What you already know on bending is sufficient to enable the basic engineering design of static beam structures. Basic engineering design is primarily concerned with:

- Controlling elastic deformations
- Maintaining structures within their elastic range

However, for safety reasons, it is essental for engineers to be able to predict how a structure would fail if unexpected loading condiitons occur.

0.1 Introduction

0.1.1 Elastic and Plastic Regimes

Stressing within the **elastic regime**:

- Material returns to original state upon removal of external actions
- Deformation depends solely upon stress and not upon load history

Exceeding the elastic regime, **plastic regime** is reached:

- Permanent distortions take place in the material
- Deformation depends upon stress and load history

Equations of solid mechanics apply to both cases.

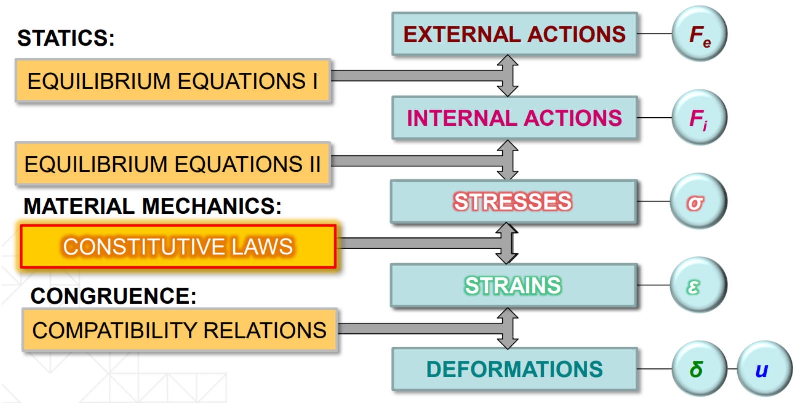


Figure 1: Solid Mechanics Equations: The relationship between actions, stresses, strains and deformations

0.1.2 Stress-Strain Plastic Relationship

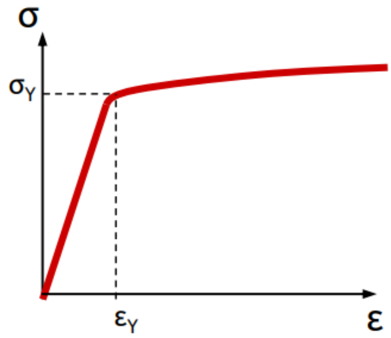


Figure 2: Typical elasto-plastic material

Typical elasto-plastic material:

- Region 1 - linear elastic behaviour up to yield stress σ_y
- Region 2 – non linear development with strain-hardening

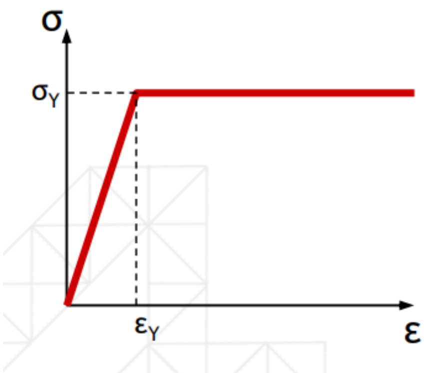


Figure 3: Perfectly elasto-plastic material

Perfectly elasto-plastic material:

- Region 1 - linear elastic behaviour up to yield stress σ_y
- Region 2 – strain increases at constant stress

Structural steels are elastoplastic materials and can be modelled as perfectly elastoplastic (neglecting the strain-hardening is conservative for safety).

0.2 Plastic Theory of Collapse

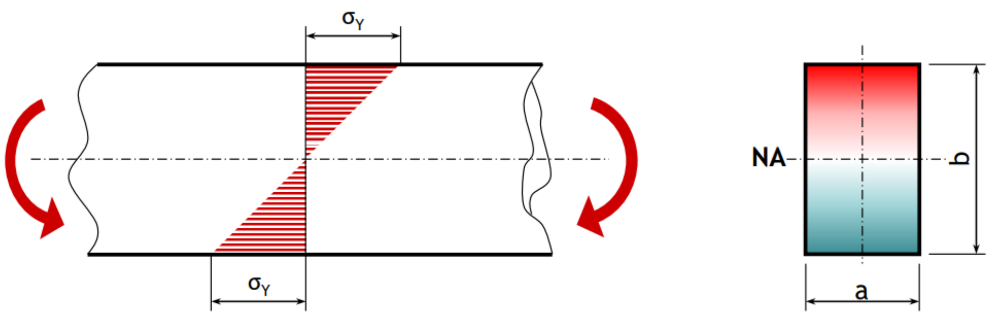
0.2.1 Bending and Plastic Collapse

Bending moment is by far the most relevant of the internal forces, since it produces the largest levels of deformations and stress into the beam. Therefore, **plastic collapse** of beam structures is commonly associated with **plastic bending**. Assumptions:

- Material is perfectly elasto-plastic. (in the plastic region stress will be constant)
- Yield stress is the same in tension and compression
- Transverse cross-sections remain plane (strain is proportional to the distance from the NA)
- When a cross-section is fully plastic (plastic hinge), its resisting moment remains constant until collapse of the whole structure
- Loads increase monotonically

0.2.2 Elastic Bending Moment

Consider a beam of rectangular cross-section, subjected to pure bending.



Maximum stress reaches the elastic limit σ_y when the bending moment is equal to:

$$M = \frac{EI}{R} \tag{1}$$

$$\sigma = \frac{Eb}{2R} \tag{2}$$

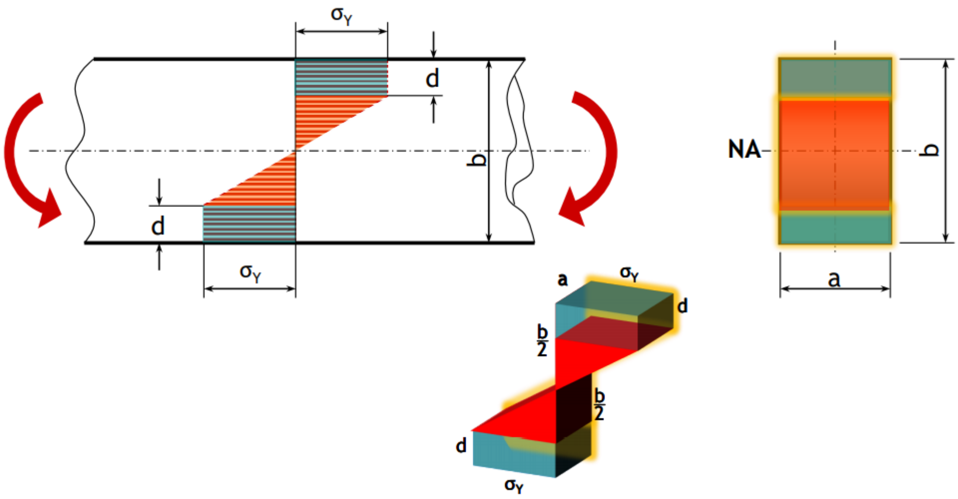
$$I = \frac{ab^3}{12} \tag{3}$$

$$\therefore \text{The yield bending moment} : M_Y = \sigma_Y \frac{ab^2}{6} \tag{4}$$

At the elastic moment, all fibres are still in the elastic condition. If moment increases further, external fibres exceed the elastic limit and yield: deformation increases but stress keeps constant and equal to σ_y. With increasing bending moment, the plastic region penetrates deeper toward the Neutral Axis (NA).

0.2.3 Elasto-Plastic Bending Moment

Consider a beam of rectangular cross-section, subjected to pure bending.



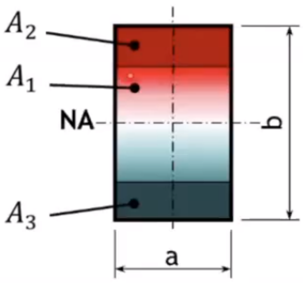
The moment is given by:

$$M = \frac{\sigma I}{h} \tag{5}$$

$$I = ? \tag{6}$$

$$\therefore \text{Elasto-plastic bending moment} : M = \frac{\sigma_Y a b^2}{6}\left[1 + 2\frac{d}{b}\left(1 - \frac{d}{b}\right)\right] \tag{7}$$

The derivation of the bending moment:



The cross-section is divided into 3 parts:

1. Elastic part in the middle (A_1)
2. Plastic part at the top (A_2)
3. Plastic part at the bottom (A_3)

$$M = \int_A \mathrm{d}M = \int_A h \cdot \sigma \,\mathrm{d}A \tag{8}$$

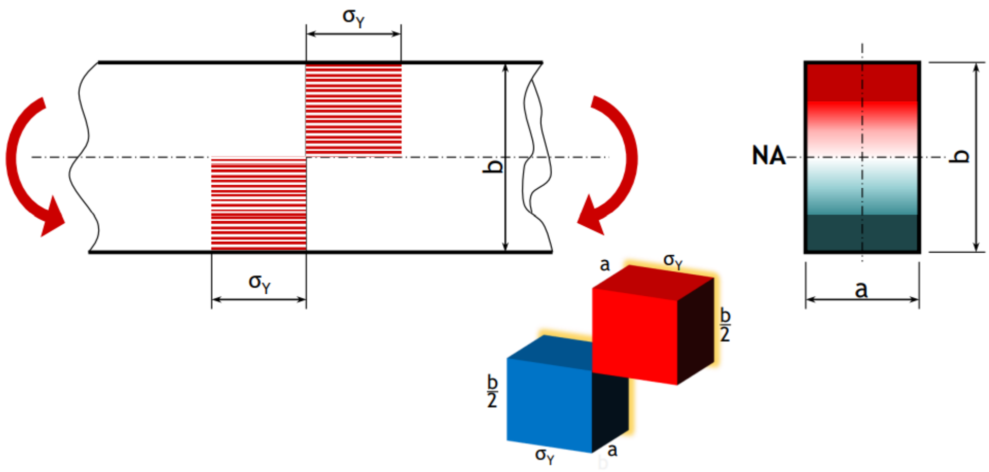
$$= \int_{A_1} \frac{\sigma_Y h}{b/2 - d} h \,\mathrm{d}A + \int_{A_2} \sigma_Y h \,\mathrm{d}A + \int_{A_3} \sigma_Y h \,\mathrm{d}A \tag{9}$$

$$= \int_{-b/2+d}^{b/2-d} \frac{\sigma_Y a}{b/2 - d} h^2 \,\mathrm{d}h + \int_{b/2-d}^{b/2} \sigma_Y a h \,\mathrm{d}h + \int_{-b/2}^{-b/2+d} \sigma_Y a h \,\mathrm{d}h \tag{10}$$

$$= \frac{\sigma_Y a b^2}{6}\left[1 + 2\frac{d}{b}\left(1 - \frac{d}{b}\right)\right] \tag{11}$$

0.2.4 Plastic Bending Moment

Consider a beam of rectangular cross-section, subjected to pure bending.



If bending moment keeps increasing, the plastic region propagate to the NA. In this case the moment is equal to:

$$\text{Plastic bending moment} : M_P = \sigma_Y \frac{ab^2}{4} \tag{12}$$

In practice, the plastic moment is given by the product of the yield stress by (the 1st moment of area (of the cross section) above the plastic NA + the 1st moment of area below the plastics NA).

0.2.5 Shape Factor

The yielding and plastic bending moments are different. In this case (rectangular cross-section) they are:

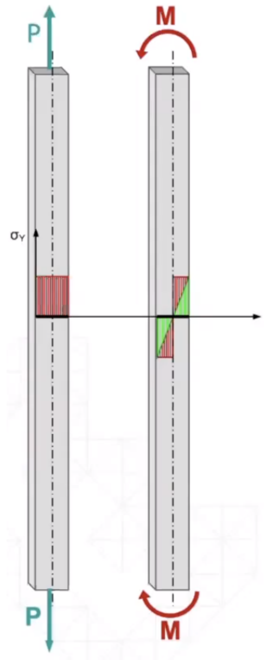
$$M_Y = \sigma_Y \frac{ab^2}{6} \qquad M_P = \sigma_Y \frac{ab^2}{4} \tag{13}$$

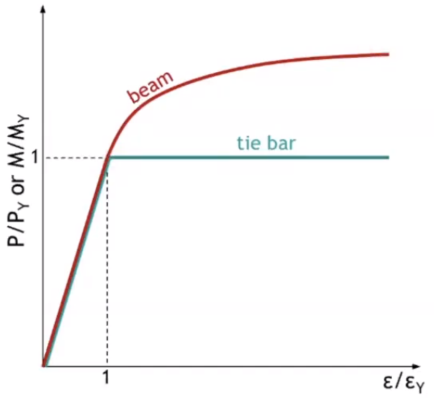
The ratio $f = \frac{M_P}{M_Y}$ between the yielding bending moment and the plastic bending moment is solely a function of the shape of the cross-section, and it is called **shape factor**. In this case (rectangular cross-section) it is:

$$f = \frac{M_P}{M_Y} = \frac{\sigma_Y ab^2}{4} \frac{6}{\sigma_Y ab^2} = 1.5 \tag{14}$$

The shape factor is an indicator of the reserve strength that the beam can offer after yielding first begins.

0.2.6 Plastic Collapse Under Bending



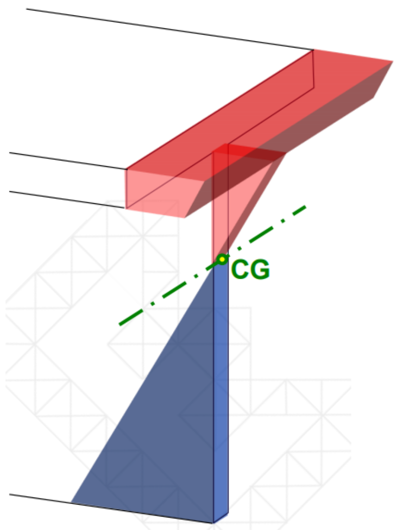


Contrary to the case of normal load, the plastic collapse of a beam under pure bending occurs at a moment greater than the yield moment.

For a rectangular cross section, it will occur at a moment 50% higher than the one that initiate yielding.

0.2.7 Neutral Axis with Asymmetry

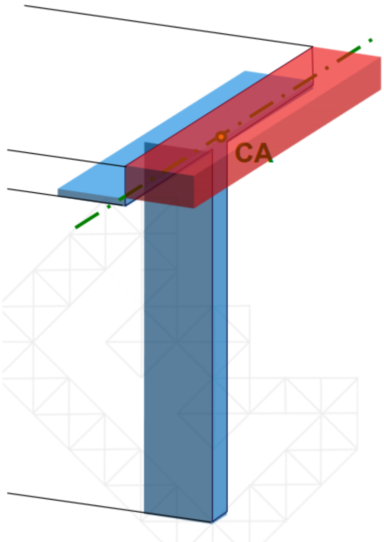
In the case of asymmetrical sections (though still singly symmetric), the analysis becomes more complex:



Equilibrium of forces in the ELASTIC case:

$$F = \int_A \sigma \cdot \mathrm{d}A = \int_A \frac{M}{I} h \cdot \mathrm{d}A = \frac{M}{I} \int_A h \cdot \mathrm{d}A = 0 \tag{15}$$

The $\int_A h \cdot \mathrm{d}A$ is the 1st moment of area. The **elastic neutral axis** corresponds to the centroid of the section (centre of gravity).



Equilibrium of forces in the PLASTIC case:

$$F = \int_A \sigma_Y \cdot \mathrm{d}A = \int_{A_T} \sigma_Y \cdot \mathrm{d}A - \int_{A_C} \sigma_Y \cdot \mathrm{d}A = 0 \tag{16}$$

$$\therefore \sigma_Y A_T = \sigma_Y A_C \tag{17}$$

$$\therefore A_T = A_C \tag{18}$$

The **plastic neutral axis** corresponds to the centre of area (belongs to the line that divides the section into 2 equal areas).

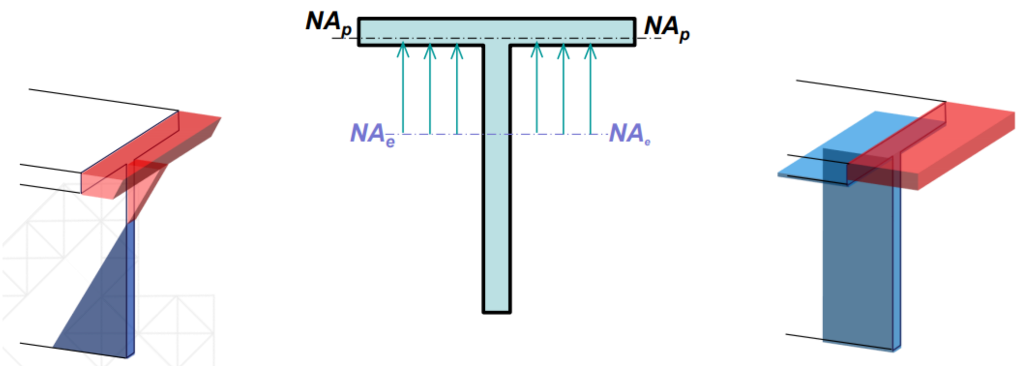
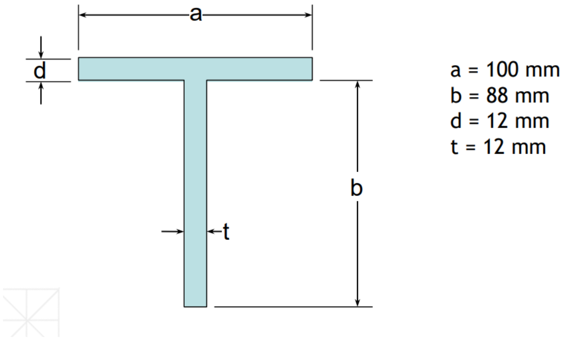


Figure 4: In the elasto-plastic phase, the neutral axis shifts from the centroid (centre of gravity) to the centre of area of the section

0.2.8 Example: Shape factor for T-beam

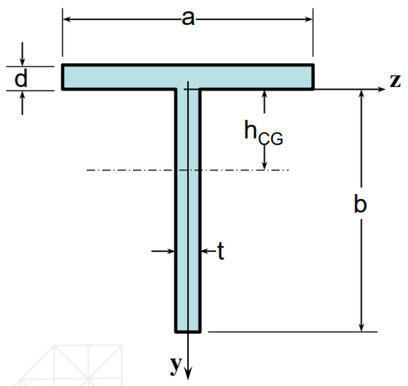


Yield moment M_Y and plastic moment M_P need to be calculated:

$$M_Y = \frac{\sigma_Y \cdot I}{h_{max}} \tag{19}$$

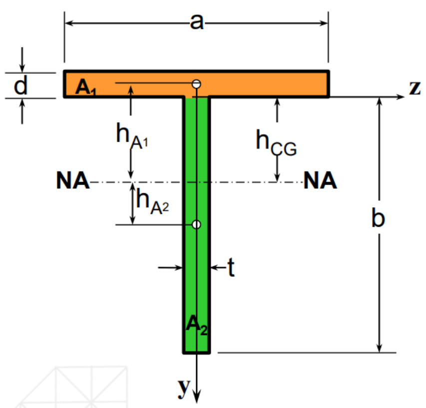
$$M_P = \sigma_Y (Q_1 + Q_2) \tag{20}$$

Elastic Neutral Axis:



$$\text{Centre of Gravity} = h_{CG} = \frac{\int_A h \cdot \mathrm{d}A}{A} \tag{21}$$

2nd Moment of Area:



$$h_{A1} = h_{CG} + \frac{d}{2} = 23.4\text{mm} \tag{22}$$

$$h_{A2} = \frac{b}{2} - h_{CG} = -26.6\text{mm} \tag{23}$$

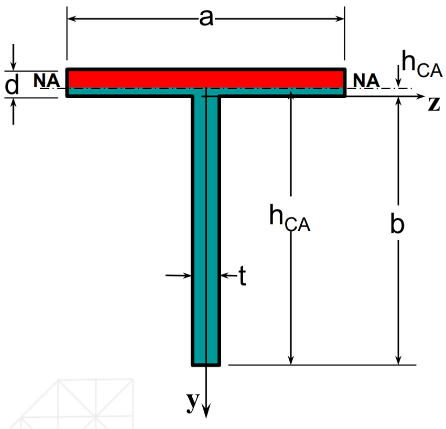
2nd moment of area (parallel axis theorem):

$$I_{zz} = (I_{A_1} + A_1 \cdot h_{A_1}^2) + (I_{A_2} + A_2 \cdot h_{A_2}^2) = 2.10 \cdot 10^6\text{mm}^4 \tag{24}$$

Yield Moment:

$$M_Y = \frac{\sigma_Y \cdot I_{zz}}{h_{CG}} = 29745 \cdot \sigma_Y \tag{25}$$

Plastic Neutral Axis:

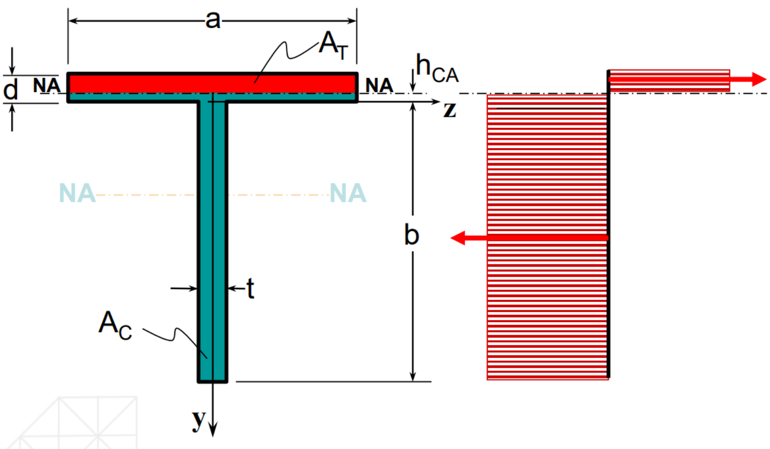


$$A_T = A_C = \frac{A_{total}}{2} \tag{26}$$

$$\text{Centre of Area} = h_{CA} = 0.72\text{mm} \tag{27}$$

Where A_T and A_C represent the areas of the top and bottom parts.

1st Moment of Area:



1st moment of the areas in tension & compression:

$$Q_{A_T} = \int_{A_T} h \cdot \mathrm{d}A = 6362\text{mm}^3 \tag{28}$$

$$Q_{A_C} = \int_{A_C} h \cdot \mathrm{d}A = 46846\text{mm}^3 \tag{29}$$

Plastic Moment:

$$M_P = \sigma_Y(Q_{A_T} + Q_{A_C}) = \sigma_Y \cdot 6362 + \sigma_Y \cdot 46846 = 53208 \cdot \sigma_Y \tag{30}$$

Shape Factor:

Calculated yield and plastic moments:

$$M_Y = \frac{\sigma_Y \cdot I_{zz}}{y_{CG}} = 29745 \cdot \sigma_Y \tag{31}$$

$$M_P = \sigma_Y(Q_{A_T} + Q_{A_C}) = 53208 \cdot \sigma_Y \tag{32}$$

$$\therefore \text{Shape Factor:} f = \frac{M_P}{M_Y} = 1.79 \tag{33}$$

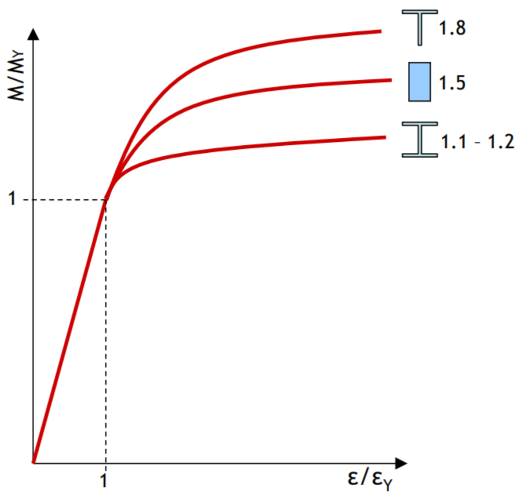


Figure 5: The shape factors of Rectangular, T-shaped and I-shaped cross sections