0.1 Thin-walled cylinders

Thin walled cylinders have a wall thickness t much smaller than the cylinder radius R (at least one twentieth).
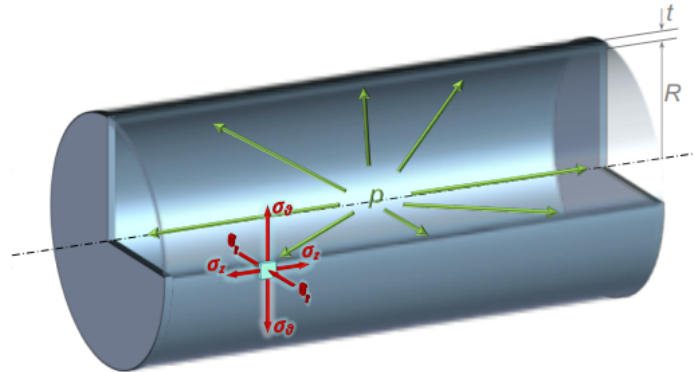

Figure 1:

The **hoop stresses** σ_θ can be considered **constant** across the wall thickness t. The **radial stresses** σ_t is **negligible** in comparison to the hoop stresses σ_θ. Under these conditions, the state of stress at each point of the cylinder can be estimated to good accuracy by simple equilibrium considerations.

Figure 2: Hoop stress - equilibrium of transversal forces.

$$p \cdot l \cdot 2R = 2\left(\sigma_\theta t l\right) \sigma_\theta = \frac{pR}{t} \tag{1}$$

Under these conditions, the state of stress at each point of the cylinder can be estimated to good accuracy by simple equilibrium considerations.
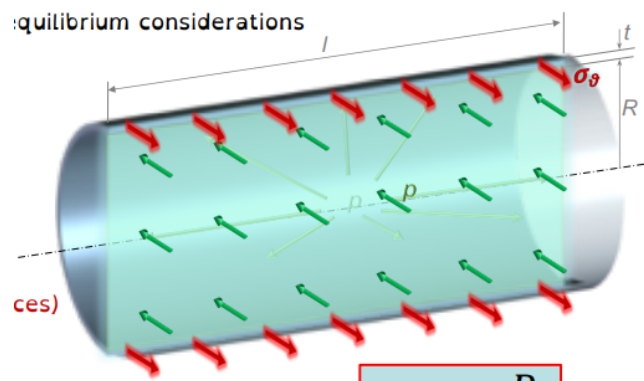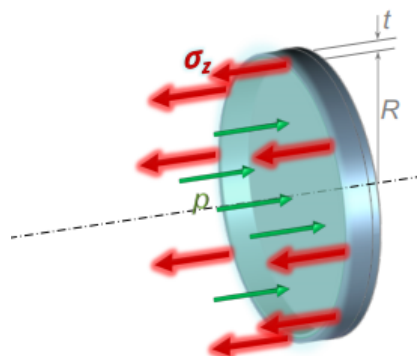
Figure 3: Longitudinal stress - equilibrium of axial forces.

$$p \cdot \pi R^2 = \sigma_z \cdot 2\pi R \cdot t \tag{2}$$

$$\sigma_z = \frac{pR}{2t} \tag{3}$$
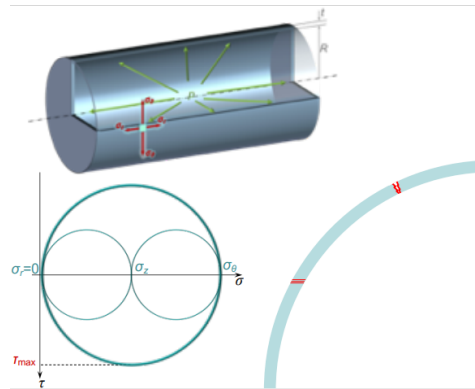

Figure 4:

$$\sigma_\theta = \frac{pR}{t} \tag{4}$$

$$\sigma_z = \frac{pR}{2t} = \frac{\sigma_\theta}{2} \tag{5}$$

0.2 Thick-walled cylinders - radial and hoop stresses

0.2.1 Failure of thick wall cylinders

Figure 5:

When thickness increases with respect to radius, the radial variation of hoop stress becomes considerable and the radial stress is no longer available.

0.2.2 Radial and hoop stresses & strains

For the determination of radial and circumferential stresses and strains, equilibrium equations on their own are not sufficient.
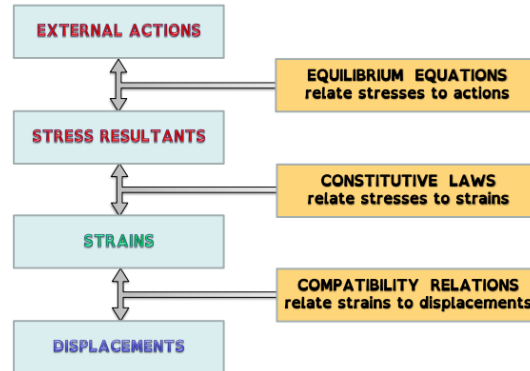
0.2.3 Solid mechanics equations

Figure 6:

0.2.4 Assumptions

For long cylinder , far from the ends, we can assume that the deformation of the cylinder is symmetrical respect to the axis:

- Cross-sections remain plane when subjected to pressure: longitudinal strain ϵ_z is independent of radius. i.e. $\epsilon_z = $ constant

- Displacements in each cross-section are purely radial

- Displacement is constant with circumferential co-ordinate and varies only with radial co-ordinate i.e. $u = f(r)$

- σ_r, σ_θ and σ_z are principal stress

- Cross-sections remain plane when subjected to pressure: longitudinal strain ϵ_z is independent of radius. i.e. $\epsilon_z = $ constant
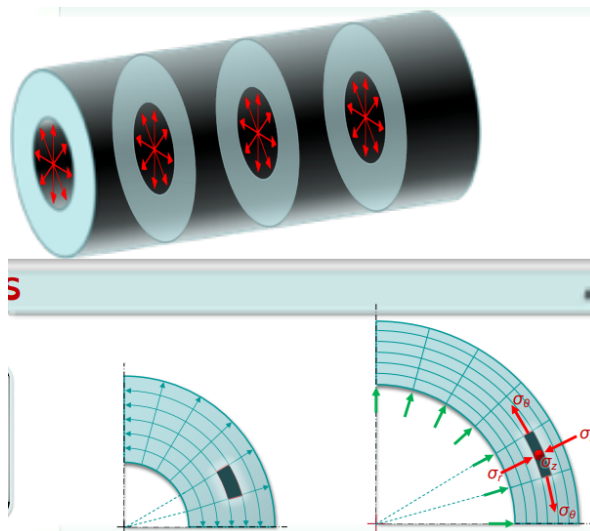
0.2.5 Equilibrium equations
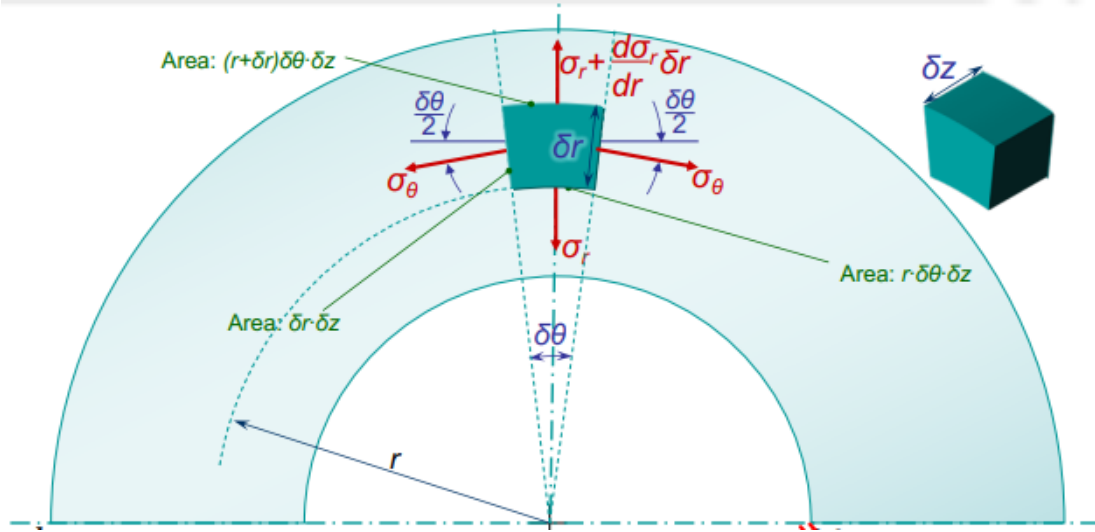
Figure 8:

The equilibrium of radial forces is given by the following equation:

$$-\sigma_r \left(r \cdot \delta\theta \cdot \delta z\right) + \left(\sigma_r + \frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\delta r\right) \left[(r + \delta r) \cdot \delta\theta \cdot \delta z\right] - 2\sigma_\theta \left(\delta r \cdot \delta z\right) \cdot \sin\frac{\delta\theta}{2} = 0 \quad (6)$$

$$\rightarrow -\sigma_r r + \sigma_r r + \sigma_r \delta r + r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\delta r + \frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\delta r^2 - \sigma_\theta \delta r = 0 \quad (7)$$

$$\rightarrow \sigma_r + r\frac{\mathrm{d}\theta_r}{\mathrm{d}r} - \sigma_\theta = 0 \quad (8)$$

0.2.6 Compatibility relations

Figure 9:

4

The radial strain is given by the following equation:

$$\epsilon_r = \frac{\overline{A'B'} - \overline{AB}}{\overline{AB}} \tag{9}$$
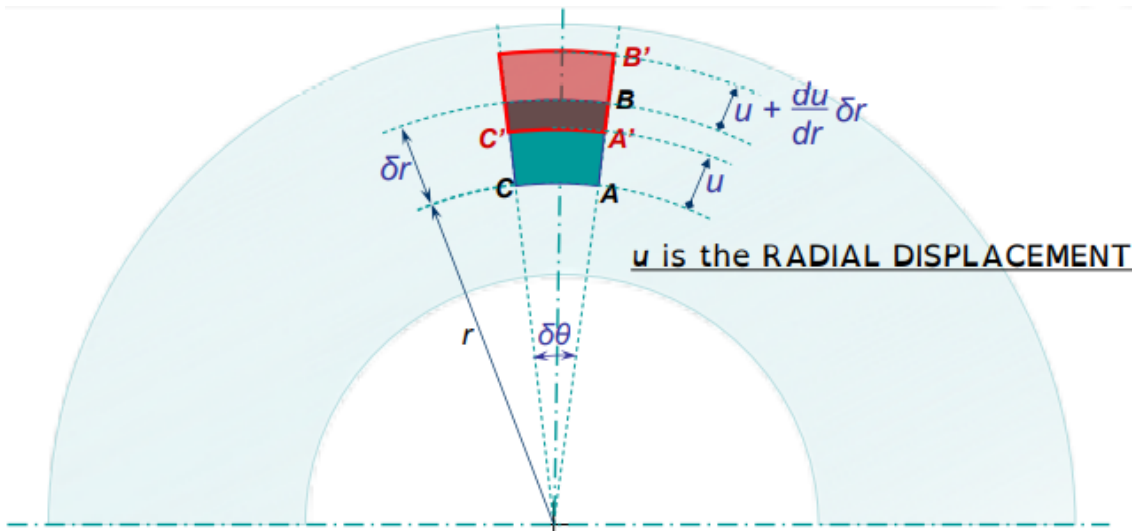
with:

$$\overline{AB} = \delta r \tag{10}$$

and

$$\overline{A'B'} = \overline{AB} + \overline{BB'} - \overline{AA'} = \delta r + \left(u + \frac{\mathrm{d}u}{\mathrm{d}r}\delta r\right) - u = \delta r + \frac{\mathrm{d}u}{\mathrm{d}r}\delta r \tag{11}$$

Leading to:

$$\varepsilon_r = \frac{\left(\delta r + \frac{\mathrm{d}u}{\mathrm{d}r}\delta r\right) - \delta r}{\delta r} \tag{12}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{13}$$

0.2.7   Constitutive laws

Uniaxial:

$$
\begin{aligned}
&\varepsilon_1 = \frac{\sigma_1}{E} &\quad &\sigma_1 = E\varepsilon_1 \\
&\varepsilon_2 = -v\varepsilon_1 &\quad &\sigma_2 = 0 \\
&\varepsilon_3 = -v\varepsilon_1 &\quad &\sigma_3 = 0
\end{aligned} \tag{14}
$$

Biaxial:

$$
\begin{aligned}
&\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - v\sigma_2\right) &\quad &\sigma_1 = \frac{E}{1-v^2}\left(\varepsilon_1 + v\varepsilon_2\right) \\
&\varepsilon_2 = \frac{1}{E}\left(\sigma_2 - v\sigma_1\right) &\quad &\sigma_2 = \frac{E}{1-v^2}\left(\varepsilon_2 + v\varepsilon_1\right) \\
&\varepsilon_3 = -\frac{v}{E}\left(\sigma_1 + \sigma_2\right) &\quad &\sigma_3 = 0
\end{aligned} \tag{15}
$$

Triaxial:

$$
\begin{aligned}
&\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - v\left(\sigma_2 + \sigma_3\right)\right] &\quad &\sigma_1 = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_1 + v\left(\varepsilon_2 + \varepsilon_3\right)\right] \\
&\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - v\left(\sigma_3 + \sigma_1\right)\right] &\quad &\sigma_2 = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_2 + v\left(\varepsilon_1 + \varepsilon_3\right)\right] \\
&\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - v\left(\sigma_1 + \sigma_2\right)\right] &\quad &\sigma_3 = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_3 + v\left(\varepsilon_1 + \varepsilon_2\right)\right]
\end{aligned} \tag{16}
$$

The analytical solution of the stress distribution is possible, in simple form, only in the *case of biaxial state of stress*.

Cylinders with internal pressure

A pressure difference can be maintained inside the cylinder in one of the following methods:
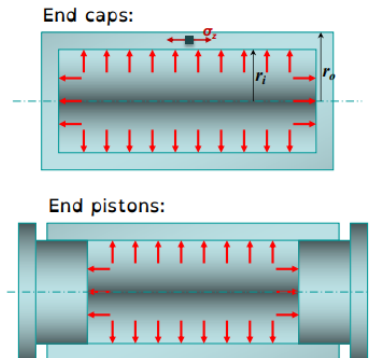
Only in the case of end pistons, can we find an analytical solution for the stress distributions in thick cylinders: no axial force acts on the cylinder and the state of stress is plane.
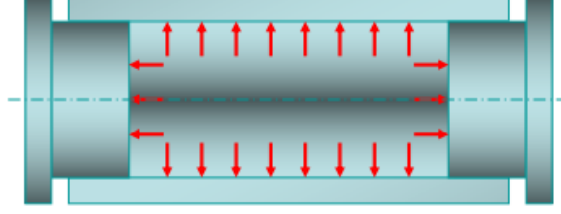

Figure 11:

Plane stress state

Biaxial:

$$
\begin{array}{ll}
\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - v\sigma_2\right) & \sigma_1 = \frac{E}{1-v^2}\left(\varepsilon_1 + v\varepsilon_2\right) \\
\varepsilon_2 = \frac{1}{E}\left(\sigma_2 - v\sigma_1\right) & \sigma_2 = \frac{E}{1-v^2}\left(\varepsilon_2 + v\varepsilon_1\right) \\
\varepsilon_3 = -\frac{v}{E}\left(\sigma_1 + \sigma_2\right) & \sigma_3 = 0
\end{array}
\tag{17}
$$

Cylindrical

In the biaxial case, in a cylindrical coordinate system r, θ, , z.

$$
\begin{array}{ll}
\varepsilon_r = \frac{1}{E}\left(\sigma_r - v\sigma_\theta\right) & \sigma_r = \frac{E}{1-v^2}\left(\varepsilon_r + var\right) \\
\varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - v\sigma_r\right) & \sigma_\theta = \frac{E}{1-v^2}\left(\varepsilon_\theta + v\varepsilon_r\right) \\
\varepsilon_z = -\frac{v}{E}\left(\sigma_r + \sigma_\theta\right) & \sigma_z = 0
\end{array}
\tag{18}
$$

0.2.8 Solid mechanics equations

Equilibrium equations relate stresses to actions:

$$
\frac{\mathrm{d}\theta_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\theta}{r} = 0
\tag{19}
$$

Constitutive laws relate stresses to strains:

$$
\sigma_r = \frac{E}{1-v^2}\left(\varepsilon_r + v\varepsilon_\theta\right)
\tag{20}
$$

$$
\sigma_\theta = \frac{E}{1-v^2}\left(\varepsilon_\theta + v\varepsilon_r\right)
\tag{21}
$$

Compatibility relations relate strains to displacements:

$$
\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r}
\tag{22}
$$

$$
\varepsilon_\theta = \frac{u}{r}
\tag{23}
$$

Combining the equations together, we arrive at Euler's differential equation:

$$
\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{1}{r}\cdot\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2} = 0
\tag{24}
$$

$$
r^2 u'' + r u' - u = 0
\tag{25}
$$

Euler's differential equation is characterised by the fact that its coefficients depend on the variable r. The general solution to Euler's differential equation is:

$$u = Ar + \frac{B}{r} \qquad u' = A - \frac{B}{r^2} \tag{26}$$

The constitutive laws hence become:

$$\sigma_r = \frac{E}{1 - v^2}\left(\frac{\mathrm{d}u}{\mathrm{d}r} + v\frac{u}{r}\right) \tag{27}$$

$$\sigma_\theta = \frac{E}{1 - v^2}\left(\frac{u}{r} + v\frac{\mathrm{d}u}{\mathrm{d}r}\right) \tag{28}$$

$$\sigma_r = \frac{E}{1 - v^2}\left[A - \frac{B}{r^2} + v\left(A + \frac{B}{r^2}\right)\right] = C - \frac{D}{r^2} \tag{29}$$

$$\sigma_\theta = \frac{E}{1 - v^2}\left[A + \frac{B}{r^2} + v\left(A - \frac{B}{r^2}\right)\right] = C + \frac{D}{r^2} \tag{30}$$

Note that:

$$\sigma_r + \sigma_\theta = 2C = \text{const} \tag{31}$$

Boundary conditions

$$\sigma_r = C - \frac{D}{r^2} = \frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} - \frac{(p_i - p_0)\, r_i^2 r_0^2}{\left(r_0^2 - r_i^2\right) r^2} \tag{32}$$

$$\sigma_\theta = C + \frac{D}{r^2} = \frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} + \frac{(p_i - p_0)\, r_i^2 r_0^2}{\left(r_0^2 - r_i^2\right) r^2} \tag{33}$$

If we have an internal pressure 'p_i' and an external pressure is 'p_0':

$$\sigma_r\left(r_i\right) = -p_i = C - \frac{D}{r_i^2} \tag{34}$$

$$C = \frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} \tag{35}$$

$$\sigma_r\left(r_i\right) = -p_0 = C - \frac{D}{r_0^2} \tag{36}$$

$$D = \frac{(p_i - p_0)\, r_i^2 r_0^2}{\left(r_0^2 - r_i^2\right) r^2} \tag{37}$$

Radial and hoop stresses

Lame's equations

$$\sigma_r = \frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} - \frac{(p_i - p_0)\, r_i^2 r_0^2}{\left(r_0^2 - r_i^2\right) r^2} \tag{38}$$

$$\sigma_\theta = \frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} + \frac{(p_i - p_0)\, r_i^2 r_0^2}{\left(r_0^2 - r_i^2\right) r^2} \tag{39}$$
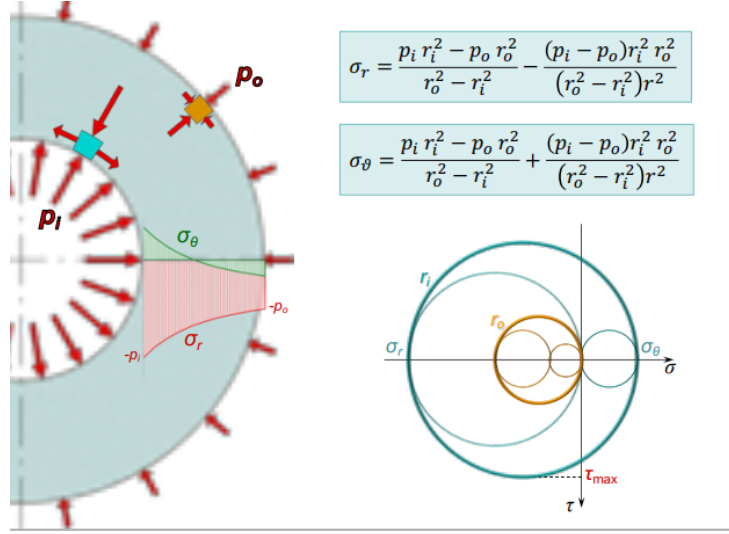
0.2.9   Stress variation

General case

$$\sigma_r = \frac{p_i\, r_i^2 - p_o\, r_o^2}{r_o^2 - r_i^2} - \frac{(p_i - p_o)r_i^2\, r_o^2}{(r_o^2 - r_i^2)r^2}$$

$$\sigma_\vartheta = \frac{p_i\, r_i^2 - p_o\, r_o^2}{r_o^2 - r_i^2} + \frac{(p_i - p_o)r_i^2\, r_o^2}{(r_o^2 - r_i^2)r^2}$$

Figure 12:

p_i only

$$\sigma_r = \frac{p_i\, r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right)$$

$$\sigma_\vartheta = \frac{p_i\, r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right)$$

$$\sigma_r(r_i) = -p_i$$

$$\sigma_\vartheta(r_i) = p_i\frac{r_i^2 + r_o^2}{r_o^2 - r_i^2}$$

$$\sigma_r(r_o) = 0$$

$$\sigma_\vartheta(r_o) = 2\frac{p_i\, r_i^2}{r_o^2 - r_i^2}$$

Figure 13:

p_0 **only**

$$\sigma_r = -\frac{p_o\, r_o^2}{r_o^2 - r_i^2}\left(1 - \frac{r_i^2}{r^2}\right)$$

$$\sigma_\vartheta = -\frac{p_o\, r_o^2}{r_o^2 - r_i^2}\left(1 + \frac{r_i^2}{r^2}\right)$$

$$\sigma_r(r_i) = 0$$

$$\sigma_\vartheta(r_i) = -\frac{2p_o r_o^2}{r_o^2 - r_i^2}$$

$$\sigma_r(r_o) = -p_o$$

$$\sigma_\vartheta(r_o) = -p_o\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2}$$
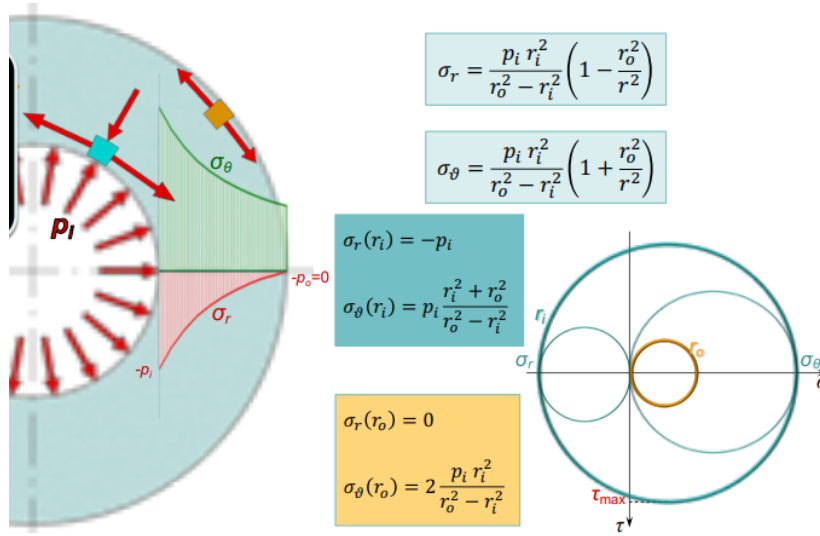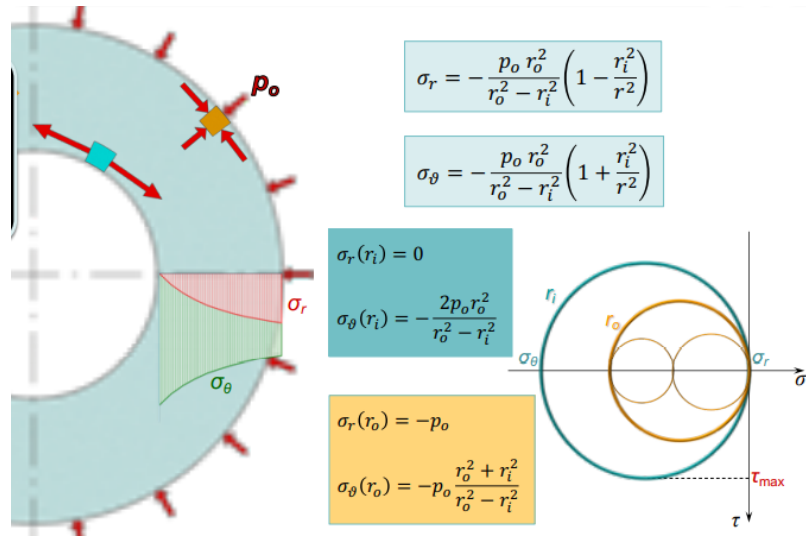
Figure 14:

0.3 Thick-walled cylinders - longitudinal stress

0.3.1 Cylinders with end-caps

How do we deal with the case of end caps pistons? (Triaxial stress state)

9