## 0.1 Exercise 2 - Fluid tutorial group B

A viscous fluid of constant density, $\rho$, and kinematic viscosity, $\nu$, flows over a flat plate inclined at an angle $\alpha$ and moving with a constant velocity $V_w$. The flow is stationary and no pressure gradients are applied. The only body force acting on the fluid is due to gravity, $g$. You can assume zero velocity component in the direction orthogonal to the plate and negligible air resistance at the interface between the viscous fluid and air ($y = h$).

### 0.1.1 Determine by means of the Navier-Stokes equation the velocity profile, $u(y)$

NSE (x,y):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + f_x \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] + f_y \tag{2}$$

- $\frac{\partial}{\partial t} = 0$ steady flow
- $v = 0$ negligible
- $\frac{\partial}{\partial x}$ no variation in flow along plate length

$$0 + 0 + 0 = 0 + \nu\frac{\partial^2 u}{\partial y^2} + g\sin\alpha \tag{3}$$

$$0 + 0 + 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g\cos\alpha \tag{4}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{g\sin(\alpha)}{\nu} \tag{5}$$

$$\frac{\partial p}{\partial y} = -\rho g\cos\alpha \tag{6}$$

Integrating equation (5) once:

$$\int\left(\frac{\partial^2 u}{\partial y^2}\right)\mathrm{d}y = \int\left(-\frac{g\sin\alpha}{\nu}\right)\mathrm{d}y \frac{\partial u}{\partial y} = -\frac{g\sin\alpha}{\nu}y + C \tag{7}$$

Apply the following boundary equation, due to 0 air resistance at $y = h$ boundary:

$$\tau(h) = \mu \frac{\partial u}{\partial y}\bigg|_{y=h} = 0 \therefore 0 = -\frac{g \sin \alpha}{\nu} h + C C = \frac{g \sin \alpha}{\nu} h \tag{8}$$

Integrate equation (8) once more:

$$\int \left(\frac{\partial u}{\partial y}\right) \mathrm{d}y = \int \left(-\frac{g \sin \alpha}{\nu} y + \frac{g \sin \alpha}{\nu} h\right) \mathrm{d}y \tag{9}$$

$$u(y) = -\frac{g \sin \alpha}{2\nu} y^2 + \frac{g \sin \alpha}{\nu} h y + D \tag{10}$$

$$y(0) = -V_w = D \tag{11}$$

$$u(y) = -\frac{g \sin \alpha}{2\nu} y^2 + \frac{g \sin \alpha}{\nu} h y - V_w \tag{12}$$

$$u(y) = \frac{g \sin \alpha}{\nu} y \left(h - \frac{1}{2} y\right) - V_w \tag{13}$$

### 0.1.2 If the net flow rate across the fluid height is zero, determine the corresponding plate velocity, $V_{wo}$, as a function of $\alpha$

Volume flow rate:

$$\int_o^h u(y)\,\mathrm{d}y = \int_o^h \left(\frac{g \sin \alpha}{\nu} \left(hy - \frac{1}{2} y^2\right) - V_w\right) \mathrm{d}y \tag{14}$$

$$= \left[\frac{g \sin \alpha}{\nu} y \left(\frac{h y^2}{2} - \frac{1}{6} y^3\right) - V_w y\right]_0^h \tag{15}$$

$$= \frac{g \sin \alpha h^3}{3\nu} - V_w h \tag{16}$$

This is equal to 0 for $V_{wo}$, hence:

$$V_{wo} = \frac{g \sin \alpha h^2}{3\nu} \tag{17}$$

### 0.1.3 For the condition found above determine the $y$ coordinate corresponding to $y = 0$. Sketch three velocity profiles for $V_{wo} < V_w$, $V_{w0} = V_w$ and $V_{wo} > V_w$ and comment them

Combining above equations for $u(y)$, we get:

$$u(y) = \frac{g \sin \alpha}{\nu} y \left( -\frac{1}{2} y^2 + hy - \frac{1}{3} h^2 \right) \tag{18}$$

Set this equal to 0 and solve:

$$0 = -\frac{1}{2} y^2 + hy - \frac{1}{3} h^2 \tag{19}$$

$$y = \frac{3 - \sqrt{3}}{3} \cdot h \tag{20}$$

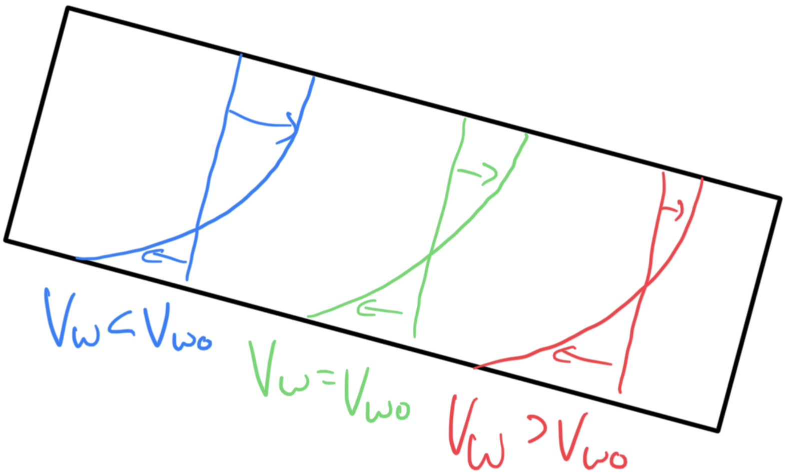


Figure 1: We can see that in the blue one velocity is increased towards the air boundary. As we move along to the point where $V_w = V_{wo}$, we see that the point where $u = 0$ is $\frac{1}{2}h$. Finally we see the velocity highest close to the plate boundary.

### 0.1.4 If the flat plate surface is $S$, determine the force, $F$, and power, $P$, required to move it with constant velocity $V_{wo}$

Using tensor equation:

$$\tau = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \rho g h \sin \alpha \tag{21}$$

Power is simply $F \times A \times V$:

$$P = \tau S V_w = \rho g h \sin \alpha S V_w \tag{22}$$