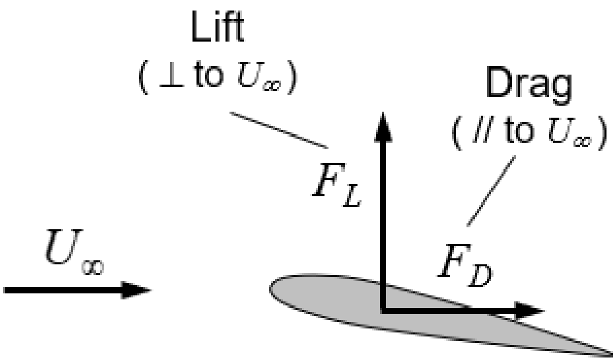
0.1 Lift and drag

Typical forces of interest for bodies in a flow are **drag** and **lift**. We can represent these in dimensionless form:

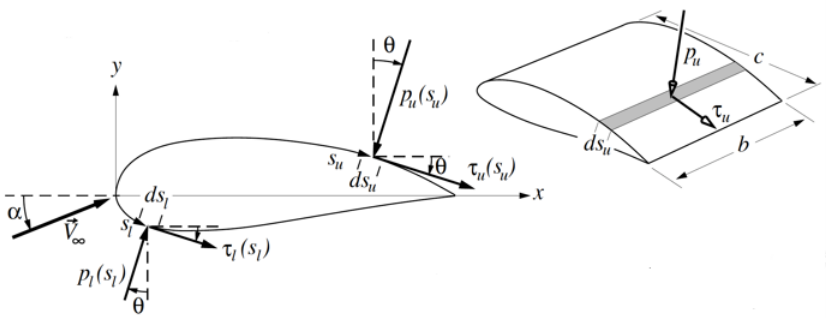
$$\text{Drag coefficient: } c_D = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 S} \tag{1}$$

$$\text{Lift coefficient: } c_L = \frac{F_L}{\frac{1}{2}\rho U_\infty^2 S} \tag{2}$$

Where S is a representative area for the body, determined by convention.



0.2 Pressure and frictional force distribution



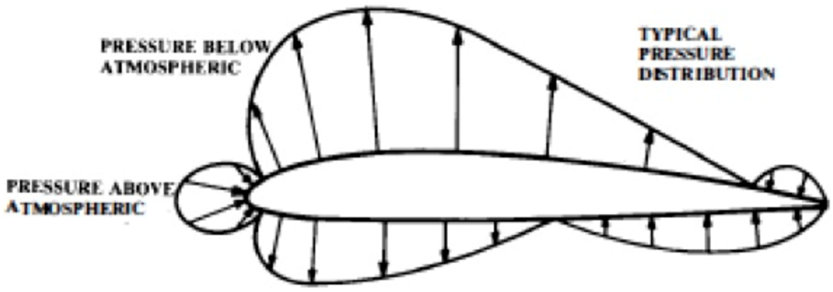
$$L = -\int_S \left(p(\hat{n}\cdot\hat{j})\right)\,\mathrm{d}S + \int^S \left(\vec{\tau}\cdot\hat{j}\right)\,\mathrm{d}S \tag{3}$$

$$D = -\int \left(p(\hat{n}\cdot\hat{j})\right)\,\mathrm{d}S + \int \left(\vec{\tau}\cdot\hat{i}\right)\,\mathrm{d}S \tag{4}$$

To determine the lift and drag coefficients c_L and c_D, we are interested in the pressure distribution over the airfoil, or more specifically in the local pressure difference from the stream pressure p_∞.

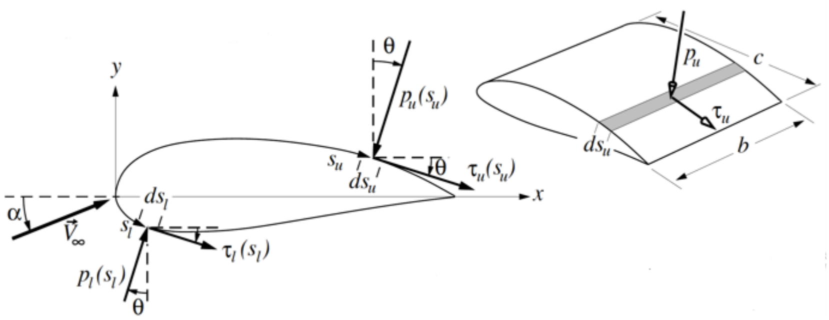
$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} \tag{5}$$

Free stream pressure and velocity are p_∞ and V_∞.



- Local suction (depression): $c_p < 0$ Vectors point away from the airfoil surface

- Local pushing: $c_p > 0$ Vectors point towards the airfoil surface



$$L = -\int_S \left(p\hat{n}\cdot\hat{j}\right)\,\mathrm{d}S = \tag{6}$$

$$c_L = -\frac{1}{S}\int_S \left(\frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2}\hat{n}\cdot\hat{j}\right)\,\mathrm{d}S = -\frac{1}{S}\int_S \left(c_p\hat{n}\cdot\hat{j}\right)\,\mathrm{d}S \tag{7}$$

The lift coefficient per unit of span-wise length is:

$$c_L' - \frac{1}{c}\int_c^0 \left(c_p\hat{n}\cdot\hat{j}\right)\,\mathrm{d}x \tag{8}$$

0.3 Rearrangement of momentum equation - x direction

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{9}$$

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+v\frac{\partial u}{\partial y}-v\frac{\partial u}{\partial y} \tag{10}$$

$$=-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial u^2}{\partial x}+\frac{\partial v^2}{\partial x}\right) \tag{11}$$

$$=-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2) \tag{12}$$

(u^2+v^2) is the total kinetic energy of the fluid particle. The derivative is the element that takes into the account the variation of this kinetic energy. $\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$ relates to the rotation of the particle. This rotation is related to the difference of velocity gradient.

$$\rho\left[-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2)\right]=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{13}$$

$$-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}+\frac{u^2+v^2}{2}\right)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{14}$$

Our Bernoulli term in the above equation is $\left(\frac{p}{\rho}+\frac{u^2+v^2}{2}\right)$, gravitational energy is negligible. $\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$ is an anti-clockwise rotation. Hence, the vorticity component in the z direction is:

$$\omega_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \tag{15}$$

Our final momentum equations in x and y are:

$$-v\omega_z=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}+\frac{u^2+v^2}{2}\right)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{16}$$

$$u\omega_z=-\frac{\partial}{\partial y}\left(\frac{p}{\rho}+\frac{u^2+v^2}{2}\right)+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{17}$$

We can make some assumptions:

- Inviscid flow - $\nu = 0$ (this may be realistic in some parts of a fluid domain but in real life, inviscid fluids do not exist)
- Irrotational flow - $\omega_z = 0$

This reduces our equations to:

$$0 = -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + 0 \tag{18}$$

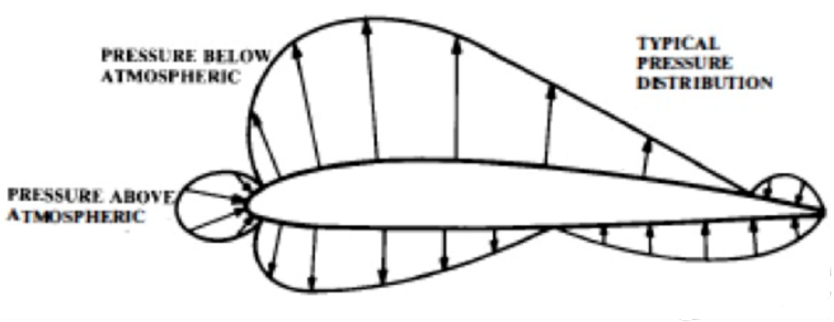
$$0 = -\frac{\partial}{\partial y}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + 0 \tag{19}$$

0.4 Application of Bernoulli

$$p_\infty + \frac{1}{2}\rho V_\infty^2 = p + \frac{1}{2}\rho(u^2+v^2) = \text{constant} \tag{20}$$

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{u^2+v^2}{V_\infty^2} = 1 - \frac{||V||^2}{V_\infty^2} \tag{21}$$

If $c_p < 0$, $||V|| > V_\infty$ and vice versa. If a fluid particle enters a region where c_p is negative it is accelerated and when c_P is positive it will lose velocity relative to the free stream.



Extending this to 3D, we can derive the 3D vorticity equation. We sum the momentum equations with the assumptions above and take the ijk components as so:

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} \tag{22}$$

If $\vec{\omega} = 0$ then a potential function, $\phi(x, y, z)$ exists such:

$$\begin{cases} u = \frac{\partial \phi(x,y,z)}{\partial x} \\ v = \frac{\partial \phi(x,y,z)}{\partial y} \\ w = \frac{\partial \phi(x,y,z)}{\partial z} \end{cases} \tag{23}$$

By plugging in these equations into our continuity equation, we get:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y} + \frac{\partial^2 \phi}{\partial z} \tag{24}$$

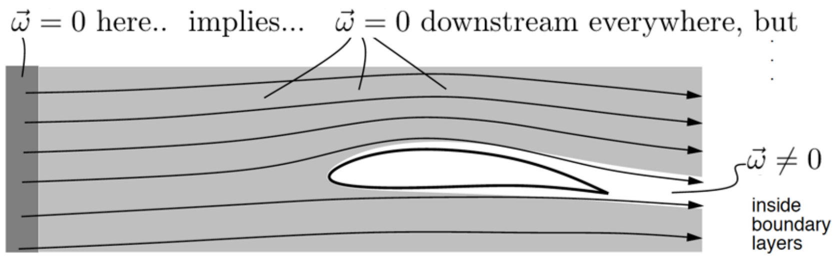
Conservation of momentum (Bernoulli term) is simply:

$$p + \frac{1}{2}\rho(u^2 + v^2) = \text{constant} \tag{25}$$

0.5 Applicability of irrotational flow

The flow domain can be subdivided into two parts.

- **Irrotational flow region**, outside of the boundary layer, Bernoulli equation, potential flow and stream function apply.
- **Boundary layer**, layer where all vorticity is confined. The friction shear the airfoil surface acts as a source of vorticity.



In the case where we consider our fluid inviscid, there is no shear stress being applied on the fluid by the airfoil. We need to understand how the no-slip condition changes. We still need a boundary condition to know the value of ϕ in our flow domain.

0.6 No-Slip condition for inviscid flow

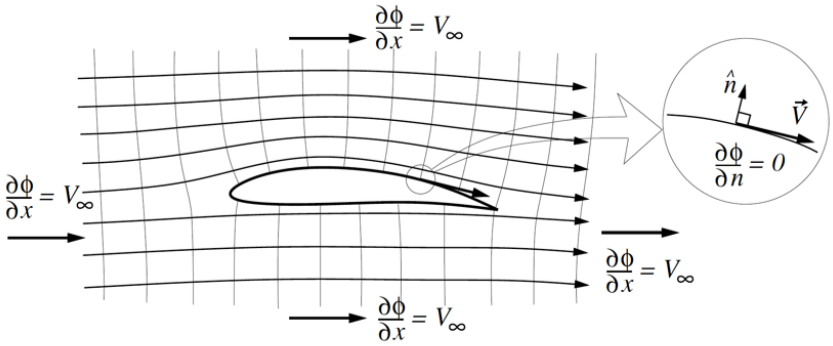
Viscous fluid

$$\nu \neq 0 \rightarrow \vec{V} = 0 \tag{26}$$

Inviscid flow

$$\nu = 0 \rightarrow V_n = \vec{V} \cdot \hat{n} = \frac{\partial \phi}{\partial n} = 0 \tag{27}$$

In essence, with inviscid flow, we are accepting that there is some movement on the boundary, however this is only parallel to the surface. n is the direction orthogonal to the boundary.



0.7 Stream function

In a 2D flow a stream function, $\psi(x, y)$, can be defined which is always aligned/parallel with the local velocity vector and visualise a streamline. Different streamlines are identified with different values of $\psi(x, y)$

$$\begin{cases} u = \frac{\partial \psi(x,y)}{\partial x} \\ v = \frac{\partial \psi(x,y)}{\partial y} \end{cases} \tag{28}$$

Iso-potential lines and streamlines are orthogonal to each other. Streamlines visualise the trajectory of a particle in the field.

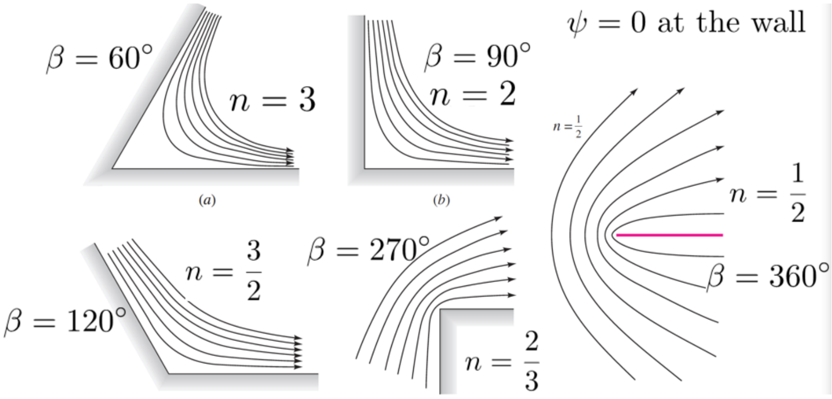
0.8 Potential flow past bodies

Flow fields for which an incompressible fluid is assumed to be frictionless and the motion to be irrotational are commonly referred to as **potential** flows. Paradoxically, potential flows can be simulated by a slow moving, viscous flow between closely spaced parallel plates.

0.9 Flow around a corner of arbitrary angle, β

Considering a radial coordinate system:

$$\phi = Ar^n \cos(n\theta) \tag{29}$$
$$\psi = Ar^n \sin(n\theta) \tag{30}$$



0.10 Cylindrical coordinates

3D vorticity equation:

$$\vec{\omega} = \omega_r \hat{i}_r + \omega_\theta \hat{i}_\theta + \omega_z \hat{i}_z \tag{31}$$
$$\vec{\omega} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{i}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{i}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\hat{i}_z \tag{32}$$

Potential flow function and stream function:

$$\text{Potential flow}\begin{cases} u_r = \frac{\partial \phi}{\partial r} \\ u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \\ u_z = \frac{\partial \phi}{\partial z} \end{cases} \quad \text{Stream function}\begin{cases} u_\theta = -\frac{\partial \psi}{\partial r} \\ u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \end{cases} \tag{33}$$

Conservation of mass (continuity equation):

$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{34}$$