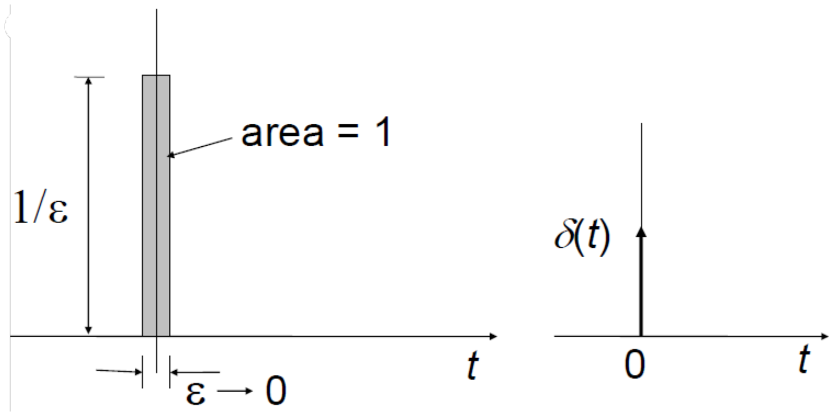
0.1 Impulse functions/responses

0.1.1 Impulse response of a system: Dirac delta function

A useful tool in analysing the transient response of a system is the impulse signal, a unit (amplitude = 1) pulse infinitesimally small, with area = 1. Formally this is known as the **Dirac delta function (impulse function)**.



The Dirac delta function is a non-physical, singularity function with the follow definition:

$$\delta(t) = \begin{cases} 0 \text{ for } t \neq 0 \\ \text{undefined for } t = 0 \end{cases} \tag{1}$$

but with the requirement that

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \tag{2}$$

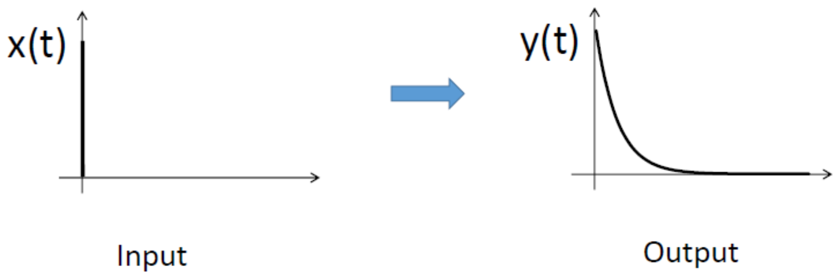
so taking the Laplace transform of this is also just 1

$$\mathcal{L}(\delta(t)) = \int_{0^-}^{\infty} \delta(t)e^{-st}\,\mathrm{d}t = 1 \tag{3}$$

Thus, the impulse response of the system is equal to the transfer function and from this it can be shown that **any** arbitrary signal can be described as a summation of impulse responses.

0.1.2 Impulse response on a first order system

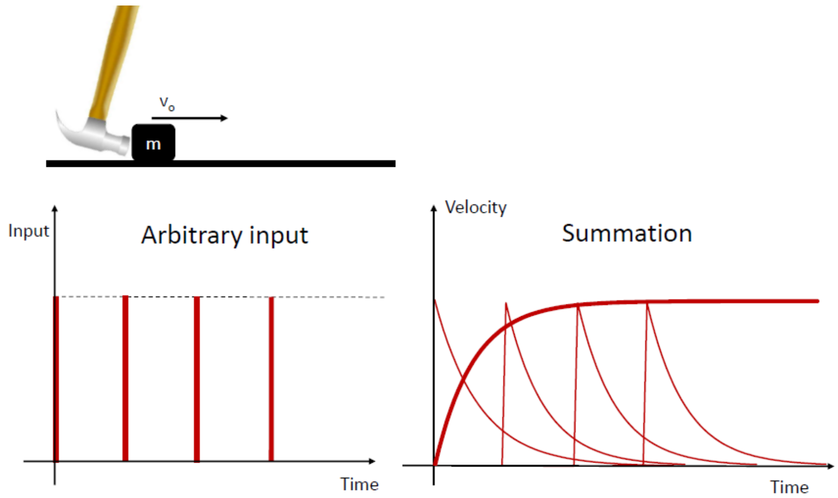
Take a first order response for example, the transfer function and thus the impulse response looks like this:



Due to our LTI assumptions:

- Scaling the input scales the output
- Superposition of inputs equals superposition of outputs
- Time invariance

0.1.3 Impulse response of a system



Time vs Frequency Domain

- u is the impulse function to the system
- h is called the impulse response of the system

- H is called the transfer function (TF) of the system

$$y(t) = \int_0^\infty h(\tau)u(t-\tau)\,\mathrm{d}\tau = \int_0^\infty h(\tau - t)u(\tau)\,\mathrm{d}\tau \tag{4}$$

with $0 \leq \tau \leq t$

$$y(t) = h(t) \cdot u(t) \tag{5}$$

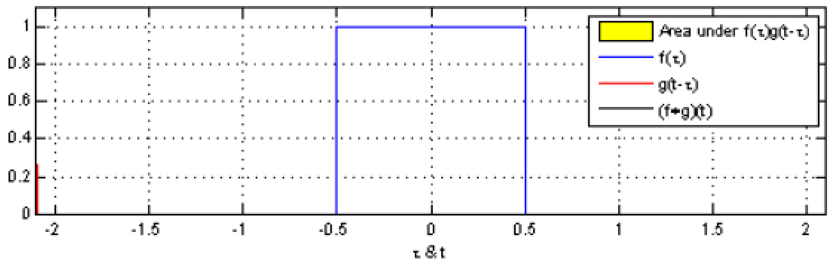
This is called convolution.

$$Y(s) = H(s) \cdot U(s) \tag{6}$$

This is called multiplication.

Convolution example

Essentially, the steps for convolving two signals are to first reflect the signal g, then offset the reflected signal. Then calculate the area under the graph for every offset, by sliding $-g$. The convolution at each time point is equal to the area under the intersection of functions. For two pulses, the result is a triangle wave:



However, the calculations to obtain this result in the time domain are complicated, but are only multiplication in the Laplace domain.

Getting the Time Response

The procedure to describe the time response for LTI systems is thus:

- Express the input, $u(t)$, in Laplace notation, $U(s)$
- Use this to find output $Y(s)$, usually by multiplying $U(s)$ by the transfer function $Y(s) = U(s)G(s)$
- Use inverse Laplace transforms (from tables) to express $Y(s)$ as a function of time, $y(t)$

0.2 Input functions: Impulse - Step - Ramp

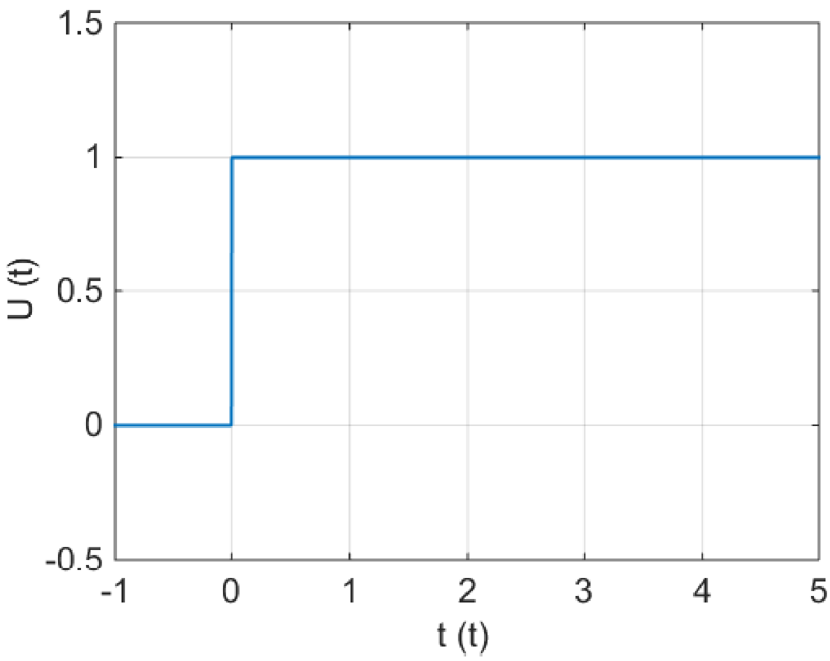
Now we will consider some standard inputs and look at the response of first and second order systems:

- Impulse
  - The Laplace transform is 1, so the response to an impulse is by the definition the transfer function
- Step
- Ramp

There are many others, particularly sinusoidal inputs or other discontinuous inputs, which are important in control loops, but we will focus on the two classic examples.

0.2.1 Step input

A step input is a discontinuous function, which is zero for all negative values of t and 1 for all positive values.



Laplace transform

$$x(t) = U(t) \tag{7}$$
$$\mathcal{L}\left\{U(t)\right\} = \int_{0^-}^{\infty} e^{-st}\,\mathrm{d}t = \left[-\frac{1}{s}e^{-st}\right]_{0^-}^{\infty} \to \frac{1}{s} \tag{8}$$
$$\tag{9}$$

Or for a gain of A

$$x(t) = AU(t) \tag{10}$$
$$\mathcal{L}\left\{AU(t)\right\} = \frac{A}{s} \tag{11}$$

Applications

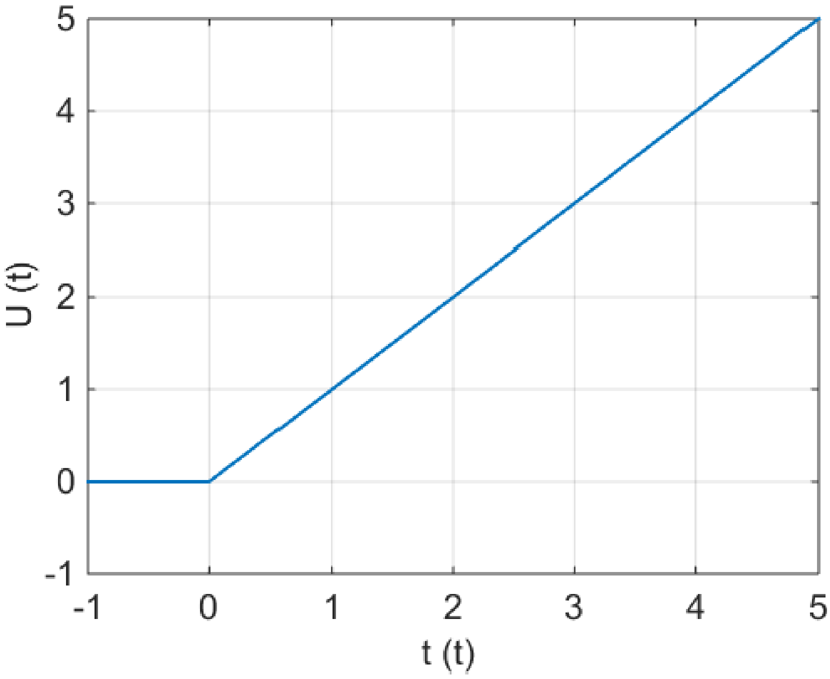
The step response is extremely useful in control theory for describing the behaviour of the system. In part because it incorporates the "transient" behaviour - from the sudden change from zero to one, as well as the "steady state" behaviour as the system settles down to a single value. IT also replicates many real world control applications such as:

- Position control - move to a $X = 10$mm position and stay
- Speed control - go to 33 RPM
- Temperature - heat element on 3D printer to 230 °C

Also, unlike the Dirac impulse - it is physically realisable.

0.2.2 Ramp input

A ramp input has a value of t for all t values above zero and zero elsewhere, often is scaled by a gain A.



Laplace transform

$$x(t) = at \tag{12}$$

$$\mathcal{L}\{at\} \int_{0^-}^{\infty} ate^{-st}\,\mathrm{d}t = -a\left[\frac{t}{s}e^{-st}\right]_0^{\infty} + a\int_0^{\infty} \frac{1}{s}e^{-st}\,\mathrm{d}t \tag{13}$$

$$= a\left[-\frac{1}{s^2}e^{-st}\right] \rightarrow \frac{a}{s^2} \tag{14}$$

Applications

Ramp inputs are useful in understanding the steady state behaviour of a system i.e. when t goes to infinity. Practical examples of control applications using ramp inputs are

- Servo motors - shaft **position** rather than speed
- Ovens for PCB manufacturing etc. - strict linear **profile** of temperature required as opposed to "get to this temperature quickly"
- CNC milling machine, move in X direction and constant rate