UCL Mechanical Engineering 2020/2021
ENGF0004 Coursework 1

NCWT3

January 11, 2021

1 Question One

a

Proof. Left hand side:

$$\sum_{n=0}^{\infty}\left(\frac{k-1}{k}\right)^n = 1+\frac{k-1}{k}+\frac{(k-1)^2}{k^2}+... \tag{1.1}$$

$$a=1,\ r=\frac{k-1}{k} \tag{1.2}$$

$\frac{k-1}{k}$ is always less than 1 for $k>1$. Hence:

$$S_{\infty,LHS}=\frac{a}{1-r}=\frac{1}{1-\frac{k-1}{k}}=\frac{k}{k-k+1} \tag{1.3}$$

$$S_{\infty,LHS}=k \tag{1.4}$$

Right hand side:

$$(k-1)\sum_{n=0}^{\infty}\left(\frac{1}{k}\right)^n=(k-1)\left[1+\frac{1}{k}+\frac{1}{k^2}+...\right] \tag{1.5}$$

$$a=1,\ r=\frac{1}{k} \tag{1.6}$$

$\frac{1}{k}$ is always less than 1 for $k>1$. Hence:

$$S_{\infty,RHS}=\frac{k-1}{1-\frac{1}{k}}=\frac{k(k-1)}{k-1} \tag{1.7}$$

$$S_{\infty,RHS}=k \tag{1.8}$$

$$\tag{1.9}$$

LHS = RHS (for $k>1$). □

b

We are given:

$$f(x)=\frac{x}{\sqrt{1-x}} \tag{1.10}$$

$$f(x)=x(1-x)^{-\frac{1}{2}} \tag{1.11}$$

Differentiating three times yields:

$$f'(x) = (1-x)^{-\frac{1}{2}} + \frac{x}{2}(1-x)^{-\frac{3}{2}} \tag{1.12}$$

$$f''(x) = \frac{1}{2}(1-x)^{-\frac{3}{2}} + \frac{1}{2}(1-x)^{-\frac{3}{2}} + \frac{3x}{4}(1-x)^{-\frac{5}{2}} \tag{1.13}$$

$$= (1-x)^{-\frac{3}{2}} + \frac{3x}{4}(1-x)^{-\frac{5}{2}} \tag{1.14}$$

$$f'''(x) = \frac{3}{2}(1-x)^{-\frac{5}{2}} + \frac{3}{4}(1-x)^{-\frac{5}{2}} + \frac{15x}{8}(1-x)^{-\frac{7}{2}} \tag{1.15}$$

$$= \frac{9}{4}(1-x)^{-\frac{5}{2}} + \frac{15x}{8}(1-x)^{-\frac{7}{2}} \tag{1.16}$$

Inputting $x = 0$:

$$f(0) = 0 \cdot (1-0)^{-\frac{1}{2}} = 0 \tag{1.17}$$

$$f'(0) = (1-0)^{-\frac{1}{2}} + \frac{0}{2}(1-0)^{-\frac{3}{2}} = 1 \tag{1.18}$$

$$f''(0) = (1-0)^{-\frac{3}{2}} + \frac{3 \cdot 0}{4}(1-0)^{-\frac{5}{2}} = 1 \tag{1.19}$$

$$f'''(0) = \frac{9}{4}(1-0)^{-\frac{5}{2}} + \frac{15 \cdot 0}{8}(1-0)^{-\frac{7}{2}} = \frac{9}{4} \tag{1.20}$$

General form of Maclaurin series:

$$f(x) \approx f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... \tag{1.21}$$

Inputting the above variables into Eq.1.21:

$$f(x) \approx x + \frac{x^2}{2} + \frac{3x^3}{8} \tag{1.22}$$

c

i

We are given:

$$E = \frac{kq}{x^2} \tag{1.23}$$

Sum of electric fields due to both charged particles is:

$$E = \frac{ke}{(x-r)^2} - \frac{ke}{(x+r)^2} \tag{1.24}$$

$$= ke\left[\frac{1}{x^2\left(1-\frac{r}{x}\right)^2} - \frac{1}{x^2\left(1+\frac{r}{x}\right)^2}\right] \tag{1.25}$$

$$E = \frac{ke}{x^2}\left[(1-\gamma)^{-2} - (1+\gamma)^{-2}\right] \tag{1.26}$$

Where $\gamma = \frac{r}{x}$.

ii

Calculation of constants to be used in Maclaurin series expansion:

$$
\begin{aligned}
f(\gamma) &= (1-\gamma)^{-2} & f(0) &= 1\\
f'(\gamma) &= 2\,(1-\gamma)^{-3} & f'(0) &= 2\\
f''(\gamma) &= 6\,(1-\gamma)^{-4} & f''(0) &= 6\\
f'''(\gamma) &= 24\,(1-\gamma)^{-5} & f'''(0) &= 24
\end{aligned}
$$

$$
\begin{aligned}
g(\gamma) &= (1+\gamma)^{-2} & g(0) &= 1\\
g'(\gamma) &= -2\,(1+\gamma)^{-3} & g'(0) &= -2\\
g''(\gamma) &= 6\,(1+\gamma)^{-4} & g''(0) &= 6\\
g'''(\gamma) &= -24\,(1+\gamma)^{-5} & g'''(0) &= -24
\end{aligned}
$$

Inputting the above variables into Eq.1.21:

$$f(\gamma) \approx 1 + \frac{2\gamma}{1!} + \frac{6\gamma^2}{2!} + \frac{24\gamma^3}{3!} + ... \tag{1.27}$$

$$f(\gamma) \approx 1 + 2\gamma + 3\gamma^2 + 4\gamma^3 \tag{1.28}$$

$$g(\gamma) \approx 1 - \frac{2\gamma}{1!} + \frac{6\gamma^2}{2!} - \frac{24\gamma^3}{3!} + ... \tag{1.29}$$

$$g(\gamma) \approx 1 - 2\gamma + 3\gamma^2 - 4\gamma^2 \tag{1.30}$$

Substitution:

$$E \approx \frac{ke}{x^2}\left[f(\gamma) - g(\gamma)\right] \tag{1.31}$$

$$\approx \frac{ke}{x^2}\left[1 + 2\gamma + 3\gamma^2 + 4\gamma^3 - 1 + 2\gamma - 3\gamma^2 + 4\gamma^3\right] \tag{1.32}$$

$$\approx \frac{ke}{x^2}\left[4\gamma + 8\gamma^3\right] \tag{1.33}$$

$$E \approx \frac{4ke}{x^2}\left[\gamma + 2\gamma^3\right] \tag{1.34}$$

iii

$y = 0.01$. Exact:

$$E_E = \frac{ke}{x^2}\left[(1-0.01)^{-2} - (1+0.01)^{-2}\right] \tag{1.35}$$

$$E_E = \frac{ke}{x^2}\left[0.0400080012\right] \tag{1.36}$$

$$\tag{1.37}$$

Approximation:

$$E_A = \frac{4ke}{x^2}\left[0.01 + 2\,(0.01)^3\right] \tag{1.38}$$

$$E_A = \frac{ke}{x^2}\left[0.040008\right] \tag{1.39}$$

Percentage error:

$$\frac{E_E - E_A}{E_E} \cdot 100 = \frac{0.0400080012 - 0.040008}{0.0400080012} \cdot 100 = 3.0 \times 10^6\ \%\ \text{error (2sf)} \tag{1.40}$$

d

We are given:

$$y'' - 2y' + y = te^t \tag{1.41}$$
$$y(0) = 0,\ y'(0) = 1 \tag{1.42}$$

Laplace transformation (from tables):

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{te^t\} \tag{1.43}$$
$$s^2Y(s) - sy(0) - y'(0) - 2\left(sY(s) - y(0)\right) + Y(s) = \frac{1!}{(s-1)^2} \tag{1.44}$$
$$s^2Y(s) - 1 - 2(sY(s) - 1) + Y(s) = \frac{1}{(s-1)^2} \tag{1.45}$$
$$Y(s)\left[s^2 - 2s + 1\right] - 1 = \frac{1}{(s-1)^2} \tag{1.46}$$
$$Y(s) = \frac{1}{(s-1)^2} + 1 \tag{1.47}$$
$$Y(s) = \frac{1}{(s-1)^4} + \frac{1}{(s-1)^2} \tag{1.48}$$

Returning to time domain. From tables:

$$L^{-1}\left[\frac{n!}{(s-a)^n}\right] = t^n e^{at} \tag{1.49}$$
$$L^{-1}\left[\frac{1}{(s-1)^2}\right] = te^t \tag{1.50}$$
$$\frac{1}{6}L^{-1}\left[\frac{3!}{(s-1)^2}\right] = \frac{1}{6}t^3e^t \tag{1.51}$$
$$y(t) = \frac{1}{6}t^3e^t + te^t \tag{1.52}$$

e

i

$a = 1 \therefore -3 \leq t \leq 3$. Sketch:

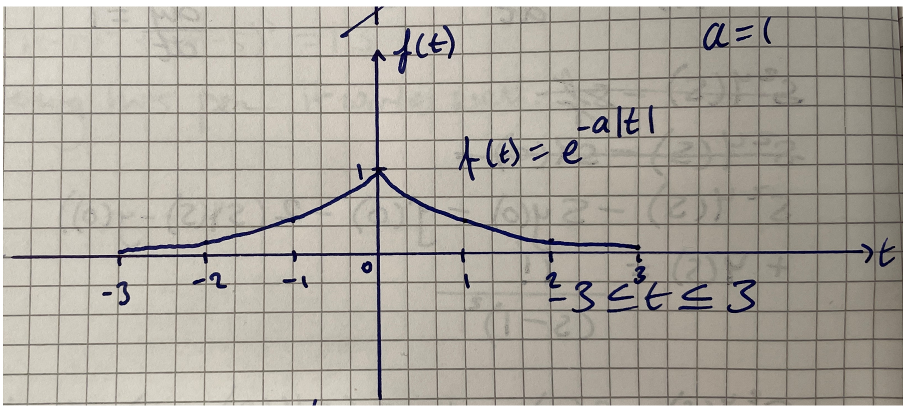

Figure 1:

ii

Let $z = \infty$.

$$F(u) = \lim_{z\to\infty} \int_{t=-z}^{0} e^{at} e^{-j2\pi ut}\ \mathrm{d}t + \lim_{z\to\infty} \int_{t=0}^{z} e^{-at} e^{-j2\pi ut}\ \mathrm{d}t \tag{1.53}$$

$$= \lim_{z\to\infty} \int_{t=-\infty}^{0} e^{t(a-j2\pi u)}\ \mathrm{d}t + \lim_{z\to\infty} \int_{t=0}^{\infty} e^{-t(a+j2\pi u)}\ \mathrm{d}t \tag{1.54}$$

$$F(u) = \lim_{z\to\infty} \left[\frac{1}{(a-j2\pi u)} e^{t(a-j2\pi u)} \Big|_{t=-z}^{0} \right] + \lim_{z\to\infty} \left[\frac{1}{-(a+j2\pi u)} e^{-t(a+j2\pi u)} \Big|_{t=0}^{z} \right] \tag{1.55}$$

Applying limits:

$$\lim_{z\to\infty} \left[\frac{1}{(a-j2\pi u)} \left[1 - e^{-z(a-j2\pi u)}\right] \right] = \left[\frac{1}{(a-j2\pi u)} [1-0] \right] = \frac{1}{a-j2\pi u} \tag{1.56}$$

$$\lim_{z\to\infty} \left[\frac{-1}{(a+j2\pi u)} \left[e^{-z(a+j2\pi u)} - 1\right] \right] = \left[\frac{-1}{(a+j2\pi u)} [0-1] \right] = \frac{1}{a+j2\pi u} \tag{1.57}$$

$$F(u) = \frac{1}{a-j2\pi u} + \frac{1}{a+j2\pi u} \tag{1.58}$$

$$= \frac{a+j2\pi u + a - j2\pi u}{a^2+4\pi^2u^2} \tag{1.59}$$

$$F(u) = \frac{2a}{a^2+4\pi^2u^2} \tag{1.60}$$

iii

Substituting $\omega = 2\pi u$, $\omega^2 = 4\pi^2u^2$:

$$F(\omega) = \frac{2a}{a^2+\omega^2} \tag{1.61}$$

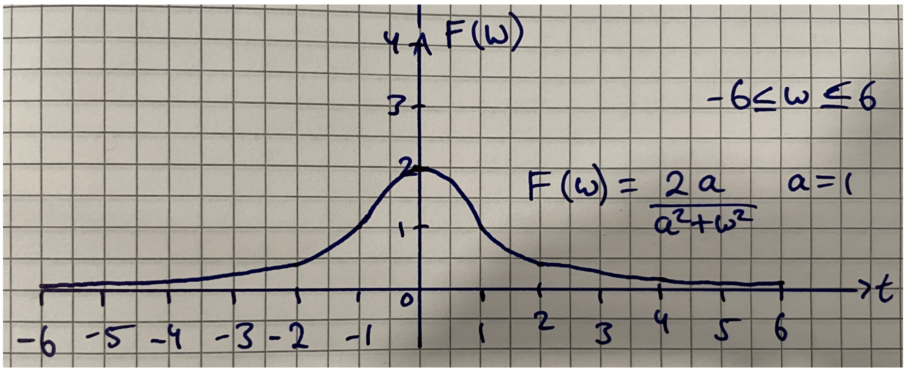

Figure 2:

iv

Full width at half maximum of $F(u)$ can be calculated as:

$$\frac{1}{2} = e^{-at} \tag{1.62}$$
$$\ln\left(\frac{1}{2}\right) = -at \tag{1.63}$$
$$-\ln(2) = at \tag{1.64}$$
$$t = \frac{\ln(2)}{a} \rightarrow \text{HWHM} \tag{1.65}$$
$$\therefore \text{FWHM} = \frac{2\ln(2)}{a} \tag{1.66}$$

Full width at half maximum of $F(\omega)$ can be calculated as:

$$\frac{1}{2}\cdot\frac{2}{a} = 2\frac{a}{a^2+\omega^2} \tag{1.67}$$
$$\frac{1}{a} = \frac{2a}{a^2+\omega^2} \tag{1.68}$$
$$\omega^2 + a^2 = 2a^2 \tag{1.69}$$
$$\omega = a \rightarrow \text{HWHM} \tag{1.70}$$
$$\therefore \text{FWHM} = 2a \tag{1.71}$$

v

Product of FWHMs:

$$2a\cdot\frac{2\ln(2)}{a} = 4\ln(2) \tag{1.72}$$

The product has no a term, thus there is no dependence on the parameter.

2 Question Two

a

i

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{2.1}$$

We know that $u(x,t) = X(x)T(t)$. Substituting:

$$X(x)T'(t) = kX''(x)T(t) \tag{2.2}$$
$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = \mu \tag{2.3}$$
$$X''(x) = \mu X(x) \tag{2.4}$$
$$T'(t) = \mu kT(t) \tag{2.5}$$

where μ is an arbitrary constant. If we define μ as negative then we obtain:

$$-X''(x) = \mu X(x) \tag{2.6}$$
$$T'(t) = -\mu k T(t) \tag{2.7}$$

ii

μ can be either positive, zero or negative. Let us consider these cases.

Case 1: positive constant $\mu = \lambda^2 > 0$

$$\frac{X''(x)}{X(x)} = \lambda^2 \tag{2.8}$$
$$X''(x) - \lambda^2 X(x) = 0 \tag{2.9}$$

Auxiliary equation:

$$m^2 - \lambda^2 = 0 \tag{2.10}$$
$$m_1 = \lambda,\ m_2 = -\lambda \tag{2.11}$$

Real and distinct roots. Hence:

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \tag{2.12}$$

Applying boundary conditions for x. Applying $u(0,t) = 0$ implies $X(0)T(t) = 0$. Substituting $x = 0$ in $X(x)$ gives:

$$X(0) = A + B \tag{2.13}$$
$$(A + B)\,T(t) = 0 \tag{2.14}$$
$$T(t) = 0 \text{ or } A + B = 0 \tag{2.15}$$

$T(t) = 0$ leads to a trivial solution of $u(x,t) = X(x)T(t) = 0$. We also have $A + B = 0$ or $A = -B$:

$$X(x) = A\left(e^{\lambda x} - e^{-\lambda x}\right) \tag{2.16}$$

Applying $u(l,t) = 0$ implies $X(l)T(t) = 0$. Substituting $x = l$ in $X(x)$ gives: $X(l) = A\left(e^{\lambda l} - e^{-\lambda l}\right)$

$$A\left(e^{\lambda l} - e^{-\lambda l}\right)T(t) = 0 \tag{2.17}$$
$$A = 0,\ T(t) = 0, \text{ or } e^{\lambda l} - e^{-\lambda l} = 0 \tag{2.18}$$

$T(t) = 0$ and $A = 0$ both will lead to a trivial solution of $u(x,t) = X(x)T(t) = 0$. $e^{\lambda l} - e^{-\lambda l} = 0$ is only true if $\lambda = 0$. However, the assumption in this case is $\lambda = \sqrt{\mu}$ where μ is a positive constant ($\lambda > 0$). Therefore, this boundary condition cannot be met and there are no useful solutions from Case 1.

Case 2: zero constant $\lambda = 0$

$$\frac{X''(x)}{X(x)} = \lambda = 0 \tag{2.19}$$
$$X''(x) = 0 \tag{2.20}$$
$$\int X''(x)\,\mathrm{d}x = \int \mathrm{d}x \tag{2.21}$$
$$X'(x) = C \tag{2.22}$$
$$\int X'(x) = \int C\,\mathrm{d}x \tag{2.23}$$
$$X(x) = Cx + D \tag{2.24}$$

where C and D are constants of integration. Applying boundary conditions for x. Applying $u(0,t) = 0$ implies $X(0)T(t) = 0$. Substituting $x = 0$ in $X(x)$ gives:

$$X(0) = D \tag{2.25}$$
$$DT(t) = 0T(t) = 0 \text{ or } D = 0 \tag{2.26}$$

$T(t) = 0$ will lead to a trivial solution of $u(x,t) = X(x)T(t) = 0$. Therefore, $D = 0$.

$$X(x) = Cx \tag{2.27}$$

Applying $u(l,t) = 0$ implies $X(l)X(t) = 0$. Substituting $x = l$ in $X(x)$ gives:

$$X(l) = Cl \tag{2.28}$$
$$ClT(t) = 0 \tag{2.29}$$
$$T(t) = 0 \text{ or } C = 0 \tag{2.30}$$

$T(t) = 0$ and $C = 0$ both will lead to a trivial solution of $u(x,t) = X(x)T(t) = 0$. Case 2 only produces the trivial solution of $u(x,t) = X(x)T(t) = 0$ and does not produce any useful solutions.

Case 3: negative constant $\mu = \lambda^2 < 0$

$$\frac{X''(x)}{X(x)} = -\lambda^2 \tag{2.31}$$
$$X''(x) + \lambda^2 X(x) = 0 \tag{2.32}$$

Auxiliary equation:

$$m^2 + \lambda^2 = 0 \tag{2.33}$$
$$m_1 = i\lambda, \; m_2 = -i\lambda \tag{2.34}$$

Complex and distinct roots. Hence:

$$X(x) = Ce^{i\lambda x} + De^{-i\lambda x} \tag{2.35}$$

Using Euler's formula and expanding:

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x) \tag{2.36}$$

where $A = C + D$ and $B = i(C - D)$. Applying boundary conditions for x. Applying $u(0,t) = 0$ implies $X(0)T(t) = 0$. Substituting $x = 0$ in $X(x)$ gives:

$$X(0) = A \tag{2.37}$$
$$AT(t) = 0 \tag{2.38}$$
$$T(t) = 0, \; A = 0 \tag{2.39}$$

$T(t) = 0$ will lead to a trivial solution of $u(x,t) = X(x)T(t) = 0$. Therefore, $A = 0$

$$X(x) = B\sin(\lambda x) \tag{2.40}$$

Applying $u(l,t) = 0$ implies $X(l)T(t) = 0$. Substituting $x = l$ in $X(x)$ gives:

$$X(l) = B\sin(\lambda l) \tag{2.41}$$
$$D\sin(\lambda l)\,T(t) = 0 \tag{2.42}$$
$$T(t) = 0, \; B = 0 \text{ or } \sin(\lambda l) = 0 \tag{2.43}$$

$T(t) = 0$ will lead to a trivial solution of $u(x,t) = X(x)T(t) = 0$. $B = 0$ will lead to a trivial solution of $u(x,t) = X(x)T(t) = 0$.

$$\sin(\lambda l) = 0 \tag{2.44}$$
$$\lambda l = n\pi \text{ for } n = 1, 2, 3, ... \tag{2.45}$$
$$\lambda_n = \frac{n\pi}{l} \tag{2.46}$$
$$\mu_n = -(\lambda_n)^2 = -\left(\frac{n\pi}{l}\right)^2 \tag{2.47}$$

We can denote B_n to represent the various constants that correspond to each value of μ. Therefore:

$$X(x) = B_n \sin\left(\frac{n\pi x}{l}\right) \text{ for } n = 1, 2, 3, ... \tag{2.48}$$

ODE in $T(t)$:

$$\frac{T'(t)}{kT(t)} = \mu_n = -\left(\lambda_n^2\right) \tag{2.49}$$
$$\int \left(\frac{T'(t)}{T(t)}\right) \mathrm{d}t = \int (\mu_n k)\, \mathrm{d}t \tag{2.50}$$

In $T(t) = \mu_n kt + d_n$ where d_n is a constant of integration for the differing values of each value of μ_n.

$$T(t) = A_n e^{-\lambda_n^2 kt} \tag{2.51}$$

where $A_n = e^{d_n}$. Therefore:

$$u(x,t) = X(x)T(t) = c_n e^{-\lambda_n^2 kt} \sin\left(\frac{n\pi x}{l}\right) \tag{2.52}$$

where $c_n = A_n B_n$. Let $u_n(x,t) e^{-\lambda_n^2 kt} \sin\left(\frac{n\pi x}{l}\right)$. The principle of superposition gives:

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) \tag{2.53}$$

Applying the initial condition $u(x,0) = f(x)$ $0 \le x \le l$. We have:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \tag{2.54}$$

iii

Using Fourier series:

$$c_n = \frac{2}{l} \int_0^l \left(f(x) \sin(n\pi x)\right) \mathrm{d}x \tag{2.55}$$

We are given:

$$u(x,0) = f(x) = x^2, \ 0 \le x \le l \tag{2.56}$$
$$u(0,t) = u(l,t) = 0, \ t > 0 \tag{2.57}$$
$$\tag{2.58}$$

Let $a = \frac{n\pi}{l}$. Substituting the above into Fourier:

$$c_n = \frac{2}{l}\int_0^l \left(x^2 \sin(ax)\right)\, \mathrm{d}x \tag{2.59}$$

$$\tag{2.60}$$

Integration by parts once. $u = x^2$, $u' = 2x$, $v = -\frac{\cos(ax)}{n\pi}$ and $v' = \sin(ax)$.

$$c_n = \frac{2}{l}\left[-\frac{x^2\cos(ax)}{a} + \frac{2}{a}\int_0^l \left(x\cos(ax)\right)\, \mathrm{d}x\right]_0^l \tag{2.61}$$

Integration by parts twice. $u = x$, $u = 1$, $v = \frac{\sin(ax)}{a}$ and $v' = \cos(ax)$.

$$c_n = \frac{2}{l}\left[-\frac{x^2\cos(ax)}{a} + \frac{2}{a}\left[\frac{x\sin(ax)}{a} - \frac{1}{a}\int_0^l \left(\sin(ax)\right)\, \mathrm{d}x\right]_0^l\right]_0^l \tag{2.62}$$

$$= 2\left[-\frac{x^2\cos(n\pi x)}{n\pi} + \frac{2}{n\pi}\left[\frac{x\sin(n\pi x)}{n\pi} - \frac{1}{n\pi}\left[-\frac{\cos(n\pi x)}{n\pi}\right]\right]_0^l\right]_0^l \tag{2.63}$$

$$= \frac{2}{l}\left[-\frac{x^2\cos(ax)}{n\pi} + \frac{2x\sin(ax)}{a^2} + \frac{2\cos(ax)}{a^3}\right]_0^l \tag{2.64}$$

$$= \frac{2}{l}\left[-\frac{l^3\cos(n\pi x)}{n\pi} + \frac{2l^3\sin(n\pi x)}{n^2\pi^2} + \frac{2l^3\cos(n\pi x)}{n^3\pi^3}\right]_0^l \tag{2.65}$$

$$= \frac{2l^2}{n\pi}\left[-\cos(n\pi x) + \frac{2\sin(n\pi x)}{n\pi} + \frac{2\cos(n\pi x)}{n^2\pi^2}\right]_0^l \tag{2.66}$$

$$c_n = \frac{2l^2}{n\pi}\left[-\cos(n\pi) + \frac{2\sin(n\pi)}{n\pi} + \frac{2\left(\cos(n\pi) - 1\right)}{n^2\pi^2}\right] \tag{2.67}$$

Substituting:

$$\therefore u_n(x,t) = \sum_{n=1}^{\infty}\left(\frac{2l^2}{n\pi}\left[-\cos(n\pi) + \frac{2\sin(n\pi)}{n\pi} + \frac{2\left(\cos(n\pi)-1\right)}{n^2\pi^2}\right]\cdot \sin\left(\frac{n\pi x}{l}\right)\cdot e^{\frac{n^2\pi^2}{l^2}kt}\right) \tag{2.68}$$

When n is even:

$$u_n(x,t) = \sum_{n=1}^{\infty}\left(-\frac{2l^2}{n\pi}\cdot \sin\left(\frac{n\pi x}{l}\right)\cdot e^{\frac{n^2\pi^2}{l^2}kt}\right) \tag{2.69}$$

When n is odd:

$$u_n(x,t) = \sum_{n=1}^{\infty}\left(-\frac{2l^2}{n\pi}\left[1 - \frac{4}{n^2\pi^2}\right]\cdot \sin\left(\frac{n\pi x}{l}\right)\cdot e^{\frac{n^2\pi^2}{l^2}kt}\right) \tag{2.70}$$

b

i

Taylor series expansion for $u = u(x, y)$

$$u(x+h,y) = u(x,y) + h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + ... \tag{2.71}$$

$$u(x-h,y) = u(x,y) - h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + ... \tag{2.72}$$

$$u(x,y+k) = u(x,y) + k\frac{\partial u}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 u}{\partial y^2} + \frac{k^3}{3!}\frac{\partial^3 u}{\partial y^3} + ... \tag{2.73}$$

$$u(x,y-k) = u(x,y) - k\frac{\partial u}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 u}{\partial y^2} - \frac{k^3}{3!}\frac{\partial^3 u}{\partial y^3} + ... \tag{2.74}$$

Summing Eq.2.71 and Eq.2.72 yields:

$$u(x+h,y) + u(x-h,y) = 2u(x,y) + h^2\frac{\partial^2 u}{\partial x^2} + h^2 O(h^2) \tag{2.75}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + O(h^2) \tag{2.76}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} \tag{2.77}$$

Summing Eq.2.73 and Eq.2.74 yields:

$$u(x,y+k) + u(x,y-k) = 2u(x,y) + k^2\frac{\partial^2 u}{\partial y^2} + k^2 O(h^2) \tag{2.78}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u(x,y+k) - 2u(x,y) + u(x,y-k)}{k^2} + O(k^2) \tag{2.79}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u(x,y+k) - 2u(x,y) + u(x,y-k)}{k^2} \tag{2.80}$$

ii

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = xe^y \tag{2.81}$$

On our mesh sizing of $h = \delta x = 0.4$ and $k = \delta y = 0.2$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = ihe^{jk} \tag{2.82}$$

$$\frac{25}{4}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + 25\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) = xe^y \tag{2.83}$$

$$\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + 4\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) = \frac{4}{25}xe^y \tag{2.84}$$

$$u_{i+1,j} - 2u_{i,j} + u_{i+1,j} + 4u_{i,j+1} - 8u_{i,j} + 4u_{i,j-1} = \frac{4}{25}xe^y \tag{2.85}$$

$$10u_{i,j} = u_{i+1,j} + u_{i+1,j} + 4u_{i,j+1} + 4u_{i,j-1} - \frac{4}{25}xe^y \tag{2.86}$$

$$u_{i,j} = \frac{1}{10}\left(u_{i+1,j} + u_{i+1,j} + 4u_{i,j+1} + 4u_{i,j-1} - \frac{4}{25}xe^y\right) \tag{2.87}$$

where Eq.2.87 is our computational molecule.

iii

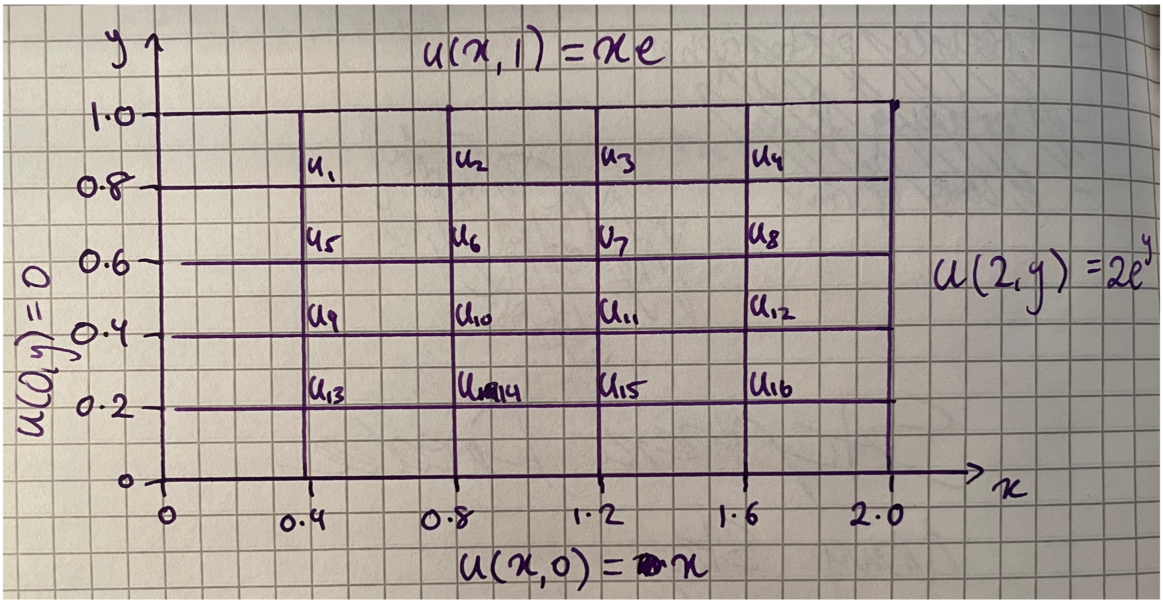

Figure 3:

16 difference equations:

$$10u_1 = u_2 + 0 + 4(0.4)e + 4u_5 - 0.16(0.4)e^{0.8} \tag{2.88}$$
$$-10u_1 + u_2 + 4u_5 = 0.064e^{0.8} - 1.6e \tag{2.89}$$

$$10u_2 = u_3 + u_1 + 4(0.8)e + 4u_6 - 0.16(0.8)e^{0.8} \tag{2.90}$$
$$-10u_2 + u_3 + u_1 + 4u_6 = 0.128e^{0.8} - 3.2e \tag{2.91}$$

$$10u_3 = u_4 + u_2 + 4(1.2)e + 4u_7 - 0.16(1.2)e^{0.8} \tag{2.92}$$
$$-10u_3 + u_4 + u_2 + 4u_7 = 0.192e^{0.8} - 4.8e \tag{2.93}$$

$$10u_4 = 2e^{0.8} + u_3 + 4(1.6)e + 4u_8 - 0.16(1.6)e^{0.8} \tag{2.94}$$
$$-10u_4 + u_3 + 4u_8 = -1.744e^{0.8} - 6.4e \tag{2.95}$$

$$10u_5 = u_6 + 0 + 4u_1 + 4u_9 - 0.16(0.4)e^{0.6} \tag{2.96}$$
$$-10u_5 + u_6 + 4u_1 + 4u_9 = 0.064e^{0.6} \tag{2.97}$$

$$10u_6 = u_7 + u_5 + 4u_2 + 4u_{10} - 0.16(0.8)e^{0.6} \tag{2.98}$$
$$-10u_6 + u_7 + u_5 + 4u_2 + 4u_{10} = 0.128e^{0.6} \tag{2.99}$$

$$10u_7 = u_8 + u_6 + 4u_3 + 4u_{11} = 0.16(1.2)e^{0.6} \tag{2.100}$$
$$-10u_7 + u_8 + u_6 + 4u_3 + 4u_{11} = 0.192e^{0.6} \tag{2.101}$$

$$10u_8 = 2e^{0.6} + u_7 + 4u_4 + 4u_{12} - 0.16(1.6)e^{0.6} \tag{2.102}$$
$$-10u_8 + u_7 + 4u_4 + 4u_{12} = -1.744e^{0.6} \tag{2.103}$$

$$10u_9 = u_{10} + 0 + 4u_5 + 4u_{13} - 0.16(0.4)e^{0.4} \tag{2.104}$$
$$-10u_9 + u_{10} + 4u_5 + 4u_{13} = 0.064e^{0.4} \tag{2.105}$$

$$10u_{10} = u_{11} + u_9 + 4u_6 + 4u_{14} - 0.16(0.8)e^{0.4} \tag{2.106}$$
$$-10u_{10} + u_{11} + u_9 + 4u_6 + 4u_{14} = 0.128e^{0.4} \tag{2.107}$$

$$10u_{11} = u_{12} + u_{10} + 4u_7 + 4u_{15} - 0.16(1.2)e^{0.4} \tag{2.108}$$
$$-10u_{11} + u_{12} + u_{10} + 4u_7 + 4u_{15} = 0.192e^{0.4} \tag{2.109}$$

$$10u_{12} = 2e^{0.4} + u_{11} + 4u_8 + 4u_{16} - 0.16(1.6)e^{0.4} \tag{2.110}$$
$$-10u_{12} + u_{11} + 4u_8 + 4u_{16} = -1.744e^{0.4} \tag{2.111}$$

$$10u_{13} = u_{14} + 0 + 4u_9 + 4(0.4) = 0.16(0.4)e^{0.2} \tag{2.112}$$
$$-10u_{13} + u_{14} + 4u_9 = 0.064e^{0.2} - 1.6 \tag{2.113}$$

$$10u_{14} = u_{15} + u_{13} + 4u_{10} + 4(0.8) - 0.16(0.8)e^{0.2} \tag{2.114}$$
$$-10u_{14} + u_{15} + u_{13} + 4u_{10} = 0.128e^{0.2} - 3.2 \tag{2.115}$$

$$10u_{15} = u_{16} + u_{14} + 4u_{11} + 4(1.2) = 0.16(1.2)e^{0.2} \tag{2.116}$$
$$-10u_{15} + u_{16} + u_{14} + 4u_{11} = 0.192e^{0.2} - 4.8 \tag{2.117}$$

$$10u_{16} = 2e^{0.2} + u_{15} + 4u_{12} + 4(1.6) - 0.16(1.6)e^{0.2} \tag{2.118}$$
$$-10u_{16} + u_{15} + 4u_{12} = -1.744e^{0.2} - 6.4 \tag{2.119}$$

iv & v

MATLAB code used to solve the system of equations above.

```
clc
clear
close all

%question 2biv
A = [-10 1 0 0 4 0 0 0 0 0 0 0 0 0 0 0;
     1 -10 1 0 0 4 0 0 0 0 0 0 0 0 0 0;
     0 1 -10 1 0 0 4 0 0 0 0 0 0 0 0 0;
     0 0 1 -10 0 0 0 4 0 0 0 0 0 0 0 0;
     4 0 0 0 -10 1 0 0 4 0 0 0 0 0 0 0;
     0 4 0 0 1 -10 1 0 0 4 0 0 0 0 0 0;
     0 0 4 0 0 1 -10 1 0 0 4 0 0 0 0 0;
     0 0 0 4 0 0 1 -10 0 0 0 4 0 0 0 0;
     0 0 0 0 4 0 0 0 -10 1 0 0 4 0 0 0;
     0 0 0 0 0 4 0 0 1 -10 1 0 0 4 0 0;
     0 0 0 0 0 0 4 0 0 1 -10 1 0 0 4 0;
     0 0 0 0 0 0 0 4 0 0 1 -10 0 0 0 4;
     0 0 0 0 0 0 0 0 4 0 0 0 -10 1 0 0;
     0 0 0 0 0 0 0 0 0 4 0 0 1 -10 1 0;
     0 0 0 0 0 0 0 0 0 0 4 0 0 1 -10 1;
     0 0 0 0 0 0 0 0 0 0 0 4 0 0 1 -10]; %LHS

b =[0.064*(exp(0.8))-1.6*exp(1);
    0.128*(exp(0.8))-3.2*exp(1);
    0.192*(exp(0.8))-4.8*exp(1);
    -1.744*(exp(0.8))-6.4*exp(1);
    0.064*(exp(0.6));
    0.128*(exp(0.6));
    0.192*(exp(0.6));
    -1.744*(exp(0.6));
    0.064*(exp(0.4));
    0.128*(exp(0.4));
    0.192*(exp(0.4));
    -1.744*(exp(0.4));
    0.064*(exp(0.2))-1.6;
    0.128*(exp(0.2))-3.2;
    0.192*(exp(0.2))-4.8;
    -1.744*(exp(0.2))-6.4]; %RHS

w = A\b; %inverts A and multiplies with B

%question 2bv
x = [0.4;0.8;1.2;1.6;0.4;0.8;1.2;1.6;0.4;0.8;1.2;1.6;0.4;0.8;1.2;1.6]; %x values
y = [0.8;0.8;0.8;0.8;0.6;0.6;0.6;0.6;0.4;0.4;0.4;0.4;0.2;0.2;0.2;0.2]; %y values
wE = x.*(exp(y)); %calculate exact solution
d = w-wE; %calculates difference between calculated and exact values
```

The values for the variables w, wE and d are shown below.

w	wE	$d \cdot 10^4$
0.8904	0.8902	1.9117
1.7808	1.7804	3.7120
2.6711	2.6706	5.1109
3.5613	3.5609	5.1313
0.7291	0.7288	2.6628
1.4582	1.4577	5.1474
2.1872	2.1865	7.0008
2.9160	2.9154	6.7963
0.5969	0.5967	2.4853
1.1939	1.1935	4.7943
1.7908	1.7902	6.4859
2.3875	2.3869	6.2168
0.4887	0.4886	1.5553
0.9774	0.9771	3.0022
1.4660	1.4657	4.0711
1.9546	1.9542	3.9375

Table 1: Values for w, wE and d.

As seen in the table above, our error term is on the magnitude of 10^{-4} which is a relatively tiny error. Hence, our approximation is useful.