# MECH0013: Mechanics of Solids and Structures
Topic Notes

UCL

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# Contents

# Course Outline

## Contact Information

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## Methods of Assessment

- 50% Final Assessment
- 20% Coursework 1
- 20% Coursework 2
- 10% Quizzes (Best 3/5 Each Term)

## Topics to be Covered

- Bending of beams
- Plastic collapse
- Energy methods
- Beam buckling
- Principal stresses and failure criteria
- Axisymmetric stress and strain

# Chapter 1

# Basic Concepts of Structure Mechanics

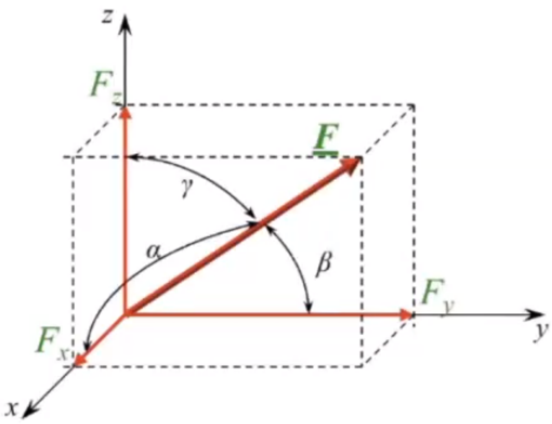
09/10/2020

## 1.1 Actions and Deformations

### 1.1.1 Vector Quantities

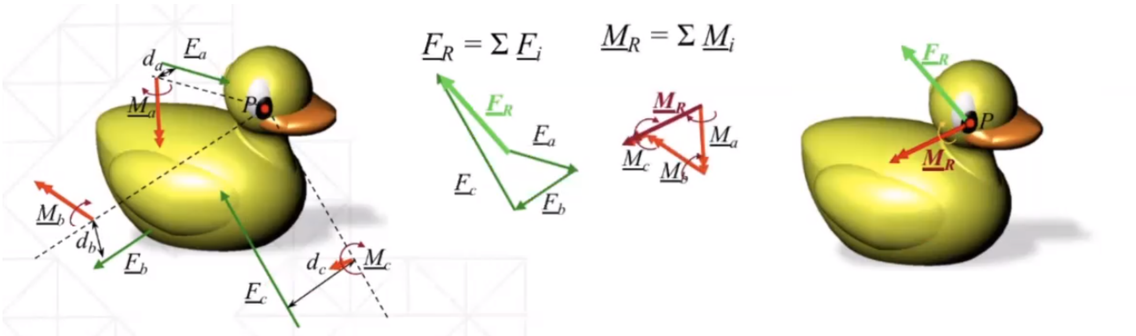
A vector is a quantity defined by **magnitude** and **direction**. Mechanical actions (forces and moments) can be represented as **vectors**.

Vector quantities can be decomposed in components, that can be conveniently oriented with the Cartesian reference system.




$$F_x = F \cdot \cos\alpha \qquad M_x = M \cdot \cos\alpha \tag{1.1.1}$$
$$F_y = F \cdot \cos\beta \qquad M_y = M \cdot \cos\beta \tag{1.1.2}$$
$$F_z = F \cdot \cos\gamma \qquad M_z = M \cdot \cos\gamma \tag{1.1.3}$$

On the other hand, a set of vector forces can be composed in a resultant force applied to any point P, and the moment they produce about P.




$$\vec{F_R} = \sum \vec{F_i} \qquad \vec{M_R} = \sum \vec{F_i} \tag{1.1.4}$$

$$F_R = \sqrt{(F_x)^2 + (F_y)^2 + (F_z)^2} \tag{1.1.5}$$

$$M_R = \sqrt{(M_x)^2 + (M_y)^2 + (M_z)^2} \tag{1.1.6}$$

### 1.1.2 Equilibrium State

If a configuration is in equilibrium, the resultant of all external forces and moments is zero. This can be expressed mathematically in the following 6 equations:

$$\begin{array}{ccc} \sum F_x = 0 & \sum F_y = 0 & \sum F_z = 0 \\ \sum M_x = 0 & \sum M_y = 0 & \sum M_z = 0 \end{array}$$

These equations have to be valid for the entire body, and for any of its portions.

### 1.1.3 Deformations

Mechanical actions produce **deformations** in the body. These can be:

- Tension
- Compression
- Bending
- Twisting

These deformations translate into local strains and are opposed and balanced by internal reaction forces (and stresses), that guarantee the structural congruence of the body.

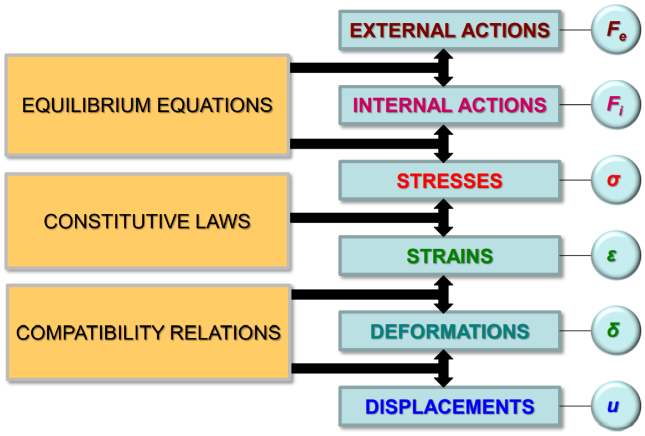
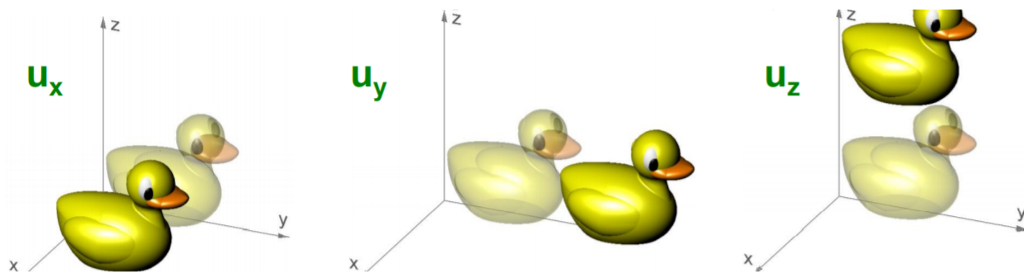



Figure 1.1: Solid Mechanics Equation: When dealing with mechanical action problems, the actions listed in the flowchart above occur, starting with external/internal forces and ending with displacements/deformations

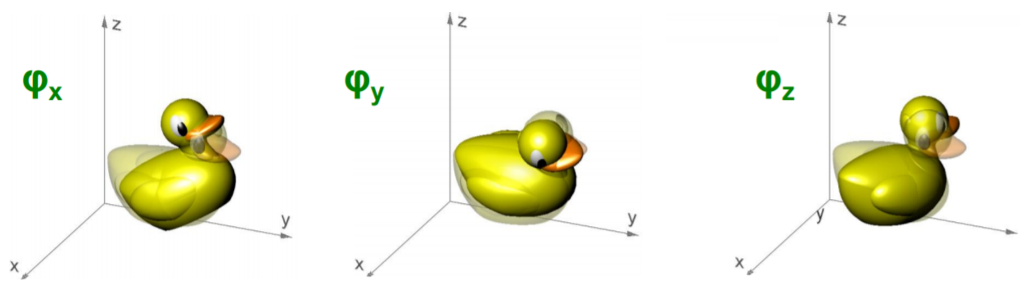
## 1.2 Degree of Freedom and Supports

We define **degree of freedom** of a system as all the basic kinematical parameters (or all the forms of movement) allowed. A rigid body in the space, in a coordinate system, has 6 degrees of freedom:

3 translations along the coordinate axes $x$, $y$ and $z$




3 rotations about the coordinate axes $x$, $y$ and $z$




The total translational and rotational movement of an object can be shown with the following expression:

$$\vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \\ \phi_x \\ \phi_y \\ \phi_z \end{bmatrix}$$

In a 2D plane, the degree of freedom reduces to pnly 3 variables:

$$\vec{u} = \begin{bmatrix} u_x \\ u_y \\ \phi_z \end{bmatrix}$$

### 1.2.1 Constraint

We define **constraint** as a limitation of the degree of freedom of the system. The most common constraints are:

- Supports providing the required reacting forces to maintain overall equilibrium
- Connections providing reaction forces between two components of the system

The table below summarizes the different types of supports (constraints) that will be used throughout the course:

| Fixed | Rotating | Roller | Sliding |
|---|---|---|---|
|  |  |  |  |
| $\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ | $\vec{u} = \begin{bmatrix} 0 \\ 0 \\ \phi \end{bmatrix}$ | $\vec{u} = \begin{bmatrix} u_x \\ 0 \\ \phi \end{bmatrix}$ | $\vec{u} = \begin{bmatrix} u_x \\ 0 \\ 0 \end{bmatrix}$ |
| $\vec{R} = \begin{bmatrix} R_x \\ R_y \\ M_z \end{bmatrix}$ | $\vec{R} = \begin{bmatrix} R_x \\ R_y \\ 0 \end{bmatrix}$ | $\vec{R} = \begin{bmatrix} 0 \\ R_y \\ 0 \end{bmatrix}$ | $\vec{R} = \begin{bmatrix} 0 \\ R_y \\ M \end{bmatrix}$ |

## 1.3 Beams and Sign Conventions

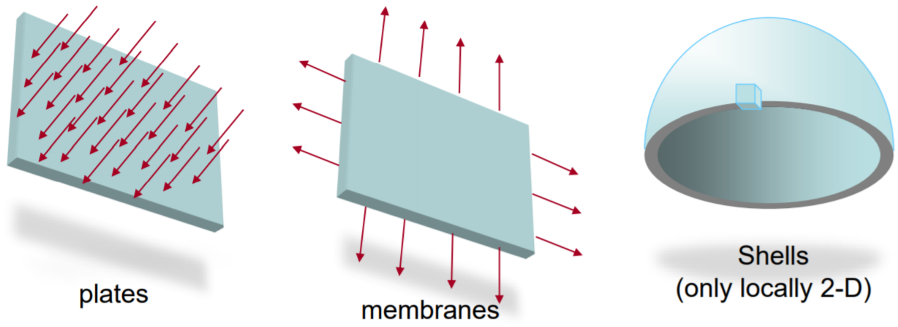
Structures are sets of solid bodies components with the function of carrying loads. All solid bodies are 3-dimensional, however, often it is possible to identify some dimension that is more relevant. Many structures can be analysed as bi-dimensional (2D) or mono-dimensional (1D).

### 1.3.1 Types of Structures

#### Bi-dimensional Structures

If one of the dimensions is negligible compared to the other two, the structure can be studied as bi-dimensional. Some examples are:

- Plates
- Membranes
- Shells (Only locally 2D)

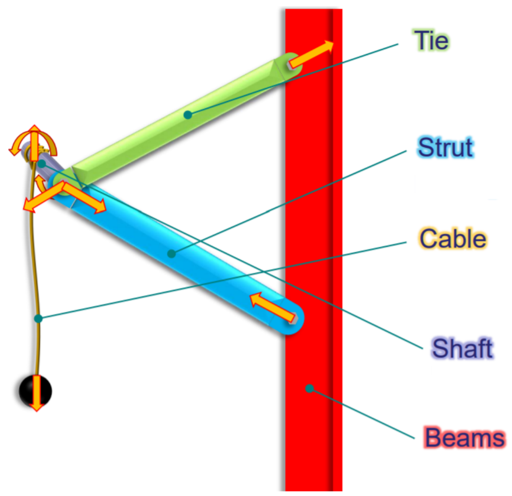



For shells to be considered bi-dimensional, they need to be looked at very closely where they can resemble a plate. The difference between plates and membranes is the bending rigidity; force is required to bend plates while membranes are really floppy.

**Mono-dimensional Structures**

If two of the dimensions are negligible compared to the other one, the structure can be studied as mono-dimensional. Some examples are:

- Tie - Prevents two parts of the structure from moving away
- Strut - Prevents two parts of the structure from moving forward
- Cable - Flexible string that stands only tensile loads
- Shaft - Is used for the transmission of torque
- Beams - Can carry also transverse loads




### 1.3.2 Beams

The generic mono-dimensional components of structures, able to carry also transverse loads are called **beams**. Beams are between the most common and important components in structures. In order to study beams as mono-dimensional structures, all mechanical actions have to act on the **centre of gravity (CG)** of the beam section. If there is a case where a force isn't acting on the CG, it will be converted to act on it, so that the beam can be analysed in a simple manner.

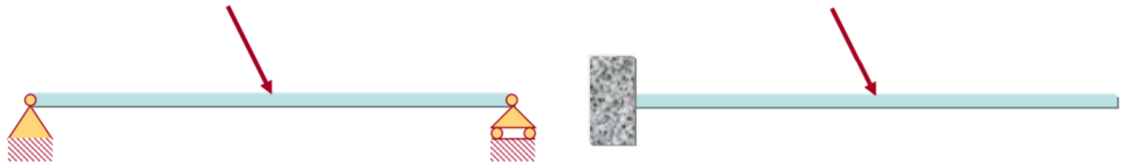
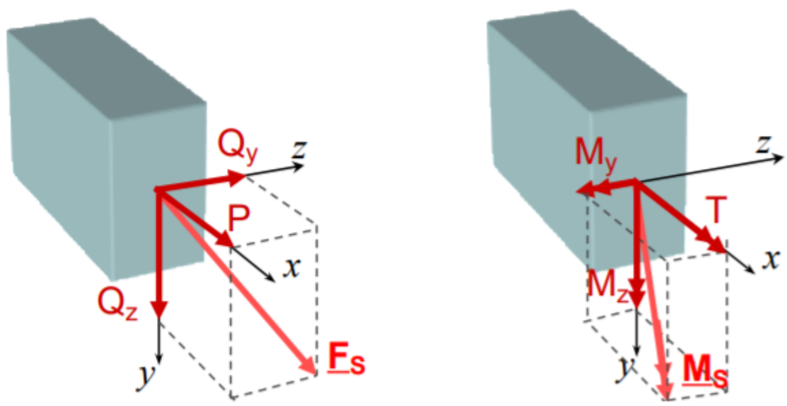


Figure 1.2: On the left: Simply supported beam (rotating support + roller support), On the right: Cantilever beam (with a fixed end)

### 1.3.3 Internal Forces

Each point of the beam is characterised by a specific set of internal forces. We consider a cross section of the beam to investigate these forces.

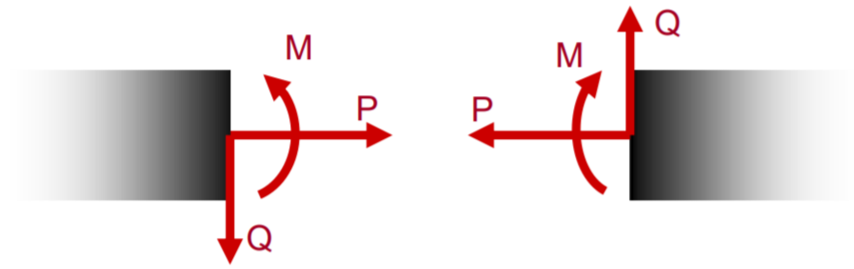



- $P$ - Longitudinal force
- $Q_y$ - $y$ shear forces
- $Q_z$ - $z$ shear forces
- $T$ - Torque
- $M_y$ - $y$ bending moment
- $M_z$ - $z$ bending moment

The internal forces at every point of the beam can be characterised with the following expression:

$$\vec{F} = \begin{bmatrix} P \\ Q_y \\ Q_z \\ \\ T \\ M_y \\ M_z \end{bmatrix}$$

In a 2D plane, it simplifies to:




- $P$ - Longitudinal force
- $Q$ - Shear forces
- $M$ - Bending moment

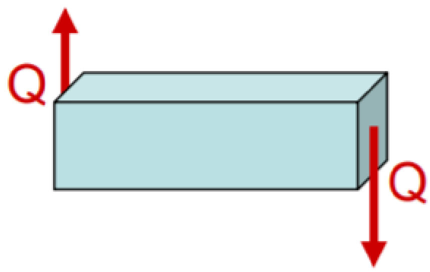
$$\vec{F} = \begin{bmatrix} P \\ Q \\ M \end{bmatrix}$$

### 1.3.4 Sign Conventions

**Longitudinal Force**


The direction of pulling is considered to be positive.
The direction of compression is considered to be negative.

**Shear Force**

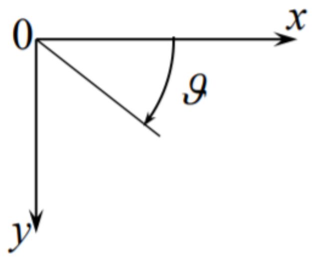



The left side pointing upwards and right side pointing downwards are taken as positive.
The left side pointing downwards and right side pointing upwards are taken as negative.

**Bending Moment**

If the bending moment is making the beam a upwards concave shape (like U), it is considered to be positive.
If the bending moment is making the beam a downwards concave shape, it is considered to be negative.

**Reference System**




$x$ axis (horizontal direction) to the right is taken as positive.
$y$ axis (vertical direction) downwards is taken as positive.

# Chapter 2

# Bending of Beams

09/10/2020

## 2.1 Internal Forces and Diagrams

### 2.1.1 Bending Moment

The **bending moment** is by far the most relevant of the internal forces, since it produces the largest levels of deformations and stress into the beam. Therefore, it is essential to be able to determine the distribution of the bending moment along the members, in order to assess the mechanical and functional safety of the structure.

### 2.1.2 Diagrams and Determination of Internal Forces

To determine the internal forces of a body, and draw the relevant diagrams, the following steps are followed:

#### Step 1

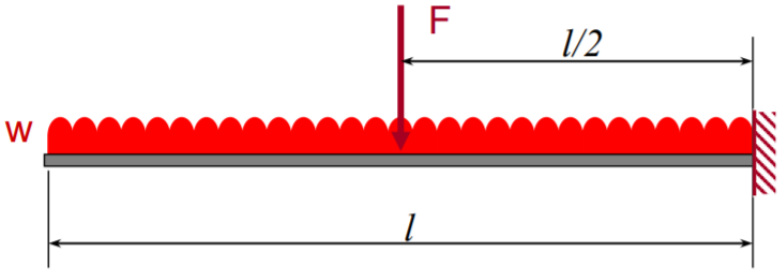
Apply to the body the force and moment equilibrium equations to find support reactions (it is possible only if the system is statically determinate)

#### Step 2

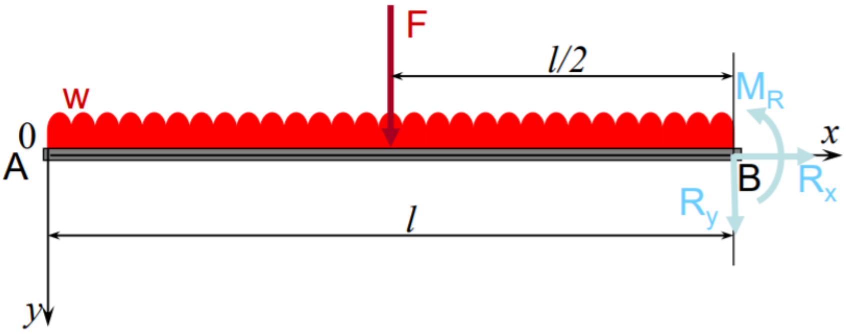
Imagine to cut the beam at a section at a distance $x$ from the beam extreme. Balance the forces and moments of the rigid body by applying the required forces and moment at the cut section

- The **Shear Force** on any given section of a structural member is the algebraic sum of the forces to **one side only** of the section considered.
- The **Bending Moment** on any given section of a structural member is the algebraic sum of the moments of all the forces to **one side only** of the section, about the section
- The maximum value of bending moment occurs at the point where the Shear Force is zero

**Example: Cantilever beam having combined concentrated and distributive loads**




**Determination of Support Reactions:**




$$\vec{R_B} = \begin{bmatrix} R_x \\ R_y \\ M \end{bmatrix}$$

$$\sum F_x : R_x = 0 \tag{2.1.1}$$

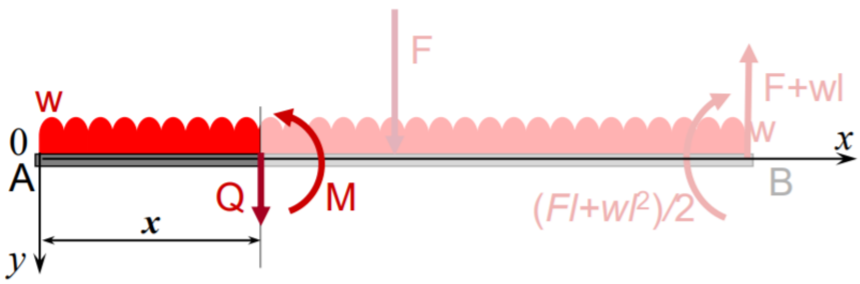
$$\sum F_y : R_y + F + wl = 0 \tag{2.1.2}$$

$$R_y = -(F + wl) \tag{2.1.3}$$

$$\sum M : M - F\frac{l}{2} - wl\frac{l}{2} = 0 \tag{2.1.4}$$

$$M = F\frac{l}{2} + wl\frac{l}{2} = \frac{Fl + wl^2}{2} \tag{2.1.5}$$

**Determination of internal forces (from $x = 0$ to $x = \frac{l}{2}$)**



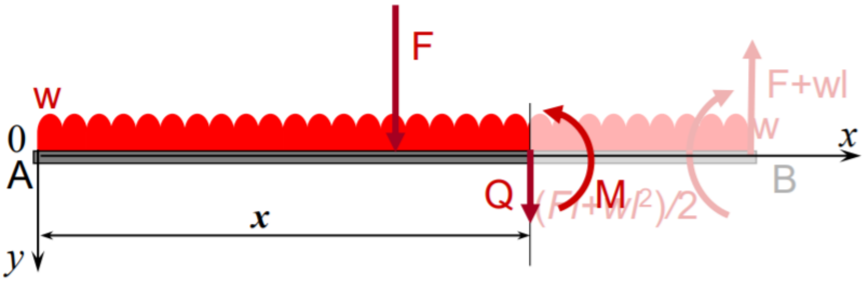
$$\sum F_y : Q + wx = 0 \tag{2.1.6}$$
$$Q = -wx \tag{2.1.7}$$

$Q$ varies linearly: it is zero at $x = 0$ and is $\frac{-wl}{2}$ at $x = \frac{l}{2}$.

$$\sum M : M_x + wx\frac{x}{2} = 0 \tag{2.1.8}$$
$$M_x = \frac{-wx^2}{2} \tag{2.1.9}$$

$M$ varies parabolically: it is zero at $x = 0$ and $\frac{-wl^2}{8}$ at $x = \frac{l}{2}$.

**Determination of internal forces (from $x = \frac{l}{2}$ to $x = l$)**



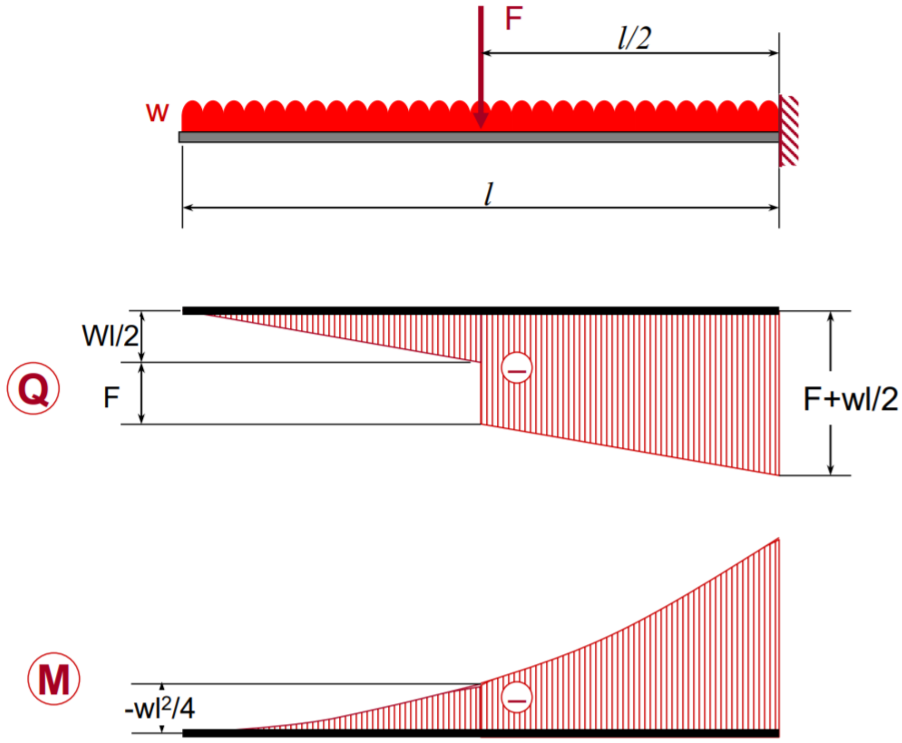

$$\sum F_y : Q + wx + F = 0 \tag{2.1.10}$$
$$Q = -wx - F \tag{2.1.11}$$

$Q$ varies linearly between $x = \frac{l}{2}$ (where it is $-(\frac{wl}{2} + F)$) and $x = l$ (where it is $-(F + wl)$)

$$\sum M : M_x + wx\frac{x}{2} + F(x - \frac{l}{2}) = 0 \tag{2.1.12}$$
$$M_x = \frac{-wx^2}{2} - F(x - \frac{l}{2}) \tag{2.1.13}$$

At $x = \frac{l}{2}$, $M$ distribution changes into a parabola with a steeper slope

**Diagram of Internal Forces**

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## 2.2 Derivation of Bending Equations

### 2.2.1 Equilibrium – Beam Bending Equations

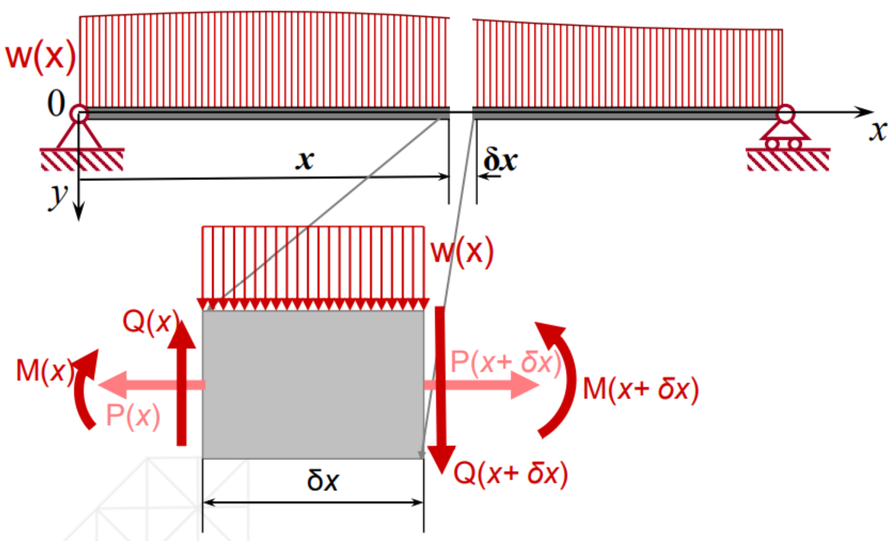



Figure 2.1: Horizontal beam with vertical distributed load. We focus a section of the beam with the very small length $\delta x$.

**y-direction**

$$Q(x+\delta x) - Q(x) + w(x)\delta x = 0 \tag{2.2.1}$$

**x-direction**

$$M(x+\delta x) - M(x) - Q(x)\delta x + w(x)\cdot\delta x\cdot\frac{\delta x}{2} = 0 \tag{2.2.2}$$

We assume that $\delta x$ is so small that no matter the distribution of the load on the beam, $w(x)$ is constant along the investigated beam section.

Rearranging the terms in the $y$ direction yields:

$$\frac{Q(x+\delta x) - Q(x)}{\delta x} = -w(x) \tag{2.2.3}$$

Taking $\delta x$ to the smallest limit (0), the above equation can be written as:

$$\delta x = 0 \tag{2.2.4}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -w(x) \tag{2.2.5}$$

$$w(x) = -\frac{\mathrm{d}Q}{\mathrm{d}x} \tag{2.2.6}$$

The load $w$ in the $y$ direction is the derivative of the shear force $Q$. Integrating equation (2.2.6):

$$Q = -\int w\,\mathrm{d}x \tag{2.2.7}$$

We consider the bending moment at the distance $\delta x$ (right-side for the example). Rearranging the terms in the $x$ direction yields:

$$\frac{M(x+\delta x) - M(x)}{\delta x} = Q(x) - \frac{1}{2}w(x)\delta x \tag{2.2.8}$$

Taking $\delta x$ to the smallest limit (0), the above equation can be written as:

$$\delta x = 0 \tag{2.2.9}$$

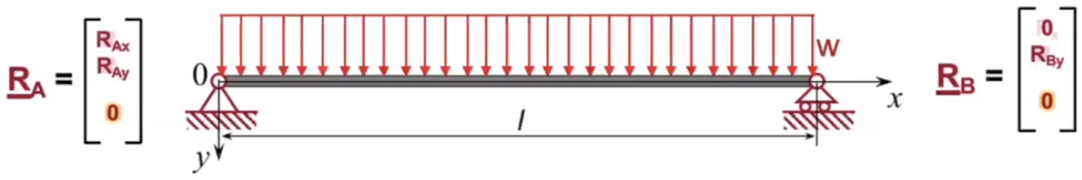
$$Q(x) = \frac{\mathrm{d}M}{\mathrm{d}x} \tag{2.2.10}$$

$$\tag{2.2.11}$$

The shear force $Q$ is the derivative of the bending Moment $M$. Integrating equation (2.2.11):

$$M = \int Q\,\mathrm{d}x \tag{2.2.12}$$

**Example:**




$$
\begin{aligned}
Q &= -\int w \, \mathrm{d}x \tag{2.2.13}\\
&= -wx + Q_0 \tag{2.2.14}\\
& \tag{2.2.15}\\
M &= \int Q \, \mathrm{d}x \tag{2.2.16}\\
&= \int (-wx + Q_0) \, \mathrm{d}x \tag{2.2.17}\\
&= -\frac{1}{2}wx^2 + Q_0 \cdot x + M_0 \tag{2.2.18}
\end{aligned}
$$

Applying the boundary conditions:

$$
\begin{aligned}
M(0) = 0 \rightarrow -\frac{1}{2}w \cdot 0^2 + Q_0 \cdot 0 + M_0 = 0 \tag{2.2.19}\\
M_0 = 0 \tag{2.2.20}\\
\tag{2.2.21}\\
M(l) = 0 \rightarrow -\frac{1}{2}wl^2 + Q_0 \cdot l + 0 = 0 \tag{2.2.22}\\
Q_0 = \frac{1}{2}wl \tag{2.2.23}
\end{aligned}
$$

Overall:

$$
\begin{aligned}
Q &= -wx + \frac{1}{2}wl \tag{2.2.24}\\
M &= -\frac{1}{2}wx^2 + \frac{1}{2}wlx \tag{2.2.25}
\end{aligned}
$$

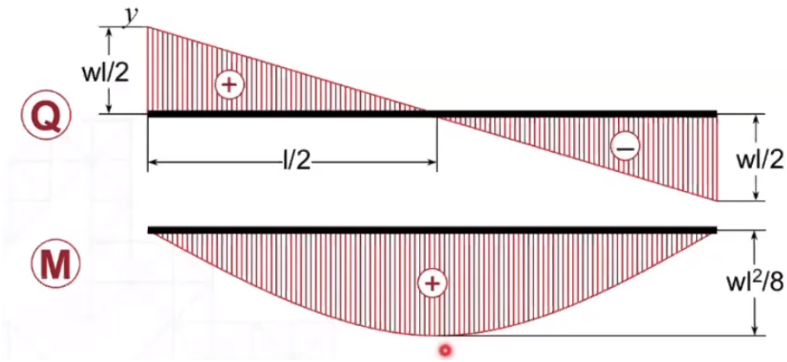



Figure 2.2: The shear force and bending moment varying along $x$

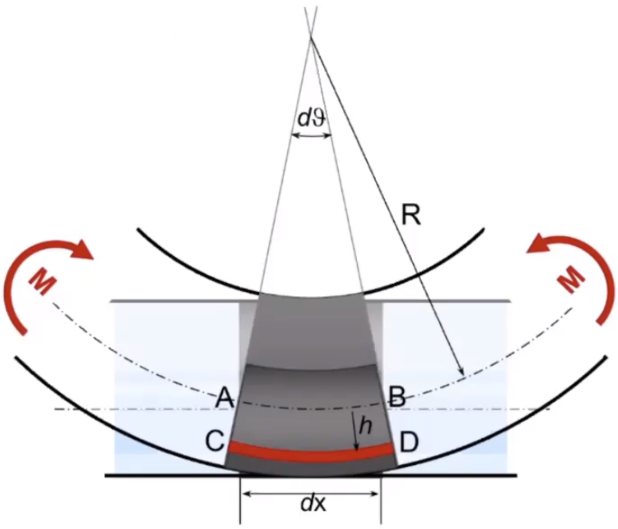
## 2.3 Differential Equations for Deflection

### 2.3.1 Theory of Pure Bending

- The beam is initially straight and unstressed;
- The beam material is perfectly homogeneous and isotropic;
- Plane cross-sections remain plane before and after bending;
- Every cross-section in the beam is symmetrical about the plane of bending;
- There is no resultant force perpendicular to any cross-section.
- The elastic limit is nowhere exceeded;
- Young's Modulus for the material is the same in tension and compression;

### 2.3.2 Compatibility - Strains in Pure Bending

Consider a portion of the length $\mathrm{d}x$ from the beam subject to uniform bending.




Lower fibres stretch and upper fibres shorten. Hence, since cross-sections remain plane, there must be a plane where the fibre elongation is zero. This is called **neutral plane** (it is a plane because section is symmetrical) and the intersection with the plane of bending is called **neutral axis**.

If we indicate with $R$ the radius of curvature of the neutral axis, along the neutral plane:

$$\overline{AB} = \widehat{AB} \tag{2.3.1}$$

$$\widehat{AB} = \mathrm{d}x = R\,\mathrm{d}\theta \tag{2.3.2}$$

For a generic plane CD, distant $h$ from N.A.:

$$\widehat{CD} = (R+h)\,\mathrm{d}\theta \tag{2.3.3}$$

The longitudinal strain of CD is given by:

$$\epsilon(h) = \frac{elongation}{initial\ length} = \frac{\widehat{CD} - \overline{CD}}{\overline{CD}} \tag{2.3.4}$$

$$\tag{2.3.5}$$

$$= \frac{(R+h)\cdot \mathrm{d}\theta - R\cdot \mathrm{d}\theta}{R\cdot \mathrm{d}\theta} = \frac{h}{R} \tag{2.3.6}$$

$$\tag{2.3.7}$$

$$\epsilon(h) = \frac{h}{R} \tag{2.3.8}$$

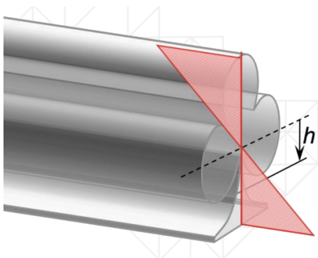



Figure 2.3: Strains are distributed linearly across the section

Stresses and strains can be associated with each other. For most materials, under small deformation:

$$\sigma = E \cdot \epsilon \tag{2.3.9}$$

Where:

- $\sigma$ is the stress
- $E$ is the Young Modulus
- $\epsilon$ is the strain

Therefore, from equations (2.3.8) and (2.3.9):

$$\sigma_x(h) = E\frac{h}{R} \tag{2.3.10}$$

### 2.3.3 Constitutive - Stress-Curvature Relation

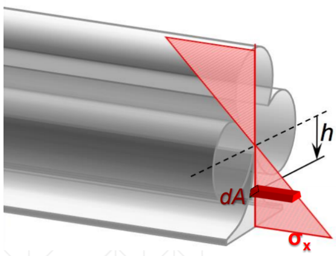
Stress is also distributed linearly across the section, being 0 at the neutral plane and maximum (in tension and compression) at the outer surfaces, where the distance from the neutral plane is maximum.

$$\sigma_{min} = E\frac{h_{min}}{R} \tag{2.3.11}$$

$$\sigma_{max} = E\frac{h_{max}}{R} \tag{2.3.12}$$

The minimum value is at the compression and the maximum value is at the tension. This just follows the convention; the tension is taken as positive and compression is taken as negative, in terms of the stress direction.

### 2.3.4 Equilibrium - Force-Stress Relation




Considering an elemental area d$A$, the force associated with bending stress is:

$$\mathrm{d}F_x = \sigma_x \cdot \mathrm{d}A \tag{2.3.13}$$

$$= E\frac{h}{R} \cdot \mathrm{d}A \tag{2.3.14}$$

For the force equilibrium of the entire section:

$$\int_A \mathrm{d}F_x = \int_A E\frac{h}{R} \cdot \mathrm{d}A = 0 \tag{2.3.15}$$

$$\tag{2.3.16}$$

$E$ and $R$ are constants, so they don't affect the integral and can be taken out. Hence the first moment of area is:

$$\int_A h \cdot \mathrm{d}A = 0 \tag{2.3.17}$$

The first moment of area of a section is zero if it is calculated about the centroid. **The neutral axis corresponds to the centroid of the section.**

### 2.3.5 Equilibrium - Bending-Stress Relation

Considering an elemental area d$A$, the internal moment produced by the bending stress is:

$$\begin{aligned} \mathrm{d}M &= \sigma_x \cdot h \cdot \mathrm{d}A && (2.3.18) \\ &= \mathrm{d}F_x \cdot h && (2.3.19) \\ &= E\frac{h}{R} \cdot h \cdot \mathrm{d}A && (2.3.20) \end{aligned}$$

Moment of the entire section:

$$M = \int_A \mathrm{d}M = \int_A E\frac{h^2}{R} \cdot \mathrm{d}A = \frac{E}{R}\int_A h^2 \cdot \mathrm{d}A \tag{2.3.21}$$

The second moment of area is:

$$\int_A h^2 \cdot \mathrm{d}A = I \tag{2.3.22}$$

The second moment of area represents how easy or difficult it is to bend a beam, depending on the shape of the cross section. The overall relation is:

$$M = \frac{EI}{R} \tag{2.3.23}$$

This defines how the cross section, elastic modulus, curvature, and bending moment are related to each other.

### 2.3.6 Solid Mechanics Equations

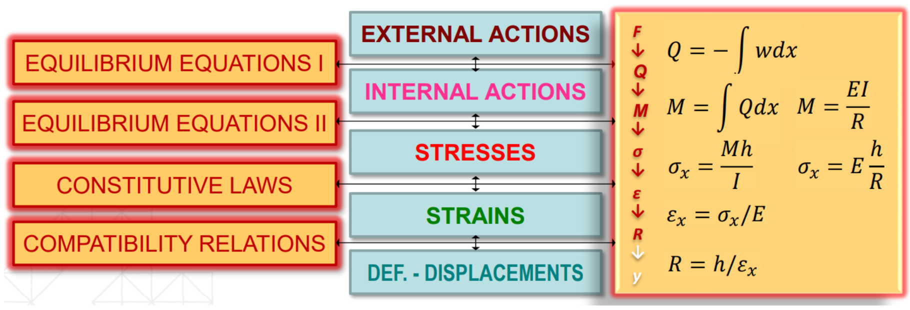



Figure 2.4: The relationships between Force, Shear Force, Bending Moment, Stress, Strain, Curvature

### 2.3.7 Geometric - Slope-Deflection Relation

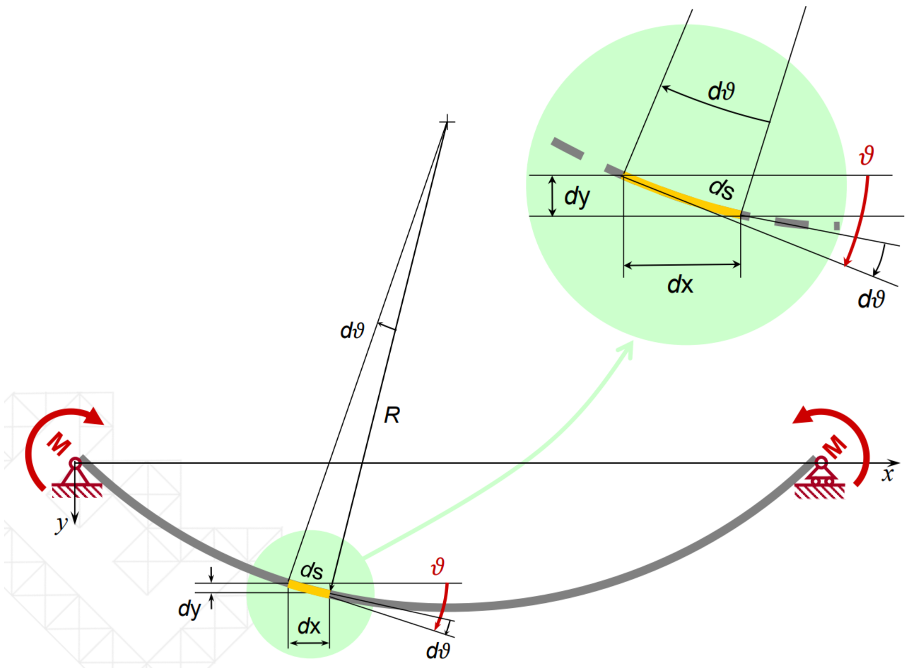



Figure 2.5: The geometry of the beam under bending. The direction of the angle $\theta$ is based on the "right-hand rule".

For infinitesimal deformations:

$$\mathrm{d}x \approx \mathrm{d}s = -R \cdot \mathrm{d}\theta \tag{2.3.24}$$

$$\frac{1}{R} = -\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{2.3.25}$$

The negative sign comes from the situation that the $x$ axis direction is towards right while the angle ($\theta$) direction is to the left. Hence, they are opposite. Since:

$$M = \frac{EI}{R} \tag{2.3.26}$$

$$\rightarrow M = -EI\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{2.3.27}$$

Assuming anlge $\theta$ is small:

$$\theta \approx \tan(\theta) = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{2.3.28}$$

- The load $w$ in the $y$ direction is the derivative of the shear force $Q$.
- The shear force $Q$ is the derivative of the bending moment $M$.
- The bending moment $M$ is proportional to the derivative of the slope $\theta$.
- The slope $\theta$ is the derivative of the deflection $y : \theta = \frac{\mathrm{d}y}{\mathrm{d}x}$

### 2.3.8 Stresses Due to Shear Force

The shear force also produces stresses into the section (shear stresses). However:

- They are much lower than bending stresses
- They are zero at surfaces, where bending stresses are maximum
- Their effect on the deformation is negligible compared to bending stresses

In general, neglecting the shear stresses due to the shear force is a good approximation for both the calculation of failure and deflections.

### 2.3.9 Summary of Beam Bending Equations

**Deflection:**

$$y \tag{2.3.29}$$

**Slope:**

$$\theta = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{2.3.30}$$

**Bending Moment:**

$$M = -EI\frac{\mathrm{d}\theta}{\mathrm{d}x} = -EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \tag{2.3.31}$$

**Shear Force:**

$$Q = \frac{\mathrm{d}M}{\mathrm{d}x} = -EI\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} \tag{2.3.32}$$

**Load Distribution:**

$$w = -\frac{\mathrm{d}Q}{\mathrm{d}x} = EI\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} \tag{2.3.33}$$

The equations above give a scenario of being given deflection $y$, and eventually finding load distribution $w$. However, the inverse can also occur where load distribution $w$ is given, and the $y$ can be found through integration:

**Load Distribution**

$$w \tag{2.3.34}$$

**Shear Force:**

$$Q = -\int w \cdot \mathrm{d}x \tag{2.3.35}$$

**Bending Moment:**

$$M = \int Q \cdot \mathrm{d}x = -\int\int w \cdot \mathrm{d}x\,\mathrm{d}x \tag{2.3.36}$$

**Slope:**

$$\theta = -\frac{1}{EI}\int M \cdot \mathrm{d}x = \frac{1}{EI}\int\int\int w \cdot \mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x \tag{2.3.37}$$

**Deflection:**

$$y = \int \theta \cdot \mathrm{d}x = \frac{1}{EI}\int\int\int\int w \cdot \mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x \tag{2.3.38}$$

### 2.3.10 Direct Integration Method

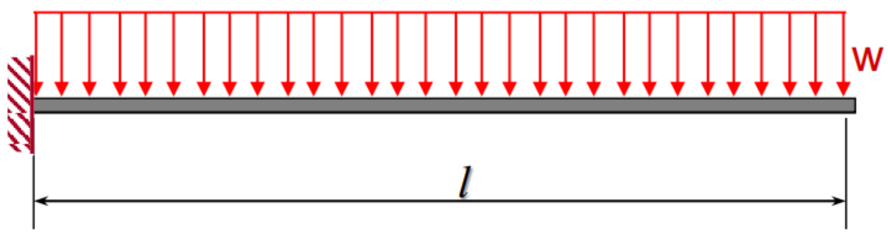
**STEP 1: Determination of Support Reactions**

Apply to the body the force and moment equilibrium equations to find support reactions (it is possible only if the system is statically determinate).

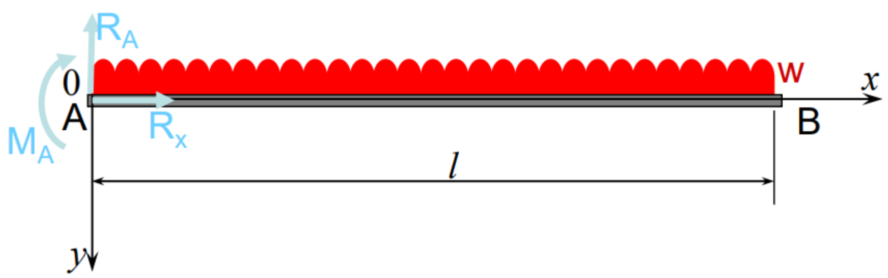
**STEP 2: Determination of Deflection**

1. Write bending moment expression
2. Use double integration on bending moment expression. This would result in 2 constants of integration.
3. Use boundary conditions to determinate constant of integration.

**Example: Uniformly Distributed Load on a Cantilever Beam**




**Determination of Support Reactions:**




$$\vec{R_A} = \begin{bmatrix} R_x \\ R_y \\ M_A \end{bmatrix}$$

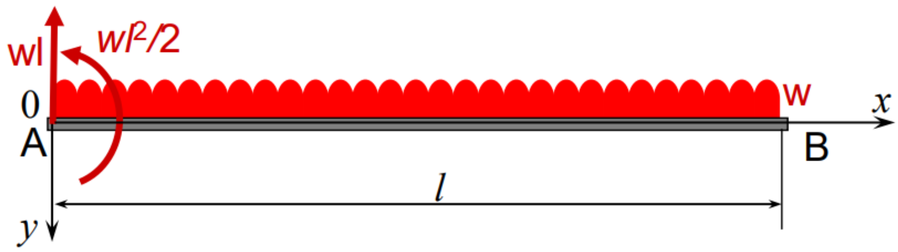
$$\sum F_x : R_x = 0 \tag{2.3.39}$$

$$\sum F_y : R_y = wl \tag{2.3.40}$$

$$\sum M : M_A + wl\frac{l}{2} = 0 \tag{2.3.41}$$

$$M_A = -\frac{wl^2}{2} \tag{2.3.42}$$

**Determination of Deflection:**




Direct Integration:

$$Q = -\int w \cdot \mathrm{d}x = -wx + Q_0 \tag{2.3.43}$$

$$\tag{2.3.44}$$

$$M = \int Q \cdot \mathrm{d}x \tag{2.3.45}$$

$$= \int (-wx + Q_0)\,\mathrm{d}x \tag{2.3.46}$$

$$= -\frac{1}{2}wx^2 + Q_0x + M_0 \tag{2.3.47}$$

Boundary Conditions:

$$Q(0) = R_y = wl \tag{2.3.48}$$

$$\rightarrow Q_0 = wl \tag{2.3.49}$$

$$M(0) = M_A = -\frac{1}{2}wl^2 \tag{2.3.50}$$

$$\rightarrow M_0 = -\frac{1}{2}wl^2 \tag{2.3.51}$$

Therefore:

$$Q = -w(x + l) \tag{2.3.52}$$

$$M = -\frac{1}{2}wx^2 + wlx - \frac{1}{2}wl^2 \tag{2.3.53}$$

Relating Deflection to the Bending Moment:

$$M = -\frac{1}{2}wx^2 + wlx - \frac{1}{2}wl^2 \tag{2.3.54}$$

$$\theta = -\frac{1}{EI}\int M \cdot \mathrm{d}x = -\frac{1}{EI}\left(-\frac{1}{6}wx^3 + \frac{1}{2}wlx^2 - \frac{1}{2}wl^2x\right) + \theta_0 \tag{2.3.55}$$

$$y = \int \theta \cdot \mathrm{d}x = -\frac{1}{EI}\left(-\frac{1}{24}wx^4 + \frac{1}{6}wlx^3 - \frac{1}{4}wl^2x^2\right) + \theta_0 x + y_0 \tag{2.3.56}$$

$$\tag{2.3.57}$$

Boundary Conditions:

$$\theta(0) = 0 \rightarrow \theta_0 = 0 \tag{2.3.58}$$

$$y(0) = 0 \rightarrow y_0 = 0 \tag{2.3.59}$$

Therefore:

$$y = \frac{1}{EI}\left(\frac{1}{24}wx^4 - \frac{1}{6}wlx^3 + \frac{1}{4}wl^2x^2\right) \tag{2.3.60}$$

$$y_{max} = y(l) = \frac{wl^4}{8EI} \tag{2.3.61}$$

# Chapter 3

# Beam buckling

# Chapter 4

# Plastic Collapse

# Chapter 5

# Principal Stresses and Failure Criteria

# Chapter 6

# Axisymmetric Stress and Strain

# Chapter 7

# Energy Methods