

UCL Mechanical Engineering 2020/2021

ENGF0004 Coursework 1

NCWT3

January 7, 2021

1 Question One

a

Proof. Left hand side:

$$\sum_{n=0}^{\infty} \left(\frac{k-1}{k} \right)^n = 1 + \frac{k-1}{k} + \frac{(k-1)^2}{k^2} + \dots \quad (1.1)$$

$$a = 1, \quad r = \frac{k-1}{k} \quad (1.2)$$

$\frac{k-1}{k}$ is always less than 1 for $k > 1$. Hence:

$$S_{\infty, LHS} = \frac{a}{1-r} \quad (1.3)$$

$$= \frac{1}{1 - \frac{k-1}{k}} \quad (1.4)$$

$$= \frac{k}{k - k + 1} \quad (1.5)$$

$$S_{\infty, LHS} = k \quad (1.6)$$

Right hand side:

$$(k-1) \sum_{n=0}^{\infty} \left(\frac{1}{k} \right)^n = (k-1) \left[1 + \frac{1}{k} + \frac{1}{k^2} + \dots \right] \quad (1.7)$$

$$a = 1, \quad r = \frac{1}{k} \quad (1.8)$$

$\frac{1}{k}$ is always less than 1 for $k > 1$. Hence:

$$S_{\infty, RHS} = \frac{k-1}{1 - \frac{1}{k}} \quad (1.9)$$

$$= \frac{k(k-1)}{k-1} \quad (1.10)$$

$$S_{\infty, RHS} = k \quad (1.11)$$

$$(1.12)$$

LHS = RHS (for $k > 1$). □

b

We are given:

$$f(x) = \frac{x}{\sqrt{1-x}} \quad (1.13)$$

$$f(x) = x(1-x)^{-\frac{1}{2}} \quad (1.14)$$

Differentiating three times yields:

$$f'(x) = (1-x)^{-\frac{1}{2}} + \frac{x}{2}(1-x)^{-\frac{3}{2}} \quad (1.15)$$

$$f''(x) = \frac{1}{2}(1-x)^{-\frac{3}{2}} + \frac{1}{2}(1-x)^{-\frac{3}{2}} + \frac{3x}{4}(1-x)^{-\frac{5}{2}} \quad (1.16)$$

$$= (1-x)^{-\frac{3}{2}} + \frac{3x}{4}(1-x)^{-\frac{5}{2}} \quad (1.17)$$

$$f'''(x) = \frac{3}{2}(1-x)^{-\frac{5}{2}} + \frac{3}{4}(1-x)^{-\frac{5}{2}} + \frac{15x}{8}(1-x)^{-\frac{7}{2}} \quad (1.18)$$

$$= \frac{9}{4}(1-x)^{-\frac{5}{2}} + \frac{15x}{8}(1-x)^{-\frac{7}{2}} \quad (1.19)$$

Inputting $x = 0$:

$$f(0) = 0 \cdot (1-0)^{-\frac{1}{2}} \quad (1.20)$$

$$= 0 \quad (1.21)$$

$$f'(0) = (1-0)^{-\frac{1}{2}} + \frac{0}{2}(1-0)^{-\frac{3}{2}} \quad (1.22)$$

$$= 1 \quad (1.23)$$

$$f''(0) = (1-0)^{-\frac{3}{2}} + \frac{3 \cdot 0}{4}(1-0)^{-\frac{5}{2}} \quad (1.24)$$

$$= 1 \quad (1.25)$$

$$f'''(0) = \frac{9}{4}(1-0)^{-\frac{5}{2}} + \frac{15 \cdot 0}{8}(1-0)^{-\frac{7}{2}} \quad (1.26)$$

$$= \frac{9}{4} \quad (1.27)$$

General form of Maclaurin series:

$$f(x) \approx f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots \quad (1.28)$$

Inputting the above variables into Eq.1.28:

$$f(x) \approx x + \frac{x^2}{2} + \frac{3x^3}{8} \quad (1.29)$$

c

i

We are given:

$$E = \frac{kq}{x^2} \quad (1.30)$$

Sum of electric fields due to both charged particles is:

$$E = \frac{ke}{(x-r)^2} - \frac{ke}{(x+r)^2} \quad (1.31)$$

$$= ke \left[\frac{1}{x^2 \left(1 - \frac{r}{x}\right)^2} - \frac{1}{x^2 \left(1 + \frac{r}{x}\right)^2} \right] \quad (1.32)$$

$$E = \frac{ke}{x^2} \left[(1-y)^{-2} - (1+y)^{-2} \right] \quad (1.33)$$

Where $y = \frac{r}{x}$.

ii

Calculation of constants to be used in Maclaurin series expansion:

$$\begin{aligned} f(y) &= (1-y)^{-2} & f(0) &= 1 \\ f'(y) &= 2(1-y)^{-3} & f'(0) &= 2 \\ f''(y) &= 6(1-y)^{-4} & f''(0) &= 6 \\ f'''(y) &= 24(1-y)^{-5} & f'''(0) &= 24 \end{aligned}$$

$$\begin{aligned} g(y) &= (1+y)^{-2} & g(0) &= 1 \\ g'(y) &= -2(1+y)^{-3} & g'(0) &= -2 \\ g''(y) &= 6(1+y)^{-4} & g''(0) &= 6 \\ g'''(y) &= -24(1+y)^{-5} & g'''(0) &= -24 \end{aligned}$$

Inputting the above variables into Eq.1.28:

$$f(y) \approx 1 + \frac{2y}{1!} + \frac{6y^2}{2!} + \frac{24y^3}{3!} + \dots \quad (1.34)$$

$$f(y) \approx 1 + 2y + 3y^2 + 4y^3 \quad (1.35)$$

$$g(y) \approx 1 - \frac{2y}{1!} + \frac{6y^2}{2!} - \frac{24y^3}{3!} + \dots \quad (1.36)$$

$$g(y) \approx 1 - 2y + 3y^2 - 4y^3 \quad (1.37)$$

Substitution:

$$E \approx \frac{ke}{x^2} [f(y) - g(y)] \quad (1.38)$$

$$\approx \frac{ke}{x^2} [1 + 2y + 3y^2 + 4y^3 - 1 + 2y - 3y^2 + 4y^3] \quad (1.39)$$

$$\approx \frac{ke}{x^2} [4y + 8y^3] \quad (1.40)$$

$$E \approx \frac{4ke}{x^2} [y + 2y^3] \quad (1.41)$$

iii

$y = 0.01$. Exact:

$$E_E = \frac{4ke}{x^2} \left[(1 - 0.01)^{-2} - (1 + 0.01)^{-2} \right] \quad (1.42)$$

$$E_E = \frac{4ke}{x^2} [0.0400080012] \quad (1.43)$$

$$(1.44)$$

Approximation:

$$E_A = \frac{4ke}{x^2} \left[0.01 + 2(0.01)^3 \right] \quad (1.45)$$

$$E_A = \frac{4ke}{x^2} [0.010002] \quad (1.46)$$

Percentage error:

$$\frac{E_A}{E_E} \cdot 100 = \frac{0.0400080012 - 0.010002}{0.0400080012} \cdot 100 = 75\% \text{ error (2sf)} \quad (1.47)$$

d

We are given:

$$y'' - 2y' + y = te^t \quad (1.48)$$

$$y(0) = 0, \quad y'(0) = 1 \quad (1.49)$$

Laplace transformation (from tables):

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{te^t\} \quad (1.50)$$

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + Y(s) = \frac{1!}{(s-1)^2} \quad (1.51)$$

$$s^2Y(s) - 1 - 2(sY(s) - 1) + Y(s) = \frac{1}{(s-1)^2} \quad (1.52)$$

$$Y(s) \left[s^2 - 2s + 1 \right] - 1 = \frac{1}{(s-1)^2} \quad (1.53)$$

$$Y(s) = \frac{1}{(s-1)^2} + 1 \quad (1.54)$$

$$Y(s) = \frac{1}{(s-1)^4} + \frac{1}{(s-1)^2} \quad (1.55)$$

Returning to time domain. From tables:

$$L^{-1} \left[\frac{n!}{(s-a)^n} \right] = t^n e^{at} \quad (1.56)$$

$$L^{-1} \left[\frac{1}{(s-1)^2} \right] = te^t \quad (1.57)$$

$$\frac{1}{6} L^{-1} \left[\frac{3!}{(s-1)^2} \right] = \frac{1}{6} t^3 e^t \quad (1.58)$$

$$y(t) = \frac{1}{6} t^3 e^t + te^t \quad (1.59)$$

e

i

$a = 1 \therefore -3 \leq t \leq 3$. Sketch:

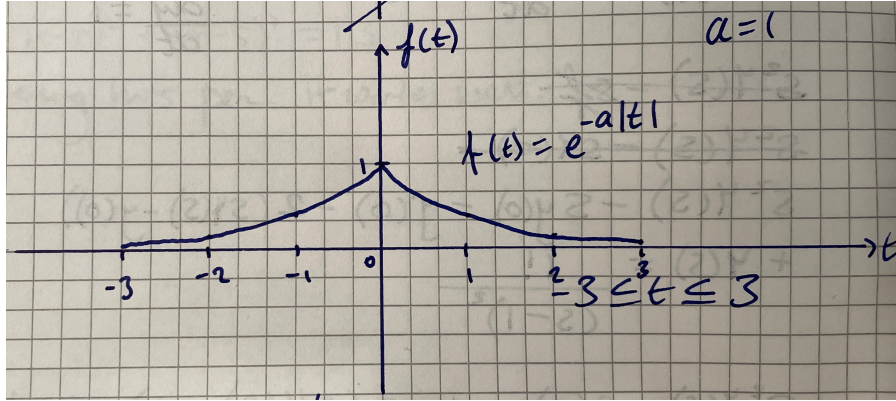


Figure 1:

ii

Let $z = \infty$.

$$F(u) = \lim_{z \rightarrow \infty} \int_{t=-z}^0 e^{at} e^{-j2\pi ut} dt + \lim_{z \rightarrow \infty} \int_{t=0}^z e^{-at} e^{-j2\pi ut} dt \quad (1.60)$$

$$= \lim_{z \rightarrow \infty} \int_{t=-\infty}^0 e^{t(a-j2\pi u)} dt + \lim_{z \rightarrow \infty} \int_{t=0}^{\infty} e^{-t(a+j2\pi u)} dt \quad (1.61)$$

$$F(u) = \lim_{z \rightarrow \infty} \left[\frac{1}{(a-j2\pi u)} e^{t(a-j2\pi u)} \Big|_{t=-z}^0 \right] + \lim_{z \rightarrow \infty} \left[\frac{1}{-(a+j2\pi u)} e^{-t(a+j2\pi u)} \Big|_{t=0}^z \right] \quad (1.62)$$

Applying limits:

$$\lim_{z \rightarrow \infty} \left[\frac{1}{(a-j2\pi u)} \left[1 - e^{-z(a-j2\pi u)} \right] \right] = \left[\frac{1}{(a-j2\pi u)} [1 - 0] \right] = \frac{1}{a-j2\pi u} \quad (1.63)$$

$$\lim_{z \rightarrow \infty} \left[\frac{-1}{(a+j2\pi u)} \left[e^{-z(a+j2\pi u)} - 1 \right] \right] = \left[\frac{-1}{(a+j2\pi u)} [0 - 1] \right] = \frac{1}{a+j2\pi u} \quad (1.64)$$

$$F(u) = \frac{1}{a-j2\pi u} + \frac{1}{a+j2\pi u} \quad (1.65)$$

$$= \frac{a+j2\pi u + a-j2\pi u}{a^2 + 4\pi^2 u^2} \quad (1.66)$$

$$F(u) = \frac{2a}{a^2 + 4\pi^2 u^2} \quad (1.67)$$

iii

Substituting $\omega = 2\pi u$, $\omega^2 = 4\pi^2 u^2$:

$$F(\omega) = \frac{2a}{a^2 + \omega^2} \quad (1.68)$$

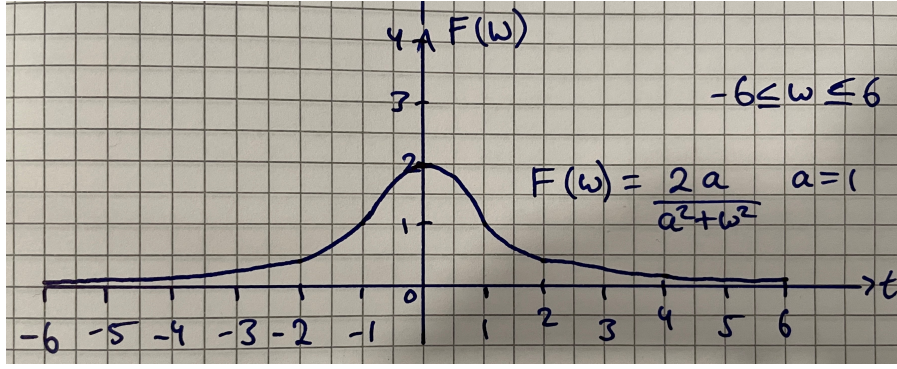


Figure 2:

iv

Full width at half maximum of $F(u)$ can be calculated as:

$$\frac{1}{2} = e^{-at} \quad (1.69)$$

$$\ln\left(\frac{1}{2}\right) = -at \quad (1.70)$$

$$-\ln(2) = at \quad (1.71)$$

$$t = \frac{\ln(2)}{a} \rightarrow \text{HWHM} \quad (1.72)$$

$$\therefore t = \frac{2 \ln(2)}{a} \rightarrow \text{FWHM} \quad (1.73)$$

Full width at half maximum of $F(\omega)$ can be calculated as:

$$\frac{1}{2} \cdot \frac{2}{a} = 2 \frac{a}{a^2 + \omega^2} \quad (1.74)$$

$$\frac{1}{a} = \frac{2a}{a^2 + \omega^2} \quad (1.75)$$

$$\omega^2 + a^2 = 2a^2 \quad (1.76)$$

$$\omega = a \rightarrow \text{HWHM} \quad (1.77)$$

$$\therefore \omega = 2a \rightarrow \text{FWHM} \quad (1.78)$$

v

Product of FWHMs:

$$2a \cdot \frac{2 \ln(2)}{a} = 4 \ln(2) \quad (1.79)$$

The product has no a term, thus there is no dependence on the parameter.

2 Question Two

a

i

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad (2.1)$$

We know that $u(x, t) = X(x)T(t)$. Substituting:

$$X(x)T'(t) = kX''(x)T(t) \quad (2.2)$$

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\mu \quad (2.3)$$

$$-X''(x) = \mu X(x) \quad (2.4)$$

$$T'(t) = -\mu kT(t) \quad (2.5)$$

where μ is an arbitrary constant.

ii

μ can be either positive, zero or negative. Let us consider these cases.

Case 1: positive $\mu = \lambda^2 > 0$

$$\frac{X''(x)}{X(x)} = \lambda^2 \quad (2.6)$$

$$X''(x) - \lambda^2 X(x) = 0 \quad (2.7)$$

Auxiliary equation:

$$m^2 - \lambda^2 = 0 \quad (2.8)$$

$$m_1 = \lambda, \quad m_2 = -\lambda \quad (2.9)$$

Real and distinct roots. Hence:

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad (2.10)$$

Applying boundary conditions for x . Applying $u(0, t) = 0$ implies $X(0)T(t) = 0$. Substituting $x = 0$ in $X(x)$ gives:

$$X(0) = A + B \quad (2.11)$$

$$(A + B)T(t) = 0 \quad (2.12)$$

$$T(t) = 0 \text{ or } A + B = 0 \quad (2.13)$$

$T(t) = 0$ leads to a trivial solution of $u(x, t) = X(x)T(t) = 0$. We also have $A + B = 0$ or $A = -B$:

$$X(x) = A(e^{\lambda x} - e^{-\lambda x}) \quad (2.14)$$

Applying $u(l, t) = 0$ implies $X(l)T(t) = 0$. Substituting $x = l$ in $X(x)$ gives: $X(l) = A(e^{\lambda l} - e^{-\lambda l})$

$$A(e^{\lambda l} - e^{-\lambda l})T(t) = 0 \quad (2.15)$$

$$A = 0, \quad T(t) = 0, \text{ or } e^{\lambda l} - e^{-\lambda l} = 0 \quad (2.16)$$

$T(t) = 0$ and $A = 0$ both will lead to a trivial solution of $u(x, t) = X(x)T(t) = 0$. $e^{\lambda l} - e^{-\lambda l} = 0$ is only true if $\lambda = 0$. However, the assumption in this case is $\lambda = \sqrt{\mu}$ where μ is a positive constant ($\lambda > 0$). Therefore, this boundary condition cannot be met and there are no useful solutions from Case 1.

Case 2: zero constant $\lambda = 0$

$$\frac{X''(x)}{X(x)} = \lambda = 0 \quad (2.17)$$

$$X''(x) = 0 \quad (2.18)$$

$$\int X''(x) dx = \int dx \quad (2.19)$$

$$X'(x) = C \quad (2.20)$$

$$\int X'(x) = \int C dx \quad (2.21)$$

$$X(x) = Cx + D \quad (2.22)$$

where C and D are constants of integration. Applying boundary conditions for x . Applying $u(0, t) = 0$ implies $X(0)T(t) = 0$. Substituting $x = 0$ in $X(x)$ gives:

$$X(0) = D \quad (2.23)$$

$$DT(t) = 0T(t) = 0 \text{ or } D = 0 \quad (2.24)$$

$T(t) = 0$ will lead to a trivial solution of $u(x, t) = X(x)T(t) = 0$. Therefore, $D = 0$.

$$X(x) = Cx \quad (2.25)$$

Applying $u(l, t) = 0$ implies $X(l)X(t) = 0$. Substituting $x = l$ in $X(x)$ gives:

$$X(l) = Cl \quad (2.26)$$

$$ClT(t) = 0 \quad (2.27)$$

$$T(t) = 0 \text{ or } C = 0 \quad (2.28)$$

$T(t) = 0$ and $C = 0$ both will lead to a trivial solution of $u(x, t) = X(x)T(t) = 0$. Case 2 only produces the trivial solution of $u(x, t) = X(x)T(t) = 0$ and does not produce any useful solutions.

iii

Using Fourier series:

$$c_n = \frac{2}{l} \int_0^l (f(x) \sin(n\pi x)) dx \quad (2.29)$$

We are given:

$$u(x, 0) = f(x) = x^2, \quad 0 \leq x \leq l \quad (2.30)$$

$$u(0, t) = u(l, t) = 0, \quad t > 0 \quad (2.31)$$

$$(2.32)$$

Let $a = \frac{n\pi}{l}$. Substituting the above into Fourier:

$$c_n = \frac{2}{l} \int_0^l (x^2 \sin(ax)) dx \quad (2.33)$$

$$(2.34)$$

Integration by parts once. $u = x^2$, $u' = 2x$, $v = -\frac{\cos(ax)}{n\pi}$ and $v' = \sin(ax)$.

$$c_n = \frac{2}{l} \left[-\frac{x^2 \cos(ax)}{a} + \frac{2}{a} \int_0^l (x \cos(ax)) \, dx \right]_0^l \quad (2.35)$$

Integration by parts twice. $u = x$, $u' = 1$, $v = \frac{\sin(ax)}{a}$ and $v' = \cos(ax)$.

$$c_n = \frac{2}{l} \left[-\frac{x^2 \cos(ax)}{a} + \frac{2}{a} \left[\frac{x \sin(ax)}{a} - \frac{1}{a} \int_0^l (\sin(ax)) \, dx \right]_0^l \right]_0^l \quad (2.36)$$

$$= 2 \left[-\frac{x^2 \cos(n\pi x)}{n\pi} + \frac{2}{n\pi} \left[\frac{x \sin(n\pi x)}{n\pi} - \frac{1}{n\pi} \left[-\frac{\cos(n\pi x)}{n\pi} \right]_0^l \right]_0^l \right]_0^l \quad (2.37)$$

$$= \frac{2}{l} \left[-\frac{x^2 \cos(ax)}{n\pi} + \frac{2x \sin(ax)}{a^2} + \frac{2 \cos(ax)}{a^3} \right]_0^l \quad (2.38)$$

$$= \frac{2}{l} \left[-\frac{l^3 \cos(n\pi x)}{n\pi} + \frac{2l^3 \sin(n\pi x)}{n^2 \pi^2} + \frac{2l^3 \cos(n\pi x)}{n^3 \pi^3} \right]_0^l \quad (2.39)$$

$$= \frac{2l^2}{n\pi} \left[-\cos(n\pi x) + \frac{2 \sin(n\pi x)}{n\pi} + \frac{2 \cos(n\pi x)}{n^2 \pi^2} \right]_0^l \quad (2.40)$$

$$c_n = \frac{2l^2}{n\pi} \left[-\cos(n\pi) + \frac{2 \sin(n\pi)}{n\pi} + \frac{2 (\cos(n\pi) - 1)}{n^2 \pi^2} \right] \quad (2.41)$$

Substituting:

$$\therefore u_n(x, t) = \sum_{n=1}^{\infty} \left(\frac{2l^2}{n\pi} \left[-\cos(n\pi) + \frac{2 \sin(n\pi)}{n\pi} + \frac{2 (\cos(n\pi) - 1)}{n^2 \pi^2} \right] \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot e^{\frac{n^2 \pi^2}{l^2} kt} \right) \quad (2.42)$$

When n is even:

$$u_n(x, t) = \sum_{n=1}^{\infty} \left(-\frac{2l^2}{n\pi} \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot e^{\frac{n^2 \pi^2}{l^2} kt} \right) \quad (2.43)$$

When n is odd:

$$u_n(x, t) = \sum_{n=1}^{\infty} \left(-\frac{2l^2}{n\pi} \left[1 - \frac{4}{n^2 \pi^2} \right] \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot e^{\frac{n^2 \pi^2}{l^2} kt} \right) \quad (2.44)$$

b

i

ii

iii

iv

v

3 EXTRA STUFF

q2aii

Where μ is the eigenvalue. Returning to our PDE in Eq.2.1, if u_1 and u_2 are solutions to the PDE, so are C_1u_1 and C_2u_2 :

$$\frac{\partial C_1u_1}{\partial t} = k \frac{\partial^2 C_1u_1}{\partial x^2} \quad (3.1)$$

$$\frac{\partial u_1}{\partial t} = k \frac{\partial^2 u_1}{\partial x^2} \quad (3.2)$$

Similarly $C_1u_1 + C_2u_2$ will satisfy the PDE:

$$U = C_1u_1 + C_2u_2 + \dots + C_nu_n \quad (3.3)$$

We can also form the following equation:

$$A\underline{X} = \mu\underline{X} \quad (3.4)$$

Where $A = n \times n$ matrix, $\underline{X} = n \times 1$ column vector and μ is a scalar.

$$A\underline{X} - \mu\underline{X} = 0 \quad (3.5)$$

$$(A - \mu I) \underline{X} = 0 \quad (3.6)$$

We have:

$$\det(A - \mu I) \underline{X} = 0 \quad (3.7)$$

$$|A - \mu I| = 0 \quad (3.8)$$

Where \underline{X} is the eigenvector.

$$X''(x) + \mu X(x) = 0 \quad (3.9)$$

$$\text{Let } X(x) = e^{mx} \quad (3.10)$$

$$X'(x) = me^{mx} \quad (3.11)$$

$$X''(x) = m^2e^{mx} \quad (3.12)$$

$$m^2 + \mu = 0 \quad (3.13)$$

$$m = \pm j\sqrt{\mu} \quad (3.14)$$

$$(3.15)$$

General solution:

$$X(x) = A \cos(\sqrt{\mu}x) + B \sin(\sqrt{\mu}x) \quad (3.16)$$

Boundary conditions:

$$u(0, t) = 0 \quad (3.17)$$

$$X(0)T(t) = 0 \quad (3.18)$$

$$(3.19)$$

We require that $X(0) = 0$. Hence:

$$X(0) = A \cos(0) + B \sin(0) \quad (3.20)$$

$$\therefore A = 0 \quad (3.21)$$

$$X(x) = B \sin(\sqrt{\mu}x) \quad (3.22)$$

$$u(l, t) = 0 \quad (3.23)$$

$$X(l) = B \sin(\sqrt{\mu}l) = 0 \quad (3.24)$$

$$\sqrt{\mu}l = n\pi \text{ where } n = 1, 2, 3, \dots \quad (3.25)$$

$$\mu = \frac{n^2\pi^2}{l^2} \quad (3.26)$$

Returning to Eq.??:

$$\frac{dT}{dt} = -k\mu T \quad (3.27)$$

$$\int \left(\frac{1}{T} \frac{dT}{dt} \right) dt = -k\mu \int dt \quad (3.28)$$

$$\ln T = -k\mu t + \ln B \quad (3.29)$$

$$\ln \left(\frac{T}{B} \right) = -k\mu t \quad (3.30)$$

$$\frac{T}{B} = e^{-k\mu t} \quad (3.31)$$

$$T = Be^{-k\mu t}, \quad \mu = \frac{n^2\pi^2}{l^2} \quad (3.32)$$

$$\therefore T = Be^{-k\frac{n^2\pi^2}{l^2}t} \quad (3.33)$$

$u(x, t) = X(x)T(t)$. Hence, by utilising principle of superposition:

$$u_n(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{l}x\right) e^{-k\frac{n^2\pi^2}{l^2}t} \quad (3.34)$$

We are given that $u(x, 0) = f(x)$. Hence, $f(x)$ is:

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{l}x\right) \quad (3.35)$$