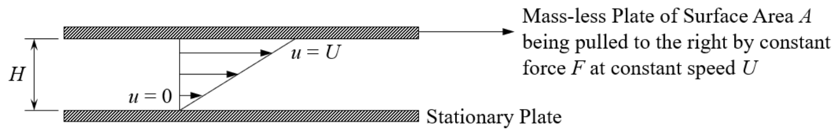
0.1 Constitutive equations

We want to find a way to link the stress tensor τ with the velocity field i.e. $\tau = f(u,\ v,\ w)$.



The angle of deformation $\Delta\theta$ can be used to derive the following:

$$\tan \Delta\theta = \frac{u \cdot \Delta T}{H} \tag{1}$$

$$\tan \Delta\theta = \mathrm{d}\theta = \frac{u \cdot t}{H} \to \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{u}{H} \tag{2}$$

$$\tau = \frac{F}{A} \propto \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{u}{H} \tag{3}$$

$$\tau = \mu \frac{\mathrm{d}\theta}{\mathrm{d}t} = \mu \frac{u}{H} \tag{4}$$

$$\tau = \mu \frac{\mathrm{d}u}{\mathrm{d}y} \tag{5}$$

- τ is the shear stress
- $\frac{\mathrm{d}u}{\mathrm{d}y}$ is the shear rate
- μ is the dynamic viscosity and has units $\mathrm{N\,s\,m^{-2}}$
- $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity and has units $\mathrm{m^2\,s^{-1}}$

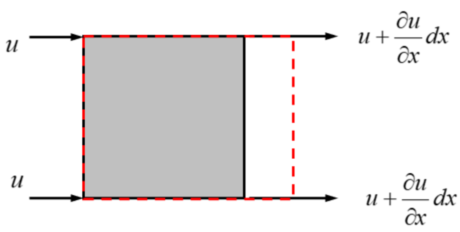
For Newtonian fluids, μ is constant. In the case above, our stress tensor is τ_{yx}, hence:

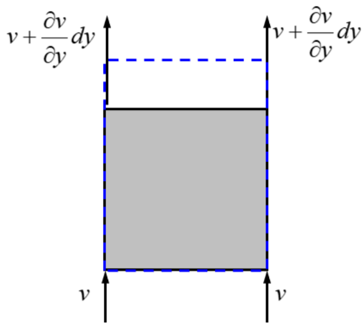
$$\tau_{yx} = \mu \frac{\partial u}{\partial y} \tag{6}$$

Our velocity gradient can be defined as:

$$\nabla \vec{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{7}$$

The left diagonal components are the normal deformation, orthogonal to the surface.





A simplified way of writing these left diagonal terms is

$$\nabla \cdot \vec{V} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{8}$$

The repeated index i means sum in the x, y and z directions.

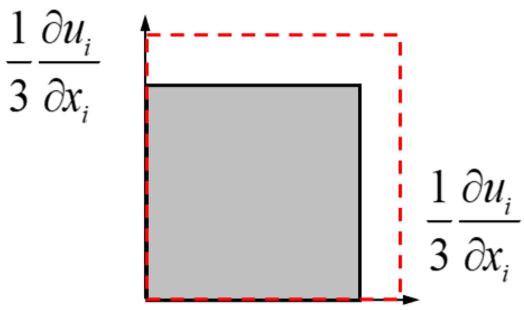
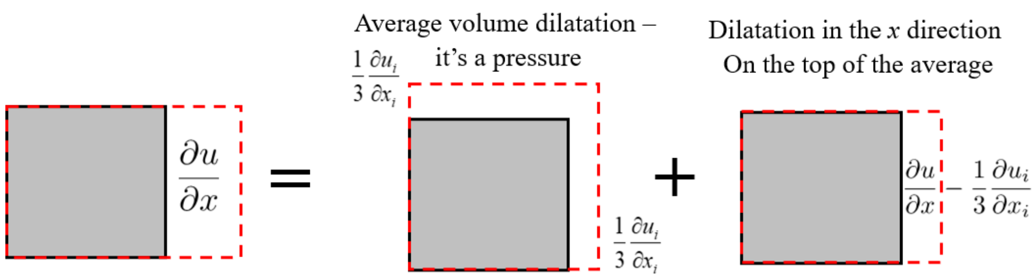


Figure 1: 1/3 symbolises the average deformation in x, y and z.

To find $\frac{\partial u}{\partial x}$, we can do the following

$$\frac{\partial u}{\partial x} = \frac{1}{3}\frac{\partial u_i}{\partial x_i} + \left(\frac{\partial u}{\partial x} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\right) \tag{9}$$



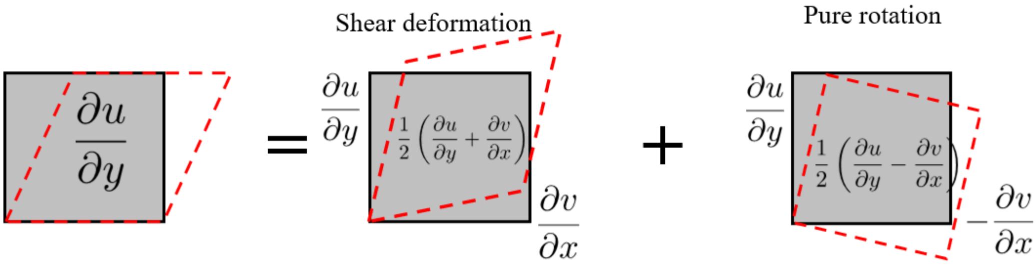
This can be also done for the other two orthogonal directions

$$\frac{\partial v}{\partial y} = \frac{1}{3}\frac{\partial u_i}{\partial x_i} + \left(\frac{\partial v}{\partial y} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\right) \tag{10}$$

$$\frac{\partial w}{\partial z} = \frac{1}{3}\frac{\partial u_i}{\partial x_i} + \left(\frac{\partial w}{\partial z} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\right) \tag{11}$$

Let us consider another term, such as $\frac{\partial u}{\partial y}$. We can define this as a component of deformation and rotation of the fluid particle.

$$\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \tag{12}$$



Example

$$\frac{\partial u}{\partial y} = 3 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 2.5 + 0.5 \tag{13}$$

$$\frac{\partial v}{\partial x} = 2 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 2.5 - 0.5 \tag{14}$$