

0.1 Exercise 2 - Fluid tutorial group B

A viscous fluid of constant density, ρ , and kinematic viscosity, ν , flows over a flat plate inclined at an angle α and moving with a constant velocity V_w . The flow is stationary and no pressure gradients are applied. The only body force acting on the fluid is due to gravity, g . You can assume zero velocity component in the direction orthogonal to the plate and negligible air resistance at the interface between the viscous fluid and air ($y = h$).

0.1.1 Determine by means of the Navier-Stokes equation the velocity profile, $u(y)$

NSE (x,y):

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] + f_x \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] + f_y \quad (2)$$

- $\frac{\partial}{\partial t} = 0$ steady flow
- $v = 0$ negligible
- $\frac{\partial}{\partial x}$ no variation in flow along plate length

$$0 + 0 + 0 = 0 + \nu \frac{\partial^2 u}{\partial y^2} + g \sin \alpha \quad (3)$$

$$0 + 0 + 0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} - g \cos \alpha \quad (4)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{g \sin(\alpha)}{\nu} \quad (5)$$

$$\frac{\partial p}{\partial y} = -\rho g \cos \alpha \quad (6)$$

Integrating equation (5) once:

$$\int \left(\frac{\partial^2 u}{\partial y^2} \right) dy = \int \left(-\frac{g \sin \alpha}{\nu} \right) dy \frac{\partial u}{\partial y} = -\frac{g \sin \alpha}{\nu} y + C \quad (7)$$

Apply the following boundary equation, due to 0 air resistance at $y = h$ boundary:

$$\tau(h) = \mu \frac{\partial u}{\partial y} \Big|_{y=h} = 0 \therefore 0 = -\frac{g \sin \alpha}{\nu} h + C = \frac{g \sin \alpha}{\nu} h \quad (8)$$

Integrate equation (8) once more:

$$\int \left(\frac{\partial u}{\partial y} \right) dy = \int \left(-\frac{g \sin \alpha}{\nu} y + \frac{g \sin \alpha}{\nu} h \right) dy \quad (9)$$

$$u(y) = -\frac{g \sin \alpha}{2\nu} y^2 + \frac{g \sin \alpha}{\nu} hy + D \quad (10)$$

$$y(0) = -V_w = D \quad (11)$$

$$u(y) = -\frac{g \sin \alpha}{2\nu} y^2 + \frac{g \sin \alpha}{\nu} hy - V_w \quad (12)$$

$$u(y) = \frac{g \sin \alpha}{\nu} y \left(h - \frac{1}{2}y \right) - V_w \quad (13)$$

0.1.2 If the net flow rate across the fluid height is zero, determine the corresponding plate velocity, V_{wo} , as a function of α

Volume flow rate:

$$\int_0^h u(y) dy = \int_0^h \left(\frac{g \sin \alpha}{\nu} \left(hy - \frac{1}{2}y^2 \right) - V_w \right) dy \quad (14)$$

$$= \left[\frac{g \sin \alpha}{\nu} y \left(\frac{hy^2}{2} - \frac{1}{6}y^3 \right) - V_w y \right]_0^h \quad (15)$$

$$= \frac{g \sin \alpha h^3}{3\nu} - V_w h \quad (16)$$

This is equal to 0 for V_{wo} , hence:

$$V_{wo} = \frac{g \sin \alpha h^2}{3\nu} \quad (17)$$

0.1.3 For the condition found above determine the y coordinate corresponding to $u = 0$. Sketch three velocity profiles for $V_{wo} < V_w$, $V_{wo} = V_w$ and $V_{wo} > V_w$ and comment them

Combining above equations for $u(y)$, we get:

$$u(y) = \frac{g \sin \alpha}{\nu} y \left(-\frac{1}{2}y^2 + hy - \frac{1}{3}h^2 \right) \quad (18)$$

Set this equal to 0 and solve:

$$0 = -\frac{1}{2}y^2 + hy - \frac{1}{3}h^2 \quad (19)$$

$$y = \frac{3 - \sqrt{3}}{3} \cdot h \quad (20)$$

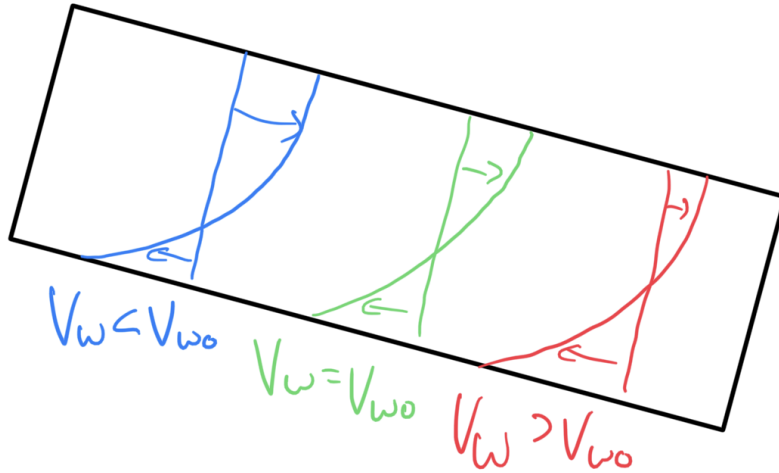


Figure 1: We can see that in the blue one velocity is increased towards the air boundary. As we move along to the point where $V_w = V_{wo}$, we see that the point where $u = 0$ is $\frac{1}{2}h$. Finally we see the velocity highest close to the plate boundary.

0.1.4 If the flat plate surface is S , determine the force, F , and power, P , required to move it with constant velocity V_{wo}

Using tensor equation:

$$\tau = \mu \frac{\partial u}{\partial y} \Big|_{y=0} = \rho g h \sin \alpha \quad (21)$$

Power is simply $F \times A \times V$:

$$P = \tau S V_w = \rho g h \sin \alpha S V_w \quad (22)$$