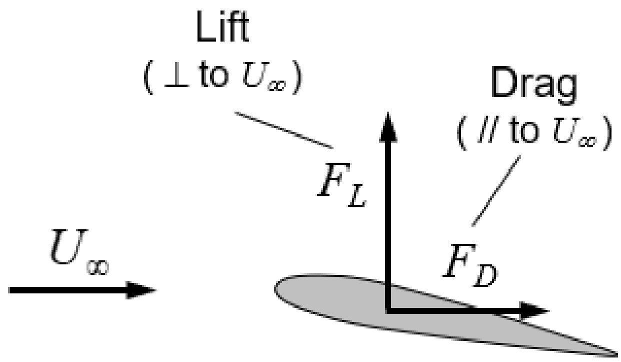
0.1 Lift and drag

Typical forces of interest for bodies in a flow are **drag** and **lift**. We can represent these in dimensionless form:

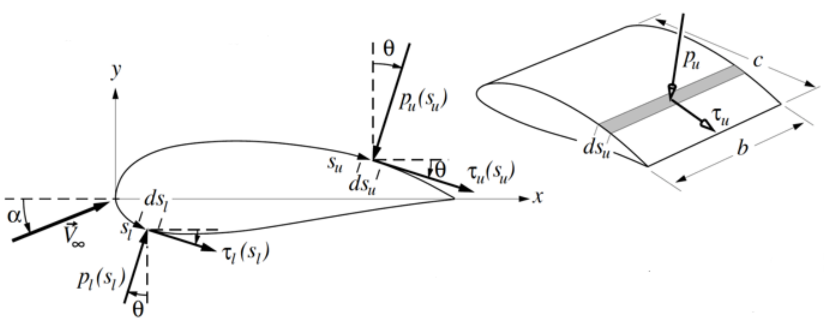
$$\text{Drag coefficient: } c_D = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 S} \tag{1}$$

$$\text{Lift coefficient: } c_L = \frac{F_L}{\frac{1}{2}\rho U_\infty^2 S} \tag{2}$$

Where S is a representative area for the body, determined by convention.



0.2 Pressure and frictional force distribution



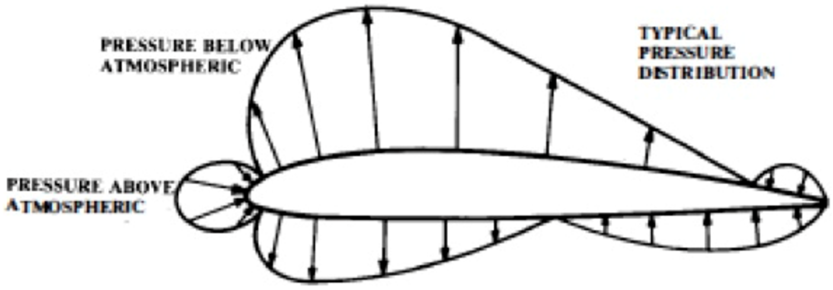
$$L = -\int_S \left(p(\hat{n} \cdot \hat{j})\right) \mathrm{d}S + \int^S \left(\vec{\tau} \cdot \hat{j}\right) \mathrm{d}S \tag{3}$$

$$D = -\int \left(p(\hat{n} \cdot \hat{j})\right) \mathrm{d}S + \int \left(\vec{\tau} \cdot \hat{i}\right) \mathrm{d}S \tag{4}$$

To determine the lift and drag coefficients c_L and c_D, we are interested in the pressure distribution over the airfoil, or more specifically in the local pressure difference from the stream pressure p_∞.

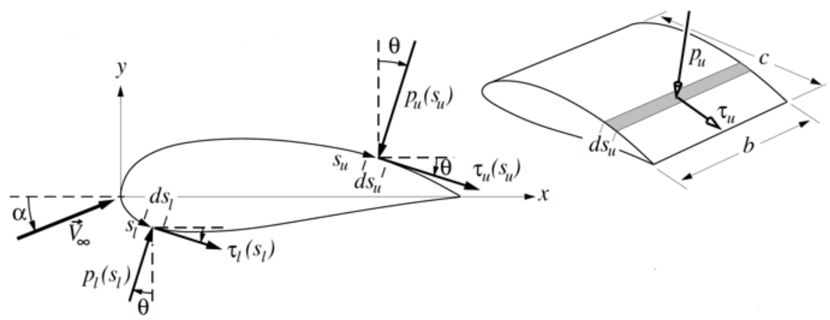
$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} \tag{5}$$

Free stream pressure and velocity are p_∞ and V_∞.



- Local suction (depression): $c_p < 0$ Vectors point away from the airfoil surface

- Local pushing: $c_p > 0$ Vectors point towards the airfoil surface



$$L = -\int_S \left(p\hat{n} \cdot \hat{j}\right) \, \mathrm{d}S = \tag{6}$$

$$c_L = -\frac{1}{S}\int_S \left(\frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2}\hat{n} \cdot \hat{j}\right) \mathrm{d}S = -\frac{1}{S}\int_S \left(c_p\hat{n} \cdot \hat{j}\right) \mathrm{d}S \tag{7}$$

The lift coefficient per unit of span-wise length is:

$$c_L' - \frac{1}{c}\int_c^0 \left(c_p\hat{n} \cdot \hat{j}\right) \mathrm{d}x \tag{8}$$

0.3 Rearrangement of momentum equation - x direction

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{9}$$

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+v\frac{\partial u}{\partial y}-v\frac{\partial u}{\partial y} \tag{10}$$

$$=-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial u^2}{\partial x}+\frac{\partial v^2}{\partial x}\right) \tag{11}$$

$$=-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2) \tag{12}$$

(u^2+v^2) is the total kinetic energy of the fluid particle. The derivative is the element that takes into the account the variation of this kinetic energy. $\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$ relates to the rotation of the particle. This rotation is related to the difference of velocity gradient.

$$\rho\left[-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2)\right]=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{13}$$

$$-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}+\frac{u^2+v^2}{2}\right)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{14}$$

Our Bernoulli term in the above equation is $\left(\frac{p}{\rho}+\frac{u^2+v^2}{2}\right)$, gravitational energy is negligible. $\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$ is an anti-clockwise rotation. Hence, the vorticity component in the z direction is:

$$\omega_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \tag{15}$$

Our final momentum equations in x and y are:

$$-v\omega_z=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}+\frac{u^2+v^2}{2}\right)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{16}$$

$$u\omega_z=-\frac{\partial}{\partial y}\left(\frac{p}{\rho}+\frac{u^2+v^2}{2}\right)+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{17}$$

We can make some assumptions:

- Inviscid flow - $\nu = 0$ (this may be realistic in some parts of a fluid domain but in real life, inviscid fluids do not exist)
- Irrotational flow - $\omega_z = 0$

This reduces our equations to:

$$0 = -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \frac{u^2 + v^2}{2}\right) + 0 \tag{18}$$

$$0 = -\frac{\partial}{\partial y}\left(\frac{p}{\rho} + \frac{u^2 + v^2}{2}\right) + 0 \tag{19}$$

0.4 Application of Bernoulli

$$p_\infty + \frac{1}{2}\rho V_\infty^2 = p + \frac{1}{2}\rho(u^2 + v^2) = \text{constant} \tag{20}$$

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{u^2 + v^2}{V_\infty^2} = 1 - \frac{||V||^2}{V_\infty^2} \tag{21}$$