

UCL Mechanical Engineering 2020/2021

ENGF0004 Coursework 2

NCWT3

April 10, 2021

1 Question 1

1.1 a

For the line integral to be independent from the path of integration, the following conditions must be fulfilled:

$$I = \int_A^B \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) \quad (1.1)$$

$$P(x, y) = \frac{\partial u}{\partial x} \text{ and } Q(x, y) = \frac{\partial u}{\partial y} \quad (1.2)$$

$$\frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x} \quad (1.3)$$

Considering the integral:

$$I = \int_A^B \left[e^{-\alpha xy} \left(\frac{\alpha - 2}{x} \right) dx - \frac{1}{\alpha y} (e^{-\alpha xy} - 1) dy \right] \quad (1.4)$$

$$P(x, y) = e^{-\alpha xy} \left(\frac{\alpha - 2}{x} \right) \text{ and } Q(x, y) = -\frac{1}{\alpha y} (e^{-\alpha xy} - 1) \quad (1.5)$$

$$\frac{\partial P(x, y)}{\partial y} = -\alpha x \left(\frac{\alpha - 2}{x} \right) e^{-\alpha xy} = (2\alpha - \alpha^2) e^{-\alpha xy} \quad (1.6)$$

$$\frac{\partial Q(x, y)}{\partial x} = -\frac{1}{\alpha y} (-\alpha y) e^{-\alpha xy} = e^{-\alpha xy} \quad (1.7)$$

$$\therefore 2\alpha e^{-\alpha xy} - \alpha^2 e^{-\alpha xy} = e^{-\alpha xy} \quad (1.8)$$

$$e^{-\alpha xy} (\alpha^2 - 2\alpha + 1) = 0 \quad (1.9)$$

$$e^{-\alpha xy} = 0 \rightarrow \text{no solutions} \quad (1.10)$$

$$(\alpha - 1)^2 = 0 \quad (1.11)$$

$$\alpha = 1 \quad (1.12)$$

1.2 b

Calculating the line integral of 1.13 from $O(0, 0)$ to $A(1, e - 1)$ along $y = e^x - 1$:

$$I = \int_O^A (ye^{-2x}) (dx + dy) \quad (1.13)$$

$$y = e^x - 1 \quad (1.14)$$

$$dy = e^x dx \quad (1.15)$$

$$I = \int_0^1 \left((e^x - 1)(e^{-2x}) + (e^x - 1)(e^{-2x})(e^x) \right) dx \quad (1.16)$$

$$= \int_0^1 (e^{-x} - e^{-x} - e^{-2x} + 1) dx \quad (1.17)$$

$$= \int_0^1 (1 - e^{-2x}) dx \quad (1.18)$$

$$= \left[x + \frac{e^{-2x}}{2} \right]_0^1 \quad (1.19)$$

$$= 1 + \frac{e^{-2}}{2} - 0 - \frac{1}{2} \quad (1.20)$$

$$I = \frac{1}{2} (e^{-2} + 1) \quad (1.21)$$

1.3 c

1.3.1 i

$$\underline{F}(x, y, z) = \begin{pmatrix} \frac{y}{x^2} \\ \frac{x}{y^2} \\ 0 \end{pmatrix} \quad (1.22)$$

$$\nabla \cdot \underline{F} = \left(\frac{\partial}{\partial x} \right) \cdot \begin{pmatrix} \frac{y}{x^2} \\ \frac{x}{y^2} \\ 0 \end{pmatrix} \quad (1.23)$$

$$= \frac{\partial}{\partial x} \left(\frac{y}{x^2} \right) + \frac{\partial}{\partial y} \left(\frac{x}{y^2} \right) \quad (1.24)$$

$$= -\frac{2y}{x^3} - \frac{2x}{y^3} \quad (1.25)$$

$$= -2 \left(\frac{y}{x^3} + \frac{x}{y^3} \right) \quad (1.26)$$

1.3.2 ii

$$I = \int_1^2 \int_1^2 \left(-2 \left(\frac{y}{x^3} + \frac{x}{y^3} \right) \right) dx dy \quad (1.27)$$

$$= \int_1^2 \left[-2 \left(\frac{y}{-2x^2} + \frac{x^2}{2y^3} \right) \right]_1^2 dy \quad (1.28)$$

$$= \int_1^2 \left[-2 \left(-\frac{y}{8} + \frac{2}{y^3} + \frac{y}{2} - \frac{1}{2y^3} \right) \right] dy \quad (1.29)$$

$$= \int_1^2 \left(-\frac{3y}{4} - \frac{3}{y^3} \right) dy \quad (1.30)$$

$$= \left[-\frac{3y^2}{8} + \frac{3}{2y^2} \right]_1^2 \quad (1.31)$$

$$= -\frac{3}{2} + \frac{3}{8} + \frac{3}{8} - \frac{3}{2} \quad (1.32)$$

$$I = -\frac{9}{4} \quad (1.33)$$

1.4 d

1.4.1 i

$$I = \int (\sin x \cos y dy + \cos x \sin y dx) \quad (1.34)$$

$$y = 0 \quad dy = 0 \quad (1.35)$$

$$I_{AB} = \int_{x=0}^{\pi} (\sin x) dx = [-\cos x]_0^{\pi} = 2 \quad (1.36)$$

$$x = \pi \quad dx = 0 \quad (1.37)$$

$$I_{BC} = \int_{y=0}^{\pi} (-\sin y) dy = [\cos y]_0^{\pi} = -2 \quad (1.38)$$

$$\therefore I = I_{AB} + I_{BC} = 2 - 2 = 0 \quad (1.39)$$

1.4.2 ii

$$I = \int (\sin x \cos y dy + \cos x \sin y dx) \quad (1.40)$$

$$y = x \quad dy = dx \quad (1.41)$$

$$I_{AC} = \int_0^{\pi} (\sin x \cos x + \sin x \cos x) dx \quad (1.42)$$

$$= \int_0^{\pi} (\sin (2x)) dx \quad (1.43)$$

$$I_{AC} = \left[-\frac{1}{2} \cos (2x) \right]_0^{\pi} = \frac{1}{2} - \frac{1}{2} = 0 \quad (1.44)$$

$$(1.45)$$

1.5 e

1.5.1 i

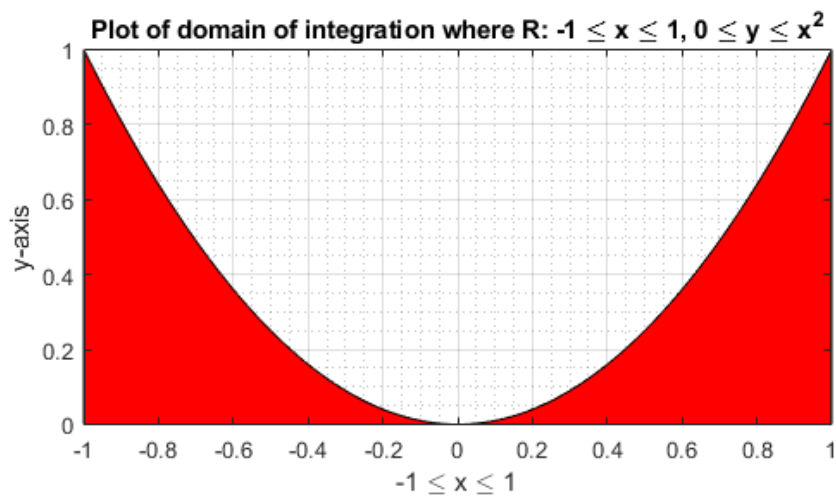


Figure 1: Domain of integration where $R: -1 \leq x \leq 1, 0 \leq y \leq x^2$.

```
1  clf
2  x=-1:0.001:1;
3  y=x.^2;
4  A=area(x,y);
5  set(A(1),'FaceColor','red');
6  axis('image');
7  xlabel('-1 \leq x \leq 1')
8  ylabel('y-axis')
9  title('Plot of domain of integration where R: -1 \leq x \leq 1, 0 \leq y
        \leq x^2')
10 grid on
11 grid minor
```

1.5.2 ii