27/10/2020

0.1 Uniform Flow

Cartesian Coordinates:

$$\phi = V_\infty \left[x\cos(\alpha) + y\sin(\alpha)\right] \tag{1}$$

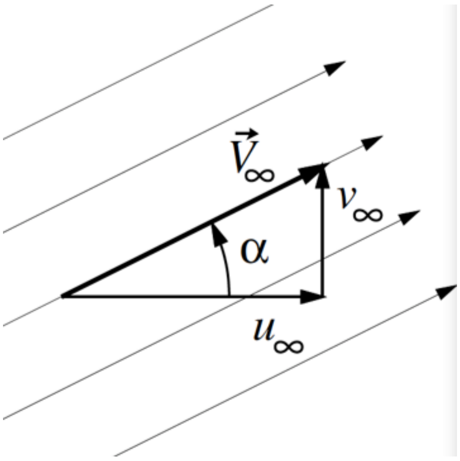
$$\psi = V_\infty \left[y\cos(\alpha) - x\sin(\alpha)\right] \tag{2}$$

The conservation of mass is balanced:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

The flow is irrotational:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{4}$$



Cylindrical Coordinates:

$$\phi(r,\theta) = V_\infty r\cos(\theta - \alpha) \tag{5}$$

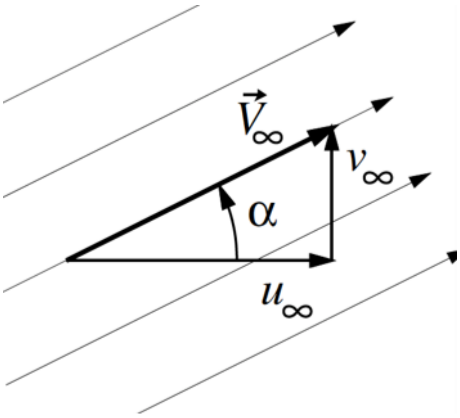
$$\psi(r,\theta) = V_\infty r\sin(\theta - \alpha) \tag{6}$$

The conservation of mass is satisfied for cylindrical coordinates:

$$\nabla \cdot \overrightarrow{V} = \frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \tag{7}$$

$$= \frac{\partial r(v_\infty \cos(\theta - \alpha))}{\partial r} - \frac{\partial v_\infty \sin(\theta - \alpha)}{\partial \theta} \tag{8}$$

$$v_\infty \cos(\theta - \alpha) - v_\infty \cos(\theta - \alpha) = 0 \tag{9}$$



0.2 Source/Sink Flow

Cartesian Coordinates:

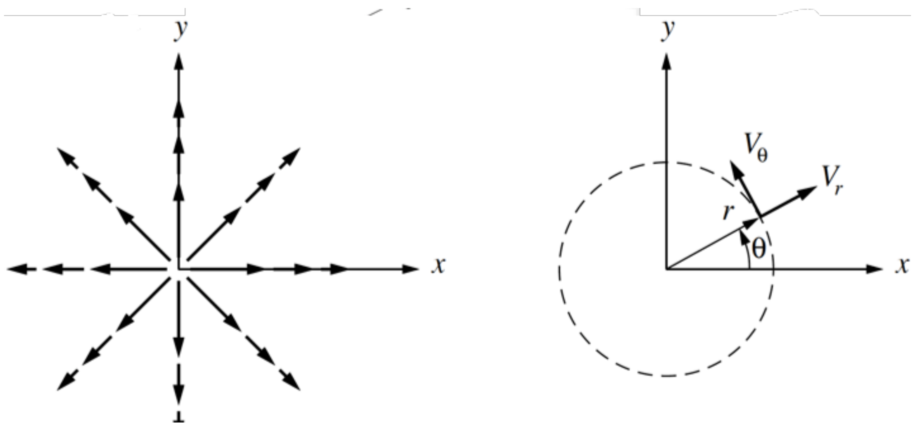
$$\phi = \frac{\Lambda}{2\pi} \ln(\sqrt{x^2 + y^2}) \tag{10}$$

$$\psi = \frac{\Lambda}{2\pi} \arctan\left(\frac{y}{x}\right) \tag{11}$$

Cylindrical Coordinates:

$$\phi = \frac{\Lambda}{2\pi} \ln(r) \tag{12}$$

$$\psi = \frac{\Lambda}{2\pi}\theta \tag{13}$$



In cylindrical coordinates, we do not have a θ component as it is moving radially outwards from a source.

$$u_r = \frac{\partial \phi}{\partial r} = \frac{\Lambda}{2\pi r} \tag{14}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \tag{15}$$

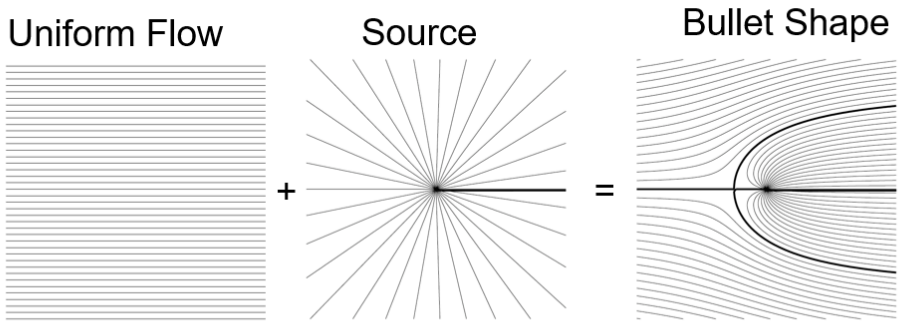
Here we can see the magnitude of the velocity is dependent on $\frac{1}{r}$. If $\Lambda > 0$, we have a source and if $\Lambda < 0$, we have a sink.

0.3 Uniform Flow + Source

$$\phi(r,\theta) = \frac{\Lambda}{2\pi} \ln(r) + V_\infty r \cos\theta \tag{16}$$

$$\psi(r,\theta) = \frac{\Lambda}{2\pi}\theta + V_\infty r \sin\theta \tag{17}$$

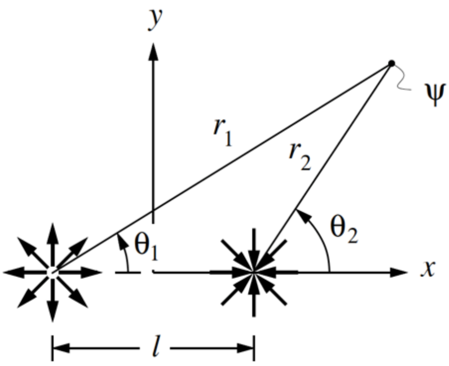
Stream function $\psi(r,\theta)$ of:

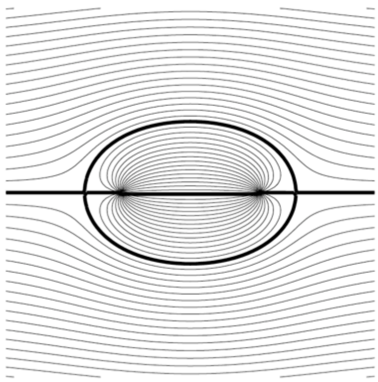


0.4 Uniform Flow + Source + Sink

$$\phi - V_\infty r \cos\theta + \frac{\Lambda}{2\pi}(\ln(r_1) - \ln(r_2)) \tag{18}$$

$$\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) \tag{19}$$



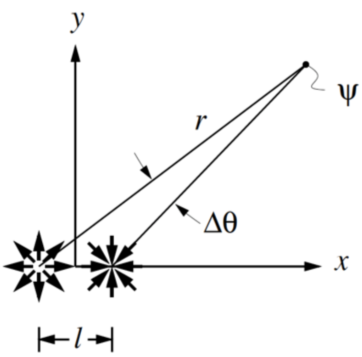


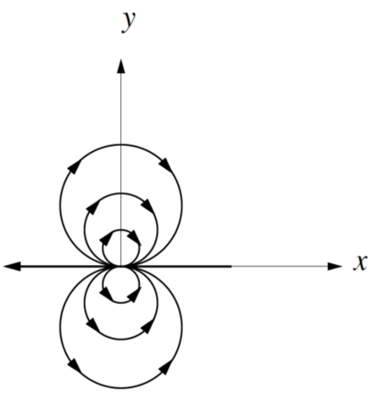
0.5 Doublet

Consider a pair of source and sink of $\pm\Lambda$ who are l apart and $l \times \Lambda =$ constant.

$$\psi = \lim_{l \to 0} \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) = -\frac{k}{2\pi}\frac{\sin\theta}{r} \tag{20}$$

$$\phi = \frac{k}{2\pi}\frac{\cos\theta}{r} \tag{21}$$

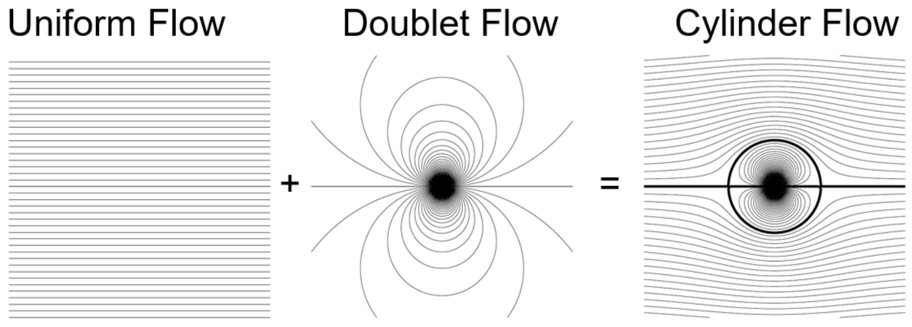




0.6 Cylinder (Uniform Flow + Doublet)

$$\phi = V_\infty r \cos\theta + \frac{k}{2\pi}\frac{\cos\theta}{r} \tag{22}$$

$$\psi = V_\infty r \sin\theta - \frac{k}{2\pi}\frac{\sin\theta}{r} \tag{23}$$



The radius of the cylinder can be derived as so:

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty \cos\theta - \frac{k}{2\pi}\frac{\cos\theta}{r^2} \tag{24}$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -\left(V_\infty \sin\theta + \frac{k}{2\pi}\frac{\sin\theta}{r^2}\right) \tag{25}$$

On the cylinder, $\vec{u} \cdot \hat{n} = 0$

$$\hat{n} = \hat{i}_r \rightarrow u_r(R) = 0 \tag{26}$$

$$V_\infty \cos\theta - \frac{k}{2\pi}\frac{\cos\theta}{R^2} = 0 \tag{27}$$

$$R = \sqrt{\frac{k}{2\pi V_\infty}} \tag{28}$$

We can rewrite ϕ and ψ

$$\phi = V_\infty r \cos\theta \left(1 + \frac{R^2}{r^2}\right) \tag{29}$$

$$\psi = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right) \tag{30}$$

On the cylinder surface, $r = R$ and inputting this into ψ:

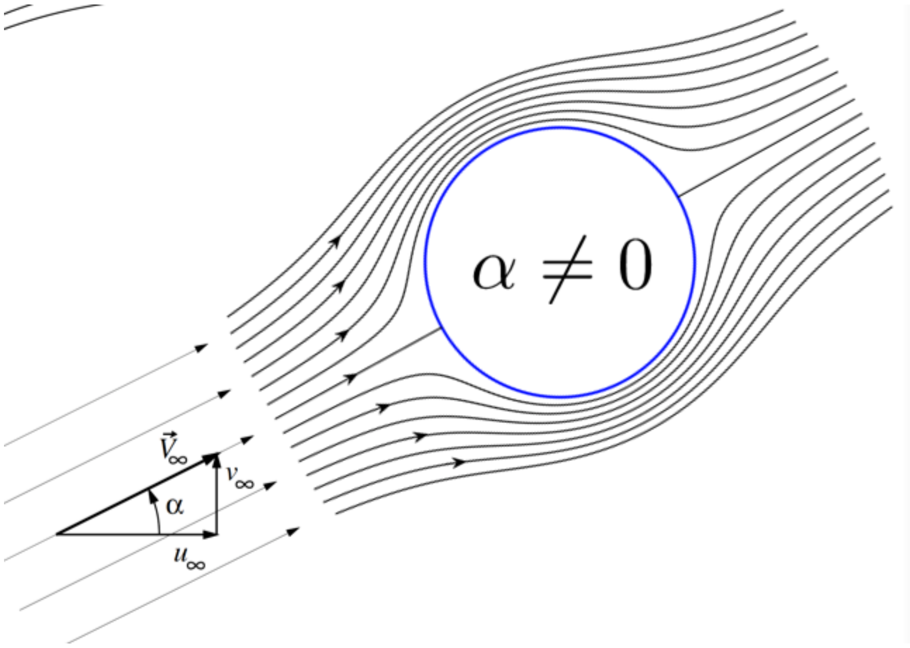
$$\psi = 0 \tag{31}$$

0.7 Uniform Stream with Varying Direction

All we need to do to generalise our equations a bit more is to rewrite our equations with an extra angular term, α:

$$\phi = V_\infty r \cos(\theta - \alpha) \left(1 + \frac{R^2}{r^2}\right) \tag{32}$$

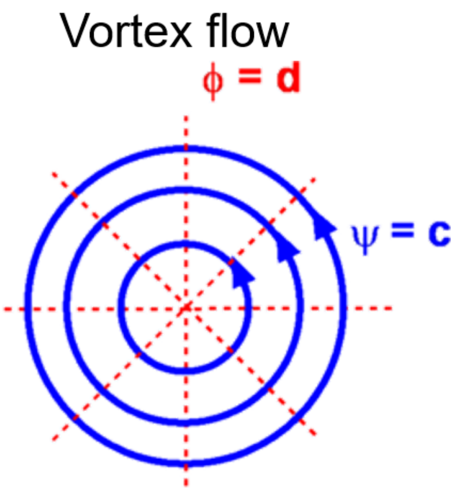
$$\psi = V_\infty r \sin(\theta - \alpha) \left(1 - \frac{R^2}{r^2}\right) \tag{33}$$



0.8 Adding Circulation with a Vortex Flow

$$\phi = -\frac{\Gamma}{2\pi}\theta \tag{34}$$
$$\psi = \frac{\Gamma}{2\pi}\ln\left(\frac{r}{R}\right) \tag{35}$$
$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \tag{36}$$
$$u_\theta = -\frac{\partial \psi}{\partial r} = -\frac{\Gamma}{2\pi r} \tag{37}$$

Where $\Gamma < 0$ is anti-clockwise motion and $\Gamma > 0$ is clockwise motion.



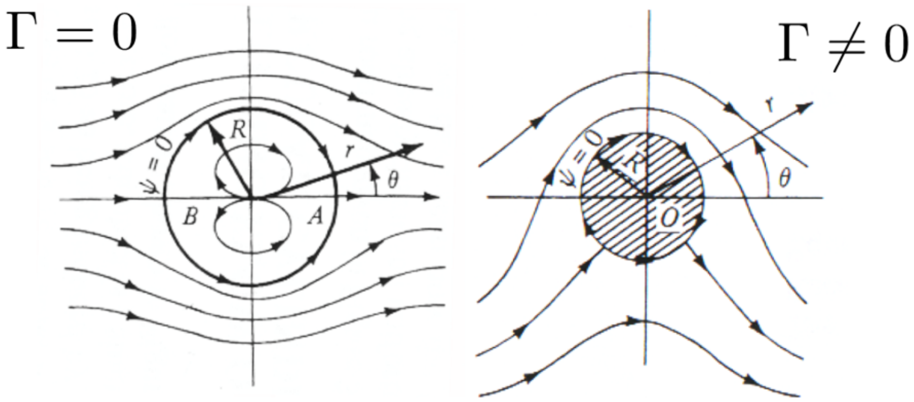
0.9 Cylinder with a Vortex Flow

$$\psi = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi} \ln\left(\frac{r}{R}\right) \tag{38}$$

$$\phi = V_\infty r \cos\theta \left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi}\theta \tag{39}$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty \cos\theta \left(1 - \frac{R^2}{r^2}\right) \tag{40}$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -V_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right) \tag{41}$$



0.10 Lift and Drag of a Cylinder with Circulation

Apply Bernoulli:

$$p_\infty + \frac{1}{2}\rho V_\infty^2 = p(r,\theta) + \frac{1}{2}\rho(u_r^2 + u_\theta^2) \tag{42}$$

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{u_r^2 + u_\theta^2}{V_\infty^2} \tag{43}$$

On the cylinder surface: $u_r = 0$

$$c_p(R,\theta) = 1 - \frac{u_\theta^2}{V_\infty^2} = 1 - \frac{(2V_\infty \sin\theta + \frac{\Gamma}{2\pi R})^2}{V_\infty^2} \tag{44}$$

$$= 1 - \left(4\sin^2(\theta) + \frac{\Gamma^2}{4\pi^2 V_\infty^2 R^2} + \frac{2\Gamma\sin(\theta)}{V_\infty \pi R}\right) \tag{45}$$

0.11 Lift of the Cylinder

We need to calculate c_p on the surface of the cylinder.

$$L = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(c_p \cdot \hat{n} \cdot \hat{j} R\right) \mathrm{d}\theta \tag{46}$$

$$= -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(c_p \sin\theta R\right) \mathrm{d}\theta \tag{47}$$

$$L = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(1 - \frac{(2V_\infty \sin\theta + \frac{\Gamma}{2\pi R})^2}{V_\infty^2}\right) \sin\theta R \,\mathrm{d}\theta \tag{48}$$

Expanding the integral out, we arrive at:

$$L = -\frac{1}{2}\rho V_\infty^2 \left[\int_0^{2\pi} \left(1 - \frac{\Gamma^2}{4\pi^2 R^2 V_\infty^2}\right) \cdot \sin\theta \cdot R \,\mathrm{d}\theta + \int_0^{2\pi} -4\sin\theta^3 \cdot R\,\mathrm{d}\theta + \int_0^{2\pi} -\frac{2\Gamma}{\pi R V_\infty} \cdot \sin\theta^2 \cdot R\,\mathrm{d}\theta\right] \tag{49}$$

Because the first term and the second term have an odd power of sin , when we integrate these, they will have negligible outcome on the lift of the cylinder. We can reduce our equation to:

$$L = -\frac{1}{2}\rho V_\infty^2 \left[\int_0^{2\pi} -\frac{2\Gamma}{\pi R V_\infty} \cdot \sin\theta^2 \cdot R\,\mathrm{d}\theta\right] \tag{50}$$

$$= \frac{\rho V_\infty \Gamma}{\pi} \int_0^{2\pi} \sin\theta^2 \,\mathrm{d}\theta \tag{51}$$

$$= \frac{\rho V_\infty \Gamma}{\pi} \left[\frac{1}{2}\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} \tag{52}$$

$$L = \rho V_\infty \Gamma \tag{53}$$

We can see here that our vortex factor Γ has a proportional effect on our lift force. We can see an example of this in real life when a football is kicked. When the ball is kicked in such a way that it has a spin, we see the ball curves in certain directions.

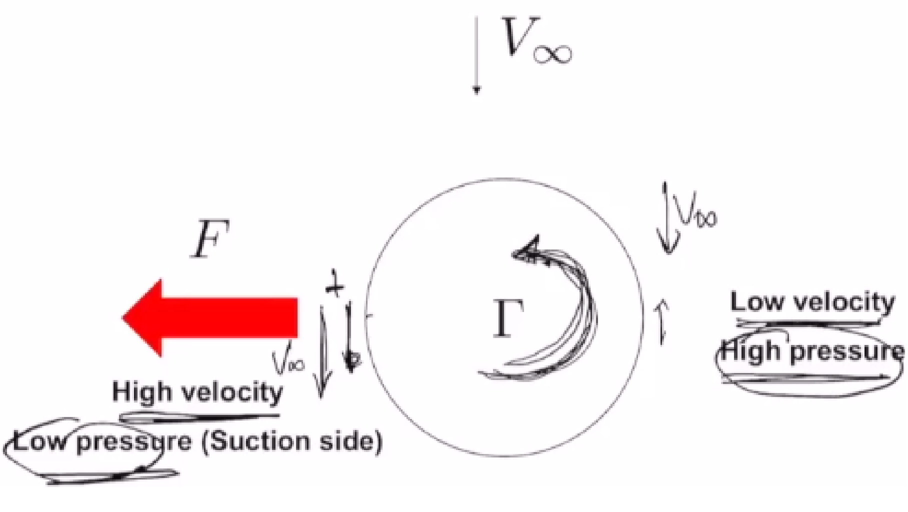
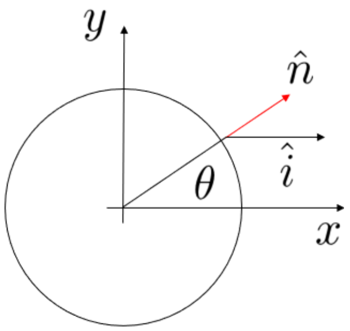


Figure 1: We can see that when the ball spins, the velocity from the free stream and the vortex combine to create regions of low and high pressure. This creates a net force, leading to a suction effect.

0.12 Drag of a cylinder



$$D = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} c_p \vec{n}\cdot\hat{i} R \,\mathrm{d}\theta \tag{54}$$

$$= -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} c_p \cos\theta R \,\mathrm{d}\theta \tag{55}$$

$$D = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(1 - \frac{\left(2V_\infty \sin\theta + \frac{\Gamma}{2\pi R}\right)^2}{V_\infty^2}\right) \cos\theta R \,\mathrm{d}\theta \tag{56}$$

The first term has an odd power of cosine, and so is negligible. The second term of the drag integral is:

$$\int_0^{2\pi} -4\sin\theta^2 \cos\theta R \,\mathrm{d}\theta = -4R \left(\frac{1}{3}\sin\theta^3\right)_p^{2\pi} = 0 \tag{57}$$

The third term of the drag integral is:

$$\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} -\frac{2\Gamma}{\pi R V_\infty} \sin\theta\cos\theta R \,\mathrm{d}\theta = -\frac{\rho V_\infty \Gamma}{\pi}\left(-\frac{1}{2}\cos 2\theta\right)_0^{2\pi} = 0 \tag{58}$$

Therefore, we see that our drag is in fact:

$$D = 0 \tag{59}$$

This is due to our assumption that our flow is inviscid and that the pressure forces are symmetrical to the left and right sides to the y axis. Therefore the net forces acting on the x direction is zero. This is where our model starts failing and is called the D'Alambert Paradox.

0.13 Inviscid and Viscous Flow past a body

High Re implies that the magnitude of the inertia forces are much greater than the magnitudes of the viscous forces in a system. This might imply that the effects of viscosity are insignificant compared with the inertia forces, but this would be a dangerous conclusion. Compare theory for zero viscosity with experiment for high Re flow past a cylinder:

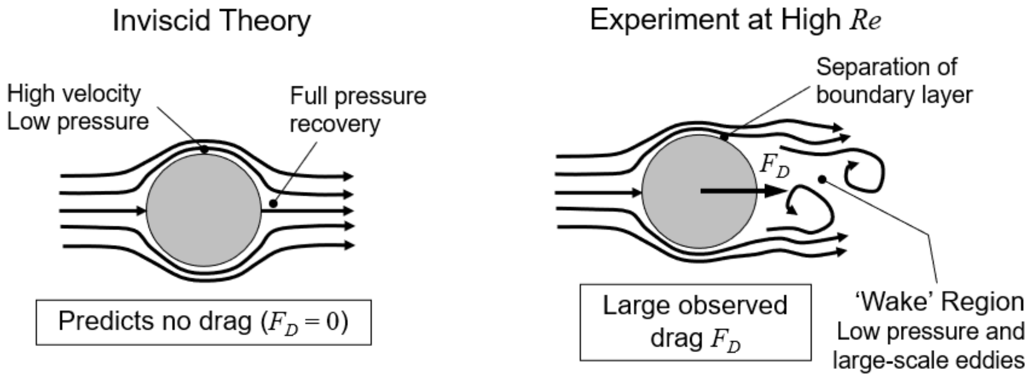
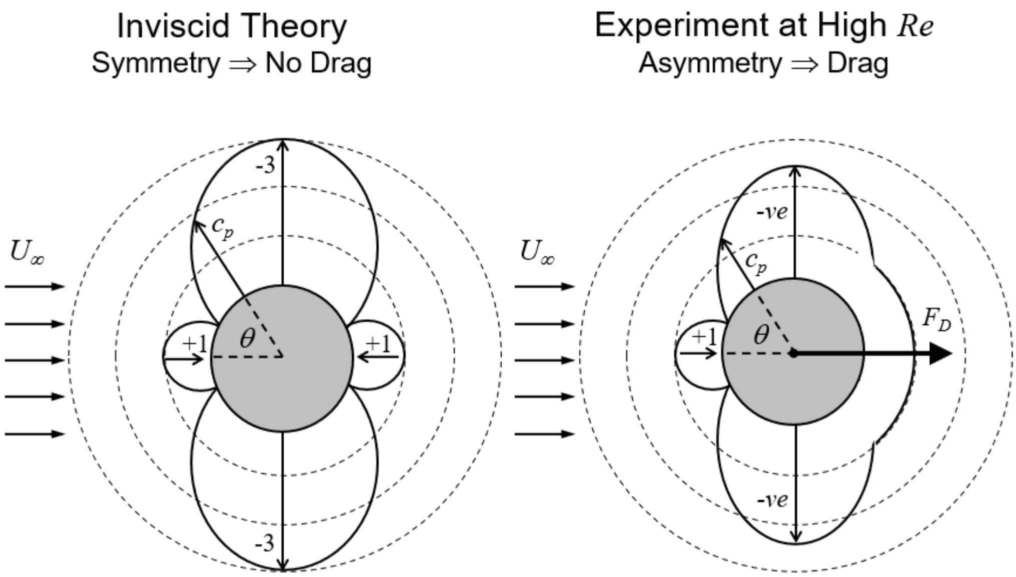
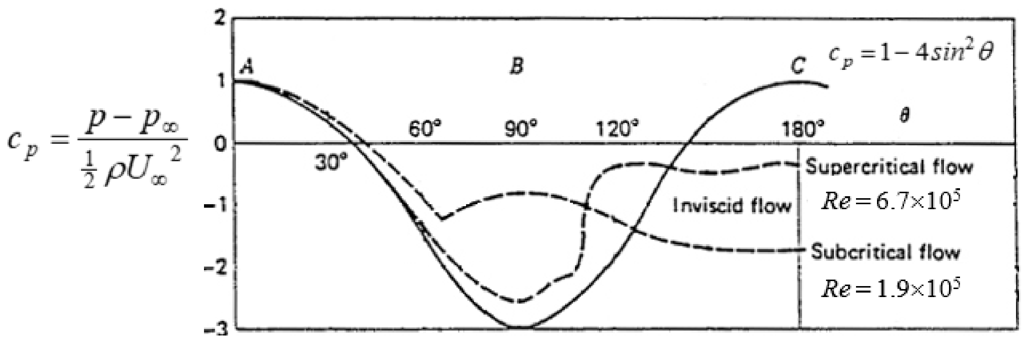


Figure 2: We have a net force in our experiment due to the fact that a net force is created from the low pressure wake region and the high pressure front side of the cylinder.

0.14 Flow past a cylinder - pressure coefficient



We can plot the pressure coefficient and see the difference between the inviscid theory and an experiment at high Re.



For a long circular cylinder, the lift coefficient c_L and the form drag coefficient to

c_D are related to c_p by:

$$c_L = \frac{1}{2}\int_0^{2\pi} c_p \sin\theta \,\mathrm{d}\theta \tag{60}$$

$$c_D = \frac{1}{2}\int_0^{2\pi} c_p \cos\theta \,\mathrm{d}\theta \tag{61}$$

Some examples of viscous flow past bodies: