MECH0009 Topic Notes

UCL

HD

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Contents

1 Fluid machines and performance of pumps **4**

1.1 Fluid Machines 4

1.1.1 Positive-displacement pumps 5

1.1.2 Dynamic pumps 5

1.1.3 Positive-displacement reciprocation pumps 6

1.1.4 Positive-displacement rotary pumps 7

1.1.5 Positive-displacement peristaltic pumps 8

1.1.6 Roto-dynamic pumps 8

1.2 Dynamic pump performance 9

1.2.1 The Bernoulli equation 9

1.2.2 Hydraulic heads 10

1.2.3 Power and efficiency 11

1.2.4 Performance curves 11

2 Centrifugal pumps and Blade design **14**

2.1 Centrifugal pumps 14

2.1.1 Functional principle 14

2.1.2 Functional components 14

2.1.3 Pumps arrangement 15

2.1.4 End-suction pump 16

2.1.5 The Impeller 17

2.1.6 Flow into the impeller 18
2.1.7 Euler's turbomachine equation 22
2.1.8 Flow into the impeller 22

Chapter 1

Fluid machines and performance of pumps

1.1 Fluid Machines

Fluid machines are devices that operate by exchanging energy with a fluid. Fluid is a phase of matter unable to support a shear stress in static equilibrium. Fluids do not resist to deformation and take the shape of the containers (they flow). Fluids are also divided into liquids and gases. **Pumps** add energy to the fluid, resulting in an increase in fluid pressure across the machine. **Turbines** extract energy from the fluid, resulting in a decrease in fluid pressure across the machine.

Fluid machines are classified on the basis of the movement producing the energy transfer.

- **Reciprocating machines** - energy is supplied or extracted by the reciprocating motion of a plunger or piston
- **Turbomachines** - energy is supplied or extracted by a rotating shaft

Fluid machines can also be classified on the basis of the way of treating the volume of fluid.

- **Positive-displacement machines** - energy is supplied or extracted to/from a closed volume of fluid
- **Dynamic machines** - energy is supplied or extracted to/from an open volume (usually through rotating blades)

Taking into account of both classifications, pumps can be divided into three main categories:

- **Reciprocating positive-displacement machines** - energy is supplied or extracted by the reciprocating motion of a piston acting on the boundaries of a closed volume of a fluid
- **Positive-displacement rotary machines** - energy is supplied or extracted by the movement of rotating components acting on a closed volume of fluid
- **Roto-dynamic machines** - energy is supplied or extracted to an open volume by the movement of rotating blades

1.1.1 Positive-displacement pumps

Advantages

- They are less traumatic for the fluid - are ideal to handle with shear sensitive fluids such as those in the food industry or blood (all pumping devices in animals are positive-displacement pumps)
- If well sealed, they are self-priming pumps - they can create a vacuum pressure at their inlet, even when dry (empty of liquid), sufficient to lift a liquid for several meters below the pump
- They run at relatively low speed - they need much less speed than dynamic pumps to operate at similar loads (fatigue and wearing are more reduced)

Disadvantages

- Their volume flow rate does not change unless operating velocity is changed - usually it is not simple because AC electric motors are designed to operate at one or more fixed speeds
- If obstructed, they can create very high pressure at the outlet side - this may overload the driving motors, that may fail (overpressure protection is required)
- They may deliver a discontinuous flow (pulsatile) - this may be unacceptable for some applications

1.1.2 Dynamic pumps

Advantages

- They are mechanically simple - because of their simplicity, they are relatively economic and low maintenance
- Tolerances are not critical - since they are momentum based, they operate with a large range of mechanical tolerances. For the same reason, they are suitable for low viscosity fluids.

- Provide a steady flow at the outlet - contrary to many positive displacement pumps

Disadvantages

- They are not suitable for highly viscous fluids - due to the large amount of work required to rotate the impellers in this case
- They produce high levels of shear on the fluid - they should not be used to pump shear sensitive fluids (such as blood)
- They are not self-priming pumps - if empty, they do not create vacuum pressure sufficient to rise fluids for meters

	positive-displ. reciprocating	positive-displ. rotary	rotodynamic
continuous flow		✓	✓
low starting torque			✓
treats generic fluids	✓		
self priming	✓		
simple and economic		✓	✓
electric motor/turbine			✓

1.1.3 Positive-displacement reciprocation pumps

Positive-displacement pumps supply energy to the fluid by the movement of the boundaries of a closed volume: Fluid is sucked into an expanding volume and then pushed along as that volume contracts, producing an increase in fluid pressure.

For example, in a reciprocating positive displacement piston pump, the piston moves backwards, reducing the fluid pressure in the cylinder. The pressure decrease causes an inlet check valve to open, so that the fluid is sucked through the inlet check valve into the expanding volume. Then the piston moves forward, pushing the fluid volume and the increasing the fluid pressure. The pressurised liquid closes the inlet valve and lifts an outlet valve, leaving the chamber at higher pressure. Energy is supplied by the reciprocating motion of a piston acting on a closed volume of fluid.

Technical characteristics

- Capacity - is low and discontinuous (pulsatile)
- Net head - high (even at low speeds)
- Efficiency - independent of capacity and head and good for any liquid

- Adjustment of capacity - obtainable by sending back part of outlet flow to the inlet
- Treatable liquids - viscous, hot and chemically aggressive and shear sensitive
- Self-priming - when dry can generate negative pressures that lift the liquid to the pump
- Starting torque - high and similar to the working torque
- Flow continuity - flow is discontinuous and needs a damping volume for smoothing
- Obstruction - can generates unacceptable high outlet pressures (safety valves required)
- Dimensions - relatively large, heavy and expensive

1.1.4 Positive-displacement rotary pumps

Positive displacement pumps supply energy to the fluid by the movement of the boundaries of a closed volume. Fluid is sucked into an expanding volume and then pushed along as that volume contracts, producing an increase in fluid pressure.

Lobe, gear and screw rotary pumps: volumes of fluid are trapped in the pockets between the lobes (or the gear tooth spaces) and the casing. With the rotation of the loves (or gears) the liquid is forced to travel in one direction around the interior of the casing and is pushed through the outlet port under pressure. Lobes are driven by external timing gears and as a result the lobes do not make contact. Liquid travels around the interior of the casing in the pockets between the lobes and the casing, meshing of the lobes forces liquid through the outlet port under pressure. Energy is supplied by the movement of rotating components acting on a closed volume of fluid.

Technical characteristics

- Capacity - depends on the shaft speed
- Net head - high (even at low speeds) and independent from of shaft speed
- Efficiency - depends on the internal tolerance and their change during use
- Flow continuity - flow is uniform and continuous
- Do not need valves - the closed volume is trapped by the rotating components
- Dimensions - compact, light and relatively cheap
- Starting torque - high and similar to the working required torque
- Treatable liquids - viscous but free from sand, fibres or impurities

- Adjustment of capacity - requires change of operating speed (not always possible)

1.1.5 Positive-displacement peristaltic pumps

Positive-displacement pumps supply energy to the fluid by the movement of the boundaries of a closed volume. Fluid is sucked into an expanding volume and then pushed along as that volume contracts, producing an increase in fluid pressure.

Peristaltic pumps: It is based on the principle of ”peristalsis” (alternating contraction and relaxation of muscles around a tube). A smooth wall flexible tube is squeezed along its length, positively displacing the fluid contained within the tube. The tube’s restitution after the squeeze creates a vacuum, which sucks more fluid into the tube, causing a gentle pumping action with minimal damage to the media inside the tube.

1.1.6 Roto-dynamic pumps

Dynamic pumps supply energy to an open volume of fluid. In the case of roto-dynamic pumps, this is done by imparting to the flowing fluid a momentum through the rotation of the blades (impeller blades or rotor blades). (There are also non rotary dynamic pumps such as jet pumps or electromagnetic pumps.) The rotating blades are called **impeller blades** (or **rotor blades**) in the case of pumps. If the blades have a casing around the turbomachine it is called **enclosed** or **ducted**. If there is no casing, it is called **open**.

Axial flow pumps: pressure is developed by the propelling or lifting action of the blades of the impeller on the liquid. These act as rotating wings, and produce force through application of both Bernoulli’s principle and Newton’s third law, generating a difference in pressure the forward and rear surfaces of the airfoil-shaped blades. Axial pumps give much lower pressures than centrifugal pumps.

Radial flow pumps: radial flow pumps are centrifugal pumps in which the pressure is developed wholly by centrifugal force: the fluid enters the pump near the centre of the impeller and is moved to its outside diameter by the rotating motion of the impeller. This allows centrifugal pumps to produce continuous flows at high pressure.

Mixed flow pump: are intermediate between the previous two cases. The pressure is developed partly by centrifugal force and partly by the lift of the vanes of the impeller on the liquid.

Technical characteristics

- Capacity - increases with the operating speed and reduces with the fluid viscosity

- Net head - increases with the operating speed^2 and reduces with the fluid viscosity
- Efficiency - depends on the capacity and head and decreases with the fluid viscosity
- Required torque - depends on the operating speed
- Starting torque - ver low, compared to the operating torque
- Dimensions - compact light and economic
- Operating devices - electro-motors or turbo-motors can be used
- Flow continuity - flow is steady
- Tolerances - are not critical
- Adjustment of capacity - affects the efficiency
- Treatable fluids - they are not suitable for viscous fluids and shear sensitive fluids

1.2 Dynamic pump performance

1.2.1 The Bernoulli equation

The **Bernoulli equation** is an approximate relation between pressure(p), velocity (v) and elevation (z).

$$\frac{p}{\rho} + \frac{v^2}{2} + gz = \text{constant (along a streamline)} \tag{1.1}$$

The sum of flow, kinetic and potential energies of a fluid particle is constant along a streamline during steady flow. Between two points on the same streamline, the Bernoulli equation can be rewritten as:

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 \tag{1.2}$$

The Bernoulli equation is accurate only if:

- The flow is steady
- The flow is incompressible
- Frictional effects are negligible

1.2.2 Hydraulic heads

Dividing each term of the Bernoulli equation by the gravity acceleration gm each term has the dimension of length.

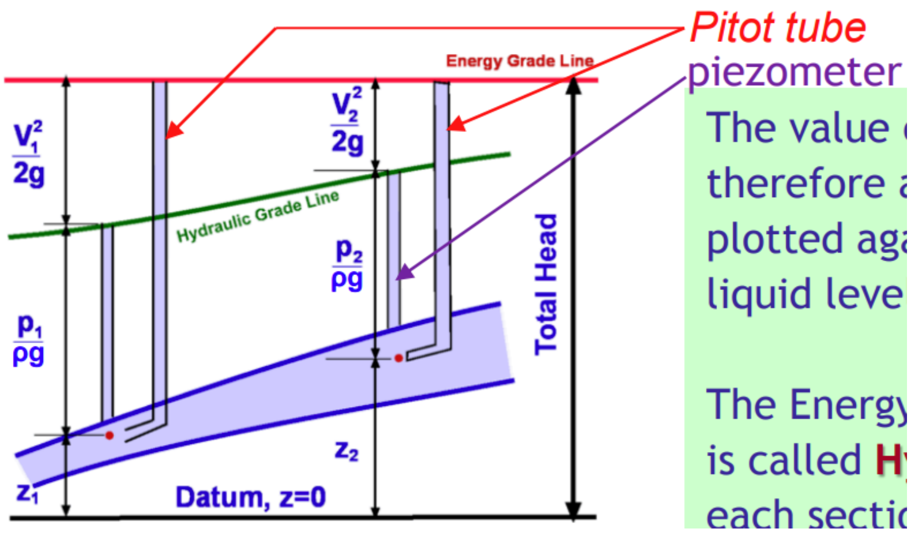
$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = h = \text{constant (along a streamline)} \tag{1.3}$$

Each term represents some kind of **head** of the flowing fluid.

- $\frac{p}{\rho g}$ is called **pressure head** and represents the height of a fluid column that produces the static pressure p
- $\frac{v^2}{2g}$ is called **velocity head** and represents the height needed for a fluid to reach the velocity v during frictionless free fall
- z is called the **elevation head** and represents the potential energy of a fluid
- h is the **total head** for the flow and is constant along a streamline

Net head of the pump

Tje value of total energy is constant and is therefore a straight line (Energy Grade Line) when plotted against distance. It corresponds to the liquid level in a Pitot tube. The Energy Grade Line take away the velocity head is called the Hydraulic Grade Line (it corresponds at each section to the liquid level in a piezometer) The Bernoulli equation can also be expressed as: the sum of the pressure, velocity and elevation heads along a streamline is constant during steady flow (when the compressibility and the frictional effects are negligible).



If a pump is present in the circuit, it will deliver energy to the fluid, raising the total head of a quantity H, corresponding to the net head of the pump. The Bernoulli equation between the inlet and the outlet of the pump can be written as:

$$\frac{p_{in}}{\rho g} + \frac{v_{in}^2}{2g} + z_{in} + H = \frac{p_{out}}{\rho g} + \frac{v_{out}^2}{2g} + z_{out} \tag{1.4}$$

Therefore, the net head of the pump is given by:

$$H = \left(\frac{p}{\rho g} + \frac{v^2}{2g} + z\right)_{out} - \left(\frac{p}{\rho g} + \frac{v^2}{2g} + z\right)_{in} = EGL_{out} - EGL_{in} \tag{1.5}$$

1.2.3 Power and efficiency

The net head of the pump is proportional to the useful power delivered to the fluid, that is called **water horsepower** (whp).

$$whp = Q \cdot p = \rho g Q \cdot H \tag{1.6}$$

(where Q is the volume flow rate).

Due to the irreversible energy losses of the pump, this energy will be less then the supplied power supplied to the pump, called brake horsepower (bhp) and equal to:

$$bhp = \omega T_{\text{shaft}} \tag{1.7}$$

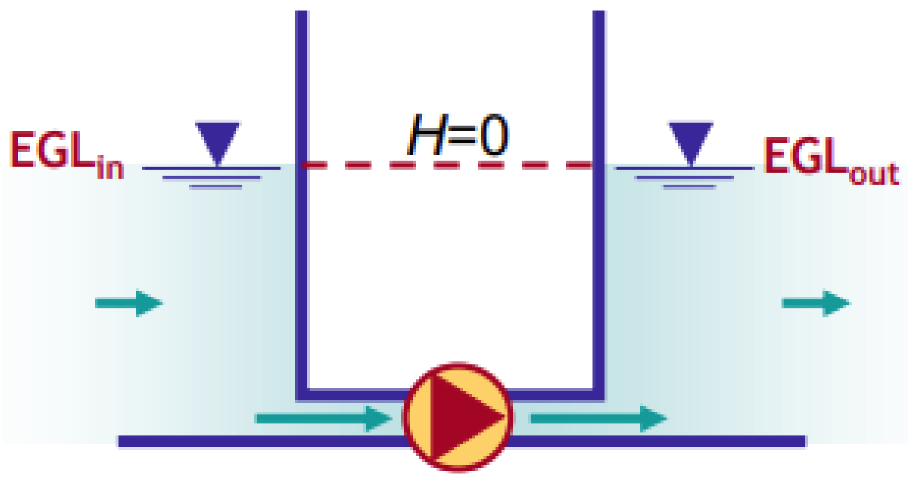
(where ω is the rotational speed of the shaft and T_{shaft} is the torque applied to the shaft).

THe **pump efficiency** (η_{pump}) is defined as the ratio of **useful power** and **supplied power** and is equal to:

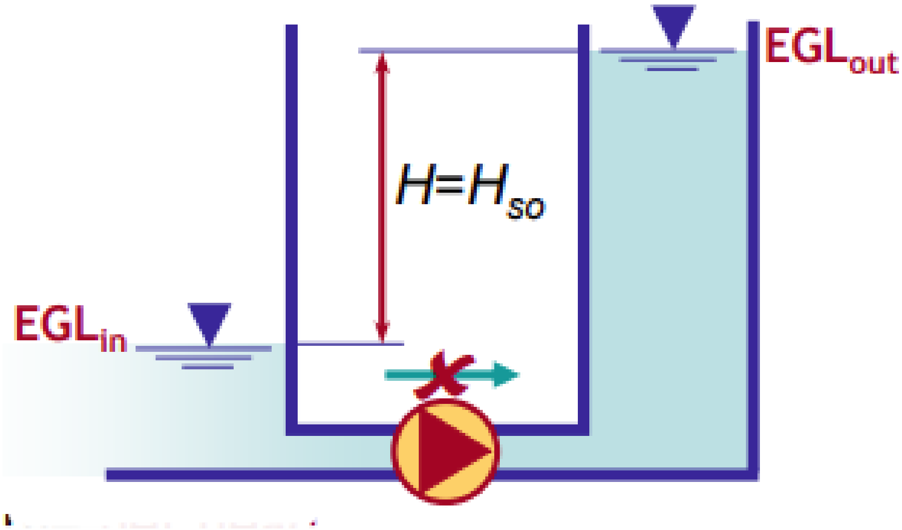
$$\eta_{\text{pump}} = \frac{\rho g Q}{\omega T_{\text{shaft}}} H \tag{1.8}$$

1.2.4 Performance curves

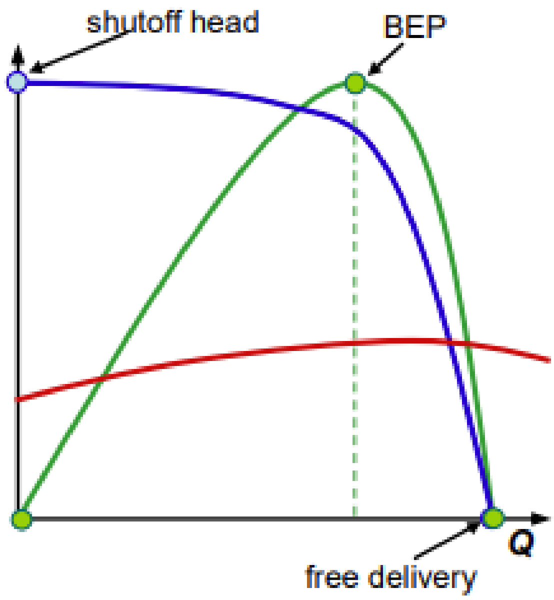
The pump's **free delivery** corresponds to the maximum flow rate and occurs when there is no flow restriction at the pump inlet or outlet (there is no load on the pump). This condition is associated with largest capacity but zero head.



At the other extreme, the pump's **shut-off head** corresponds to a large head (not necessarily the highest) and occurs when the outlet pressure is so high as to block off the outlet port (there is no flow rate). This condition is associated with zero capacity but with a large net head. Efficiency depends on the pump size, type, design and finishes. Larger pumps and smoother impeller finishes increase the efficiency.

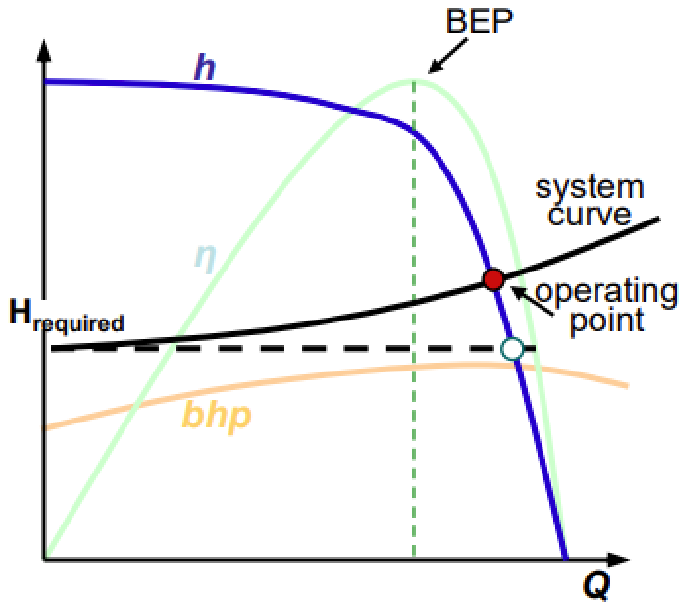


Consider a specific rotational speed of the pump. The curve of the net head as a function of the capacity is equal to the shut-off head for $Q = 0$ and equal 0 for at free delivery. The curve of the efficiency (green) as function of the capacity is equal to zero at zero capacity (shut-off head) and free delivery, and reaches the maximum somewhere in between, at a point called **best efficiency point** (BEP). Commonly, the curve of the of the brake horsepower is included in the diagram that is provided by the pump manufacturer.



The diagram, generated by experimental tests performed by the pump manufacturer at a specific speed, describes the operating properties of the pump and is known as the pump's **characteristic curves**. Under steady conditions, the pump can operate along its performance curves.

Commonly, the pump is used to provide a required head to a specific system, including all the piping and equipment from an inlet section (often the fluid surface of the suction) to an outlet section (often the fluid surface of the discharge tank).



Each piping system is characterised by hydraulic losses, due to pipe friction, valves and other fittings, entrance and exit losses and variation of pipe diameters. Losses depend on the flow conditions (in general, the increase with the square of the flow rate). Therefore, the required pump head varies with the capacity, describing a plot called the **system curve**. The system operating conditions may change over time, therefore all possible system curves should be considered in order to select the most suitable pump.

The **operating point** of the pumping system corresponds to the point where the net head curve and the system curve intersect. If possible, the operating point should be kept close to the best efficiency point.

Chapter 2

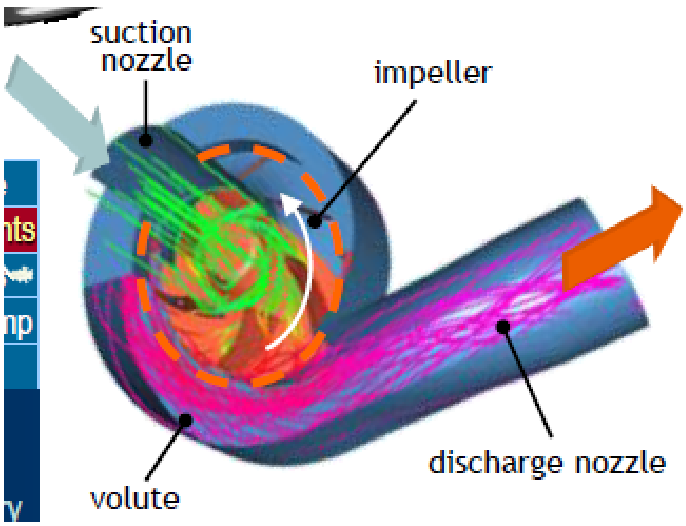
Centrifugal pumps and Blade design

2.1 Centrifugal pumps

2.1.1 Functional principle

Centrifugal pump's working principle can be understood very simply if you think about what happens to the liquid level when you steer a liquid in a glass: the free surface of the liquid becomes concave, and rises from the centre to the periphery - you are raising the flow head at the periphery.

2.1.2 Functional components



The fluid enters axially and receives tangential and radial velocity by momentum transfer from rotating blades. The centrifugal force provides further radial velocity.

Then the flow is collected and decelerated, in order to increase its pressure before it leaves the pump. The process produces depression at the pump axis, which sucks new fluid. The functional components of centrifugal pumps are:

- **The suction nozzle**: leads the fluid from pump inlet to the impeller eye. It is usually a cylindrical or truncated conical tube and can have stationary vanes to induce some spinning to the suction flow

- **The impeller**: it is rotating component that provides momentum to the fluid. It can have many different features, that we will analyse in more detail later on

- **The volute (or scroll)**: decelerates the fluid, converting the momentum into pressure, and delivers it to the discharge pipe. It has a gradually enlarging section (which gives its snail-shape) and can have stationary blades

2.1.3 Pumps arrangement

We have already mentioned the main functional limitations of roto-dynamic pumps over positive-displacement pumps:

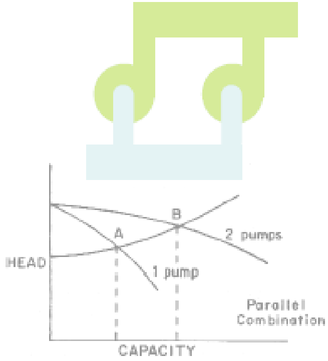
- Adjustment o capacity requires changes in the operating speed and affects the efficiency

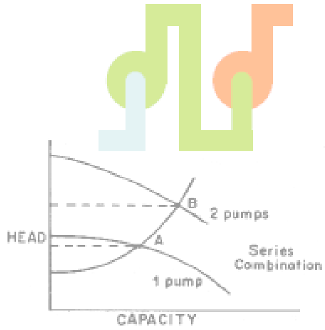
- The net head is lower and its adjustment requires significant changes in the operating speed and efficiency

These limitations can be overcome by using two or more pumps in **parallel** or in **series**.

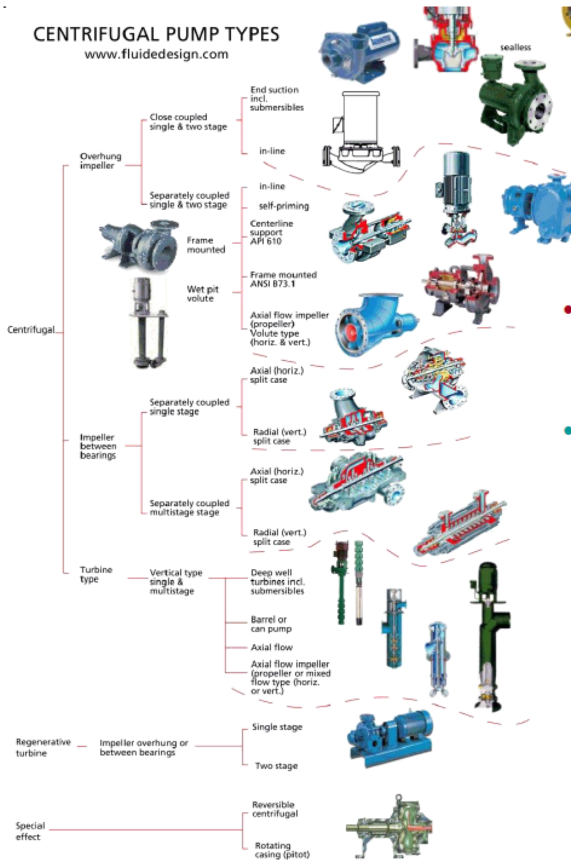
- **Pumps operating in parallel**: two or more pumps are used to draw fluid from the same source and discharge it to a single pipe. This arrangement allows variable discharge arrangement in terms of capacity

- **Pumps operating in series**: the discharge from one pump is piped into the suction nozzle of a flowing pump. Each pump adds energy to the same fluid, increasing the final net head. Efficiency decrease for more than 10-12 impellers





2.1.4 End-suction pump



The different operating requirements have produced a large number of possible centrifugal pump designs. We will consider the most common version, which is called **end-suction centrifugal pump**, and consists of the following parts:

- A rotating component called rotor (red)
  - Impeller blades, hub and shroud
  - Shaft
- A stationary component (green)
  - Suction nozzle
  - Scroll casing (or volute casing)
  - Pump casing
  - Back cover
  - Bearings

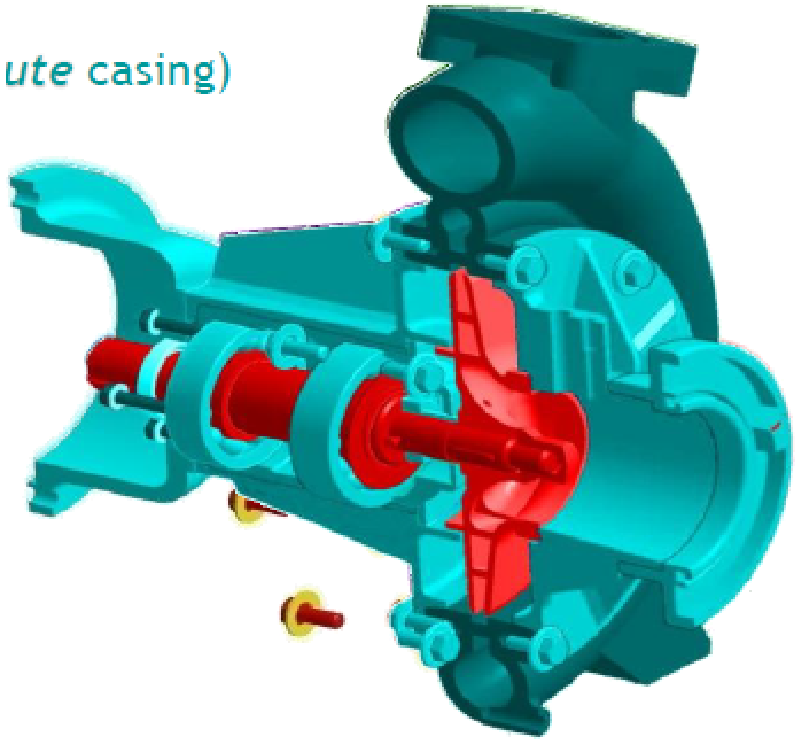


Figure 2.1: End-suction centrifugal pump

2.1.5 The Impeller

The **impeller** is the rotating component that provides the momentum to the fluid. It can be classified based on the major direction of the flow at the exit from the blades.

- Axial flow - the flow leaves the impeller parallel to the axis of rotation
- Mixed flow - the flow leaves the impeller at an angle with the axis of rotation
- Radial flow - the flow leaves the impeller orthogonal to the axis of rotation

It can be classified based on the suction type.

- Single suction - the liquid inlet to the impeller is on one side only
- Double suction - the liquid inlet to the impeller is symmetrical from two sides. The provides a very stable hydraulic performance because the hydraulic forces are balanced

It can be classified based on the presence of shrouds.

- Open impeller - blades are supported almost entirely by the impeller hub. It is primarily applied to clean, non-abrasive, low-horsepower applications and operates at a high efficiency
- Enclosed impeller - incorporates a full front and back shroud. Fluid flows without hydraulic interaction with the casing walls. The friction caused by the small gap between shroud and casing reduces efficiency
- Semi-closed impeller - incorporates a single shroud, usually located on the back of the impeller. It is more efficient than an enclosed impeller (lower disc friction and tighter axial clearances). Compared to an open impeller it can be adjusted axially to compensate for casing wear. We will be designing one of these

It can be classified based on the blade geometry.

- Backward inclined - have the highest efficiency (the fluid experiences the minimum amount of turning) and intermediate pressure rise
- Radial - produce the highest pressure rise. Pressure rise drops suddenly after the BEP. Efficiency is intermediate
- Forward inclined - produce a lower but more constant pressure rise over different flow rates. Efficiency is lower

2.1.6 Flow into the impeller

The design of the blades has to take into account the velocity flow through the impeller: The flow field into the impeller is:

- Unsteady
- Fully three-dimensional
- May be compressible

Assumptions:

- Steady flow
- We do not consider axial velocity - we consider only the normal (n) and tangential (t) components in the impeller plane
- We consider incompressible flow
- No leakage - in practice leakage will be considered by increasing the volume flow rate using in the design calculation
- We consider the flow to be always tangential to the blades surface when viewed from a reference frame rotating with the blades (shockless entry and no flow separation)

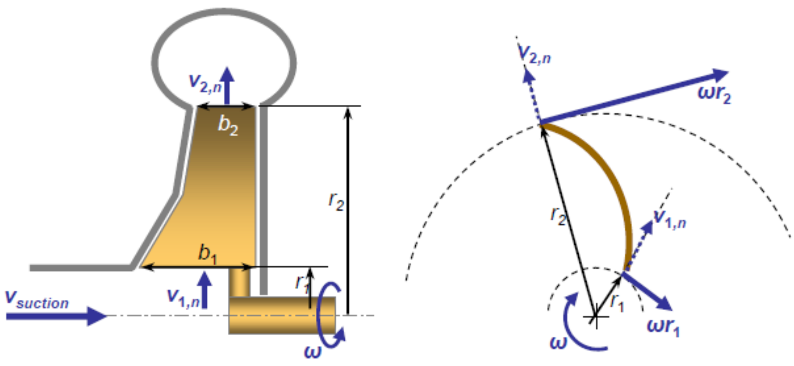
Efficient designs are aimed at accommodating the fluid flow into the impeller. The flow velocities into the impeller ensure the requirements to be met (are linked to Q and H) and are the combination of:

- Velocity of the fluid 'relative' to the blades
- Velocity of the blades

Velocities of the blades are known both in direction and magnitude, everywhere in the impeller.

- Direction: orthogonal to the radius (tangential direction)
- Magnitude: depending on the angular velocity ω of the impeller and on the local radius r.

$$v_{r,\text{ blade}} = \omega r \tag{2.1}$$
$$V_{1,\text{ blade}} = \omega r_1 \tag{2.2}$$
$$V_{2,\text{ blade}} = \omega r_2 \tag{2.3}$$
$$\tag{2.4}$$



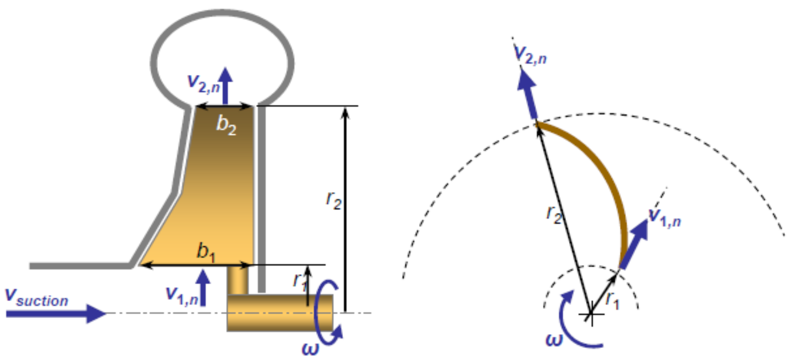
Velocities of the flow relative to the blades, we know the radial component and the direction. Due to conservation of mass and incompressibility assumption. The flow rate at each radius has to be the same (equal to the suction flow rate).

$$\text{if } Q = A_{\text{suction}} \times v_{\text{suction}} \tag{2.5}$$

$$Q = 2\pi r_1 b_1 \times v_{1,n} = 2\pi r_2 b_2 \times v_{2,n} \tag{2.6}$$

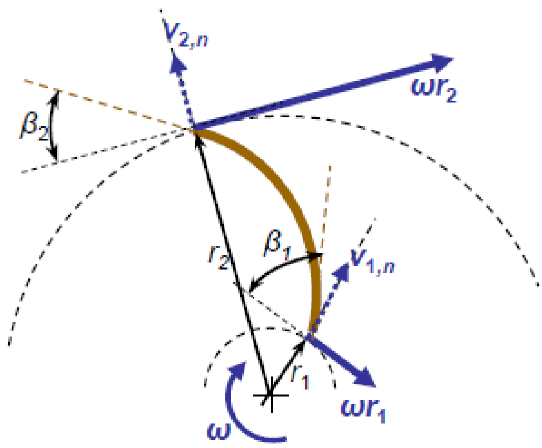
$$v_{2,n} = v_{1,n} \frac{r_1 b_1}{r_2 b_2} \tag{2.7}$$

where subscript n signifies the normal component.



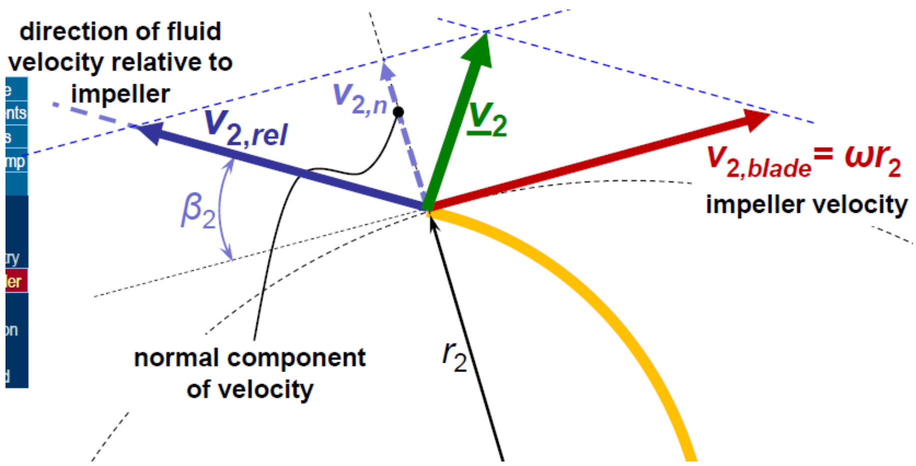
- Direction: due to no-separation condition, direction is parallel to the blade surfaces

- At the inlet and outlet edges ($v_{1,rel}$ and $v_{2,rel}$) they form angles with the tangential direction respectively equal to β_1 and β_2



The leading edge angle β_1 is defined as the blade angle relative to the reverse tangential direction at radius r_1 and the trailing edge angle β_2 as the blade angle relative to the reverse tangential direction at radius r_2. We can combine all information and apply the parallelogram rule to calculate

- The magnitude of $v_{1,rel}$ and $v_{2,rel}$
- The absolute fluid velocities V_1 and V_2 in direction and magnitude



The normal and tangential components of the absolute velocity are:

$$v_{2,n} = v_{2,rel} \sin\beta \tag{2.8}$$

$$v_{2,t} = (v_{2,\mathrm{blade}} - v_{2,rel}\cos\beta) = \omega r_2 - v_{2,rel}\cos\beta \tag{2.9}$$

The normal components of the inlet and outlet velocities are directly related to the capacity through the conservation equation. The tangential components of the inlet and outlet velocities are directly related to the torque and to the required head through the Euler's turbomachine equation. A similar equation is found at the blade inlet.

2.1.7 Euler's turbomachine equation

To calculate the torque T_{shaft} on the rotating shaft, the angular momentum relation is used. The starting point to obtain it is Newton's second law: the amount of force exerted upon the object is directly proportional to the rate of change in the momentum of the object.

$$\vec{F} = \frac{d}{dt}(m\vec{v}) \tag{2.10}$$

The moment of a force $\vec{F}$ about a point O is given by $\vec{T} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the vector from O to $\vec{F}$ called radius vector (where $\times$ is cross product). From the two expressions:

$$\vec{T} = \vec{r} \times \frac{d}{dt}(mv) \tag{2.11}$$

We define a vector $\vec{L}$ called angular momentum as vector product of the radius vector to the object and the linear momentum of the object.

$$\vec{L} = \vec{r} \times m\vec{v} \tag{2.12}$$

Upon taking the temporal differential of $\vec{L}$, we have:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times \frac{d}{dt}(m\vec{v}) \tag{2.13}$$

But $\frac{d\vec{r}}{dt} = \vec{v}$ and the cross product of parallel vectors is 0. Therefore:

$$\vec{T} = \frac{d\vec{L}}{dt} \tag{2.14}$$

The rate of change of angular momentum of an object about a fixed point is equal to the torque applied to the object.

2.1.8 Flow into the impeller

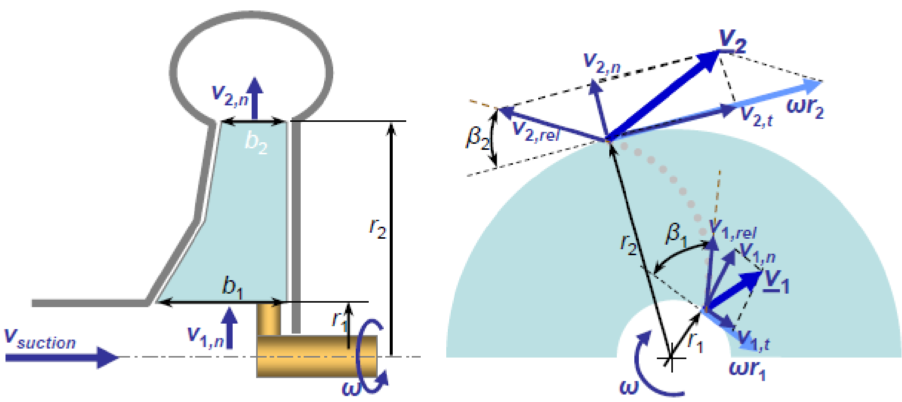


Figure 2.2: Consider the control volume described by the rotating impeller blades.

We can calculate the torque T_{shaft} on the rotating shaft from the Euler turbomachine equation:

$$T_{\text{shaft}} = \rho Q(r_s v_{2,t} - r_1 v_{1,t}) \tag{2.15}$$

Assuming no losses, the efficiency is equal to one and therefore the ideal net head of the pump can be calculated as:

$$\eta_{\text{pump}} = \frac{\rho g Q}{\omega T_{\text{shaft}}} H = 1 \tag{2.16}$$

$$\rho g Q H = \omega T_{\text{shaft}} = \omega \rho Q(r_2 v_{2,t} - r_1 v_{1,t}) \tag{2.17}$$

$$H = \frac{1}{g}(\omega r_2 v_{2,t} - \omega r_1 v_{1,t}) \tag{2.18}$$

The operating conditions of the pump are related to the flow velocities into the impeller:

$$\text{Flow rate: } Q = 2\pi r_1 b_1 b_{1,n} = 2\pi r_2 b_2 v_{2,n} \tag{2.19}$$

$$\text{Ideal head: } H = \frac{1}{g}(\omega r_2 v_{2,t} - \omega r_1 v_{1,t}) \tag{2.20}$$

Since the flow velocities into the impeller depend on the blades angles (e.g. leading and trailing edge angles) the impeller cna be designed in order to match required operative conditions.

$$v_{1,2,n} = v_{1,2,rel} \sin \beta_{1,2} \tag{2.21}$$

$$v_{1,2,n} = (v_{1,2,\text{blade}} - v_{1,2,rel} \cos \beta_{1,2}) = \omega r_{1,2} - v_{1,2,rel} \cos \beta_{1,2} \tag{2.22}$$

$Q,\ H,\ v_n,\ v_t$ are in terms of $r,\ b,\ \omega,\ \beta,\ v_{rel}$