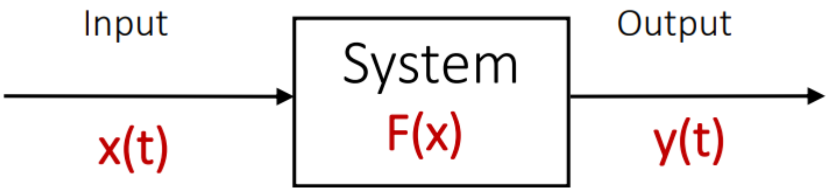
Previously, we have considered systems in the general sense, with a generic function block approach.



Now we will investigate how we model physical systems and obtain the function block $F(x)$ for electrical and mechanical components.

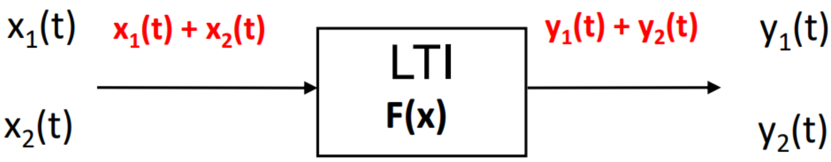
0.1 Linear Term Invariant (LTI) systems

The focus of this course and indeed much of control theory itself focuses on modelling physical systems as **linear** and **time invariant**. LTI systems have three key properties:

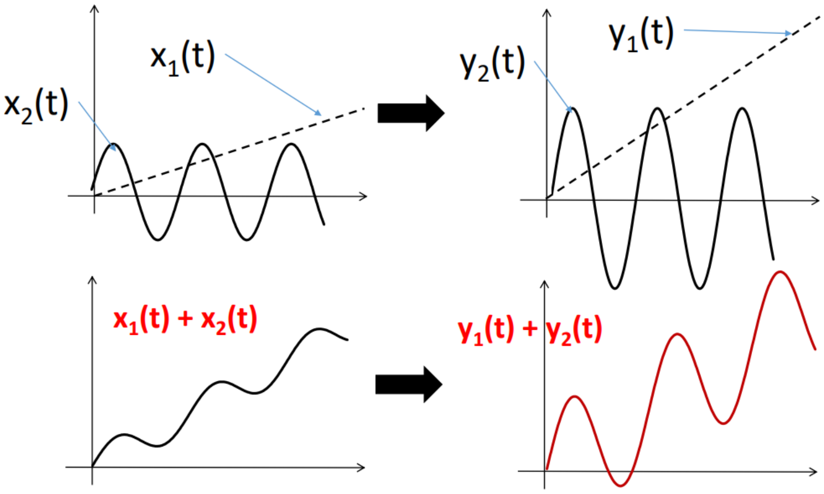
- Obey principle of superposition
- Homogeneity
- Time invariance

0.1.1 Superposition

If input $x_1(t)$ produces output $y_1(t)$ and input $x_2(t)$ produces $y_2(t)$, then input $x_1(t) + x_2(t)$ produces output $y_1(t) + y_2(t)$

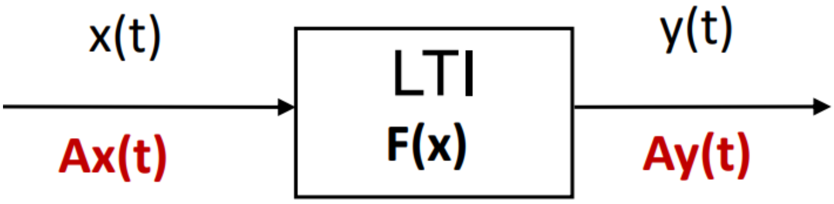


Say for a system which doubles the input $F(x) = 2x$

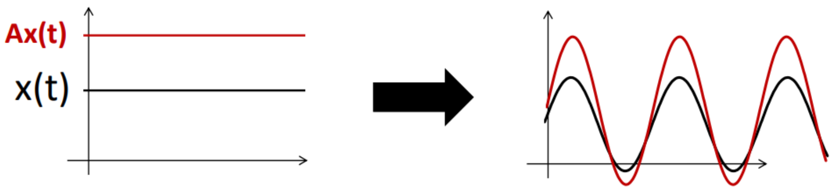


0.1.2 Homogeneity

If the input to the system $x(t)$ is scaled by a magnitude scale factor A, then the output $y(t)$ is also scaled by the same factor.

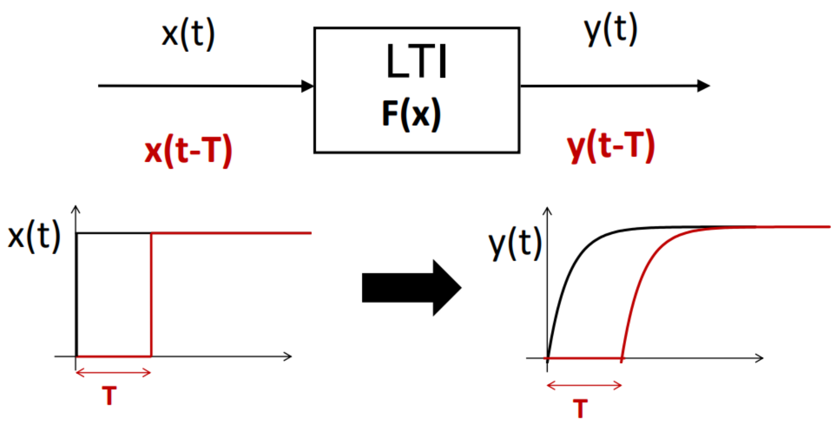


For example, consider a system which generates a sine wave at a given amplitude, with a set frequency:



0.1.3 Time invariance

If input is applied at time $t = 0$ or T s from now, the output is identical with the exception of a delay of T s.



Are these models suitable for physical systems?

These three requirements, whilst simple, are so stringent that **almost no physical LTI system truly exists**. Consider a car engine - the performance deteriorates over time, to stretch it further, would you expect a system to give the same output after a time delay T of 10 years? Even simple systems such as a resistor in an electrical circuit have non-linearities - a scaling factor A could be chosen for $x(t)$ which would mean too much current flows and the resistor melts.

Most practical systems are not linear, but often we can assume they behave linearly **under certain conditions/assumptions**. Linear systems are **much** easier to solve! There are **analytic** solutions with standard tools used solve the equations. Whereas for non-linear problems it is often necessary to solve them numerically.

0.1.4 Linearisation: Example 1

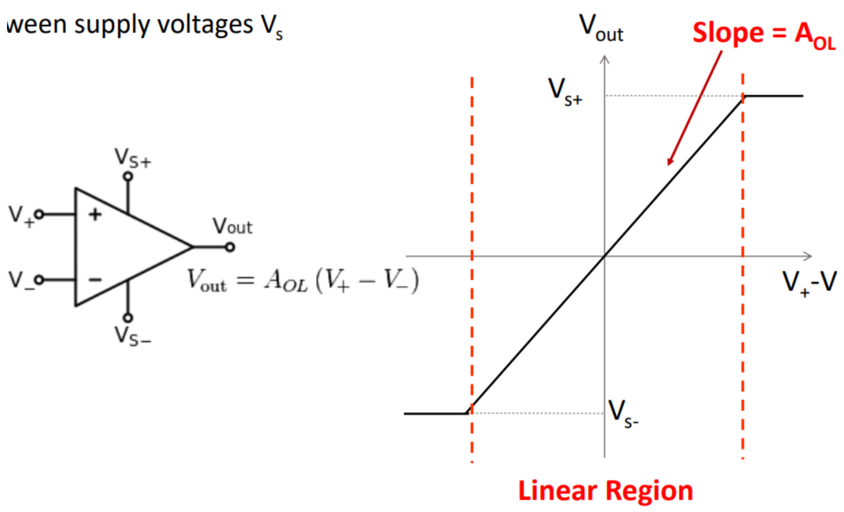
For a simple system such as a spring, across all possible compressions or extensions the response is non-linear:

$$F = -kX \tag{1}$$

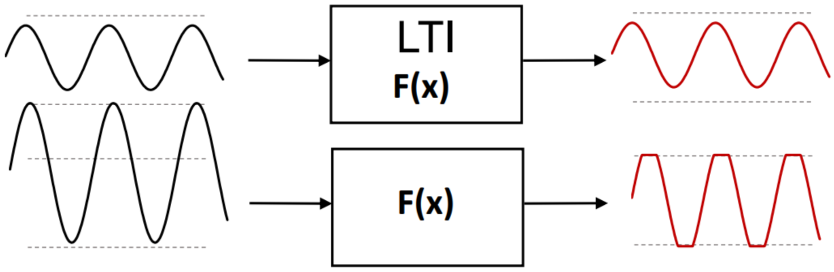
Hookes law is only a linear approximation of the true response. However, if we choose the operating range of the spring correctly, the response is within the linear region. This approximation is **valid**.

Linearisation: Example 2

Similarly an op-amp has a **linear region** where the output signals are between supple voltages V_S



Keep signals within these regions and the linear assumption holds:

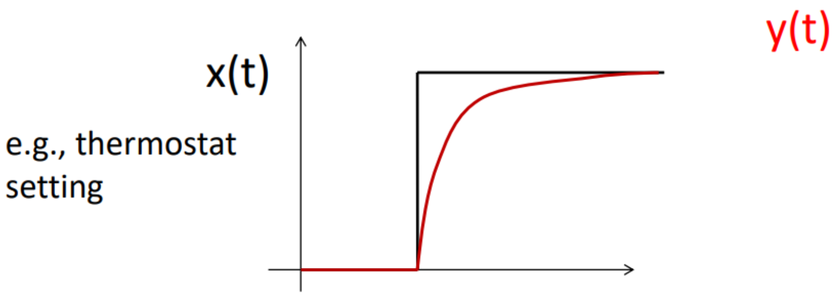


Add LTI example

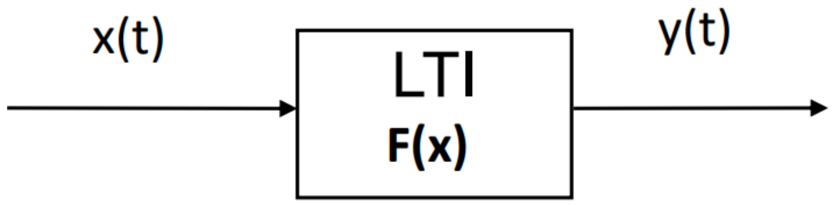
0.2 Dynamic systems - Laplace Transform

0.2.1 Dynamic systems as ODEs

Ideal systems would respond **instantaneously** to inputs, however real world systems require some time to adjust to changes and are thus known as **dynamic systems** as the output changes over time.



As we are intersected in describing something that **changes** with time, it is useful to express the function block of the system $F(t)$ as an ordinary differential equation (ODE).



$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + ... + a_2\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a_1\frac{\mathrm{d}y}{\mathrm{d}t} + a_0 = bx \tag{2}$$

- x is input function or forcing function
- y is output
- n is **order** of the ODE
- a_0... are coefficients. These **completely characterise the system**

0.2.2 Laplace Transforms

Because of our **linear assumptions** we can use Laplace transforms to simplify solving the ODEs. The Laplace transforms of a signal (function) x is the function $X = \mathcal{L}(x)$ defined by

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,\mathrm{d}t \tag{3}$$

For those $s \in \mathbf{C}$ for which the integral makes sense.

- X is a complex-valued function of complex numbers
- s is called the (complex) **frequency variable** with units s^{-1}, t is called the **time variable** (in sec); st is unitless
- $s = \sigma + j\omega$

As we shall see:

- Differential operators are replaced with algebraic variables
- Algebraic equations are much easier to manipulate & solve
- Standard forms exist for many physical systems

Laplace Transforms: Example 1

Let's fine Laplace transform $x(t) = e^t$:

$$X(e^t) = \int_{0^-}^{\infty} e^t e^{-st}\,\mathrm{d}t \tag{4}$$

$$X(e^t) = \int_{0^-}^{\infty} e^{(1-s)t}\,\mathrm{d}t \tag{5}$$

$$Xe^t = \frac{1}{1-s}e^{(1-s)t}\Big|_{0^-}^{\infty} \tag{6}$$

$$X(e^t) = \frac{1}{1-s} \times 0 - \frac{1}{1-s} \times 1 = \frac{1}{s-1} \tag{7}$$

Laplace Transforms: Example 2

Constant or **unit step** $x(t) = 1$ (for $t \geq 0$)

$$X(s) = \int_{0^-}^{\infty} 1e^{-st}\,\mathrm{d}t \tag{8}$$

$$X(s) = \int = -\frac{1}{s}e^{-st}\Big|_{0^-}^{\infty} \tag{9}$$

$$X(s) = -\frac{1}{s} \times 0 - (-\frac{1}{s}) \times 1 = \frac{1}{s} \tag{10}$$

Laplace Transforms: Example 3

Sinusoid: first express $x(t) = \cos(\omega t)$ as:

$$x(t) = \frac{1}{2}e^{i\omega t} + \frac{1}{2}e^{-i\omega t} \text{ (Euler's formula)} \tag{11}$$

$$X(s) = \int_{0^-}^{\infty} e^{-st}\left(\frac{1}{2}e^{i\omega t} + \frac{1}{2}e^{-i\omega t}\right)\mathrm{d}t \tag{12}$$

$$X(s) = \frac{1}{2}\int_{0^-}^{\infty} e^{(-s+i\omega)t}\,\mathrm{d}t + \frac{1}{2}\int_{0^-}^{\infty} e^{(-s-i\omega)t}\,\mathrm{d}t \tag{13}$$

$$X(s) = \frac{1}{2}\frac{1}{s-i\omega} + \frac{1}{2}\frac{1}{s+i\omega} = \frac{s}{s^s+\omega^s} \tag{14}$$

You can look up these transforms in a table. The Laplace variable, s, can be considered to represent the differential operator (very useful for control engineering):

$$s \equiv \frac{\mathrm{d}}{\mathrm{d}t} \tag{15}$$

$$\frac{1}{s} \equiv \int_{0^-}^{\infty} \mathrm{d}t \tag{16}$$

Laplace transform of time derivative $\frac{\mathrm{d}x}{\mathrm{d}t}$:

$$L\left\{\frac{\mathrm{d}x}{\mathrm{d}t}\right\} = \int_{0^-}^{\infty} \frac{\mathrm{d}x}{\mathrm{d}t} e^{-st}\,\mathrm{d}t \tag{17}$$

Integrating by parts

$$L\left\{\frac{\mathrm{d}x}{\mathrm{d}t}\right\} = s\int_{0^-}^{\infty} x(t)e^{-st}\,\mathrm{d}t + \left[x(t)e^{-st}\right]_{0^-}^{\infty} \tag{18}$$

The initial condition $x(0^-)$ is often zero in practice

$$L\left\{\frac{\mathrm{d}x}{\mathrm{d}t}\right\} = sX(s) - x(0^-) = sX(s) \tag{19}$$

We can substitute this result to solve higher order derivatives:

$$L\left\{\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right\} = sL\left\{\frac{\mathrm{d}x}{\mathrm{d}t}\right\} - \frac{difx}{\mathrm{d}t}(0^-) \tag{20}$$

$$L\left\{\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right\} = s^2X(s) - sx(0^-) - \frac{\mathrm{d}X}{\mathrm{d}t}(0^-) = s^2X(s) \tag{21}$$

So more generally, with all initial conditions set to zero:

$$L\left\{\frac{d^nx}{dt^n}\right\} = s^nX(s) \tag{22}$$

0.2.3 Transfer Functions

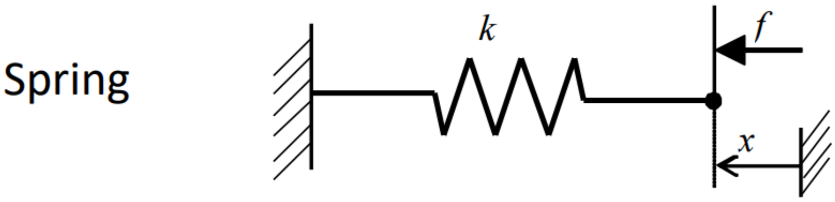
After we have taken the Laplace transform of the differential equation of a system, it's useful to rearrange to give the system **output** as the product of the system **input** and the system **transfer function**.

The transfer function of a linear system is defined as the **ratio** of the Laplace transform of the output variable to the Laplace transform of the input variable, with all the initial conditions assumed to be zero.

0.2.4 Transfer Functions of Mechanical Components

Spring

Convention is input force, output displacement. Balance **forces**



Time domain equation (Hooke's Law):

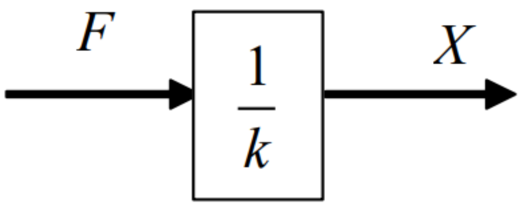
$$f(x) - kx \tag{23}$$

k is stiffness in $\mathrm{N\,m^{-1}}$. Laplace domain equation:

$$F(s) = kX(s) \tag{24}$$

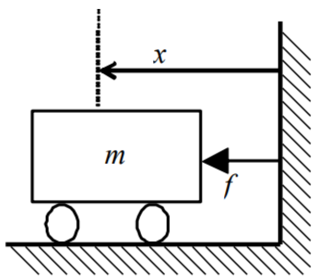
Transfer function:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{k} \tag{25}$$



Mass

Inertial load - mass



Time domain equation from Newton's 2nd law of motion

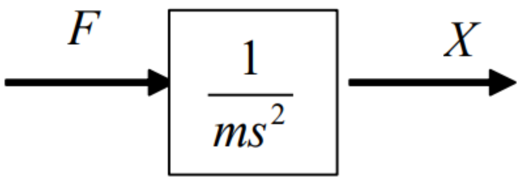
$$f(t) = m\frac{\mathrm{d}^2x(t)}{dt^2} \tag{26}$$

Laplace domain equation

$$F(s) = ms^2X(s) \tag{27}$$

Transfer function

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2} \tag{28}$$



Damper

Below is a dashpot - a viscous damper. They resist motion through friction. The damping coefficient is in terms of c with units $\mathrm{N\,m^{-1}\,s^{-1}}$. Time domain equation:

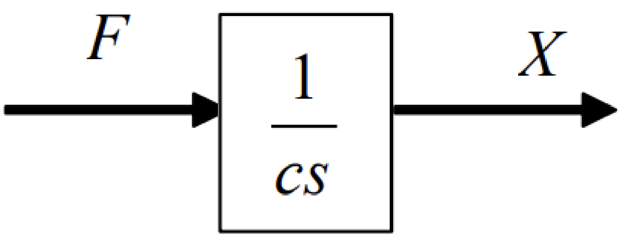
$$f(x) = x\frac{\mathrm{d}x}{\mathrm{d}t} = csx \tag{29}$$

Laplace domain equation:

$$F(s) = csX(s) \tag{30}$$

Transfer function

$$\frac{X(s)}{F(s)} = \frac{1}{cs} \tag{31}$$



0.2.5 Combining components

Springs

Springs in parallel

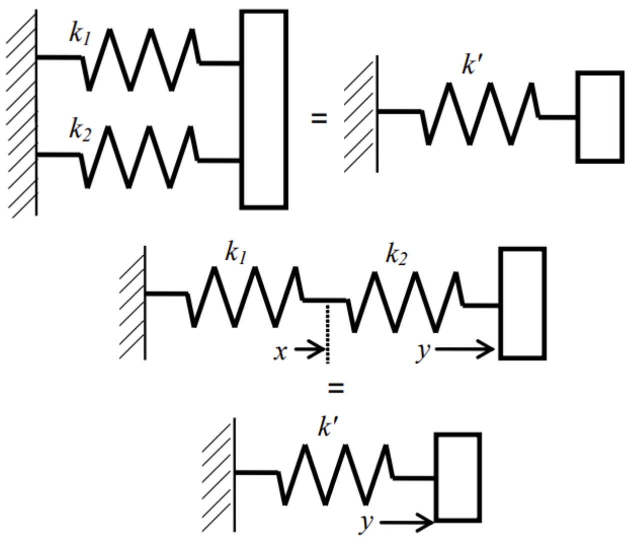
$$k' - k_1 + k_2 \tag{32}$$

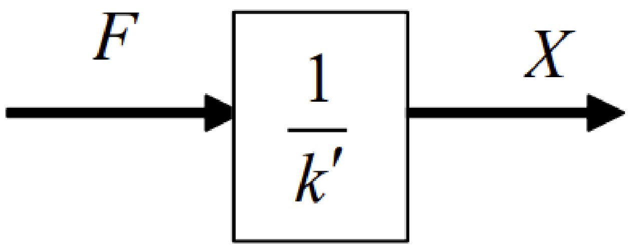
$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{k'} \tag{33}$$

Springs in series

$$\frac{1}{k'} = \frac{1}{k_1} + \frac{1}{k_2} \tag{34}$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{k'} \tag{35}$$





Dashpots behave in a similar way to springs. Parallel:

$$c' = c_1 + c_2 \tag{36}$$

and in series:

$$\frac{1}{c'} = \frac{1}{c_1} + \frac{1}{c_2} \tag{37}$$

Resistor

Convention is input voltage, output current. Balance voltages. Ohms law is $V = IR$. Time domain:

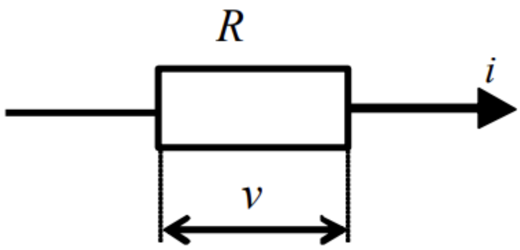
$$v(t) = i(t)R \tag{38}$$

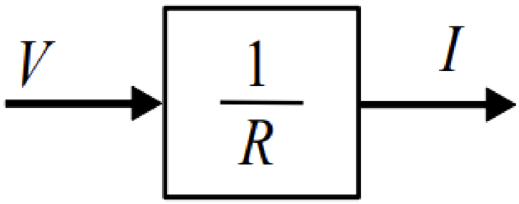
Laplace domain:

$$V(s) = I(s)R \tag{39}$$

Transfer function:

$$G(s) = \frac{I(s)}{V(s)} = \frac{1}{R} \tag{40}$$





Capacitor

Either definition of current/voltage relationship gives same result. Time domain:

$$i(t) = C\frac{\mathrm{d}v}{\mathrm{d}t} \tag{41}$$

$$v(t) = \frac{1}{C}\int i(t)\,\mathrm{d}t \tag{42}$$

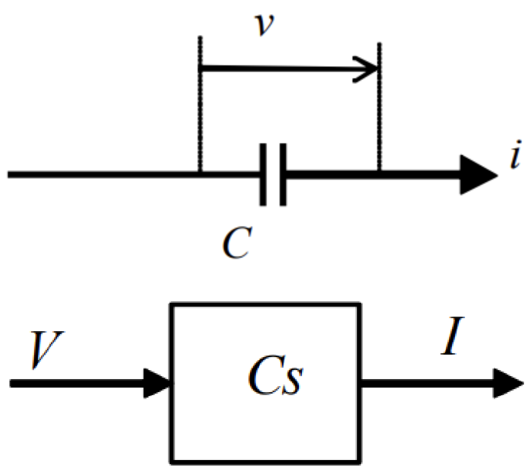
Laplace domain:

$$I(s) = CsV(s) \tag{43}$$

$$V(s) = \frac{1}{C}\frac{1}{s}I(s) \tag{44}$$

Transfer function:

$$G(s) = \frac{I(s)}{V(s)} = Cs \tag{45}$$



Inductor

An inductor resists changes of current by generating a voltage in opposition via magnetic induction. From Faraday's Law:

$$v(t) = L\frac{\mathrm{d}i}{\mathrm{d}t} \tag{46}$$

Laplace domain

$$V(s) = LsI(s) \tag{47}$$

Transfer function:

$$G(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls} \tag{48}$$