

MECH0005 Topic Notes

UCL

First Edition

Hasha Dar

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Chapter 1

Fluids

1.1 Fluid Properties

Fluids have many properties including:

- Surface Tension - σ .
- Viscosity - μ .
- Compressibility.
- Density ρ .
- Temperature.
- Pressure - P .

What is a fluid?

A fluid is a substance that deforms continuously when acted on by a shearing stress of any magnitude.

This includes gases and liquids but not silly putty, gels or glass. Some soft materials such as toothpaste will only start to flow once a critical shear stress has been reached. This is the study of rheology and not studied in this module.

In classical fluid mechanics, we treat fluids as a continuum because there is an extremely large number of particles. Thus, we can conclude that fluid parameter such as pressure and density vary continuously throughout the fluid.

1.1.1 Shear strain for solid bodies

What is shear strain?

Shear strain is the change in angle as an element experience a force tangential to its surface.

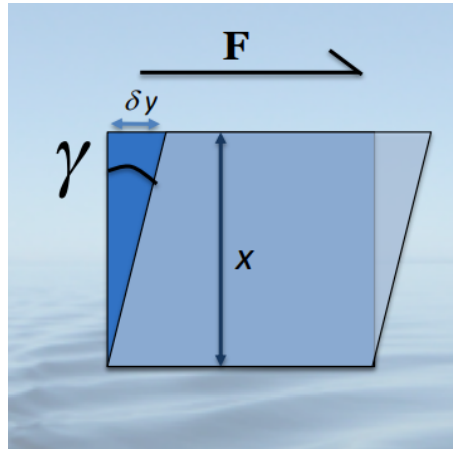


Figure 1.1: Force diagram on a section of fluid.

For a fluid, we are interested in the rate of shear. A specific tangential force will cause a fixed amount of shear strain per unit time. If the speed of the top layer is u_y , the shear rate is $\delta u_y / \delta x$.

$$\tan(\gamma) = \frac{\delta y}{x} \approx \gamma \quad (1.1.1)$$

1.1.2 Shear stress on a solid body

For a solid, the shear force is the force applied tangentially to a surface.

- *Shear stress* is the tangential force per unit area τ .
- *Normal stress* is the perpendicular force per unit area σ .

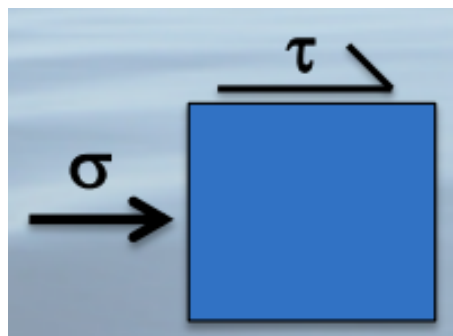


Figure 1.2: Stresses on a solid body.

1.1.3 Simple laminar flow case

Consider flat layers of fluid sliding over each other. The sideways velocity of the fluid changes as you move away from the stationary boundary. The velocity of the y direction, u_y is changing with position, x .

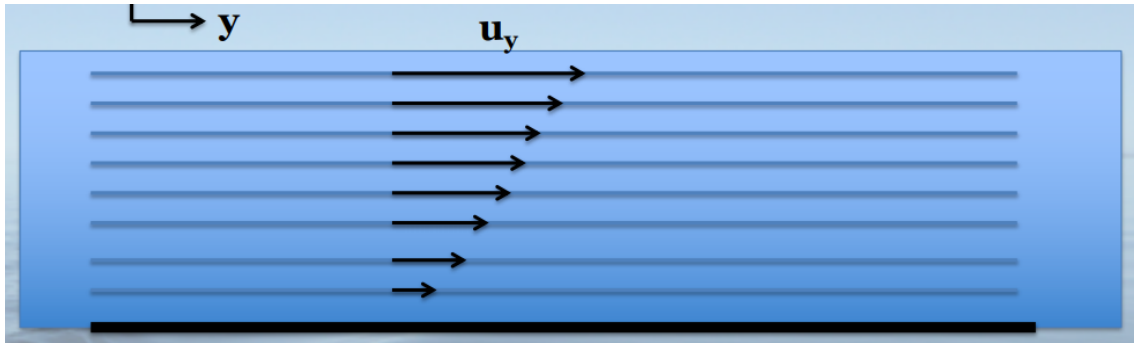


Figure 1.3: Laminar flow diagram.

1.1.4 Viscosity

$$\tau = \mu \frac{du_y}{dx} \quad (1.1.2)$$

Where:

- τ = Shear stress = force/area.
- μ = Dynamic viscosity.
- du_y/dx = Shear rate.
- Units: Velocity change per unit perpendicular distance.

Viscosity is the constant of proportionality, telling us how much shear stress is required to produce a given shear rate. Shear stress can be different at different places in the same fluid. Equation (2) is Newton's equation for viscosity and he assumed that the viscosity was a constant. However, this is not always true and the fluids for which that does not apply are called *non-Newtonian fluids*. An example of a shear *thickening* fluid is corn-starch, i.e. viscosity increases with shear rate. An example of shear *thinning* fluid is blood or ketchup, i.e. viscosity decreases with shear rate. We cannot use laminar flow analysis to study blood vessels because they constrict and dilate as fluids pass through - the rigid boundary condition does not apply (or you would not be able to feel your pulse.) We will only be studying Newtonian fluids and rigid boundaries.

Viscosity changes as a function of temperature and pressure. In liquids, when the

temperature increases, the viscosity decreases. In gases, the reverse occurs, with viscosity increasing with temperature. Relatively, pressure has a weak effect on the viscosity of a fluid.

The standard symbol for the viscosity as defined above is μ and this is known as the *dynamic viscosity* (ratio of shear stress to the shear rate) with units $\text{kg m}^{-1} \text{s}^{-1}$ or Pa.s. This will be our definition of viscosity. For some applications it is easier to do calculations in terms of the viscosity per unit density:

$$v = \frac{\mu}{\rho} \quad (1.1.3)$$

1.1.5 Surface tension

What is surface tension?

Surface tension is the effective tangential force per unit length (N m^{-1}) exerted on an imaginary line through a surface, due to intermolecular interaction.

Surface tension is always considered perpendicular to the line it's pulling on. In this case, we want the force exerted on the thread. If the thread has length L , the total force on one side of the soap film is σL , so the total force from both sides is $2\sigma L$. Surface tension is what allows a drop of liquid to take its unique shape, when there are no external forces present.

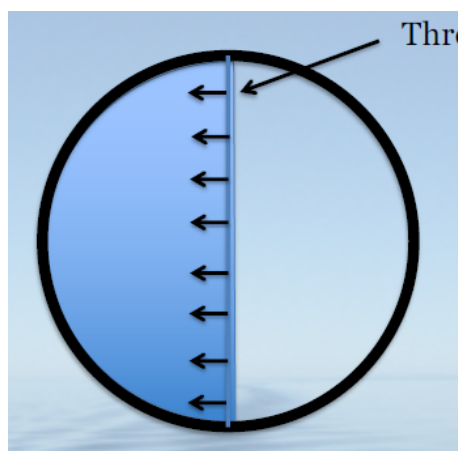


Figure 1.4: Soap film diagram.

1.1.6 Capillary action

This is caused by two processes:

- *Adhesion* between the water molecules and the tube wall.
- *Cohesion* between water molecules.

However, the surface tension (cohesion) must still be strong enough to drag the fluid up the pipe. When the angle of contact between a solid and a liquid is 90 degrees, then the cohesive force = the adhesive force. The liquid will rise by itself up the

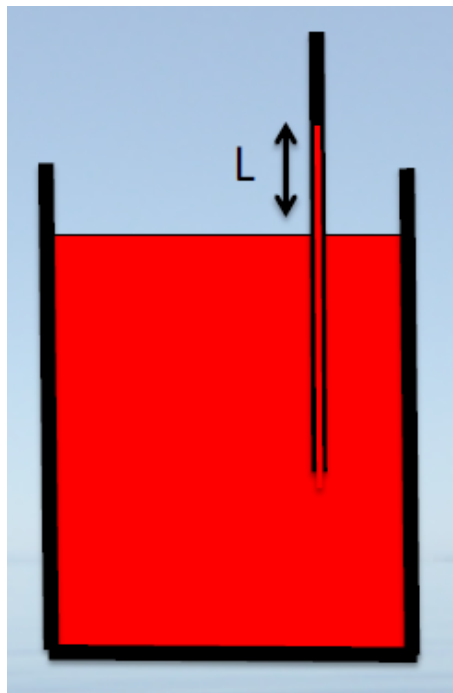


Figure 1.5: Capillary action.

tube and this gives us a crude method for calculating surface tension. The weight of the water must be balanced by the surface tension forces at the water surface.

$$F = 2\pi\sigma = L\pi r^2\rho g \quad (1.1.4)$$

$$\sigma = \frac{1}{2}\rho r g L \quad (1.1.5)$$

1.1.7 Pressure

What is Pascal's Law?

The pressure at a point in a fluid at rest or in motion is independent of direction as long as there are no shearing forces present.

The volume of a fluid under pressure will depend on the magnitude of the pressure. However, for many liquids (e.g. water), the change in volume is so small that it can be considered *negligible*. Even in everyday situations, water obviously is not really incompressible because sound waves can travel through it. For a truly incompressible fluid, the speed of sound would be infinite. Seawater density varies with temperature, pressure and salinity but below the top one kilometre, where salinity and temperature are nearly uniform, there is almost no variation in density with depth. The relationship between the pressure, volume and temperature of a fluid is described by an *equation of state*. Any equation that relates pressure and volume tells us something about the compressibility of a fluid. Pressure is also the same in all directions from our continuum assumption. We are interested in

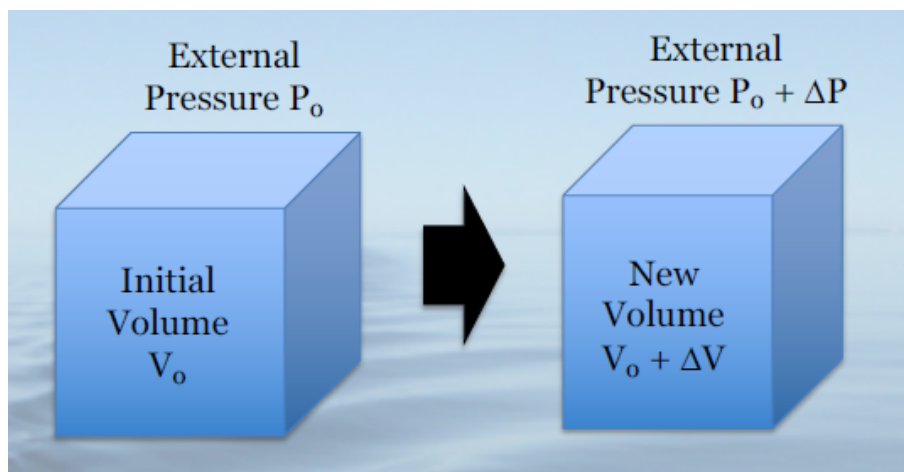


Figure 1.6: The compressibility of a fluid.

finding dV/dP which is (almost) always negative. A fluid is considered *compressible* when its volume varies significantly with temperature and pressure in the parameter space of interest. If there is no significant variation, the fluid can be considered *incompressible*. We will only be covering compressible fluids.

1.1.8 Example Question: Cartesian diver

Consider a closed vessel that is completely full of water but with flexible sides (plastic bottle). Inside is a cylinder that is closed at the top but open at the bottom. It is partly full of air and has some extra mass added at the base. The plastic parts of the diver have a volume of $1.067 \times 10^{-6} \text{ m}^3$ and a density of 2400 kg m^{-3} . At

atmospheric pressure, the length of the air column is 1.5 cm and the cross-sectional area is $1 \times 10^{-4} \text{ m}^2$. The temperature in the room is 20°C .

Question: How much pressure must be applied to the bottle to make the diver sink?

We must ask an important question: is the total density of the diver and the air and the water greater than or less than the density of the water around it? We know that $\rho_w = 1000 \text{ kg/m}^3$ and if the water is compressible, the total density of the whole diver is given by:

$$\rho_D = \frac{\rho_A l_1 A + \rho_w l_2 A + M_D}{(l_1 + l_2)A} \quad (1.1.6)$$

Where M_D is the mass of the plastic parts of the diver. We can see that the diver

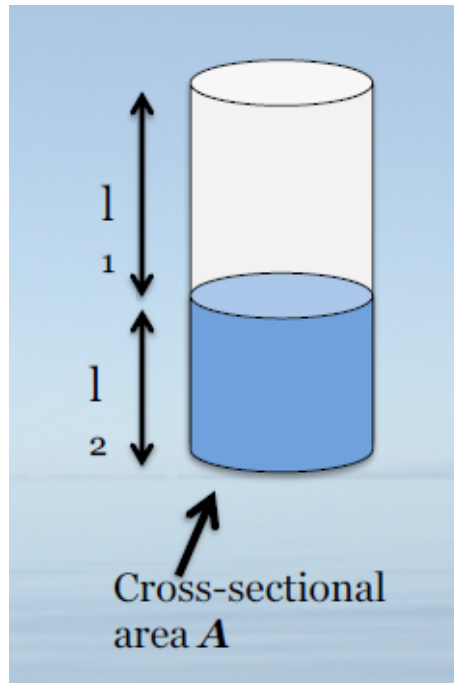


Figure 1.7: Cartesian diver.

will sink if $\rho_D > \rho_w$ and it rises if $\rho_D < \rho_w$.

Method: set the densities to be equal and deduce an expression for the pressure in this situation.

$$\rho_w = \rho_D = \frac{\rho_A l_1 A + \rho_w l_2 A + \rho_P V_P}{(l_1 + l_2)A + V_P} \quad (1.1.7)$$

To calculate the density of air, we need the equation of state. Assume that $R_g = 287 \text{ J kg}^{-1} \text{ K}^{-1}$

$$P = \rho_A R_g T \quad (1.1.8)$$

The necessary pressure is determined by the necessary density of air. So, we start

by rearranging the equation to specify the density of air.

$$\rho_A = \frac{1}{l_1 A} (\rho_w ((l_1 + l_2)A + V_P) - \rho_w l_2 A - \rho_P V_P) \quad (1.1.9)$$

$$\rho_A = \frac{1}{l_1 A} (\rho_w ((l_1 A + V_P)) - \rho_P V_P) \quad (1.1.10)$$

We can now use the equation of state to find an expression for the pressure.

$$P = \frac{R_g T}{l_1 A} (\rho_w ((l_1 A + V_P)) - \rho_P V_P) \quad (1.1.11)$$

From here we can substitute our values to arrive at an answer of $P = 3.48 \times 10^5 \text{ N m}^{-2}$ or approximately 3.5 atmospheres. In many situations, we assume that liquids are incompressible. However, this is an *assumption* that must be justified.

1.2 Buoyancy and stability

The resultant force acts at the centre of gravity of the fluid, the centroid of the volume. This is called the centre of buoyancy. Whatever is filling the space also has a force acting on it: it's weight. This acts at its centre of mass. The resultant force is given by:

$$F_{\uparrow} - F_{\downarrow} = (\rho_w - \rho_o)gV \quad (1.2.1)$$

An object is neutrally buoyant if the weight and the buoyant forces are equal, so

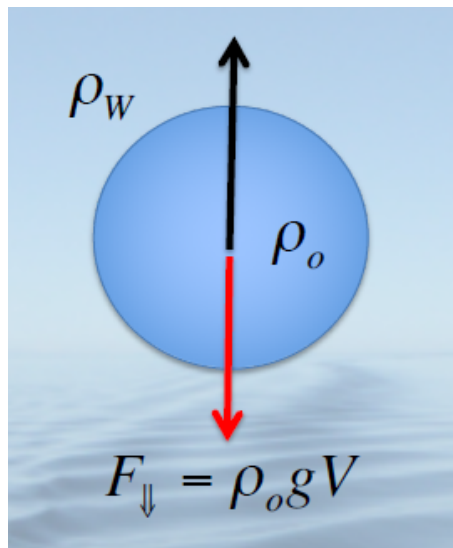


Figure 1.8: Buoyancy forces.

the system is in equilibrium. INSERT EXAMPLE MERT

1.2.1 Instability

Consider a submerged ping pong ball which is half filled with plasticine and the other half just filled with air. Assume that the ball overall is neutrally buoyant. The

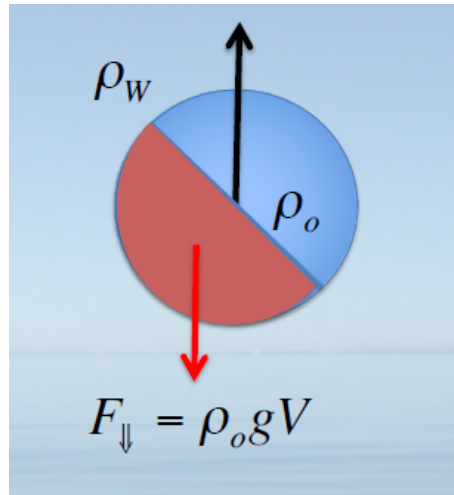


Figure 1.9: Unstable buoyancy forces.

forces are equal and opposite but there is a *couple*. The ball will rotate in response to the couple until the forces are aligned. At this point, the object is stable. The buoyancy and the weight do not necessarily have the same line of action. We must also consider whether the object will maintain its orientation.

- *Stable* equilibrium: when displaced slightly, there is a net force acting to push the object back to its initial orientation.
- *Unstable* equilibrium: when displaced slightly, there is a net force acting to push the object further from its initial orientation.
- *Neutral* equilibrium: when displaced slightly, there is no net force.

If the centre of buoyancy is above the centre of gravity, the object is in a stable equilibrium.

1.2.2 Centre of mass

The centre of mass is calculated in an analogous way to the centroid of the volume. If we split any object in half with a flat plane that passes through its centre of mass,

there will be equal mass in both halves.

$$x_{CoM} = \frac{1}{m} \int_m x dm \quad (1.2.2)$$

$$y_{CoM} = \frac{1}{m} \int_m y dm \quad (1.2.3)$$

$$z_{CoM} = \frac{1}{m} \int_m z dm \quad (1.2.4)$$

Where m is the total mass of the object.

In a semi-submerged body, the upward thrust is equal to the weight of the *displaced* fluid. For a body to be in vertical equilibrium, the immersed body must generate a buoyancy force equal and opposite to the weight of the object.

1.3 Dimensionless Numbers

1.3.1 Notation

We denote a dimension by using a capital letter in square brackets. Here are some common dimensions.

- $[M]$ - mass (unit: kilogram).
- $[T]$ - time (unit: second).
- $[L]$ - length (unit: metre).
- $[\Theta]$ - temperature (unit: kelvin).

Thus, we can derive that the dimensions of acceleration (which has units m s^{-2}) $[L][T]^{-2}$. Some dimensions of common measurements are shown below:

- Force - $[M][L][T]^{-2}$.
- Energy - $[M][L]^2[T]^{-2}$.

We use dimensional analysis to check derivations. The dimensions of both sides of any equation must match. Physical constants also often have units associated with them - these must also be considered. Some variables are dimensionless such as the

Reynolds number.

$$Re = \frac{\rho l u}{\mu} \quad (1.3.1)$$

$$[Re] = \frac{[M][L]^{-3} \cdot [L] \cdot [L][T]^{-1}}{[L][M][T]^{-2} \cdot [T][L]^{-2}} = \frac{[M][L]^{-1}[T]^{-1}}{[M][L]^{-1}[T]^{-1}} \quad (1.3.2)$$

Hence, we can see that the Reynolds number is a dimensionless quantity as all the dimensions cancel.

1.3.2 Example

We can use dimensional analysis to derive basic forms of equations. We want to work out the pressure drop as oil flows through a pipe. Let us consider the parameters this may depend on.

- Viscosity - $[M][L]^{-1}[T]^{-1}$.
- Pipe length - $[L]$.
- Pipe diameter - $[L]$.
- Velocity - $[L][T]^{-1}$.
- Pressure - $[M][L]^{-1}[T]^{-2}$.

Next we can assume that the pressure is a function of the other four. Some combination of the others must have the same dimension as the quantity we want.

$$[M][L]^{-1}[T]^{-2} = ([M][L]^{-1}[T]^{-1})^\alpha \cdot ([L])^\beta \cdot ([L])^\gamma \cdot ([L][T]^{-1})^\delta \quad (1.3.3)$$

$$[L] : -1 = -\alpha + \beta + \gamma + \delta \quad (1.3.4)$$

$$[M] : 1 = \alpha \quad (1.3.5)$$

$$[T] : -2 = -\alpha - \delta \quad (1.3.6)$$

$$\alpha = 1, \delta = 1, \beta + \gamma = -1 \quad (1.3.7)$$

So it must be true that:

$$\Delta P = \mu \cdot v \cdot l^\beta \cdot D^\gamma \quad (1.3.8)$$

Where $\beta + \gamma = -1$ The actual answer for laminar flow is:

$$\Delta P = \frac{2\mu L v}{D^2} \quad (1.3.9)$$

This sort of analysis is useful for checking on the functional form of relationships, but it won't give you the exact relationship, or the value of any dimensionless constants

involved.

1.3.3 Similarity

- Geometrical similarity: fixed ratio of lengths.
- Kinematic similarity: fixed ratio of velocities.
- Dynamic similarity - fixed ratio of forces.

Note on inertia: Inertia is not a force. However, for considering its importance to dynamic similarity, we can use the force needed to slow down a moving object. So we quantify inertia for these purposes as ma , from $F = ma$. Since the forces on flow change fluid motion, we use this often.

Dynamic similarity: viscosity

Compare the inertia "force" and the viscous force for a fluid:

$$\frac{[Inertia\ force]}{[Viscous\ force]} = \frac{\rho L^2 u^2}{\mu u L} = \frac{\rho L u}{\mu} \quad (1.3.10)$$

The Reynolds number is something very specific - it allows us to calculate the ratio of inertial and viscous forces in order to check for dynamical similarity.

- Honey: $Re \approx 1.3 \times 10^{-4}$
- Tea: $Re \approx 1100$

Therefore, they are not dynamically similar with respect to viscosity. For complete dimensional similarity, we must match the Reynolds number with the Froude number. If the same working fluid is used for the model and the prototype it is not possible to match the Reynolds number and the Froude number except if the model and the prototype have the same length. To achieve complete dynamic similarity between geometrically similar flows, it is necessary to duplicate the values of the independent dimensionless groups; by so doing the value of the dependent parameter is then duplicated. This is important because measured values of drag from model test could be scaled to predict drag for the operating conditions of the prototype. INSERT EXAMPLE MERT

1.3.4 Dimensionless groups

We have identified some dimensionless groups such as Reynolds number and Froude number. There are many more such as:

- Bond number: ratio of gravitational to surface tension forces.
- Capillary number: ratio of surface tension to viscous forces.
- Euler number: ratio of pressure force to inertial force.
- Grashof number: ratio of buoyancy to viscous forces.
- Cauchy number: ratio of inertial to elastic forces.
- Weber number: ratio of inertial to surface tension forces.

1.4 Integration over areas

1D integrations allows you to sum a function with respect to one variable.

$$I = \int_{x_0}^{x_1} F(x) dx \quad (1.4.1)$$

Where x_0 and x_1 are the lower and upper limits respectively. If our function varies with two variables, we have $F(x, y)$. In order to integrate this, we need to use double integration.

$$I = \int \int_x^y F(x, y) dx dy \quad (1.4.2)$$

The product $dx dy$ is the area of a small element on the x-y plane and we may also label it dA , where A represents the area. Because A represents the area, you may also see this written like this:

$$I = \int_A F(x, y) dA \quad (1.4.3)$$

These two notations are equivalent. The subscript \int_A just means an integration over all A , just as \int_x means an integration over all x . In most of the problems you will deal with, the order of the integration is not important – it doesn't matter whether you integrate over x first or y first. By convention, the inner one is done first, so

$$I = \int \int_x^y F(x, y) dx dy \quad (1.4.4)$$

implies that the integration will be done in the x -direction first, then in the y -direction.

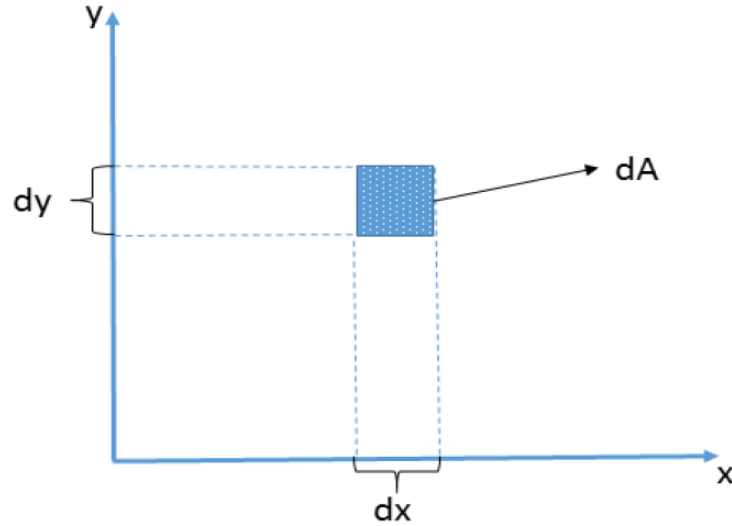


Figure 1.10: The resulting area dA when utilising double integration.

Consider this function

$$F(x, y) = 3x^2 + 2y \quad (1.4.5)$$

We want to sum this quantity over the x-y plane. For example, $F(x, y)$ could represent the volume flow through that element of the plane per unit time. Adding up that quantity for every element of the plane will give us the volume flow through the entire area, per unit time.

$$I = \int_{Y_0}^{Y_1} (2x^2 + 2y) dx dy \quad (1.4.6)$$

We carry out the inner integration first, treating y as though it were a constant:

$$I = \int_{Y_0}^{Y_1} (x^3 + 2yx)_{X_0}^{X_1} dy = \int_{Y_0}^{Y_1} ((X_1^3 + 2yX_1) - (X_0^3 + 2yX_0)) dy \quad (1.4.7)$$

Then we integrate this with respect to dy :

$$I = \int_{Y_0}^{Y_1} ((X_1^3 - X_0^3) + 2y(X_1 - X_0)) dy = (y(X_1^3 - X_0^3) + y^2(X_1 - X_0))_{Y_0}^{Y_1} \quad (1.4.8)$$

$$I = (Y_1 - Y_0)(X_1^3 - X_0^3) + (Y_1^2 - Y_0^2)(X_1 - X_0) \quad (1.4.9)$$

This is not always the most efficient way to integrate over an area. We can construct elements dA in whatever shape is most convenient. To find the area of a rectangle, we could integrate over all the elements $dA = dx dy$. But because of the symmetry

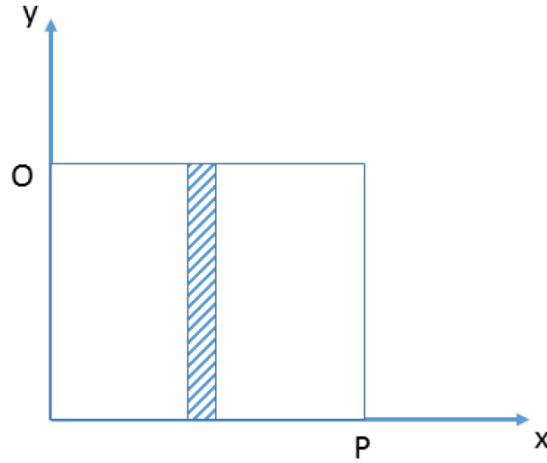


Figure 1.11: A rectangle.

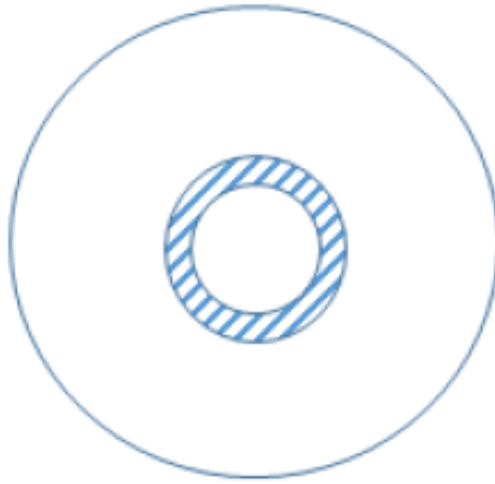


Figure 1.12: A circle.

of the problem, we can also choose a strip to integrate over:

$$I = \int_x O dx \quad (1.4.10)$$

This means adding up all the strips, each one of area $O dx$. For a circle, we can split the area into concentric rings: The area of this ring is $2\pi dr$. To calculate the area of the circle, we can just add up all the rings out to same radius R :

$$A = \int_0^R (2\pi r) dr = \int_0^R dA = 2\pi \int_0^R (r) dr = 2\pi \left[\frac{1}{2} r^2 \right]_0^R = \pi R^2 \quad (1.4.11)$$

Now consider a pipe with a circular cross-section. Water is flowing steadily through the pipe, but it has a different velocity at different radii. Suppose that the velocity

is given by:

$$v(r) = (R - r)^2 = r^2 - 2Rr + R^2 \quad (1.4.12)$$

where R is the radius of the pipe. Consider a ring of radius r and dr wide. The volume of water passing through this ring per unit time is:

$$V(r) = v(r)dA \quad (1.4.13)$$

the velocity (is the distance travelled in 1 second) multiplied by the area of the ring. We want to find the total volume flowing through the pipe per unit time, so we want to add up the flow through each ring.

$$Flow = \int_0^R v(r)dA \quad (1.4.14)$$

We know that $dA = 2\pi r dr$, and we know $v(r)$:

$$F = \int_0^R ((r^2 - 2Rr + R^2)2\pi r)dr \quad (1.4.15)$$

$$= 2\pi \int (r^3 - 2Rr^2 + R^2r)dr \quad (1.4.16)$$

$$= 2\pi \left[\frac{1}{4}r^4 - \frac{2}{3}Rr^3 + \frac{1}{2}R^2r^2 \right]_0^R \quad (1.4.17)$$

$$= \frac{\pi R^4}{6} \quad (1.4.18)$$

By choosing dA appropriately, we can integrate quantities over specific areas. You'll learn to choose the most efficient dA with experience.

1.5 Coordinate systems and velocity fields

Fluid flow can be classified in four ways:

1. Viscous/inviscid - when the viscosity of a fluid is equal to zero e.g. viscous - blood flow through a capillary, honey and inviscid - tornado, stirring tea.
2. Laminar and turbulent flow - Reynold's number. This is a measure used to determine whether a flow is likely to be laminar or turbulent.
3. Compressible or incompressible - density is constant/does not change with time e.g. compressible - clouds, gas turbines and incompressible - oil pipelines, water dams.
4. Steady and unsteady flow - the velocity at a particular point in space does

not change with time and eddy formations or the opposite with eddies and turbulence.

1.5.1 Coordinate systems and labelling

There are various notations used.

Cartesian coordinates

Position: x, y, z

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\underline{i} + y\underline{j} + z\underline{k}$$

Velocity: u, v, w

$$\underline{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u\underline{i} + v\underline{j} + w\underline{k}$$

Forces

Forces can either be applied to the body or on one of its surfaces.

- Body forces - The force acts at the centre of each increment of mass e.g. gravity.
- Surface flows - The force acts at the surface of each increment of mass e.g. shear stress.

As there are two types of forces, we need a consistent labelling system to distinguish them.

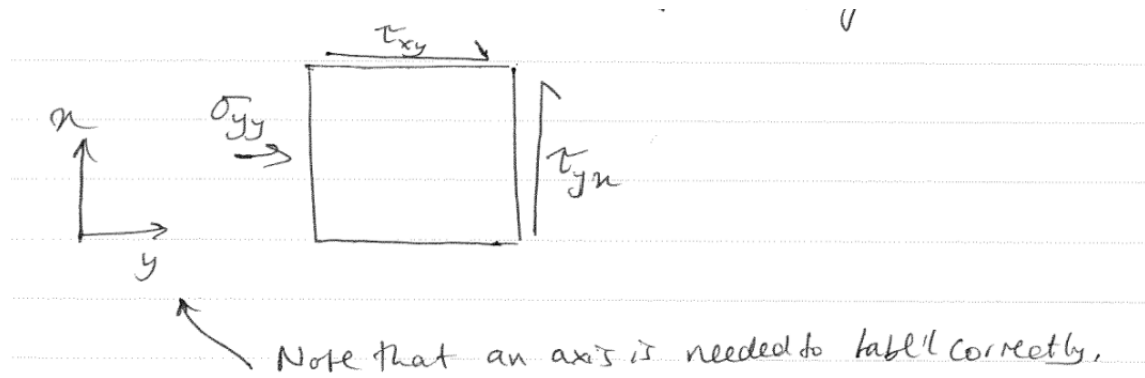


Figure 1.13: Force Diagram on an increment of fluid

σ_{yy} The first subscript refers to the face that the force is applied to. The second subscript is the direction the force acts in.

Incremental element of volume

Consider an incremental volume with dimensions: dx , dy , dz .

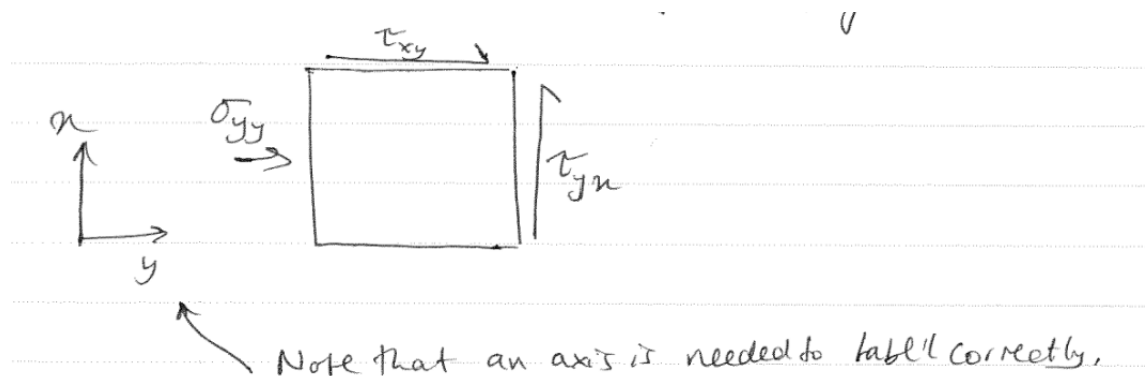


Figure 1.14: Force Diagram on an increment of fluid

There is a variation of fluid speeds in this box. Each component of the velocity can vary in each of these directions. The x-component of velocity, u , could vary with x , y or z independently.

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$$

and the same goes to v and w .

1.5.2 Velocity Fields

A field is a three dimensional description of a field parameter in space. For example:

$$\underline{v} = u \begin{pmatrix} x \\ y \\ z \end{pmatrix} \underline{i} + v \begin{pmatrix} x \\ y \\ z \end{pmatrix} \underline{j} + w \begin{pmatrix} x \\ y \\ z \end{pmatrix} \underline{k}$$

More generally, the flow field can be a function of time, t (this is unsteady flow.)

$$\underline{v} = u \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \underline{i} + v \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \underline{j} + w \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \underline{k}$$

You can also get acceleration fields. Velocity and acceleration fields are vector fields as it has magnitude and direction. A scalar field only has magnitude e.g. Pressure, temperature and density fields.

1.5.3 Eulerian and Lagrangian flow descriptions

There are two general approaches in analysing fluid mechanics problems:

1. Eulerian method - this uses fields to describe the flow at a specific position in time and space.
2. Lagrangian method - this describes what happens to an individual particle with time g.g ocean drifters, measuring conditions as they float freely through the oceans.

1.5.4 One-, two- and three- dimensional flows

Most flows are 3D flows, However, if one of the velocity components may be small (in some sense) relative to the other components then it may be reasonable to neglect the smaller components and assume 2D flow. That is $\underline{v} = u\underline{i} + v\underline{j}$. Sometimes two of the components may be negligible so it can be approximated to one dimensional flow. That is $\underline{v} = u\underline{i}$.

For many flows, one dimensional flow will provide a reasonable approximation,

whereas in others would produce erroneous results.

$$\text{Steady flow} \rightarrow \frac{\partial v}{\partial t} = 0$$

$$\text{Unsteady flow} \rightarrow \frac{\partial v}{\partial t} \neq 0$$

1.5.5 Streamlines, streaklines and pathlines

- Pathlines - the line traced out by a given particle as it flows from one point to another.
- Streaklines - A line that is created by particles in a flow that have previously passed through a common point.
- Streamlines - A line that is tangent to the velocity to the velocity field (if flow is steady).

If the flow is steady nothing at a fixed point changes with time, so the streamlines are fixed in space. If unsteady, the streamlines will change with time. Streamlines are obtained by integration.

$$\frac{dy}{dx} = \frac{v}{u}$$

v and u are usually functions of x and y and so integration after separating variables can be done and graphs drawn for various constant of integration. If steady, streamline = streakline.

1.5.6 The acceleration field

To apply Newton's second law, we need the acceleration. For the Lagrangian method, $a = a(t)$ for each particle. For the Eulerian description, we can find the acceleration field, which provides the acceleration of given points in a field. We will discuss how to get this if the velocity field is known.

1.5.7 The material derivatives

Consider a fluid particle moving along its pathline. In general, the particles velocity, denoted V_A for particle A, is a function of location and time. That is

$$V_A = V_A(x_A, y_A, z_A, t) = V_A[x_A(t), y_A(t), z_A(t), t] = V_A[x_A, y_A, z_A, t]$$

By definition, acceleration is the rate of change of velocity. We have to apply the chain rule as acceleration may change due to a change in position or time.

$$\begin{aligned}
 \underline{a}_A(t) &= \frac{dV_A}{dt} = \frac{\partial V_A}{\partial t} \frac{dt}{dt} + \frac{\partial V_A}{\partial x} \frac{dx_A}{dt} + \frac{\partial V_A}{\partial y} \frac{dy_A}{dt} + \frac{\partial V_A}{\partial z} \frac{dz_A}{dt} \\
 &= \frac{\partial V_A}{\partial t} + \frac{\partial V_A}{\partial x} \frac{dx_A}{dt} + \frac{\partial V_A}{\partial y} \frac{dy_A}{dt} + \frac{\partial V_A}{\partial z} \frac{dz_A}{dt} \\
 &\text{since } u_A = \frac{dx_A}{dt}, v_A = \frac{dy_A}{dt}, w_A = \frac{dz_A}{dt} \\
 \underline{a}_A &= \frac{\partial V_A}{\partial t} + \frac{\partial V_A}{\partial x} u_A + \frac{\partial V_A}{\partial y} v_A + \frac{\partial V_A}{\partial z} w_A
 \end{aligned}$$

This applies for any project and so we can drop the subscript.

$$\underline{a} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} u + \frac{\partial V}{\partial y} v + \frac{\partial V}{\partial z} w$$

This is a vector result where scalar components can be written as:

$$\begin{aligned}
 a_x &= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\
 a_y &= \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\
 a_z &= \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}
 \end{aligned}$$

All of this is written in shorthand as:

$$\underline{a} = \frac{Dv}{Dt}$$

Where the operator $\frac{D}{Dt}$ is the material derivative.

$$\frac{D}{Dt}() = \left[\frac{\partial ()}{\partial t} \right] (A) + \left[u \frac{\partial ()}{\partial x} + v \frac{\partial ()}{\partial y} + w \frac{\partial ()}{\partial z} \right] (B)$$

Where A is the local derivative and B is the convective derivative.

If the parameter involved is acceleration, term (A) is called *local acceleration* for steady flow this is equal to zero. $\frac{\partial v}{\partial t} = 0$. This physically means there is no change in flow parameter at a fixed point in space if the flow is steady.

The collection of terms labelled (B) are called the convective derivative. It represents the fact that a flow property associated with a fluid particle may vary because of the motion of the particle from one place to another. This contribution occurs when

the parameter of concern is acceleration, then section (B) is called the convective acceleration.

1.5.8 Streamline coordinates

in the streamline C.S. the flow is described in terms of one coordinate along the streamlines, denoted s , and the second coordinate normal to the streamlines, denoted n . Unit vectors in this direction are denoted by \hat{s} and \hat{n} . The flow plane is covered by an orthogonal curved net of coordinate lines. s and n are always perpendicular, but the lines of constant s or constant n are not necessarily straight. Note that velocity is always tangent to the s direction. Thus:

$$\underline{v} = v\hat{s}$$

For steady, 2D flow, the acceleration is:

$$\underline{a} = \frac{Dv}{Dt} = a_s\hat{s} + a_n\hat{n}$$

Where $a_s\hat{s}$ is the acceleration in the streamline direction and $a_n\hat{n}$ is the acceleration in the normal direction.

In general, for steady flow, both the speed and the flow direction are a function of location.

$$v = v(s, n) \text{ and } \hat{s} = \hat{s}(s, n)$$

For a given particle, the value dt s changes with time but the value of n remains fixed as the particle flows along a streamline defined by $n = \text{constant}$. (note: pathlines = streamlines during steady flow). Thus:

$$\underline{a} = \frac{Dv}{Dt} = \frac{D}{Dt}(v\hat{s}) = \frac{Dv}{Dt}\hat{s} + v\frac{D\hat{s}}{Dt}$$

Applying the material derivative,

$$\underline{a} = \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial s} \frac{ds}{dt} + \frac{\partial v}{\partial n} \frac{dn}{dt} \right) \hat{s} + v \left(\frac{\partial \hat{s}}{\partial t} + \frac{\partial \hat{s}}{\partial s} \frac{ds}{dt} + \frac{\partial \hat{s}}{\partial n} \frac{dn}{dt} \right)$$

For steady flow, this simplifies too:

$$\underline{a} = \left(v \frac{\partial v}{\partial s} \right) \hat{s} + v \left(v \frac{\partial \hat{s}}{\partial s} \right)$$

This equation can also be rewritten as:

$$\underline{a} = v \frac{\partial v}{\partial s} \underline{\hat{s}} + \frac{v^2}{R} \underline{\hat{n}}$$

Or,

$$a_s = v \frac{\partial v}{\partial s}, \quad a_n = \frac{v^2}{R} \quad (1.5.1)$$

Where a_s is the streamwise/acceleration convective and a_n is the normal/acceleration centrifugal.

1.6 Control-volume approach

Recall from thermodynamics that $\dot{m}_{in} = \dot{m}_{out}$ for a control volume (A c.v. allows heat, work and mass transfer across its boundary). In fluid dynamics different terminology is usually used:

- Closed system = control mass = system (in fluid mechanics).
- Open system = control volume.

We also know that $\frac{dm_{cv}}{dt} = \dot{m}_{in} - \dot{m}_{out}$ from thermodynamics. For a *system*, $\left(\frac{dm}{dt}\right)_{sys} = 0$ since a system is a control mass.

1.7 Simple Reynolds transport theorem

$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} + \dot{B}_{out} - \dot{B}_{in} \quad (1.7.1)$$

Where $\frac{\partial B_{cv}}{\partial t}$ is how the property changes inside and $\dot{B}_{out} - \dot{B}_{in}$ is what is flowing in and out.

$$\dot{B}_{in} = b_{in} \dot{m}_{in} = b_{in} \rho_{in} v_{in} A_{in} \text{ and} \quad (1.7.2)$$

$$\dot{B}_{out} = b_{out} \dot{m}_{out} = b_{out} \rho_{out} v_{out} A_{out} \quad (1.7.3)$$

$$\therefore \frac{DB_{sys}}{Dt} = \left(\frac{\partial B}{\partial t} \right)_{cv} + (b\rho v A)_{out} - (b\rho v A)_{in} \quad (1.7.4)$$

This makes an assumption: that the flow is linear across the cross-section, whereas

in reality this is not the case.

$$\dot{m} = \rho v A \rightarrow \text{unrealistic} \quad (1.7.5)$$

$$\dot{m} = \int_A (\rho v) dA \rightarrow \text{realistic} \quad (1.7.6)$$

Equation 1.7.6 is the integral over different sections and encapsulates that the velocity is changing and that the density may also be changing. If the velocity of the fluid is not in the direction of the pipe, then you have to only multiply by a component.

$$\dot{m} = \int_A (\rho v \cos \theta) dA \quad (1.7.7)$$

Instead of mass, let's do it for \dot{B}_{in} and \dot{B}_{out} .

$$\dot{B}_{out} = \int_A (b \rho v \cos \theta) dA \quad (1.7.8)$$

$$\text{where } \rho v \cos \theta dA = \dot{m} \quad (1.7.9)$$

Let us define a unit normal vector \hat{n} on the exit. Thus:

$$\dot{B}_{out} = \int_A (b \rho \bar{v} \cdot \hat{n}) dA \quad (1.7.10)$$

$$\bar{v} \cdot \hat{n} = |\bar{v}| \cos \theta \text{ hence} \quad (1.7.11)$$

$$= \dot{B}_{out} - \dot{B}_{in} \quad (1.7.12)$$

As the cross product would produce a negative if the angle is more than 90° .

$$\therefore \dot{B}_{out} - \dot{B}_{in} = \int_A (b \rho \bar{v} \cdot \hat{n}) dA \quad (1.7.13)$$

$$\therefore \left(\frac{DB}{Dt} \right)_{sys} = \left(\frac{\partial B}{\partial t} \right)_{cv} + \int (b \rho \bar{v} \cdot \hat{n}) dA \quad (1.7.14)$$

1.8 Reynolds transport theorem: RTT

Where:

- \hat{n} = unit vector.
- dA = area element.
- dV = volume element.
- B = an extensive property.
- b = an intensive property.

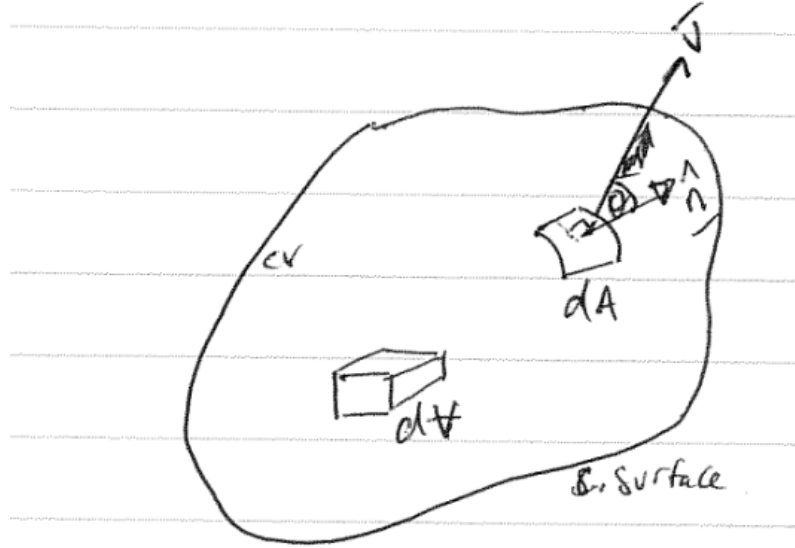


Figure 1.15: Diagram to show the Reynolds Transport Theorem.

As a reminder the SRTT is:

$$\left(\frac{DB}{Dt} \right)_{sys} = \left(\frac{\partial B}{\partial t} \right)_{cv} + \dot{B}_{out} - \dot{B}_{in} \quad (1.8.1)$$

The general form of the RTT is:

$$\left(\frac{DB}{Dt} \right)_{sys} = \frac{\partial}{\partial t} \int_{cv} (\rho b) dV + \int_{cs} (\rho b \vec{v} \cdot \hat{n}) dA \quad (1.8.2)$$

ρdV = mass of dV then take an integral to calculate total mass. The change in the property of the C.V. $\int_{cs} (\rho b \vec{v} \cdot \hat{n}) dA$ tells us how much B is changing per second. The net flow out of the C.S.

We apply the RTT to 3 cases:

1. Mass conservation.
2. Newton's second law.
3. Energy conservation (first law of thermodynamics).

1.8.1 RTT and conservation of mass

The property of consideration is mass:

$$B = \text{mass} = M \therefore b = \frac{B}{\text{mass}} = \frac{M}{M} = 1 \quad (1.8.3)$$

Applying these substitutions into RTT

$$\frac{DM_{sys}}{Dt} = \frac{\partial}{\partial t} \int_{cv} (\rho) dV + \int_{cs} (\rho \bar{v} \cdot \hat{n}) dA \quad (1.8.4)$$

By definition, a system = closed system = control mass.

$$\therefore \frac{DM_{sys}}{Dt} = 0 \quad (1.8.5)$$

$$0 = \frac{\partial M_{cv}}{\partial t} + \int_{cs} (\rho v_n) dA \quad (1.8.6)$$

$$\therefore \frac{\partial M_{cv}}{\partial t} = - \int_{cs} (\rho v_n) dA \quad (1.8.7)$$

Note: $\int_{cs} (\rho v_n) dA$ is the net mass flow rate through the C.S. out.

1.8.2 Example Q

INSERT

1.8.3 RTT and Newton's second law

$$\left. \begin{array}{l} \underline{B} = m\bar{v} = \text{linear momentum} \\ \underline{b} = \bar{v} = \text{velocity} \end{array} \right\} \text{Vectors} \quad (1.8.8)$$

Substituting this into RTT gives (replacing b with \bar{v}).

$$\frac{D}{Dt}(m\bar{v})_{sys} = \frac{\partial}{\partial t} \int_{cv} (\rho \underline{v}) dV + \int_{cs} (\rho \bar{v}(\bar{v} \cdot \hat{n})) dA = ? \quad (1.8.9)$$

Where $\frac{D}{Dt}(m\bar{v})_{sys}$ is the acceleration. M is a constant and hence $m \frac{D}{Dt} \bar{v} = \bar{a} \times m$. Thus, the $? = \sum \bar{F} \leftarrow$ sum of all the forces by Newton's second law. Hence:

$$\frac{D}{Dt}(m\bar{v})_{sys} = \frac{\partial}{\partial t} \int_{cv} (\rho \underline{v}) dV + \int_{cs} (\rho \bar{v}(\bar{v} \cdot \hat{n})) dA = \sum \bar{F} \quad (1.8.10)$$

This is a vector equation and can be split up into x, y and z components. $\frac{\partial}{\partial t} \int_{cv} (\rho \underline{v}) dV$ is the rate of change of linear momentum inside the C.V. $\int_{cs} (\rho \bar{v}(\bar{v} \cdot \hat{n})) dA$ is the net flow of linear momentum through C.S.

Now

1.9 Laminar flow and boundary layers

1.9.1 Boundary conditions

You will generally only come across two types of boundary conditions:

1. No-slip condition: at the point where the fluid touches the boundary, the velocity of the fluid along the boundary is zero. This is the case where the boundary is a solid.
2. Free surface: there are no viscous forces on the fluid where it touches the boundary, so the speed can take any value. This is the case where there is a liquid-gas boundary and may be called (e.g. for a stream) *open-channel flow*.

1.9.2 Laminar flow calculations

Consider laminar flow between two flat plates. Split the flow into sheets of thickness d_{in} . The top plate moves at constant speed v and the lower plate is stationary. Here boundary condition (1) applies i.e. fluid at the boundary moves at the speed of the boundary. The top plate has an area A and the sideways force being applied to it is $F \therefore \tau = \frac{F}{A}$. If the flow is steady, the forces must balance for each layer. However, the liquid layers must transmit the stress so τ is constant throughout the depth. Let us apply our boundary conditions:

$$u_y(0) = 0$$

$$u_y(D) = v$$

where D is the vertical distance between the upper and lower plate.

$$\therefore \tau = \mu \frac{du_y}{dx}$$

$$\left. \begin{array}{l} \frac{\tau}{\mu} dx = du_y \\ \frac{\tau}{\mu} \int dx = \int du_y \\ k + \frac{\tau}{\mu} x = u_y \end{array} \right\} \begin{array}{l} u_y = \frac{\tau}{\mu} x + k \\ 0 = \frac{\tau}{\mu} \cdot 0 + k \therefore k = 0 \\ \text{also } \frac{\tau}{\mu} = \frac{v}{D} \rightarrow \tau = \frac{v}{D} \mu \end{array}$$

$$\therefore u_y = \frac{v}{D} x$$

1.9.3 Laminar flow between solid boundaries

In laminar flow, individual particles of fluid follow paths that do not cross those of neighbouring particles. However, there is still a velocity gradient across the flow. Laminar flow is not normally found except in the neighbourhood of a solid boundary, the retarding effect of which causes the transverse velocity gradient.

$$\tau = \mu \left(\frac{\partial u}{\partial y} \right)$$

Where τ is the resultant shear stress, μ is the dynamic viscosity and $\left(\frac{\partial u}{\partial y} \right)$ is the rate at which the velocity u increases with the coordinate y perpendicular to the velocity.

We can use the above to find the required shear stress, if we know the speed the fluid layer thickness and the viscosity:

$$\tau = \frac{v}{D} \mu$$

Flow between flat plates example

Consider two flat plates stuck together with a layer of honey of uniform thickness. Assume laminar flow between the plates. How hard must you pull sideways to separate them at a speed of 1 cm s^{-1} ?

- Viscosity of honey = 100 Pa s
- Thickness $D = 0.003 \text{ m}$
- Speed $v = 0.01 \text{ m s}^{-1}$
- Width = 0.02 m
- Area (A) = width \times length (l)

Assuming that there end effects can be neglected, the stress is equal to the force applied divided by the area. Thus, we can see that the required force decreases as

the plates separate.

$$\begin{aligned}\tau &= \frac{F}{A} \\ &= \frac{v}{D}x \\ F &= \frac{v}{D}\mu \cdot A \\ F &= \frac{0.01}{0.003}100 \cdot 0.02 \cdot l \\ F &= 6.6l\end{aligned}$$

1.9.4 Finding shear stress

The velocity field in a sample of seawater is given below, where A and B are constants:

$$u_y = A(B^2 - x^2)$$

Tiny sea creatures called dinoflagellates emit light when the shear stress in the water around passes a critical level. This is a major cause of bioluminescence at the surface of the ocean. For one particular species, the critical stress level is $\tau_c = 1\text{N m}^{-2}$. The water viscosity is $\mu = 1.2 \times 10^{-3}\text{Pa s}$ and the constants are $A = -200$, $B = 0.1$. At what x-position in this water sample will the dinoflagellates start to glow?

Remembering: $\tau = \mu \frac{du_y}{dx}$, we can solve for the case τ_c .

$$\begin{aligned}u_y &= -200(0.01 - x^2) \\ u_y &= -2 + 200x^2 \\ \frac{du_y}{dx} &= 400x \\ 1 &= 1.2 \cdot 10^{-3} \cdot 400x \\ x &= 2.08\text{m}\end{aligned}$$

1.9.5 Laminar flow in pipes

What is the velocity at different radii in the pipe? We have circular symmetry and a no-slip boundary condition. Flow is being driven by a pressure difference between two ends of the tube. Note that the velocity and the pressure gradient vector point in opposite directions because fluid flows from high to low pressure. We will assume that the flow is fully developed i.e. that the velocity profile is constant along the

pipe axis. Consider one cylinder of fluid with an edge at constant radius r . the total pipe radius is R . There is a pressure difference between the two ends of the tube. The total pressure difference is ΔP over a length L of tube. The total resultant force parallel to the tube axis must be $F \cdot A$.

$$\begin{aligned} Force &= \Delta P \pi r^2 \\ Force &= \frac{dP}{dx} L \pi r^2 \end{aligned}$$

The force provided by the pressure difference must be equal to the shear stress on the outside of the cylinder, since the forces are balanced.

$$Force = 2\tau \pi r L$$

Equating these two forces:

$$\begin{aligned} \frac{dP}{dx} L \pi r^2 &= 2\tau \pi r L \\ \tau &= \frac{r}{2} \frac{dP}{dx} \end{aligned}$$

We can relate the shear stress at the cylinder surface to the velocity change with radius there:

$$\begin{aligned} \tau &= \mu \frac{du}{dr} \\ \tau &= \mu \frac{du}{dr} \\ &= \frac{r}{2} \frac{dP}{dx} \\ \frac{du}{dr} &= \frac{r}{2\mu} \frac{dP}{dx} \end{aligned}$$

Now we can integrate to u :

$$\begin{aligned} u &= \int \left(\frac{r}{2\mu} \frac{dP}{dx} \right) dr \\ u &= \frac{1}{2\mu} \frac{dP}{dx} \int (r) dr \\ u &= \frac{1}{2\mu} \frac{dP}{dx} \frac{1}{2} r^2 + k \end{aligned}$$

Applying the no-slip boundary condition, $u(R) = 0$:

$$\begin{aligned}0 &= \frac{1}{2\mu} \frac{dP}{dx} \frac{1}{2} R^2 + k \\k &= \frac{-1}{4\mu} \frac{dP}{dx} R^2 \\\therefore u &= \frac{-1}{4\mu} \frac{dP}{dx} (R^2 - r^2)\end{aligned}$$

How can we use this? The maximum speed is at the centre of the pipe, where $r = 0$. We know the flow speed with radius, so we can integrate this to find the total volume flow.

$$\begin{aligned}u_{max} &= \frac{-1}{4\mu} \frac{dP}{dx} R^2 \\Q &= \int_0^R (u(r) \cdot 2\pi r) dr \\Q &= \int_0^R \left(\frac{-1}{4\mu} \frac{dP}{dx} (R^2 - r^2) \cdot 2\pi r \right) dr\end{aligned}$$

The average flow speed is the total volume flow rate divided by the cross sectional area:

$$u_{mean} = \frac{Q}{\pi r^2}$$

Laminar flow through pipes of circular cross section is often called Hagan-Poiseuille flow.

1.10 Fluids formulas

1.10.1 Density

$$v = \frac{1}{\rho}$$

1.10.2 Specific weight

$$\gamma = \rho g$$

1.10.3 Specific gravity

$$SG = \frac{\rho}{\rho_{H_2O@4^\circ C}}$$

1.10.4 Ideal gas law

$$P = \rho RT$$

$$Pv = RT$$

$$PV = nR_u T$$

1.10.5 Kinematic viscosity

$$v = \frac{\mu}{\rho}$$

1.10.6 Surface tension

$$\sigma = \frac{F}{L}$$

1.10.7 Capillary action

$$h = \frac{2\sigma \cos \theta}{\gamma R}$$

1.10.8 Centre of mass

$$\begin{aligned}\bar{x} &= \frac{\int_m (x) dm}{m} \\ &= \frac{\int_m (x) dm}{\int_m dm}\end{aligned}$$

1.10.9 Stable equilibrium

The centre of gravity is directly below the centre of buoyancy.

1.10.10 Reynolds Number

$$Re = \frac{\rho Lu}{\mu}$$

$$Re < 2000 \rightarrow \text{Laminar}$$

$$Re > 2000 \rightarrow \text{Turbulent}$$

1.10.11 Material derivative

$$\begin{aligned}\frac{D}{Dt}() &= \frac{\partial}{\partial t}() + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \\ \underline{a} = \frac{D}{Dt}(\underline{v}) &= \frac{\partial \underline{v}}{\partial t} + u \frac{\partial \underline{v}}{\partial x} + v \frac{\partial \underline{v}}{\partial y} + w \frac{\partial \underline{v}}{\partial z}\end{aligned}$$

1.10.12 Streamline coordinates

$$\begin{aligned}\underline{a} = \frac{D\underline{v}}{Dt} &= a_s \hat{s} + a_n \hat{n} \\ &= v \frac{\partial \underline{v}}{\partial s} \hat{s} + \frac{v^2}{R} \hat{n} \\ a_s &= \frac{-v \sin \theta - \frac{\partial P}{\partial s}}{\rho} = v \frac{\partial v}{\partial s} \\ a_n &= \frac{-v \cos \theta - \frac{\partial P}{\partial n}}{\rho} = \frac{v^2}{R}\end{aligned}$$

1.10.13 RTT

$$\begin{aligned}\left(\frac{DB}{Dt}\right)_{sys} &= \frac{\partial}{\partial t} \int_{cv} (\rho b) dV + \int_{cs} (\rho b(\underline{v} \cdot \hat{n})) dA \\ \text{When } B=M \rightarrow 0 &= \frac{\partial}{\partial t} \int_{cv} (\rho) dV + \int_{cs} (\rho(\underline{v} \cdot \hat{n})) dA \\ \text{When } \underline{B} = m\underline{v} \rightarrow \sum F_{sys} &= \frac{\partial}{\partial t} \int_{cv} (\rho \underline{v}) dV + \int_{cs} (\rho \underline{v}(\underline{v} \cdot \hat{n})) dA\end{aligned}$$

1.10.14 Propeller stages

1. Bernoulli equation between 1 and 2.

$$P_1 = P_2 + 0.5\rho(v_2^2 - v_1^2)$$

2. Bernoulli equation between 3 and 4.

$$P_4 = P_3 + 0.5\rho(v_3^2 - v_4^2)$$

3. $P_1 = P_4$.

$$\therefore \Delta P = P_3 - P_2 = 0.5\rho(v_4^2 - v_1^2)$$

4. $F = \Delta P \times A$.

$$\therefore F_{thrust} = \Delta P \times A = 0.5\rho A(v_4^2 - v_1^2)$$

5. Apply momentum equation.

$$\begin{aligned}\sum F_x &= \int_{inlet} \rho v_1(-v_1)dA + \int_{outlet} \rho v_4(v_4)dA \\ &= -\dot{m}_1 v_1 + \dot{m}_2 v_4 \\ &= \dot{m}(v_4 - v_1) \text{ (assume steady state, no viscous force, neglect g)}\end{aligned}$$

6. At the propeller, $\dot{m} = \rho A_P v_p = \rho A v_p$.

$$\therefore F_{thrust} = \rho A v_p (v_4 - v_1)$$

7. $0.5\rho A(v_4^2 - v_1^2) = \rho A v_p (v_4 - v_1)$.

$$\therefore v_p = 0.5(v_4 + v_1)$$

8. Work out v_p and use it to work out thrust.

9. $P = F \times v$.

1.10.15 Moving control volume

$$\underline{v} = \underline{v}_{CV} + \underline{W}$$

Where \underline{v} is the absolute velocity, \underline{v}_{CV} is the velocity of the control volume and \underline{W} is the velocity relative to the control volume.

Conservation of mass

$$0 = \frac{\partial}{\partial t} \int_{CV} (\rho) dV + \int_{CV} (\rho(\underline{W} \cdot \underline{\hat{n}})) dA$$

Conservation of momentum

$$\sum \underline{F}_{sys} = \int_{CV} (\rho \underline{W}(\underline{W} \cdot \underline{\hat{n}})) dA$$

When steady state and in an inertial reference frame.

1.10.16 Energy equation

$$\left[\sum \dot{Q}_{\text{net in}} + \sum \dot{W}_{\text{net in}} \right]_{CV} = \frac{\partial}{\partial t} \int_{CV} (e\rho) dV + \int (e\rho(\underline{v} \cdot \underline{\hat{n}})) dA$$

Or,

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}} = \frac{\partial}{\partial t} \int_{CV} (e\rho) dV + \int_{CS} \left(u + \frac{P}{\rho} + \frac{v^2}{2} + gz \right) \rho(\underline{v} \cdot \underline{\hat{n}}) dA$$

When steady flow and only one stream,

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}} = \dot{m} \left[u_{\text{out}} - u_{\text{in}} + \left(\frac{P}{\rho} \right)_{\text{out}} - \left(\frac{P}{\rho} \right)_{\text{in}} + \frac{v_{\text{out}}^2 - v_{\text{in}}^2}{2} + g(z_{\text{out}} - z_{\text{in}}) \right]$$

In the Bernoulli form:

$$\frac{P_{\text{out}}}{\rho} + \frac{v_{\text{out}}^2}{2} + gz_{\text{out}} = \frac{P_{\text{in}}}{\rho} + \frac{v_{\text{in}}^2}{2} + gz_{\text{in}} + W_{\text{shaft net in}} - \text{loss, which is:}$$

$$\text{loss} = u_{\text{out}} - u_{\text{in}} - q_{\text{net in}}$$

1.10.17 Shaft work efficiency

$$\eta = \frac{W_{\text{shaft net in}} - \text{loss}}{W_{\text{shaft net in}}} \quad \text{Where loss} = u_{\text{out}} - u_{\text{in}} - q_{\text{net in}}$$

1.10.18 Laminar flow and boundary layers

$$u_y(0) = 0$$

$$u_y(D) = v$$

$$\tau = \mu \frac{du_y}{dx}$$

$$\frac{\tau}{\mu} \int du = \int du_y$$

$$\frac{\tau}{\mu} x + k = u_y$$

When $x = 0$, $u_y = 0 \rightarrow k = 0$

$$\therefore u_y = \frac{\tau}{\mu} x$$

$$u_y = \frac{v}{D} x$$

1.10.19 Flow through porous media

$$\bar{u} = -C \frac{\partial P}{\partial x}$$

1.10.20 Newtonian fluids

$$\bar{u} \propto \frac{1}{\mu}$$

1.10.21 Hagen-Poiseuille law

$$\begin{aligned}\Delta P &= \frac{dP}{dx} L \\ \tau &= \mu \frac{du_y}{dx} = \mu \frac{du_y}{dr} \\ F &= \tau \cdot 2\pi r \cdot L \\ F &= \left(\frac{dP}{dx} \cdot L \right) \pi r^2 \\ u &= -\frac{1}{4\mu} \frac{dP}{dx} (R^2 - r^2) \\ u_{mean} = \bar{u} &= \frac{Q}{\text{Area}} = -\frac{R^2}{8\mu} \frac{dP}{dx} \\ u_{mean} &= -\frac{1}{4\mu} \frac{dP}{dx} R^2 \\ Q &= -\frac{\pi R^4}{8\mu} \frac{dP}{dx} \\ \text{note } u_{mean} = \bar{u} &= \frac{u_{max}}{2}\end{aligned}$$

1.10.22 Boundary layer

- Laminar region, thickness increases as $x^{0.5}$.
- Turbulent region, thickness increases as $x^{0.8}$
- Boundary is taken to be at the contour where the velocity is 99% of main flow.

1.10.23 Displacement thickness

$$S^* = \int_0^\infty \left(1 - \frac{u}{u_m} \right) dy$$

1.10.24 Momentum thickness

$$\theta = \int_0^\infty \frac{u}{u_m} \left(1 - \frac{u}{u_m} \right) dy$$

Chapter 2

Thermodynamics

2.1 Second Law of Thermodynamics

First, let us remind ourselves of the first law of thermodynamics and some of its limitations.

Energy can neither be created or destroyed during a process; it can only change forms.

A certain energy balance will hold when a system undergoes change or a thermodynamic process.

- But does not give information on whether the change of state or the process is at all feasible or not.
- It cannot indicate whether a metallic bar of uniform temperature can spontaneously become warmer at one end and cooler at the other.
- However, if that process did occur, all that law can state is that the energy gained at one end would exactly equal the energy lost at the other.

2.1.1 Introduction to the second law

The second law of thermodynamics provides the criterion as to the *probability* of various processes.

Spontaneous processes in nature occur only in one direction.

Heat flows from a body at high temperature to a body at low temperature; water always flows downwards etc. The spontaneity of the process is due to finite driving potential, sometimes called 'force' or 'cause', e.g. a temperature or concentration gradient or an electric potential. What happens as a result of this finite driving potential is called the 'flux' or the 'current' or the 'effect' (heat transfer, mass transfer, flow of electric current). This directional law puts a limitation on energy transformation other than that imposed by the first law. The second law also asserts that energy has *quality* as well as *quantity*. The first law is concerned with the quantity of energy and the transformations of energy from one form to another, with no regard to its quality.

2.1.2 Qualitative difference between heat and work

There is also a qualitative difference between heat and work. Energy supplied as work can be completely converted to heat, e.g. paddle wheel work on a liquid in an adiabatic vessel. However, the complete conversion of heat into work is not possible, thus making heat and work not completely interchangeable forms of energy. Also, we considered the problem of a simple steam power plant and by using the Steady Flow Energy Equation (i.e. first law) and the properties of steam, were able to calculate the work done and heat transfers for individual components. However, we are not yet able to understand ways of improving steam engine efficiency.

The second law is based on experimental observation and was the result of the question, 'how efficient can one make a steam engine?' From now we will start by considering engines and define them with the precision that thermodynamics requires. The sort of steam engines we shall discuss are heat devices in boxes with no fluid entering or leaving but with just heat and work crossing the boundaries.

2.1.3 Thermodynamic cycles and thermal reservoirs

Thermodynamic cycles consist of a system, a cold reservoir and a hot reservoir. Reservoirs are regions outside a system that are so large that their intensive properties remain constant. Thermal reservoirs are bodies that exchange an infinite amount of heat with the system. The temperature of a thermal reservoir never changes. For example: Earth's atmosphere, large bodies of water, vapour condensing at a constant pressure. A *heat sink* absorbs heat energy. A *heat source* transfers energy to the system.

2.1.4 Heat engine

A heat engine (or Cyclic Heat Power Plant - CHPP) is a continuously operating thermodynamic system at the boundary of which there are heat and work transfers.

Notes:

- 'Continuously operating' means that the state of the system exhibits only periodic (cyclic) changes.
- The heat engine is a thermodynamic system and so no matter crosses the boundary e.g. simple steam power plant and closed-cycle gas turbine plant.

As we know that heat transfer to work does not occur. However devices like heat engines have been created, which are special devices which are used to produce work from heat. All heat engines differ but can be characterised by the following:

- They receive heat from a high temperature source (solar energy, oil furnace, nuclear reactor, etc.)
- They convert part of this heat to work (usually in the form of a rotating shaft.)
- They reject the remaining heat to a low temperature sink (the atmosphere, body of water, etc.)
- They operate on a cycle.

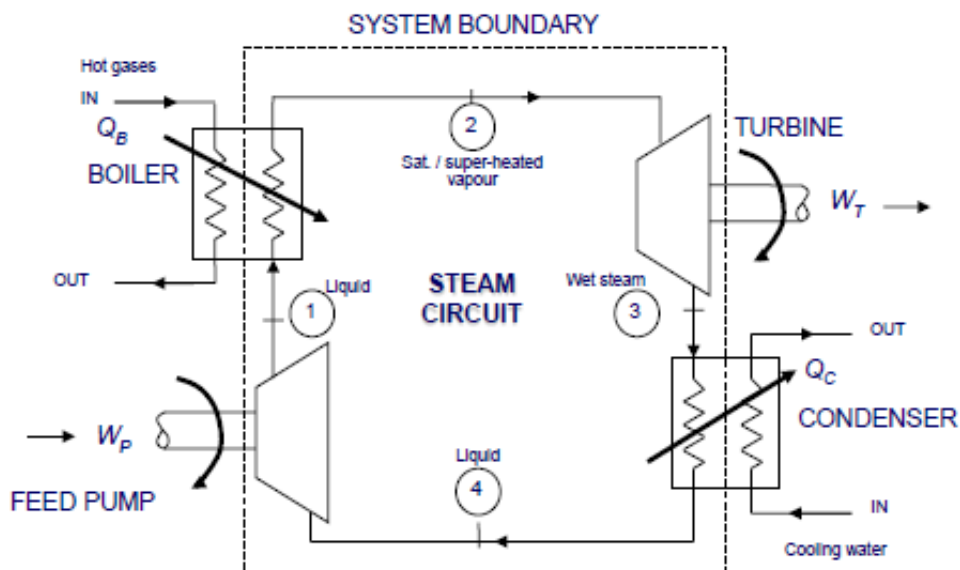


Figure 2.1: A simple steam power plant

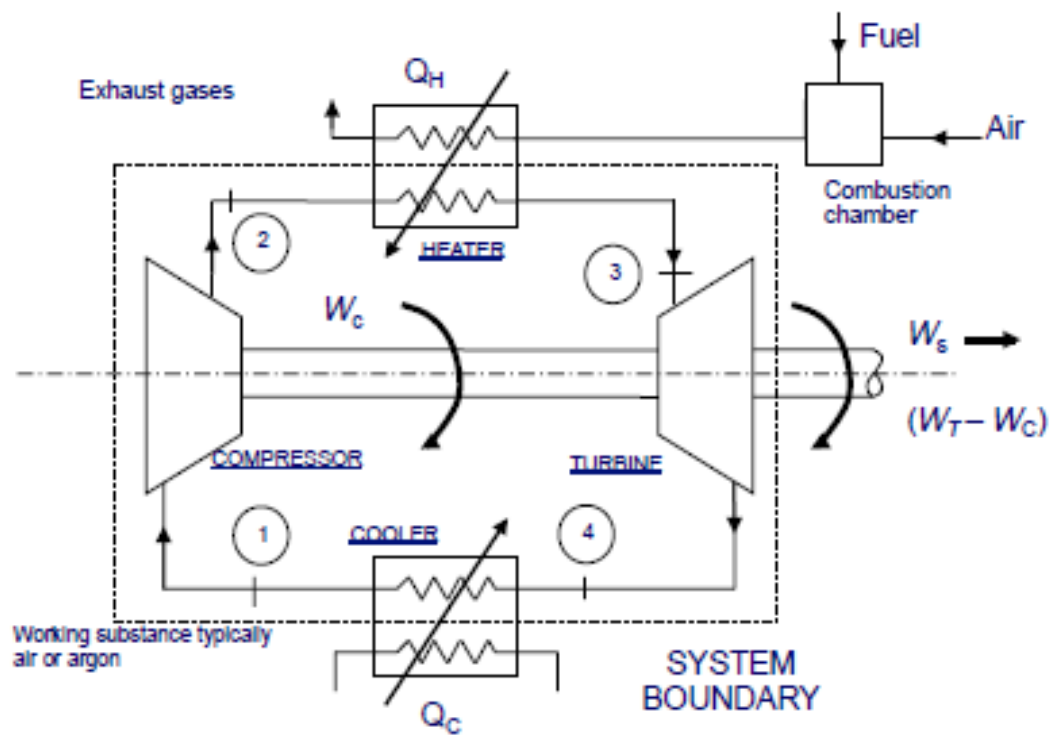


Figure 2.2: A closed cycle gas power plant

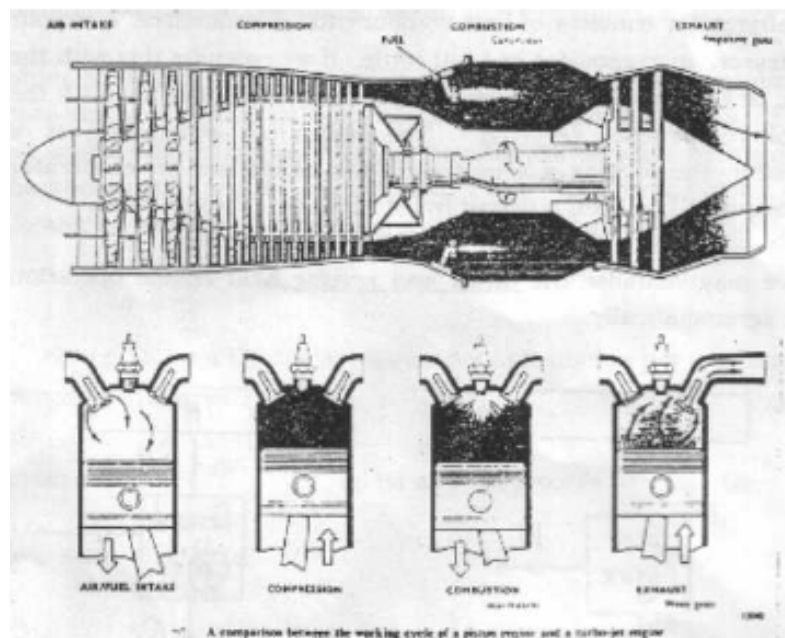


Figure 2.3: A diesel engine

Diesel engines (Figure 2.3) (and internal combustion engines generally) are not heat engines (CHPP) because matter crosses its boundaries. A jet engine is also not a

CHPP because matter, air, fuel and exhaust cross the boundary of the system.

2.1.5 Steam power plant

The work producing device that best suits this definition of a heat engine is the steam power plant, which is an external combustion engine.

1. Combustion takes place outside.
2. Transferred to steam as heat.
3. Passed through various devices that transfer its energy to work.

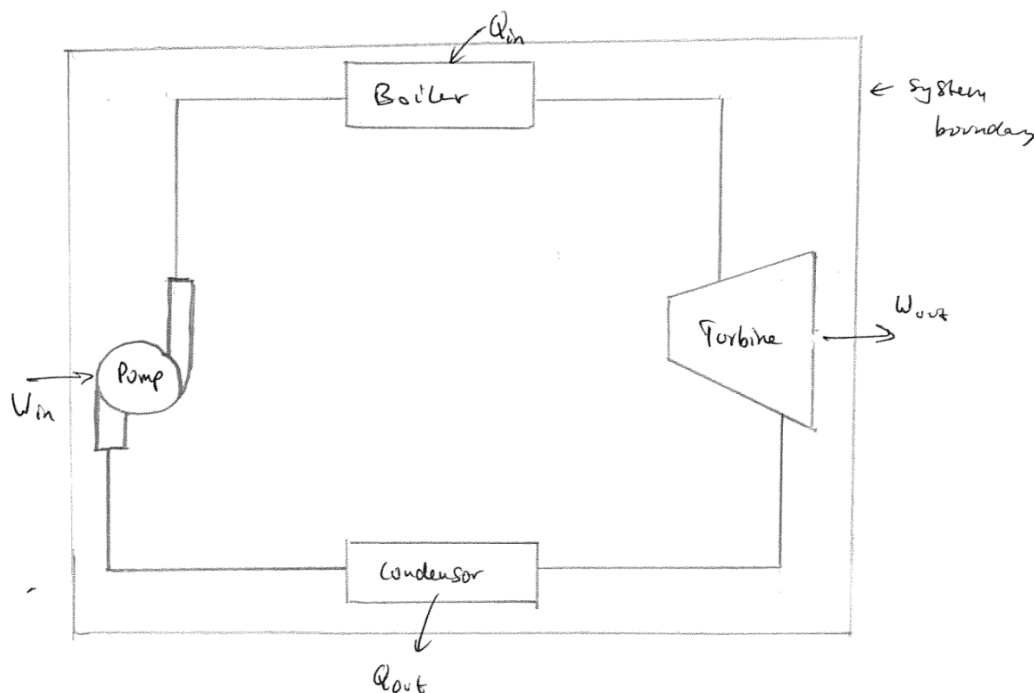


Figure 2.4: Steam power plant operation

Figure 2.4 shows a simplified diagram of a steam plant. From this we can see that $W_{net\ out} = W_{out} - W_{in}$ (kJ). Considering that the change in internal energy is zero for cycles, we can derive, $W_{net\ out} = Q_{in} - Q_{out}$ (kJ).

2.1.6 Thermal efficiency of direct engines / steam power plants

For heat engines, the desired output is the net work output and the required input is the amount of heat supplied to the working fluid. Thus the thermal efficiency is expressed as:

$$\text{Thermal efficiency} = \frac{\text{Net work output}}{\text{Total heat input}} \quad (2.1.1)$$

$$\eta = \frac{W_{\text{net out}}}{Q_{\text{in}}} \quad (2.1.2)$$

$$\eta = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}} \quad (2.1.3)$$

Since cyclic devices at practical interest operate between a high temperature T_H and a low temperature medium T_L , we define these two quantities:

- Q_H = magnitude of heat transfer between the cyclic device and the high temperature medium T_H .
- Q_L = magnitude of heat transfer between the cyclic device and the low temperature medium T_L .

Thus, the previous equations can be written as follows:

$$W_{\text{net out}} = Q_H - Q_L \quad (2.1.4)$$

$$\eta = 1 - \frac{Q_L}{Q_H} \quad (2.1.5)$$

Thermal efficiencies of work producing devices are relatively low. Ordinary spark ignition automobile engines have a thermal efficiency of about 25%. Gas steam power plants reach only 60%.

2.1.7 Can we save Q_{out} ?

In a steam power plant, the condenser is the device where large quantities of waste heat is rejected to rivers, lakes or the atmosphere. One may ask, can we not save all this waste energy? The answer is a firm *no*. Without a heat rejection process in the condenser, the cycle cannot be completed. Cyclic devices such as steam power plants cannot run continuously, unless the cycle is completed.

2.1.8 The Kelvin-Planck statement

The Kelvin-Planck statement of the second law of thermodynamics is expressed as follows:

It is impossible for any device that operates on a cycle to receive heat from a single reservoir and produce a net amount of work.

A heat engine must exchange heat with a low temperature sink as well as a high temperature source to keep operating. This implies that it is impossible to build a heat engine that has a thermal efficiency of 100%. Complete conversion of heat into work is *not possible*. This is contrary to the fact that 100% of work can be transferred to heat. This is the directional implication of the second law. This is not due to frictional/non-adiabatic effects. It is a *necessity*.

Proof on page 128 of Cengel (insert edition.)

2.1.9 Refrigerators and heat pumps

From experience, heat always flows from high temperature to low temperature. These reverse process, however, cannot occur by itself. This reverse process requires special devices called refrigerators. Refrigerators are cyclic devices and the working fluid is called a refrigerant. A frequently used refrigeration cycle is the 'vapour-compression refrigeration cycle.' This involves a compressor, a condenser, an expansion valve and an evaporator.

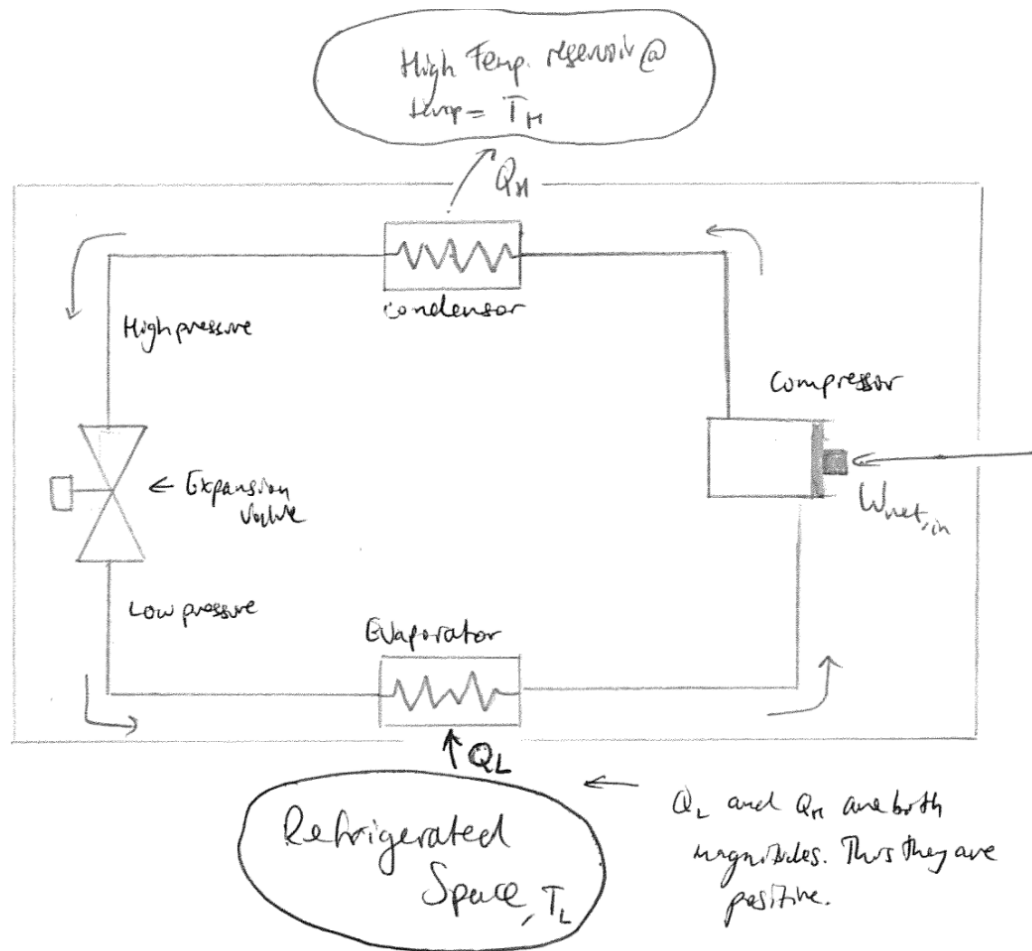


Figure 2.5: Refrigeration cycle

2.1.10 Coefficient of performance

The efficiency of a refrigerator is called the coefficient of performance (COP), denoted by COP_R . The objective of a refrigerator is to: remove heat from the refrigerated space by being provided with a work input $W_{\text{net in}}$. Thus:

$$\text{COP}_R = \frac{Q_L}{W_{\text{net in}}} = \frac{\dot{Q}_L}{\dot{W}_{\text{net in}}} \quad (2.1.6)$$

Also, since $W_{\text{net in}} = Q_H - Q_L$

$$\text{COP}_R = \frac{Q_L}{Q_H - Q_L} \quad (2.1.7)$$

$$\text{COP}_R = \frac{1}{\frac{Q_H}{Q_L} - 1} \quad (2.1.8)$$

COP can also be greater than unity. This means the heat removed can be greater than the work input. Whereas thermal efficiency can never be greater than 1.

2.1.11 Heat pumps

Another device that transfers heat from a low temperature medium to a high temperature one is a heat pump. Here are some differences between refrigerators and heat pumps.

- Refrigerators:
 - Maintain the refrigerated space at a low temperature by removing heat from it.
 - This extracted heat is then discharged to a high temperature medium out of necessity.
- Heat pumps:
 - Maintain a heated space at a high temperature.
 - This is accomplished by absorbing heat from a low temperature source e.g. well, water, cold air and then supplying this heat to the high temperature medium such as a house.

A heat pump runs on the same cycle as a refrigerator. The measure of performance of a heat pump is also expressed in terms of coefficient of performance COP_{HP} .

$$\text{COP}_{HP} = \frac{Q_H}{W_{\text{net in}}} = \frac{Q_H}{Q_H - Q_L} = \frac{1}{1 - \frac{Q_L}{Q_H}} \quad (2.1.9)$$

$$\text{COP}_{HP} = \text{COP}_R + 1 \quad (2.1.10)$$

This relation shows us that $\text{COP}_{HP} > \text{COP}_R$ at all times as COP_R is always positive.

2.1.12 Air conditioners

Air conditioners are basically refrigerators where refrigerated space is a room instead of a food compartment. The same air conditioning unit can be used as a heat pump in winter by installing it backwards. In this mode, the unit absorbs heat from the cold outside and delivers it to the room.

2.1.13 The Clausius statement

The Kelvin-Planck statement was for direct heat engines. The Clausius statement is for reversed heat engines. The statement is as follows:

It is impossible to construct a device that operates in a cycle and produces no effect other than the transfer of heat from a low temperature body to a higher temperature body.

This means that it is impossible to construct a refrigerator that operates without an input of work. $W_{\text{net in}} \neq 0 \therefore \text{COP}_R \neq 0$.

2.1.14 Reversible and irreversible processes

As a result of the second law we know that the complete conversion of heat to work is impossible. Thus, an efficiency of 100% cannot be achieved for a heat engine. Then what is the maximum efficiency of a heat engine/CHPP? Before we answer this question, we need to define an idealised process first, which is called a reversible process. Reversible process:

A process that can be reversed without having any trace to the surroundings. Both the system and the surroundings are returned to their initial states at the end of the reverse process. Thus, for the combined forward and reverse processes the *net heat* and *net work* transfer is *zero*.

Reversible processes do not occur in nature naturally. They are merely an idealisation of real processes. They can be approximated by actual devices but can never be achieved. We consider them even though they are impossible, as they are easy to analyse and they can be used for comparison.

2.1.15 Irreversibilities

The factors that cause a process to be irreversible are called irreversibilities. They include:

- Friction.

- Unrestrained expansion.
- Mixing of two fluids.
- Heat transfer across a finite temperature difference.
- Electrical resistance.
- Inelastic deformation of solids.
- Chemical reactions.

A reversible process includes none of these. There are three types of reversible process:

- Externally reversible.
- Internally reversible.
- Totally reversible.

Externally reversible

No irreversibilities exist in the surroundings. Heat transfer can occur between the system and the surroundings but only with an infinitesimal temperature difference. There may still be irreversibilities with the system.

Internally reversible

No irreversibilities exist within the system. The system moves slowly and without friction through a series of equilibrium states. Irreversibilities may exist in the surroundings usually due to heat transfer through a finite temperature difference.

Totally reversible

A process is called totally reversible, or simply reversible, if it involves no irreversibilities within the system or its surroundings. A totally reversible process involves no heat transfer through a finite temperature difference, no non quasi-equilibrium changes and no friction or other dissipative effects.

2.2 The Carnot cycle

A quick recap of the content covered so far.

Heat engines are cyclic devices. The working fluid returns back to its original state at the end of each cycle. In part one (of the cycle), there is work done by the fluid and in part two, work is done on the working fluid. This difference between these two parts is the net work delivered by the engine. Cycle efficiency can be maximised by using processes that require the least amount of work. This is achieved by using *reversible processes*. Reversible cycles cannot be achieved in practice because the irreversibilities associated with each process cannot be eliminated. However, they provide upper limits to the performance of real cycles. The most famous reversible cycle is the Carnot cycle. A theoretical heat engine that operates on a Carnot cycle is called a Carnot heat engine. It is composed of four reversible processes: two isothermal and two adiabatic. It can be executed either in a closed system or a steady flow system.

Consider a system that consists of a gas contained in an adiabatic piston-cylinder device. The insulation of the cylinder head is such that it may be removed to bring the cylinder into contact with reservoirs to provide heat transfer. The four reversible processes that make up the Carnot cycle are as follows:

A: Reversible isothermal expansion (Process 1-2, constant T_H)

1. The gas is at temperature T_H .
2. The cylinder head is in close contact with a source at temperature T_H .
3. The gas is allowed to expand slowly and does work on the surrounding.
4. But as the temperature drops by dT , some heat is transferred from the source to the gas, raising the temperature back to T_H . Thus, the temperature remains constant at T_H .

Since the temperature difference never exceeds dT , it is a reversible heat transfer process. It continues until reaching position 2. The total amount of heat transferred to the gas during this process is Q_H .

B: Reversible adiabatic expansion (Process 2-3, temperature drops from $T_H \rightarrow T_L$)

1. The heat source in A is removed and the insulation is added, so the system becomes adiabatic.
2. The gas expands slowly, doing work on the surroundings until its temperature drops from T_H to T_L .

The piston is assumed to be frictionless and the process to be quasistatic equilibrium, so the process is reversible and adiabatic.

C: Reversible isothermal compression (Process 3-4, $T_L = \text{constant}$)

1. Insulation is removed and replaced with a low temperature sink at temp T_L .
2. The piston is pushed inwards by an external force, doing work on the gas.
3. As the gas is compressed, its temperature tends to rise.
4. If it rises by dT , it drops back to T_L as heat is transferred from the gas to the sink.
5. Thus, temperature remains constant at T_L .

This is a reversible heat transfer process as the temperature difference never exceeds dT . The amount of heat rejected is Q_L .

D: Reversible adiabatic compression (Process 4-1, temperature rises from $T_L \rightarrow T_H$)

1. Insulation is added back.
2. Gas is compressed.
3. The gas returns to its original state.
4. Temperature rises from T_L to T_H which completes the cycle.

All of the steps of the Carnot cycle can be summarised on a P-V graph (Figure 2.6). Being a reversible cycle, the Carnot cycle is the most efficient cycle operating between two specified temperature limits.

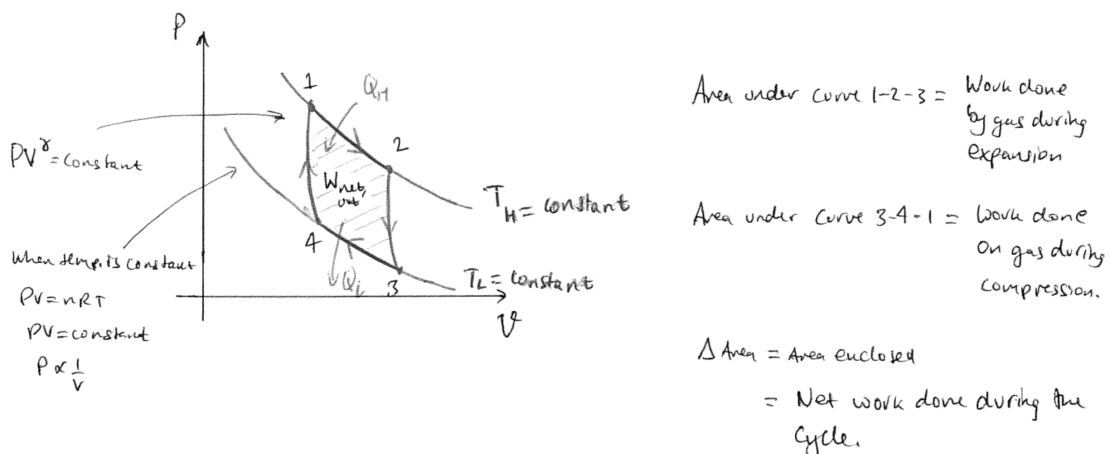


Figure 2.6: Carnot cycle

2.2.1 Reversed Carnot cycle

The Carnot cycle is totally reversible. Thus, if each process is reversed, it becomes the Carnot refrigeration cycle. Heat and work interactions are reversed.

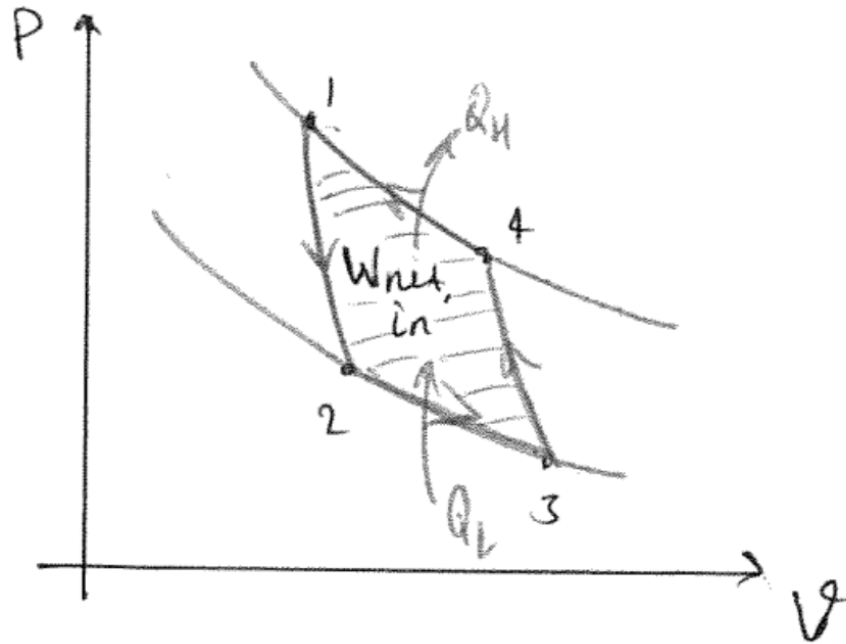


Figure 2.7: Reversed Carnot cycle

2.2.2 The Carnot principles

The second law of thermodynamics puts limits on the operation of cyclic devices as expressed by the Kelvin-Planck and Clausius statements. A heat engine cannot operate by exchanging heat with a single reservoir and a refrigerator cannot operate without a net energy input from an external source. These bring rise to the Carnot principles:

1. The efficiency of an irreversible heat engine is always less than the efficiency of a reversible one operating between the same two reservoirs.
2. The efficiency of all reversible heat engines operating between the same two reservoirs are the same.

Proof for statement 1 (of Carnot principles)

insert proof

2.2.3 The thermodynamic temperature cycle

insert proof

$$\left(\frac{Q_M}{Q_L}\right)_{rev} = \frac{T_H}{T_L} \leftarrow \text{use Kelvin} \quad (2.2.1)$$

2.2.4 The Carnot heat engines

Any heat engine has a thermal efficiency of η_{th} given by the formula

$$\eta_{th} = 1 - \frac{Q_L}{Q_H} \quad (2.2.2)$$

For reversible heat engines, such as a Carnot engine, this becomes:

$$\eta_{th, rev} = 1 - \frac{T_L}{T_H} \quad (2.2.3)$$

Using the formula shown in the previous part. This is called the Carnot efficiency, since the Carnot heat engine is the best known reversible engine. This is the highest efficiency a heat engine operating between two thermal energy reservoirs at temperature T_H and T_L can have. If you calculate the η_{th} of a heat engine, their nature can be deduced as follows:

$$\eta_{th} < \eta_{th, rev} \rightarrow \text{Irreversible heat engine} \quad (2.2.4)$$

$$\eta_{th} = \eta_{th, rev} \rightarrow \text{Reversible heat engine} \quad (2.2.5)$$

$$\eta_{th} > \eta_{th, rev} \rightarrow \text{Impossible heat engine} \quad (2.2.6)$$

From the equation $\eta_{th, rev} = 1 - \frac{T_L}{T_H}$, we can see that the efficiency of a Carnot heat engine increases as T_H is increased, or as T_L is decreased. As T_L decreases so does the amount of heat rejected. The same is also true for actual heat engines. The thermal efficiency of actual heat engines can be maximised by:

- Supplying heat to the engine at the highest possible temperature (limited by material strength).
- And rejecting heat from the engine at the lowest possible temperature (lim-

ited by the temperature of the cooling medium such as rivers, lakes or the atmosphere).

2.2.5 The Carnot refrigerator and heat pump

The COP of any heat pump and refrigerator is given by:

$$\text{COP}_R = \frac{1}{\frac{Q_H}{Q_L} - 1} \quad (2.2.7)$$

$$\text{COP}_{HP} = \frac{1}{1 - \frac{Q_L}{Q_H}} \quad (2.2.8)$$

Using the thermodynamic scale formulas, the COP for heat pumps and refrigerators running on a reverse Carnot cycle is:

$$\text{COP}_{R, rev} = \frac{1}{\frac{T_H}{T_L} - 1} \quad (2.2.9)$$

$$\text{COP}_{HP, rev} = \frac{1}{1 - \frac{T_L}{T_H}} \quad (2.2.10)$$

These are the highest COP that a refrigerator or a heat pump operating between the temperature limits at T_L and T_H can have.

$$\text{COP}_R = \begin{cases} < \text{COP}_{R, rev} \\ = \text{COP}_{R, rev} \\ > \text{COP}_{R, rev} \end{cases} \quad \& \quad \text{COP}_{HP} = \begin{cases} < \text{COP}_{HP, rev} \rightarrow \text{Irreversible} \\ = \text{COP}_{HP, rev} \rightarrow \text{Reversible} \\ > \text{COP}_{HP, rev} \rightarrow \text{Impossible} \end{cases} \quad (2.2.11)$$

2.3 Entropy

An important inequality that has major consequences in thermodynamics is the *Clausius inequality*.

$$\oint \frac{\delta Q}{T} \geq 0 \quad (2.3.1)$$

The cyclic integral of $\frac{\delta Q}{T}$ is always less than or equal to zero. This inequality is valid for all cycles reversible or irreversible. $\frac{\delta Q}{T}$ is the sum of all differential amounts of heat transfer to or from a system, divided by the temperature of the boundary.

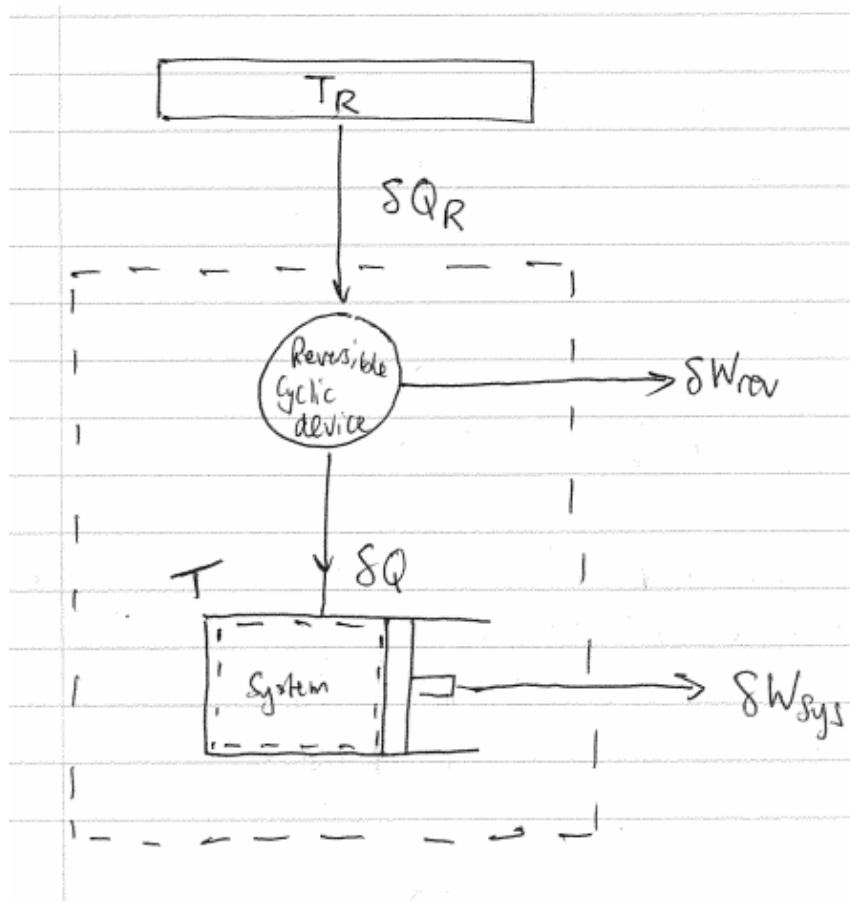


Figure 2.8: Diagram to show proof of Clausius inequality

2.3.1 Proof of the Clausius inequality

Consider a system connected to a thermal energy reservoir at constant temperature T_R . Through a *reversible* cycle device. Where δQ_R is the heat supplied to the cyclic device and δQ is the heat rejected by the device to a system which has a temperature T .

The reversible cyclic device produced work = $\delta W_{text{rmrev}}$ and the system produced work = δW_{sys} . Applying energy balance to the entire combined system gives:

$$dE = \delta Q_R - \delta W_{sys} - \delta W_{rev} \quad (2.3.2)$$

$$\delta W_C = \delta W_{sys} - \delta W_{rev} \quad (2.3.3)$$

$$dE = \delta Q_R - \delta W_C \quad (2.3.4)$$

$$\delta W_C = \delta Q_R - dE_C \quad (2.3.5)$$

Since the cyclic device is a reversible one:

$$\therefore \frac{\delta Q_R}{\delta Q} = \frac{T_R}{T} \leftarrow \text{thermodynamic scale} \quad (2.3.6)$$

$$\therefore \delta Q_R = T_R \frac{\delta Q}{T} \quad (2.3.7)$$

Substituting equation (2.3.7) into (2.3.5) gives:

$$\delta W_C = T_R \frac{\delta Q}{T} - dE_C \quad (2.3.8)$$

Let the system undergo a cycle while the cyclic device undergoes an integral number of cycles. Thus the relation becomes:

$$W_C = T_R \oint \frac{\delta Q}{T} \quad (2.3.9)$$

The net work for the combined cycle is $W_C = \oint \delta W_C$ and $\oint dE = 0$ as the net change in energy, which is a property, during a cycle is zero.

Now since the Kelvin-Planck statement states that:

No system can produce a net amount of work while operating in a cycle and exchanging heat with a single thermal reservoir.

And the combined system is exchanging heat with a single thermal reservoir and producing work, hence this must be impossible. Thus,

$$T_R \oint \frac{\delta Q}{T} \leq 0 \quad (2.3.10)$$

Since T_R is positive dividing by it gives:

$$\oint \frac{\delta Q}{T} \leq 0 \quad (2.3.11)$$

If no reversibilities occur within the system as well as the reversible cyclic device then the cycle undergone by the combined system is internally reversible. Thus, it cannot be reverted.

In the reversed cycle case, all quantities have the same magnitude but opposite sign.

$$\therefore W_C = T_R \oint \frac{\delta Q}{T} \quad (2.3.12)$$

We cannot be negative in the reversed case as that would break the Kelvin-Planck

statement and W_C cannot be positive either,

$$\therefore T_R \oint \frac{\delta Q}{T} = 0 \quad (2.3.13)$$

Since T_R cannot be zero, hence

$$\oint \left(\frac{\delta Q}{T} \right)_{\text{int rev}} = 0 \quad (2.3.14)$$

Thus,

$$\oint \left(\frac{\delta Q}{T} \right)_{\text{int rev}} = 0 \text{ and } \oint \frac{\delta Q}{T} \leq 0 \quad (2.3.15)$$

The *inequality* holds for the totatlly or just internally reversible cycles and the inequality for irreversible ones.

We have proven that $\oint \left(\frac{\delta Q}{T} \right)_{\text{int rev}} = 0$ and $\oint \frac{\delta Q}{T} \leq 0$. here we have a quantity whose cyclic integral is zero. A quantity whose cyclic integral is zero depends on the state only and no the process path, is a property. Therefore, the quantity $\left(\frac{\delta Q}{T} \right)_{\text{int rev}}$ must represent a property in the differential form. This is designated the name entropy, S .

$$ds = \left(\frac{\delta Q}{T} \right)_{\text{int rev}} \text{ kJ K}^{-1} \quad (2.3.16)$$

Entropy is an extensive property of a system and is sometimes referred as total entropy. Entropy per junit mass is designated s and is an intensive property and has the unit $\text{kJ K}^{-1} \text{kg}^{-1}$. The entropy change of a system during a process can be determined by integration of equation (2.3.16) between initial and final states.

$$\Delta S = S_2 - S_1 = \int_1^2 \left(\frac{\delta Q}{T} \right)_{\text{int rev}} \quad (2.3.17)$$

Here we have defined the change in entropy instead of the entropy itself. Engineers are usually concerned with the change in entropy. This reference states are used in tabulated data. Entropy is a property; thus, it has fixed values at fixed states. Thus, ΔS between two specified states is the same no matter what path, reversible or irreversible is followed during a process. Also note that $\int \frac{\delta Q}{T}$ gives us ΔS only if the integration is carried out along an internally reversible path between the two states. Therefore, even for irreversible processes, the entropy change should be determined by carrying out this integration along an imaginary internally reversible path between the specified states.

Special case: internally reversible isothermal heat transfer process

Isothermal heat transfer processes are internally reversible. Let us perform the integration

$$\Delta S = \int_1^2 \left(\frac{\delta Q}{T} \right)_{\text{int rev}} \quad (2.3.18)$$

$$= \int_1^2 \left(\frac{\delta Q}{T_0} \right)_{\text{int rev}} \quad (2.3.19)$$

$$= \int_1^2 \frac{1}{T_0} (\delta Q)_{\text{int rev}} \quad (2.3.20)$$

$$\therefore \Delta S = \frac{Q}{T_0} \text{ kJ K}^{-1} \quad (2.3.21)$$

ΔS can be positive or negative. This depends on the direction of Q . Losing heat reduces entropy.

2.3.2 The increase of entropy principle

Consider a cycle. It has two processes:

- Process 1-2: could be reversible or irreversible.
- Process 2-1: Internally reversible.

The Clausius inequality states:

$$\oint \frac{\delta Q}{T} \leq 0 \quad (2.3.22)$$

Or,

$$\int_1^2 \frac{\delta Q}{T} + \int_1^2 \left(\frac{\delta Q}{T} \right)_{\text{int rev}} \leq 0 \quad (2.3.23)$$

$$\int_1^2 \frac{\delta Q}{T} + (S_1 - S_2) \leq 0 \quad (2.3.24)$$

$$S_2 - S_1 \geq \int_1^2 \frac{\delta Q}{T} \quad (2.3.25)$$

When written in the differential form:

$$ds \geq \frac{\delta Q}{T} \quad (2.3.26)$$

Where T is the thermodynamic temperature at the boundary. δQ is the heat transferred between the system and surroundings. ds is the differential change in energy.

When reversible $ds = \frac{\delta Q}{T}$. When irreversible $ds \geq \frac{\delta Q}{T}$. This equation shows that:

Change in entropy of a closed system during an irreversible process is *always greater* than the integral of $\frac{\delta Q}{T}$ evaluated for that process.

This inequality shows that during an irreversible process, the entropy change of a closed system is *always greater* than the entropy of transfer by heat. For a reversible process, $\Delta S = \int_1^2 \frac{\delta Q}{T}$, which represents the entropy transfer by heat. Thus, some *entropy generation* must be occurring. This generated entropy is denoted as S_{gen} .

$$\therefore S_2 - S_1 \geq \int_1^2 \frac{\delta Q}{T} \quad (2.3.27)$$

$$\therefore \Delta S_{\text{sys}} \geq \int_1^2 \frac{\delta Q}{T} \quad (2.3.28)$$

$$\therefore \Delta S_{\text{sys}} = \int_1^2 \frac{\delta Q}{T} + S_{\text{gen}} \quad (2.3.29)$$

S_{gen} is always positive or zero. This depends upon the process, hence not a property.

When a system is isolated, adiabatic and closed. The heat transfer is zero i.e. $\delta Q = 0$.

$$\therefore \Delta S_{\text{sys}} = \Delta S_{\text{isolated}} \geq \int_1^2 0 \quad (2.3.30)$$

$$\therefore \Delta S_{\text{isolated}} \geq 0 \Delta S_{\text{isolated}} = S_{\text{gen}} \quad (2.3.31)$$

This is the increase of entropy principle. The entropy of an isolated system during a process always increases or in the limiting case of a reversible process, remains constant. Note that in the absence of any heat transfer, entropy change is due to irreversibilities only and their effect is only and always an increase in entropy.

Entropy is an extensive property. Thus,

$$\sum_{i=1}^N \Delta S_i = \Delta S_{\text{total}} \quad (2.3.32)$$

i.e. the use of the change in entropies of the parts of the system is equal to the change in entropy of a system. An isolated system can consist of a number of subsystems. A system and its surroundings, for example, constitute an isolated system since both can be enclosed by a sufficiently large arbitrary boundary at which there is no heat, work or mass transfer. Thus, a system and its surroundings can be considered as two subsystems of an isolated system. Since $\Delta S_{\text{isolated}} = S_{\text{gen}} \geq 0$ but

$\Delta S_{\text{isolated}} = \Delta S_{\text{total}} = \Delta S_{\text{sys}} + \Delta S_{\text{surr}}$. Therefore:

$$S_{\text{gen}} = \Delta S_{\text{total}} = \Delta S_{\text{sys}} + \Delta S_{\text{surr}} \geq 0 \quad (2.3.33)$$

When considering a reversible process. Then $\Delta S_{\text{sys}} + \Delta S_{\text{surr}} = 0 = S_{\text{gen}}$. The inequality holds for irreversible process. Note that S_{gen} can never be negative, but ΔS_{sys} can be negative and ΔS_{surr} can be negative. The sum of the two must be positive or zero. For example, consider the following system.

$$S_{\text{gen}} = \Delta S_{\text{total}} \quad (2.3.34)$$

$$= \Delta S_{\text{sys}} + \Delta S_{\text{surr}} \quad (2.3.35)$$

$$= -2 + 3 \quad (2.3.36)$$

$$S_{\text{gen}} = 1 \text{ kJ K}^{-1} \quad (2.3.37)$$

$$S_{\text{gen}} = \begin{cases} > 0 \text{ Irreversible process} \\ = 0 \text{ Reversible process} \\ < 0 \text{ Impossible process} \end{cases} \quad (2.3.38)$$

Example question

Using the increase in entropy principle, show that conduction with finite temperature difference is an irreversible process.

$$S_{\text{gen}} = \Delta S_{\text{total}} = \Delta S_{\text{high temp. reservoir}} + \Delta S_{\text{low temp. reservoir}} \geq 0 \quad (2.3.39)$$

$$\Delta S_{\text{total}} = \int \left(\frac{-SQ}{T_H} \right) + \int \frac{SQ}{T_L} \quad (2.3.40)$$

$$= -\frac{1}{T_H} \int SQ + \frac{1}{T_L} \int SQ \quad (2.3.41)$$

$$-\frac{Q}{T_H} + \frac{Q}{T_L} \geq 0 \quad (2.3.42)$$

$$S_{\text{gen}} = \frac{-Q}{T_H} + \frac{Q}{T_L} \geq 0 \quad (2.3.43)$$

$$\text{since } T_H > T_L \quad (2.3.44)$$

$$\therefore S_{\text{gen}} > 0 \rightarrow \text{irreversible process} \quad (2.3.45)$$

2.3.3 Entropy change of pure substances

The entropy of a pure substance is determined from tables in the same manner as other properties such as volume, internal energy and enthalpy. The value of entropy

at a specified state is determined just like any other property.

- Compressed liquids \rightarrow from tables (if available).
- Saturated liquid S_P : from tables.
- Saturated mixture: $S = S_P + xS_{fg}$.
- Saturated vapour S_g : from tables.
- Superheated vapour: from tables.

If the compressed liquid data is not available, the entropy of the compressed liquid can be approximated by the saturated liquid at the same temperature.

$$S_{\text{compressed @ T, P}} = S_{f@T} \quad (2.3.46)$$

Note that $\Delta S = m\Delta S = m(S_2 - S_1)$.

2.3.4 T-S diagram characteristics

Constant volume lines are steeper than the constant pressure lines. Constant pressure lines are parallel to the constant temperature lines in the saturated liquid-vapour mixture region. Constant pressure lines almost coincide with the saturated liquid line in the compressed liquid region.

2.3.5 Isentropic processes

Entropy can be changed by: heat transfer or irreversibilities. During a process that is internally reversible and adiabatic, the entropy of a fixed mass does not change.

Entropy remains constant = isentropic process $\rightarrow \Delta S = 0$ or $S_2 = S_1$

A reversible adiabatic process \rightarrow isentropic

However, isentropic \rightarrow reversible adiabatic (Which usually means an internally reversible, adiabatic process)

2.3.6 Property diagrams involving entropy

Two diagrams commonly used in the second-law analysis are the temperature - entropy and the enthalpy - entropy diagrams. Remember:

$$\frac{SQ_{\text{int rev}}}{T} = dS \quad (2.3.47)$$

$$\therefore SQ_{\text{int rev}} = TdS \quad (2.3.48)$$

$$\therefore Q_{\text{int rev}} = \int_1^2 TdS \text{ (kJ)} \quad (2.3.49)$$

This integral computes the area under a process curve on a T-S diagram. This represents the heat transfer during an internally reversible process.

Area = heat transfer for processes that are internally/totally reversible. (This area has no meaning for irreversible processes.)

It can also be written in specific form

$$q_{\text{int rev}} = \int_1^2 TdS \text{ (kJ kg}^{-1}\text{)} \quad (2.3.50)$$

$$sq_{\text{int rev}} = Tds \text{ (kJ kg}^{-1}\text{)} \quad (2.3.51)$$

For internally reversible isothermal processes, T is constant: T_0 therefore,

$$Q_{\text{int rev isothermal}} = T_0\Delta S \text{ (kJ)} \quad (2.3.52)$$

$$q_{\text{int rev isothermal}} = T_0\Delta s \text{ (kJ kg}^{-1}\text{)} \quad (2.3.53)$$

An isentropic process on a T-S diagram is easily recognised as a *vertical* line segment. Isentropic means no heat transfer, thus area under the graph must be zero. Another diagram commonly used in engineering is the enthalpy - entropy diagram, which is quite valuable in the analysis of steady flow devices such as turbines, compressors and nozzles. In analysing the steady flow of steam through an adiabatic turbine for example, the graph produced is shown below. This graph is also called the Mollier Diagram.

2.3.7 T-S diagram of the Carnot cycle

The Carnot cycle is made up of two, reversible isothermal $CT = \text{constant}$ S processes and two isentropic (adiabatic and $S = \text{constant}$) processes. These four processes form

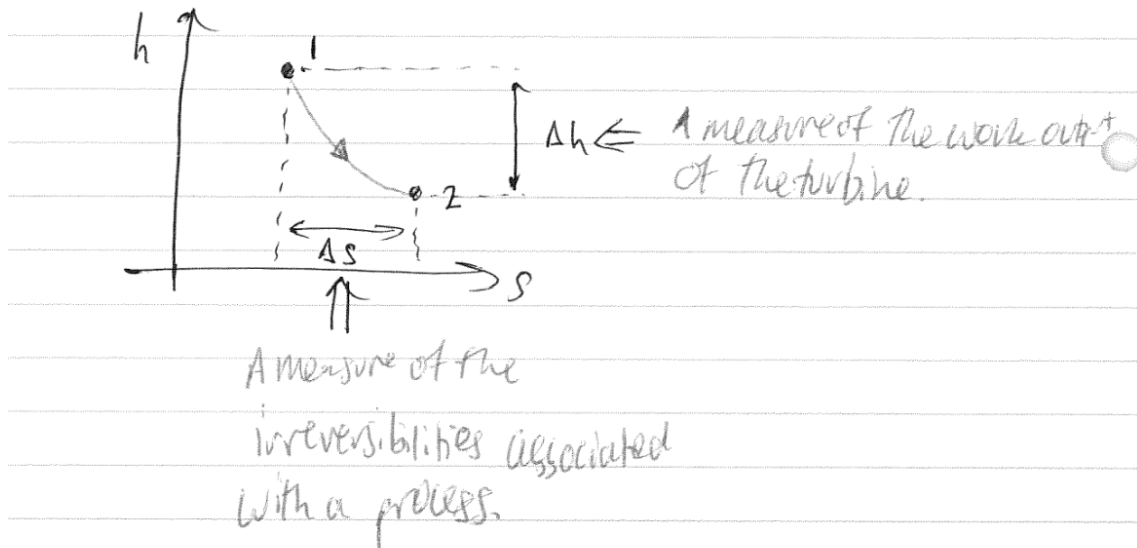


Figure 2.9: Mollier Diagram

a rectangle on the T-S diagram.

2.3.8 The Tds relations

The first law states:

$$\Delta E = \Delta U + \Delta KE + \Delta PE \quad (2.3.54)$$

For closed systems that are stationary, $\Delta KE = \Delta PE = 0$ therefore,

$$\Delta E = \Delta U \quad (2.3.55)$$

$$\Delta U = \Delta Q + \Delta W (+\Delta E_{mass} = 0) \quad (2.3.56)$$

$$\Delta U = \Delta Q + \Delta W \quad (2.3.57)$$

$$\Delta U = Q_{\text{net in}} - W_{\text{net out}} \quad (2.3.58)$$

In differential form and for: a simple compressible substance, an internally reversible process,

$$dU = SQ_{\text{int rev}} - SW_{\text{int rev out}} \quad (2.3.59)$$

But $SQ_{\text{int rev}} = TdS$ and $SW_{\text{int rev out}} = PdV$ thus,

$$dU = TdS - PdV \quad (2.3.60)$$

$$TdS = dU + PdV \text{ (kJ)} \quad (2.3.61)$$

$$Tds = du + Pdv \text{ (kJ kg}^{-1}\text{)} \quad (2.3.62)$$

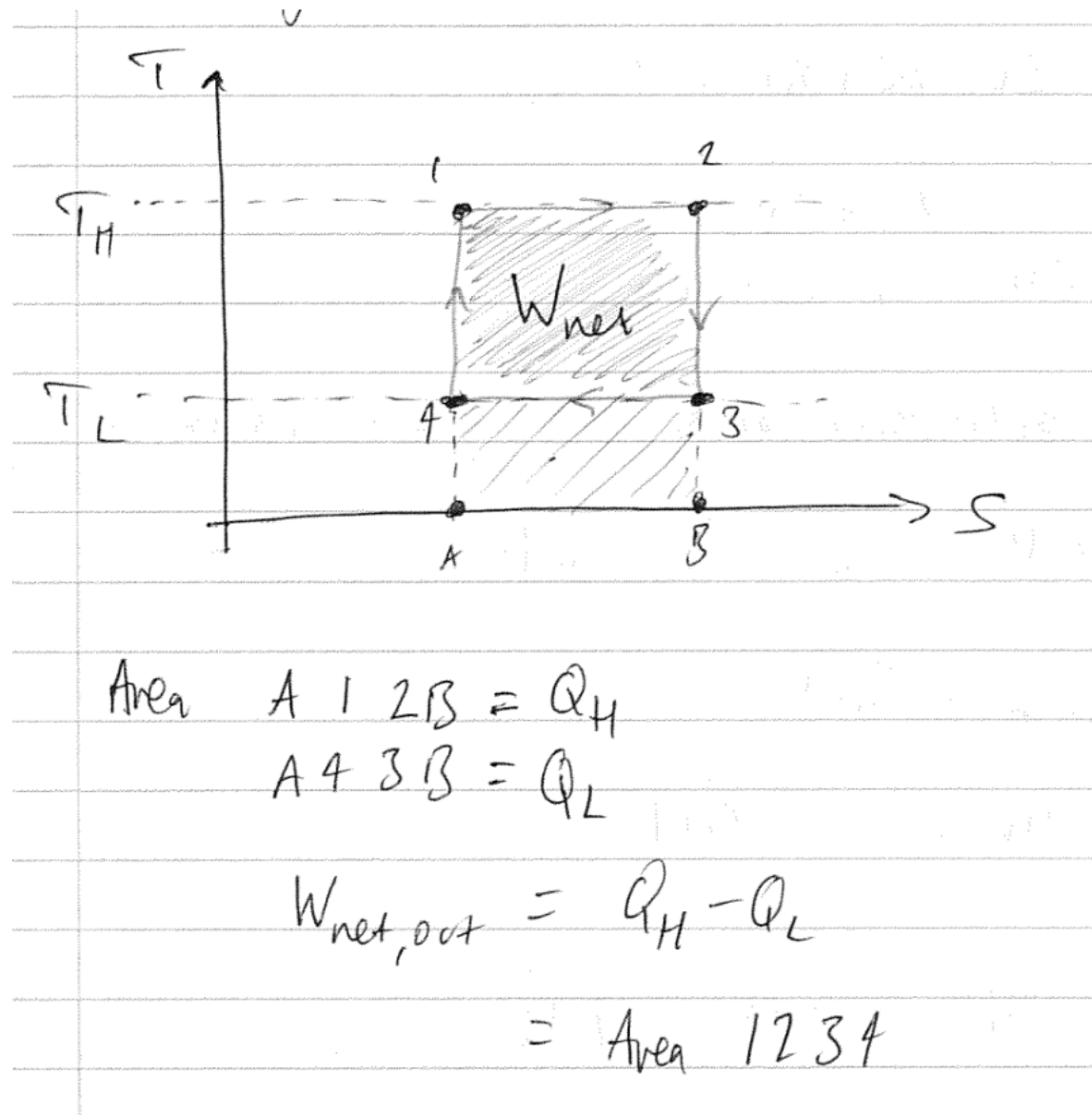


Figure 2.10: T-S diagram for the Carnot cycle

These are known as the first Tds relations.

Since $h = u + Pv$, differentiating both sides yields:

$$dh = d[u + Pv] \quad (2.3.63)$$

$$= du + d[Pv] \quad (2.3.64)$$

$$dh = du + Pdv + dPv \quad (2.3.65)$$

But the first Tds relation states that $Tds = du + Pdv$ therefore,

$$du = Tds - Pdv \quad (2.3.66)$$

$$\therefore dh = Tds + vdP \quad (2.3.67)$$

$$Tds = dh - vdP \quad (2.3.68)$$

This is the second Tds relation.

To summarise, the Tds relations are shown below.

1. $Tds = dh - vdP$
2. $Tds = du + Pdv$

These equations were derived with internally reversible processes but since the entropy change between two states is independent of the process the system undergoes, hence the equations are valid for both irreversible and reversible processes. These are also valid for the change of states of a substance in a closed or an open system. Explicit relations for differential changes in entropy are obtained by solving for ds in the Tds relations.

$$ds = \frac{du}{T} + \frac{Pdv}{T} \text{ and} \quad (2.3.69)$$

$$ds = \frac{dh}{T} - \frac{vdP}{T} \quad (2.3.70)$$

ΔS can be found by *integrating* these equations between final and initial states. To perform these integrations we must know the relationship between du or dh and temperature. For an ideal gas:

$$du = c_V dT, \quad dh = c_P dT \quad (2.3.71)$$

The equation of state of the substance is given by:

$$Pv = RT \text{ (for ideal gases)} \quad (2.3.72)$$

For other substances, we have to rely on tabulated data.

2.3.9 Entropy change of liquids and solids

$$ds = \frac{du}{T} + \frac{Pdv}{T} \quad (2.3.73)$$

But for liquids and solids, they can be approximated as incompressible substances since their specific volumes remain nearly constant during a process. Thus $dv \approx 0$ for liquids and solids.

$$\therefore ds = \frac{du}{T} \quad (2.3.74)$$

Performing the integration gives:

$$\Delta s = s_2 - s_1 = \int_1^2 ds = \int_1^2 \frac{du}{T} \quad (2.3.75)$$

But $du = cdT$ where $c = c_P = c_V$ for liquids and solids.

$$\Delta s = s_2 - s_1 = \int_1^2 \frac{cdT}{T} = \int_1^2 \frac{c(T)dt}{T} \quad (2.3.76)$$

$$\approx c_{avg} \ln\left(\frac{T_2}{T_1}\right) \quad (2.3.77)$$

$$\Delta s = c_{avg} \ln\left(\frac{T_2}{T_1}\right) \text{ kJ kg}^{-1} \text{ K}^{-1} \quad (2.3.78)$$

Where c_{avg} is the average specific heat capacity over the interval. Note that the entropy change of a truly incompressible substance depends only on temperature.

The equation $s_2 - s_1 \approx c_{avg} \ln\left(\frac{T_2}{T_1}\right)$ can be used with liquids and solids to reasonable accuracy (liquids and solids only). However, for liquids that expand considerably with temperature, it may be necessary to consider the effects of volume change in calculations. This is especially the case when the temperature change is large. Applying this to isentropic processes; for liquids and solids only (incompressible):

$$s_2 - s_1 = 0 = C_{avg} \ln\left(\frac{T_2}{T_1}\right) \quad (2.3.79)$$

$$\ln(T_2) - \ln(T_1) = 0 \quad (2.3.80)$$

$$\ln(T_2) = \ln(T_1) \quad (2.3.81)$$

$$T_2 = T_1 \quad (2.3.82)$$

Hence, for truly incompressible substances that undergo an isentropic process, it is also isothermal.

incompressible + isentropic \rightarrow isothermal

This behaviour is closely operate by liquids and solids.

2.3.10 Entropy change of ideal gases

Remember that

$$ds = \frac{dh}{T} - \frac{vdP}{T} \quad (2.3.83)$$

But for ideal gases:

$$dh = c_P dT \quad (2.3.84)$$

$$\therefore ds = \frac{c_P dT}{T} - \frac{vdP}{T} \quad (2.3.85)$$

But the ideal gas relation gives:

$$Pv = RT \quad (2.3.86)$$

$$v = \frac{RT}{P} \quad (2.3.87)$$

$$\therefore ds = \frac{C_P dT}{T} - \frac{RT dP}{PT} \quad (2.3.88)$$

$$ds = \frac{C_P dT}{T} - \frac{R dP}{P} \quad (2.3.89)$$

Integrating both sides gives:

$$\Delta s = s_2 - s_1 = \int_1^2 c_P(T) \frac{dT}{T} - R \ln \left(\frac{P_2}{P_1} \right) \quad (2.3.90)$$

Also $ds = \frac{du}{T} + \frac{Pdv}{T}$ But for ideal gases:

$$du = c_V dT \quad (2.3.91)$$

$$\therefore ds = \frac{c_V dT}{T} + \frac{Pdv}{T} \quad (2.3.92)$$

But the ideal gas relation gives:

$$Pv = RT \quad (2.3.93)$$

$$P = \frac{RT}{v} \quad (2.3.94)$$

$$\therefore ds = \frac{C_V dT}{T} - \frac{RT dv}{vT} \quad (2.3.95)$$

$$ds = \frac{C_V dT}{T} - \frac{R dv}{v} \quad (2.3.96)$$

Integrating both sides gives:

$$\Delta s = s_2 - s_1 = \int_1^2 c_V(T) \frac{dT}{T} - R \ln \left(\frac{v_2}{v_1} \right) \quad (2.3.97)$$

For all ideal gases except monatomic gases, the specific heats depend on the temperature. Thus, the integrals on the previous pages cannot be performed unless the dependencies of c_V and c_P on temperature is known. Even when the $c_V(T)$ and $c_P(T)$ functions are available, long integrals make this approach impractical; this leaves two choices:

1. Assume constant specific heats
2. Evaluate those integrals once and tabulate the results

The relationships allow us to form formulas for Δs , in terms of temperature and pressure and temperature and specific volume. We can now derive a formula for Δs in terms of pressure and specific volume.

$$ds = \frac{c_P dT}{T} - \frac{R dP}{P} \quad (2.3.98)$$

$$\Delta s = mc_P \int_1^2 \frac{dT}{T} - mR \int_1^2 \frac{dP}{P} \text{ assuming } c_P \text{ is constant} \quad (2.3.99)$$

$$\Delta s = mc_P \ln \left(\frac{T_2}{T_1} \right) - mR \ln \left(\frac{P_2}{P_1} \right) \quad (2.3.100)$$

$$\text{since } R = c_P - c_V \quad (2.3.101)$$

$$\Delta s = mc_P \ln \left(\frac{T_2}{T_1} \right) - mc_P \ln \left(\frac{P_2}{P_1} \right) + mc_V \ln \left(\frac{P_2}{P_1} \right) \quad (2.3.102)$$

$$= mc_P \ln \left(\frac{T_2}{P_2} \cdot \frac{P_1}{T_1} \right) + mc_V \ln \left(\frac{P_2}{P_1} \right) \quad (2.3.103)$$

$$\Delta s = mc_P \ln \left(\frac{v_2}{v_1} \right) + mc_V \ln \left(\frac{P_2}{P_1} \right) \quad (2.3.104)$$

2.4 Gas power cycles

2.4.1 Thermodynamic cycles

Two important areas of application of thermodynamics are power generation and refrigeration. Both are accomplished by systems that operate on a thermodynamic cycle.

- Power cycles: operated as engines.
- Refrigeration cycles: operated as refrigerators, air conditioners, heat pumps, etc.
- Gas cycles: the working fluid remains in the gaseous phase throughout the entire cycle.
- Vapour cycles: the working fluid exists in the vapour phase during part of the cycle and in the liquid phase during another part.
- Closed cycles: the working fluid is returned to the initial state at the end of the cycle and is recirculated.
- Open cycles: the working fluid is renewed at the end of the cycle instead of being recirculated.

2.4.2 Heat engines

In automobile engines, the combustion gases are exhausted and replaced by the fresh air-fuel mixture at the end of each cycle. The engine operates on a mechanical cycle but the working fluid does not go through a complete thermodynamic cycle.

- Internal combustion engine: heat is supplied by burning the fuel within the system boundaries e.g. automobile engines.
- External combustion engine: heat is supplied to the working fluid from an external source such as a furnace e.g. steam power plants.

2.4.3 Basic considerations in the analysis of power cycles

The cycles encountered in actual devices are difficult to analyse because of the presence of complicating effects such as friction and the absence of sufficient time for establishment of the equilibrium conditions during the cycle. Thus, we utilise

some idealisations. An ideal cycle is an actual cycle stripped of all its internal irreversibilities, producing a graph made up totally of internally reversible processes. The ideal cycle can be used to show the general characteristics. Recall,

$$\eta_{th} = \frac{W_{net}}{Q_{in}} \text{ or} \quad (2.4.1)$$

$$\eta_{th} = \frac{w_{net}}{q_{in}} \quad (2.4.2)$$

Also, heat engines on a totally reversible cycle (such as the Carnot cycle) have the

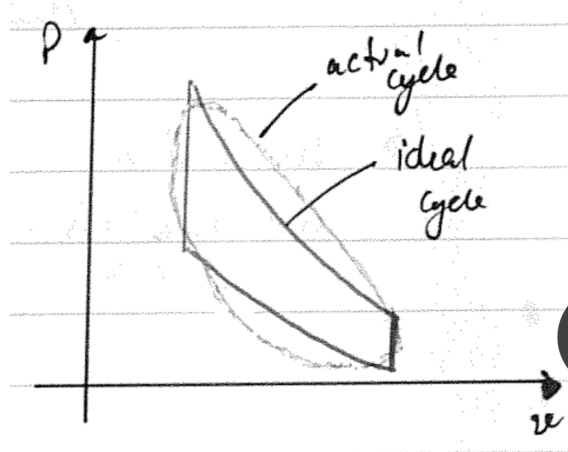


Figure 2.11: How an ideal cycle compares to the actual cycle.

highest thermal efficiency of all heat engines operating between the same temperature levels. The question arises: why do we not use the Carnot cycle? This is due to hardware. It is difficult to implement it. Thus, other ideal cycles are used.

$$\eta_{th, \text{totally reversible}} > \eta_{th, \text{internally reversible}} = \eta_{th, \text{ideal gas}} \quad (2.4.3)$$

This applies between the same temperature limits. We also have:

$$\eta_{th, \text{ideal cycle}} > \eta_{th, \text{actual cycle}} \quad (2.4.4)$$

The idealisations and simplifications commonly employed in the analysis of power cycles can be summarised as follows:

1. The cycle does not involve any friction. Thus, the working fluid does not experience any pressure drop as it flows in pipes or devices such as heat exchangers.
2. All expansion and compression processes take place in a quasi - equilibrium manner.
3. The pipes connecting various components of a system are well insulated and heat transfer through them is negligible.

Kinetic energy and potential energy are also generally neglected as their changes are negligible.

$$\text{KE and PE neglected for } \left\{ \begin{array}{l} \text{turbines} \\ \text{compressors} \\ \text{pumps} \end{array} \right. \quad (2.4.5)$$

$$\text{KE neglected for } \left\{ \begin{array}{l} \text{condensers} \\ \text{boilers} \\ \text{mixing chambers} \end{array} \right. \quad (2.4.6)$$

$$\text{KE not neglected for } \left\{ \begin{array}{l} \text{nozzles} \\ \text{diffusers} \end{array} \right. \quad (2.4.7)$$

Also, note that for P-V and T-S diagrams, the area enclosed, for both, is the net work produced during a cycle.

2.4.4 T-S diagram tips and tricks

- A heat - addition process proceeds in the direction of increasing entropy.
- A heat - rejection process proceeds in the direction of decreasing entropy.
- An isentropic (internally reversible, adiabatic) process proceeds at constant entropy.
- The area under the process curve on a T-S diagram represents the heat transfer for that process.

2.4.5 The Carnot cycle and its value in engineering

$$\eta_{th, \text{Carnot}} = 1 - \frac{Q_L}{Q_H} = 1 - \frac{T_L}{T_H} \quad (2.4.8)$$

Thermal efficiency increases with an increase in the average temperature at which heat is supplied to the system, or with a decrease in the average temperature at which heat is rejected from the system. The highest temperature in the cycle is limited by the maximum temperature that the components of the heat engine, such as the piston or the turbine blades, can withstand. The lowest temperature is limited by the temperature of the cooling medium utilised in the cycle such as a lake, a river or the atmospheric air.

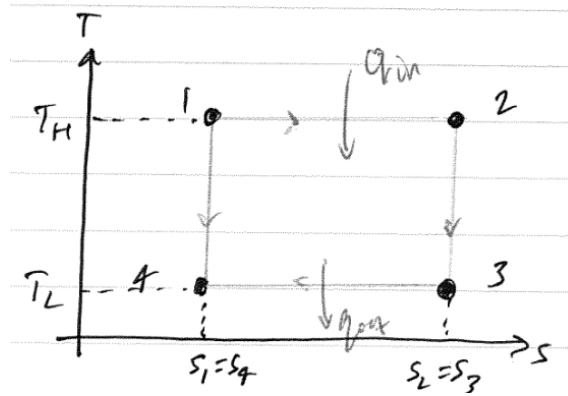


Figure 2.12: T-S diagram for Carnot cycle

Proof question

Show that the thermal efficiency of a Carnot cycle operating between the temperature limits T_H and T_L is solely a function of these two temperature and is given by equation 2.4.8.

Carnot cycle is:

1. Isothermal heat addition.
2. Isentropic expansion.
3. Isothermal heat rejection.
4. Isentropic compression.

$$q_{in} = \text{Area under graph as processes are totally reversible} \quad (2.4.9)$$

$$= T_H(S_2 - S_1) = T_H(S_3 - S_4) \quad (2.4.10)$$

$$q_{out} = \text{Area under process 3-4} \quad (2.4.11)$$

$$= T_L(S_3 - S_4) = T_L(S_2 - S_1) \quad (2.4.12)$$

$$\eta_{th} = 1 - \frac{q_{out}}{q_{in}} = 1 - \frac{T_L(S_2 - S_1)}{T_H(S_2 - S_1)} \quad (2.4.13)$$

$$\eta_{th, \text{const}} = 1 - \frac{T_L}{T_H} \quad (2.4.14)$$

2.4.6 Air-standard assumptions

Even though internal combustion engines operate on a mechanical cycle (the piston returns to its starting position at the end of each revolution), the working fluid does not undergo a complete thermodynamic cycle. It is thrown out of the engine at some

point in the cycle (as exhaust gases) instead of being returned to its initial state. Working on an open cycle is the characteristic of all internal combustion engines. To reduce the analysis to a manageable level, we utilise the following approximations, commonly known as the *air-standard assumptions*. Those cycles that run this are called air-standard cycles.

1. The working fluid is air, which continuously circulates in a closed loop and always behaves as an ideal cycle.
2. All the processes are internally reversible.
3. The combustion process is replaced by heat-addition process from an external source.
4. The exhaust process is replaced by a heat rejection process that restores the working fluid to its initial state.

Also we assume that air has constant specific heats whose value is determined at room temperature (25°C). When this assumption is applied, true air-standard assumptions are called the cold-air-standard assumptions.

2.4.7 An overview of reciprocating engines

The basic components of a reciprocating engine are shown in the diagrams.

- TDC: top dead center - position of the piston where it forms the smallest volume.
- BDC: bottom dead center - position of the piston where it forms the largest volume.
- Stroke: distance between TDC and BDC.
- Bore: diameter of the piston (internal diameter of the cylinder).
- Clearance volume: minimum volume formed.
- Displacement volume: the volume displaced by the piston as it moves between TDC and BDC.

The ratio of the maximum volume formed in the cylinder to the (minimum) clearance volume is called the compression ratio, r , of the engine.

$$\text{Compression ratio: } r = \frac{V_{max}}{V_{min}} = \frac{V_{BDC}}{V_{TDC}} \quad (2.4.15)$$

$$\text{Displacement volume} = \text{no. of cylinders} \times \text{stroke length} \times \text{bore area} \quad (2.4.16)$$

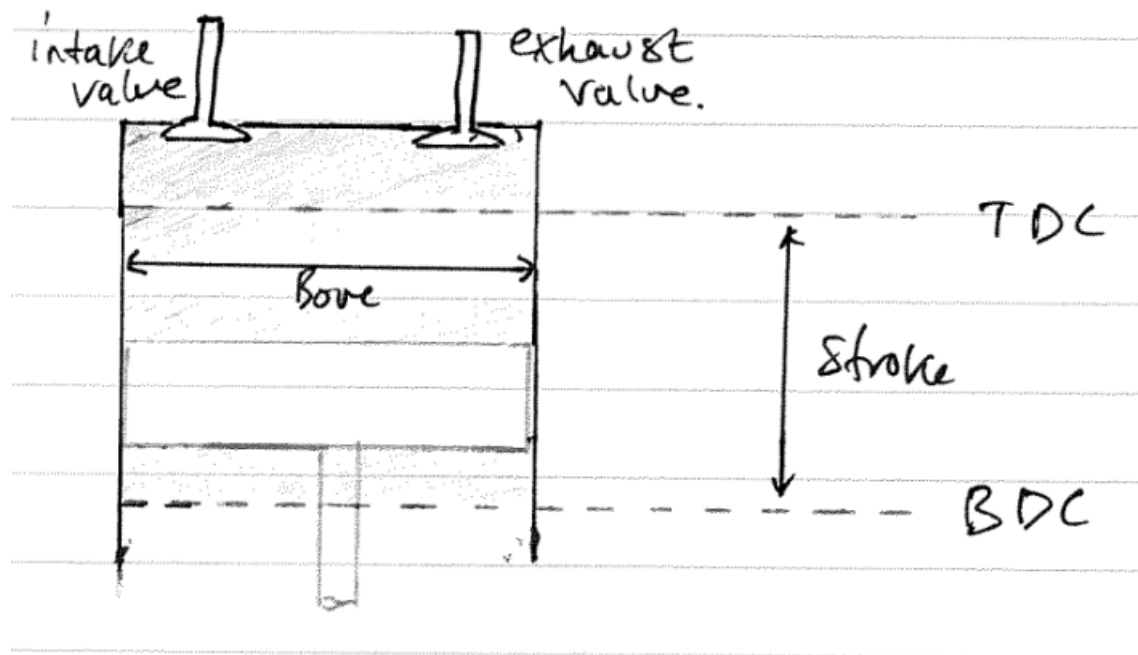


Figure 2.13: Basic components of a reciprocating engine.

The MEP (mean effective pressure) is a constant theoretical pressure, that if it acts on the piston to produce work, it would produce the same amount as that during an actual cycle.

$$\therefore W_{net} = \text{MEP} \times \text{piston area} \times \text{stroke} \quad (2.4.17)$$

$$\text{or } W_{net} = \text{MEP} \times \text{displacement volume} \quad (2.4.18)$$

$$(2.4.19)$$

or rearranging for MEP gives:

$$\text{MEP} = \frac{W_{net}}{(V_{max} - V_{min})} = \frac{w_{net}}{(v_{max} - v_{min})} \quad (2.4.20)$$

Engines with larger MEP values deliver more net work per cycle and thus perform better.

- Reciprocating engines
 - SI (spark ignition) engines - combustion is initiated using a spark plug. An example of this is the Otto cycle.
 - CI (compression ignition) engines - self-ignited as compression of gas (air-fuel) mixture raises the temperature. an example of this is the Diesel cycle.

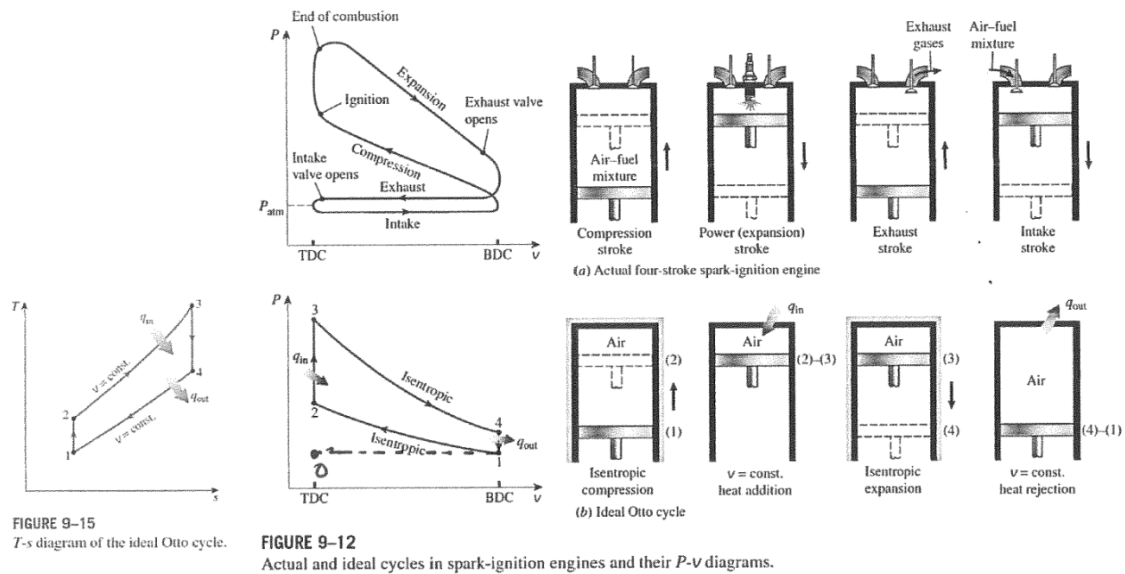


Figure 2.14: T-S and P-V diagrams for Otto cycle, alongside mechanical cycle breakdown for the ideal and actual cases.

2.4.8 Otto cycle: ideal cycle for SI engines

In most SI engines, the piston executes four complete strokes (two mechanical cycles) within the cylinder and the crankshaft completes two revolutions for each thermodynamic cycle. These engines are called four-stroke IC engines. The air-standard Otto cycle is described below in terms of processes.

- Process 1-2: isentropic compression.
- Process 2-3: constant volume heat addition.
- Process 3-4: isentropic expansion.
- Process 4-1: constant volume heat removal

In these diagrams, the Otto cycle replaces the combustion process of the actual engine operation with the constant - volume heat addition process. The constant-volume heat removal process replaces the exhaust blowdown. The original figure of the P-V diagram only shows two strokes of the engine. The intake and exhaust strokes are the missing ones. That is accounted for by drawing the graph with a process 0-1 added (the blue line).

2.4.9 Efficiency of the Otto cycles

Lets apply the first law to an Otto cycle during each process separately:

$$\Delta U = \Delta Q + \Delta W \text{ assuming } \Delta PE = \Delta KE \approx 0 \quad (2.4.21)$$

$$\Delta u = \Delta q + \delta w \quad (2.4.22)$$

$$\Delta u = (q_{in} - q_{out}) + (w_{in} - w_{out}) \quad (2.4.23)$$

For the heat transfer processes (process 2-3 and process 4-1), $\Delta w = 0$ as there is no boundary work as it is a constant volume process. Therefore for process 4-1 and 2-3:

$$\Delta u = q_{in} - q_{out} \quad (2.4.24)$$

For process 4-1: heat removal

$$\Delta u = -q_{out} \quad (2.4.25)$$

$$u_1 - u_4 = -q_{out} \quad (2.4.26)$$

$$q_{out} = u_4 - u_1 \quad (2.4.27)$$

$$q_{out} = c_V(T_4 - T_1) \quad (2.4.28)$$

For process 2-3: heat addition

$$\Delta u = q_{in} \quad (2.4.29)$$

$$q_{in} = u_3 - u_2 \quad (2.4.30)$$

$$q_{in} = c_V(T_3 - T_2) \quad (2.4.31)$$

$$\therefore \eta_{th, otto} = \frac{w_{net}}{q_{in}} = 1 - \frac{q_{out}}{q_{in}} = 1 - \frac{T_4 - T_1}{T_3 - T_2} = 1 - \frac{T_4 \left(\frac{T_4}{T_1} - 1 \right)}{T_3 \left(\frac{T_3}{T_4} - 1 \right)} \quad (2.4.32)$$

But processes 1-2 and 3-4 are isentropic (adiabatic, internally reversible) and $v_2 = v_3$ and $v_4 = v_1$. From the definition of adiabatic processes:

$$\frac{T_1}{T_2} = \left(\frac{v_2}{v_1}\right)^{\gamma-1} \quad (2.4.33)$$

$$\frac{T_4}{T_3} = \left(\frac{v_3}{v_4}\right)^{\gamma-1} \quad (2.4.34)$$

$$\therefore \frac{T_1}{T_2} = \frac{T_4}{T_3} \quad (2.4.35)$$

$$\therefore \eta - th, otto = 1 - \frac{T_4 \left(\frac{T_4}{T_1} - 1\right)}{T_3 \left(\frac{T_3}{T_4} - 1\right)} \quad (2.4.36)$$

$$\eta_{th, otto} = 1 - \frac{T_3}{T_4} \left[\frac{\frac{T_4}{T_3} \frac{T_3}{T_2} \frac{T_2}{T_1} - 1}{\frac{T_3}{T_2} - 1} \right] \quad (2.4.37)$$

The compression ratio r is

$$r = \frac{V_{max}}{V_{min}} = \frac{V_1}{V_2} = \frac{V_4}{V_3} \quad (2.4.38)$$

Since it is a fixed mass:

$$r = \frac{v_1}{v_2} = \frac{v_4}{v_3} \quad (2.4.39)$$

Remember process 2-3 and 4-1 are constant volume processes. Hence,

$$Pv = mRt \quad (2.4.40)$$

$$\frac{v}{mR} = \frac{T}{P} \quad (2.4.41)$$

$$\therefore \frac{T_2}{P_2} = \frac{T_3}{P_3} \rightarrow \frac{P_3}{P_2} = \frac{T_3}{T_2} \text{ and} \quad (2.4.42)$$

$$\therefore \frac{T_4}{P_4} = \frac{T_1}{P_1} \rightarrow \frac{P_1}{P_4} = \frac{T_1}{T_4} \quad (2.4.43)$$

Process 1-2 and 3-4 are adiabatic (isentropic):

$$Pv = mRT \quad (2.4.44)$$

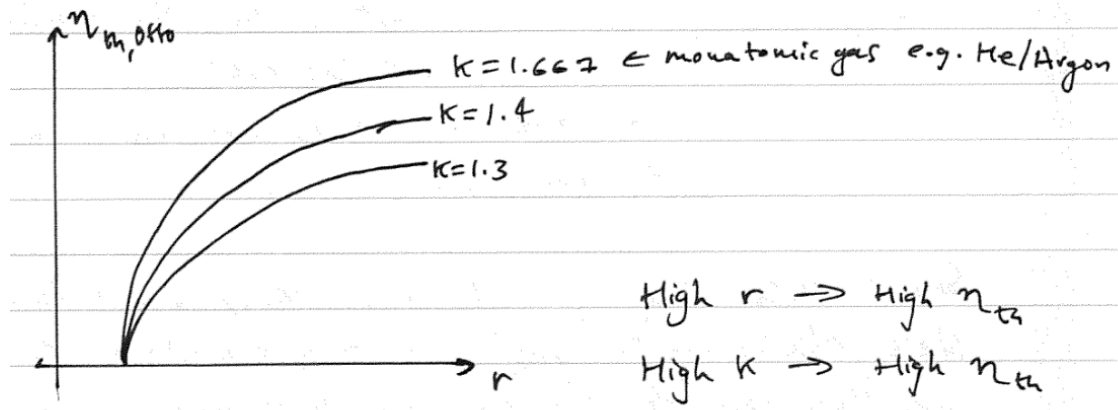
$$\frac{Pv}{T} = \text{constant} \quad (2.4.45)$$

$$\therefore \frac{P_1 v_1}{T_1} = \frac{P_2 v_2}{T_2} \quad (2.4.46)$$

$$\frac{P_2}{P_1} = \frac{T_2 v_1}{T_1 v_2} \quad (2.4.47)$$

$$\frac{T_2}{T_1} = \frac{P_2}{P_1} \cdot \frac{v_2}{v_1} \quad (2.4.48)$$

$$\frac{T_2}{T_1} = \frac{P_2}{P_1} \cdot r^{-1} \quad (2.4.49)$$

Figure 2.15: Graph to show r against $\eta_{th, otto}$.

Similarly,

$$Pv = mRT \quad (2.4.50)$$

$$\frac{Pv}{T} = \text{constant} \quad (2.4.51)$$

$$\therefore \frac{P_4 v_4}{T_4} = \frac{P_3 v_3}{T_3} \quad (2.4.52)$$

$$\frac{P_4}{P_3} = \frac{T_4 v_3}{T_3 v_4} \quad (2.4.53)$$

$$\frac{T_3}{T_4} = \frac{P_3}{P_4} \cdot \frac{v_3}{v_4} \quad (2.4.54)$$

$$\frac{T_3}{T_4} = \frac{P_3}{P_4} \cdot r \quad (2.4.55)$$

Hence,

$$\eta_{th, otto} = 1 - \frac{T_4}{T_3} \left[\frac{\frac{T_3}{T_2} - 1}{\frac{T_3}{T_1} - 1} \right] \quad (2.4.56)$$

$$\eta_{th, otto} = 1 - \frac{T_4}{T_3} \quad (2.4.57)$$

$$\eta_{th, otto} = 1 - \left(\frac{1}{r} \right)^{\gamma-1} \quad (2.4.58)$$

$$\eta_{th, otto} = 1 - \frac{1}{r^{\gamma-1}} \quad (2.4.59)$$

$$\text{where } r = \frac{v_1}{v_2} = \frac{v_4}{v_3} \quad (2.4.60)$$

For higher efficiency, higher compression ratios are required. However, increase in pressure ratios would increase the air-fuel temperature above the temperature at which the mixture can auto-ignite. This would result in '*engine knock*' (auto ignition

that occurs prematurely, produces an audible noise) reducing the performance of the engine. In order to avoid such situations, additives are generally added which increase the auto-ignition temperature.

2.4.10 Diesel cycle: ideal cycle for CI engines

The diesel cycle is the ideal cycle for CI reciprocating engines. It differs to SI engines mainly in the method of initiating combustion.

SI engines have an air-fuel mixture which is compressed below auto-ignition temperature and ignited with a spark. CI engines compress air to a temperature greater than the auto-ignition temperature of the fuel. Fuel is injected into hot compressed air and combustion is initiated.

Since in diesel engines, only air is compressed during the compression stroke, eliminating the possibility of auto-ignition. Therefore, diesel engines are designed to operate on much higher compression ratios. Assumptions for Diesel cycle.

- Fixed amount of air (ideal gas) for working fluid.
- Combustion process replace by constant pressure heat addition.
- Intake and exhaust not considered, cycle completed with constant volume heat removal with piston at BDC.
- All processes considered internally reversible.

The air-standard Diesel cycle is described below in terms of processes.

- Process 1-2: isentropic compression.
- Process 2-3: isobaric heat addition.
- Process 3-4: isentropic expansion.
- Process 4-1: isochoric heat removal.

Lets apply the first law to a diesel cycle during each process separately.

$$\Delta u = (q_{in} - q_{out}) + (w_{in} - w_{out}) \quad (2.4.61)$$

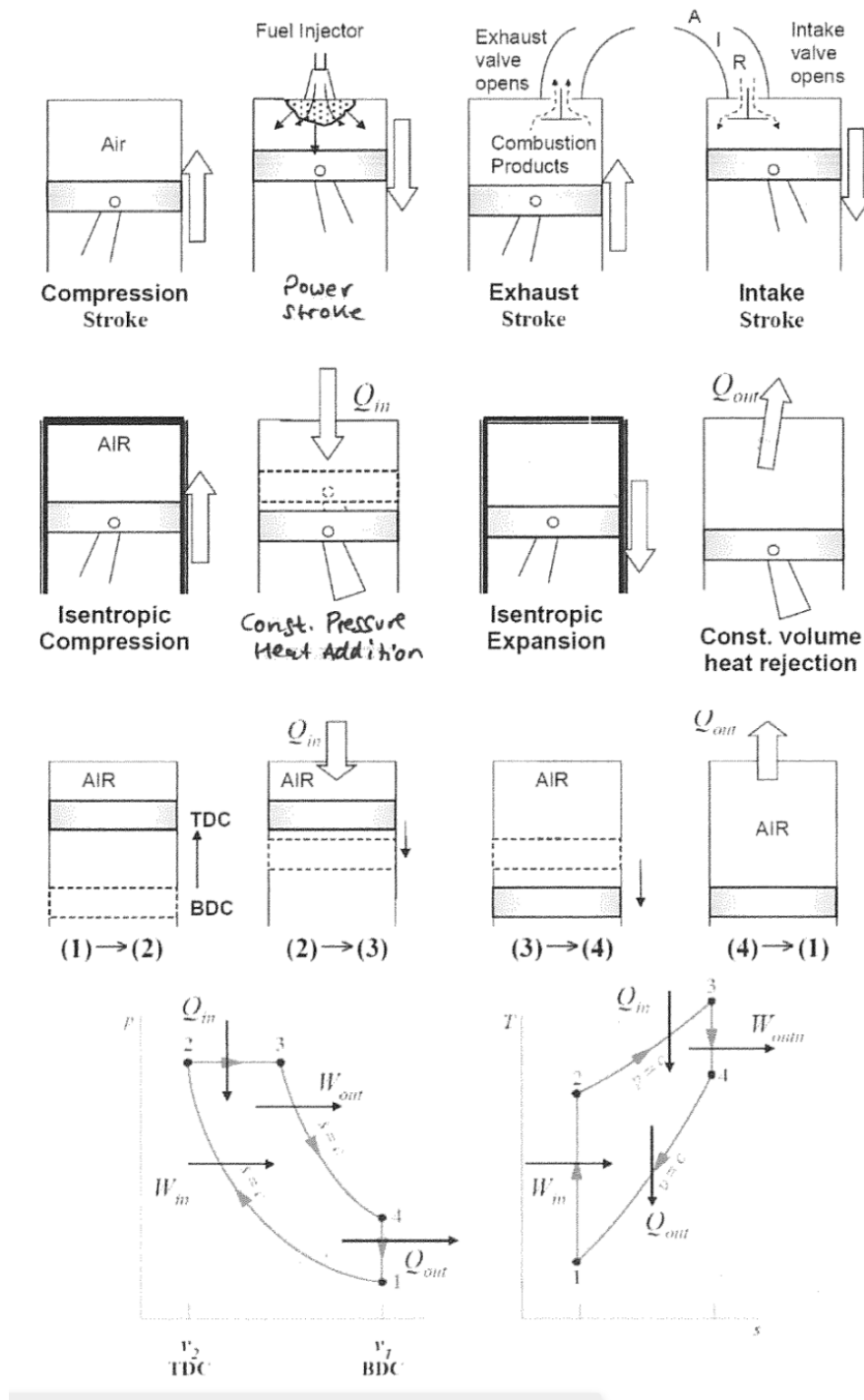


Figure 2.16: Diagram showing mechanical positions of piston at each process step along with P-v and T-s diagrams

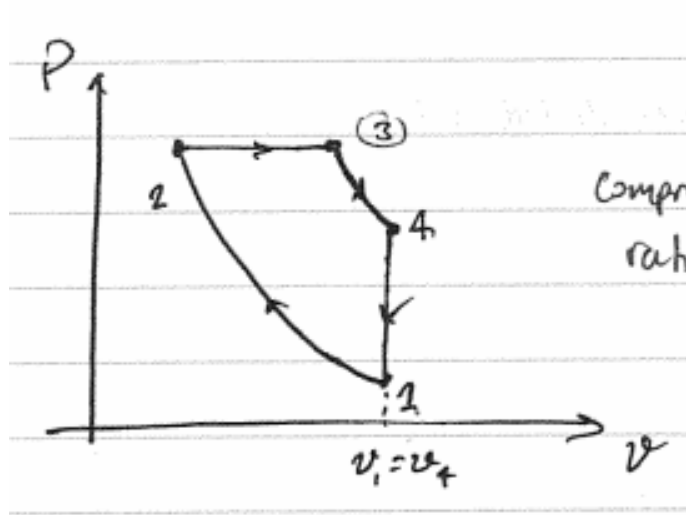


Figure 2.17: PV diagram for Diesel cycle.

For process 2-3: constant pressure heat addition.

$$u_3 - u_2 = q_{in} - w_{b, out} \quad (2.4.62)$$

$$q_{in} = u_3 - u_2 + \int_2^3 P dv \quad (2.4.63)$$

$$q_{in} = u_3 - u_2 + P_2(v_3 - v_2) \quad (2.4.64)$$

$$q_{in} = h_3 - h_2 \quad (2.4.65)$$

$$q_{in} = c_P(T_3 - T_2) \quad (2.4.66)$$

For process 4-1: isochoric heat rejection.

$$u_1 - u_4 = -q_{out} \quad (2.4.67)$$

$$q_{out} = u_4 - u_1 \quad (2.4.68)$$

$$q_{out} = c_V(T_4 - T_1) \quad (2.4.69)$$

Given,

$$\text{Compression ratio} \rightarrow r_v = \frac{v_1}{v_2} \quad (2.4.70)$$

$$\text{Expansion ratio} \rightarrow r_e = \frac{v_4}{v_3} \quad (2.4.71)$$

$$\text{Cut off ratio} \rightarrow r_c = \frac{v_3}{v_2} \quad (2.4.72)$$

$$r_e \cdot r_c = \frac{v_4}{v_3} \cdot \frac{v_3}{v_2} = \frac{v_4}{v_2} = \frac{v_1}{v_2} = r_v \quad (2.4.73)$$

$$\therefore r_v = r_c \cdot r_e \quad (2.4.74)$$

Process 1-2: isentropic compression.

$$\text{adiabatic, hence } \frac{T_1}{T_2} = \left(\frac{V_2}{V_1} \right)^{\gamma-1} = \frac{1}{r_V^{\gamma-1}} \quad (2.4.75)$$

$$T_1 = \frac{T_2}{r_V^{\gamma-1}} \quad (2.4.76)$$

Process 2-3: isobaric heat addition

$$Pv = RT \quad (2.4.77)$$

$$\frac{P}{R} = \frac{T}{v} = \text{constant} \quad (2.4.78)$$

$$\frac{T_2}{v_2} = \frac{T_3}{v_3} \quad (2.4.79)$$

$$\therefore T_2 = T_3 \frac{v_2}{v_3} \quad (2.4.80)$$

$$T_2 = \frac{T_3}{v_c} \quad (2.4.81)$$

Process 3-4: isentropic heat expansion.

$$\frac{T_4}{T_3} = \left(\frac{v_3}{v_4} \right)^{\gamma-1} = \frac{1}{r_e^{\gamma-1}} \quad (2.4.82)$$

$$T_4 = \frac{T_3}{r_e^{\gamma-1}} \quad (2.4.83)$$

Thermal efficiency can be calculated as,

$$\eta_{th, diesel} = 1 - \frac{1}{\gamma} \frac{1}{r_v^{\gamma-1}} \left[\frac{r_c^{\gamma-1}}{r_c - 1} \right] \quad (2.4.84)$$

2.4.11 Otto and Diesel comparison

$$\eta_{th, diesel} = 1 - \frac{1}{\gamma} \frac{1}{r_v^{\gamma-1}} \left[\frac{r_c^{\gamma-1}}{r_c - 1} \right] \quad (2.4.85)$$

$$\eta_{th, otto} = 1 - \frac{1}{r^{\gamma-1}} \quad (2.4.86)$$

$$\text{for } r_v = \frac{v_1}{v_2} > 1 \quad (2.4.87)$$

$$\frac{1}{\gamma} \left[\frac{r_c^{\gamma-1}}{r_c - 1} \right] > 1 \quad (2.4.88)$$

Therefore, the efficiency of the diesel cycle is less than that of the Otto cycle for the same compression ratio. However, the advantages of Diesel over petrol engines

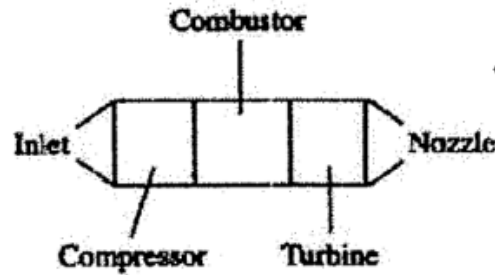


Figure 2.18: Sketch of the jet engine components.

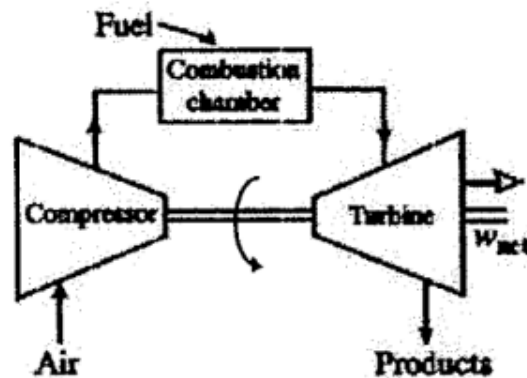


Figure 2.19: Open cycle operation for Joule cycle.

is that they can operate at higher compression ratios without auto-ignition.

$$\eta_{th, otto} > \eta_{th, diesel} \quad (2.4.89)$$

Also if r_c decreases, then $\eta_{th, diesel}$ increases. When the limiting case of $r_c = 1$ occurs, the quantity in the brackets becomes 1 and hence $\eta_{th, diesel} = \eta_{th, otto}$.

2.4.12 Brayton cycle (Joule cycle)

The Brayton cycle (Joule cycle) represents the operation of a gas turbine engine. Gas turbines usually operate on an open cycle, as shown in Figure 2.19. Fresh air is drawn into the compressor, where its temperature and pressure are. The high-pressure air proceeds into the combustion chamber, where the fuel is burned at constant pressure. The resulting high-temperature gases then enter the turbine where they expand to the atmospheric pressure while producing power. The exhaust gases leaving the turbine are thrown out (not recirculated), causing the cycle to be classified as an open cycle. The open gas-turbine cycle described above can be modelled as a closed cycle, as shown in Figure 2.20, by utilising the air-standard assumptions. Here

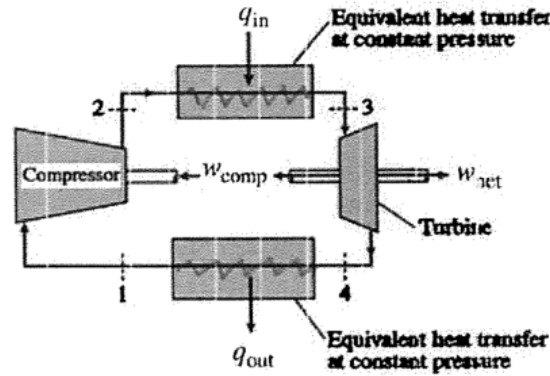


Figure 2.20: Closed cycle operation for Joule cycle.

the compression and expansion processes remain the same but the combustion process is replaced by an isobaric heat addition process from an external source and the exhaust process is replaced by an isobaric heat rejection process to the ambient air. The ideal cycle that the working fluid undergoes in this closed loop is the Joule cycle. It consists of four processes:

- Process 1-2: isentropic compression in the inlet and compressor.
- Process 2-3: isobaric heat addition.
- Process 3-4: isentropic expansion, in the turbine and exhaust nozzle, with which we
 - take some work out of the air and use it to drive the compressor and
 - take the remaining work out and use it to accelerate fluid for jet propulsion, or to turn a generator for electrical power generation.
- Process 4-1: isobaric heat rejection.

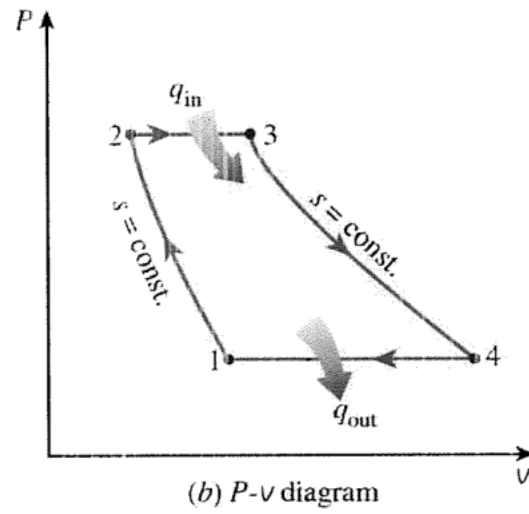


Figure 2.21: PV diagram for Joule cycle.

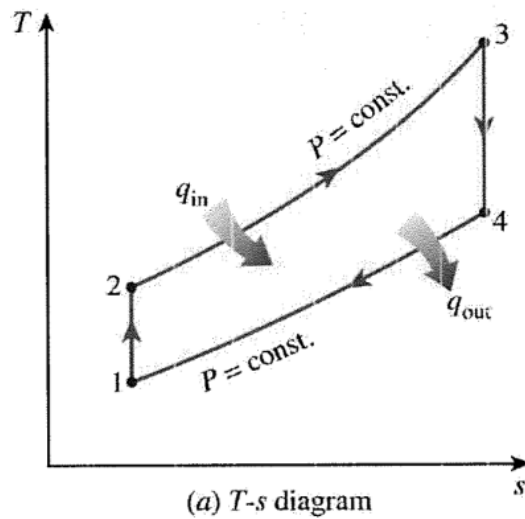


Figure 2.22: TS diagram for Joule cycle.

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