MECH0005 Topic Notes

UCL

First Edition

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Contents

Chapter 1

Fluids

1.1 Fluid Properties

Fluids have many properties including:

- Surface Tension - σ.
- Viscosity - μ.
- Compressibility.

- Density ρ.
- Temperature.
- Pressure - P.

What is a fluid?

A fluid is a substance that deforms continuously when acted on by a shearing stress of any magnitude.

This includes gases and liquids but not silly putty, gels or glass. Some soft materials such as toothpaste will only start to flow once a critical shear stress has been reached. This is the study of rheology and not studied in this module.

In classical fluid mechanics, we treat fluids as a continuum because there is an extremely large number of particles. Thus, we can conclude that fluid parameter such as pressure and density vary continuously throughout the fluid.

1.1.1 Shear strain for solid bodies

What is shear strain?

Shear strain is the change in angle as an element experience a force
tangential to its surface.
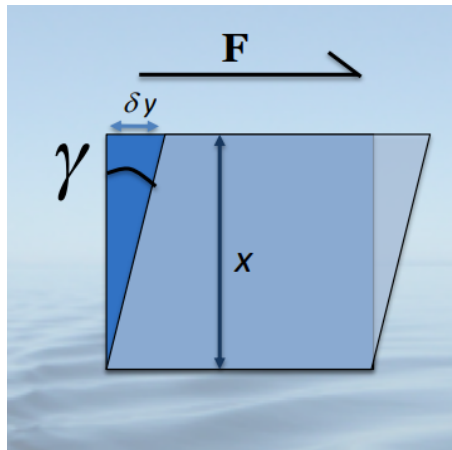
For a fluid, we are interested in the rate of shear. A specific tangential force will cause a fixed amount of shear strain per unit time. If the speed of the top layer is u_y, the shear rate is $\delta u_y / \delta x$.

$$tan(\gamma) = \frac{\delta y}{x} \approx \gamma \qquad (1.1.1)$$

1.1.2 Shear stress on a solid body

For a solid, the shear force is the force applied tangentially to a surface.

Shear stress is the tangential force per unit area τ.

Normal stress is the perpendicular force per unit area σ.
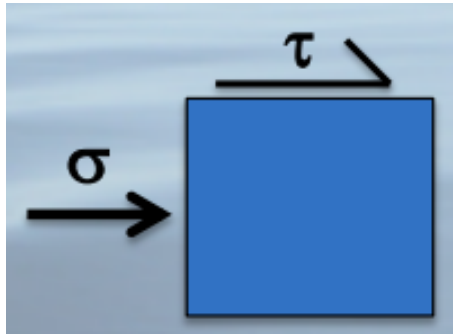
1.1.3   Simple laminar flow case

Consider flat layers of fluid sliding over each other. The sideways velocity of the fluid changes as you move away from the stationary boundary. The velocity of the y direction, u_y is changing with position, x.
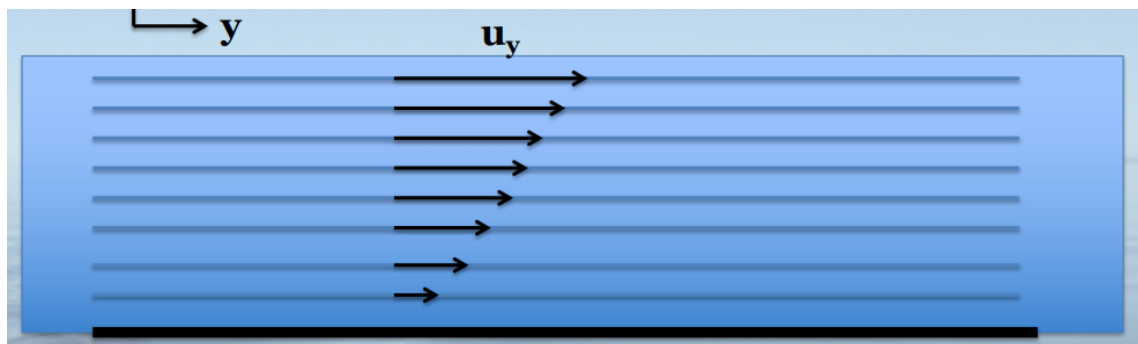
1.1.4   Viscosity

$$\tau = \mu \frac{du_y}{dx} \tag{1.1.2}$$

Where:

- τ = Shear stress = force/area.
- μ = Dynamic viscosity.
- du_y/dx = Shear rate.

- Units: Velocity change per unit perpendicular distance.

Viscosity is the constant of proportionality, telling us how much shear stress is required to produce a given shear rate. Shear stress can be different at different places in the same fluid. Equation (2) is Newton's equation for viscosity and he assumed that the viscosity was a constant. However, this is not always true and the fluids for which that does not apply are called *non-Newtonian fluids*. An example

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of a shear *thickening* fluid is corn-starch, i.e. viscosity increases with shear rate. An example of shear *thinning* fluid is blood or ketchup, i.e. viscosity decreases with shear rate. We cannot use laminar flow analysis to study blood vessels because they constrict and dilate as fluids pass through - the rigid boundary condition does not apply (or you would not be able to feel your pulse.) We will only be studying Newtonian fluids and rigid boundaries.

Viscosity changes as a function of temperature and pressure. In liquids, when the temperature increases, the viscosity decreases. In gases, the reverse occurs, with viscosity increasing with temperature. Relatively, pressure has a weak effect on the viscosity of a fluid.

The standard symbol for the viscosity as defined above is μ and this is known as the *dynamic viscosity* (ratio of shear stress to the the shear rate) with units $\mathrm{kg\,m^{-1}\,s^{-1}}$ or $\mathrm{Pa\,s}$. This will be our definition of viscosity. For some applications it is easier to do calculations in terms of the viscosity per unit density:

$$v = \frac{\mu}{\rho} \tag{1.1.3}$$

1.1.5 Surface tension

What is surface tension?

Surface tension is the effective tangential force per unit length $(\mathrm{N\,m^{-1}})$
exerted on an imaginary line through a surface, due to intermolecular
interaction.

Surface tension is always considered perpendicular to the line it's pulling on. In this case, we want the force exerted on the thread. If the thread has length L, the total force on one side of the soap film is σL, so the total force from both sides is $2\sigma L$.
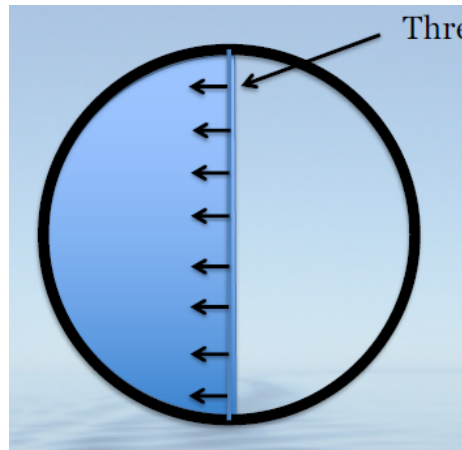
Surface tension is what allows a drop of liquid to take its unique shape, when there are no external forces present.

1.1.6 Capillary action

This is caused by two processes:

- *Adhesion* between the water molecules and the tube wall.
- *Cohesion* between water molecules.

However, the surface tension (cohesion) must still be strong enough to drag the fluid up the pipe. When the angle of contact between a solid and a liquid is 90 degrees, then the cohesive force = the adhesive force.
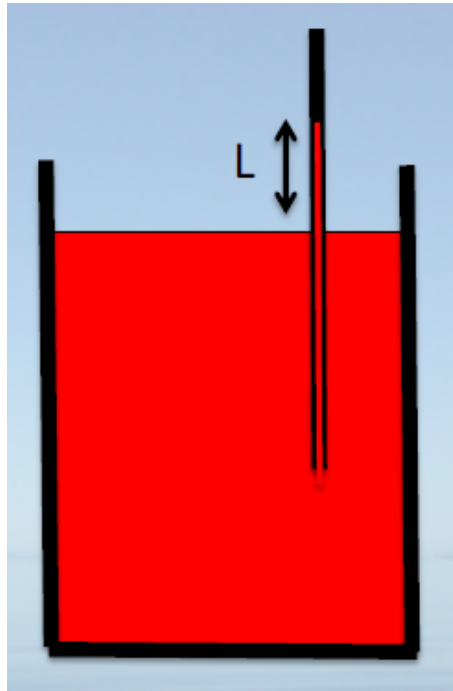
The liquid will rise by itself up the tube and this gives us a crude method for calculating surface tension. The weight of the water must be balanced by the surface tension forces at the water surface.

$$F = 2\pi\sigma = L\pi r^2 \rho g \qquad (1.1.4)$$

$$\sigma = \frac{1}{2}\rho r g L \qquad (1.1.5)$$

1.1.7 Pressure

What is Pascal's Law?

> The pressure at a point in a fluid at rest or in motion is independent of
> direction as long as there are no shearing forces present.

The volume of a fluid under pressure will depend on the magnitude of the pressure. However, for many liquids (e.g. water), the change in volume is so small that it can be considered *negligible.* Even in everyday situations, water obviously is

not really incompressible because sound waves can travel through it. For a truly incompressible fluid, the speed of sound would be infinite. Seawater density varies with temperature, pressure and salinity but below the top one kilometre, where salinity and temperature are nearly uniform, there is almost no variation in density with depth. The relationship between the pressure, volume and temperature of a fluid is described by an *equation of state*. Any equation that relates pressure and volume tells us something about the compressibility of a fluid. Pressure is also the same in all directions from our continuum assumption.
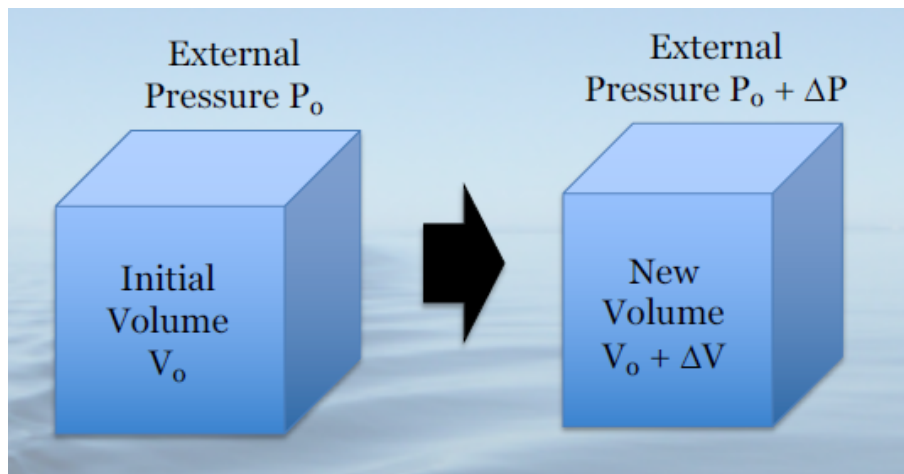
We are interested in finding dV/dP which is (almost) always negative. A fluid is considered *compressible* when its volume varies significantly with temperature and pressure in the parameter space of interest. If there is no significant variation, the fluid can be considered *incompressible*. We will only be covering compressible fluids.

1.1.8 Example Question: Cartesian diver

Consider a closed vessel that is completely full of water but with flexible sides (plastic bottle). Inside is a cylinder that is closed at the top but open at the bottom. It is partly full of air and has some extra mass added at the base. The plastic parts of the diver have a volume of 1.067×10^{-6} m^3 and a density of 2400 kg m^{-3}. At atmospheric pressure, the length of the air column is 1.5 cm and the cross-sectional area is 1×10^{-4} m^2. The temperature in the room is 20°C.

Question: How much pressure must be applied to the bottle to make the diver sink?

We must ask an important question: is the total density of the diver and the air and the water greater than or less than the density of the water around it? We know that $\rho_w = 1000$ kg/m^3 and if the water is compressible, the total density of

the whole diver is given by:

$$\rho_D = \frac{\rho_A l_1 A + \rho_w l_2 A + M_D}{(l_1 + l_2) A}$$

Where M_D is the mass of the plastic parts of the diver.

Cross-sectional
area A
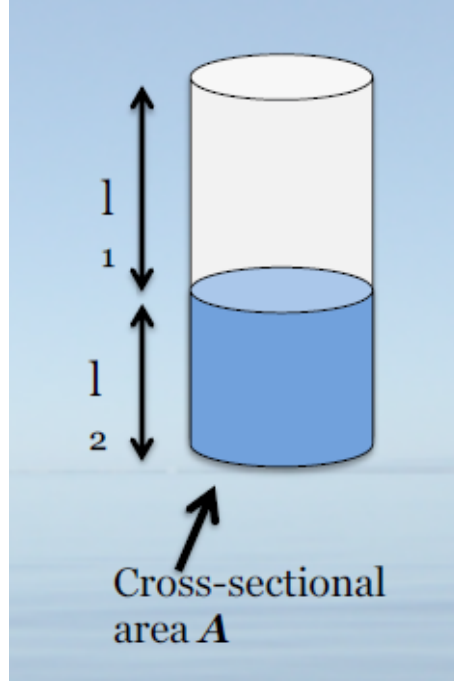
We can see that the diver will sink if $\rho_D > \rho_w$ and it rises if $\rho_D < \rho_w$.

Method: set the densities to be equal and deduce an expression for the pressure in this situation.

$$\rho_w = \rho_D = \frac{\rho_A l_1 A + \rho_w l_2 A + \rho_P V_P}{(l_1 + l_2) A + V_P}$$

To calculate the density of air, we need the equation of state. Assume that $R_g = 287$ $\mathrm{J\,kg^{-1}\,K^{-1}}$

$$P = \rho_A R_g T$$

The necessary pressure is determined by the necessary density of air. So, we start by rearranging the equation to specify the density of air.

$$\rho_A = \frac{1}{l_1 A} \left(\rho_w \left((l_1 + l_2) A + V_P \right) - \rho_w l_2 A - \rho_P V_P \right)$$

$$\rho_A = \frac{1}{l_1 A} \left(\rho_w \left((l_1 A + V_P) \right) - \rho_P V_P \right)$$

We can now use the equation of state to find an expression for the pressure.

$$P = \frac{R_g T}{l_1 A} \left(\rho_w \left((l_1 A + V_P) \right) - \rho_P V_P \right)$$

From here we can substitute our values to arrive at an answer of $P = 3.48 \times 10^5$ $\mathrm{N\,m^{-2}}$ or approximately 3.5 atmospheres.

In many situations, we assume that liquids are incompressible. However, this is an *assumption* that must be justified.

1.2   Buoyancy and stability

The resultant force acts at the centre of gravity of the fluid, the centroid of the volume. This is called the centre of buoyancy. Whatever is filling the space also has a force acting on it: it's weight. This acts at its centre of mass. The resultant force is given by:

$$F_\uparrow - F_\downarrow = (\rho_w - \rho_0) give \tag{1.2.1}$$
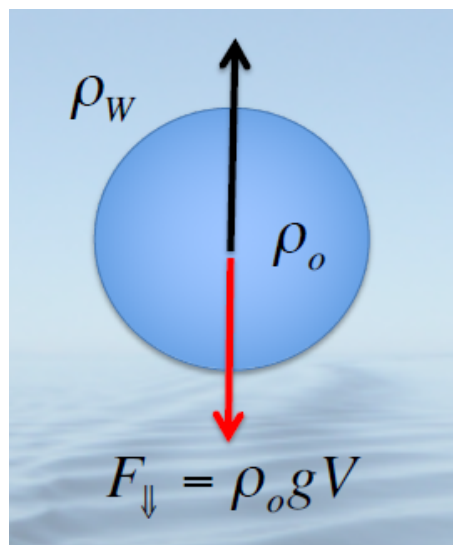
An object is neutrally buoyant if the weight and the buoyant forces are equal, so the system is in equilibrium. INSERT EXAMPLE MERT

1.2.1   Instability

Consider a submerged ping pong ball which is half filled with plasticine and the other half just filled with air.
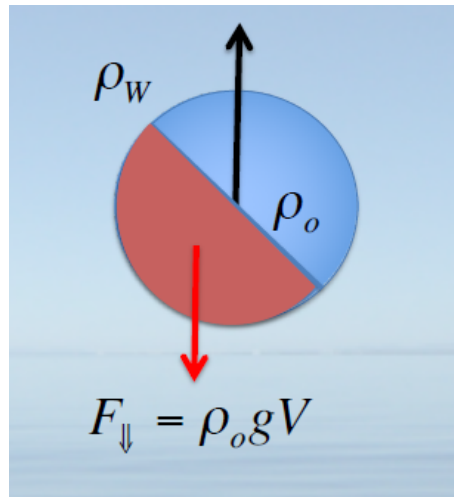
Assume that the ball overall is neutrally buoyant. The forces are equal and opposite but there is a *couple*. The ball will rotate in response to the couple until the forces are aligned. At this point, the object is stable. The buoyancy and the weight do not necessarily have the same line of action. We must also consider whether the object will maintain its orientation.

- *Stable* equilibrium: when displaced slightly, there is a net force acting to push the object back to its initial orientation.

- *Unstable* equilibrium: when displaced slightly, there is a net force acting to push the object further from its initial orientation.

- *Neutral* equilibrium: when displaced slightly, there is no net force.

If the centre of buoyancy is above the centre of gravity, the object is in a stable equilibrium.

1.2.2   Centre of mass

The centre of mass is calculated in an analogous way to the centroid of the volume. If we split any object in half with a flat plane that passes through its centre of mass, there will be equal mass in both halves.

$$x_{CoM} = \frac{1}{m} \int_m x \, dm$$

$$y_{CoM} = \frac{1}{m} \int_m y \, dm$$

$$z_{CoM} = \frac{1}{m} \int_m z \, dm$$

Where m is the total mass of the object.

In a semi-submerged body, the upward thrust is equal to the weight of the *displaced* fluid. For a body to be in vertical equilibrium, the immersed body must generate a buoyancy force equal and opposite to the weight of the object.

1.3 Dimensionless Numbers

1.3.1 Notation

We denote a dimension by using a capital letter in square brackets. Here are some common dimensions.

- [M] - mass (unit: kilogram).
- [T] - time (unit: second).
- [L] - length (unit: metre).
- [Θ] - temperature (unit: kelvin).

Thus, we can derive that the dimensions of acceleration (which has units $\mathrm{m\,s^{-2}}$) $[L][T]^{-2}$. Some dimensions of common measurements are shown below:

- Force - $[M][L][T]^{-2}$.
- Energy - $[M][L]^2[T]^{-2}$.

We use dimensional analysis to check derivations. The dimensions of both sides of any equation must match. Physical constants also often have units associated with them - these must also be considered. Some variables are dimensionless such as the Reynolds number.

$$Re = \frac{\rho l u}{\mu}$$

$$[Re] = \frac{[M][L]^{-3} \cdot [L] \cdot [L][T]^{-1}}{[L][M][T]^{-2} \cdot [T][L]^{-2}} = \frac{[M][L]^{-1}[T]^{-1}}{[M][L]^{-1}[T]^{-1}}$$

Hence, we can see that the Reynolds number is a dimensionless quantity as all the dimensions cancel.

1.3.2 Example

We can use dimensional analysis to derive basic forms of equations. We want to work out the pressure drop as oil flows though a pipe. Let us consider the parameters this may depend on.

- Viscosity - $[M][L]^{-1}[T]^{-1}$.
- Pipe length - $[L]$.
- Pipe diameter - $[L]$.
- Velocity - $[L][T]^{-1}$.
- Pressure - $[M][L]^{-1}[T]^{-2}$.

Next we can assume that the pressure is a function of the other four. Some combination of the others must have the same dimension as the quantity we want.

$$[M][L]^{-1}[T]^{-2} = ([M][L]^{-1}[T]^{-1})^{\alpha} \cdot ([L])^{\beta} \cdot ([L])^{\gamma} \cdot ([L][T]^{-1})^{\delta}$$

$$[L] : -1 = -\alpha + \beta + \gamma + \delta$$

$$[M] : 1 = \alpha$$

$$[T] : -2 = -\alpha - \delta$$

$$\alpha = 1, \ \delta = 1, \ \beta + \gamma = -1$$

So it must be true that:
$$\Delta P = \mu \cdot v \cdot I^{\beta} \cdot D^{\gamma}$$

Where $\beta + \gamma = -1$ The actual answer for laminar flow is:

$$\Delta P = \frac{2\mu L v}{D^2}$$

This sort of analysis is useful for checking on the functional form of relationships, but it won't give you the exact relationship, or the value of any dimensionless constants involved.

1.3.3 Similarity

- Geometrical similarity: fixed ratio of lengths.
- Kinematic similarity: fixed ratio of velocities.
- Dynamic similarity - fixed ratio of forces.

Note on inertia: Inertia is not a force. However, for considering its importance to dynamic similarity, we can use the force needed to slow down a moving object. So we quantify inertia for these purposes as ma, from $F = ma$. Since the forces on flow change fluid motion, we use this often.

Dynamic similarity: viscosity

Compare the inertia "force" and the viscous force for a fluid:

$$\frac{[Inertia\ force]}{[Viscous\ force]} = \frac{\rho L^2 u^2}{\mu u L} = \frac{\rho L u}{\mu}$$

The Reynolds number is something very specific - it allows us to calculate the ratio of inertial and viscous forces in order to check for dynamical similarity.

- Honey: Re $\approx 1.3 \times 10^{-4}$
- Tea: Re ≈ 1100

Therefore, they are not dynamically similar with respect to viscosity. For complete dimensional similarity, we must match the Reynolds number with the Froude number. If the same working fluid is used for the model and the prototype it is not possible to match the Reynolds number and the Froude number except if the model and the prototype have the same length. To achieve complete dynamic similarity between geometrically similar flows, it is necessary to duplicate the values of the independent dimensionless groups; by so doing the value of the dependent parameter is then duplicated. This is important because measured values of drag from model test could be scaled to predict drag for the operating conditions of the prototype. INSERT EXAMPLE MERT

1.3.4   Dimensionless groups

We have identified some dimensionless groups such as Reynolds number and Froude number. There are many more such as:

- Bond number: ratio of gravitational to surface tension forces.
- Capillary number: ratio of surface tension to viscous forces.
- Euler number: ratio of pressure force to inertial force.
- Grashof number: ratio of buoyancy to viscous forces.

- Cauchy number: ratio of inertial to elastic forces.
- Weber number: ratio of inertial to surface tension forces.

1.4   Buckhingham Pi

insert theory here

1.5   Coordinate systems and velocity fields

Fluid flow can be classified in four ways:

1. Viscous/inviscid - when the viscosity of a fluid is equal to zero e.g. viscous - blood flow through a capillary, honey and inviscid - tornado, stirring tea.
2. Laminar and turbulent flow - Reynold's number. This is a measure used to determine whether a flow is likely to be laminar or turbulent.
3. Compressible or incompressible - density is constant/does not change with time e.g. compressible - clouds, gas turbines and incompressible - oil pipelines, water dams.
4. Steady and unsteady flow - the velocity at a particular point in space does not change with time and eddy formations or the opposite with eddies and turbulence.

1.5.1   Coordinate systems and labelling

There are various notations used.

Cartesian coordinates

Position: x, y, z

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\underline{i} + y\underline{j} + z\underline{k}$$

Velocity: u, v, w

$$\underline{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u\underline{i} + v\underline{j} + w\underline{k}$$

Forces

Forces can either be applied to the body or on one of its surfaces.

- Body forces - The force acts at the centre of each increment of mass e.g. gravity.
- Surface flows - The force acts at the surface of each increment of mass e.g. shear stress.

As there are two types of forces, we need a consistent labelling system to distinguish them.
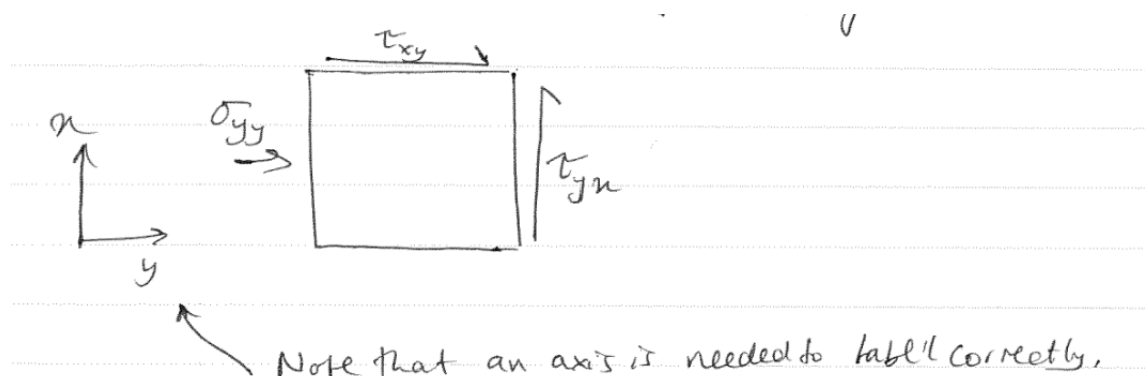
Figure 1.1: Force Diagram on an increment of fluid

$\sigma_y y$ The first subscript refers to the face that the force is applied to. The second subscript is the direction the force acts in.

Incremental element of volume

Consider an incremental volume with dimensions: dx, dy, dz.

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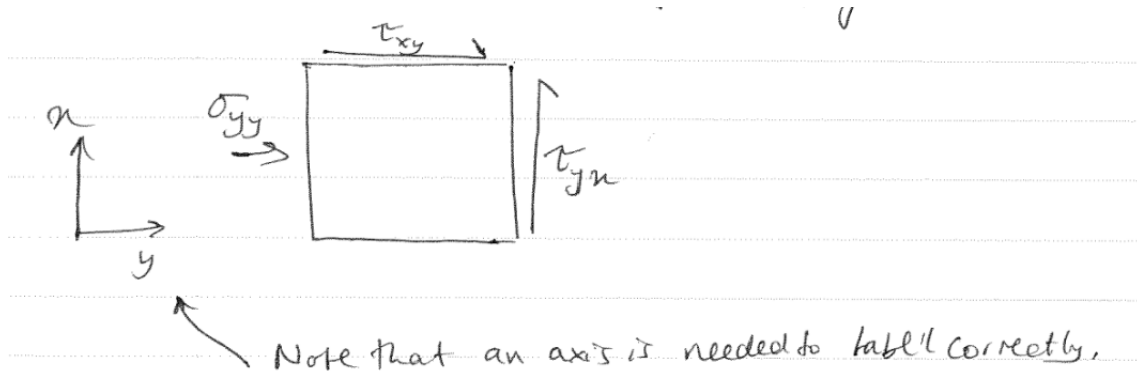
Figure 1.2: Force Diagram on an increment of fluid

There is a variation of fluid speeds in this box. Each component of the velocity can vary in each of these directions. The x-component of velocity, n, could vary with x, y or z independently.

$$\frac{\partial n}{\partial x}, \quad \frac{\partial n}{\partial y}, \quad \frac{\partial n}{\partial z}$$

and the same goes to v and w.

1.5.2   Velocity Fields

A field is a three dimensional description of a field parameter in space. For example:

$$\underline{v} = u\begin{pmatrix} x \\ y \\ z \end{pmatrix}\underline{i} + v\begin{pmatrix} x \\ y \\ z \end{pmatrix}\underline{j} + w\begin{pmatrix} x \\ y \\ z \end{pmatrix}\underline{k}$$

More generally, the flow field can be a function of time, t (this is unsteady flow.)

$$\underline{v} = u\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}\underline{i} + v\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}\underline{j} + w\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}\underline{k}$$

You can also get acceleration fields. Velocity and acceleration fields are vector fields as it has magnitude and direction. A scalar field only has magnitude e.g. Pressure, temperature and density fields.

1.5.3 Eularian and Lagrangian flow descriptions

There are two general approaches in analysing fluid mechanics problems:

1. Eularian method - this uses fields to describe the flow at a specific position in time and space.
2. Lagrangian method - this describes what happens to an individual particle with time g.g ocean drifters, measuring conditions as they float freely through the oceans.

1.5.4 One-, two- and three- dimensional flows

Most flows are 3D flows, However, if one of the velocity components may be small (in some sense) relative to the other components then it may be reasonable to neglect the smaller components and assume 2D flow. That is $\underline{v} = u\underline{i} + v\underline{j}$. Sometimes two of the components may be negligible so it can be approximated to one dimensional flow. That is $\underline{v} = u\underline{i}$.

For many flows, one dimensional flow will provide a reasonable approximation, whereas in others would produce erroneous results.

$$\text{Steady flow} \to \frac{\partial v}{\partial t} = 0$$

$$\text{Unsteady flow} \to \frac{\partial v}{\partial t} \neq 0$$

1.5.5 Streamlines, streaklines and pathlines

- Pathlines - the line traced out by a given particle as it flows from one point to another.
- Streaklines - A line that is created by particles in a flow that have previously passed through a common point.
- Streamlines - A line that is tangent to the velocity to the velocity field (if flow is steady).

If the flow is steady nothing at a fixed point changes with time, so the streamlines are fixed in space. If unsteady, the streamlines will change with time. Streamlines are obtained by integration.

$$\frac{dy}{dx} = \frac{v}{u}$$

v and u are usually functions of x and y and so integration after separating variables can be done and graphs drawn for various constant of integration. If steady, streamline = streakline.

1.5.6 The acceleration field

To apply Newton's second law, we need the acceleration. For the Lagrangian method, $a = a(t)$ for each particle. For the Eularian description, we can find the acceleration field, which provides the acceleration of given points in a field. We will discuss how to get this if the velocity field is known.

1.5.7 The material derivatives

Consider a fluid particle moving along its pathline. In general, the particles velocity, denoted V_A for particle A, is a function of location and time. That is

$$V_A = V_A(V_A, \ b) = V_A \left[x_A(t), \ y_A(t), \ z_A(t), t \right] = V_A \left[x_A, \ y_A, \ z_A, \ t \right]$$

By definition, acceleration is the rate of change of velocity. We have to apply the chain rule as acceleration may change due to a change in position or time.

$$\underline{a}_A(t) = \frac{dV_A}{dt} = \frac{\partial V_A}{\partial t}\frac{dt}{dt} + \frac{\partial V_A}{\partial x}\frac{dx_A}{dt} + \frac{\partial V_A}{\partial y}\frac{dy_A}{dt} + \frac{\partial V_A}{\partial z}\frac{dz_A}{dt}$$

$$= \frac{\partial V_A}{\partial t} + \frac{\partial V_A}{\partial x}\frac{dx_A}{dt} + \frac{\partial V_A}{\partial y}\frac{dy_A}{dt} + \frac{\partial V_A}{\partial z}\frac{dz_A}{dt}$$

$$\text{since } u_A = \frac{dx_A}{dt}, \ v_A = \frac{dy_A}{dt}, \ w_A = \frac{dz_A}{dt}$$

$$\underline{a}_A = \frac{\partial V_A}{\partial t} + \frac{\partial V_A}{\partial x}u_A + \frac{\partial V_A}{\partial y}v_A + \frac{\partial V_A}{\partial z}w_A$$

This applies for any project and so we can drop the subscript.

$$\underline{a} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}u + \frac{\partial V}{\partial y}v + \frac{\partial V}{\partial z}w$$

This is a vector result where scalar components can be written as:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

All of this is written in shorthand as:

$$\underline{a} = \frac{Dv}{Dt}$$

Where the operator $\frac{D}{Dt}$ is the material derivative.

$$\frac{D}{Dt}() = \left[\frac{\partial()}{\partial t}\right](A) + \left[u\frac{\partial()}{\partial x} + v\frac{\partial()}{\partial y} + w\frac{\partial()}{\partial z}\right](B)$$

Where A is the local derivative and B is the convective derivative.

If the parameter involved is acceleration, term (A) is called *local acceleration* for steady flow this is equal to zero. $\frac{\partial v}{\partial t} = 0$. This physically means there is no change in flow parameter at a fixed point in space if the flow is steady.

The collection of terms labelled (B) are called the convective derivative. It represents the fact that a flow property associated with a fluid particle may vary because of the motion of the particle from one place to another. This contribution occurs when the parameter of concern is acceleration, then section (B) is called the convective acceleration.

1.5.8   Streamline coordinates

in the streamline C.S. the flow is described in terms of one coordinate along the streamlines, denoted s, and the second coordinate normal to the streamlines, denoted n. Unit vectors in this direction are denoted by $\underline{\hat{s}}$ and $\underline{\hat{n}}$. The flow plane is covered by an orthogonal curved net of coordinate lines. s and n are always perpendicular, but the lines of constant s or constant n are not necessarily straight. Note that velocity is always tangent to the s direction. Thus:

$$\underline{v} = v\underline{\hat{s}}$$

For steady, 2D flow, the acceleration is:

$$\underline{a} = \frac{Dv}{Dt} = a_s\hat{s} + a_n\hat{n}$$

Where $a_s\hat{s}$ is the acceleration in the streamline direction and $a_n\hat{n}$ is the acceleration in the normal direction.

In general, for steady flow, both the speed and the flow direction are a function of location.

$$v = v(s, n) \text{and} \underline{\hat{s}} = \underline{\hat{s}}(s, n)$$

For a given particle, the value dt s changes with time but the value of n remains fixed as the particle flows along a streamline defined by n = constant. (note: pathlines = streamlines during steady flow). Thus:

$$\underline{a} = \frac{Dv}{Dt} = \frac{D}{Dt}(v\hat{s}) = \frac{Dv}{Dt}\underline{\hat{s}} + v\frac{D\underline{\hat{s}}}{Dt}$$

Applying the material derivative,

$$\underline{a} = \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial n}\frac{dn}{dt}\right)\hat{s} + v\left(\frac{\partial \underline{\hat{s}}}{\partial t} + \frac{\partial \underline{\hat{s}}}{\partial s}\frac{ds}{dt} + \frac{\partial \underline{\hat{s}}}{\partial n}\frac{dn}{dt}\right)$$

For steady flow, this simplifies too:

$$\underline{a} = \left(v\frac{\partial v}{\partial s}\right)\underline{\hat{s}} + v\left(v\frac{\partial \underline{\hat{s}}}{\partial s}\right)$$

This equation can also be rewritten as:

$$\underline{a} = v\frac{\partial v}{\partial s}\underline{\hat{s}} + \frac{v^2}{R}\underline{\hat{n}}$$

Or,

$$a_s = v\frac{\partial v}{\partial s}, \ a_n = \frac{v^2}{R}$$

Where a_s is the streamwise/acceleration convective and a_n is the normal/acceleration centrifugal.

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1.6 Control-volume approach

Recall from thermodynamics th at $\dot{m}_{in} = \dot{m}_{out}$ for a control volume (A c.v. allows heat, work and mass transfer across its boundary). In fluid dynamics different terminology is usually used:

- Closed system = control mass = system (in fluid mechanics).

- Open system = control volume.

We also know that $\frac{dm_{cv}}{dt} = \dot{m}_{in} - \dot{m}_{out}$ from thermodynamics. For a *system*, $\left(\frac{dm}{dt}\right)_{sys} = 0$ since a system is a control mass.

IGNORE

1.7 Laminar flow and boundary layers

1.7.1 Boundary conditions

You will generally only come across two types of boundary conditions:

1. No-slip condition: at the point where the fluid touches the boundary, the velocity of the fluid along the boundary is zero. This is the case where the boundary is a solid.
2. Free surface: there are no viscous forces on the fluid where it touches the boundary, so the speed can take any value. This is the case where there is a liquid-gas boundary and may be called (e.g. for a stream) *open-channel flow*.

1.7.2 Laminar flow calculations

Consider laminar flow between two flat plates. Split the flow into sheets of thickness d_{in}. The top plate moves at constant speed v and the lower plate is stationary. Here boundary condition (1) applies i.e. fluid at the boundary moves at the speed of the boundary. The top plate has an area A and the sideways force being applied to it is $F \therefore \tau = \frac{F}{A}$. If the flow is steady, the forces must balance for each layer. However, the liquid layers must transmit the stress so τ is constant throughout the depth. Let us apply our boundary conditions:

$$u_y(0) = 0$$
$$u_y(D) = v$$

where D is the vertical distance between the upper and lower plate.

$$\therefore \tau = \mu \frac{du_y}{dx}$$

$$\left.\begin{array}{l} \frac{\tau}{\mu}dx = du_y \\ \frac{\tau}{\mu}\int dx = \int du_y \\ k + \frac{\tau}{\mu}x = u_y \end{array}\right\} \begin{array}{l} u_y = \frac{\tau}{\mu}x + k \\ 0 = \frac{\tau}{\mu}\cdot 0 + k \therefore k = 0 \\ \text{also } \frac{\tau}{\mu} = \frac{v}{D} \to \tau = \frac{v}{D}\mu \end{array}$$

$$\therefore u_y = \frac{v}{D}x$$

1.7.3   Laminar flow between solid boundaries

In laminar flow, individual particles of fluid follow paths that do not cross those of neighbouring particles. However, there is still a velocity gradient across the flow. Laminar flow is not normally found except in the neighbourhood of a solid boundary, the retarding effect of which causes the transverse velocity gradient.

$$\tau = \mu\left(\frac{\partial u}{\partial y}\right)$$

Where τ is the resultant shear stress, μ is the dynamic viscosity and $\left(\frac{\partial u}{\partial y}\right)$ is the rate at which the velocity u increases with the coordinate y perpendicular to the velocity.

We can use the above to find the required shear stress, if we know the speed the fluid layer thickness and the viscosity:

$$\tau = \frac{v}{D}\mu$$

Flow between flat plates example

Consider two flat plates stuck together with a layer of honey of uniform thickness. Assume laminar flow between the plates. How hard must you pull sideways to separate them at a speed of 1 cm s^{-1}?

- Viscosity of honey $= 100 \text{ Pa s}$
- Thickness $D = 0.003$ m
- Speed $v = 0.01 \text{ m s}^{-1}$
- Width $= 0.02$ m
- Area $(A) = $ width $\times$ length (l)

Assuming that there end effects can be neglected, the stress is equal to the force applied divided by the area. Thus, we can see that the required force decreases as

the plates separate.

$$\tau = \frac{F}{A}$$
$$= \frac{v}{D}x$$
$$F = \frac{v}{D}\mu \cdot A$$
$$F = \frac{0.01}{0.003}100 \cdot 0.02 \cdot l$$
$$F = 6.6l$$

1.7.4 Finding shear stress

The velocity field in a sample of seawater is given below, where A and B are constants:

$$u_y = A(B^2 - x^2)$$

Tiny sea creatures called dinoflagellates emit light when the shear stress in the water around passes a critical level. This is a major cause of bioluminescence at the surface of the ocean. For one particular species, the critical stress level is $\tau_c = 1\,\mathrm{N\,m^{-2}}$. The water viscosity is $\mu = 1.2 \times 10^{-3}\mathrm{Pa\,s}$ and the constants are $A = -200,\ B = 0.1$. At what x-position in this water sample will the dinoflagellates start to glow?

Remembering: $\tau = \mu\frac{du_y}{dx}$, we can solve for the case τ_c.

$$u_y = -200(0.01 - x^2)$$
$$u_y = -2 + 200x^2$$
$$\frac{du_y}{dx} = 400x$$
$$1 = 1.2 \cdot 10^{-3} \cdot 400x$$
$$x = 2.08\mathrm{m}$$

1.7.5 Laminar flow in pipes

What is the velocity at different radii in the pipe? We have circular symmetry and a no-slip boundary condition. Flow is being driven by a pressure difference between two ends of the tube. Note that the velocity and the pressure gradient vector point in opposite directions because fluid flows from high to low pressure. We will assume that the flow is fully developed i.e. that the velocity profile is constant along the

pipe axis. Consider one cylinder of fluid with an edge at constant radius r. the total pipe radius is R. There is a pressure difference between the two ends of the tube. The total pressure difference is ΔP over a length L of tube. The total resultant force parallel to the tube axis must be $F \cdot A$.

$$Force = \Delta P \pi r^2$$
$$Force = \frac{dP}{dx} L \pi r^2$$

The force provided by the pressure difference must be equal to the shear stress on the outside of the cylinder, since the forces are balanced.

$$Force = 2\tau \pi r L$$

Equating these two forces:

$$\frac{dP}{dx} L \pi r^2 = 2\tau \pi r L$$
$$\tau = \frac{r}{2} \frac{dP}{dx}$$

We can relate the shear stress at the cylinder surface to the velocity change with radius there:

$$\tau = \mu \frac{du}{dr}$$
$$\tau = \mu \frac{du}{dr}$$
$$= \frac{r}{2} \frac{dP}{dx}$$
$$\frac{du}{dr} = \frac{r}{2\mu} \frac{dP}{dx}$$

Now we can integrate to u:

$$u = \int \left(\frac{r}{2\mu} \frac{dP}{dr} \right) dr$$
$$u = \frac{1}{2\mu} \frac{dP}{dx} \int (r) dr$$
$$u = \frac{1}{2\mu} \frac{dP}{dx} \frac{1}{2} r^2 + k$$

Applying the no-slip boundary condition, $u(R) = 0$:

$$0 = \frac{1}{2\mu}\frac{dP}{dx}\frac{1}{2}R^2 + k$$

$$k = \frac{-1}{4\mu}\frac{dP}{dx}R^2$$

$$\therefore u = \frac{-1}{4\mu}\frac{dP}{dx}(R^2 - r^2)$$

How can we use this? The maximum speed is at the centre of the pipe, where r = 0. We know the flow speed with radius, so we can integrate this to find the total volume flow.

$$u_{max} = \frac{-1}{4\mu}\frac{dP}{dx}R^2$$

$$Q = \int_0^R (u(r) \cdot 2\pi r)\, dr$$

$$Q = \int_0^R \left(\frac{-1}{4\mu}\frac{dP}{dt}(R^2 - r^2) \cdot 2\pi r\right) dr$$

The average flow speed is the total volume flow rate divided by the cross sectional area:

$$u_{mean} = \frac{Q}{\pi r^2}$$

Laminar flow through pipes of circular cross section is often called Hagan-Poiseulle flow.

Chapter 2

Fluids Worksheets

2.1 Introduction to Fluid Dynamics

2.1.1 Question 1

a

Let us start with the equation for viscosity:

$$\tau = \mu \frac{\partial u_y}{\partial x} \tag{2.1.1}$$

Rearranging the following equation for $\partial u_y / \partial x$ gives us:

$$\frac{\partial u_y}{\partial x} = \frac{\tau}{\mu}$$

Inputting our variables:

$$\frac{\partial u_y}{\partial x} = \frac{0.4}{5} = 0.08 \text{ s}^{-1}$$

b

We need to find the shear stress exerted by the block on the oil, found using the following equation:

$$\tau = \frac{F_t}{A} \tag{2.1.2}$$

The force that the block exerts tangentially on the oil covered inclined plane surface is given by:

$$F_t = 10 \cdot g \cdot sin(20) \text{ N}$$

The area of contact between the oil and the block is

$$A = 0.1 \text{ m}^2$$

Inputting into equation (2):

$$\tau = 100 \cdot g \cdot sin(20)$$

We are told the velocity distribution through the oil is linear and hence can make the following simplification:

$$\frac{\partial u_x}{\partial y} = \frac{u_{block}}{D}$$

$$\tau = \mu \frac{u_{block}}{D}$$

$$u_{block} = \frac{\tau D}{\mu}$$

$$u_{block} = \frac{100 \cdot 9.81 \cdot sin(20) \cdot 1 \times 10^{-4}}{5}$$

$$u_{block} = 0.0067 \text{ m s}^{-1}$$

2.1.2   Question 2

Chapter 3

Thermodynamics

3.1   Second Law of Thermodynamics

First, let us remind ourselves of the first law of thermodynamics and some of its limitations.

> Energy can neither be created or destroyed during a process; it can only change forms.

A certain energy balance will hold when a system undergoes change or a thermodynamic process.

- But does not give information on whether the change of state or the process is at all feasible or not.
- It cannot indicate whether a metallic bar of uniform temperature can spontaneously become warmer at one end and cooler at the other.
- However, if that process did occur, all that law can state is that the energy gained at one end would exactly equal the energy lost at the other.

3.1.1   Introduction to the second law

The second law of thermodynamics provides the criterion as to the *probability* of various processes.

Spontaneous processes in nature occur only in one direction.

Heat flows from a body at high temperature to a body at low temperature; water always flows downwards etc. The spontaneity of the process is due to finite driving potential, sometimes called 'force' or 'cause', e.g. a temperature or concentration gradient or an electric potential. What happens as a result of this finite driving potential is called the 'flux' or the 'current' or the 'effect' (heat transfer, mass transfer, flow of electric current). This directional law puts a limitation on energy transformation other than that imposed by the first law. The second law also asserts that energy has *quality* as well as *quantity.* The first law is concerned with the quantity of energy and the transformations of energy from one form to another, with no regard to its quality.

3.1.2   Qualitative difference between heat and work

There is also a qualitative difference between heat and work. Energy supplied as work can be completely converted to heat, e.g. paddle wheel work on a liquid in an adiabatic vessel. However, the complete conversion of heat into work is not possible, thus making heat and work not completely interchangeable forms of energy. Also, we considered the problem of a simple steam power plant and by using the Steady Flow Energy Equation (i.e. first law) and the properties of steam, were able to calculate the work done and heat transfers for individual components. However, we are not yet able to understand ways of improving steam engine efficiency.

The second law is based on experimental observation and was the result of the question, 'how efficient can one make a steam engine?' From now we will start by considering engines and define them with the precision that thermodynamics requires. The sort of steam engines we shall discuss are heat devices in boxes with no fluid entering or leaving but with just heat and work crossing the boundaries.

3.1.3   Thermodynamic cycles and thermal reservoirs

Thermodynamic cycles consist of a system, a cold reservoir and a hot reservoir. Reservoirs are regions outside a system that are so large that their intensive properties remain constant. Thermal reservoirs are bodies that exchange an infinite amount of heat with the system. The temperature of a thermal reservoir never changes. For example: Earth's atmosphere, large bodies of water, vapour condensing at a constant pressure. A *heat sink* absorbs heat energy. A *heat source* transfers energy to the system.

3.1.4   Heat engine

A heat engine (or Cyclic Heat Power Plant - CHPP) is a continuously operating thermodynamic system at the boundary of which there are heat and work transfers. Notes:

- 'Continuously operating' means that the state of the system exhibits only periodic (cyclic) changes.
- The heat engine is a thermodynamic system and so no matter crosses the boundary e.g. simple steam power plant and closed-cycle gas turbine plant.

As we know that heat transfer to work does not occur. However devices like heat engines have been created, which are special devices which are used to produce work from heat. All heat engines differ but can be characterised by the following:

- They receive heat from a high temperature source (solar energy, oil furnace, nuclear reactor, etc.)
- They convert part of this heat to work (usually in the form of a rotating shaft.)
- They reject the remaining heat to a low temperature sink (the atmosphere, body of water, etc.)
- They operate on a cycle.

Diesel engines (Figure 3) (and internal combustion engines generally) are not heat engines (CHPP) because matter crosses its boundaries. A jet engine is also not a CHPP because matter, air, fuel and exhaust cross the boundary of the system.

3.1.5 Steam power plant

The work producing device that best suits this definition of a heat engine is the steam power plant, which is an external combustion engine.

1. Combustion takes place outside.
2. Transferred to steam as heat.
3. Passed through various devices that transfer its energy to work.

Figure 4 shows a simplified diagram of a steam plant. From this we can see that $W_{\text{net out}} = W_{\text{out}} - W_{\text{in}}$ (kJ). Considering that the change in internal energy is zero for cycles, we can derive, $W_{\text{net out}} = Q_{\text{in}} - Q_{\text{out}}$ (kJ).

3.1.6 Thermal efficiency of direct engines/steam power plants

For heat engines, the desired output is the net work output and the required input is the amount of heat supplied to the working fluid. Thus the thermal efficiency is expressed as:

$$\text{Thermal efficiency} = \frac{\text{Net work output}}{\text{Total heat input}}$$

$$\eta = \frac{W_{\text{net out}}}{Q_{\text{in}}}$$

$$\eta = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}}$$

Since cyclic devices at practical interest operate between a high temperature T_H and a low temperature medium T_L, we define these two quantities:

- Q_H = magnitude of heat transfer between the cyclic device and the high temperature medium T_H.

- Q_L = magnitude of heat transfer between the cyclic device and the low temperature medium T_L.

Thus, the previous equations can be written as follows:

$$W_{\text{net out}} = Q_H - Q_L$$

$$\eta = 1 - \frac{Q_L}{Q_H}$$

Thermal efficiencies of work producing devices are relatively low. Ordinary spark ignition automobile engines have a thermal efficiency of about 25%. Gas steam power plants reach only 60%.

3.1.7 Can we save Q_{out}?

In a steam power plant, the condenser is the device where large quantities of waste heat is rejected to rivers, lakes or the atmosphere. One may ask, can we not save all this waste energy? The answer is a firm *no*. Without a heat rejection process in the condenser, the cycle cannot be completed. Cyclic devices such as steam power plants cannot run continuously, unless the cycle is completed.

3.1.8 The Kelvin-Planck statement

The Kelvin-Planck statement of the second law of thermodynamics is expressed as follows:

> It is impossible for any device that operates on a cycle to receive heat from a single reservoir and produce a net amount of work.

A heat engine must exchange heat with a low temperature sink as well as a high temperature source to keep operating. This implies that it is impossible to build a heat engine that has a thermal efficiency of 100%. Complete conversion of heat into work is *not possible*. This is contrary to the fact that 100% of work can be transferred to heat. This is the directional implication of the second law. This is not due to frictional/non-adiabatic effects. It is a *necessity*.

Proof on page 128 of Cengel (insert edition.)

3.1.9 Refrigerators and heat pumps

From experience, heat always from from high temperature to low temperature. These reverse process, however, cannot occur by itself. This reverse process re-

quires special devices called refrigerators. Refrigerators are cyclic devices and the working fluid is called a refrigerant. A frequently used refrigeration cycle is the 'vapour-compression refrigeration cycle.' This involves a compressor, a condenser, an expansion valve and an evaporator.

3.1.10 Coefficient of performance

The efficiency of a refrigerator is called the coefficient of performance (COP), denoted by COP_R. The objective of a refrigerator is to: remove heat from the refrigerated space by being provided with a work input $W_{\mathrm{net\ in}}$. Thus:

$$\mathrm{COP}_R = \frac{Q_L}{W_{\mathrm{net\ in}}} = \frac{\dot{Q}_L}{\dot{W}_{\mathrm{net\ in}}}$$

Also, since $W_{\mathrm{net\ in}} = Q_{\mathrm{in}} - Q_L$

$$\mathrm{COP}_R = \frac{Q_L}{Q_H - Q_L}$$

$$\mathrm{COP}_R = \frac{1}{\frac{Q_H}{Q_L} - 1}$$

COP can also be greater than unity. This means the heat removed can be greater than the work input. Whereas thermal efficiency can never be greater than 1.

3.1.11 Heat pumps

ANother device that transfers heat from a low temperature medium to a high temperature one is a heat pump. Here are some differences between refrigerators and heat pumps.

- Refrigerators:
    - Maintain the refrigerated space at a low temperature by removing heat from it.
    - This extracted heat is then discharged to a high temperature medium out of necessity.
- Heat pumps:
    - Maintain a heated spaced at a high temperature.

– This is accomplished by absorbing heat from a low temperature source e.g. well, water, cold air and then supplying this heat to the high temperature medium such as a house.

A heat pump runs on the same cycle as a refrigerator. The measure of performance of a heat pump is also expressed in terms of coefficient of performance COP_{HP}.

$$\text{COP}_{HP} = \frac{Q_H}{W_{\text{net in}}} = \frac{Q_H}{Q_H - Q_L} = \frac{1}{1 - \frac{Q_L}{Q_H}}$$

$$\text{COP}_{HP} = \text{COP}_R + 1$$

This relation shows us that $\text{COP}_{HP} > \text{COP}_R$ at all times as COP_R is always positive.

3.1.12 Air conditioners

Air conditioners are basically refrigerators where refrigerated space is a room instead of a food compartment. The same air conditioning unit can be used as a heat pump in winter by installing it backwards. In this mode, the unit absorbs heat from the cold outside and delivers it to the room.

3.1.13 The Clausius statement

The Kelvin-Planck statement was for direct heat engines. The Clausius statement is for reversed heat engines. The statement is as follows:

It is impossible to construct a device that operates in a cycle and produces no effect other than the transfer of heat from a low temperature body to a higher temperature body.

This means that it is impossible to construct a refrigerator that operates without an input of work. $W_{\text{net in}} \neq 0 \therefore \text{COP}_R \neq 0$.

3.1.14 Reversible and irreversible processes

As a result of the second law we know that the complete conversion of heat to work is impossible. Thus, an efficiency of 100% cannot be achieved for a heat engine. Then what is the maximum efficiency of a heat engine/CHPP? Before we answer this question, we need to define an idealised process first, which is called a reversible process. Reversible process:

A process that can be reversed without having any trace to the surroundings. Both the system and the surroundings are returned to their initial states ath the end of the reverse process. Thus, for the combined forward and reverse processes the *net heat* and *net work* transfer is *zero*.

Reversible processes do not occur in nature naturally. They are merely an idealisation of real processes. They can be approximated by actual devices but can never be achieved. We consider theme even though they are impossible, as they are easy to analyse and they can be used for comparison.

3.1.15 Irreversibilities

The factors that cause a process to be irreversible are called irreversibilities. They include:

- Friction.
- Unrestrained expansion.
- Mixing of two fluids.
- Heat transfer across a finite temperature difference.
- Electrical resistance.
- Inelastic deformation of solids.
- Chemical reactions.

A reversible process includes none of these. There are three types of reversible process:

- Externally reversible.

- Internally reversible.
- Totally reversible.

Externally reversible

No irreversibilities exist in the surroundings. Heat transfer can occur between the system and the surroundings but only with an infinitesimal temperature difference. There may still be irreversibilities with the system.

Internally reversible

No irreversibilities exist within the system. The system moves slowly and without friction through a series of equilibrium states. Irreversibilities may exist in the surroundings usually due to heat transfer through a finite temperature difference.

Totally reversible

A process is called totally reversible, or simply reversible, if it involves no irreversibilities within the system or its surroundings. A totally reversible process involves no heat transfer through a finite temperature difference, no non quasi-equilibrium changes and no friction or other dissipative effects.

3.2 The Carnot cycle

A quick recap of the content covered so far.

Heat engines are cyclic devices. The working fluid returns back to its original state at the end of each cycle. In part one (of the cycle), there is work done by the fluid and in part two, work is done on the working fluid. This difference between these two parts is the net work delivered by the engine. Cycle efficiency can be maximised by using processes that require the least amount of work. This is achieved by using *reversible processes*. Reversible cycles cannot be achieved in practice because the irreversibilities associated with each process cannot be eliminated. However, they provide upper limits to the performance of real cycles. The most famous reversible cycle is the Carnot cycle. A theoretical heat engine that operates on a Carnot cycle is called a Carnot heat engine. It is composed of four reversible processes: two

isothermal and two adiabatic. It can be executed either in a closed system or a steady flow system.

Consider a system that consists of a gas contained in an adiabatic piston-cylinder device. The insulation of the cylinder head is such that it may be removed to bring the cylinder into contact with reservoirs to provide heat transfer. The four reversible processes that make up the Carnot cycle are as follows:

A: Reversible isothermal expansion (Process 1-2, constant T_H)

1. The gas is at temperature T_H.
2. The cylinder head is in close contact with a source at temperature T_H.
3. The gas is allowed to expand slowly and does work on the surrounding.
4. But as the temperature drops by dT, some heat is transferred from the source to the gas, raising the temperature back to T_H. Thus, the temperature remains constant at T_H.

Since the temperature difference never exceeds dT, it is a reversible heat transfer process. It continues until reaching position 2. The total amount of heat transferred to the gas during this process is Q_H.

B: Reversible adiabatic expansion (Process 2-3, temperature drops from $T_H \rightarrow T_L$)

1. The heat source in A is removed and the insulation is added, so the system becomes adiabatic.
2. The gas expands slowly, doing work on the surroundings until its temperature drops from T_H to T_L.

The piston is assumed to be frictionless and the process to be quasistatic equilibrium, so the process is reversible and adiabatic.

C: Reversible isothermal compression (Process 3-4, $T_L = $ constant)

1. Insulation is removed and replaced with a low temperature sink at temp T_L.
2. The piston is pushed inwards by an external force, doing work on the gas.
3. As the gas is compressed, its temperature tends to rise.
4. If it rises by dT, it drops back to T_L as heat is transferred from the gas to the sink.

5. Thus, temperature remains constant at T_L.

This is a reversible heat transfer process as the temperature difference never exceeds dT. The amount of heat rejected is Q_L.

D: Reversible adiabatic compression (Process 4-1, temperature rises from $T_L \to T_H$)

1. Insulation is added back.
2. Gas is compressed.
3. The gas returns to its original state.
4. Temperature rises from T_L to T_H which completes the cycle.

All of the steps of the Carnot cycle can be summarised on a P-V graph. Being a reversible cycle, the Carnot cycle is the most efficient cycle operating between two specified temperature limits.

3.2.1 Reversed Carnot cycle

The Carnot cycle is totally reversible. Thus, if each process is reversed, it becomes the Carnot refrigeration cycle. Heat and work interactions are reversed.

3.2.2 The Carnot principles

The second law of thermodynamics puts limits on the operation of cyclic devices as expressed by the Kelvin-Planck and Clausius statements. A heat engine cannot operate by exchanging heat with a single reservoir and a refrigerator cannot operate without a net energy input from an external source. These bring rise to the Carnot principles:

1. The efficiency of an irreversible heat engine is always less than the efficiency of a reversible one operating between the same two reservoirs.
2. The efficiency of all reversible heat engines operating between the same two reservoirs are the same.

Proof for statement 1 (of Carnot principles)

insert proof

3.2.3 The thermodynamic temperature cycle

insert proof

$$\left(\frac{Q_M}{Q_L}\right)_{rev} = \frac{T_H}{T_L} \leftarrow \text{use Kelvin}$$

3.2.4 The Carnot heat engines

Any heat engine has a thermal efficiency of η_{th} given by the formula

$$\eta_{th} = 1 - \frac{Q_L}{Q_H}$$

For reversible heat engines, such as a Carnot engine, this becomes:

$$\eta_{th,\ rev} = 1 - \frac{T_L}{T_H}$$

Using the formula shown in the previous part. This us called the Carnot efficiency, since the Carnot heat engine is the best known reversible engine. This is the highest efficiency a heat engine operating between two thermal energy reservoirs at temperature T_H and T_L can have. If you calculate the η_{th} of a heat engine, their nature can be deduced as follows:

$$\eta_{th} < \eta_{th,\ rev} \rightarrow \text{Irreversible heat engine}$$

$$\eta_{th} = \eta_{th,\ rev} \rightarrow \text{Reversible heat engine}$$

$$\eta_{th} > \eta_{th,\ rev} \rightarrow \text{Impossible heat engine}$$

From the equation $\eta_{th,\ rev} = 1 - \frac{T_L}{T_H}$, we can see that the efficiency of a Carnot heat engine increases as T_H is increased, or as T_L is decreased. As T_L decreases so does the amount of heat rejected. The same is also true for actual heat engines. The thermal efficiency of actual heat engines can be maximised by:

- Supplying heat to the engine at the highest possible temperature (limited by material strength).

- And rejecting heat from the engine at the lowest possible temperature (limited by the temperature of the cooling medium such as rivers, lakes or the atmosphere).

3.2.5 The Carnot refrigerator and heat pump

The COP of any heat pump and refrigerator is given by:

$$\text{COP}_R = \frac{1}{\frac{Q_H}{Q_L} - 1}$$

$$\text{COP}_{HP} = \frac{1}{1 - \frac{Q_L}{Q_H}}$$

Using the thermodynamic scale formulas, the COP for heat pumps and refrigerators running on a reverse Carnot cycle is:

$$\text{COP}_{R,\ rev} = \frac{1}{\frac{T_H}{T_L} - 1}$$

$$\text{COP}_{HP,\ rev} = \frac{1}{1 - \frac{T_L}{T_H}}$$

These are the highest COP that a refrigerator or a heat pump operating between the temperature limits at T_L and T_H can have.

$$\text{COP}_R = \begin{cases} < \text{COP}_{R,\ rev} \\ = \text{COP}_{R,\ rev} \\ > \text{COP}_{R,\ rev} \end{cases} \quad \& \quad \text{COP}_{HP} = \begin{cases} < \text{COP}_{HP,\ rev} \rightarrow \text{Irreversible} \\ = \text{COP}_{HP,\ rev} \rightarrow \text{Reversible} \\ > \text{COP}_{HP,\ rev} \rightarrow \text{Impossible} \end{cases}$$
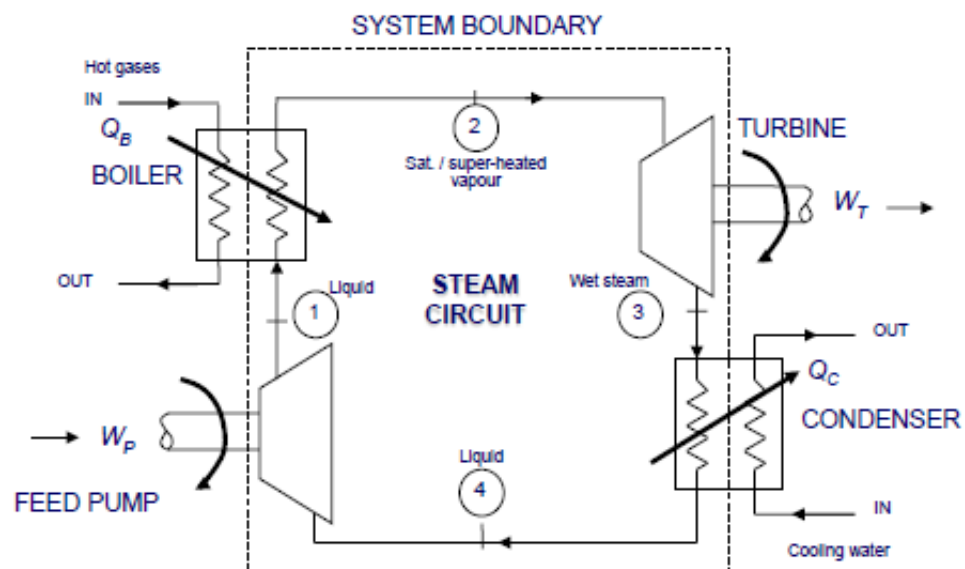
3.3   Figures
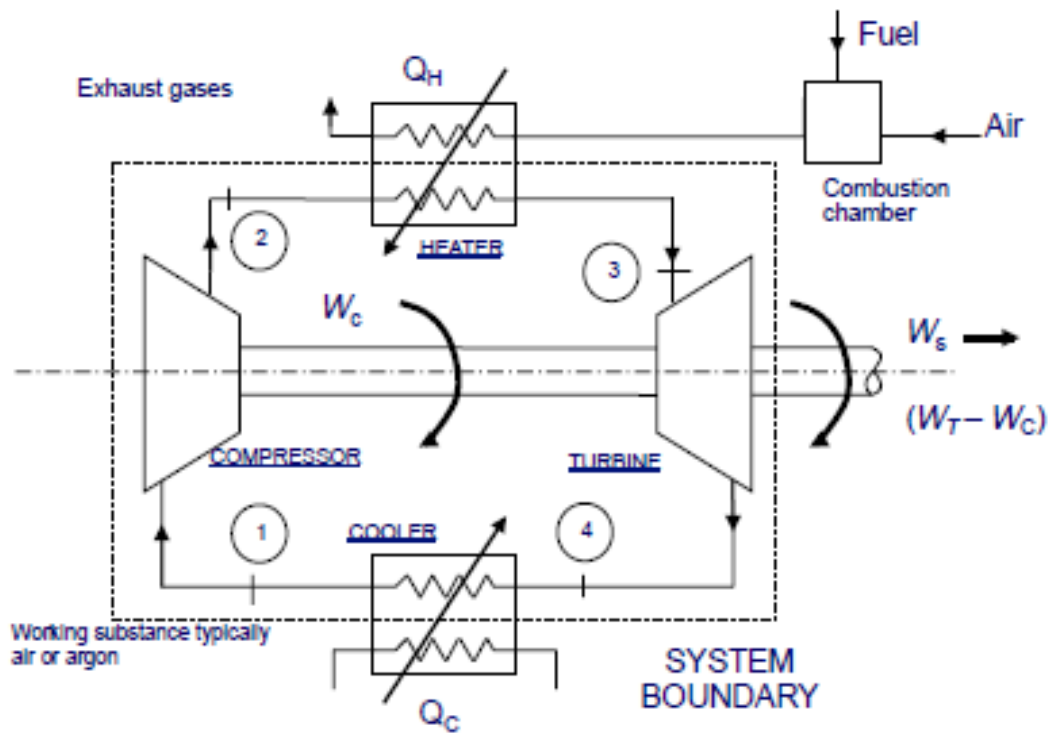

Figure 3.1: A simple steam power plant

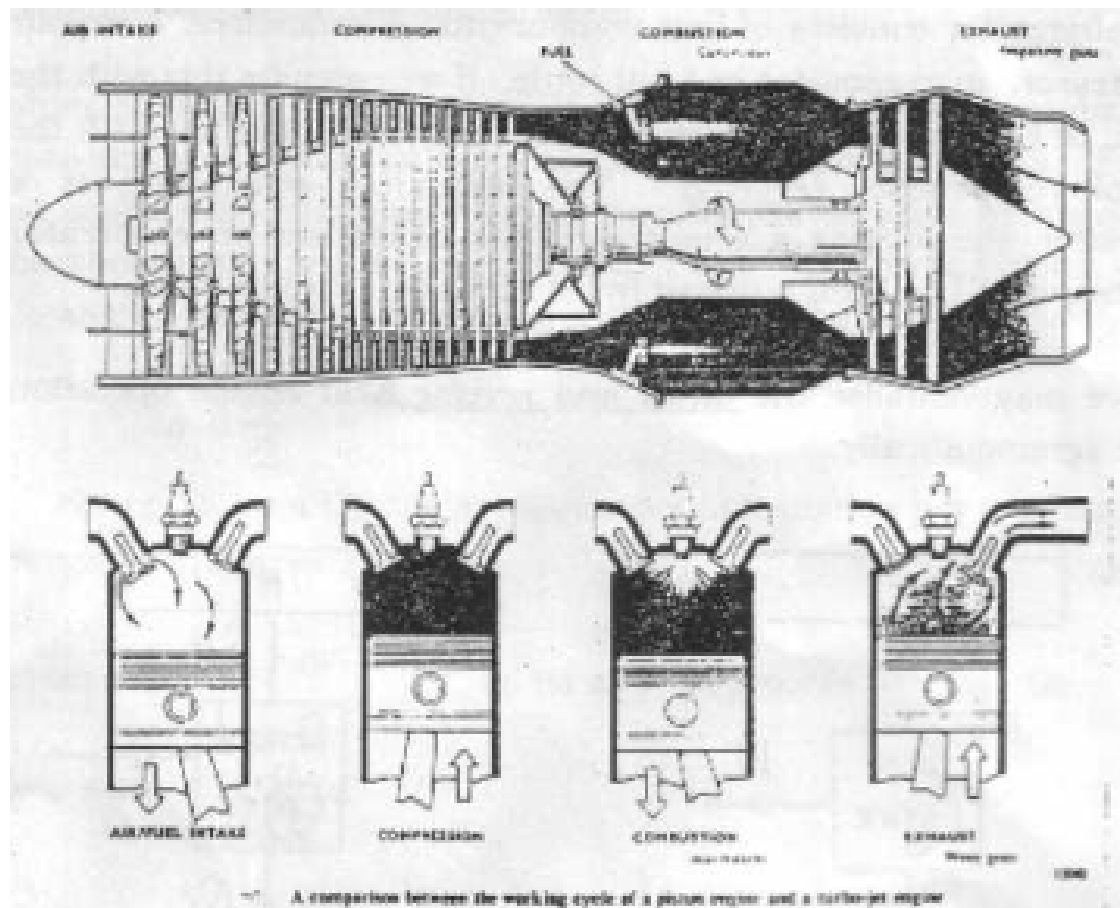Figure 3.2: A closed cycle gas power plant

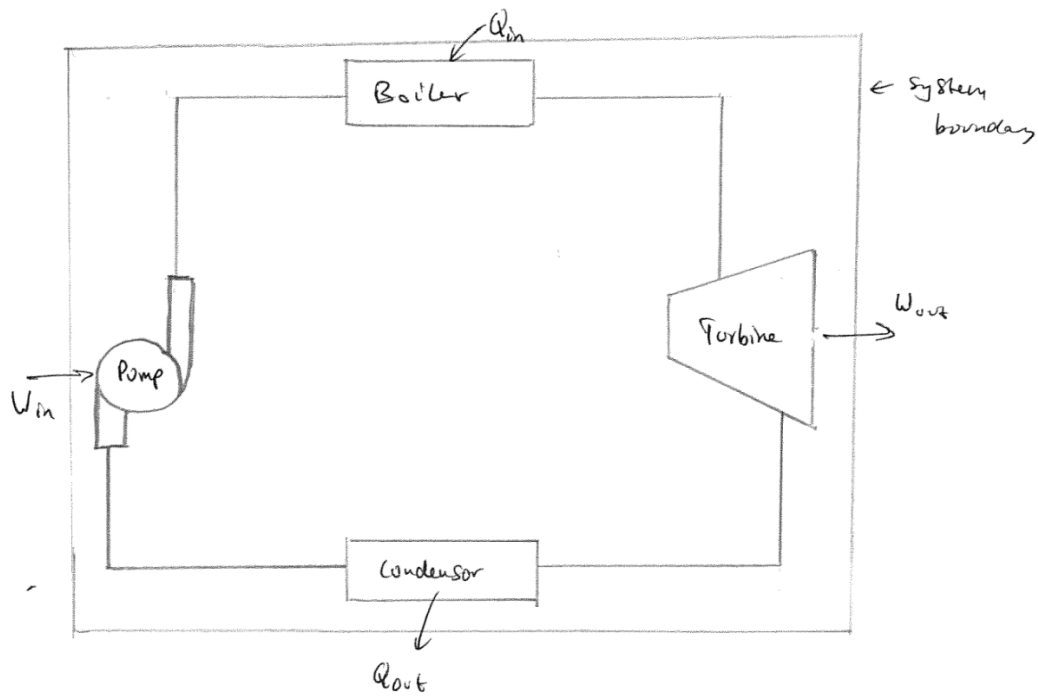Figure 3.3: A diesel engine

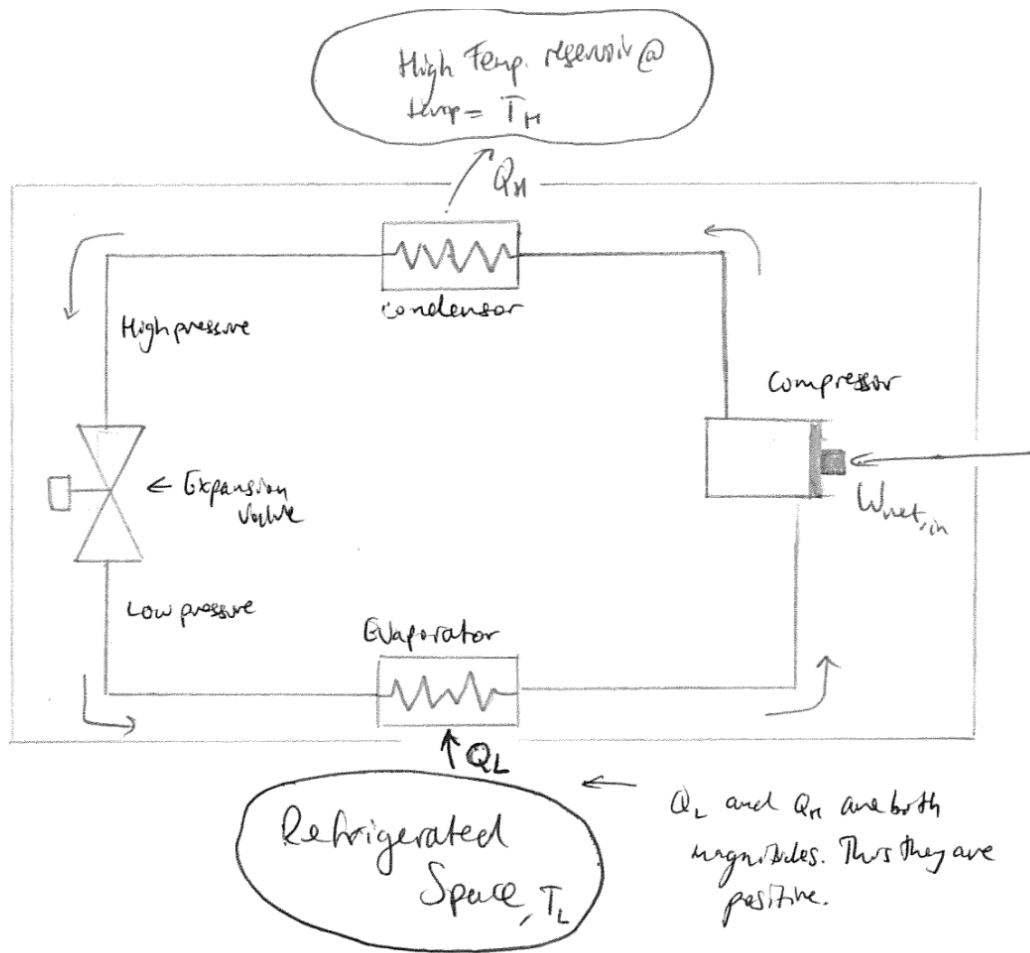Figure 3.4: Steam power plant operation
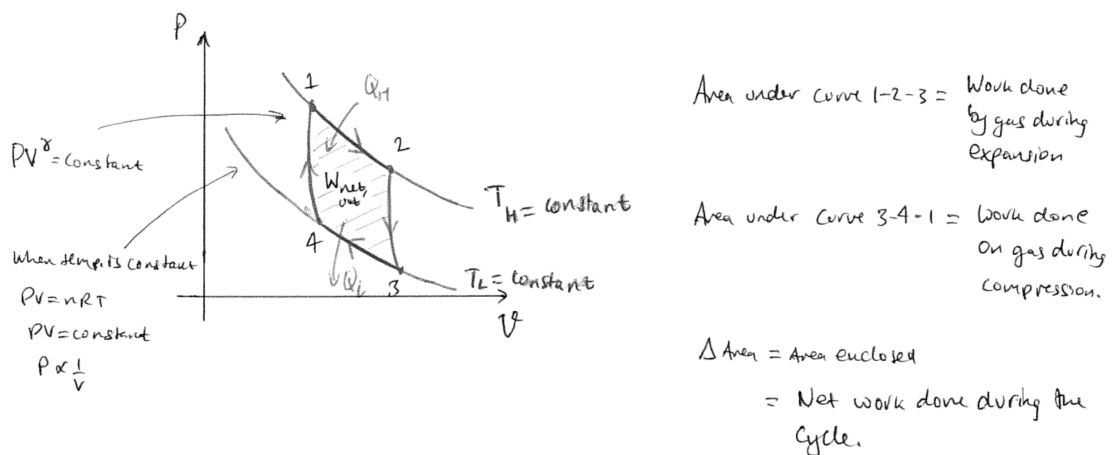
Figure 3.5: Refrigeration cycle
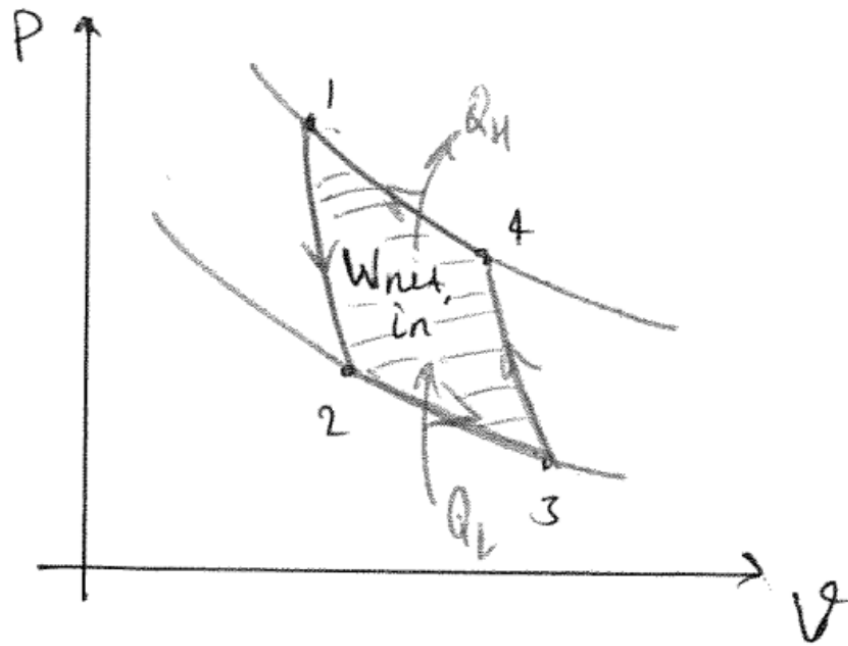

Figure 3.6: Carnot cycle

Figure 3.7: Reversed Carnot cycle

3.4 Entropy

An important inequality that has major consequences in thermodynamics is the *Clausius inequality*.

$$\oint \frac{\delta Q}{T} \geq 0 \tag{3.4.1}$$

The cyclic integral of $\frac{\delta Q}{T}$ is always less than or equal to zero. This inequality is valid for all cycles reversible or irreversible. $\frac{\delta Q}{T}$ is the sum of all differential amounts of heat transfer to or from a system, divided by the temperature of the boundary.

3.4.1 Proof of the Clausius inequality

INSERT PROOF

3.4.2 The increase of entropy principle

Consider a cycle. It has two processes:

- Process 1-2: could be reversible or irreversible.
- Process 2-1L Internally reversible.

The Clausius inequality states:

$$\oint \frac{\delta Q}{T} \geq 0 \tag{3.4.2}$$

Or,

$$\int_1^2 \frac{\delta Q}{T} + \int_1^2 \left(\frac{\delta Q}{T}\right)_{\text{int rev}} \geq 0 \tag{3.4.3}$$

$$\int_1^2 \frac{\delta Q}{T} + (S_1 - S_2) \geq 0 \tag{3.4.4}$$

$$S_2 - S_1 \leq \int_1^2 \frac{\delta Q}{T} \tag{3.4.5}$$

When written in the differential form:

$$ds \leq \frac{\delta Q}{T} \tag{3.4.6}$$

Where T is the thermodynamic temperature at the boundary. δQ is the heat transferred between the system and surroundings. ds is the differential change in energy. When reversible $ds = \frac{\delta Q}{T}$. When irreversible $ds \leq \frac{\delta Q}{T}$. This equation shows that:

Change in entropy of a closed system during an irreversible process is *always greater* than the integral of $\frac{SQ}{T}$ evaluated for that process.

This inequality shows that during an irreversible process, the entropy change of a closed system is *always greater* than the entropy of transfer by heat. For a reversible process, $\Delta S = \int_1^2 \frac{\delta Q}{T}$, which represents the entropy transfer by heat. Thus, some *entropy generation* must be occurring. This generated entropy is denoted as S_{gen}.

$$\therefore S_2 - S_1 \geq \int_1^2 \frac{\delta Q}{T} \tag{3.4.7}$$

$$\therefore \Delta S_{\text{sys}} \geq \int_1^2 \frac{\delta Q}{T} \tag{3.4.8}$$

$$\therefore \Delta S_{\text{sys}} = \int_1^2 \frac{\delta Q}{T} + S_{\text{gen}} \tag{3.4.9}$$

S_{gen} is always positive or zero. This depends upon the process, hence not a property.

When a system is isolated, adiabatic and closed. The heat transfer is zero i.e.

$SQ = 0.$

$$\therefore \Delta S_{\text{sys}} = \Delta S_{\text{isolated}} \geq \int_1^2 0 \tag{3.4.10}$$

$$\therefore \Delta S_{\text{isolated}} \geq 0 \Delta S_{\text{isolated}} = S_{\text{gen}} \tag{3.4.11}$$

This is the increase of entropy principle. The entropy of an isolated system during a process always increases or in the limiting case of a reversible process, remains constant. Note that in the absence of any heat transfer, entropy change is due to irreversibilities only and their effect is only and always an increase in entropy.

Entropy is an extensive property. Thus,

$$\sum_{i=1}^{N} \Delta S_i = \Delta S_{\text{total}} \tag{3.4.12}$$

i.e. the use of the change in entropies of the parts of the system is equal to the change in entropy of a system. An isolated system can consist of a number of subsystems. A system and its surroundings, for example, constitute an isolated system since both can be enclosed by a sufficiently large arbitrary boundary at which there is no heat, work or mass transfer. Thus, a system and its surroundings can be considered as two subsystems of an isolated system. Since $\Delta S_{\text{isolated}} = S_{\text{gen}} \geq 0$ but $\Delta S_{\text{isolated}} = \Delta S_{\text{total}} = \Delta S_{\text{sys}} + \Delta S_{\text{surr}}$. Therefore:

$$S_{\text{gen}} = \Delta S_{\text{total}} = \Delta S_{\text{sys}} + \Delta S_{\text{surr}} \geq 0 \tag{3.4.13}$$

When considering a reversible process. Then $\Delta S_{\text{sys}} + \Delta S_{\text{surr}} = 0 = S_{\text{gen}}$. The inequality holds for irreversible process. Note that S_{gen} can never be negative, but ΔS_{sys} can be negative and ΔS_{surr} can be negative. The sum of the two must be positive or zero. For example, consider the following system.

$$S_{\text{gen}} = \Delta S_{\text{total}} \tag{3.4.14}$$

$$= \Delta S_{\text{sys}} + \Delta S_{\text{surr}} \tag{3.4.15}$$

$$= -2 + 3 \tag{3.4.16}$$

$$S_{\text{gen}} = 1 \text{ kJ K}^{-1} \tag{3.4.17}$$

$$S_{\text{gen}} = \begin{cases} > 0 \text{ Irreversible process} \\ = 0 \text{ Reversible process} \\ < 0 \text{ Impossible process} \end{cases} \tag{3.4.18}$$

Example question

Using the increase in entropy principle, show that conduction with finite temperature difference is an irreversible process.

$$S_{\text{gen}} = \Delta S_{\text{total}} = \Delta S_{\text{high temp. reservoir}} + \Delta S_{\text{low temp. reservoir}} \geq 0 \tag{3.4.19}$$

$$\Delta S_{\text{total}} = \int \left(\frac{-SQ}{T_H} \right) + \int \frac{SQ}{T_L} \tag{3.4.20}$$

$$= -\frac{1}{T_H} \int SQ + \frac{1}{T_L} \int SQ \tag{3.4.21}$$

$$-\frac{Q}{T_H} + \frac{Q}{T_L} \geq 0 \tag{3.4.22}$$

$$S_{\text{gen}} = \frac{-Q}{T_H} + \frac{Q}{T_L} \geq 0 \tag{3.4.23}$$

$$\text{since } T_H > T_L \tag{3.4.24}$$

$$\therefore S_{\text{gen}} > 0 \rightarrow \text{irreversible process} \tag{3.4.25}$$

3.4.3 Entropy change of pure substances

The entropy of a pure substance us determined from tables in the same manner as other properties such as volume, internal energy and enthalpy. The value of entropy at a specified state is determined just like any other property.

- Compressed liquids $\rightarrow$ from tables (if available).
- Saturated liquid S_P: from tables.
- Saturated mixture: $S = S_P + xS_{fg}$.
- Saturated vapour S_g: from tables.
- Superheated vapour: from tables.

If the compressed liquid data is not available, the entropy of the compressed liquid can be approximated by the saturated liquid at the same temperature.

$$S_{\text{compressed @ T, P}} = S_{f@T} \tag{3.4.26}$$

Note that $\Delta S = m\Delta S = m(S_2 - S_1)$.

3.4.4   T-S diagram characteristics

Constant volume lines are steeper than the constant pressure lines. Constant pressure lines are parallel to the constant temperature lines in the saturated liquid-vapour mixture region. Constant pressure lines almost coincide with the saturated liquid line in the compressed liquid region.

3.4.5   Isentropic processes

Entropy can be changed by: heat transfer or irreversibilities. During a process that is internally reversible and adiabatic, the entropy of a fixed mass does not change.

Entropy remains constant = isentropic process $\rightarrow \Delta S = 0$ or $S_2 = S_1$

A reversible adiabatic process $\rightarrow$ isentropic

However, isentropic $\rightarrow$ reversible adiabatic (Which usually means an internally reversible, adiabatic process)

3.4.6   Property diagrams involving entropy

Two diagrams commonly used in the second-law analysis are the temperature - entropy and the enthalpy - entropy diagrams. Remember:

$$\frac{SQ_{\text{int rev}}}{T} = dS \tag{3.4.27}$$

$$\therefore SQ_{\text{int rev}} = TdS \tag{3.4.28}$$

$$\therefore Q_{\text{int rev}} = \int_1^2 TdS \ (\text{kJ}) \tag{3.4.29}$$

This integral computes the area under a process curve on a T-S diagram. This represents the heat transfer during an internally reversible process.

Area = heat transfer for processes that are internally/totally reversible. (This area has no meaning for irreversible processes.)
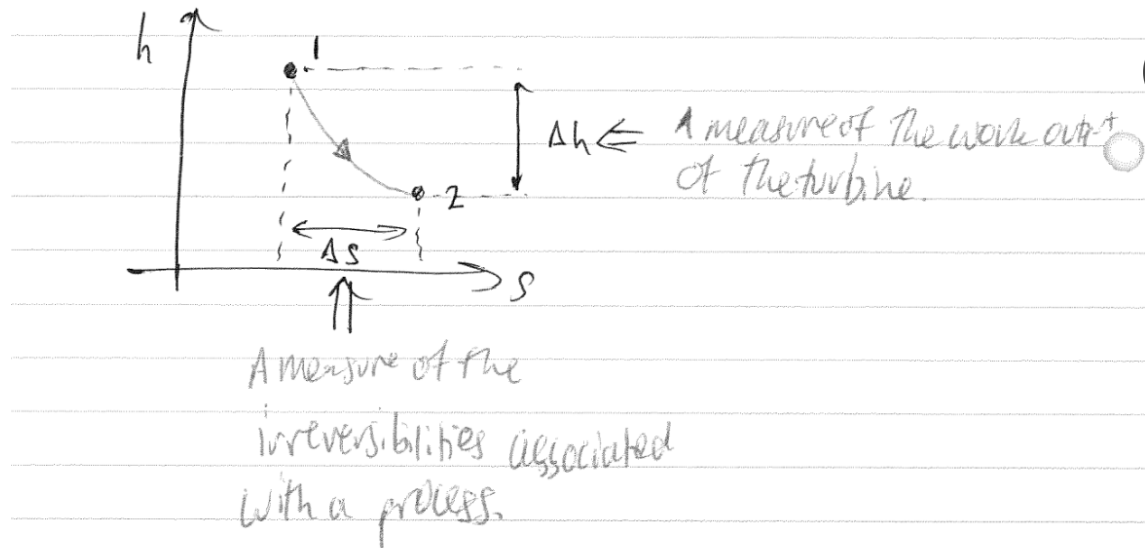
Figure 3.8: Mollier Diagram

It can also be written in specific form

$$q_{\text{int rev}} = \int_1^2 TdS \ (\text{kJ kg}^{-1})$$ (3.4.30)

$$sq_{\text{int rev}} = Tds \ (\text{kJ kg}^{-1})$$ (3.4.31)

For internally reversible isothermal processes, T is constant: T_0 therefore,

$$Q_{\text{int rev isothermal}} = T_0 \Delta S \ (\text{kJ})$$ (3.4.32)

$$q_{\text{int rev isothermal}} = T_0 \Delta s \ (\text{kJ kg}^{-1})$$ (3.4.33)

An isentropic process on a T-S diagram is easily recognised as a *vertical* line segment. Isentropic means no heat transfer, thus area under the graph must be zero. Another diagram commonly used in engineering is the enthalpy - entropy diagram, which is quite valuable in the analysis of steady flow devices such as turbines. compressors and nozzles. In analysing the steady flow of steam through an adiabatic turbine for example, the graph produced is show below. This graph is also called the Mollier Diagram.

3.4.7 T-S diagram of the Carnot cycle

The Carnot cycle is made up of two, reversible isothermal CT = constant S processes and two isentropic (adiabatic and S = constant) processes. These four processes form a rectangle on the T-S diagram.

Area $A\ 1\ 2B = Q_H$

$A\ 4\ 3\ B = Q_L$
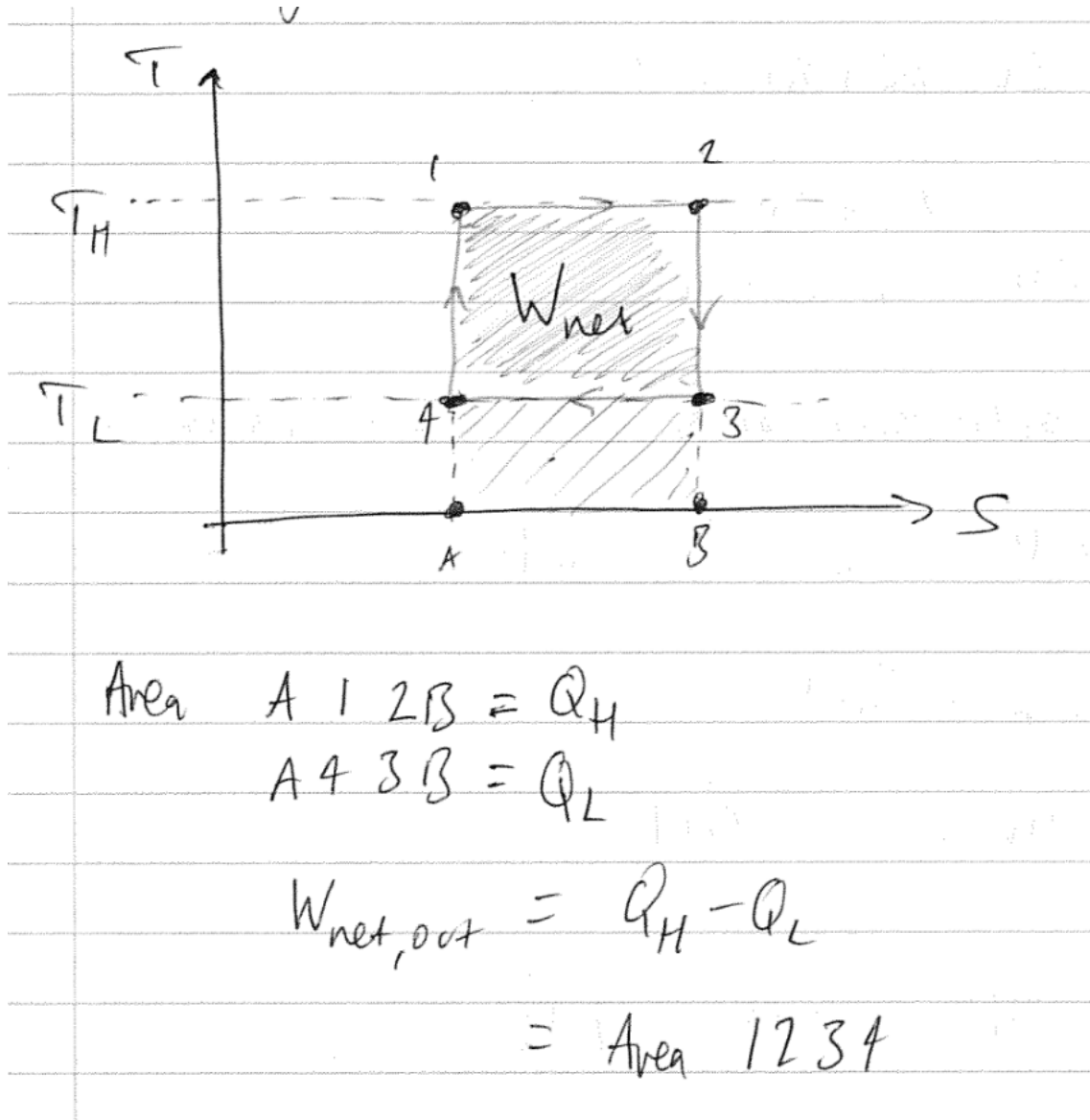
$W_{net,out} = Q_H - Q_L$

$= $ Area $1\ 2\ 3\ 4$

Figure 3.9: T-S diagram for the Carnot cycle

3.4.8 The Tds relations

The first law states:
$$\Delta E = \Delta U + \Delta KE + \Delta PE \tag{3.4.34}$$

For closed systems that are stationary, $\Delta KE = \Delta PE = 0$ therefore,

$$\Delta E = \Delta U \tag{3.4.35}$$

$$\Delta U = \Delta Q + \Delta W (+\Delta E_{mass} = 0) \tag{3.4.36}$$

$$\Delta U = \Delta Q + \Delta W \tag{3.4.37}$$

$$\Delta U = Q_{\text{net in}} - W_{\text{net out}} \tag{3.4.38}$$

In differential form and for: a simple compressible substance, an internally reversible process,
$$dU = SQ_{\text{int rev}} - SW_{\text{int rev out}} \tag{3.4.39}$$

But $SQ_{\text{int rev}} = TdS$ and $SW_{\text{int rev out}} = PdV$ thus,

$$dU = TdS - PdV \tag{3.4.40}$$

$$TdS = dU + PdV \text{ (kJ)} \tag{3.4.41}$$

$$Tds = du + Pdv \text{ (kJ kg}^{-1}) \tag{3.4.42}$$

These are known as the first Tds relations.

Since $h = u + Pv$, differentiating both sides yields:

$$dh = d[u + Pv] \tag{3.4.43}$$

$$= du + d[Pv] \tag{3.4.44}$$

$$dh = du + Pdv + dPv \tag{3.4.45}$$

But the first Tds relation states that $Tds = du + Pdv$ therefore,

$$du = Tds - Pdv \tag{3.4.46}$$

$$\therefore dh = Tds + vdP \tag{3.4.47}$$

$$Tds = dh - vdP \tag{3.4.48}$$

This is the second Tds relation.

To summarise, the Tds relations are shown below.

1. Tds = dh - vdP

2. Tds = du + Pdv

These equations were derived with internally reversible processes but since the entropy change between two states is independent of the process the system undergoes, hence the equations are valid for both irreversible and reversible processes. These are also valid for the change of states of a substance in a closed or an open system. Explicit relations for differential changes in entropy are obtained by solving for ds in the Tds relations.

$$ds = \frac{du}{T} + \frac{Pdv}{t} \text{ and} \tag{3.4.49}$$

$$ds = \frac{dh}{T} - \frac{vdP}{T} \tag{3.4.50}$$

ΔS can be found by *integrating* these equations between final and initial states. To perform these integrations we must know the relationship between du or dh and temperature. For an ideal gas:

$$du = c_V dT, \ dh = c_P dt \tag{3.4.51}$$

The equation of state of the substance is given by:

$$Pv = RT(\text{for ideal gases}) \tag{3.4.52}$$

For other substances, we have to rely on tabulated data.

3.4.9   Entropy change of liquids and solids

$$ds = \frac{du}{T} + \frac{Pdv}{t} \tag{3.4.53}$$

But for liquids and solids, they can be approximated as incompressible substances since their specific volumes remain nearly constant during a process. Thus $dv \approx 0$ for liquids and solids.

$$\therefore ds = \frac{du}{t} \tag{3.4.54}$$

Performing the integration gives:

$$\Delta s = s_2 - s_1 = \int_1^2 ds = \int_1^2 \frac{du}{T} \tag{3.4.55}$$

But $du = cdT$ where $c = c_P = c_V$ for liquids and solids.

$$\Delta s = s_2 - s_1 = \int_1^2 \frac{cdT}{T} = \int_1^2 \frac{c(T)dt}{T} \tag{3.4.56}$$

$$\approx c_{avg}ln(\frac{T_2}{T_1}) \tag{3.4.57}$$

$$\Delta s = c_{avg}ln(\frac{T_2}{T_1}) \text{ kJ kg}^{-1} \tag{3.4.58}$$

Where c_{avg} is the average specific heat capacity over the interval. Note that the entropy change of a truly incompressible substance depends only on temperature.

The equation $s_2 - s_1 \approx c_{avg}ln(\frac{T_2}{T_1})$ can be used with liquids and solids to reasonable accuracy (liquids and solids only). However, for liquids that expand considerablly with temperature, it may be necessary to consider the effects of volume change in calculcations. This is especially the case when the temperature change is large. Applying this to isentropic processes; for liquids and solids only (incompressible):

$$s_2 - s_1 = 0 = C_{avg}ln(\frac{T_2}{T_1}) \tag{3.4.59}$$

$$ln(T_2) - ln(T_1) = 0 \tag{3.4.60}$$

$$ln(T_2) = ln(T_1) \tag{3.4.61}$$

$$T_2 = T_1 \tag{3.4.62}$$

Hence, for truly incompressible substances that undergo an isentropic process, it is also isothermal.

$$\text{incompressible + isentropic} \rightarrow \text{isothermal}$$

This behaviour is closesly operate by liquids and solids.

3.4.10   Entropy change of ideal gases

Remember that
$$ds = \frac{dh}{T} - \frac{vdP}{T} \tag{3.4.63}$$

But for ideal gases:

$$dh = c_P dT \tag{3.4.64}$$

$$\therefore ds = \frac{c_P dT}{T} - \frac{vdP}{T} \tag{3.4.65}$$

But the ideal gas relation gives:

$$Pv = RT \tag{3.4.66}$$

$$v = \frac{RT}{P} \tag{3.4.67}$$

$$\therefore ds = \frac{C_P dT}{T} - \frac{RT dP}{PT} \tag{3.4.68}$$

$$ds = \frac{C_P dT}{T} - \frac{R dP}{P} \tag{3.4.69}$$

Integrating both sides gives:

$$Deltas = s_2 - s_1 = \int_1^2 c_P(T)\frac{dT}{T} - Rln\left(\frac{P_2}{P_1}\right) \tag{3.4.70}$$

Also $ds = \frac{du}{T} + \frac{Pdv}{T}$ But for ideal gases:

$$du = c_V dT \tag{3.4.71}$$

$$\therefore ds = \frac{c_V dT}{T} - \frac{Pdv}{T} \tag{3.4.72}$$

But the ideal gas relation gives:

$$Pv = RT \tag{3.4.73}$$

$$P = \frac{RT}{v} \tag{3.4.74}$$

$$\therefore ds = \frac{C_V dT}{T} - \frac{RT dv}{vT} \tag{3.4.75}$$

$$ds = \frac{C_V dT}{T} - \frac{R dv}{v} \tag{3.4.76}$$

Integrating both sides gives:

$$\Delta s = s_2 - s_1 = \int_1^2 c_V(T)\frac{dT}{T} - Rln\left(\frac{v_2}{v_1}\right) \tag{3.4.77}$$

For all ideal gases except monatomic gases, the specific heats depend on the temperature. Thus, the integrals on the previous pages cannot be performed unless the dependencies of c_V and c_P on temperature is known. Even when the $c_V(T)$ and $c_P(T)$ functions are available, long integrals make this approach impractical; this leaves two choices:

1. Assume constant specific heats
2. Evaluate those integrals once and tabulate the results

The relationships allow us to form formulas for Δs, in terms of temperature and pressure and temperature and specific volume. We can now derive a formula for Δs in terms of pressure and specific volume.

$$ds = \frac{c_P dT}{T} - \frac{R dP}{P} \tag{3.4.78}$$

$$\Delta s = mc_P \int_1^2 \frac{dT}{T} - mR \int_1^2 \frac{dP}{P} \text{ assuming } c_P \text{ is constant} \tag{3.4.79}$$

$$\Delta s = mc_P ln\left(\frac{T_2}{T_1}\right) - mR ln\left(\frac{P_2}{P_1}\right) \tag{3.4.80}$$

since $R = c_P - c_V$ $\tag{3.4.81}$

$$\Delta s = mc_P ln\left(\frac{T_2}{T_1}\right) - mc_P ln\left(\frac{P_2}{P_1}\right) + mc_V ln\left(\frac{P_2}{P_1}\right) \tag{3.4.82}$$

$$= mc_P ln\left(\frac{T_2}{P_2} \cdot \frac{P_1}{T_1}\right) + mc_V ln\left(\frac{P_2}{P_1}\right) \tag{3.4.83}$$

$$\Delta s = mc_P ln\left(\frac{v_2}{v_1}\right) + mc_V ln\left(\frac{P_2}{P_1}\right) \tag{3.4.84}$$