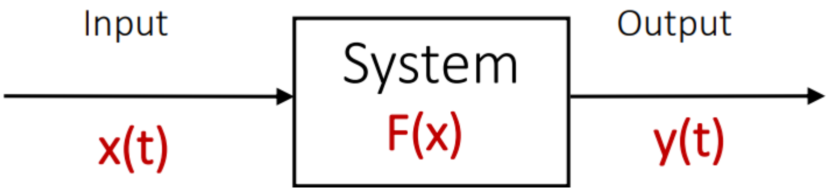
Previously, we have considered systems in the general sense, with a generic function block approach.



Now we will investigate how we model physical systems and obtain the function block $F(x)$ for electrical and mechanical components.

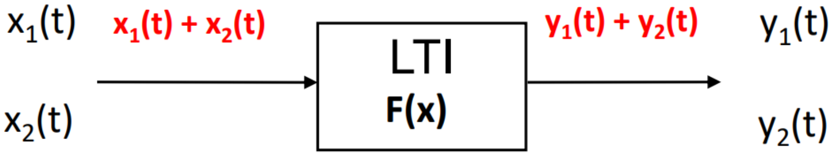
0.1 Linear Term Invariant (LTI) systems

The focus of this course and indeed much of control theory itself focuses on modelling physical systems as **linear** and **time invariant**. LTI systems have three key properties:

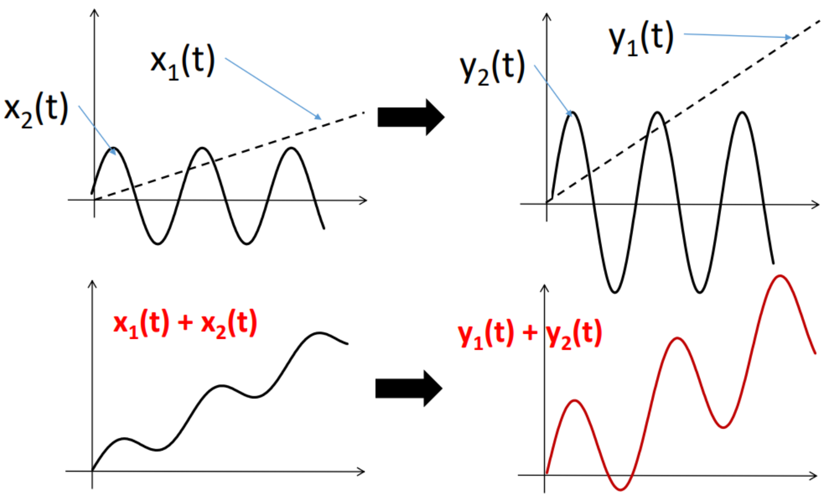
- Obey principle of superposition
- Homogeneity
- Time invariance

0.1.1 Superposition

If input $x_1(t)$ produces output $y_1(t)$ and input $x_2(t)$ produces $y_2(t)$, then input $x_1(t) + x_2(t)$ produces output $y_1(t) + y_2(t)$

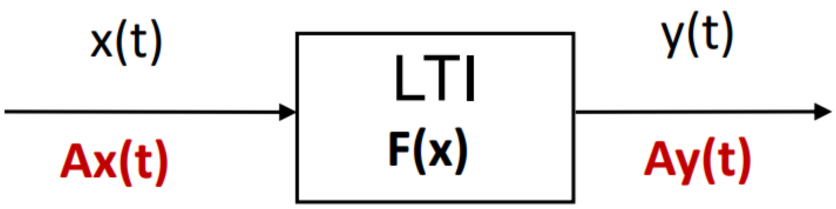


Say for a system which doubles the input $F(x) = 2x$

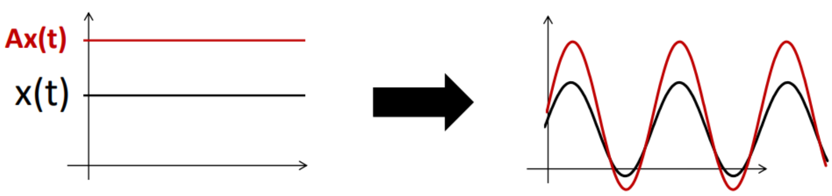


0.1.2 Homogeneity

If the input to the system $x(t)$ is scaled by a magnitude scale factor A, then the output $y(t)$ is also scaled by the same factor.

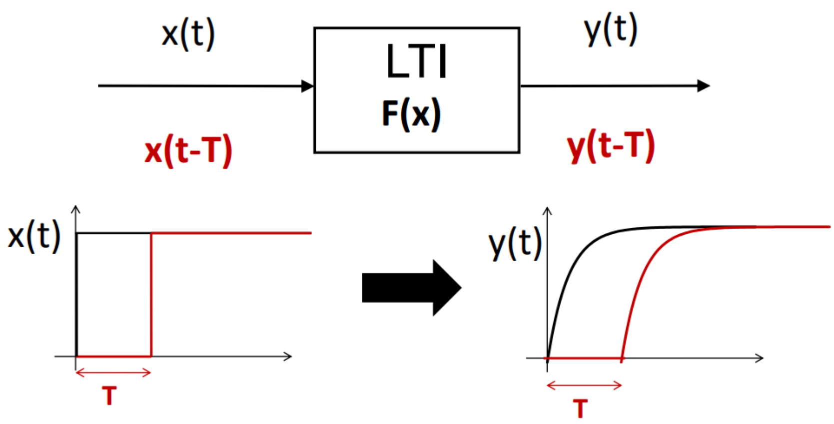


For example, consider a system which generates a sine wave at a given amplitude, with a set frequency:



0.1.3 Time invariance

If input is applied at time $t = 0$ or T s from now, the output is identical with the exception of a delay of T s.



Are these models suitable for physical systems?

These three requirements, whilst simple, are so stringent that **almost no physical LTI system truly exists**. Consider a car engine - the performance deteriorates over time, to stretch it further, would you expect a system to give the same output after a time delay T of 10 years? Even simple systems such as a resistor in an electrical circuit have non-linearities - a scaling factor A could be chosen for $x(t)$ which would mean too much current flows and the resistor melts.

Most practical systems are not linear, but often we can assume they behave linearly **under certain conditions/assumptions**. Linear systems are **much** easier to solve! There are **analytic** solutions with standard tools used solve the equations. Whereas for non-linear problems it is often necessary to solve them numerically.

0.1.4 Linearisation: Example 1

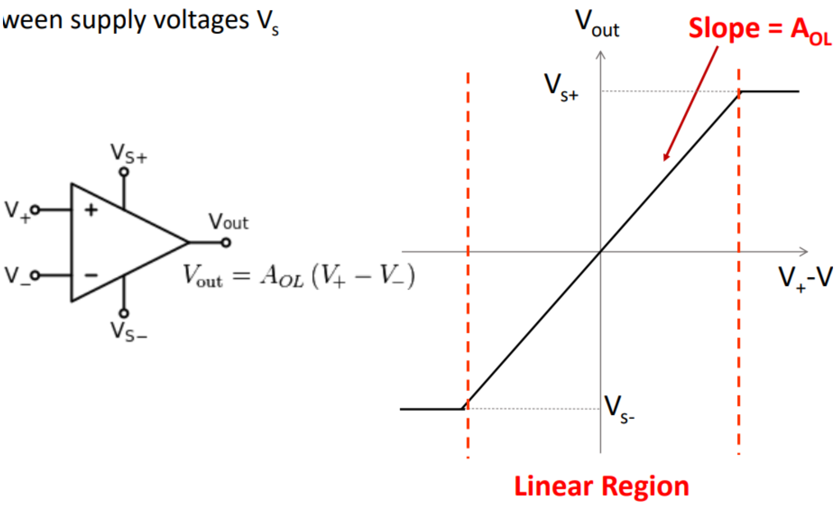
For a simple system such as a spring, across all possible compressions or extensions the response is non-linear:

$$F = -kX \tag{1}$$

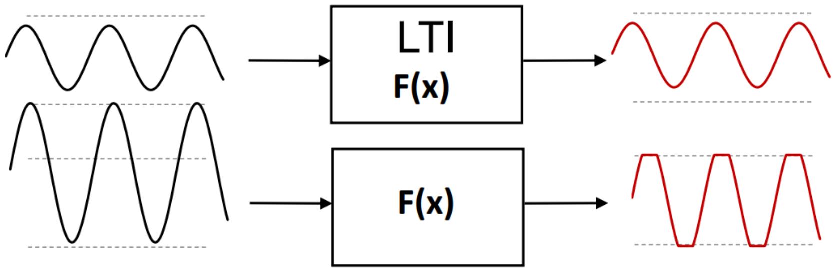
Hookes law is only a linear approximation of the true response. However, if we choose the operating range of the spring correctly, the response is within the linear region. This approximation is **valid**.

Linearisation: Example 2

Similarly an op-amp has a **linear region** where the output signals are between supple voltages V_S



Keep signals within these regions and the linear assumption holds:

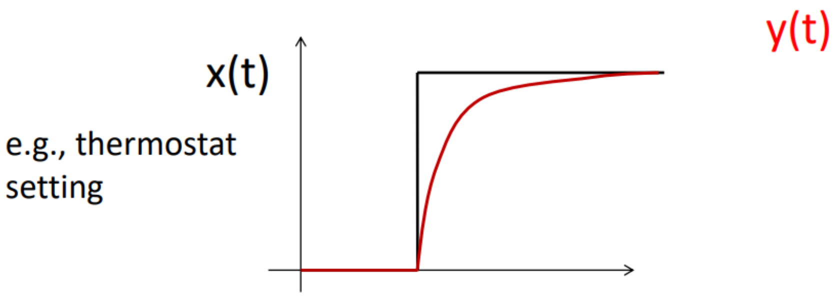


Add LTI example

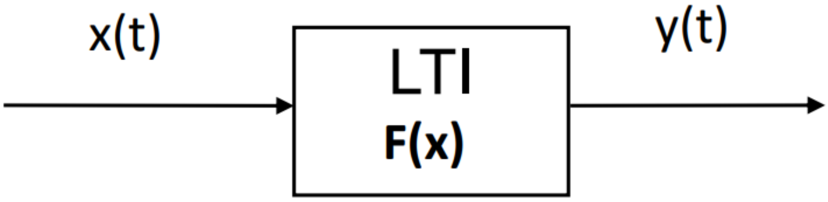
0.2 Dynamic systems - Laplace Transform

0.2.1 Dynamic systems as ODEs

Ideal systems would respond **instantaneously** to inputs, however real world systems require some time to adjust to changes and are thus known as **dynamic systems** as the output changes over time.



As we are intersected in describing something that **changes** with time, it is useful to express the function block of the system $F(t)$ as an ordinary differential equation (ODE).



$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + ... + a_2\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a_1\frac{\mathrm{d}y}{\mathrm{d}t} + a_0 = bx \tag{2}$$

- x is input function or forcing function
- y is output
- n is **order** of the ODE
- a_0... are coefficients. These **completely characterise the system**

0.2.2 Laplace Transforms

Because of our **linear assumptions** we can use Laplace transforms to simplify solving the ODEs. The Laplace transforms of a signal (function) x is the function $X = \mathcal{L}(x)$ defined by

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,\mathrm{d}t \tag{3}$$

For those $s \in \mathbf{C}$ for which the integral makes sense.

- X is a complex-valued function of complex numbers
- s is called the (complex) **frequency variable** with units s^{-1}, t is called the **time variable** (in sec); st is unitless
- $s = \sigma + j\omega$

As we shall see:

- Differential operators are replaced with algebraic variables
- Algebraic equations are much easier to manipulate & solve
- Standard forms exist for many physical systems

Laplace Transforms: Example 1

Let's fine Laplace transform $x(t) = e^t$:

$$X(e^t) = \int_{0^-}^{\infty} e^t e^{-st}\,\mathrm{d}t \tag{4}$$

$$X(e^t) = \int_{0^-}^{\infty} e^{(1-s)t}\,\mathrm{d}t \tag{5}$$

$$Xe^t = \frac{1}{1-s} e^{(1-s)t}\Big|_{0^-}^{\infty} \tag{6}$$

$$X(e^t) = \frac{1}{1-s} \times 0 - \frac{1}{1-s} \times 1 = \frac{1}{s-1} \tag{7}$$

Laplace Transforms: Example 2

Constant or **unit step** $x(t) = 1$ (for $t \geq 0$)

$$X(s) = \int_{0^-}^{\infty} 1e^{-st}\,\mathrm{d}t \tag{8}$$

$$X(s) = \int = -\frac{1}{s} e^{-st}\Big|_{0^-}^{\infty} \tag{9}$$

$$X(s) = -\frac{1}{s} \times 0 - (-\frac{1}{s}) \times 1 = \frac{1}{s} \tag{10}$$

Laplace Transforms: Example 3

Sinusoid: first express $x(t) = \cos(\omega t)$ as:

$$x(t) = \frac{1}{2} e^{i\omega t} + \frac{1}{2} e^{-i\omega t} \text{ (Euler's formula)} \tag{11}$$

$$X(s) = \int_{0^-}^{\infty} e^{-st} \left(\frac{1}{2} e^{i\omega t} + \frac{1}{2} e^{-i\omega t} \right) \mathrm{d}t \tag{12}$$

$$X(s) = \frac{1}{2} \int_{0^-}^{\infty} e^{(-s+i\omega)t}\,\mathrm{d}t + \frac{1}{2} \int_{0^-}^{\infty} e^{(-s-i\omega)t}\,\mathrm{d}t \tag{13}$$

$$X(s) = \frac{1}{2} \frac{1}{s-i\omega} + \frac{1}{2} \frac{1}{s+i\omega} = \frac{s}{s^s + \omega^s} \tag{14}$$

You can look up these transforms in a table. The Laplace variable, s, can be considered to represent the differential operator (very useful for control engineering):

$$s \equiv \frac{\mathrm{d}}{\mathrm{d}t} \tag{15}$$

$$\frac{1}{s} \equiv \int_{0^-}^{\infty} \mathrm{d}t \tag{16}$$

0.2.3 Transfer Functions

0.2.4 Combining components