0.1 Failure modes

Structural failure occurs when a structural component is unable to fulfil its designed purpose.

0.1.1 Forms of failure

- Elastic failure - excessive elastic deformation (e.g. buckling)
- Brittle failure - sudden fracture of the component
- Plastic failure - permanent variation of geometry
- Progressive failure - sudden fracture due to fatigue loading
- Creep - ductile failure due to high loads prolonged in time
- Corrosion - loss of surface integrity due to environmental factors
- Wearing - loss of surface integrity due to abrasive contact

0.1.2 Micro-structural failures

At microscopic level, most forms of mechanical failure occur according to two main mechanisms.

- Brittle failure - characterised by the breaking of the atomic bonds
- Plastic failure - characterised by the spreading of dislocations through the crystal structure

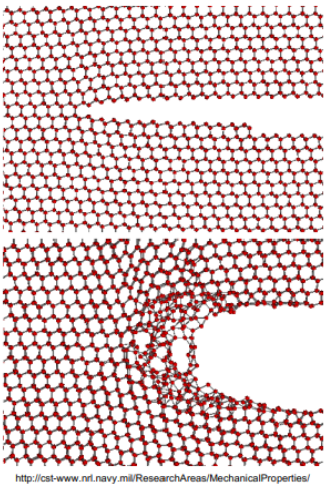

Figure 1:

Failure criteria have the function to give indication on the possibility that one of these mechanisms may occur, for a known state of stress.

0.1.3 Ductile and brittle materials

It is essential, first, to identify which of the two failing mechanisms is going to occur! In static conditions, the selection is simple: it depends essentially on the material subject to the stress state. There are materials that fail according to the brittle failure mechanism in the most of the cases, called **brittle materials**. There are materials that fail according to the plastic failure mechanism in the most of the cases, called ductile materials.

0.1.4 Brittle materials

In brittle materials, atoms have low mobility. Failure occurs without visible permanent deformations, when energy is sufficient to break the atomic bonds.

Figure 2:

0.1.5 Ductile materials

In ductile materials, for energy lower than the one producing the spreading of dislocations, atoms undergo reciprocal displacements without breaking atomic bonds: Ductile materials can store large amounts of energy in the form of *recoverable elastic distortion.* This mechanism is called **elastic behaviour.** For higher levels of energy, dislocations spread and the crystal structure changes. Macroscopically, the component undergoes *permanent deformations*, eventually leading to *fracture.*

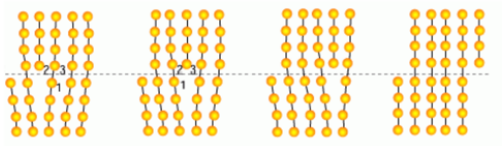

Figure 3:

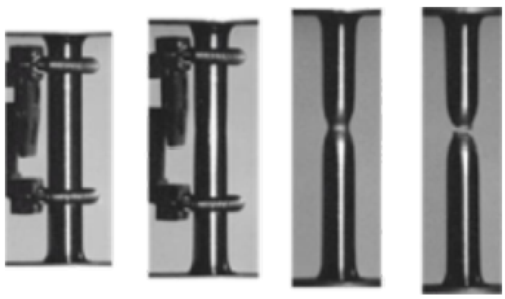

Figure 4:

0.1.6 Brittle - ductile

Brittle and ductile materials are only ideal extremes: All materials have an intermediate behaviour. Moreover, under specific states of stress, typically ductile materials can undergo brittle failure and typically brittle materials can fail plastically.

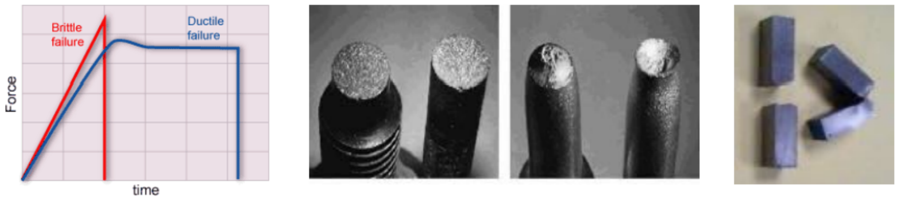

Figure 5:

0.1.7 Hydrostatic stress state

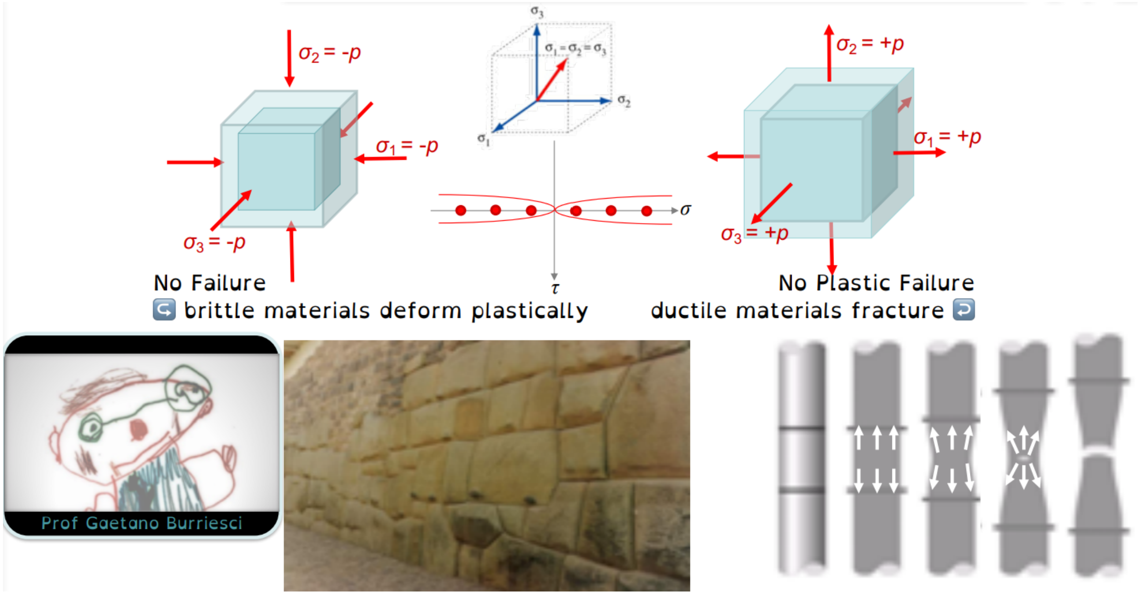

Figure 6:

0.2 Failure criteria

Experience has shown that failure consistently occurs when the *components of the stress tensor* at a point of the structure reach specific values, depending on the *material*. The state of stress can be easily determined analytically or numerically. However, since infinite different states of stress can occur, *it is impossible to analyse what is the limiting level for all cases.* A possible solution would be to convert a specific state of stress into a single equivalent scalar value, called **effective stress**, which can represent it and can be directly compared with a limit value, associated with failure.

> Failure criteria suggest **how to combine the components of a specific state to obtain an effective stress**.

Once the appropriate failure criterion has been identified, the *limit for the effective stress* can be easily measured by determining when failure occurs in a simplified case, e.g. unidirectional tensile or compressive test.

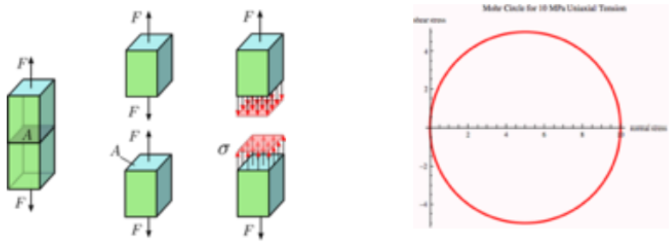

Figure 7:

For ductile materials, failure is commonly associated with yielding. For brittle materials it is associated with brittle fracture.

0.3 Maximum principal stress theory

The first failure criterion can be attributed to Galileo Galilei who observed (from the studies on stone ties) that there is a limit to the maximum principal stress.

> **Failure occurs whenever the maximum principal stress equals the strength of the material.**

The structure is safe if at all points it is:

$$\sigma_1 < \sigma_f \,\&\, \sigma_2 < \sigma_f \,\&\, \sigma_3 < \sigma_f \tag{1}$$

0.3.1 Max & min principal stress theory

The *maximum and minimum principal stress theory,* usually attributed to Rankine, includes in the previous criterion a limit to the compressive stress.

> **Failure occurs whenever the maximum principal stress equals the tensile strength or the minimum principal stress equals the compressive strength.**

$$\sigma_{fc} < \sigma_1 < \sigma_{ft} \,\&\, \sigma_{fc} < \sigma_2 < \sigma_{ft} \,\&\, \sigma_{fc} < \sigma_3 < \sigma_{ft} \tag{2}$$

Representing the points of safety in a plane σ_1, σ_2, a square is described. Representing the points of safety in a space σ_1, σ_2, σ_3, a cube is described.

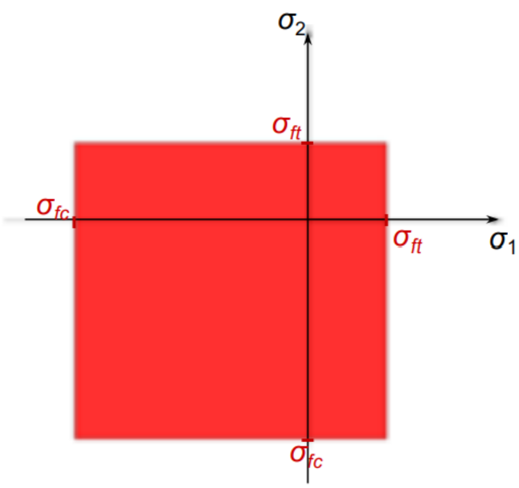

Figure 8:

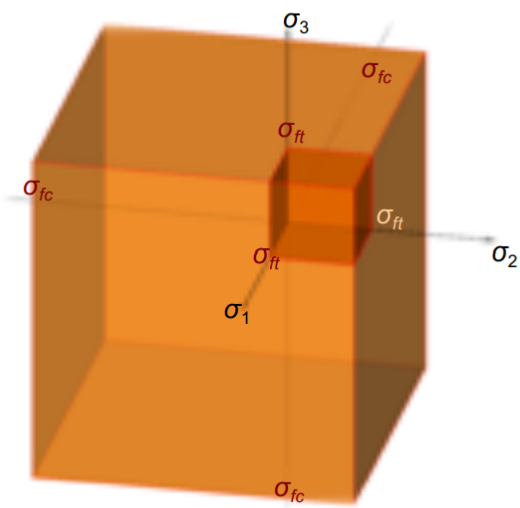

Figure 9:

In the Mohr's representation, all circles between two vertical lines of abscissas σ_{ft} and σ_{fc} described safe states of stress.

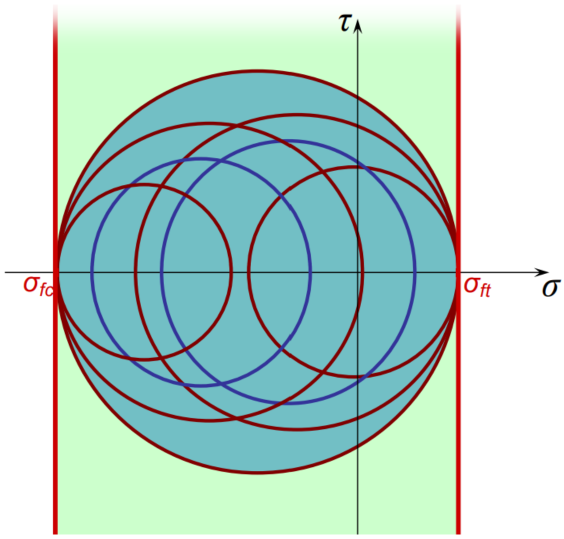

Figure 10:

The main limitations of the maximum principle stress theory are:

- interaction between the principal stress is not considered
- does not predict unlimited resistance under hydrostatic compression.

Applicability:

> **The maximum principal stress theory can be reliable only for the study of brittle materials subjected to states of stress different from the hydrostatic compression.**

0.3.2 Maximum normal strain theory

The *maximum normal strain theory*, usually attributed to Saint-Venant, is conceptually different, in that it limits the strain instead of the stress. It was developed to justify the vertical cracks that break stone columns.

> **Failure occurs whenever the maximum principal strain equals the strain corresponding to the tensile strength.**

The structure is safe if at all points it is:

$$\epsilon_1 < \epsilon_f;\ \epsilon_2 < \epsilon_f;\ \epsilon_3 < \epsilon_f \tag{3}$$

Since:

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right] \tag{4}$$

$$\epsilon_1 = \frac{1}{E}\left[\sigma_2 - \nu\left(\sigma_1 + \sigma_3\right)\right] \tag{5}$$

$$\epsilon_1 = \frac{1}{E}\left[\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right)\right] \tag{6}$$

and, from a uni-axial tensile test:

$$\epsilon_f = \frac{\sigma_f}{E} \tag{7}$$

We can derive that:

$$\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right) < \sigma_f \tag{8}$$
$$\sigma_2 - \nu\left(\sigma_1 + \sigma_3\right) < \sigma_f \tag{9}$$
$$\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right) < \sigma_f \tag{10}$$

Representing the points of safety in the plane σ_1, σ_2, a triangle is obtained. Representing the points of safety in a space σ_1, σ_2, σ_3, a pyramid of triangular equilateral section is obtained.

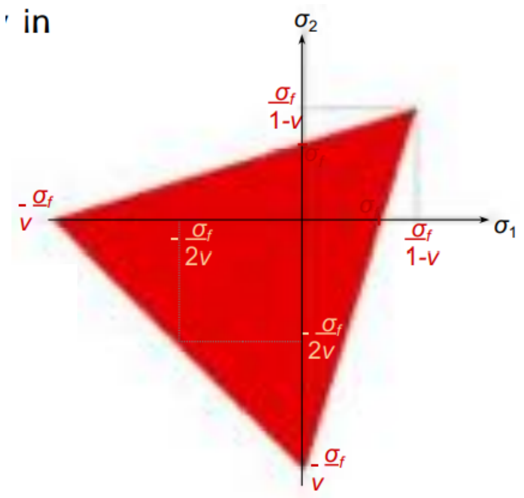

Figure 11:

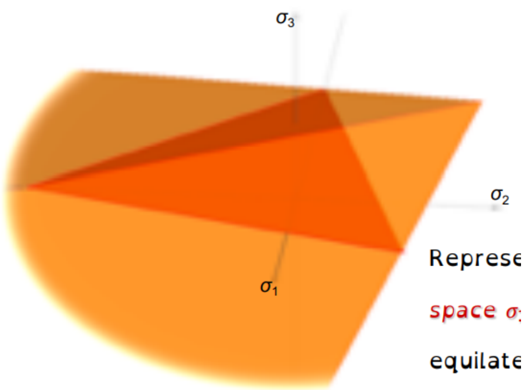

Figure 12:

The *maximum normal strain theory* was completed by Grashof, who introduced a limit to the maximum negative strain also. The main limitations of the maximum strain theory are:

- though very used in the past, it does not give results in agreement with experimental evidence
- does not predict unlimited resistance under hydrostatic compression.

Applicability:

> **the maximum normal strain theory is reliable for the study of brittle materials with tensile and compressive strengths not very different, subjected to states of stress different from the hydrostatic compression.**

It was the first principle to consider the interaction between principal stresses.

0.4 Maximum shear stress theory

0.4.1 Criteria for ductile materials

Previous criteria were developed when the materials of main interest were brittle (stones, bricks and timbers); and therefore they are suitable to describe brittle failure. When the use of metals became more relevant, new criteria had to be developed, in order to better describe the ductile failure due to yielding.

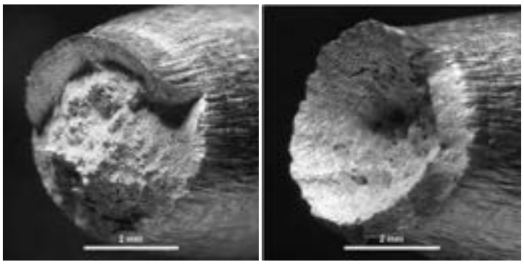

Figure 13:

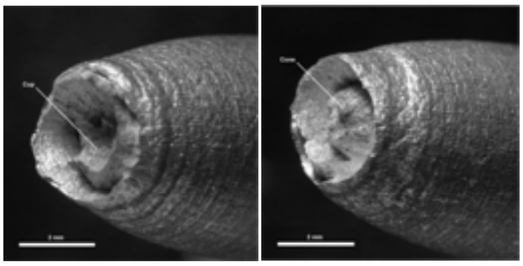

Figure 14:

The maximum-shear-stress theory (also called Tresca theory) assumes that plastic failure (yielding) is produced by the shear stresses.

Yielding occurs whenever the maximum shear stress equals the maximum shear stress τ_y produced in a tension-test specimen of the same material when that specimen begins to yield.

The structure is safe if at all points it is:

$$\tau_{max} < \tau_Y \tag{11}$$

The maximum shear stress τ_Y produced in a tension-test specimen when that specimens begins to yield is given by:

$$\tau_Y = \frac{1}{2}\sigma_Y \tag{12}$$

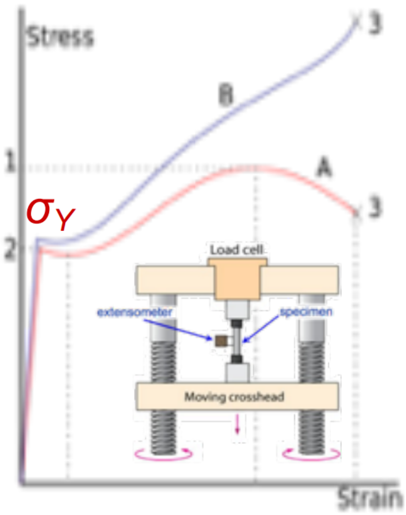

Figure 15:

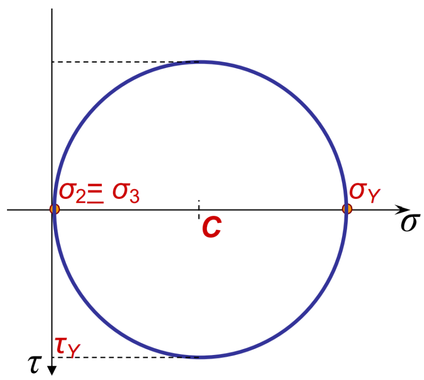

Figure 16:

Maximum shear stresses in the principal planes of the component will be the maximum of:

$$(\tau_{max})_3 = \frac{|\sigma_1 - \sigma_2|}{2} \quad (\tau_{max})_1 = \frac{|\sigma_2 - \sigma_3|}{2} \quad (\tau_{max})_2 = \frac{|\sigma_3 - \sigma_1|}{2} \tag{13}$$

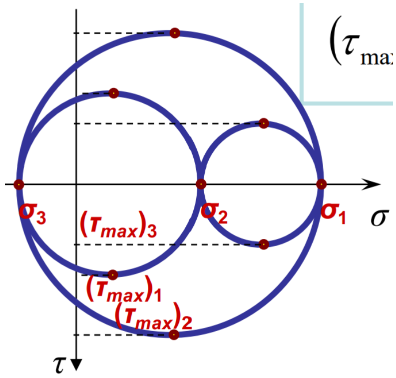

Figure 17:

Therefore:

$$|\sigma_1 - \sigma_2| \, ; \, |\sigma_2 - \sigma_3| \, ; \, |\sigma_3 - \sigma_1| < \sigma_Y \tag{14}$$

For a plane stress state ($\sigma_3 = 0$):

$$\tau_{max} < \tau_Y \rightarrow |\sigma_1 - \sigma_2| \, ; \, |\sigma_2 - \sigma_3| \, ; \, |\sigma_3 - \sigma_1| < \sigma_Y \tag{15}$$

$$|\sigma_1 - \sigma_2| < \sigma_Y \rightarrow \begin{array}{l} \sigma_2 - \sigma_Y < \sigma_1 < \sigma_2 + \sigma_Y \\ \sigma_1 - \sigma_Y < \sigma_2 < \sigma_1 + \sigma_Y \end{array} \tag{16}$$

$$|\sigma_2| < \sigma_Y \rightarrow -\sigma_Y < \sigma_2 < \sigma_Y \tag{17}$$

$$|\sigma_1| < \sigma_Y \rightarrow -\sigma_Y < \sigma_1 < \sigma_Y \tag{18}$$

An irregular hexagon is described.

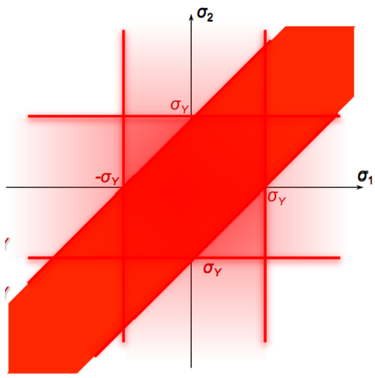

Figure 18:

For a tri-axial stress state:

$$\tau_{max} < \tau_Y \rightarrow |\sigma_1 - \sigma_2| \, ; \, |\sigma_2 - \sigma_3| \, ; \, |\sigma_3 - \sigma_1| < \sigma_Y \tag{19}$$

Representing the points of safety in a space σ_1, σ_2, σ_3, a prism of regular hexagonal section, with axis along the trisector to the planes is described.

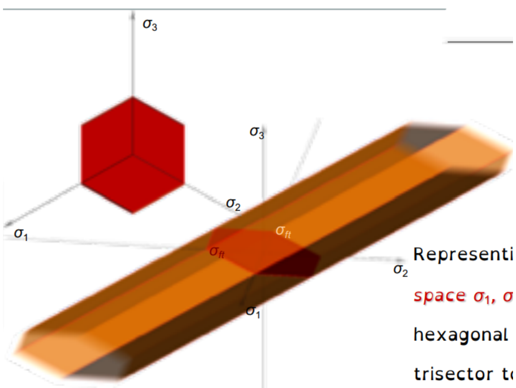

Figure 19:

In the Mohr's representation, all circles between the two horizontal lines of ordinate $\frac{1}{2}\sigma_Y$ and $-\frac{1}{2}\sigma_Y$ describe the safe state of stress.

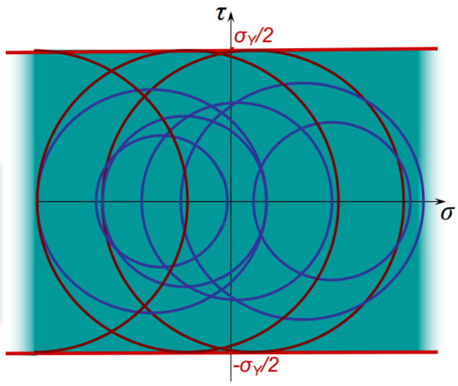

Figure 20:

Applicability:

> **The maximum shear stress theory is reliable for the study of ductile materials. For these material it is conservative (does not consider the benefit given by the intermediate principal stress). It does not limit hydrostatic compressive loads (in agreement with experimental evidence).**

The main limitation of the maximum shear stress theory is that it does not predict the failure under hydrostatic tension.

0.5 Maximum energy theories

0.5.1 Maximum strain energy theory

The criterion proposed by Beltrami assumes that a material can store a limited amount of elastic energy as strain before undergoing plastic deformations (failure).

> **Yielding occurs whenever the strain energy density equals the strain energy density that produces yielding in a tension-test specimen of the same material.**

$$\frac{\mathrm{d}W}{\mathrm{d}V} = \frac{1}{2}\left(\sigma_1 \cdot \epsilon_1 + \sigma_2 \cdot \epsilon_2 + \sigma_3 \cdot \epsilon_3\right) \tag{20}$$

$$= \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 - 2\nu\left(\sigma_1 \cdot \sigma_2 + \sigma_2 \cdot \sigma_3 + \sigma_3 \cdot \sigma_1\right)\right] \tag{21}$$

$$\begin{array}{l} \epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right] \\ \epsilon_1 = \frac{1}{E}\left[\sigma_2 - \nu\left(\sigma_1 + \sigma_3\right)\right] \\ \epsilon_1 = \frac{1}{E}\left[\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right)\right] \end{array} \quad \text{and} \quad \frac{\mathrm{d}W_Y}{\mathrm{d}V} = \frac{1}{2}\sigma_Y \cdot \epsilon_Y = \frac{1}{2}\sigma_Y \cdot \frac{\sigma_Y}{E} \tag{22}$$

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1 \cdot \sigma_2 + \sigma_2 \cdot \sigma_3 + \sigma_3 \cdot \sigma_1\right) < \sigma_Y^2 \tag{23}$$

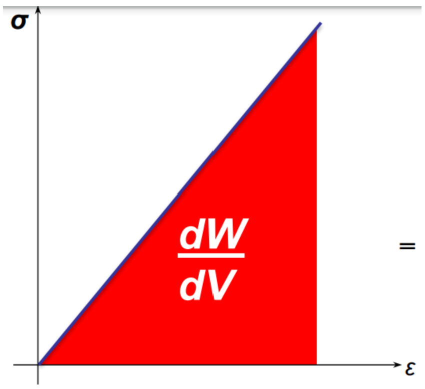

Figure 21:

Representing the points of safety in a plane σ_1, σ_2, $\sigma_3 = 0$, an ellipse is described. Representing the points of safety in a space σ_1, σ_2, σ_3, an axial-symmetric ellipsoid with an axis along the trisector to the planes is described.

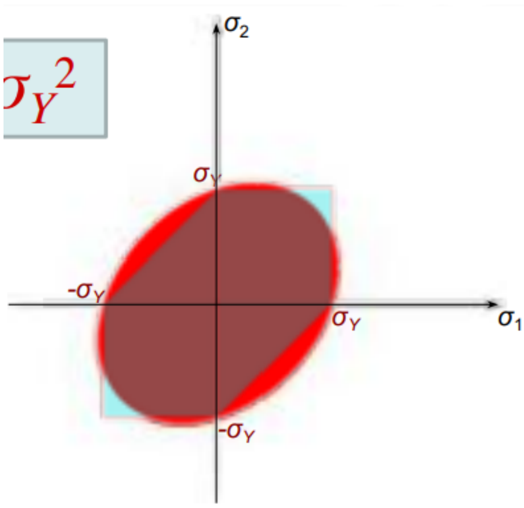

Figure 22:

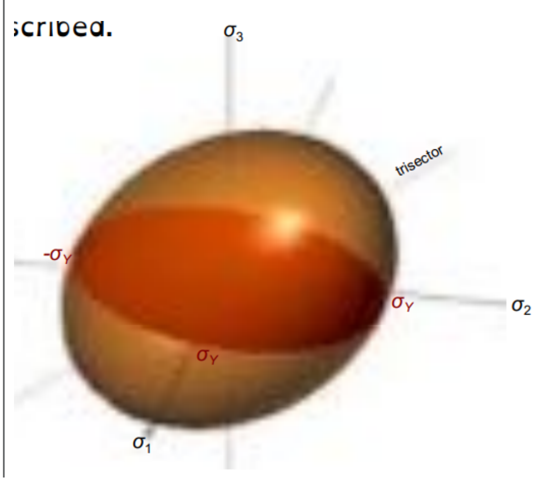

Figure 23:

Applicability:

> **The maximum strain energy theory can be used for the study of ductile materials. It is usually very conservative, especially close the hydrostatic states of stress.**

It was the first principle based on energetic considerations.

0.5.2 Distortion-energy theory

Any three-dimensional state of deformation can be modelled as the result of two components.

- a **hydrostatic component**, producing only a volume change
- a **distortional component**, producing only angular distortions, without volume changes

The criterion proposed by von Mises assumes that not all elastic-energy contributes to yielding, but only the distortional component.

> **Yielding occurs whenever the distortion strain energy density equals the distortion strain energy density that produces yielding in a tension-test specimen of the same material.**

$$\frac{\mathrm{d}W_{DY}}{\mathrm{d}V} < \frac{\mathrm{d}W_{DY}}{\mathrm{d}V}\frac{\mathrm{d}W}{\mathrm{d}V} = \frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu\left(\sigma_1\cdot\sigma_2+\sigma_2\cdot\sigma_3+\sigma_1\cdot\sigma_3\right)\right] \tag{24}$$

Hydrostatic stress component is:

$$\sigma_{oct} = \frac{\sigma_1+\sigma_2+\sigma_3}{3} \tag{25}$$

Then, strain energy associated with change of volume is:

$$\frac{\mathrm{d}W_V}{\mathrm{d}V} = \frac{1}{2}\left(\sigma_{oct}\cdot\epsilon_1+\sigma_{oct}\cdot\epsilon_2+\sigma_3\cdot\epsilon_3\right) = \frac{\sigma_{oct}}{2}\left(\epsilon_1+\epsilon_2+\epsilon_3\right) \tag{26}$$

$$= \frac{1-2\nu}{6E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+2\left(\sigma_1\cdot\sigma_2+\sigma_2\cdot\sigma_3+\sigma_1\cdot\sigma_3\right)\right] \tag{27}$$

The strain energy associated with distortion (change of shape) is:

$$\frac{\mathrm{d}W}{\mathrm{d}V}-\frac{\mathrm{d}W_V}{\mathrm{d}V} = \frac{1+\nu}{6E}\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right] \tag{28}$$

$$\frac{\mathrm{d}W_{DY}}{\mathrm{d}V} = \frac{1+\nu}{6E}2\sigma_Y^2 \rightarrow \sqrt{\frac{\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2}{2}} < \sigma_Y \tag{29}$$

Representing the points of safety in a plane σ_1, σ_2, $\sigma_3=0$, an ellipse is described. Representing the points of safety in a space σ_1, σ_2, σ_3, a cylinder of circular section, with axis along the trisection to the planes is defined.

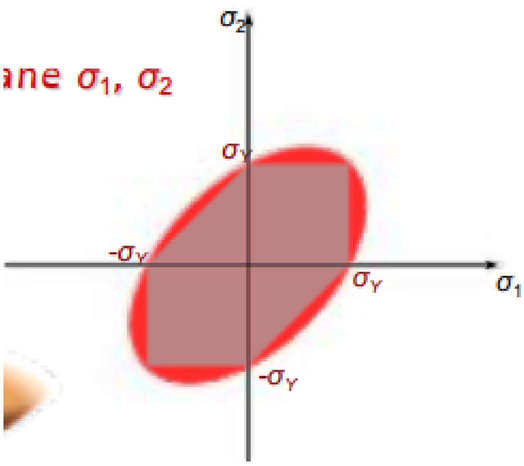

Figure 24:

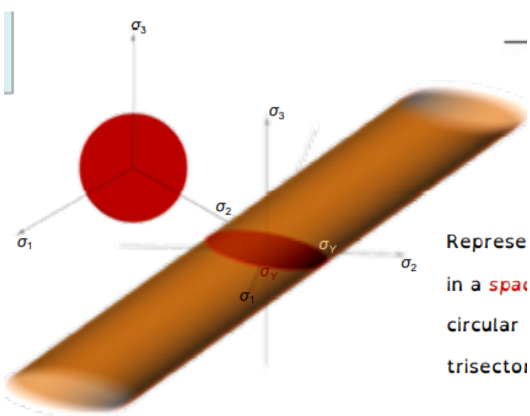

Figure 25:

In the Mohr's representation, the result is similar to the one obtained with Tresca representation, but in the region close to the τ-axis the advantage due to interaction between the principal stress becomes significant.

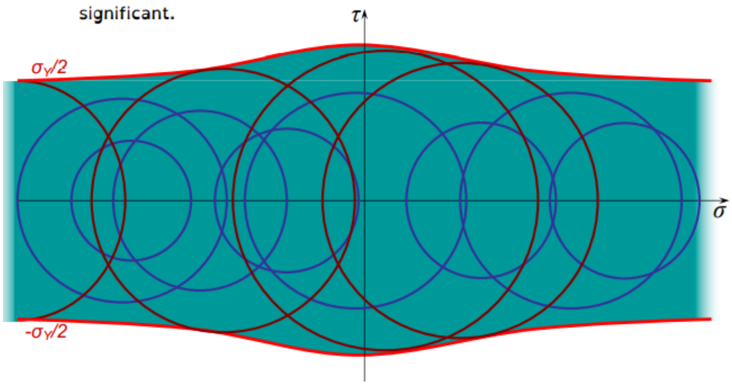

Figure 26:

Applicability:

> **The distortion-energy theory is the one that better describes the experimental behaviour of ductile materials. It does not limit hydrostatic compressive loads (in agreement with experimental evidence). Contrary to Tresca theory, it does consider the benefit given by the intermediate principal stress.**

The main limitation of the distortion-energy theory is that it does not predict the failure under hydrostatic tension.

0.5.3 Comparison with experiments

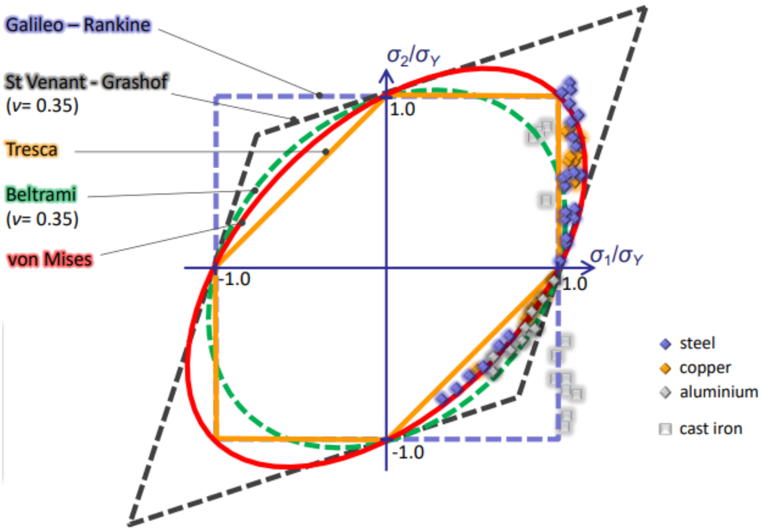

Figure 27:

0.6 Experimental criteria

0.6.1 Mohr's criterion

The failure criterion proposed by Mohr differs from previous criteria because it is purely experimental.

> **The Mohr's circles from a number of tests, under different stress conditions, can be enveloped by a failure curve. Failure occurs whenever the Mohr's circle describing the state of stress in the material becomes tangent to the failure envelope.**

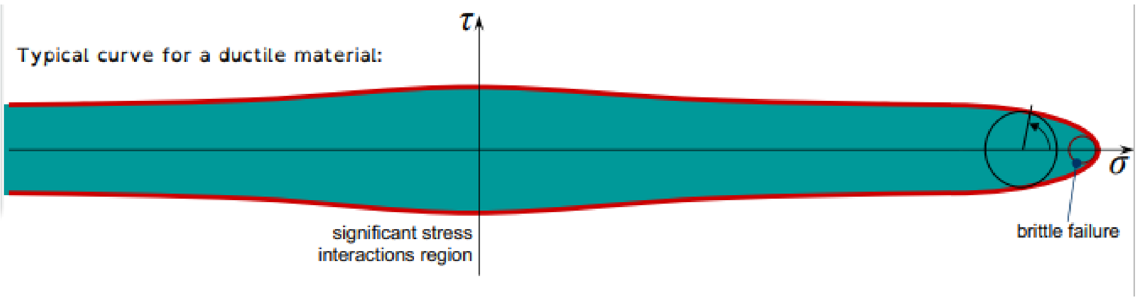

Figure 28:

In the case of brittle materials, the failure envelope has a shape similar to the one in Fig.29.

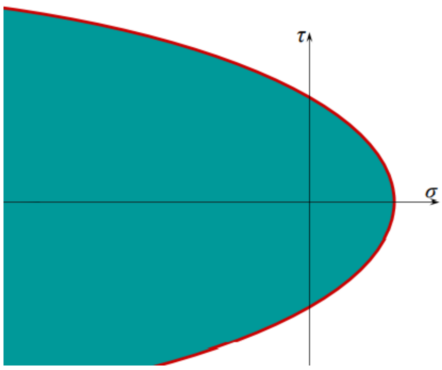

Figure 29:

0.6.2 Coulomb-Mohr's criterion

For this case, *Coulomb-Mohr* criterion uses a simplified envelope, obtained by simply performing one tensile test and one compressive test. The two Mohr's circles relative to the tests are joined with straight tangent lines.

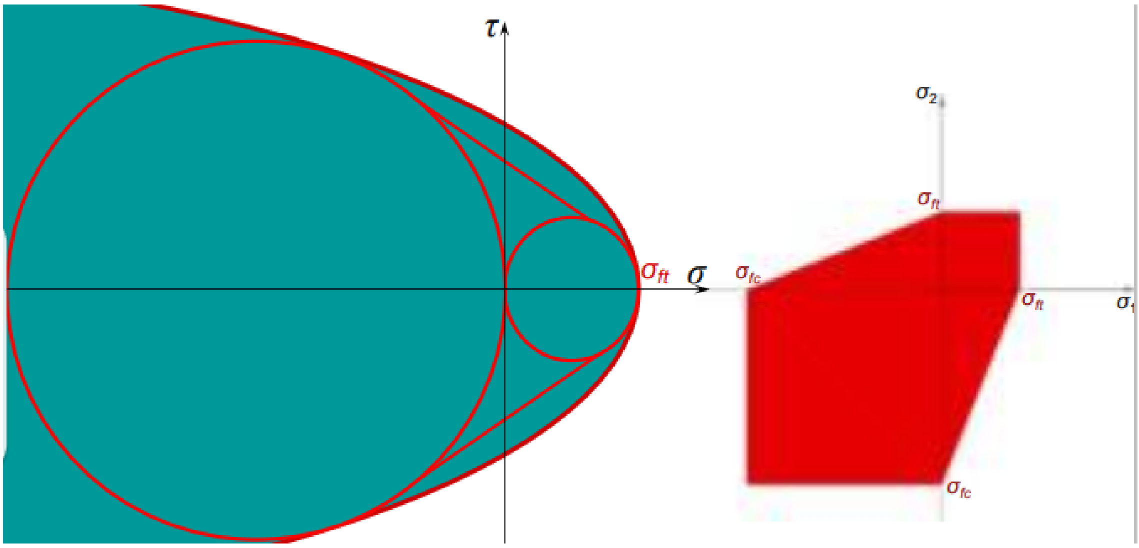

Figure 30:

Applicability:

> **The Coulomb-Mohr criterion is reliable for the study of brittle materials, and gives conservative results.**

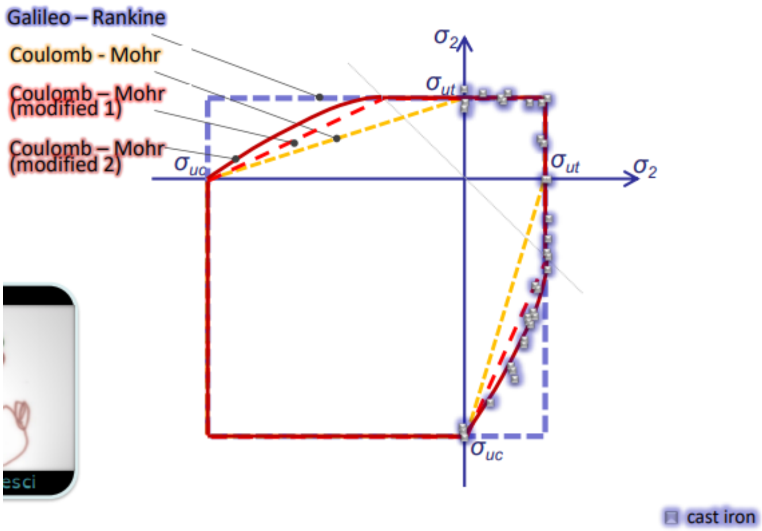

Figure 31:

0.7 Conclusion

For brittle materials, the maximum principal stress or Coulomb-Mohr criteria are most suitable. For bi-axial stress conditions, the Coulomb-Mohr modified criteria are preferable, provided that reliable test data are available for tension, compression and torsion. For ductile materials, the von Mises "distortion-energy" theory is the most reliable. The Tresca maximum shear stress theory is more conservative (i.e. the safest) and has a simple formula.