0.1 Compressible duct flows

0.1.1 Problem types

We are concerned with flow in ducts or 'long' pipes whose cross-section area is simple (rectangular / circular / conical). There are three types of problems where:

1. changes in cross-sectional area are important ($\mathrm{d}A \neq 0$).
2. frictional forces are important (**momentum is not conserved**) - *Fanno flow* (mass, energy, state).
3. heating and cooling are important (**energy is not conserved**) - *Rayleigh flow* (mass, momentum, state).

We study these effects separately and then consider them combined.

0.1.2 Influence of changes in cross-section area

To understand the physics, we simplify the analysis to gas moving without frictional forces and the addition of heat. There are two approaches to analyse the problem:

1. Differential approach - how flow properties vary along pipe (Rayleigh flow, Laval nozzle) - sometimes called a 1D model.
2. Integral approach - relationship between two flow states (Fanno flow, isothermal examples) - sometimes called a 0D model.

We have met these two approaches before:

1. Navier-Stokes equation.

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \mu \underline{\nabla}^2 \underline{u}$$

2. Momentum-integral approach.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V (\rho u) \, \mathrm{d}V = \int_S (-p\underline{I} + \underline{\tau}) \cdot \underline{\hat{n}} \, \mathrm{d}S$$

   Where the integral is taken over a volume V bounded by a surface S.

0.1.3 Recap of reference states

The stagnation values are valid for isentropic flows:

$$\frac{p}{p_0} = \left(1 + \frac{1}{2}(\gamma - 1) M^2\right)^{-\frac{\gamma}{\gamma-1}} \tag{1}$$

$$\frac{\rho}{\rho_0} = \left(1 + \frac{1}{2}(\gamma - 1) M^2\right)^{-\frac{\gamma}{\gamma-1}} \tag{2}$$

and for adiabatic flows:

$$\frac{T}{T_0} = \left(1 + \frac{1}{2}(\gamma - 1) M^2\right)^{-1} \tag{3}$$

It is important to be aware of what is conserved for adiabatic or isentropic conditions. The reference state of $M = 1$ is useful for flow in pipes where the sonic condition is common. The sonic reference condition is usually referred to as the critical condition and denoted with a '*'. Therefore:

$$\frac{p_*}{p_0} = \left(\frac{1}{2}(\gamma+1)(M=1)^2\right)^{-\frac{\gamma}{\gamma-1}} \approx 0.528 \tag{4}$$

$$\frac{\rho_*}{\rho_0} = \left(\frac{1}{2}(\gamma+1)(M=1)^2\right)^{-\frac{1}{\gamma-1}} \approx 0.91 \tag{5}$$

$$\frac{T_*}{T_0} = \left(\frac{1}{2}(\gamma+1)(M=1)^2\right)^{-1} \approx 0.833 \tag{6}$$

In duct flows, the critical state is used. p_0, ρ_0 are constant when there is no shock. T_0 is constant even with a shock.

0.1.4 Conservation of mass

The purpose is to relate A and M. The conservation of mass requires (integral or 0D approach and assuming no shocks at the moment):

$$\dot{m} = \rho u A = \text{const} = \rho^* u^* A^* \tag{7}$$

or

$$\frac{\rho}{\rho^*}\frac{u}{u^*}\frac{A}{A^*} \tag{8}$$

Substituting for our isentropic and adiabatic parts:

$$\frac{\rho}{\rho^*} = \left(\frac{1+\frac{1}{2}(\gamma-1)M^2}{\frac{1}{2}(\gamma+1)}\right)^{-\frac{1}{\gamma-1}} \tag{9}$$

$$\frac{u}{u^*} = \frac{M}{M^*}\frac{c}{c^*} = \frac{M}{M^*}\left(\frac{T}{T^*}\right)^{\frac{1}{2}} = M\left(\frac{\frac{1}{2}(\gamma+1)}{1+\frac{1}{2}(\gamma-1)M^2}\right)^{\frac{1}{2}} \tag{10}$$

The right-hand side is a constant. It is a state that may, or may not, be realised

0.1.5 Relationship between flow state and area

We are to explore some of the important relationships between the state of the flow and the cross-sectional area.

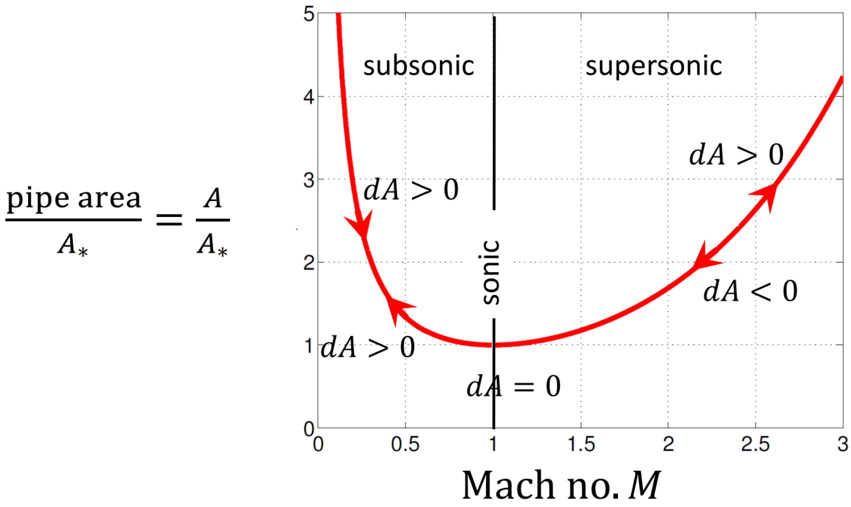

Figure 1: Graph to show pipe area vs Mach number.

$\mathrm{d}A > 0$	Subsonic	$\mathrm{d}M < 0$
$\mathrm{d}A < 0$	Subsonic	$\mathrm{d}M > 0$
$\mathrm{d}A > 0$	Supersonic	$\mathrm{d}M > 0$
$\mathrm{d}A < 0$	Supersonic	$\mathrm{d}M < 0$

Table 1: Table to show pipe area conditions and Mach number

Profound elements - 2 solutions are possible!

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1+\frac{1}{2}(\gamma-1)M^2}{\frac{1}{2}(\gamma+1)}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{11}$$

$$M = 1 \text{ at } \mathrm{d}A = 0 \text{ (point of inflection)} \tag{12}$$

0.1.6 Creating supersonic flows

Figure 2: Supersonic flow generator.

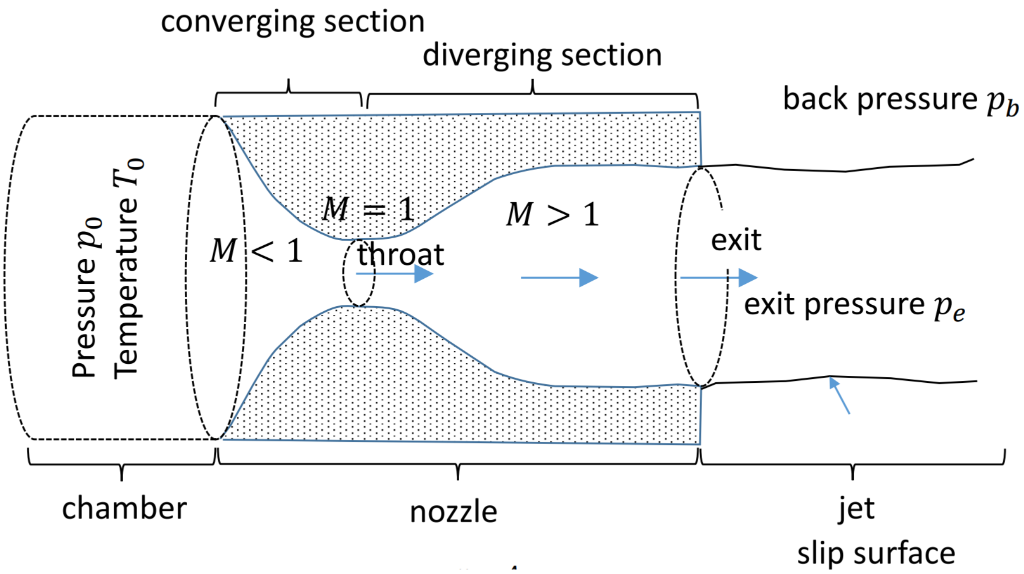

Figure 3: Geometry of supersonic flow generator.

This requires a converging, throat and diverging sections. The flow is determined by $\frac{p_b}{p_0}$, $\frac{A_e}{A_t}$.

0.1.7 Differential approach to derivation

Conservation of mass shows:

$$\rho u A = \text{const} \tag{13}$$

This can be converted to a differential form:

$$\log(\rho u A) = \log \rho + \log u + \log A = \text{const} \tag{14}$$

Taking the differential gives:

$$\frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}A}{A} = 0 \tag{15}$$

Conservation of momentum gives:

$$\rho u A \,\mathrm{d}U = -A \,\mathrm{d}p \tag{16}$$

For an isentropic flow:

$$\frac{\mathrm{d}p}{p} = -\gamma \frac{\mathrm{d}\rho}{\rho} \tag{17}$$

From the conservation of energy:

$$\frac{T}{T_0} = \left(1 + \frac{1}{2}(\gamma - 1) M^2\right)^{-1} \tag{18}$$

$$\frac{\mathrm{d}T}{T} = \frac{\frac{1}{2}(\gamma - 1)\,\mathrm{d}M^2}{1 + \frac{1}{2}(\gamma - 1) M^2} \tag{19}$$

THis gives:

$$\frac{\mathrm{d}M^2}{M^2} = -\frac{2\left(1+\frac{1}{2}(\gamma-1)M^2\right)}{1-M^2}\frac{\mathrm{d}A}{A} \tag{20}$$

Integrate:

$$\log\frac{A_2}{A_1} = \int_{A_1}^{A_2}\left(\frac{1}{A}\right)\mathrm{d}A = \int_{M_1^2}^{M_2^2}\left(\frac{1}{M^2}\right)\mathrm{d}M^2 \tag{21}$$

This can be manipulated to the formula on the crib sheet.

0.1.8 Mass flux relationship

The mass flux (per unit area) is $\frac{m}{A} = \rho u$. Its derivative w.r.t. M is:

$$\frac{\mathrm{d}}{\mathrm{d}M}\left(\frac{\rho u}{\rho_* u_*}\right) = \left(1-M^2\right)\left(1+\frac{1}{2}(\gamma-1)M^2\right)^{-\frac{\gamma-3}{2(\gamma-1)}}\left(\frac{1}{2}(\gamma+1)\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{22}$$

The maximum occurs when $M = 1$, when the flow is choked. The maximum mass flux is:

$$\dot{m}_{max} = \rho^* u^* A^* = \left(\frac{1}{\frac{1}{2}(\gamma+1)}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\left(\frac{\gamma}{RT_0}\right)^{\frac{1}{2}} A^* p_0 \tag{23}$$

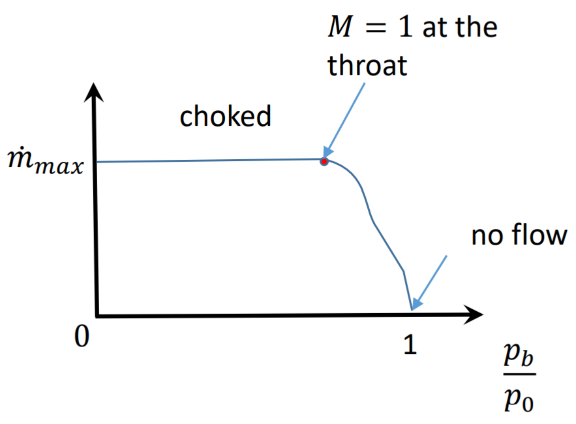

Figure 4

where p_b is the back pressure and p_0 is the reservoir pressure.

0.1.9 Types of solution available

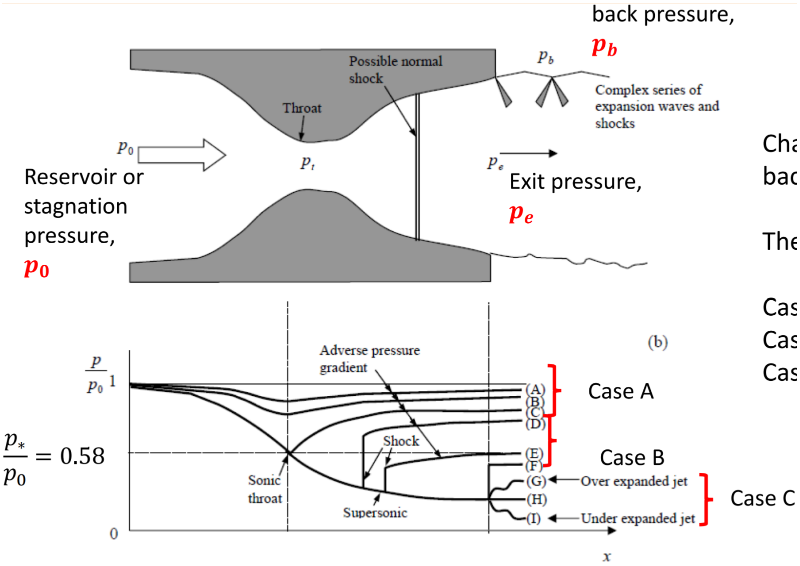

Figure 5: Overview of problem.

Characteristics depend on the value of the back pressure. There are three types of solution:

- Case A: Subsonic in whole flow ($p_e = p_b$)
- Case B: Choked but subsonic outlet ($p_e = p_b$)
- Case C: Supersonic outlet ($p_e \lessgtr p_b$)

Case A: Subsonic flow

For subsonic flow condition, $M \leq 1$ everywhere and the flow is isentropic because no shocks form. The pressure at the outlet of the duct, p_e, is the same as the back pressure, i.e:

$$p_e = p_b \tag{24}$$

The exit Mach number, M_e, is determined from the isentropic relationship:

$$M_e^2 = \frac{2}{\gamma - 1}\left(\left(\frac{p_0}{p_b}\right)^{\frac{\gamma-1}{\gamma}} - 1\right) \tag{25}$$

The critical area A^* can be determined from the crib sheet as:

$$A_* = A_e M_e \left(\frac{1 + \frac{1}{2}(\gamma - 1)M_e^2}{\frac{1}{2}(\gamma + 1)}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \tag{26}$$

The mass flux can be determined at any point along the duct. Choosing the outlet plane, then:

$$\dot{m} = \rho_e u_e A_e = \rho_0 \sqrt{\gamma R T_0} M_e A_e \left(1 + \frac{1}{2}(\gamma - 1)M_e^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \tag{27}$$

This has a maximum value when $\frac{p_b}{p_0} = \frac{p_*}{p_0}$. For a moderate drop in p_b from states A and B, the throat is still subsonic. For curve C, the area ratio is:

$$\frac{A_e}{A_t} = \frac{A_e}{A_*} \tag{28}$$

and the flow is sonic at the throat.

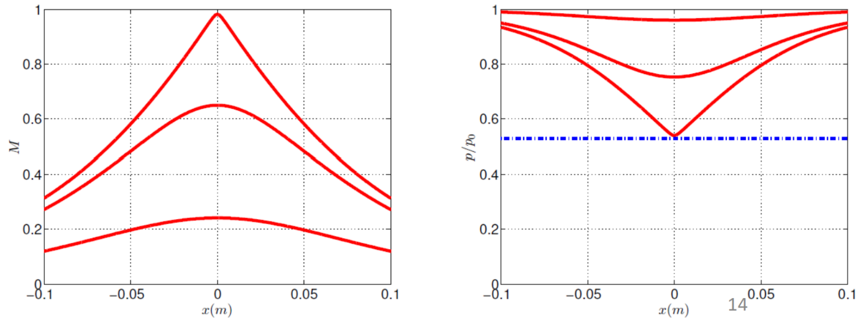

Figure 6: Mach number in the supersonic generator, case A.

Case B: Choked flow with subsonic outlet

When p_b decreases, the only way for the flow to adapt is for a shock to be created between the nozzle and the outlet. Isentropic model still applies but supplement by normal shock relationship across the shock. If $p_b < p_*$, the nozzle cannot respond because it is choked at its maximum throat mass flow. The flow switches to the supersonic condition after the throat since the flow is mass constrained. The flow passes smoothly through this transition. The only way of the pressure recovering to match the back pressure is to have a normal shock at some location in the nozzle. The exit's jets is then subsonic and is able to match the back pressure.

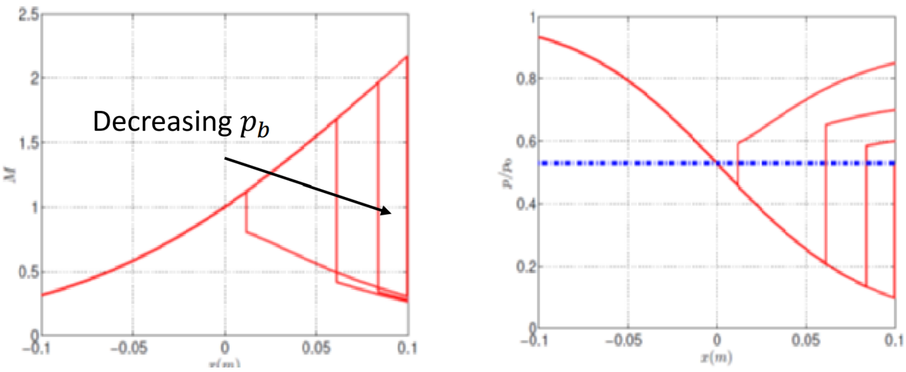

Figure 7: Mach number in supersonic generator, case B.

7 with subsonic outlet with choked flow and $\frac{p_b}{p_0} = 0.53$, 0.60, 0.70 and 0.85. Note that $\frac{p_*}{p_0} = 0.528$. After the shock A_* is different before and after the shock ($A_* = A_t$ before the shock.)
The flow characteristics are separated into three parts:

1. Choked flow to normal shock.

In this region, since the flow is choked, $A_* = A_t$. The change in Mach number and pressure are:

$$\frac{A}{A_t} = \frac{1}{M}\left(\frac{1 + \frac{1}{2}(\gamma - 1) M^2}{\frac{1}{2}(\gamma + 1)}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{29}$$

$$\frac{p}{p_0} = \left(1 + \frac{1}{2}(\gamma - 1) M^2\right)^{-\frac{\gamma}{\gamma-1}} \tag{30}$$

2. Normal Shock relationship

   If the normal shock occurs at $x = x_s$ where $M_1 = M(x_s)$ and $p_1 = p(x_s)$, the Mach number and pressure after the shock are:

$$M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1) M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)} \tag{31}$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma + 1} M_1^2 - \frac{\gamma - 1}{\gamma + 1} \tag{32}$$

   The stagnation pressure decreases after the shock and needs to be recalculated:

$$p_{20} = p_2 \left(1 + \frac{1}{2}(\gamma - 1) M_2^2\right)^{\frac{\gamma}{\gamma-1}} \tag{33}$$

   The Mach number at the exit is:

$$M_e^2 = \frac{2}{\gamma - 1}\left(\left(\frac{p_{20}}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right) \tag{34}$$

   F is required normal shock in the duct exit. At the back pressure G no single normal shock can be the job and so the flow compresses outside the exit in a complex series of oblique shocks until it matches p_b. In state F:

$$\frac{A_e}{A_t} = \frac{1}{M_e}\left(\frac{1 + \frac{1}{2}(\gamma - 1) M_e^2}{\frac{1}{2}(\gamma + 1)}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{35}$$

$$p_b = p_2 = p_1 \left(\frac{2\gamma}{\gamma + 1} M_e^2 - \frac{\gamma - 1}{\gamma + 1}\right) \tag{36}$$

$$p_1 = p_e = p_0 \left(1 + \frac{1}{2}(\gamma - 1) M_e\right)^{-\frac{\gamma}{\gamma-1}} \tag{37}$$

Case C: Supersonic outlet

The final state does not depend on the shape of the throat:

$$\frac{A_e}{A_t} = \frac{1}{M_e} \left(\frac{1 + \frac{1}{2} (\gamma - 1) M_e^2}{\frac{1}{2} (\gamma + 1)} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{38}$$

$$\frac{p_e}{p_0} = \left(1 + \frac{1}{2} (\gamma - 1) M_e^2 \right)^{-\frac{\gamma}{\gamma-1}} \tag{39}$$

$p_e = p_b$	Design pressure	
$p_e > p_b$	Under-expanded flow	Exit pressure higher than back pressure, need expansion fans to match flow.
$p_e < p_b$	Over-expanded flow	Exit pressure less than back pressure so need shocks to increase pressure.