0.1 Thin-walled cylinders

Thin walled cylinders have a wall thickness t much smaller than the cylinder radius R (at least one twentieth).

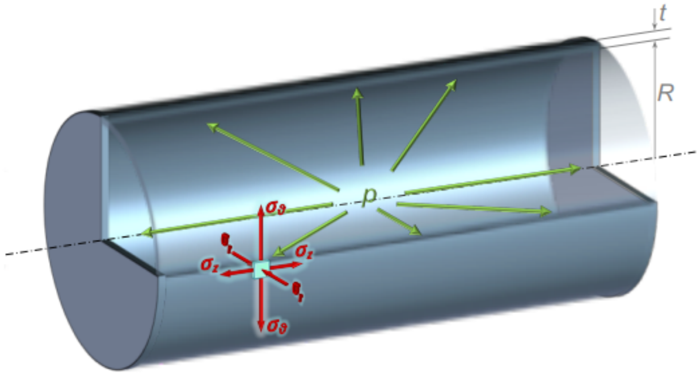


Figure 1:

The **hoop stresses** σ_θ can be considered **constant** across the wall thickness t. The **radial stresses** σ_t is **negligible** in comparison to the hoop stresses σ_θ. Under these conditions, the state of stress at each point of the cylinder can be estimated to good accuracy by simple equilibrium considerations.

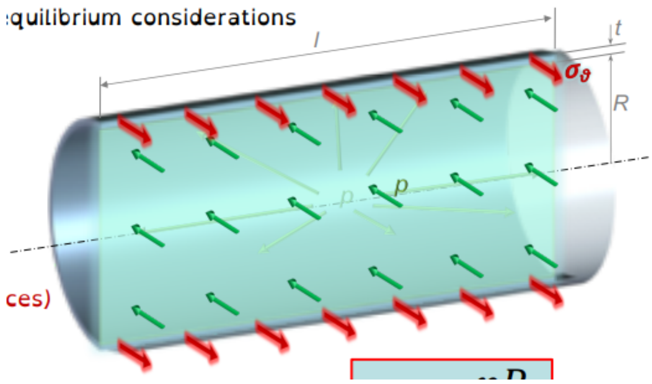


Figure 2: Hoop stress - equilibrium of transversal forces.

$$p \cdot l \cdot 2R = 2\left(\sigma_\theta t l\right)\sigma_\theta = \frac{pR}{t} \tag{1}$$

Under these conditions, the state of stress at each point of the cylinder can be estimated to good accuracy by simple equilibrium considerations.

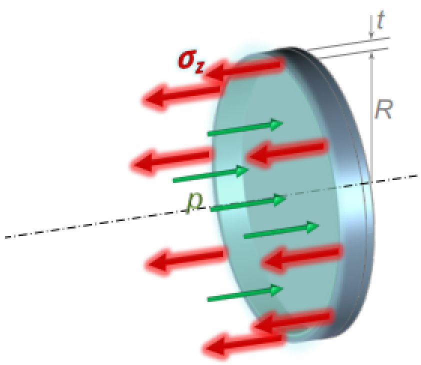


Figure 3: Longitudinal stress - equilibrium of axial forces.

$$p \cdot \pi R^2 = \sigma_z \cdot 2\pi R \cdot t \tag{2}$$

$$\sigma_z = \frac{pR}{2t} \tag{3}$$

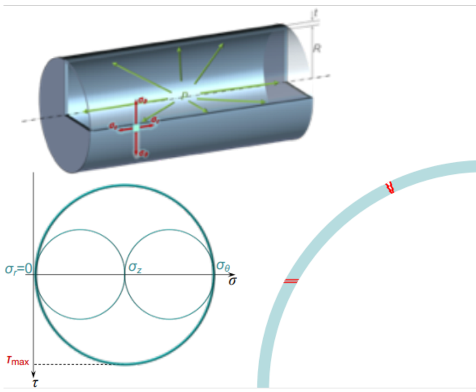


Figure 4:

$$\sigma_\theta = \frac{pR}{t} \tag{4}$$

$$\sigma_z = \frac{pR}{2t} = \frac{\sigma_\theta}{2} \tag{5}$$

0.2 Thick-walled cylinders - radial and hoop stresses

0.2.1 Failure of thick wall cylinders

Figure 5:

When thickness increases with respect to radius, the radial variation of hoop stress becomes considerable and the radial stress is no longer available.

0.2.2 Radial and hoop stresses & strains

For the determination of radial and circumferential stresses and strains, equilibrium equations on their own are not sufficient.

0.2.3 Solid mechanics equations

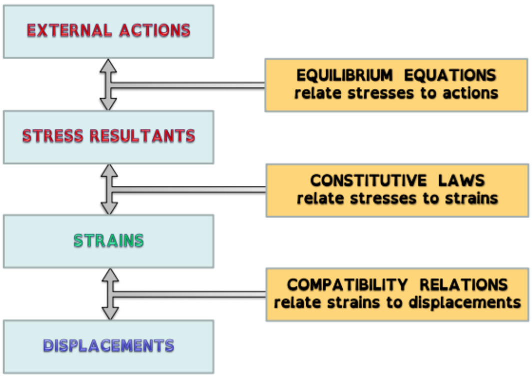


Figure 6:

0.2.4 Assumptions

For long cylinder , far from the ends, we can assume that the deformation of the cylinder is symmetrical respect to the axis:

- Cross-sections remain plane when subjected to pressure: longitudinal strain ϵ_z is independent of radius. i.e. ϵ_z = constant
- Displacements in each cross-section are purely radial
- Displacement is constant with circumferential co-ordinate and varies only with radial co-ordinate i.e. $u = f(r)$
- σ_r, σ_θ and σ_z are principal stress
- Cross-sections remain plane when subjected to pressure: longitudinal strain ϵ_z is independent of radius. i.e. ϵ_z = constant

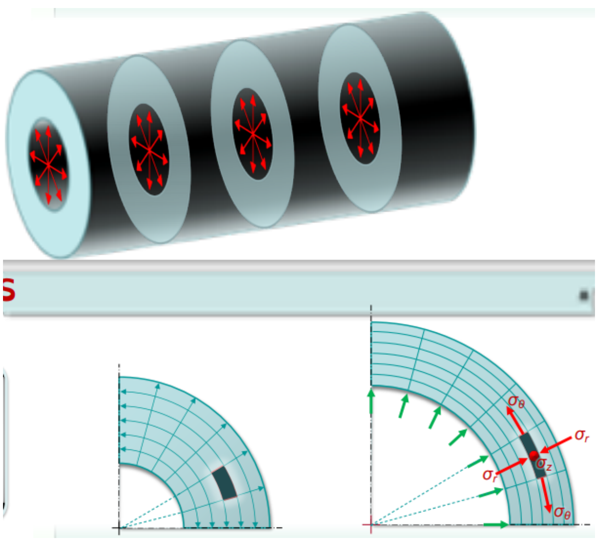


Figure 7:

0.2.5 Equilibrium equations

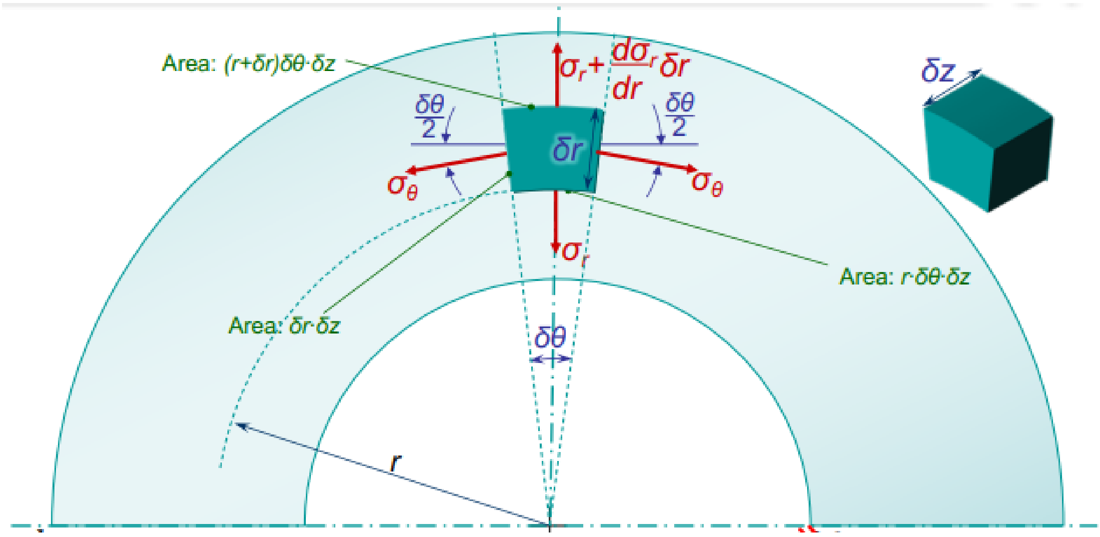


Figure 8:

The equilibrium of radial forces is given by the following equation:

$$-\sigma_r\left(r\cdot\cancel{\delta\theta}\cdot\cancel{\delta z}\right)+\left(\sigma_r+\frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\delta r\right)\left[(r+\delta r)\cdot\cancel{\delta\theta}\cdot\cancel{\delta z}\right]-\cancel{2}\sigma_\theta\left(\delta r\cdot\cancel{\delta z}\right)\cdot\cancel{\sin}\frac{\cancel{\delta\theta}}{\cancel{2}}=0 \quad (6)$$

$$\rightarrow \cancel{-\sigma_r r}+\cancel{\sigma_r r}+\sigma_r\cancel{\delta r}+r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\cancel{\delta r}+\cancel{\frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\delta r^2}-\sigma_\theta\cancel{\delta r}=0 \quad (7)$$

$$\rightarrow \sigma_r+r\frac{\mathrm{d}\theta_r}{\mathrm{d}r}-\sigma_\theta=0 \quad (8)$$

0.2.6 Compatibility relations

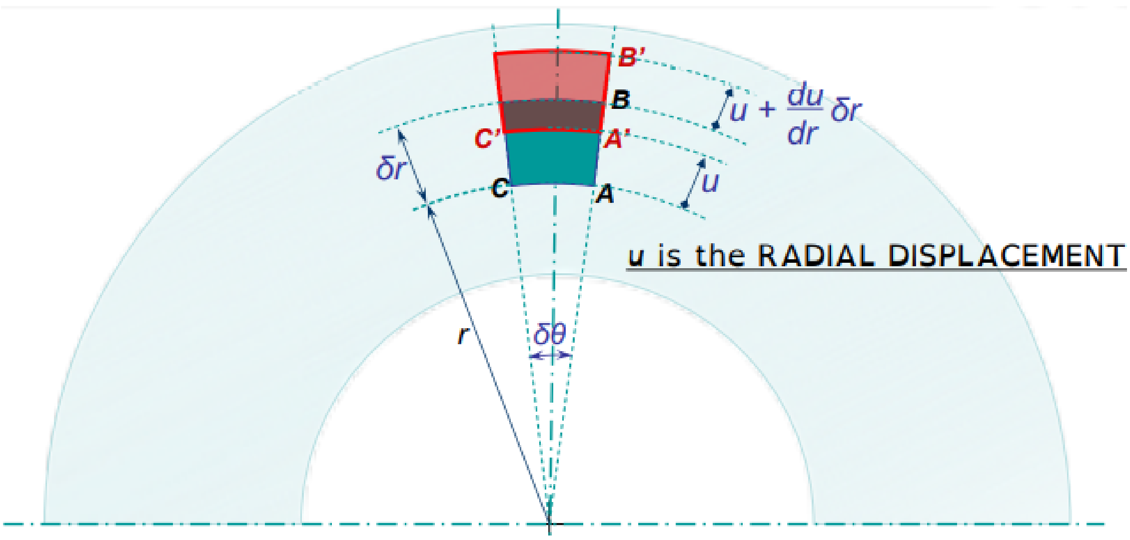


Figure 9:

The radial strain is given by the following equation:

$$\epsilon_r = \frac{\overline{A'B'} - \overline{AB}}{\overline{AB}} \tag{9}$$

with:

$$\overline{AB} = \delta r \tag{10}$$

and

$$\overline{A'B'} = \overline{AB} + \overline{BB'} - \overline{AA'} = \delta r + \left(\cancel{u} + \frac{\mathrm{d}u}{\mathrm{d}r}\delta r\right) - \cancel{u} = \delta r + \frac{\mathrm{d}u}{\mathrm{d}r}\delta r \tag{11}$$

Leading to:

$$\varepsilon_r = \frac{\left(\cancel{\delta r} + \frac{\mathrm{d}u}{\mathrm{d}r}\cancel{\delta r}\right) - \cancel{\delta r}}{\cancel{\delta r}} \tag{12}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{13}$$

0.2.7 Constitutive laws

Uniaxial:

$$\begin{array}{ll} \varepsilon_1 = \frac{\sigma_1}{E} & \sigma_1 = E\varepsilon_1 \\ \varepsilon_2 = -v\varepsilon_1 & \sigma_2 = 0 \\ \varepsilon_3 = -v\varepsilon_1 & \sigma_3 = 0 \end{array} \tag{14}$$

Biaxial:

$$\begin{array}{ll} \varepsilon_1 = \frac{1}{E}\left(\sigma_1 - v\sigma_2\right) & \sigma_1 = \frac{E}{1-v^2}\left(\varepsilon_1 + v\varepsilon_2\right) \\ \varepsilon_2 = \frac{1}{E}\left(\sigma_2 - v\sigma_1\right) & \sigma_2 = \frac{E}{1-v^2}\left(\varepsilon_2 + v\varepsilon_1\right) \\ \varepsilon_3 = -\frac{v}{E}\left(\sigma_1 + \sigma_2\right) & \sigma_3 = 0 \end{array} \tag{15}$$

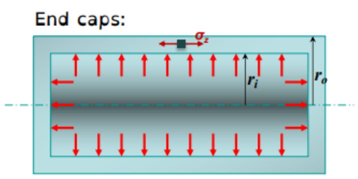
Triaxial:

$$\begin{array}{ll} \varepsilon_1 = \frac{1}{E}\left[\sigma_1 - v\left(\sigma_2 + \sigma_3\right)\right] & \sigma_1 = \frac{E}{(1+v)(1-2v)}\left[(1-v)\,\varepsilon_1 + v\left(\varepsilon_2 + \varepsilon_3\right)\right] \\ \varepsilon_2 = \frac{1}{E}\left[\sigma_2 - v\left(\sigma_3 + \sigma_1\right)\right] & \sigma_2 = \frac{E}{(1+v)(1-2v)}\left[(1-v)\,\varepsilon_2 + v\left(\varepsilon_1 + \varepsilon_3\right)\right] \\ \varepsilon_3 = \frac{1}{E}\left[\sigma_3 - v\left(\sigma_1 + \sigma_2\right)\right] & \sigma_3 = \frac{E}{(1+v)(1-2v)}\left[(1-v)\,\varepsilon_3 + v\left(\varepsilon_1 + \varepsilon_2\right)\right] \end{array} \tag{16}$$

The analytical solution of the stress distribution is possible, in simple form, only in the *case of biaxial state of stress.*

Cylinders with internal pressure

A pressure difference can be maintained inside the cylinder in one of the following methods:



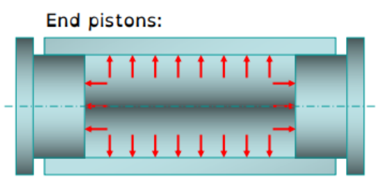


Figure 10:

Only in the case of end pistons, can we find an analytical solution for the stress distributions in thick cylinders: no axial force acts on the cylinder and the state of stress is plane.

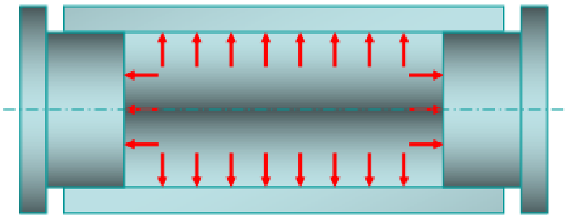


Figure 11:

Plane stress state

Biaxial:

$$\begin{array}{ll} \varepsilon_1 = \frac{1}{E}\left(\sigma_1 - v\sigma_2\right) & \sigma_1 = \frac{E}{1-v^2}\left(\varepsilon_1 + v\varepsilon_2\right) \\ \varepsilon_2 = \frac{1}{E}\left(\sigma_2 - v\sigma_1\right) & \sigma_2 = \frac{E}{1-v^2}\left(\varepsilon_2 + v\varepsilon_1\right) \\ \varepsilon_3 = -\frac{v}{E}\left(\sigma_1 + \sigma_2\right) & \sigma_3 = 0 \end{array} \tag{17}$$

Cylindrical

In the biaxial case, in a cylindrical coordinate system r, θ, , z.

$$\begin{array}{ll} \varepsilon_r = \frac{1}{E}\left(\sigma_r - v\sigma_\theta\right) & \sigma_r = \frac{E}{1-v^2}\left(\varepsilon_r + var\right) \\ \varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - v\sigma_r\right) & \sigma_\theta = \frac{E}{1-v^2}\left(\varepsilon_\theta + v\varepsilon_r\right) \\ \varepsilon_z = -\frac{v}{E}\left(\sigma_r + \sigma_\theta\right) & \sigma_z = 0 \end{array} \tag{18}$$

0.2.8 Solid mechanics equations

Equilibrium equations relate stresses to actions:

$$\frac{\mathrm{d}\theta_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{19}$$

Constitutive laws relate stresses to strains:

$$\sigma_r = \frac{E}{1-v^2}\left(\varepsilon_r + v\varepsilon_\theta\right) \tag{20}$$

$$\sigma_\theta = \frac{E}{1-v^2}\left(\varepsilon_\theta + v\varepsilon_r\right) \tag{21}$$

Compatibility relations relate strains to displacements:

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r} \tag{22}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{23}$$

Combining the equations together, we arrive at Euler's differential equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{1}{r} \cdot \frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2} = 0 \tag{24}$$

$$r^2 u'' + r u' - u = 0 \tag{25}$$

Euler's differential equation is characterised by the fact that its coefficients depend on the variable r. The general solution to Euler's differential equation is:

$$u = Ar + \frac{B}{r} u' \qquad = A - \frac{B}{r^2} \tag{26}$$

The constitutive laws hence become:

$$\sigma_r = \frac{E}{1-v^2}\left(\frac{\mathrm{d}u}{\mathrm{d}r} + v\frac{u}{r}\right) \tag{27}$$

$$\sigma_\theta = \frac{E}{1-v^2}\left(\frac{u}{r} + v\frac{\mathrm{d}u}{\mathrm{d}r}\right) \tag{28}$$

$$\sigma_r = \frac{E}{1-v^2}\left[A - \frac{B}{r^2} + v\left(A + \frac{B}{r^2}\right)\right] = C - \frac{D}{r^2} \tag{29}$$

$$\sigma_\theta = \frac{E}{1-v^2}\left[A + \frac{B}{r^2} + v\left(A - \frac{B}{r^2}\right)\right] = C + \frac{D}{r^2} \tag{30}$$

Note that:

$$\sigma_r + \sigma_\theta = 2C = \mathrm{const} \tag{31}$$

Boundary conditions

$$\sigma_r = C - \frac{D}{r^2} = \frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} - \frac{(p_i - p_0)\, r_i^2 r_0^2}{\left(r_0^2 - r_i^2\right) r^2} \tag{32}$$

$$\sigma_\theta = C + \frac{D}{r^2} = \frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} + \frac{(p_i - p_0)\, r_i^2 r_0^2}{\left(r_0^2 - r_i^2\right) r^2} \tag{33}$$

If we have an internal pressure 'p_i' and an external pressure is 'p_0':

$$\sigma_r(r_i) = -p_i = C - \frac{D}{r_i^2} \tag{34}$$

$$C = \frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} \tag{35}$$

$$\sigma_r(r_i) = -p_0 = C - \frac{D}{r_0^2} \tag{36}$$

$$D = \frac{(p_i - p_0)\, r_i^2 r_0^2}{\left(r_0^2 - r_i^2\right) r^2} \tag{37}$$

Radial and hoop stresses

Lame's equations

$$\sigma_r = \frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} - \frac{(p_i - p_0)\, r_i^2 r_0^2}{\left(r_0^2 - r_i^2\right) r^2} \tag{38}$$

$$\sigma_\theta = \frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} + \frac{(p_i - p_0)\, r_i^2 r_0^2}{\left(r_0^2 - r_i^2\right) r^2} \tag{39}$$

0.2.9 Stress variation

General case

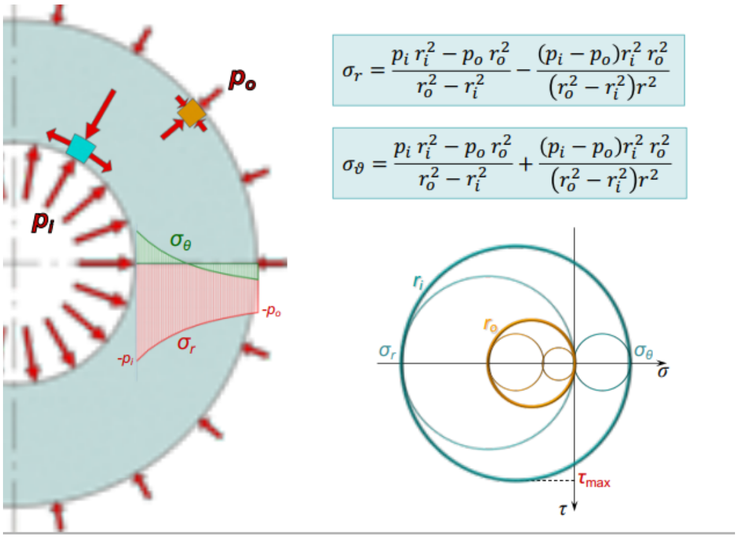


Figure 12:

p_i only

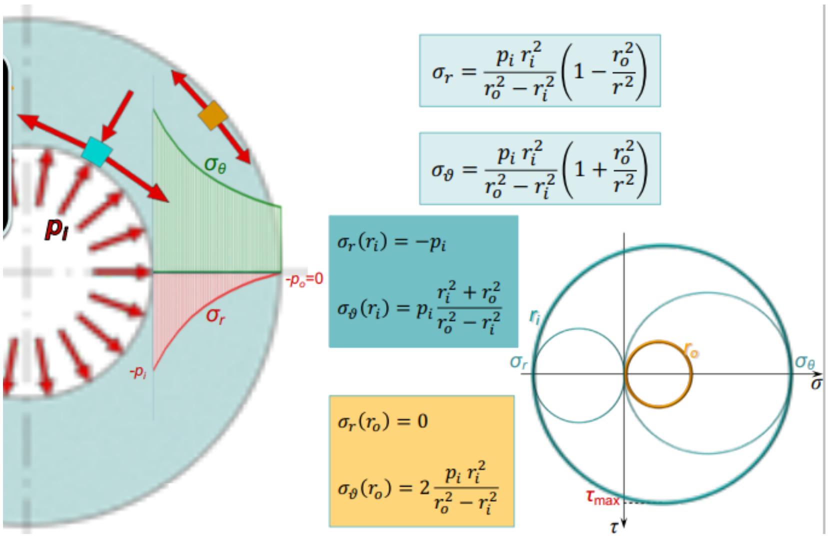


Figure 13:

p_0 only

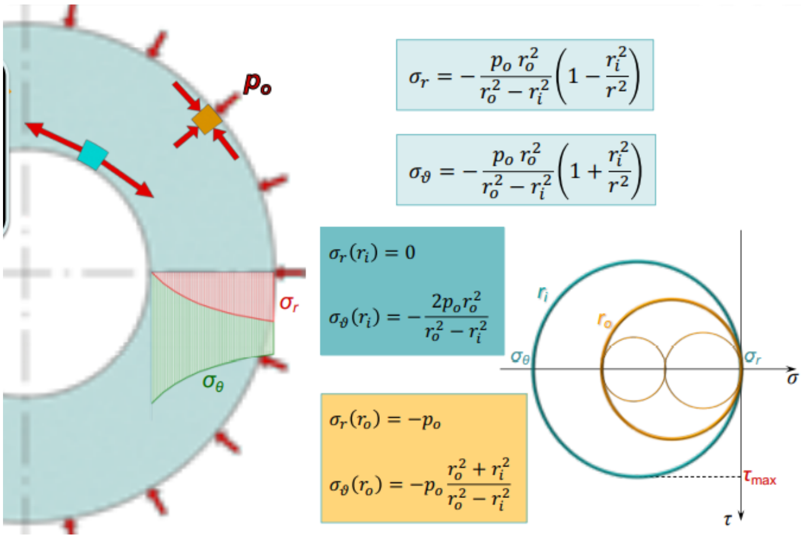


Figure 14:

0.3 Thick-walled cylinders - longitudinal stress

How do we deal with the case of end caps pistons? (Triaxial stress state). We know:

$$\sigma_r = C - \frac{D}{r^2} \tag{40}$$

$$\sigma_\theta = C + \frac{D}{r^2} \tag{41}$$

$$\sigma_r + \sigma_\theta = 2C = \text{const} \tag{42}$$

Triaxial stress state:

$$\begin{aligned} \varepsilon_r &= \tfrac{1}{E}\left[\sigma_r - v\left(\sigma_\theta + \sigma_z\right)\right] & \sigma_r &= \tfrac{E}{(1+v)(1-2v)}\left[(1-v)\,\varepsilon_r + v\left(\varepsilon_\theta + \varepsilon_z\right)\right] \\ \varepsilon_\theta &= \tfrac{1}{E}\left[\sigma_\theta - v\left(\sigma_z + \sigma_r\right)\right] & \sigma_\theta &= \tfrac{E}{(1+v)(1-2v)}\left[(1-v)\,\varepsilon_\theta + v\left(\varepsilon_r + \varepsilon_z\right)\right] \\ \varepsilon_z &= \tfrac{1}{E}\left[\sigma_z - v\left(\sigma_r + \sigma_\theta\right)\right] & \sigma_z &= \tfrac{E}{(1+v)(1-2v)}\left[(1-v)\,\varepsilon_z + v\left(\varepsilon_\theta + \varepsilon_r\right)\right] \end{aligned} \tag{43}$$

We know that ε_z, $\frac{1}{E}$ and $v\left(\sigma_r + \sigma_\theta\right)$ are constant, therefore σ_z is also constant. Longitudinal stresses can be estimated by simple equilibrium considerations.

$$p \cdot \pi r_i^2 = \sigma_z\left(\pi r_0^2 - \pi r_i^2\right) \tag{44}$$

$$\sigma_z - p\frac{r_i^2}{r_0^2 - r_i^2} \tag{45}$$

with $p = p_i - p_0$. In the case of end pistons, the longitudinal component of the stress can be superimposed.

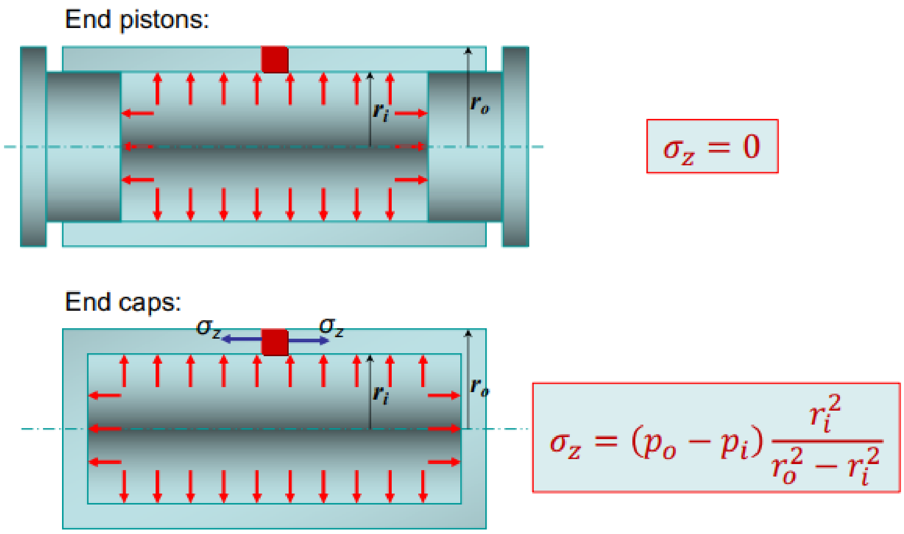


Figure 15:

0.3.1 Thick vs thin cylinders

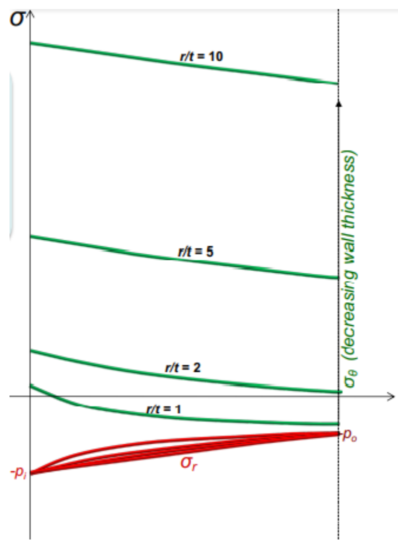


Figure 16:

Decreasing wall thickness with respect to radius, the hoop stress increases dramatically, but the radial stress does not change significantly. Assumptions of thin wall theories become more accurate:

- constant hoop stresses
- negligible radial stresses