

0.1 Uniform flow

Cartesian coordinates:

$$\phi = V_{\infty} [x \cos(\alpha) + y \sin(\alpha)] \quad (1)$$

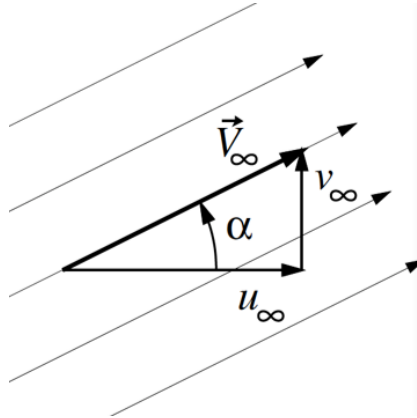
$$\psi = V_{\infty} [y \cos(\alpha) - x \sin(\alpha)] \quad (2)$$

Is the conservation of mass balanced?

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3)$$

Is the flow irrotational?

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (4)$$

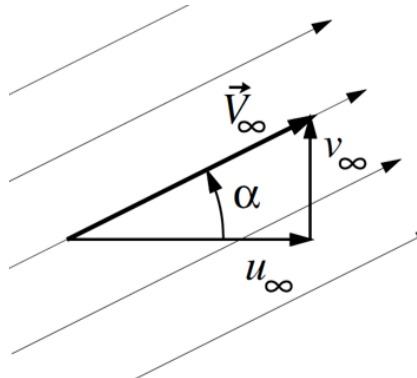


Cylindrical coordinates:

$$\phi(r, \theta) = V_{\infty} r \cos(\theta - \alpha) \quad (5)$$

$$\psi(r, \theta) = V_{\infty} r \sin(\theta - \alpha) \quad (6)$$

Check these two conditions in cylindrical coordinates. Is the conservation is mass equation balanced? Is the flow irrotational?



0.2 Source/sink flow

Cartesian coordinates:

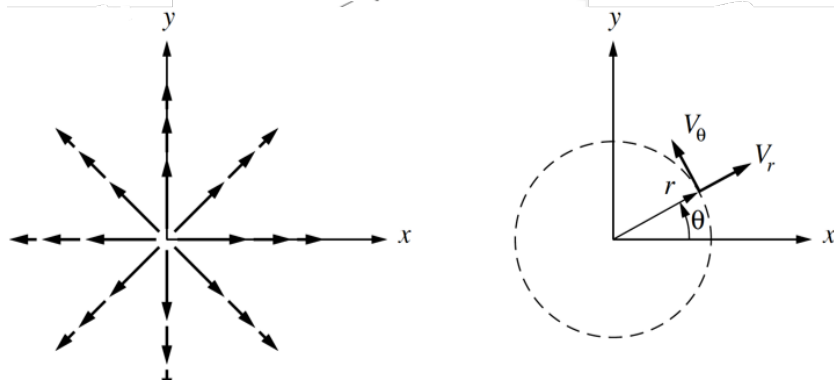
$$\phi = \frac{\Lambda}{2\pi} \ln(\sqrt{x^2 + y^2}) \quad (7)$$

$$\psi = \frac{\Lambda}{2\pi} \arctan\left(\frac{y}{x}\right) \quad (8)$$

Cylindrical coordinates

$$\phi = \frac{\Lambda}{2\pi} \ln(r) \quad (9)$$

$$\psi = \frac{\Lambda}{2\pi} \theta \quad (10)$$



In cylindrical coordinates, we do not have a θ component as it is moving radially outwards from a source.

$$u_r = \frac{\partial \phi}{\partial r} = \frac{\Lambda}{2\pi r} \quad (11)$$

$$u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0 \quad (12)$$

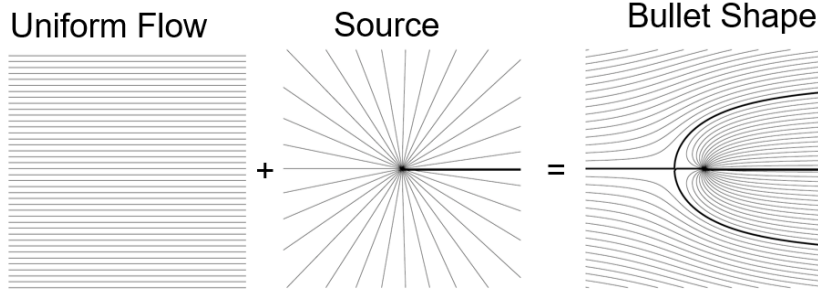
Here we can see the magnitude of the velocity is dependent on $\frac{1}{r}$. If $\Lambda > 0$, we have a source and if $\Lambda < 0$, we have a sink.

0.3 Uniform flow + source

$$\phi(r, \theta) = \frac{\Lambda}{2\pi} \ln(r) + V_\infty r \cos \theta \quad (13)$$

$$\psi(r, \theta) = \frac{\Lambda}{2\pi} \theta + V_\infty r \sin \theta \quad (14)$$

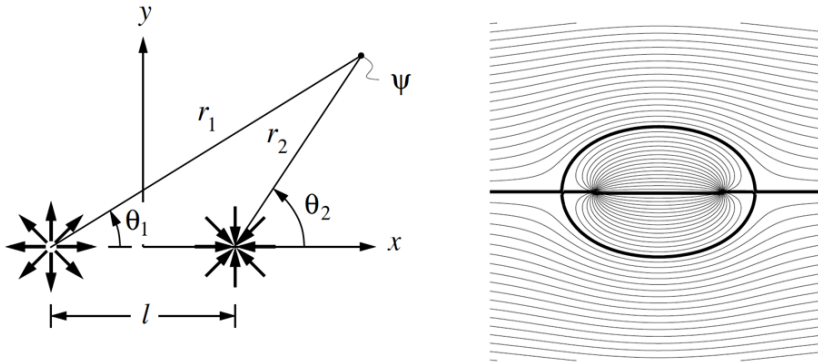
Stream function $\psi(r, \theta)$ of:



0.4 Uniform flow + source + sink

$$\phi = V_{\infty} r \cos \theta + \frac{\Lambda}{2\pi} (\ln(r_1) - \ln(r_2)) \quad (15)$$

$$\psi = V_{\infty} r \sin \theta + \frac{\Lambda}{2\pi} (\theta_1 - \theta_2) \quad (16)$$

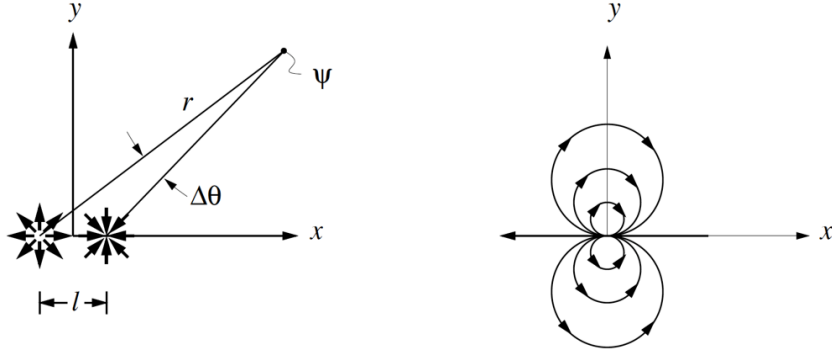


0.5 Doublet

Consider a pair of source and sink of $\pm\Lambda$ who are l apart and $l \times \Lambda = \text{constant}$.

$$\psi = \lim_{l \rightarrow 0} \frac{\Lambda}{2\pi} (\theta_1 - \theta_2) = -\frac{k}{2\pi} \frac{\sin \theta}{r} \quad (17)$$

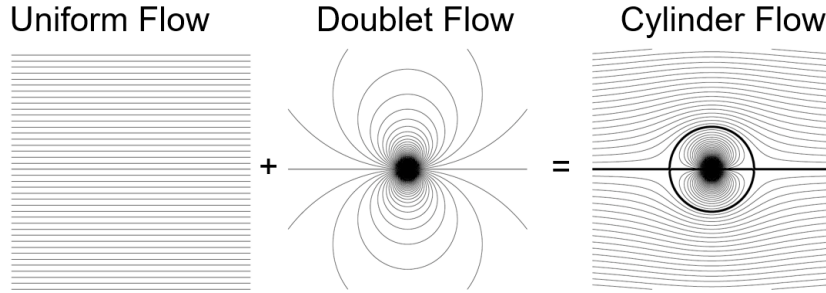
$$\phi = \frac{k}{2\pi} \frac{\cos \theta}{r} \quad (18)$$



0.6 Cylinder (uniform flow + doublet)

$$\phi = V_{\infty} r \cos \theta + \frac{k}{2\pi} \frac{\cos \theta}{r} \quad (19)$$

$$\psi = V_{\infty} r \sin \theta - \frac{k}{2\pi} \frac{\sin \theta}{r} \quad (20)$$



The radius of the cylinder can be derived as so:

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = V_{\infty} \cos \theta - \frac{k}{2\pi} \frac{\cos \theta}{r^2} \quad (21)$$

$$u_{\theta} = -\frac{\partial \psi}{\partial r} = -\left(V_{\infty} \sin \theta + \frac{k}{2\pi} \frac{\sin \theta}{r^2} \right) \quad (22)$$

On the cylinder, $\vec{u} \cdot \hat{n} = 0$

$$\hat{n} = \hat{i}_r \rightarrow u_r(R) = 0 \quad (23)$$

$$V_{\infty} \cos \theta - \frac{k}{2\pi} \frac{\cos \theta}{R^2} = 0 \quad (24)$$

$$R = \sqrt{\frac{k}{2\pi V_{\infty}}} \quad (25)$$

We can rewrite ϕ and ψ

$$\phi = V_{\infty} r \cos \theta \left(1 + \frac{R^2}{r^2} \right) \quad (26)$$

$$\psi = V_{\infty} r \sin \theta \left(1 - \frac{R^2}{r^2} \right) \quad (27)$$

On the cylinder surface, $r = R$ and inputting this into ψ :

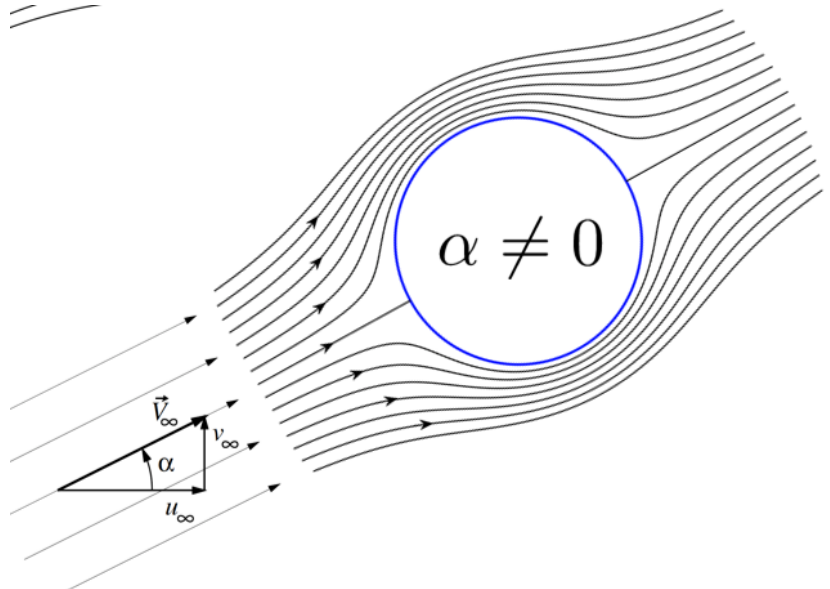
$$\psi = 0 \quad (28)$$

0.7 Uniform stream with varying direction

All we need to do to generalise our equations a bit more is to rewrite our equations with an extra angular term, α :

$$\phi = V_{\infty} r \cos(\theta - \alpha) \left(1 + \frac{R^2}{r^2} \right) \quad (29)$$

$$\psi = V_{\infty} r \sin(\theta - \alpha) \left(1 - \frac{R^2}{r^2} \right) \quad (30)$$



0.8 Adding circulation with a vortex flow

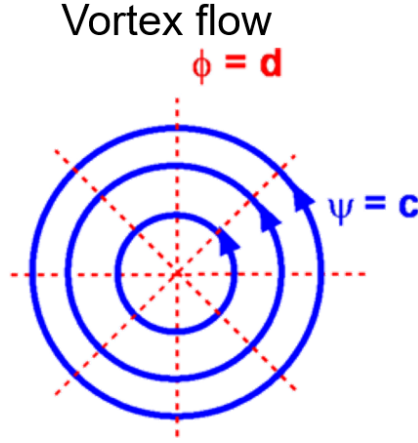
$$\phi = -\frac{\Gamma}{2\pi}\theta \quad (31)$$

$$\psi = \frac{\Gamma}{2\pi} \ln\left(\frac{r}{R}\right) \quad (32)$$

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = 0 \quad (33)$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -\frac{\Gamma}{2\pi r} \quad (34)$$

Where $\Gamma < 0$ is anti-clockwise motion and $\Gamma > 0$ is clockwise motion.



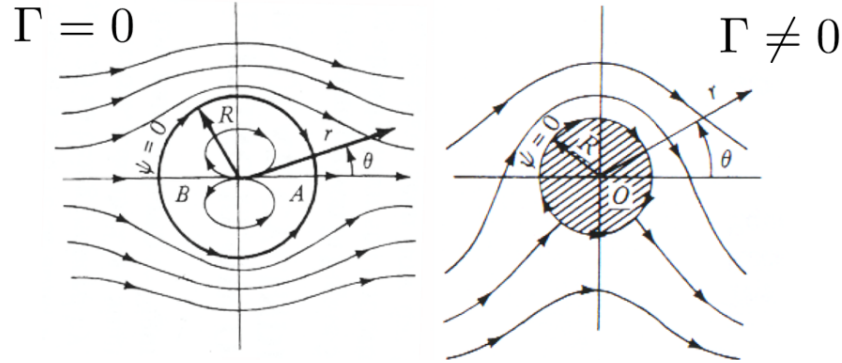
0.9 Cylinder with a vortex flow

$$\psi = V_\infty r \sin \theta \left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi} \ln\left(\frac{r}{R}\right) \quad (35)$$

$$\phi = V_\infty r \cos \theta \left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi} \theta \quad (36)$$

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = V_\infty \cos \theta \left(1 - \frac{R^2}{r^2}\right) \quad (37)$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin \theta \left(1 + \frac{R^2}{r^2}\right) \quad (38)$$



0.10 Lift and drag of a cylinder with circulation

Apply Bernoulli:

$$p_\infty + \frac{1}{2}\rho V_\infty^2 = p + \frac{1}{2}(\dot{u}_r^2 + \dot{u}_\theta^2) \quad (39)$$

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{\dot{u}_r^2 + \dot{u}_\theta^2}{V_\infty^2} \quad (40)$$

On the cylinder surface: $u_r = 0$

$$c_p(R, \theta) = 1 - \frac{\dot{u}_\theta^2}{V_\infty^2} = 1 - \frac{(2V_\infty \sin \theta + \frac{\Gamma}{2\pi R})^2}{V_\infty^2} \quad (41)$$

0.11 Lift of the cylinder

We need to calculate c_p on the surface of the cylinder.

$$L = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} (c_p \cdot \hat{n} \cdot \hat{j} R) d\theta \quad (42)$$

$$= -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} (c_p \sin \theta R) d\theta \quad (43)$$

$$L = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(1 - \frac{(2V_\infty \sin \theta + \frac{\Gamma}{2\pi R})^2}{V_\infty^2} \right) \sin \theta R d\theta \quad (44)$$