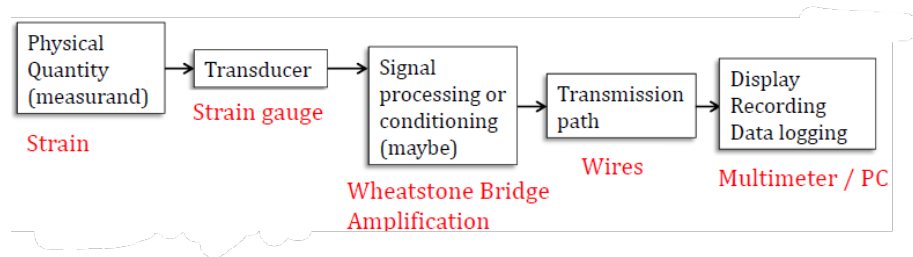
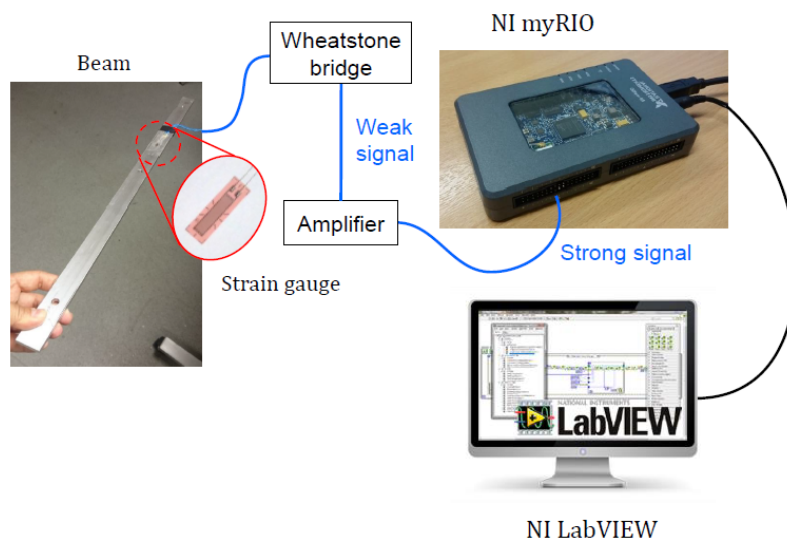


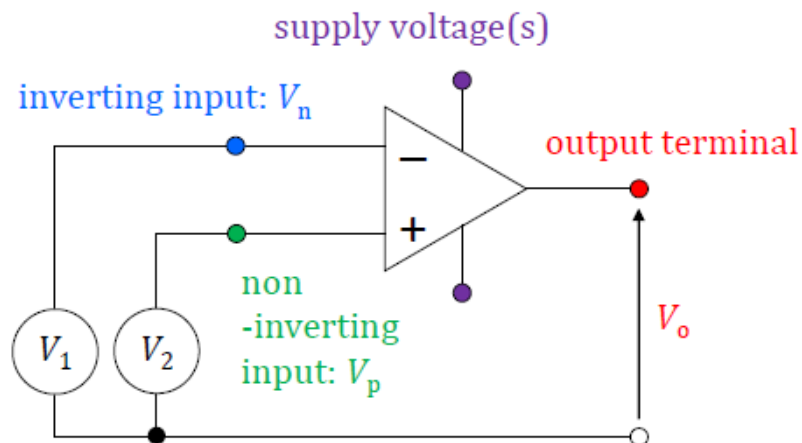
Stages in a measurement system



0.1 Amplifiers

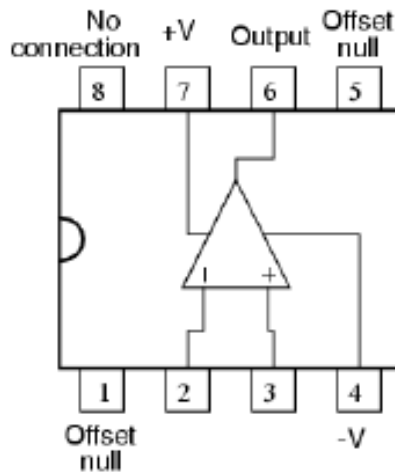


The voltage output from a transducer tends to be weak and this needs to be amplified. Operational amplifiers are normally around 1cm in size.

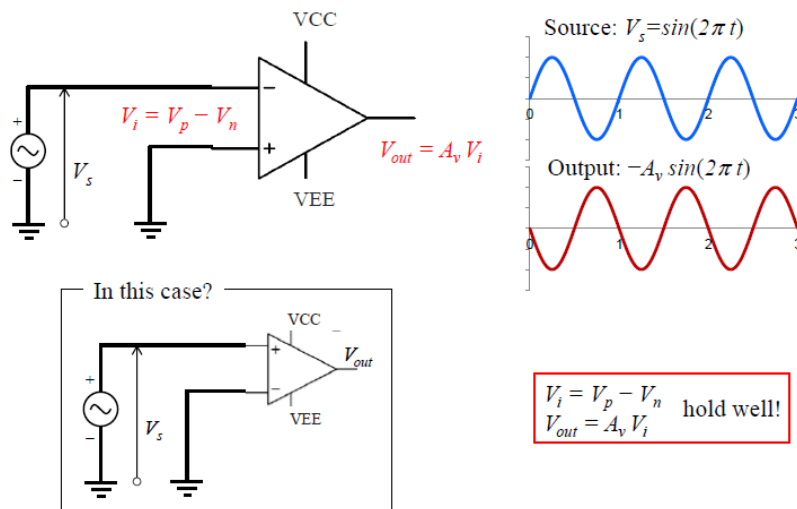


$$V_o = A_v(V_p - V_n) = A_v(V_2 - V_1) \quad (1)$$

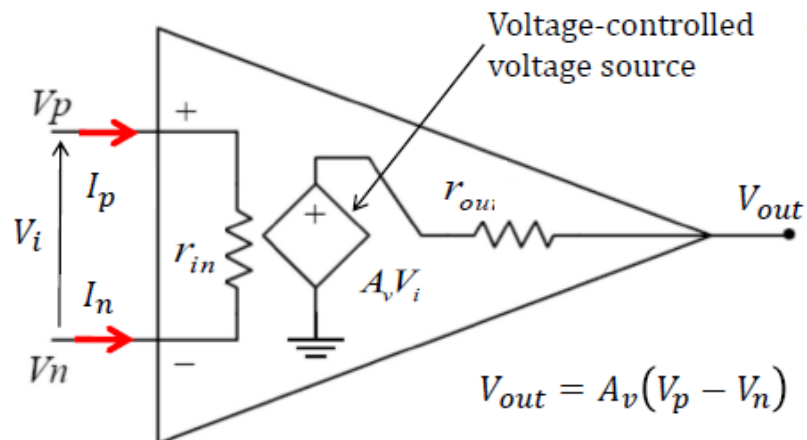
Where A_v is the gain and $(V_p - V_n)$ can be from a Wheatstone bridge. Operational amplifiers are quite an old technology (since WW2) and are still used because they are cheap and convenient.



Pin-out for a 8-pin op-amp



0.1.1 Equivalent op-amp circuit and conditions of an ideal op-amp



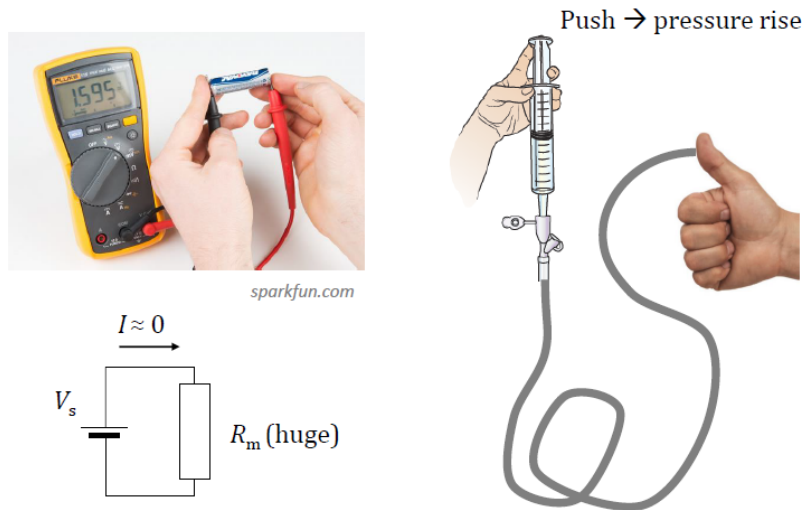
Terminology:

- Differential input voltage: $V_i = V_p - V_n$

- Input resistance: r_{in}
- Output resistance: r_{out}
- Open-circuit output voltage: V_{out}
- Differential voltage gain: A_v

Conditions of an ideal op-amp	In reality
No current into input terminals: $I_p = I_n = 0$	
Infinite input resistance: $r_{in} \rightarrow \infty$	$r_{in} > 200\text{k}\Omega$
Zero Output resistance $r_{out} = 0$	$r_{out} < 1\text{k}\Omega$
Infinite differential (or open-loop) gain $A_v \rightarrow \infty$	$A_v > 100,000$
Zero common-mode voltage gain $A_{cm} = 0$	A_v also frequency dependant

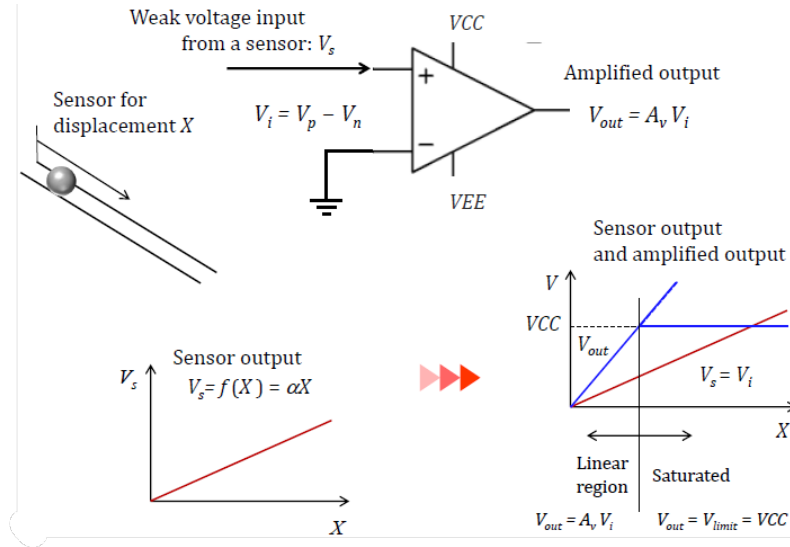
0.1.2 Non-zero voltage with zero current?



A question that is often asked is if the current is zero, how can we still have a voltage? When you connect a battery to a meter, where the meter resistance is normally huge. We can still measure the voltage. This is due to Ohm's law $I = \frac{V}{R_m}$. A good analogy to make is a plunger system, where the other end is covered. When you push down, increasing the pressure, this can still be felt on the tip of the thumb, despite there being no flow. In the same way, we can still 'feel' the voltage with 'no' current flow.

0.1.3 Saturation

In a practical scenario, we want to strengthen the weak signal from the sensor. Let us connect an input to the non-inverting terminal and the inverting input to the ground.

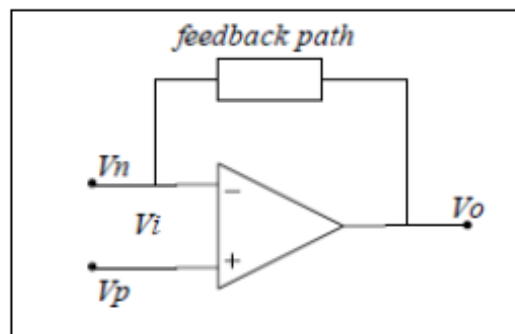


We can see that the signal is not infinitely amplified, as we are limited by the energy provided by our supply voltage. This is reflected in the graph above as the straight portion of the amplified line. This is called saturation. Due to A_v being quite large, our linear region is normally tiny. This may be a problem if we want to measure the linear region.

0.1.4 Negative feedback

So op-amps seem to be a useful device but also seem to be a bit tricky. They are much more useful when used as part of a larger circuit. Negative feedback is achieved by feeding a fraction of the output signal back to the **inverting** terminal as shown. By definition, the closed-loop gain of such a device is:

$$G = \frac{V_o}{V_i} \quad (2)$$



Negative feedback trades a reduction in gain with one that is smaller, stable and predictable, whilst giving the designer control over operational characteristics.

0.1.5 Basic feedback amplifier configurations

Only 2 resistors are required to construct a negative feedback amplifier. Note that output signal is always fed-back to the inverting input terminal.

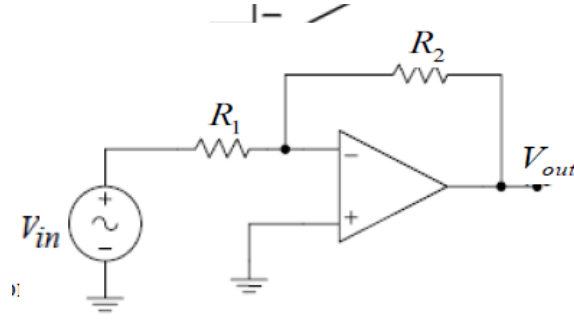


Figure 1: Inverting feedback amplifier arrangement.

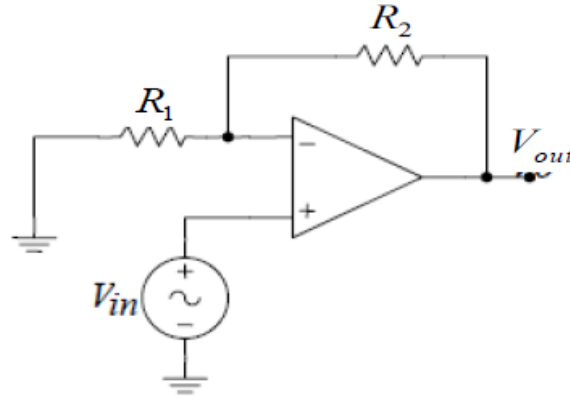
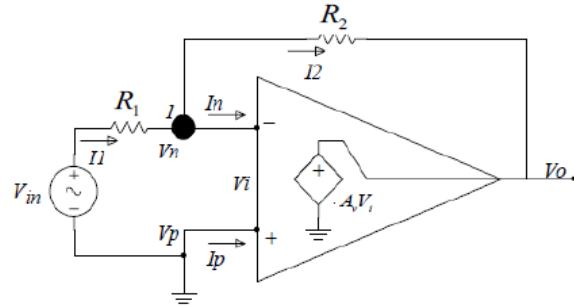


Figure 2: Non-inverting feedback amplifier arrangement.

We can see the difference between an inverting and non-inverting feedback amplifier is which input the source signal is connected to.

0.1.6 Basic inverting feedback amplifier



Let us derive its closed loop gain G . By definition:

$$V_0 = A_v(V_p - V_n) \quad (3)$$

$$V_p \rightarrow 0 \implies V_n = -\frac{V_0}{A_v} \quad (4)$$

Apply KCL to Node 1:

$$\frac{V_{in} - V_n}{R_1} - I_n - \frac{V_n - V_0}{R_2} = 0 \implies \frac{R_2}{R_1}V_{in} + v_n \left(1 + \frac{R_2}{R_1}\right) = V_0 \quad (5)$$

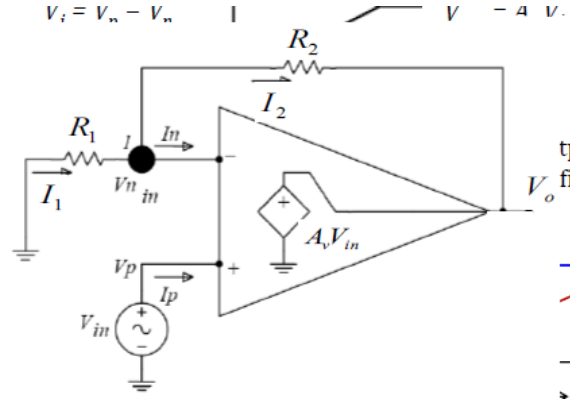
I_n is 0 from our model assumptions. Combining equations (4) and (5), we arrive at:

$$G \equiv \frac{V_0}{V_{in}} = -\frac{R_2}{R_1} \frac{1}{1 + \frac{1}{A_v} \left(1 + \frac{R_2}{R_1}\right)} \quad (6)$$

For an ideal op-amp, $A_v \rightarrow \infty$:

$$G = \frac{V_0}{V_{in}} = -\frac{R_2}{R_1} \quad (7)$$

0.1.7 Basic non-inverting feedback amplifier



Derive its closed-loop gain G . By definition:

$$V_0 = A_v(V_p - V_n) \quad (8)$$

Apply KCL to node 1

$$\frac{0 - V_n}{R_1} = \frac{V_n - V_0}{R_2} \rightarrow \frac{V_0}{R_2} = V_n \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad (9)$$

Combining equation (8) and (9), we arrive at:

$$G \equiv \frac{V_0}{V_{in}} = \frac{1 + \frac{R_2}{R_1}}{1 + \frac{1}{A_v} \left(1 + \frac{R_2}{R_1}\right)} \quad (10)$$

For an ideal amplifier, $A_v \rightarrow \infty$:

$$G = 1 + \frac{R_2}{R_1} \quad (11)$$

This is positive and always greater than 1.

0.1.8 Summary

Ideal feedback amplifier characteristics

	Inverting amplifier	Non-inverting amplifier
$G = \frac{V_0}{V_{in}}$	$-\frac{R_2}{R_1}$	$1 + \frac{R_2}{R_1}$
R_{in}	R_1	∞
R_{out}	0	0

0.1.9 Voltage amplification summary

Operational amplifier:

- The gain usually is too large (typically $\geq 100,000$) and not well defined. (V_0 cannot be larger than supply voltage)
- The gains for two op-amps of the same type can differ by a factor of two or more.
- The gain is not stable, can change with temperature, can change with changes of supply voltage.

To reduce these effects, we use negative feedback.

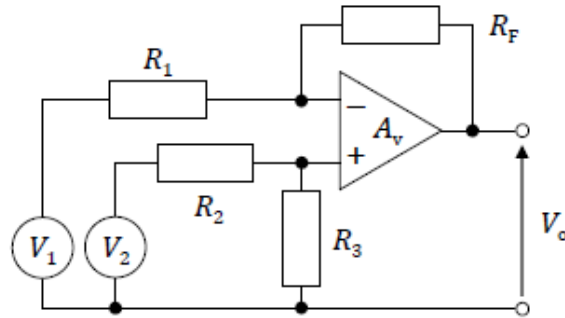


Figure 3: V_1 and V_2 could come from a Wheatstone bridge.

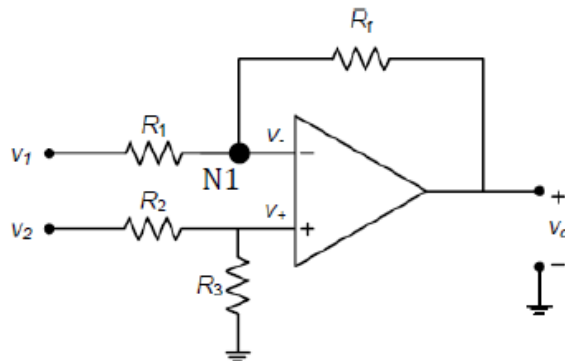
The output voltage V_0 is given as:

$$V_0 = \frac{R_F}{R_1}(V_2 - V_1) \quad (12)$$

Only when $\frac{R_3}{R_2} = \frac{R_F}{R_1}$. This is independent of A_v and the gain of this circuit, $G = \frac{R_F}{R_1}$ is well controllable.

0.1.10 Difference amplifier

Combining a non-inverting and inverting op-amp to take the difference between two inputs:



$$\frac{V_0}{A_v} = V_+ - V_- \rightarrow 0 \text{ as } A_v \rightarrow \infty \quad (13)$$

$$V_+ = \frac{R_3}{R_2 + R_3} \leftarrow \text{voltage divider} \quad (14)$$

KCL at node N1

$$\frac{V_0 - V_-}{R_f} = \frac{V_- V_1}{R_1} \rightarrow \frac{V_0}{R_f} + \frac{V_1}{R_1} = \left(\frac{1}{R_f} + \frac{1}{R_1} \right) V_- \quad (15)$$

$$\frac{V_0}{R_f} + \frac{V_1}{R_1} = \left(\frac{R_f + R_1}{R_f R_1} \right) \left(\frac{R_3}{R_2 + R_3} \right) V_2 \quad (16)$$

Combining equation (14) and (16), we arrive at:

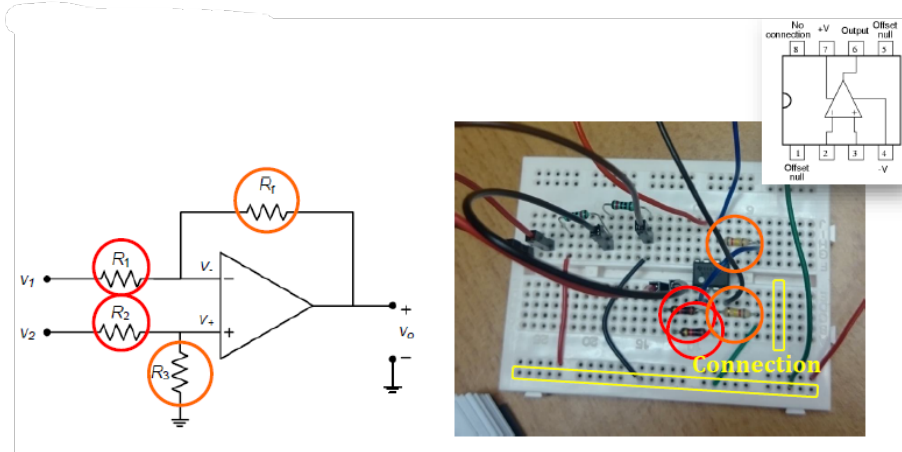
$$V_0 = R_f \left(\frac{R_f + R_1}{R_f R_1} \right) \left(\frac{R_3}{R_2 + R_3} \right) V_2 - \frac{R_f}{R_1} V_1 \quad (17)$$

$$V_0 = \frac{1 + \frac{R_f}{R_1}}{1 + \frac{R_2}{R_3}} V_2 - \frac{R_f}{R_1} V_1 \quad (18)$$

If we can make sure that $\frac{R_f}{R_1} = \frac{R_3}{R_2}$ is satisfied, we can further reduce our equation to:

$$V_0 = \frac{R_f}{R_i} (V_2 - V_1) \quad (19)$$

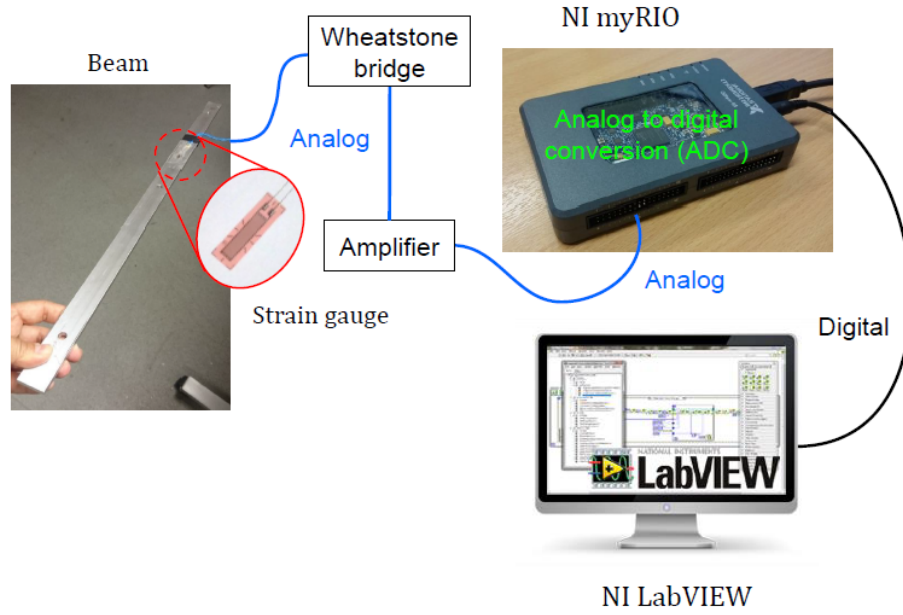
0.1.11 On a bread board



$$V_0 = \frac{R_f}{R_i} (R_2 - R_1) \quad (20)$$

if $\frac{R_f}{R_1} = \frac{R_3}{R_2}$ can be satisfied.

0.2 Analogue-Digital (AD) conversion



0.2.1 Comparator

We are using an op-amp here, because when we have a high A_v , we have a very small linear region. Hence, when we look at a signal graph, we can see this as a sort of switching device:



With very large A_v , output V_o would be: $V_o = A_v(V_p - V_n)$. Hence:

$$V_o \begin{cases} = V_{CC} & \dots \text{ if } V_p > V_n \\ = V_{EE} & \dots \text{ if } V_p < V_n \end{cases} \quad (21)$$

This essentially makes comparisons based on V_p and V_n

$$V_p > V_n ? \rightarrow \text{YES/NO} \quad (22)$$

0.2.2 Example ADC mechanism using comparators

A circuit to compare voltages:

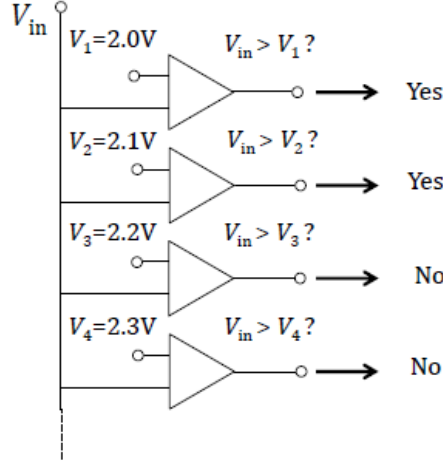


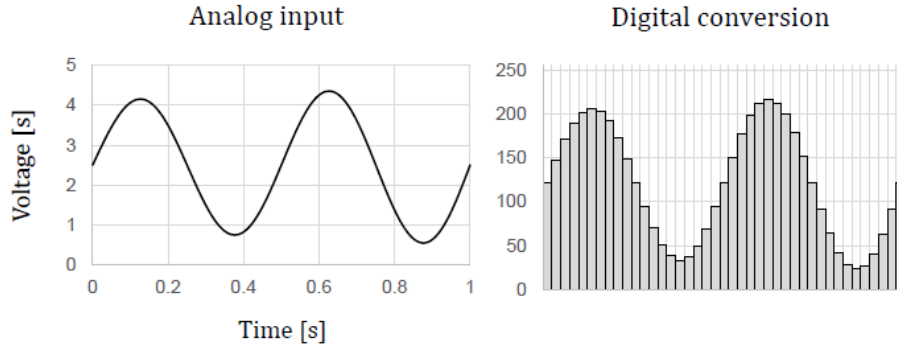
Figure 4: **Flash ADC:** number of comparisons done in parallel.

Setting up these comparators in parallel and setting a reference voltage for each, we can compare a single input and produce a digital binary signal, which tells us the range of voltages that V_{in} may occupy. In this case we can see that:

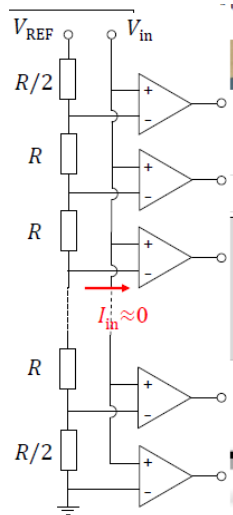
$$2.1V < V_{in} < 2.2V \quad (23)$$

Hence, V_{in} may be 2.1V, but other options are possible. When we store in a digital binary format this is called **quantisation**. When a voltage range 0-5V ($V_{REF} = 5V$) is quantised in 8-bit,

$$V_{in} = 2.14V \rightarrow V_{in, \text{quantised}} = \text{int} \left(\frac{V_{in}}{5[V]/2^8} \right) = 110 \quad (24)$$



A more practical form of a flash ADC circuit can be achieved using voltage dividers, for the k-th comparator:



$$V_{o,k} = A_v \left(V_{in} - \frac{(k + \frac{1}{2}) R}{nR} V_{REF} \right), k = 1, 2, \dots, n - 1 \quad (25)$$

n comparators allow judgement of YES/NO per every V_{REF}/n volt \rightarrow **resolution** of conversion.

- 8-bit quantisation requires $2^8 = 256$ comparators.
- 16-bit quantisation requires $2^{16} = 25536$ comparators.

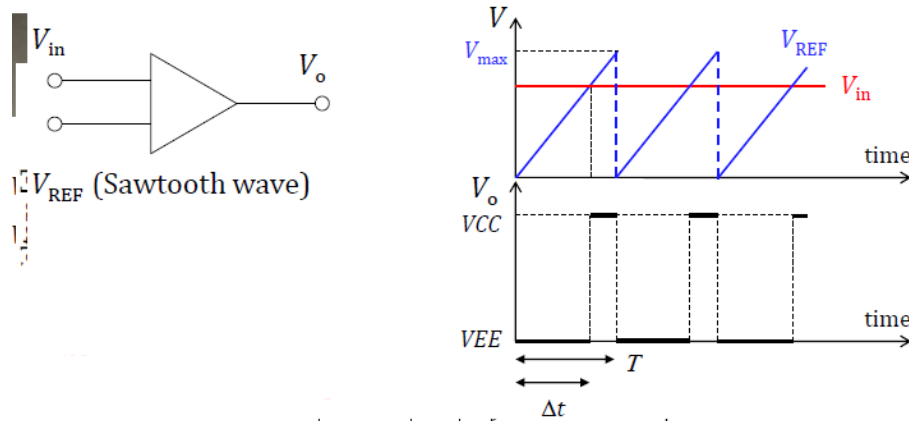
Flash ADC can work very quickly but becomes complex for high resolutions.

0.2.3 Other types of ADC

There are other types of conversion:

- Successive conversion - successively narrow down range of V_{in}
- Integrating - use input signal integration time to find V_{in}
- Ramp-compare

Ramp-comparison utilises a saw tooth wave and the time period of switching to work out what V_{in} is.



$$V_{in} = \frac{\Delta t}{T} V_{max} \quad (26)$$

Ramp-compare can be done with a simply system (only one comparator) and can be accurate but may required (relatively - you can make the frequency of the sawtooth very large) long processing time.

0.2.4 Typical issues to be considered

Resolution

Number of bits allocated to represent the input range (= interval of discrete data values).

Accuracy

How the digitised values are different from the original values. E.g. affected by accuracy of Δt measurement in ramp-compare. Though quantisation causes rounding error, it is not necessarily associated with accuracy.

Sampling rate

How many data points over time (how many ADC per time period). Affected by ADC time. Particular impact on time-dependent data/frequency (e.g. music) leading to aliasing/anti-aliasing.

Noise

Some methods (e.g. integrating) are good for noisy data.

0.3 Interface devices to PC and programming platform

0.3.1 Arduino

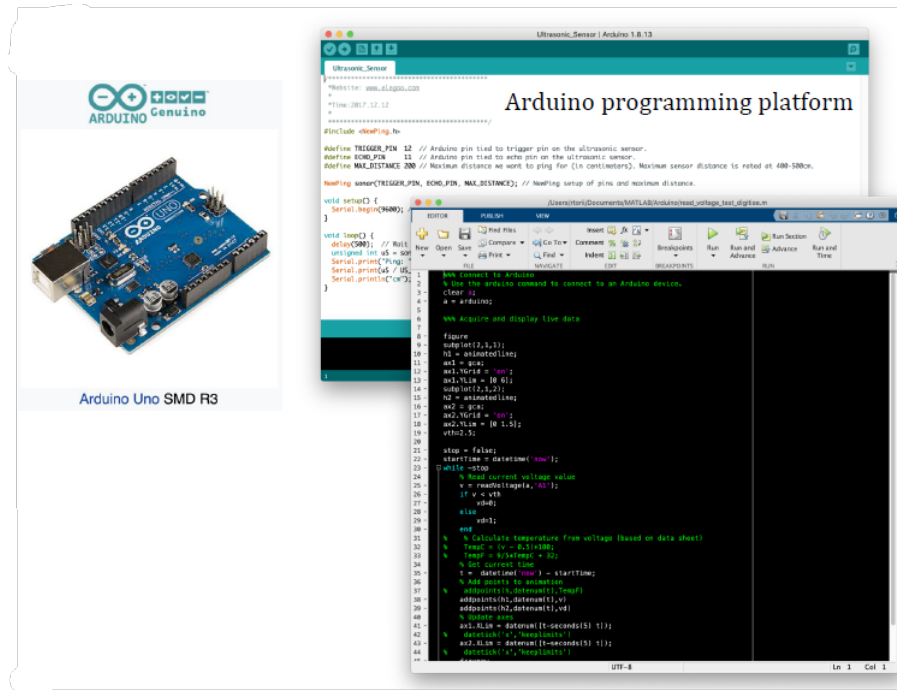
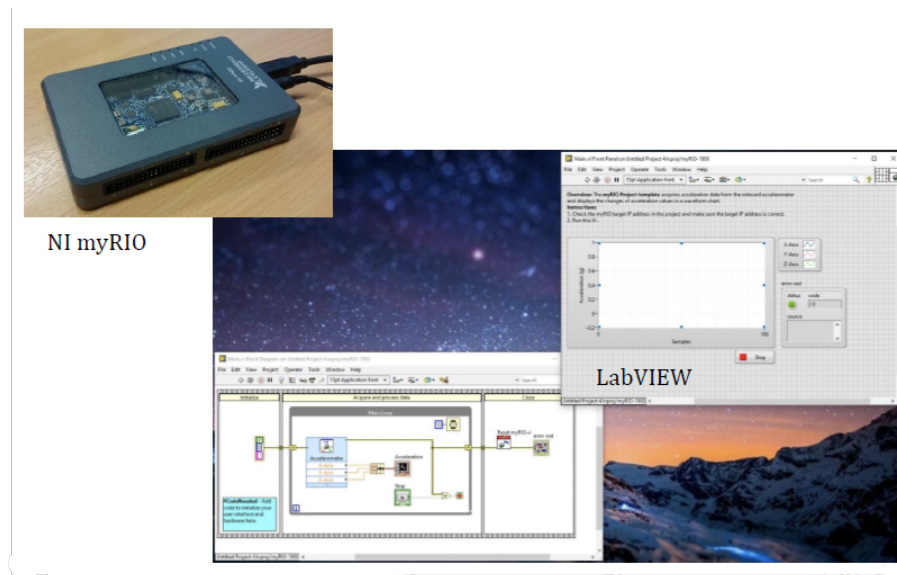


Figure 5: Can be linked up with MATLAB or any other language.

0.3.2 NI products and LabVIEW



0.4 Experimental Errors

Because of the inherent limitations of measurement equipment of techniques, there will always be some 'uncertainty' associated with experimental results. This uncer-

tainty of error conveys significant information about the experiment and the result obtained, so you will be expected to assess the error in the measurements that you perform.

Example

We want to know the volume of a cylinder based on diameter and height measurement:

$$V = \pi r^2 h = \frac{\pi h d^2}{4} \quad (27)$$

We must ask how accurate are d and h , and also how they can be measured in the calculation of V ? Suppose you are measuring a variable x :

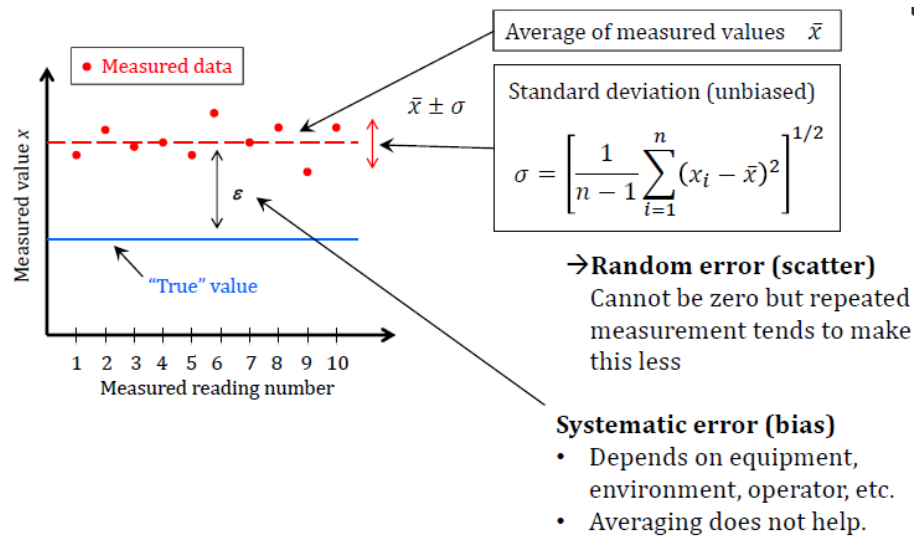


Figure 6: Actual error is mostly combination of both types of errors.

0.4.1 Systematic errors

Depends on:

- Equipment
- Environment
 - For example temperature may affect the measurement results, e.g. strain gauges. We deal with this by introducing temperature compensation.
- Operator
 - Consider measuring a sample with length L and mass M . These are well defined variables, however the thickness h will require sampling.

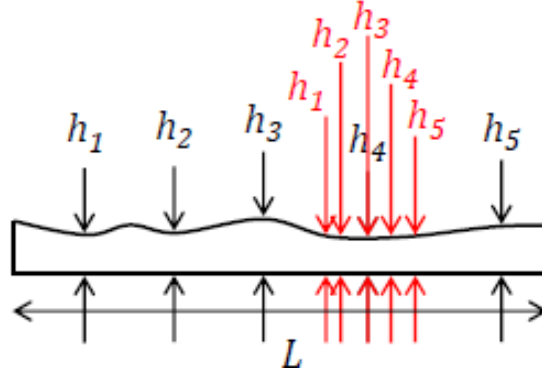


Figure 7: Sampling needs to be done carefully to ensure good representation.

Looping back to our cylinder example earlier, let us say:

$$d = 8.2 \pm 0.5\text{cm} \quad (28)$$

$$h = 9.5 \pm 0.2\text{cm} \quad (29)$$

0.4.2 Combining errors in a single quantity

Uncertainty analysis / error propagation - Estimating the overall error in the measurement system. For example, when two resistors are connected in series, what is the expected uncertainty of the total resistance?

$$R_1 = 1000 \pm 1.5\Omega \quad (30)$$

$$R_2 = 500 \pm 1.0\Omega \quad (31)$$

The straightforward option is to simply add them directly $\rightarrow 2.5\Omega$

$$R_{total} = R_1 + R_2 = 1500 \pm 2.5\Omega \quad (32)$$

The error in R_1 and R_2 are a statistical indication, whereby the error is assumed to follow a Gaussian (normal) distribution. However, simple addition of the errors is too pessimistic! The two errors are independent and unlikely to show their maximum at the same time. If the errors are **independent** and in the same **quantity**, they can be combined as:

$$E = \pm\sqrt{e_1^2 + e_2^2} \quad (33)$$

and for this resistor example, overall error is 1.8Ω . The rule above can be applied to any number of independent errors:

$$E = \pm\sqrt{e_1^2 + e_2^2 + e_e^2 + \dots} \quad (34)$$

0.4.3 Various combinations of quantities

Assuming independent measurements A and B , which have total errors ΔA and ΔB associated with them, are combined to give the result X , which has error ΔX ,

then:

$$\text{if } X = A + B \text{ or } X = A - B \longrightarrow \Delta X = \sqrt{(\Delta A)^2 + (\Delta B)^2} \quad (35)$$

$$\text{if } X = AB \text{ or } X = \frac{A}{B} \longrightarrow \frac{\Delta X}{X} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2} \quad (36)$$

$$\text{if } X = A^n \longrightarrow \frac{\Delta X}{X} = n \frac{\Delta A}{A} \quad (37)$$

$$(38)$$

Note for equation (36), we always use plus in our square root because the error only increases with addition or subtraction. When more than two quantities are involved, the equations are extended in a straightforward way:

$$\text{if } X = A + B - C + D... \longrightarrow \Delta X = \sqrt{(\Delta A)^2 + (\Delta B)^2 + (\Delta C)^2 + (\Delta D)^2} \quad (39)$$

$$\text{if } X = \frac{AB}{CD} \longrightarrow \frac{\Delta X}{X} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2 + \left(\frac{\Delta C}{C}\right)^2 + \left(\frac{\Delta D}{D}\right)^2} \quad (40)$$

Adding errors in summation (A+B) and subtraction (A-B) are simpler because they must be the same physical quantity (mass, length, etc.) with the same unit. Products and divisions are usually in different quantity and the errors need to be added in normalised fashion. Let us use these in our cylinder example now:

$$V = \pi r^2 h = \frac{\pi h d^2}{4} \quad (41)$$

$$d = 8.2 \pm 0.5 \text{cm} \quad (42)$$

$$h = 9.5 \pm 0.2 \text{cm} \quad (43)$$

$$V = 387 \text{cm}^3 \quad (44)$$

Using two of the error estimation rules in combination,

$$\frac{\Delta V}{V} = \sqrt{\left(\frac{\Delta h}{h}\right)^2 + \left(\frac{2\Delta d}{d}\right)^2} \longrightarrow \Delta V = 54 \text{cm}^3 \quad (45)$$

Therefore the volume is $V = 387 \pm 54 \text{cm}^3$

0.4.4 General rule for combining errors

Consider the problem of computing a quantity N , where N is a known function of the n independent variables $u_1, u_2, u_3, \dots, u_n$. That is:

$$N = f(u_1, u_2, u_3, \dots, u_n) \quad (46)$$

The u s are the measured quantity with errors $\pm \Delta u_1, \pm \Delta u_2, \pm \Delta u_3, \dots, \pm \Delta u_n$ which result in error in N . We could now write:

$$N \pm \Delta N = f(u_1 \pm \Delta u_1, u_2 \pm \Delta u_2, u_3 \pm \Delta u_3, \dots, u_n \pm \Delta u_n) \quad (47)$$

Expanding f in Taylor series we get:

$$f(u_1 \pm \Delta u_1, u_2 \pm \Delta u_2, u_3 \pm \Delta u_3, \dots, u_n \pm \Delta u_n) = f(u_1, u_2, u_3, \dots, u_n) + \Delta u_1 \frac{\partial f}{\partial u_1} + \Delta u_2 \frac{\partial f}{\partial u_2} + \dots + \Delta u_n \frac{\partial f}{\partial u_n} + \text{higher order terms} \quad (48)$$

Where $f(u_1, u_2, u_3, \dots, u_n)$ is our N . We can neglect our higher order terms because $\Delta u_n \ll 1$ and what we are left with is our error.

$$\text{Error} = \Delta u_1 \frac{\partial f}{\partial u_1} + \Delta u_2 \frac{\partial f}{\partial u_2} + \dots + \Delta u_n \frac{\partial f}{\partial u_n} \quad (49)$$

Considering Δu s are not absolute limits of error but rather as statistical bounds of uncertainties the proper method of combining the error is according to the root-sum square formula:

$$E = \sqrt{\left(\Delta u_1 \frac{\partial f}{\partial u_1}\right)^2 + \left(\Delta u_2 \frac{\partial f}{\partial u_2}\right)^2 + \dots + \left(\Delta u_n \frac{\partial f}{\partial u_n}\right)^2} \quad (50)$$

0.4.5 Example

The resistance of a certain size of copper wire is given as

$$R = R_0[1 + \alpha(T - 20)] \quad (51)$$

Where $R_0 = 6\Omega \pm 0.3\%$ is the resistance at 20°C , $\alpha = 0.004^\circ\text{C}^{-1} \pm 1\%$ is the temperature coefficient of resistance, and the temperature of the wire is $T = 30 \pm 1^\circ\text{C}$. Calculate the resistance of the wire and its uncertainty.

The nominal resistance is:

$$R = 6[1 + 0.004 \times (30 - 20)] = 6.24\Omega \quad (52)$$

The uncertainty is from the equation earlier,

$$\Delta R = \sqrt{\left(\Delta R_0 \frac{\partial R}{\partial R_0}\right)^2 + \left(\Delta \alpha \frac{\partial R}{\partial \alpha}\right)^2 + \left(\Delta T \frac{\partial R}{\partial T}\right)^2} \quad (53)$$

Here,

$$\frac{\partial R}{\partial R_0} = 1 + \alpha(T - 20) \quad (54)$$

$$= 1 + 0.004 \times (30 - 20) = 1.04 \quad (55)$$

$$\frac{\partial R}{\partial \alpha} = R_0(T - 20) \quad (56)$$

$$= 6 \times (30 - 20) = 60 \quad (57)$$

$$\frac{\partial R}{\partial T} = R_0 \alpha \quad (58)$$

$$= 6 \times 0.004 = 0.024 \quad (59)$$

And,

$$\Delta R_0 = 6 \times 0.003 = 0.018\Omega \quad (60)$$

$$\Delta\alpha = 0.004 \times 0.01 = 4 \times 10^{-5}\text{C}^{-1} \quad (61)$$

$$\Delta T = 1^\circ\text{C} \quad (62)$$

Thus, the uncertainty in the resistance is:

$$0.0305\Omega \text{ or } 0.49\% \quad (63)$$

0.4.6 What if we do not know the relationship between error sources and measured variables?

Example: load cell (force measurement)

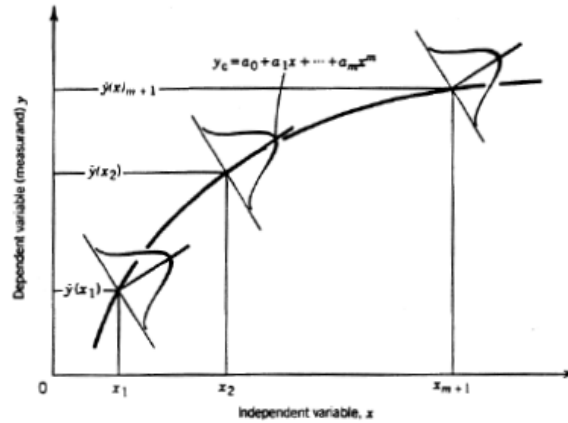
$$V_0 = f(\text{load, etc}) \quad (64)$$

Simply add relevant errors up based on the specification sheet as a safety measure.

0.5 Reference information about least-square curve fitting

0.5.1 Least square analysis

The least squares method is often used to establish the functional relationship between the dependent measured variable and the independent process variable. In most situations, the anticipated relationship is polynomial. The variation found in the measured variable is assumed to follow a normal (Gaussian) distribution about each fixed value of the independent variable.

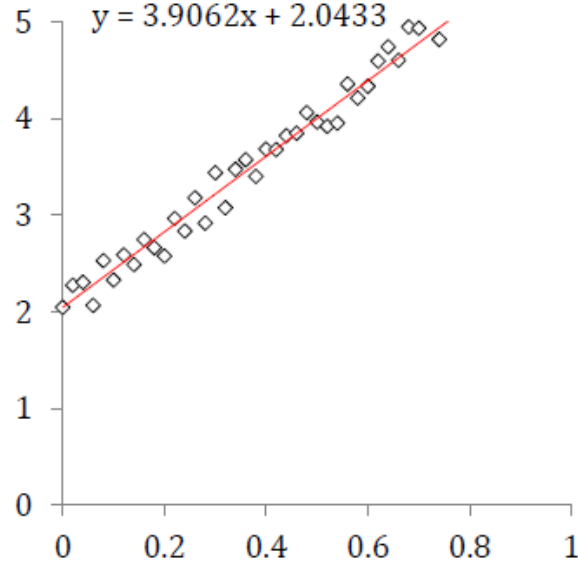


Suppose two variables x and y are measured over a range of values. We want to find their mathematical relationship. The simplest polynomial function is linear: $y = ax + b$. Least square analysis is done by minimising the quantity:

$$S = \sum_{i=1}^n [y_i - (ax_i + b)]^2 \quad (65)$$

This is the same as finding:

$$\frac{\partial S}{\partial a} = 0, \quad \frac{\partial S}{\partial b} = 0 \quad (66)$$



$$\frac{\partial S}{\partial a} = \sum_{i=1}^n (2ax_i^2 + 2bx_i - 2x_i y_i) = 0 \rightarrow a \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i \quad (67)$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^n (2ax_i + 2b - 2y_i) \rightarrow a \sum_{i=1}^n x_i + nb = \sum_{i=1}^n y_i \quad (68)$$

$$a = \frac{n \sum_{i=1}^n x_i y_i - (\sum_{i=1}^n x_i) (\sum_{i=1}^n y_i)}{n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \quad (69)$$

$$b = \frac{(\sum_{i=1}^n x_i^2) (\sum_{i=1}^n y_i) - (\sum_{i=1}^n x_i y_i) (\sum_{i=1}^n x_i)}{n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \quad (70)$$

Even with higher order polynomials, for example

$$y = ax^2 + bx + c \quad (71)$$