0.1 Reynold's number

Reynolds conducted experiments, in which he measured pressure drop and critical velocity in a variety of pipe diameters and with different fluids, and verified the importance of the parameter $\frac{\rho d\vec{U}}{\mu}$ later to be given his name. He found that, instead of a different critical velocity for each fluid and pipe, the onset of turbulence (i.e. transition) was determined by the achievement of the same critical value of Reynolds number, usually quoted as:

$$Re_{d,crit} = 2300 \tag{1}$$

We note the confirmation of the empirical results regarding critical velocity because if:

$$Re_{d,crit} = \frac{\rho d\vec{U}_{crit}}{\mu} = 2300 \tag{2}$$

then $\vec{U} \propto \frac{1}{d}$ for given ρ and μ and $\vec{U}_{crit} \propto \mu$ for given d and μ.

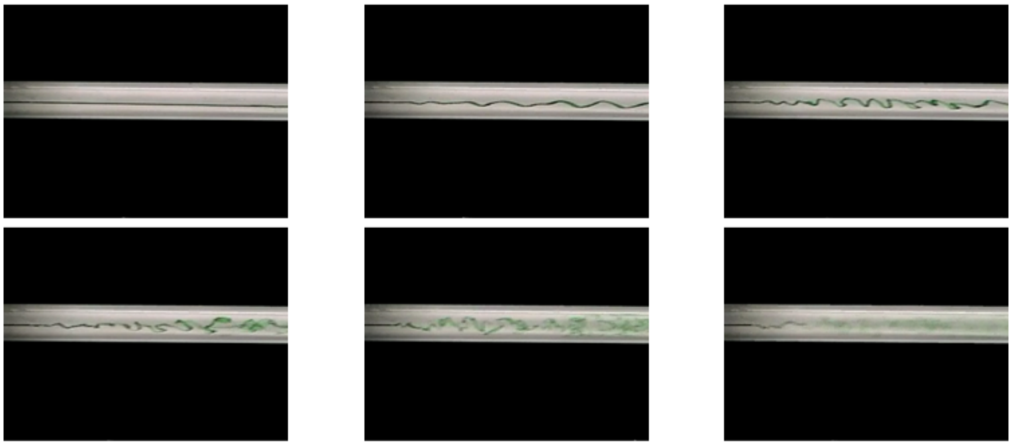

Figure 1:

- In this experiment water flows through a clear pipe with increasing speed
- Dye is injected through a small diameter tube at the left portion of the screen
- At low speed ($Re < 2300$) the flow is laminar and the dye is a straight line
- As the speed increases, the dye stream becomes wavy (oscillatory laminar flow)
- At still higher speeds ($Re > 4000$) the flow becomes turbulent and the dye stream is dispersed randomly throughout the flow

0.2 Flow in pipes

0.2.1 Entrance region

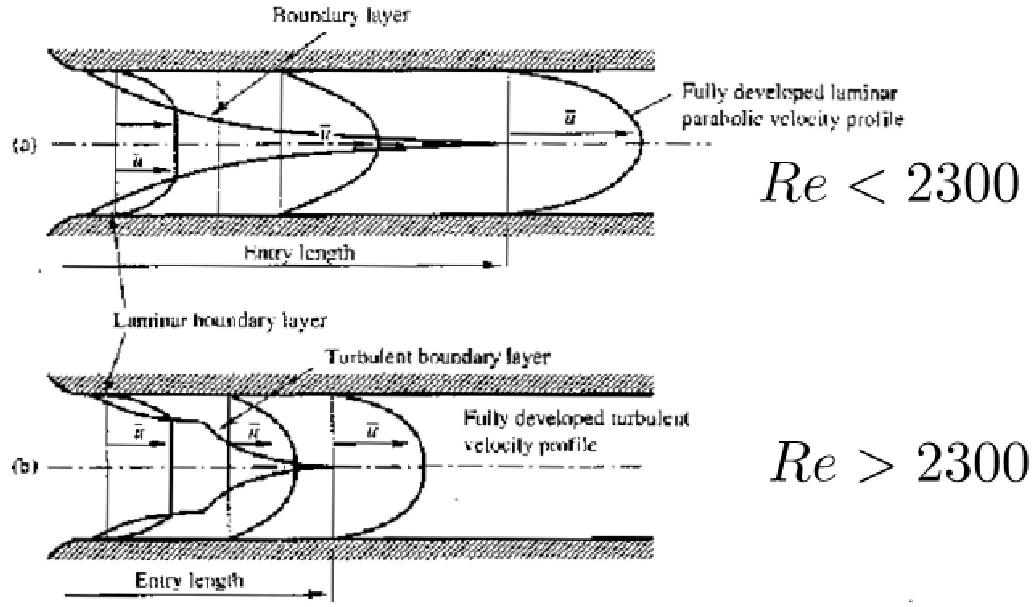

Figure 2:

Reynolds number: $Re = \frac{\rho \vec{U} D}{\mu}$ where D is the diameter.

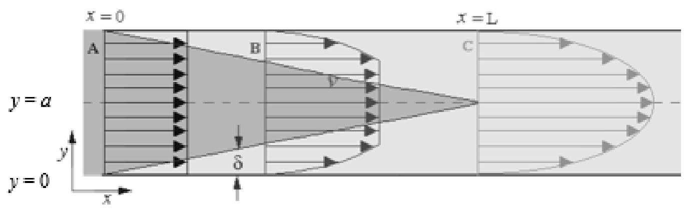

Figure 3:

- The nature of the flow in the entrance region and hence its length, depend on whether the fully-developed flow is laminar or turbulent
  - $Re_d < 2300$: Fully-developed flow should be laminar
  - $Re_d > 2300$: Fully-developed flow should be turbulent
  - $2000 < Re_d < 3000$: Transitional flow...
- Figures given for laminar entry length variety
  - *Laghaar* quotes: Entry length $= 0.057 d Re_d$
  - Rule of thumb: Entry length $= 100d$ minimum
- For turbulent flow the fully-developed state is achieved much sooner (entry length less dependent on Re_d)
  - Rule of thumb: Entry length $= 50d$ minimum

0.2.2 Shear stress

What is the shear stress distribution in fully-developed pipe flows? Consider an axial cylindrical element of length L within a pipe.

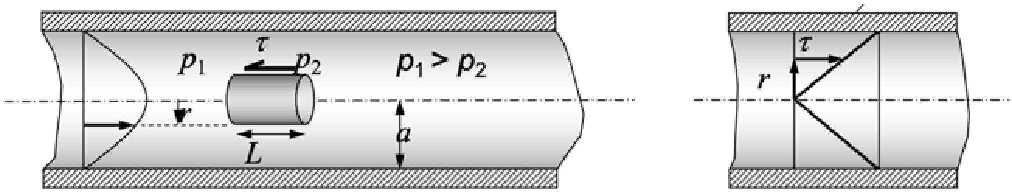

Figure 4:

The forces due to pressure and shear (from velocity gradient) must balance out:

$$p_1\pi r^2 - p_2\pi r^2 - \tau 2\pi r L = 0 \rightarrow \tau = \frac{p_1 - p_2}{L}\frac{r}{2} = \frac{\Delta p}{L}\frac{r}{2} \tag{3}$$

This relationship is valid for both laminar and turbulent motion and shows that shear stress must vary linearly with radius r in the pipe. The value of τ at the wall $(r = a)$ is: $\tau_w = \frac{\Delta p}{L}\frac{a}{2}$ so that: $\frac{\tau}{\tau_w} = \frac{r}{a}$. Wall shear stress can be measured by the pressure gradient in the pipe.

0.2.3 Velocity profile

Laminar motion ($Re < 2300$). What is the velocity profile? If the motion is laminar, the shear stress is easily related to the velocity gradient through the dynamic viscosity, μ.

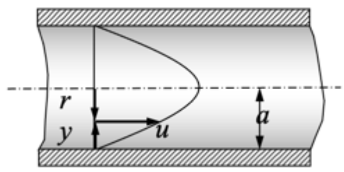

Figure 5:

$$\tau = \mu\frac{\mathrm{d}u}{\mathrm{d}y} \text{ where } y = a - r \text{ such that: } \mathrm{d}y = -\,\mathrm{d}r \tag{4}$$

- From previous result: $\tau = -\mu\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\Delta p}{L}\frac{r}{2}$
- Thus: $\mathrm{d}u = -\frac{\Delta p}{2L}\frac{r}{\mu}\,\mathrm{d}r$ which we can integrate to get: $u = -\frac{\Delta p}{4L\mu}r^2 + C$
- No slip boundary condition says: $u = 0$ at $r = a \rightarrow C = \frac{\Delta p}{4L\mu}a^2$
- Thus, finally: $u = \frac{1}{4\mu}\frac{\Delta p}{L}\left(a^2 - r^2\right)$ Parabolic velocity profile (*Hagen-Poiseuille Law*)

Segment V6.6 Laminar flow (Related to textbook section 6.9.3 - Steady, Laminar flow in circular tubes) The velocity distribution is parabolic for steady, laminar flow in circular tubes. A filament of dye is placed across a circular tube containing a very viscous liquid which is initially at rest. With the opening of a valve at the bottom of the tube the liquid starts to flow, and the parabolic velocity distribution is revealed. Although the flow is actually unsteady, it is quasi-steady since it is only slowly changing. Thus, at any instant in time the velocity distribution corresponds to the characteristic steady-flow parabolic distribution.

0.2.4 Flow rate

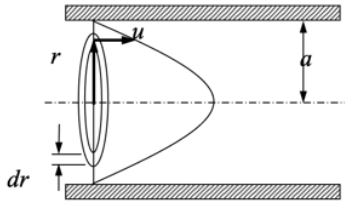

Figure 6:

Parabolic velocity profile:

$$u = \frac{1}{4\mu}\frac{\Delta p}{L}\left(a^2 - r^2\right) \tag{5}$$

The maximum velocity U_{max} occurs on the centre-line of the pipe (i.e. at $r = 0$):

$$U_{max} = \frac{1}{4\mu}\frac{\Delta p}{L}a^2 \tag{6}$$

$$\frac{u}{U_{max}} = \frac{a^2 - r^2}{a^2} \to \frac{u}{U_{max}} = 1 - \frac{r^2}{a^2} \tag{7}$$

The mean velocity/flow rate is given by:

$$\vec{U} = \frac{Q}{A} = \frac{\int_0^a (2\pi r u)\, \mathrm{d}r}{\pi a^2} \tag{8}$$

$$\vec{U} = \frac{2\pi}{\pi a^2}\int_0^a (ru)\, \mathrm{d}r = \frac{2}{a^2}\int_0^a \left(U_{\max}\left(1 - \frac{r^2}{a^2}\right) r\right) \mathrm{d}r \tag{9}$$

$$\vec{U} = \frac{2U_{max}}{a^2}\int_0^a \left(r - \frac{r^3}{a^2}\right) \mathrm{d}r \tag{10}$$

$$\vec{U} = \frac{2U_{max}}{a^2}\left[\frac{r^2}{2} - \frac{r^4}{4a^2}\right]_0^a = \frac{2U_{max}}{a^2}\left[\frac{a^2}{2} - \frac{a^4}{4a^2}\right] = \frac{U_{max}}{2} \tag{11}$$

For a parabolic profile the mean velocity is hald the maximum velocity:

$$u = \frac{1}{4\mu}\frac{\Delta p}{L}\left(a^2 - r^2\right) \qquad U_{max} = \frac{1}{4\mu}\frac{\Delta p}{L}a^2 \qquad \vec{U} = \frac{1}{8\mu}\frac{\Delta p}{L}a^2 \tag{12}$$

The volumetric flow rate is:

$$Q = \pi a^2 \vec{U} = \frac{\pi}{8\mu}\frac{\Delta p}{L}a^4 \tag{13}$$

This can be used as a basis for measuring μ if all the other parameters can be measured sufficiently accurately.

0.3 Flow in pipes

0.3.1 Friction factor

We have seen that the pressure loss in a pipe (whether laminar or turbulent) is related to the wall shear stress by:

$$\tau_w = \frac{\Delta p}{L}\frac{a}{2} \tag{14}$$

A dimensionless representation of the wall shear stress (and therefore the pressure gradient) is given by the defining a friction factor as the ratio of wall shear stress to dynamic pressure in the flow (based on mean velocity).

$$f = \frac{\tau_w}{\frac{1}{2}\rho \vec{U}^2} \tag{15}$$

0.3.2 Pressure drop due to friction

Generalised expression for pressure drop due to friction. We have seen from equilibrium considerations (for laminar or turbulent flow) that:

$$\tau_w = \frac{\Delta p}{L}\frac{a}{2} = \frac{\delta p}{L}\frac{d}{4} \tag{16}$$

Thus,

$$\Delta p = \frac{4L\tau_w}{d} \rightarrow \Delta p = \frac{4L}{d} f \frac{1}{2}\rho \vec{U}^2 \tag{17}$$

Hence:

$$h_f = \frac{\Delta p}{\rho g} \rightarrow h_f = \frac{4L}{d} f \frac{\vec{U}^2}{2g} \tag{18}$$

Eq.18 is known as *Darcy's* Formula. To work out the pressure drop due to friction in pipe flows, we need to know accurately the friction factor f.

0.4 Friction factor for laminar flow in pipes

Friction factor as a function of Reynolds number. Definition of friction factor (laminar or turbulent flow):

$$f = \frac{\tau_w}{\frac{1}{2}\rho \vec{U}^2} \tag{19}$$

For laminar flow we have:

$$u = U_{max}\left(1 - \frac{r^2}{a^2}\right) \qquad \vec{U} = \frac{U_{max}}{2} \tag{20}$$

Also:

$$\tau_w = -\mu \left[\frac{\mathrm{d}u}{\mathrm{d}r}\right]_{r=a} = -\mu \left[\frac{-2\mu U_{max} a}{a^2}\right]_{r=a} = \frac{2\mu U_{max} a}{a^2} = \frac{2\mu U_{max}}{a} = \frac{4\mu U_{max}}{d} \tag{21}$$

$$\tau_w = \frac{8\mu \vec{U}}{d} \tag{22}$$

Thus, finally:

$$f = \frac{\frac{8\mu \vec{U}}{d}}{\frac{1}{2}\rho \vec{U}^2} \rightarrow f = \frac{16\mu}{\rho d \vec{U}} \rightarrow f = \frac{16}{Re_d} \tag{23}$$

0.5 Laminar vs Turbulent profiles

Segment V8.3 Laminar/Turbulent Velocity profiles (Related to textbook section 8.3.3 - Turbulent velocity profiles) The velocity profile for laminar flow in a pipe is quite different than that for turbulent flow. An approximation to the velocity profile in a a pipe is obtained by observing the motion of a dye streak placed across the pipe. With a viscous oil at Reynolds number of about 1, viscous effects dominate and it is easy to inject a relatively straight dye streak. The resulting laminar flow profile is parabolic. With water at Reynolds number of about 10000, inertial effects dominate and it is difficult to inject a straight dye streak. It is clear, however, that the turbulent velocity profile is not parabolic, but is more nearly uniform than for laminar flow.