0.1 Momentum equation

$$\sum F_{sys} = \frac{\partial}{\partial t}\int_{CV} \underline{V}\rho d\forall + \int_{CS} \rho \underline{V}(\underline{V}\cdot\underline{n})dA \tag{1}$$

0.1.1 Vane example:

A horizontal jet of water exits a nozzle with a uniform speed of $V_1 = 3.048$ $\mathrm{m\,s^{-1}}$, strikes a vane and is turned through an angle θ. Determine the anchoring force needed to hold the vane stationary if gravity and visocus effects are negligible.

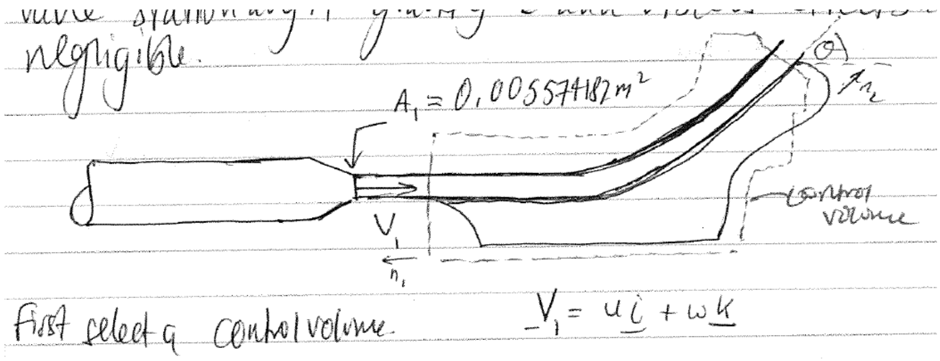


Figure 1: Water flow through vane system.

The only portions of the control surfrace across which fluid flows are section 1 (the entrance) and section 2 (the exit). Hence, the momentum equation becomes in the x and z components.

$$\sum F_x = \int_{inlet} u\rho(\underline{v}\cdot\underline{n})dA + \int_{outlet} u\rho(\underline{V}\cdot\hat{n})dA \tag{2}$$

$$\sum F_x = u_1\rho(-V_1)A_1 + u_2\rho(V_2)A_2 \tag{3}$$

$$\sum F_x = u_2\rho A_2 V_2 - i_1\rho A_1 V_1 \tag{4}$$

In the z direction:

$$\sum F_z = \int_{inlet} w\rho(\underline{V}\cdot\hat{n})dA + \int_{outlet} w\rho(\underline{V}\cdot n)dA \tag{5}$$

$$\sum F_z = w_2\rho A_2 V_2 - w_1\rho A_1 V_1 \tag{6}$$

We know that at inlet V_1 there is no vertical component, hence $w_1 = 0$ and $u_1 = V_1$. At the outlet:

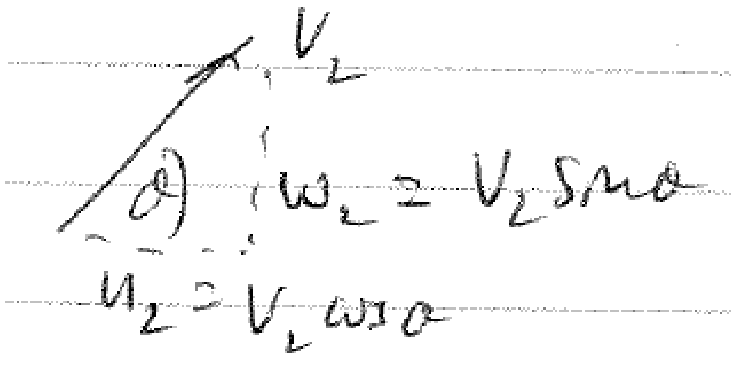


Figure 2: Velocity components at outlet.

Also lets find V_1 and V_2. From Bernoulli's equation (neglecting g, assuming incompressible and that $P_1 = P_2 = P_{atm}$), $V_1 = V_2$.

$$\therefore \sum F_z = V_2 \sin\theta \rho A_2 V_2 = V_1^2 A_2 \sin\theta \rho \tag{7}$$

$$\sum F_x = V_2 \cos\theta \rho A_2 V_2 - V_1 \rho A_1 V_1 \tag{8}$$

$$\sum F_x = V_1^2 A_2 \cos\theta \rho - V_1^2 \rho A_1 \tag{9}$$

$$\sum F_x = V_1^2 (A_2 \cos\theta \rho - \rho A_1) \tag{10}$$

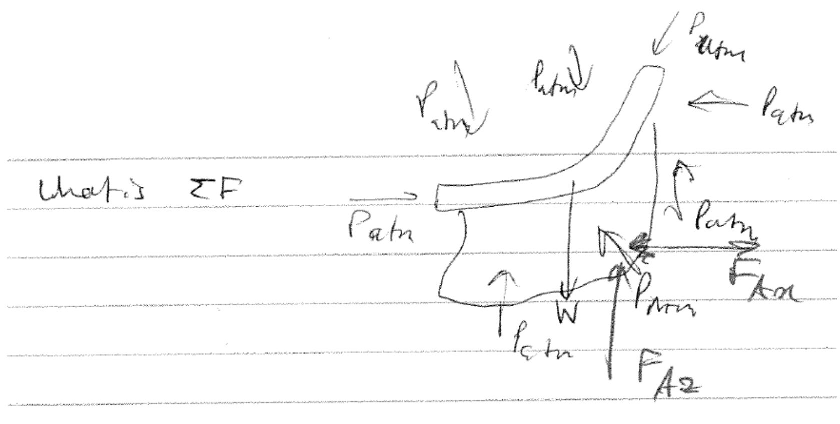


Figure 3: Forces acting on system.

Neglect w and the net force due to $P_{atm} = 0$

$$\therefore \sum F_x = F_{Ax} \tag{11}$$
$$\sum F_z = F_{Az} \tag{12}$$
$$F_{Ax} = V_1^2 A_2 \sin\theta\rho \tag{13}$$
$$F_{Az} = V_1^2 (A_z \cos\theta\rho - \rho A_1) \tag{14}$$

Also remember that,

$$\dot{m}_1 = \dot{m}_2 \tag{15}$$
$$V_1 A_1 = V_2 A_2 \text{ (incompressibility)} \tag{16}$$
$$V_1 = V_2 \therefore A_1 = A_2 \tag{17}$$
$$\therefore F_{Ax} = V_1^2 A_1 \sin\theta\rho \tag{18}$$
$$F_{Az} = \rho V_1^2 (A_1 \cos\theta - A_1) = \rho V_1^2 A_1 (\cos\theta - 1) \tag{19}$$

Plug in data to find F_{Ax} and F_{Az}.

0.2 Calculating the thrust produced by a propeller

We want to know the thrust applied by the propeller and the total work done. We will assume that there is are no losses due to friction or viscosity. There are three stages to this:

1. Use Bernoulli's equation to derive an expression for thrust.
2. Use the momentum equation to derive an independent equation for the thrust.
3. Combine these two equations to solve for the speed of the propeller.

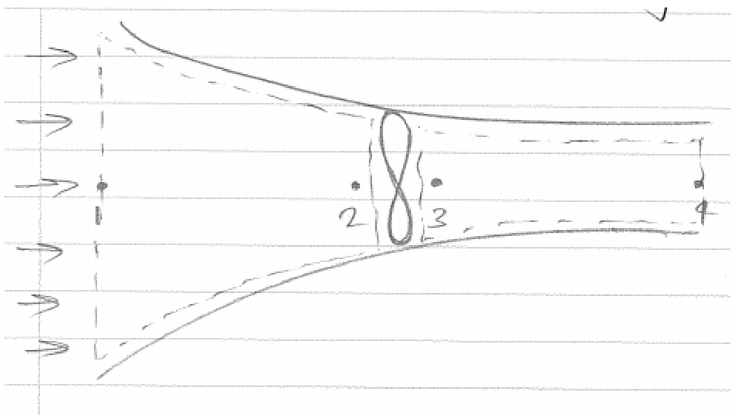


Figure 4: Diagram of a propeller with the flow going from left to right.

1. Create a control volume.
2. Apply Bernoulli's equation between points 1 and 2:

$$P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_1^2 \tag{20}$$

3. Apply Bernoulli's equation between points 3 and 4:

$$P_3 + \frac{1}{2}\rho V_3^2 = P_4 + \frac{1}{2}\rho V_4^2 \tag{21}$$

4. $P_1 = P_4 = P_{atm}$, so rearrange these two equations and equate:

$$P_1 = P_2 + \frac{1}{2}\rho(V_2^2 - V_1^2) \tag{22}$$

$$P_4 = P_3 + \frac{1}{2}\rho(V_3^2 - V_4^2) \tag{23}$$

$$P_2 + \frac{1}{2}\rho(V_2^2 - V_1^2) = P_3 + \frac{1}{2}\rho(V_3^2 - V_4^2) \tag{24}$$

5. We can now say that the velocity change across the propeller is small such that $V_2 \approx V_3$, so we can write.

$$\Delta P = P_3 - P_2 = \frac{1}{2}\rho[V_4^2 - V_1^2] = \frac{1}{2}\rho[V_4^2 - V_1^2] \tag{25}$$

$$\therefore \Delta P = \frac{1}{2}\rho(V_4^2 - V_1^2) \tag{26}$$

6. The thrust F provided by the pressure must be equal to the pressure difference multiplied by the area of the propellers.

$$\therefore F = \Delta P \times A \tag{27}$$

$$F = \frac{1}{2}\rho A(V_4^2 - V_1^2) \tag{28}$$

7. Now we use the momentum equation to derive another expression for the force. Force exerted on this central volume is given by the momentum equation.

$$\sum F = \frac{\partial}{\partial t}\int_{CV} \rho \underline{V} d\forall + \int_{CS} \rho \underline{V}(\underline{V} \cdot \underline{\hat{n}}) dA \tag{29}$$

Assume steady flow, hence $\frac{\partial}{\partial t}\int_{CV} \rho \underline{V} d\forall = 0$

$$\sum F_x = \int_{CS} \rho \underline{V}(\underline{V} \cdot \underline{\hat{n}}) dA = \int_{inlet} \rho V_1(-V_1) dA + \int_{output} \rho V_4(V_4) dA \tag{30}$$

$$= -\dot{m}_1 V_1 + \dot{m}_4 V_4 \tag{31}$$

Assume steady flow, neglect gravity and viscous forces: $\dot{m}_1 = \dot{m}_4$

$$F = \dot{m}(V_4 - V_1) \tag{32}$$

8. But at the propeller the mass flow rate equals:

$$\dot{m} = \rho A_P \times V_P = \rho A V_P \therefore F = \rho A V_P (V_4 - V_1) \tag{33}$$

9. Equate equations (28) and (33)

$$F = V_P(V_4 - V_1) = \frac{1}{2}(V_4^2 - V_1^2) \tag{34}$$

$$V_P = \frac{1}{2}\frac{(V_4 - V_1)(V_4 + V_1)}{(V_4 - V_1)} \tag{35}$$

$$V_P = \frac{1}{2}(V_4 + V_1) \tag{36}$$

10. Now using given data, you can work out V_P and thus find the thrust and the work done.

0.2.1 Moving control volume

Let us recap on the RTT:

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \rho b d\forall + \int_{CS} \rho b(\underline{V} \cdot \underline{\hat{n}})dA \tag{37}$$

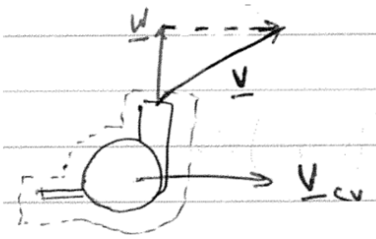
For mass conservation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho(\underline{V} \cdot \underline{\hat{n}})dA \tag{38}$$

For momentum equation:

$$\sum F_{sys} = \frac{\partial}{\partial t}\int_{CV} \rho \underline{V} d\forall + \int_{CS} \rho \underline{V}(\underline{V} \cdot \underline{\hat{n}})dA \tag{39}$$

What if the control volume is moving around, in an inertial reference frame: not accelerating? Our $\underline{V} \cdot \underline{\hat{n}}$ terms pertains to the net mass flow out relative to that boundary.



$$\underline{V} = \underline{V}_{cv} + \underline{W}$$

absolute — of CV — relative to CV

Conservation of mass

$$\text{original} \to 0 = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho(\underline{V} \cdot \underline{\hat{n}})dA \tag{40}$$

$$\text{moving inertial} \to 0 = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho(\underline{W} \cdot \underline{\hat{n}})dA \tag{41}$$

When in steady state, first term cancels,

$$0 = \int_{CS} \rho(\underline{W} \cdot \underline{\hat{n}}) dA \tag{42}$$

Mass crosses boundary due to the relative velocity, not the absolute.

Inertial moving object: conservation of momentum

$$\text{original} \rightarrow \sum F_{sys} = \frac{\partial}{\partial t} \int_{CV} \rho \underline{V} d\forall + \int_{CS} \rho \underline{V}(\underline{V} \cdot \underline{\hat{n}}) dA \tag{43}$$

Swap $\underline{V}$ for $\underline{W}$ in the second integral's bracket as the bracket relates to the mass flow out, relative to the boundary.

$$\sum F_{sys} = \frac{\partial}{\partial t} \int_{CV} \rho \underline{V} d\forall + \int_{CS} \rho \underline{V}(\underline{W} \cdot \underline{\hat{n}}) dA \tag{44}$$

Since

$$\underline{V} = \underline{V}_{CV} + \underline{W} \tag{45}$$

$$\frac{\partial}{\partial t} \int_{CV} \rho(\underline{V}_{CV} + \underline{W}) d\forall = 0 \rightarrow \text{steady state} \tag{46}$$

$$\sum F_{sys} = \int_{CS} \rho(\underline{V}_{CV} + \underline{W})(\underline{W} \cdot \underline{\hat{n}}) dA \tag{47}$$

$$\sum F_{sys} = \int_{CS} \rho \underline{W}(\underline{W} \cdot \underline{\hat{n}}) dA + \underline{V}_{CV} \int_{CS} \rho(\underline{W} \cdot \underline{\hat{n}}) dA \tag{48}$$

Second term is equal to zero due to steady state flow and mass conservation

$$\sum F_{sys} = \int_{CS} \rho(\underline{W})(\underline{W} \cdot \underline{\hat{n}}) dA \tag{49}$$

Hence, steady state conservation of momentum for a moving object in an inertial reference frame is:

$$\sum F_{sys} = \int_{CS} \rho \underline{W}(\underline{W} \cdot \underline{\hat{n}}) dA \tag{50}$$

0.2.2 The energy equation

Review:

$$\frac{DB_{sys}}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho b d\forall + \int_{CS} \rho b(\underline{V} \cdot \underline{\hat{n}})dA \tag{51}$$

Let $B = E \therefore b = \frac{E}{m} = e = u + ke + pe = u + \frac{V^2}{2} + gz$.

$$\frac{DE_{sys}}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho e d\forall + \int_{CS} \rho e(\underline{V} \cdot \underline{\hat{n}})dA \tag{52}$$

Also recall from first law of thermodynamics that, $\Delta E = \Delta KE + \Delta PE + \Delta U = \Delta Q + \Delta W$.

$$\dot{E} = \dot{Q}_{in} - \dot{Q}_{out} + \dot{W}_{in} - \dot{W}_{out} \tag{53}$$

$$\frac{DE_{sys}}{dt} = (\dot{Q}_{in} - \dot{Q}_{out}) + (\dot{W}_{in} - \dot{W}_{out}) \tag{54}$$

$$\frac{DE_{sys}}{dt} = \left(\sum \dot{Q}_{\text{net in}} + \sum \dot{W}_{\text{net in}}\right)_{sys} \tag{55}$$

For a control volume that is coincident with the system at an instant in time.

$$\left(\sum \dot{Q}_{\text{net in}} + \sum \dot{W}_{\text{net in}}\right)_{sys} = \left(\sum \dot{Q}_{\text{net in}} + \sum \dot{W}_{\text{net in}}\right)_{\text{coincident control volume}} \tag{56}$$

Hence,

$$\frac{D}{Dt}\int_{sys} e\rho d\forall = \left(\sum \dot{Q}_{\text{net in}} + \sum \dot{W}_{\text{net in}}\right)_{\text{coincident control volume}} \tag{57}$$

Equation (55) = Equation (57)

$$\therefore \left[\sum \dot{Q}_{\text{net in}} + \sum \dot{W}_{\text{net in}}\right]_{CV} = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho(\underline{V} \cdot \underline{\hat{n}})dA \tag{58}$$

0.2.3 Different forms of work

Shaft work

$$\dot{W}_{shaft} = T_{shaft}\omega \tag{59}$$

$$\dot{W}_{\text{shaft net in}} = \sum \dot{W}_{shaftin} - \sum W_{\text{shaft out}} \tag{60}$$

Flow work

$$\dot{W}_{\text{normal stress}} = \int_{CS} \sigma(\underline{V} \cdot \hat{n})dA \tag{61}$$

$$= -\int_{CS} P(\underline{V} \cdot \hat{n})dA \tag{62}$$

$$\text{where } \sigma \text{ (local normal stress)} = -P \text{ (fluid pressure)} \tag{63}$$

$$(\delta\dot{W}_{\text{normal stress}} = \delta F_{\text{normal stress}} \cdot V = \sigma\underline{\hat{n}}\delta A \cdot \underline{V} = -P\underline{V} \cdot \underline{\hat{n}}\delta A) \tag{64}$$

Therefore, equation (58) becomes,

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{net in}} = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho(\underline{V} \cdot \underline{\hat{n}})dA \tag{65}$$

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}} + \dot{W}_{\text{normal stress}} = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho(\underline{V} \cdot \underline{\hat{n}})dA \tag{66}$$

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}} - \int_{CS} P(\underline{V} \cdot \hat{n})dA = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho(\underline{V} \cdot \underline{\hat{n}})dA \tag{67}$$

Rearranging gives,

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}} = \int_{CS} P(\underline{V} \cdot \hat{n})dA + \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho(\underline{V} \cdot \underline{\hat{n}})dA \tag{68}$$

Factorise $\rho(\underline{V} \cdot \underline{\hat{n}})$

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}} = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} \rho(\underline{V} \cdot \underline{\hat{n}})(e + \frac{P}{\rho})dA \tag{69}$$

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}} = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} \rho(\underline{V} \cdot \underline{\hat{n}})(u + \frac{P}{\rho} + \frac{V^2}{2} + gz)dA \tag{70}$$

0.2.4 Application of the energy equation

When steady, $\frac{\partial}{\partial t}\int_{CV} e\rho d\forall$ cancels out or when flow is steady in the mean (cyclical). Also, in the second term, the integrand can be non zero only where the fluid crosses the surface. If the properties are all assumed to be uniformly distributed then the integration becomes simple and gives:

$$\int_{CS} (u + \frac{P}{\rho} + \frac{V^2}{2} + gz)\rho(\underline{V} \cdot \hat{n})dA \tag{71}$$

$$= \sum_{\text{flow out}} \dot{m}(u + \frac{P}{\rho} + \frac{V^2}{2} + gz) - \sum_{\text{flow in}} \dot{m}(u + \frac{P}{\rho} + \frac{V^2}{2} + gz) \tag{72}$$

If only one stream is coming in and out:

$$= \dot{m}_{out}(u + \frac{P}{\rho} + \frac{V^2}{2} + gz)_{out} - \dot{m}_{in}(u + \frac{P}{\rho} + \frac{V^2}{2} + gz)_{in} \tag{73}$$

For steady flow, $\dot{m}_{in} = \dot{m}_{out} = \dot{m}$, therefore, the energy equation becomes:

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}} = \tag{74}$$

$$\dot{m}\left[u_{out} - u_{in} + \left(\frac{P}{\rho}\right)_{out} - \left(\frac{P}{\rho}\right)_{in} + \frac{V_{out}^2 - V_{in}^2}{2} + g(z_{out} - z_{in})\right] \tag{75}$$