$Q = \frac{\mathrm{d}M}{\mathrm{d}x} = -EI\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} \tag{57}$$

Load Distribution:

$$w = -\frac{\mathrm{d}Q}{\mathrm{d}x} = EI\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} \tag{58}$$

The equations above give a scenario of being given deflection y, and eventually finding load distribution w. However, the inverse can also occur where load distribution w is given, and the y can be found through integration:

Load Distribution

$$w \tag{59}$$

Shear Force:

$$Q = -\int w \cdot \mathrm{d}x \tag{60}$$

Bending Moment:

$$M = \int Q \cdot \mathrm{d}x = -\int\int w \cdot \mathrm{d}x\,\mathrm{d}x \tag{61}$$

Slope:

$$\theta = -\frac{1}{EI}\int M \cdot \mathrm{d}x = \frac{1}{EI}\int\int\int w \cdot \mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x \tag{62}$$

Deflection:

$$y = \int \theta \cdot \mathrm{d}x = \frac{1}{EI}\int\int\int\int w \cdot \mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x \tag{63}$$

0.2.10 Direct Integration Method

STEP 1: Determination of Support Reactions

Apply to the body the force and moment equilibrium equations to find support reactions (it is possible only if the system is statically determinate).

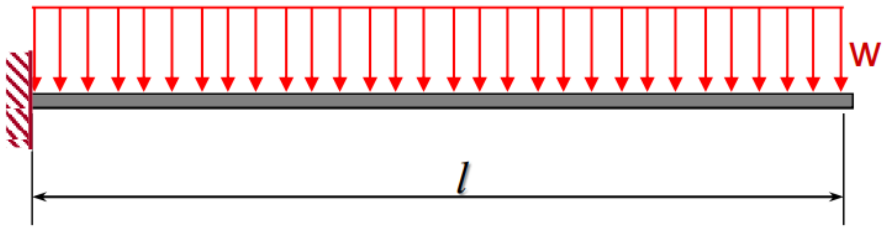
STEP 2: Determination of Deflection

1. Write bending moment expression

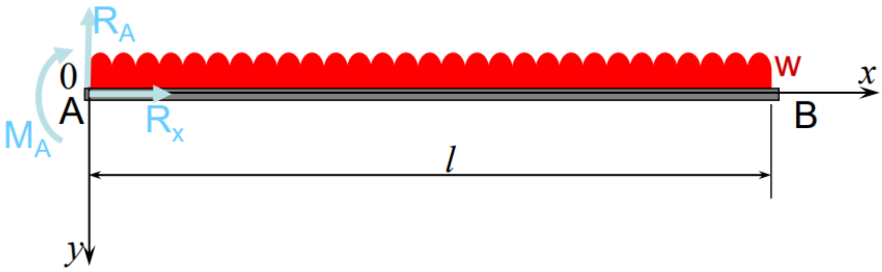
2. Use double integration on bending moment expression. This would result in 2 constants of integration.

3. Use boundary conditions to determinate constant of integration.

Example: Uniformly Distributed Load on a Cantilever Beam



Determination of Support Reactions:



$$\vec{R_A} = \begin{bmatrix} R_x \\ R_y \\ M_A \end{bmatrix}$$

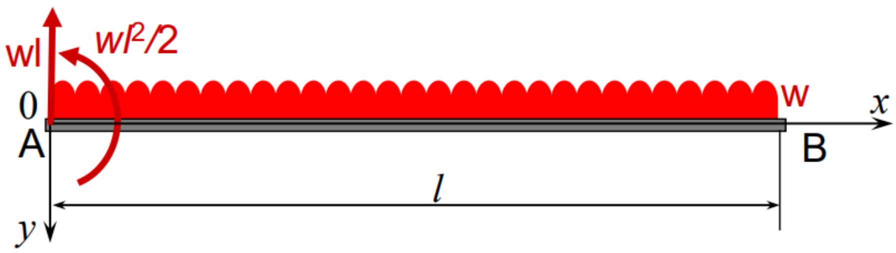
$$\sum F_x : R_x = 0 \quad (64)$$

$$\sum F_y : R_y = wl \quad (65)$$

$$\sum M : M_A + wl\frac{l}{2} = 0 \quad (66)$$

$$M_A = -\frac{wl^2}{2} \quad (67)$$

Determination of Deflection:



Direct Integration:

$$Q = -\int w \cdot \mathrm{d}x = -wx + Q_0 \tag{68}$$

$$\tag{69}$$

$$M = \int Q \cdot \mathrm{d}x \tag{70}$$

$$= \int (-wx + Q_0)\,\mathrm{d}x \tag{71}$$

$$= -\frac{1}{2}wx^2 + Q_0 x + M_0 \tag{72}$$

Boundary Conditions:

$$Q(0) = R_y = wl \tag{73}$$

$$\rightarrow Q_0 = wl \tag{74}$$

$$M(0) = M_A = -\frac{1}{2}wl^2 \tag{75}$$

$$\rightarrow M_0 = -\frac{1}{2}wl^2 \tag{76}$$

Therefore:

$$Q = -w(x + l) \tag{77}$$

$$M = -\frac{1}{2}wx^2 + wlx - \frac{1}{2}wl^2 \tag{78}$$

Relating Deflection to the Bending Moment:

$$M = -\frac{1}{2}wx^2 + wlx - \frac{1}{2}wl^2 \tag{79}$$

$$\theta = -\frac{1}{EI}\int M \cdot \mathrm{d}x = -\frac{1}{EI}\left(-\frac{1}{6}wx^3 + \frac{1}{2}wlx^2 - \frac{1}{2}wl^2x\right) + \theta_0 \tag{80}$$

$$y = \int \theta \cdot \mathrm{d}x = -\frac{1}{EI}\left(-\frac{1}{24}wx^4 + \frac{1}{6}wlx^3 - \frac{1}{4}wl^2x^2\right) + \theta_0 x + y_0 \tag{81}$$

$$\tag{82}$$

Boundary Conditions:

$$\theta(0) = 0 \rightarrow \theta_0 = 0 \tag{83}$$

$$y(0) = 0 \rightarrow y_0 = 0 \tag{84}$$

Therefore:

$$y = \frac{1}{EI}\left(\frac{1}{24}wx^4 - \frac{1}{6}wlx^3 + \frac{1}{4}wl^2x^2\right) \tag{85}$$

$$y_{max} = y(l) = \frac{wl^4}{8EI} \tag{86}$$