UCL Mechanical Engineering 2020/2021

ENGF0004 Coursework 1

NCWT3

January 7, 2021

1 Question One

a

Proof. Left hand side:

$$\sum_{n=0}^{\infty}\left(\frac{k-1}{k}\right)^n = 1 + \frac{k-1}{k} + \frac{(k-1)^2}{k^2} + ... \tag{1.1}$$

$$a = 1,\ r = \frac{k-1}{k} \tag{1.2}$$

$\frac{k-1}{k}$ is always less than 1 for $k > 1$. Hence:

$$S_{\infty,LHS} = \frac{a}{1-r} \tag{1.3}$$

$$= \frac{1}{1-\frac{k-1}{k}} \tag{1.4}$$

$$= \frac{k}{k-k+1} \tag{1.5}$$

$$S_{\infty,LHS} = k \tag{1.6}$$

Right hand side:

$$(k-1)\sum_{n=0}^{\infty}\left(\frac{1}{k}\right)^n = (k-1)\left[1 + \frac{1}{k} + \frac{1}{k^2} + ...\right] \tag{1.7}$$

$$a = 1,\ r = \frac{1}{k} \tag{1.8}$$

$\frac{1}{k}$ is always less than 1 for $k > 1$. Hence:

$$S_{\infty,RHS} = \frac{k-1}{1-\frac{1}{k}} \tag{1.9}$$

$$= \frac{k(k-1)}{k-1} \tag{1.10}$$

$$S_{\infty,RHS} = k \tag{1.11}$$

$$\tag{1.12}$$

LHS = RHS (for $k > 1$). □

b

We are given:

$$f(x) = \frac{x}{\sqrt{1-x}} \tag{1.13}$$
$$f(x) = x\,(1-x)^{-\frac{1}{2}} \tag{1.14}$$

Differentiating three times yields:

$$f'(x) = (1-x)^{-\frac{1}{2}} + \frac{x}{2}\,(1-x)^{-\frac{3}{2}} \tag{1.15}$$
$$f''(x) = \frac{1}{2}\,(1-x)^{-\frac{3}{2}} + \frac{1}{2}\,(1-x)^{-\frac{3}{2}} + \frac{3x}{4}\,(1-x)^{-\frac{5}{2}} \tag{1.16}$$
$$= (1-x)^{-\frac{3}{2}} + \frac{3x}{4}\,(1-x)^{-\frac{5}{2}} \tag{1.17}$$
$$f'''(x) = \frac{3}{2}\,(1-x)^{-\frac{5}{2}} + \frac{3}{4}\,(1-x)^{-\frac{5}{2}} + \frac{15x}{8}\,(1-x)^{-\frac{7}{2}} \tag{1.18}$$
$$= \frac{9}{4}\,(1-x)^{-\frac{5}{2}} + \frac{15x}{8}\,(1-x)^{-\frac{7}{2}} \tag{1.19}$$

Inputting $x = 0$:

$$f(0) = 0 \cdot (1-0)^{-\frac{1}{2}} \tag{1.20}$$
$$= 0 \tag{1.21}$$
$$f'(0) = (1-0)^{-\frac{1}{2}} + \frac{0}{2}\,(1-0)^{-\frac{3}{2}} \tag{1.22}$$
$$= 1 \tag{1.23}$$
$$f''(0) = (1-0)^{-\frac{3}{2}} + \frac{3 \cdot 0}{4}\,(1-0)^{-\frac{5}{2}} \tag{1.24}$$
$$= 1 \tag{1.25}$$
$$f'''(0) = \frac{9}{4}\,(1-0)^{-\frac{5}{2}} + \frac{15 \cdot 0}{8}\,(1-0)^{-\frac{7}{2}} \tag{1.26}$$
$$= \frac{9}{4} \tag{1.27}$$

General form of Maclaurin series:

$$f(x) \approx f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... \tag{1.28}$$

Inputting the above variables into Eq.1.28:

$$f(x) \approx x + \frac{x^2}{2} + \frac{3x^3}{8} \tag{1.29}$$

c

i

We are given:

$$E = \frac{kq}{x^2} \tag{1.30}$$

Sum of electric fields due to both charged particles is:

$$E = \frac{ke}{(x-r)^2} - \frac{ke}{(x+r)^2} \tag{1.31}$$

$$= ke\left[\frac{1}{x^2\left(1-\frac{r}{x}\right)^2} - \frac{1}{x^2\left(1+\frac{r}{x}\right)^2}\right] \tag{1.32}$$

$$E = \frac{ke}{x^2}\left[(1-y)^{-2} - (1+y)^{-2}\right] \tag{1.33}$$

Where $y = \frac{r}{x}$.

ii

Calculation of constants to be used in Maclaurin series expansion:

$$\begin{aligned} f(y) &= (1-y)^{-2} & f(0) &= 1 \\ f'(y) &= 2\,(1-y)^{-3} & f'(0) &= 2 \\ f''(y) &= 6\,(1-y)^{-4} & f''(0) &= 6 \\ f'''(y) &= 24\,(1-y)^{-5} & f'''(0) &= 24 \end{aligned}$$

$$\begin{aligned} g(y) &= (1+y)^{-2} & g(0) &= 1 \\ g'(y) &= -2\,(1+y)^{-3} & g'(0) &= -2 \\ g''(y) &= 6\,(1+y)^{-4} & g''(0) &= 6 \\ g'''(y) &= -24\,(1+y)^{-5} & g'''(0) &= -24 \end{aligned}$$

Inputting the above variables into Eq.1.28:

$$f(y) \approx 1 + \frac{2y}{1!} + \frac{6y^2}{2!} + \frac{24y^3}{3!} + ... \tag{1.34}$$

$$f(y) \approx 1 + 2y + 3y^2 + 4y^3 \tag{1.35}$$

$$g(y) \approx 1 - \frac{2y}{1!} + \frac{6y^2}{2!} - \frac{24y^3}{3!} + ... \tag{1.36}$$

$$g(y) \approx 1 - 2y + 3y^2 - 4y^2 \tag{1.37}$$

Substitution:

$$E \approx \frac{ke}{x^2}\left[f(y) - g(y)\right] \tag{1.38}$$

$$\approx \frac{ke}{x^2}\left[1 + 2y + 3y^2 + 4y^3 - 1 + 2y - 3y^2 + 4y^3\right] \tag{1.39}$$

$$\approx \frac{ke}{x^2}\left[4y + 8y^3\right] \tag{1.40}$$

$$E \approx \frac{4ke}{x^2}\left[y + 2y^3\right] \tag{1.41}$$

iii

$y = 0.01$. Exact:

$$E_E = \frac{4ke}{x^2}\left[(1-0.01)^{-2} - (1+0.01)^{-2}\right] \tag{1.42}$$
$$E_E = \frac{4ke}{x^2}\left[0.0400080012\right] \tag{1.43}$$
$$\tag{1.44}$$

Approximation:

$$E_A = \frac{4ke}{x^2}\left[0.01 + 2\left(0.01\right)^3\right] \tag{1.45}$$
$$E_A = \frac{4ke}{x^2}\left[0.010002\right] \tag{1.46}$$

Percentage error:

$$\frac{E_A}{E_E} \cdot 100 = \frac{0.0400080012 - 0.010002}{0.0400080012} \cdot 100 = 75\% \text{ error (2sf)} \tag{1.47}$$

d

We are given:

$$y'' - 2y' + y = te^t \tag{1.48}$$
$$y(0) = 0,\ y'(0) = 1 \tag{1.49}$$

Laplace transformation (from tables):

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{te^t\} \tag{1.50}$$
$$s^2Y(s) - sy(0) - y'(0) - 2\left(sY(s) - y(0)\right) + Y(s) = \frac{1!}{(s-1)^2} \tag{1.51}$$
$$s^2Y(s) - 1 - 2(sY(s) - 1) + Y(s) = \frac{1}{(s-1)^2} \tag{1.52}$$
$$Y(s)\left[s^2 - 2s + 1\right] = \frac{1}{(s-1)^2} \tag{1.53}$$
$$Y(s) = \frac{1}{(s-1)^2\left(s^2 - 2s + 1\right)} \tag{1.54}$$
$$= \frac{1}{(s-1)^2\,(s-1)^2} \tag{1.55}$$
$$Y(s) = \frac{1}{(s-1)^4} \tag{1.56}$$

e

i

$a = 1 \therefore -3 \leq t \leq 3$. Sketch:

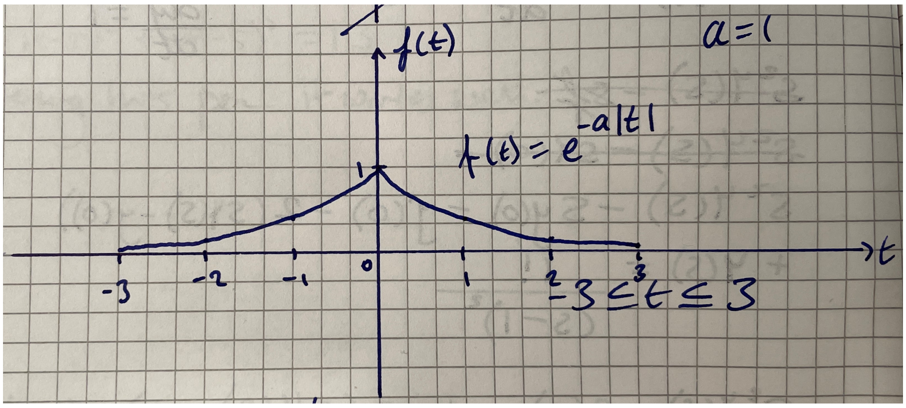

Figure 1:

ii

$$F(u) = \int_{t=-\infty}^{0} e^{at} e^{-j2\pi ut}\ \mathrm{dt} + \int_{t=0}^{\infty} e^{-at} e^{-j2\pi ut}\ \mathrm{dt} \tag{1.57}$$

$$= \int_{t=-\infty}^{0} e^{t(a-j2\pi u)}\ \mathrm{dt} + \int_{t=0}^{\infty} e^{-t(a+j2\pi u)}\ \mathrm{dt} \tag{1.58}$$

$$= \frac{1}{(a-j2\pi u)} e^{-t(a-j2\pi u)}\bigg|_{t=-\infty}^{0} + \frac{1}{-(a+j2\pi u)} e^{-t(a+j2\pi u)}\bigg|_{t=0}^{\infty} \tag{1.59}$$

$$= \frac{1}{a-j2\pi u} + \frac{1}{a+j2\pi u} \tag{1.60}$$

$$= \frac{a+j2\pi u + a - j2\pi u}{a^2+4\pi^2u^2} \tag{1.61}$$

$$F(u) = \frac{2a}{a^2+4\pi^2u^2} \tag{1.62}$$

iii

Substituting $\omega = 2\pi u$, $\omega^2 = 4\pi^2u^2$:

$$F(\omega) = \frac{2a}{a^2+\omega^2} \tag{1.63}$$

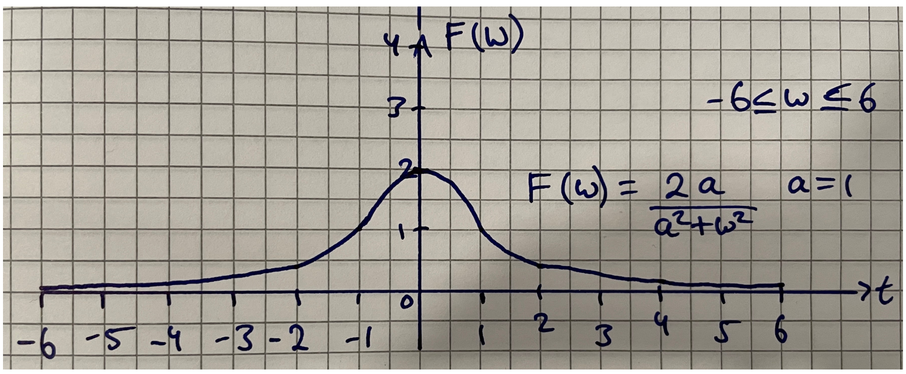

Figure 2:

iv

Full width at half maximum of $F(u)$ can be calculated as:

$$\frac{1}{2} = e^{-at} \tag{1.64}$$

$$\ln\left(\frac{1}{2}\right) = -at \tag{1.65}$$

$$-\ln(2) = at \tag{1.66}$$

$$t = \frac{\ln(2)}{a} \to \text{HWHM} \tag{1.67}$$

$$\therefore t = \frac{2\ln(2)}{a} \to \text{FWHM} \tag{1.68}$$

Full width at half maximum of $F(\omega)$ can be calculated as:

$$\frac{1}{2} \cdot \frac{2}{a} = 2\frac{a}{a^2+\omega^2} \tag{1.69}$$

$$\frac{1}{a} = \frac{2a}{a^2+\omega^2} \tag{1.70}$$

$$\omega^2 + a^2 = 2a^2 \tag{1.71}$$

$$\omega = a \to \text{HWHM} \tag{1.72}$$

$$\therefore \omega = 2a \to \text{FWHM} \tag{1.73}$$

v

Product of FWHMs:

$$2a \cdot \frac{2\ln(2)}{a} = 4\ln(2) \tag{1.74}$$

The product has no a term, thus there is no dependence on the parameter.

2 Question Two

a

i

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{2.1}$$

We know that $u(x,t) = X(x)T(t)$. Substituting:

$$X(x)T'(t) = kX''(x)T(t) \tag{2.2}$$

$$\frac{T'(t)}{kT'(t)} = \frac{X''(x)}{X(x)} = -\mu \tag{2.3}$$

$$-X''(x) = \mu X(x) \tag{2.4}$$

$$T'(t) = -\mu k T(t) \tag{2.5}$$

where μ is an arbitrary constant.

μ can be either positive, zero or negative. Let us consider these cases.

Case 1: positive $\mu = \lambda^2 > 0$

$$\frac{X''(x)}{X(x)} = \lambda^2 \tag{2.6}$$

$$X''(x) - \lambda^2 X(x) = 0 \tag{2.7}$$

Auxiliary equation:

$$m^2 - \lambda^2 = 0 \tag{2.8}$$

$$m_1 = \lambda,\ m_2 = -\lambda \tag{2.9}$$

Real and distinct roots. Hence:

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \tag{2.10}$$

Applying boundary conditions for x. Applying $u(0,t) = 0$ implies $X(0)T(t) = 0$. Substituting $x = 0$ in $X(x)$ gives:

$$X(0) = A + B \tag{2.11}$$

$$(A + B)\,T(t) = 0 \tag{2.12}$$

$$T(t) = 0 \text{ or } A + B = 0 \tag{2.13}$$

$T(t) = 0$ leads to a trivial solution of $u(x,t) = X(x)T(t) = 0$. We also have $A + B = 0$ or $A = -B$:

$$X(x) = A\left(e^{\lambda x} - e^{-\lambda x}\right) \tag{2.14}$$

Applying $u(l,t) = 0$ implies $X(l)T(t) = 0$. Substituting $x = l$ in $X(x)$ gives: $X(l) = A\left(e^{\lambda l} - e^{-\lambda l}\right)$

$$A\left(e^{\lambda l} - e^{-\lambda l}\right) T(t) = 0 \tag{2.15}$$

$$A = 0,\ T(t) = 0,\ \text{or } e^{\lambda l} - e^{-\lambda l} = 0 \tag{2.16}$$

$T(t) = 0$ and $A = 0$ both will lead to a trivial solution of $u(x,t) = X(x)T(t) = 0$. $e^{\lambda l} - e^{-\lambda l} = 0$ is only true if $\lambda = 0$. However, the assumption in this case is $\lambda = \sqrt{\mu}$ where μ is a positive constant ($\lambda > 0$). Therefore, this boundary condition cannot be met and there are no useful solutions from Case 1.

Case 2: zero constant $\lambda = 0$

$$\frac{X''(x)}{X(x)} = \lambda = 0 \tag{2.17}$$

$$X''(x) = 0 \tag{2.18}$$

$$\int X''(x)\,\mathrm{d}x = \int \mathrm{d}x \tag{2.19}$$

$$X'(x) = C \tag{2.20}$$

$$\int X'(x) = \int C\,\mathrm{d}x \tag{2.21}$$

$$X(x) = Cx + D \tag{2.22}$$

where C and D are constants of integration. Applying boundary conditions for x. Applying $u(0,t) = 0$ implies $X(0)T(t) = 0$. Substituting $x = 0$ in $X(x)$ gives:

$$X(0) = D \tag{2.23}$$
$$DT(t) = 0T(t) = 0 \text{ or } D = 0 \tag{2.24}$$

$T(t) = 0$ will lead to a trivial solution of $u(x,t) = X(x)T(t) = 0$. Therefore, $D = 0$.

$$X(x) = Cx \tag{2.25}$$

Applying $u(l,t) = 0$ implies $X(l)X(t) = 0$. Substituting $x = l$ in $X(x)$ gives:

$$X(l) = Cl \tag{2.26}$$
$$ClT(t) = 0 \tag{2.27}$$
$$T(t) = 0 \text{ or } C = 0 \tag{2.28}$$

$T(t) = 0$ and $C = 0$ both will lead to a trivial solution of $u(x,t) = X(x)T(t) = 0$. Case 2 only produces the trivial solution of $u(x,t) = X(x)T(t) = 0$ and does not produce any useful solutions.

iii

Using Fourier series:

$$c_n = \frac{2}{l}\int_0^l \big(f(x)\sin(n\pi x)\big)\ \mathrm{d}x \tag{2.29}$$

We are given:

$$u(x,0) = f(x) = x^2,\ 0 \le x \le l \tag{2.30}$$
$$u(0,t) = u(l,t) = 0,\ t > 0 \tag{2.31}$$
$$\tag{2.32}$$

Let $a = \frac{n\pi}{l}$. Substituting the above into Fourier:

$$c_n = \frac{2}{l}\int_0^l \Big(x^2\sin(ax)\Big)\ \mathrm{d}x \tag{2.33}$$
$$\tag{2.34}$$

Integration by parts once. $u = x^2$, $u' = 2x$, $v = -\frac{\cos(ax)}{n\pi}$ and $v' = \sin(ax)$.

$$c_n = \frac{2}{l}\left[-\frac{x^2\cos(ax)}{a} + \frac{2}{a}\int_0^l \big(x\cos(ax)\big)\ \mathrm{d}x\right]_0^l \tag{2.35}$$

Integration by parts twice. $u = x$, $u = 1$, $v = \frac{\sin(ax)}{a}$ and $v' = \cos(ax)$.

$$c_n = \frac{2}{l}\left[-\frac{x^2\cos(ax)}{a} + \frac{2}{a}\left[\frac{x\sin(ax)}{a} - \frac{1}{a}\int_0^l \big(\sin(ax)\big)\,\mathrm{d}x\right]_0^l\right]_0^l \tag{2.36}$$

$$= 2\left[-\frac{x^2\cos(n\pi x)}{n\pi} + \frac{2}{n\pi}\left[\frac{x\sin(n\pi x)}{n\pi} - \frac{1}{n\pi}\left[-\frac{\cos(n\pi x)}{n\pi}\right]\right]_0^l\right]_0^l \tag{2.37}$$

$$= \frac{2}{l}\left[-\frac{x^2\cos(ax)}{n\pi} + \frac{2x\sin(ax)}{a^2} + \frac{2\cos(ax)}{a^3}\right]_0^l \tag{2.38}$$

$$= \frac{2}{l}\left[-\frac{l^3\cos(n\pi x)}{n\pi} + \frac{2l^3\sin(n\pi x)}{n^2\pi^2} + \frac{2l^3\cos(n\pi x)}{n^3\pi^3}\right]_0^l \tag{2.39}$$

$$= \frac{2l^2}{n\pi}\left[-\cos(n\pi x) + \frac{2\sin(n\pi x)}{n\pi} + \frac{2\cos(n\pi x)}{n^2\pi^2}\right]_0^l \tag{2.40}$$

$$c_n = \frac{2l^2}{n\pi}\left[-\cos(n\pi) + \frac{2\sin(n\pi)}{n\pi} + \frac{2\left(\cos(n\pi) - 1\right)}{n^2\pi^2}\right] \tag{2.41}$$

Substituting:

$$\therefore u_n(x,t) = \sum_{n=1}^{\infty}\left(\frac{2l^2}{n\pi}\left[-\cos(n\pi) + \frac{2\sin(n\pi)}{n\pi} + \frac{2\left(\cos(n\pi) - 1\right)}{n^2\pi^2}\right]\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot e^{\frac{n^2\pi^2}{l^2}kt}\right) \tag{2.42}$$

When n is even:

$$u_n(x,t) = \sum_{n=1}^{\infty}\left(-\frac{2l^2}{n\pi}\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot e^{\frac{n^2\pi^2}{l^2}kt}\right) \tag{2.43}$$

When n is odd:

$$u_n(x,t) = \sum_{n=1}^{\infty}\left(-\frac{2l^2}{n\pi}\left[1 - \frac{4}{n^2\pi^2}\right]\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot e^{\frac{n^2\pi^2}{l^2}kt}\right) \tag{2.44}$$

b

i

ii

iii

iv

v

3 EXTRA STUFF

q2aii

Where μ is the eigenvalue. Returning to our PDE in Eq.2.1, if u_1 and u_2 are solutions to the PDE, so are $C_1 u_1$ and $C_2 u_2$:

$$\frac{\partial C_1 u_1}{\partial t} = k\frac{\partial^2 C_1 u_1}{\partial x^2} \tag{3.1}$$

$$\frac{\partial u_1}{\partial t} = k\frac{\partial^2 u_1}{\partial x^2} \tag{3.2}$$

Similarly $C_1 u_1 + C_2 u_2$ will satisfy the PDE:

$$U = C_1 u_1 + C_2 u_2 + ... + C_n u_n \tag{3.3}$$

We can also form the following equation:

$$A\underline{X} = \mu \underline{X} \tag{3.4}$$

Where $A = n \times n$ matrix, $\underline{X} = n \times 1$ column vector and μ is a scalar.

$$A\underline{X} - \mu\underline{X} = 0 \tag{3.5}$$

$$(A - \mu I)\,\underline{X} = 0 \tag{3.6}$$

We have:

$$\det(A - \mu I)\,\underline{X} = 0 \tag{3.7}$$

$$|A - \mu I| = 0 \tag{3.8}$$

Where $\underline{X}$ is the eigenvector.

$$X''(x) + \mu X(x) = 0 \tag{3.9}$$

$$\text{Let } X(x) = e^{mx} \tag{3.10}$$

$$X'(x) = me^{mx} \tag{3.11}$$

$$X''(x) = m^2 e^{mx} \tag{3.12}$$

$$m^2 + \mu = 0 \tag{3.13}$$

$$m = \pm j\sqrt{\mu} \tag{3.14}$$

$$\tag{3.15}$$

General solution:

$$X(x) = A\cos\left(\sqrt{\mu}x\right) + B\sin\left(\sqrt{\mu}x\right) \tag{3.16}$$

Boundary conditions:

$$u\left(0,t\right) = 0 \tag{3.17}$$
$$X(0)T(t) = 0 \tag{3.18}$$
$$\tag{3.19}$$

We require that $X(0) = 0$. Hence:

$$X(0) = A\cos\left(0\right) + B\sin\left(0\right) \tag{3.20}$$
$$\therefore A = 0 \tag{3.21}$$
$$X(x) = B\sin\left(\sqrt{\mu}x\right) \tag{3.22}$$
$$u\left(l,t\right) = 0 \tag{3.23}$$
$$X(l) = B\sin\left(\sqrt{\mu}l\right) = 0 \tag{3.24}$$
$$\sqrt{\mu}l = n\pi \text{ where } n = 1,\ 2,\ 3,\ ... \tag{3.25}$$
$$\mu = \frac{n^2\pi^2}{l^2} \tag{3.26}$$

Returning to Eq.**??**:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k\mu T \tag{3.27}$$
$$\int\left(\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t}\right)\,\mathrm{d}t = -k\mu\int\,\mathrm{d}t \tag{3.28}$$
$$\ln T = -k\mu t + \ln B \tag{3.29}$$
$$\ln\left(\frac{T}{B}\right) = -k\mu t \tag{3.30}$$
$$\frac{T}{B} = e^{--k\mu t} \tag{3.31}$$
$$T = Be^{-k\mu t},\ \mu = \frac{n^2\pi^2}{l^2} \tag{3.32}$$
$$\therefore T = Be^{-k\frac{n^2\pi^2}{l^2}t} \tag{3.33}$$

$u(x,t) = X(x)T(t)$. Hence, by utilising principle of superposition:

$$u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{l}x\right) e^{-k\frac{n^2\pi^2}{l^2}t} \tag{3.34}$$

We are given that $u(x,0) = f(x)$. Hence, $f(x)$ is:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{l}x\right) \tag{3.35}$$