MECH0011 Topic Notes

UCL

HD

November 28, 2020

Contents

Part I

Fluids

Chapter 1

Differential equations of mass, momentum and energy

1.1 Differential analysis of fluid flow

What do we want to know? The velocity field, pressures, densities and temperature everywhere and anytime. Hence, these will be a function of (x, y, z, t).

List of variables

Variable	Type	Units
$\overrightarrow{U} = u\hat{i} + v\hat{j} + w\hat{k}$ $\overrightarrow{U} = u_1\hat{i}_1 + u_2\hat{i}_2 + u_3\hat{i}_3$	Velocity/Vector	$\mathrm{m\,s^{-1}}$
p	Pressure/Scalar	$\mathrm{N\,m^{-2}}$
T	Temperature/Scalar	°C
ρ	Density/Scalar	$\mathrm{kg\,m^{-3}}$
$T = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$	Stress Tensor	$\mathrm{N\,m^{-2}}$

Since we have 12 variables, we need 12 equations to describe the fluid!

From last year, we have our conservation of mass equation

$$\frac{\partial}{\partial t}\int_V \rho \, \mathrm{d}V + \oint_S (\rho\overrightarrow{V} \cdot \hat{n}) \, \mathrm{d}S = 0$$

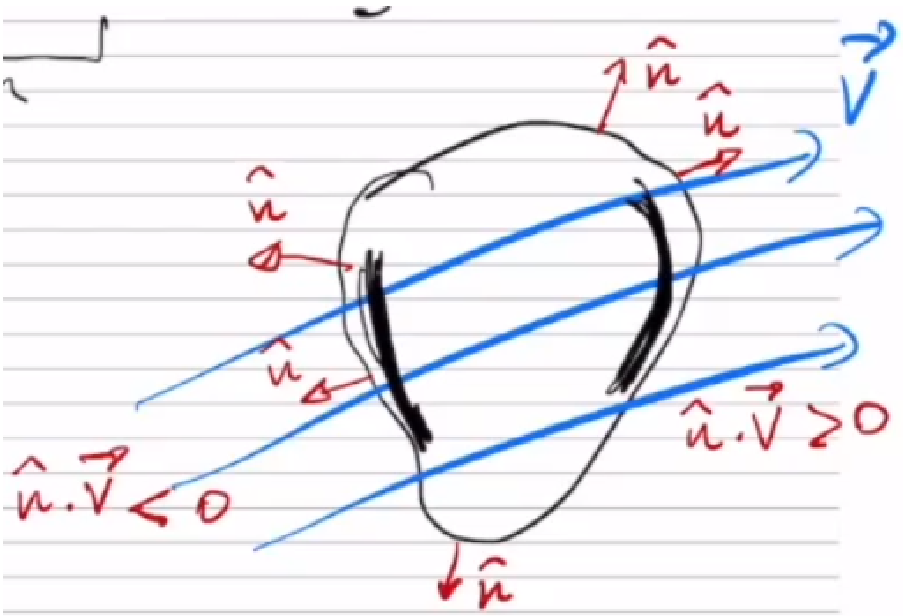


Figure 1.1: Consider $\hat{n}$ to be a vector coming out of the control volume. Depending on where $\hat{n}$ is, our dot product will either be greater than or less than 0.

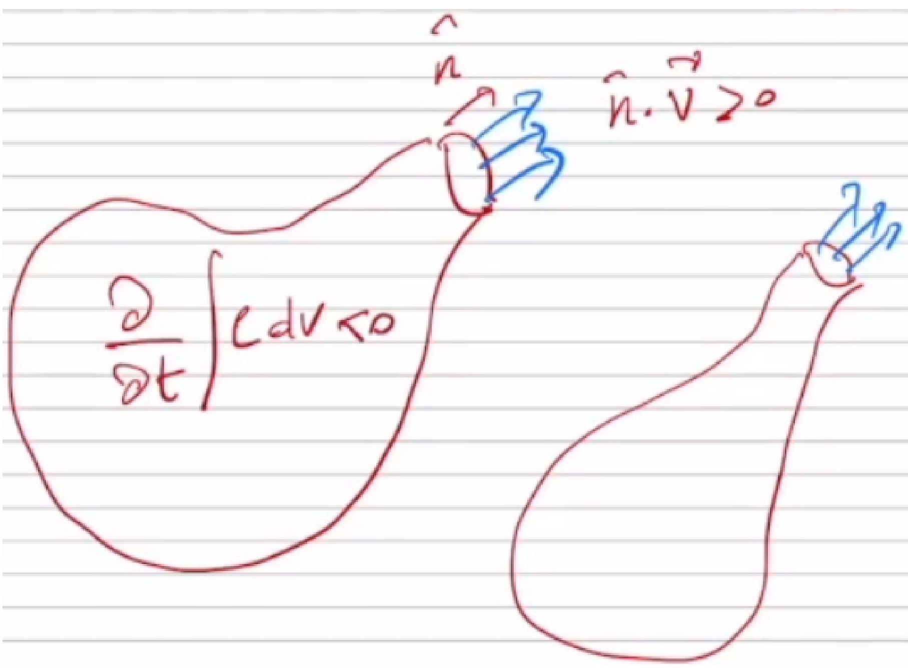
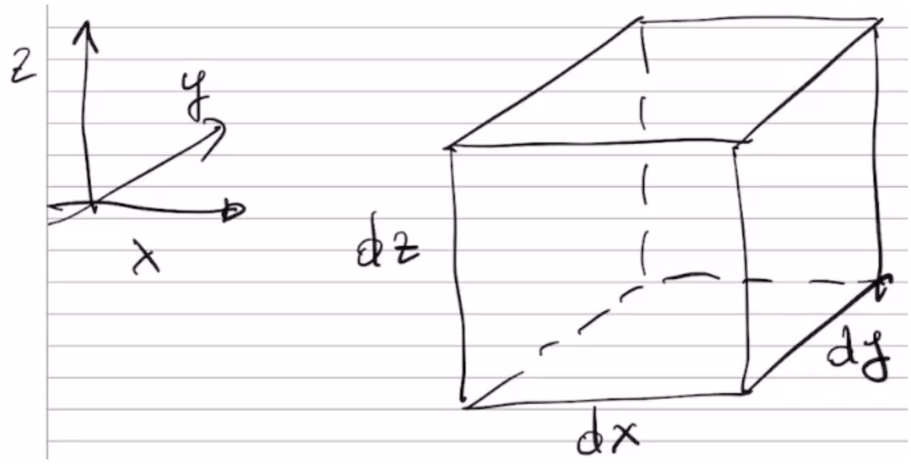


Figure 1.2: There is a velocity exiting the balloon. The amount of mass inside will decrease with time. The volume of the balloon will become smaller. This will be equal to the amount of mass which came out of the control volume (the balloon). If the second term of the continuity equation is positive, the first term must be negative.

1.2 Conservation of mass

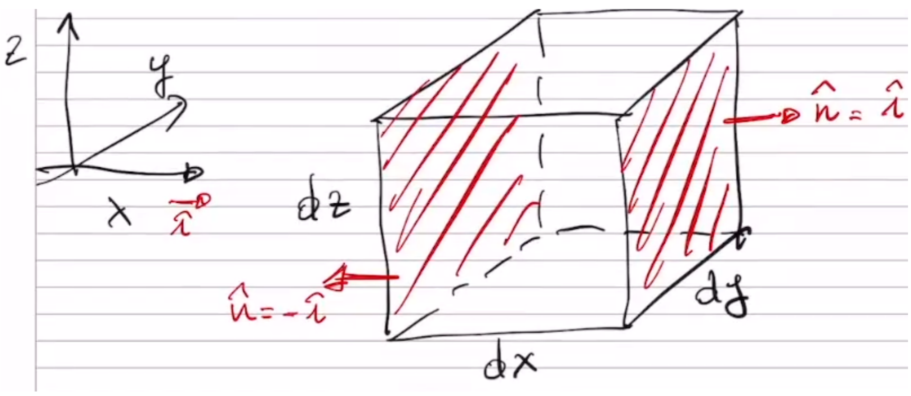
Let us consider an infinitesimally small cube:



Consider term 1 in our continuity equation - the mass variation inside the control volume.

$$\frac{\partial \rho}{\partial t} \cdot dV = \frac{\partial \rho}{\partial t} \cdot \mathrm{d}x \cdot \mathrm{d}y \cdot \mathrm{d}z$$

Consider term 2 - the contribution of mass from the sides of the cube, which are orthogonal to x, shown below.



Left side:

$$\rho \overrightarrow{V} \cdot \hat{n} \, \mathrm{d}S = \rho \overrightarrow{V} \cdot (-\hat{\imath}) \, \mathrm{d}z \, \mathrm{d}y \quad (1.1)$$
$$= -\rho u \, \mathrm{d}z \, \mathrm{d}y \quad (1.2)$$

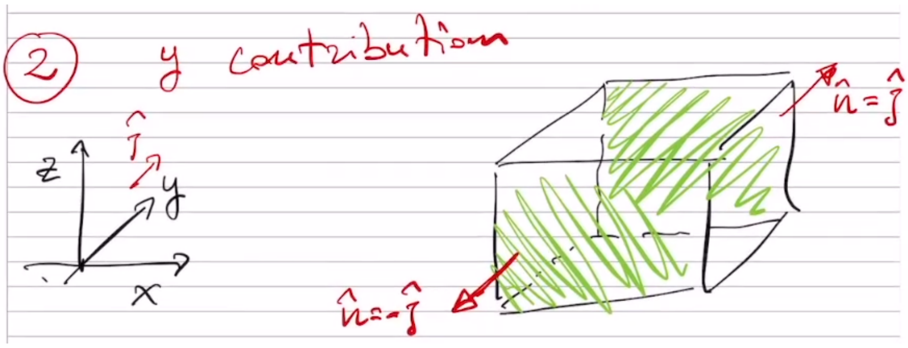
Looking from the right side, we will not have a negative (which is coming from the fact that the normal vector is going in the opposite direction to i)

$$(\rho u + \frac{\partial \rho u}{\partial x} \cdot \mathrm{d}x) \, \mathrm{d}z \, \mathrm{d}y \quad (1.3)$$

The net contribution from the orthogonal x direction is

$$= (\rho u + \frac{\partial \rho u}{\partial x} \cdot \mathrm{d}x)\,\mathrm{d}z \cdot \mathrm{d}y - \rho u\,\mathrm{d}z\,\mathrm{d}y \tag{1.4}$$
$$= \frac{\partial \rho u}{\partial x}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \tag{1.5}$$

y orthogonal contribution



Front side

$$\rho \overrightarrow{V} \cdot \hat{n}\,\mathrm{d}S = \rho \overrightarrow{V} \cdot (-\hat{j})\,\mathrm{d}x\,\mathrm{d}z \tag{1.6}$$
$$= -\rho v\,\mathrm{d}x\,\mathrm{d}z \tag{1.7}$$

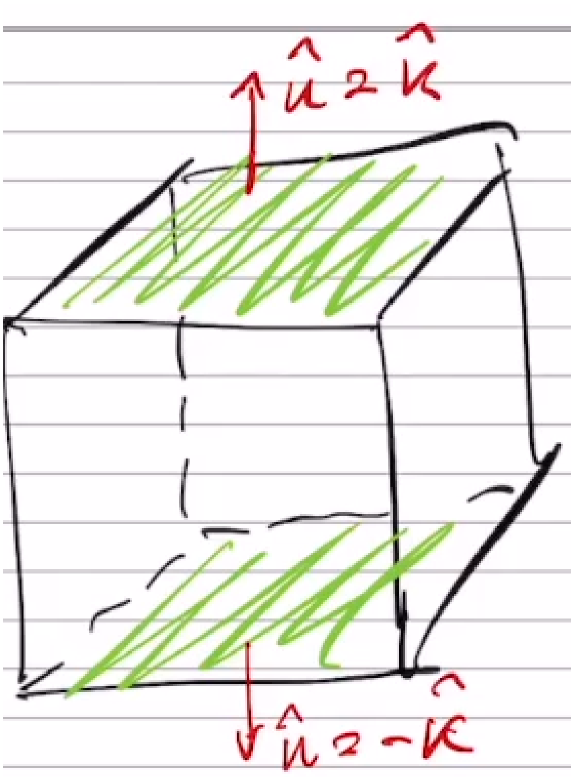
Back side

$$(\rho v + \frac{\partial \rho v}{\partial y} \cdot \mathrm{d}y)\,\mathrm{d}z\,\mathrm{d}x \tag{1.8}$$

Final contribution

$$\frac{\partial \rho v}{\partial y}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x \tag{1.9}$$

z orthogonal contribution



Bottom side

$$\rho\vec{V}\cdot\hat{n}\,\mathrm{d}S = \rho\vec{V}\cdot(-\hat{k})\,\mathrm{d}x\,\mathrm{d}y \tag{1.10}$$
$$= -\rho w\,\mathrm{d}x\,\mathrm{d}y \tag{1.11}$$

Top side

$$(\rho w + \frac{\partial \rho w}{\partial z}\cdot\mathrm{d}z)\,\mathrm{d}x\,\mathrm{d}y \tag{1.12}$$

Final contribution

$$\frac{\partial \rho w}{\partial z}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \tag{1.13}$$

If we add up all of the contributions above, we get the conservation of mass for an infinitesimal volume.

$$\frac{\partial \rho}{\partial t}\cdot\mathrm{d}x\cdot\mathrm{d}y\cdot\mathrm{d}z + \frac{\partial \rho u}{\partial x}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y + \frac{\partial \rho v}{\partial y}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x + \frac{\partial \rho w}{\partial z}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{1.14}$$

This simplifies to:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{1.15}$$

We can simplify this a bit more by introducing a term called the divergence.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0 \tag{1.16}$$

Where $\nabla\cdot(\rho\vec{V})$ is the divergence of the vector $\rho\vec{V}$. It is a scalar.

$$\frac{\partial \rho}{\partial t} + \vec{V}\cdot\nabla\rho + \rho\nabla\cdot\vec{V} = 0 \tag{1.17}$$

$\nabla \cdot \overrightarrow{V}$ is the divergence of the vector $\overrightarrow{V}$ and it is a scalar. $\nabla\rho$ is the gradient of the density ρ and is a vector. It can be expanded as:

$$\frac{\partial \rho}{\partial x}\hat{i} + \frac{\partial \rho}{\partial y}\hat{j} + \frac{\partial \rho}{\partial z}\hat{k} \tag{1.18}$$

$$\overrightarrow{V} \cdot \rho\nabla = (u\hat{i} + v\hat{j} + w\hat{k}) \cdot \left(\frac{\partial \rho}{\partial x}\hat{i} + \frac{\partial \rho}{\partial y}\hat{j} + \frac{\partial \rho}{\partial z}\hat{k}\right) = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \tag{1.19}$$

$$\rho\nabla \cdot \overrightarrow{V} = \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{1.20}$$

For steady flow:

$$\frac{\partial \rho}{\partial t} = 0 \rightarrow \nabla \cdot (\rho\overrightarrow{V}) = 0 \tag{1.21}$$

[H] For incompressible flow, the density is constant. This means all derivatives of ρ are 0. Hence, our equation reduces to:

$$\rho = \text{const} \rightarrow \nabla \cdot \overrightarrow{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1.22}$$

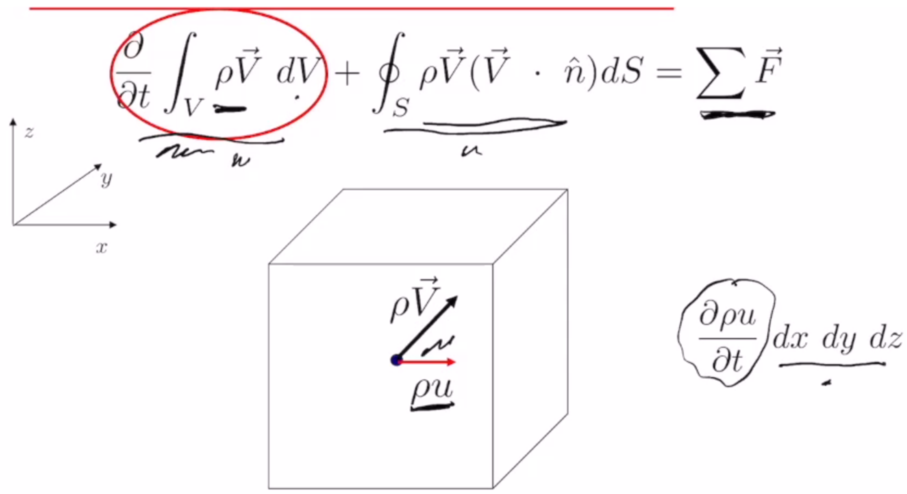
Each of these derivatives represent the stretch or compression of the fluid particle in the orthogonal direction. When these are all added up, it gives the variation in volume. If this is positive, it shows that the volume has increased with time. If ρ is constant, then the volume cannot change. Which is why equation (1.22) must equal 0.

1.3 Conservation of momentum

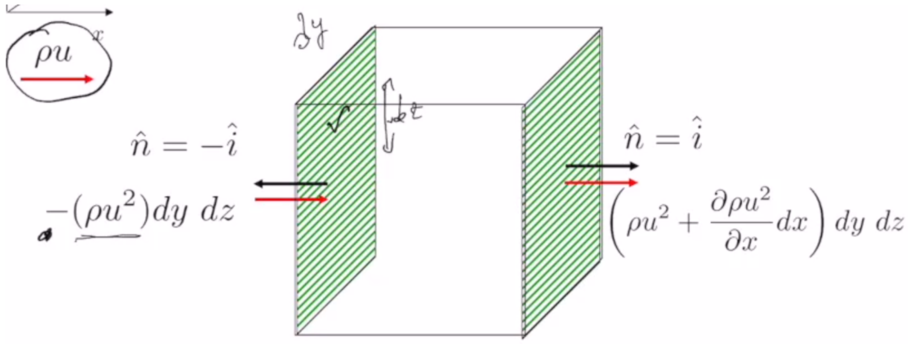
$$\frac{\partial}{\partial t}\int_V \rho\overrightarrow{V}\,\mathrm{d}V + \oint_S \rho\overrightarrow{V}(\overrightarrow{V} \cdot \hat{n})\,\mathrm{d}S = \sum \overrightarrow{F} \tag{1.23}$$

We have two types of external force that can act on our infinitesimal fluid element, **volumetric** forces (e.g. gravity) and **surface** forces (shear, pressure).

Momentum is a vector as we have to take into account momentum in 3D (first term). We want to know how these change with time. The second term looks at the flux of momentum through the sides of the control volume. In this example, we are only looking at the momentum in the x direction.

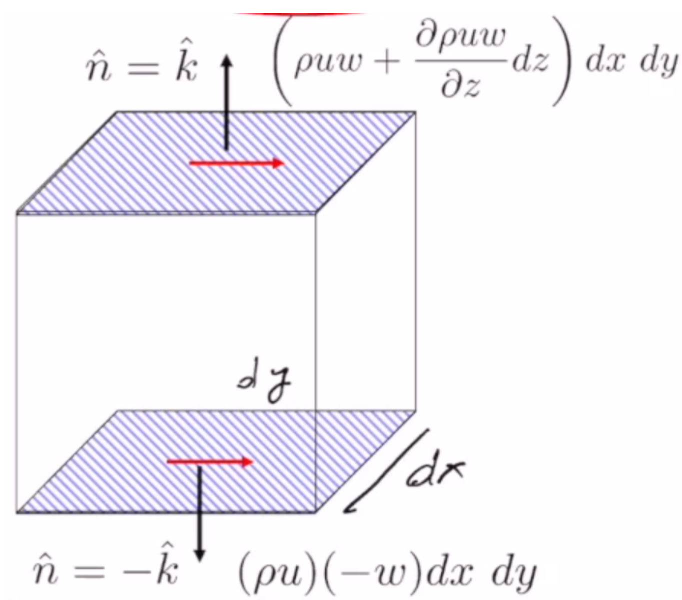


Where $\frac{\partial\rho u}{\partial t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is the momentum in x.



Adding the left and the right sides, we get

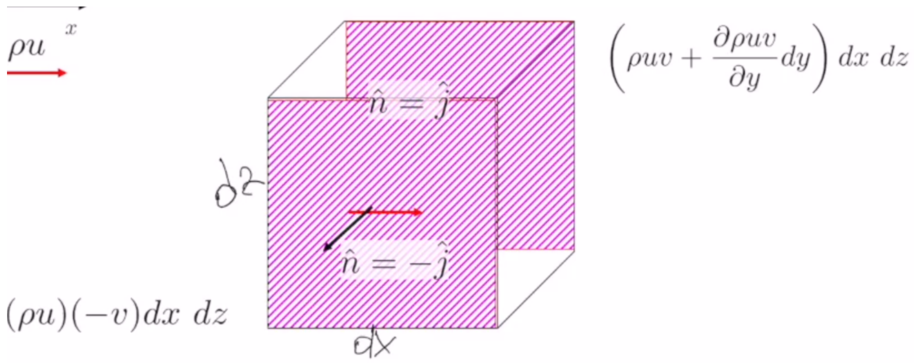
$$\rho u(\overrightarrow{V}\cdot\hat{n})\,\mathrm{d}S = \frac{\partial(\rho u^2)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.24}$$



Adding the net contribution to our equation:

$$\rho u(\overrightarrow{V}\cdot\hat{n})\,\mathrm{d}S = \frac{\partial(\rho u^2)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \frac{\partial(\rho uw)}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.25}$$

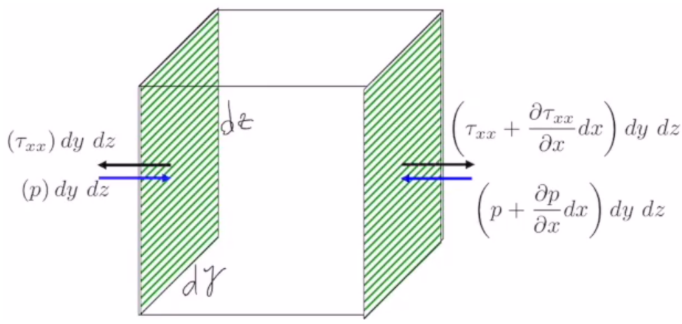
y orthogonal



Adding the net contribution to our equation:

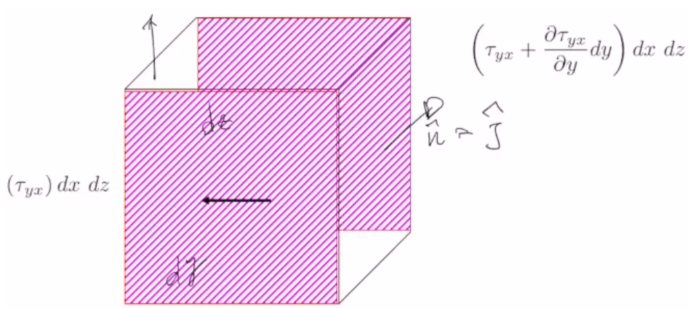
$$\rho u(\overrightarrow{V}\cdot\hat{n})\,\mathrm{d}S = \frac{\partial(\rho u^2)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \frac{\partial(\rho uw)}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \frac{\partial \rho uv}{\partial y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.26}$$

Let us look at the $\sum\overrightarrow{F}$ term. Pressure is always exerted orthogonal to a face. We also have our τ stresses acting orthogonally



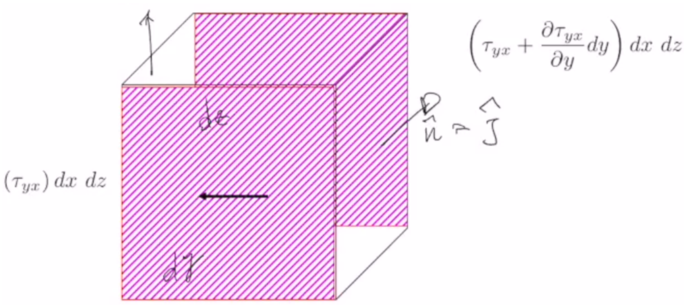
$$\sum F_x = -\left(\frac{\partial p}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \left(\frac{\partial \tau_{xx}}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.27}$$

z orthogonal shear force.



$$\sum F_x = -\left(\frac{\partial p}{\partial x}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial \tau_{xx}}{\partial x}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial \tau_{zx}}{\partial z}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{1.28}$$

y orthogonal shear force z orthogonal shear force.



$$\sum F_x = -\left(\frac{\partial p}{\partial x}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial \tau_{xx}}{\partial x}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial \tau_{zx}}{\partial z}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial \tau_{yx}}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{1.29}$$

Substituting this back into our original equation, the conservation of momentum in the x direction is:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{1.30}$$

y direction:

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho vu) + \frac{\partial}{\partial y}(\rho vv) + \frac{\partial}{\partial z}(\rho vw) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{1.31}$$

z direction (we add ρg here due to the gravitational force acting downwards).

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho wu) + \frac{\partial}{\partial y}(\rho wv) + \frac{\partial}{\partial z}(\rho ww) = -\frac{\partial p}{\partial x} - \rho g + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \tag{1.32}$$

These can be added to find that we arrive with two terms, one being the continuity equation, which must equal 0. To summarise, we have our continuity equation and momentum equations below.

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{1.33}$$

x direction momentum

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{1.34}$$

y direction momentum

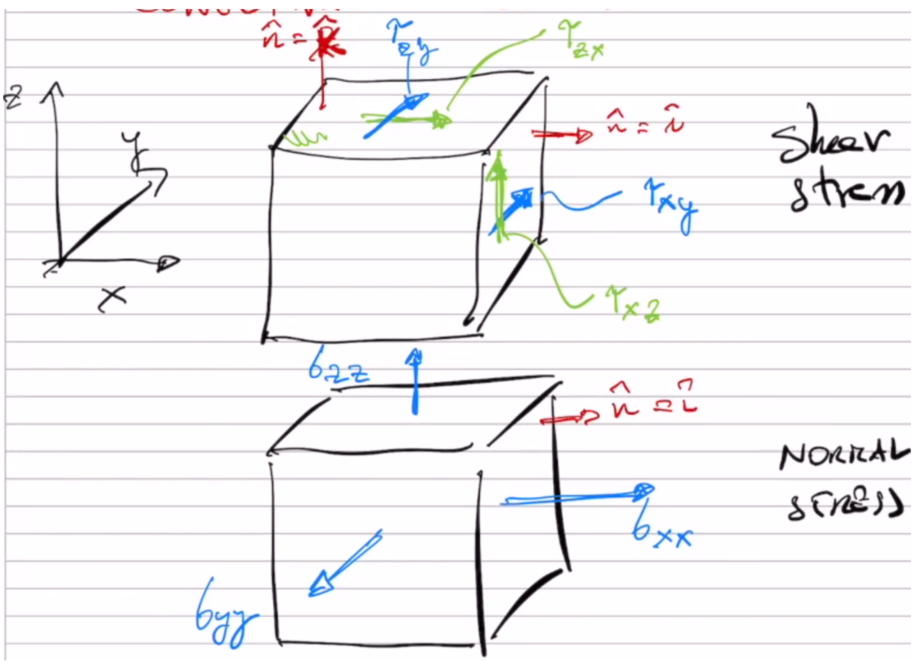
$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{1.35}$$

z direction momentum

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \tag{1.36}$$

1.4 Stress tensor notation

To identify a stress component we use a double subscript notation (tensor notation). The first subscript indicates the direction of the normal to the plane on which stress acts. Second subscript indicates the direction of the stress. Thus, the symbol τ_{ij} denotes a stress in j direction on a face normal to the i-axis.



The normal stresses two contributions are pressure p and viscous stress τ. Pressure is always negative due to it acting against the surface (if we take the arrow coming out of the surface as positive). τ accounts for the extra stress coming from viscosity.

$$\sigma_{xx} = -p + \tau_{xx} \tag{1.37}$$
$$\sigma_{yy} = -p + \tau_{yy} \tag{1.38}$$
$$\sigma_{zz} = -p + \tau_{zz} \tag{1.39}$$

Parts on the opposite sides of a stress tensor are equal.

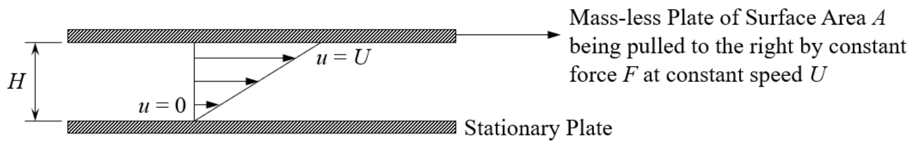
$$\tau_{xy} = \tau_{yx}, \;\; \tau_{xz} = \tau_{zx}, \;\; \tau_{yz} = \tau_{zy}$$

Chapter 2

Navier-Stokes equations

2.1 Constitutive equations

We want to find a way to link the stress tensor τ with the velocity field i.e. $\tau = f(u,\ v,\ w)$.



The angle of deformation $\Delta\theta$ can be used to derive the following:

$$\tan\Delta\theta = \frac{u \cdot \Delta T}{H} \tag{2.1}$$

$$\tan\Delta\theta = \mathrm{d}\theta = \frac{u \cdot t}{H} \rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{u}{H} \tag{2.2}$$

$$\tau = \frac{F}{A} \propto \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{u}{H} \tag{2.3}$$

$$\tau = \mu\frac{\mathrm{d}\theta}{\mathrm{d}t} = \mu\frac{u}{H} \tag{2.4}$$

$$\tau = \mu\frac{\mathrm{d}u}{\mathrm{d}y} \tag{2.5}$$

- τ is the shear stress
- $\frac{\mathrm{d}u}{\mathrm{d}y}$ is the shear rate
- μ is the dynamic viscosity and has units $\mathrm{N\,s\,m^{-2}}$
- $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity and has units $\mathrm{m^2\,s^{-1}}$

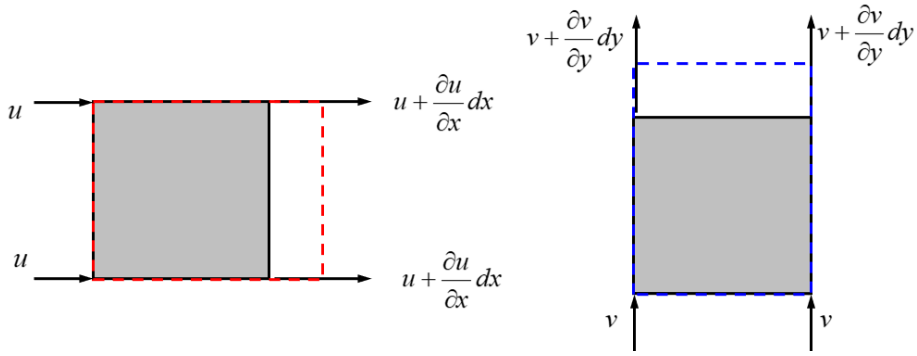
For Newtonian fluids, μ is constant. In the case above, our stress tensor is τ_{yx}, hence:

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} \tag{2.6}$$

Our velocity gradient can be defined as:

$$\nabla \vec{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{2.7}$$

The left diagonal components are the normal deformation, orthogonal to the surface.



A simplified way of writing these left diagonal terms is

$$\nabla \cdot \vec{V} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{2.8}$$

The repeated index i means sum in the x, y and z directions.

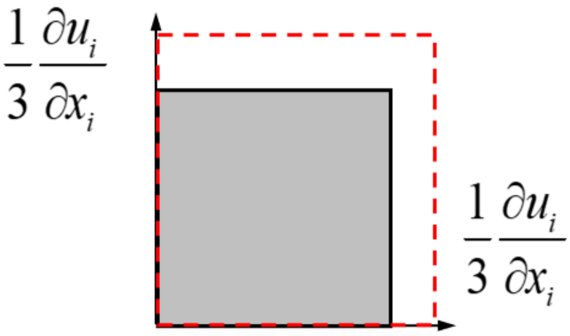
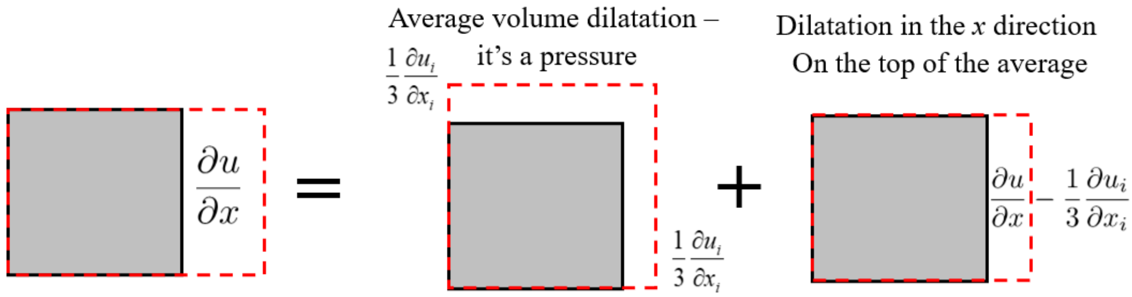


Figure 2.1: 1/3 symbolises the average deformation in x, y and z.

To find $\frac{\partial u}{\partial x}$, we can do the following

$$\frac{\partial u}{\partial x} = \frac{1}{3}\frac{\partial u_i}{\partial x_i} + \left(\frac{\partial u}{\partial x} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\right) \tag{2.9}$$



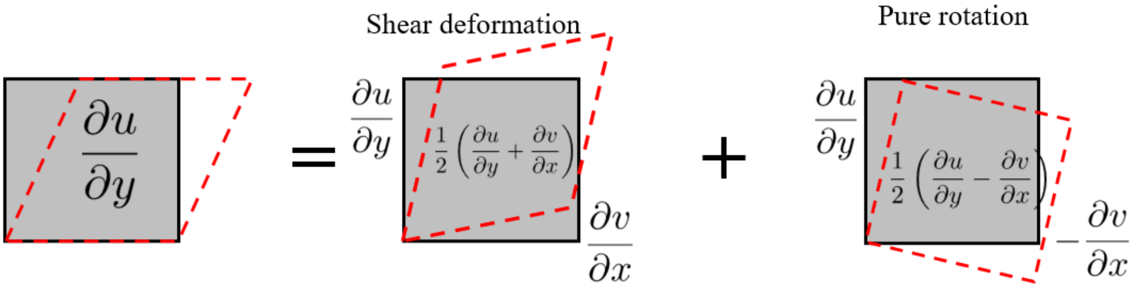
This can be also done for the other two orthogonal directions

$$\frac{\partial v}{\partial y} = \frac{1}{3}\frac{\partial u_i}{\partial x_i} + \left(\frac{\partial v}{\partial y} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\right) \tag{2.10}$$

$$\frac{\partial w}{\partial z} = \frac{1}{3}\frac{\partial u_i}{\partial x_i} + \left(\frac{\partial w}{\partial z} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\right) \tag{2.11}$$

Let us consider another term, such as $\frac{\partial u}{\partial y}$. We can define this as a component of deformation and rotation of the fluid particle.

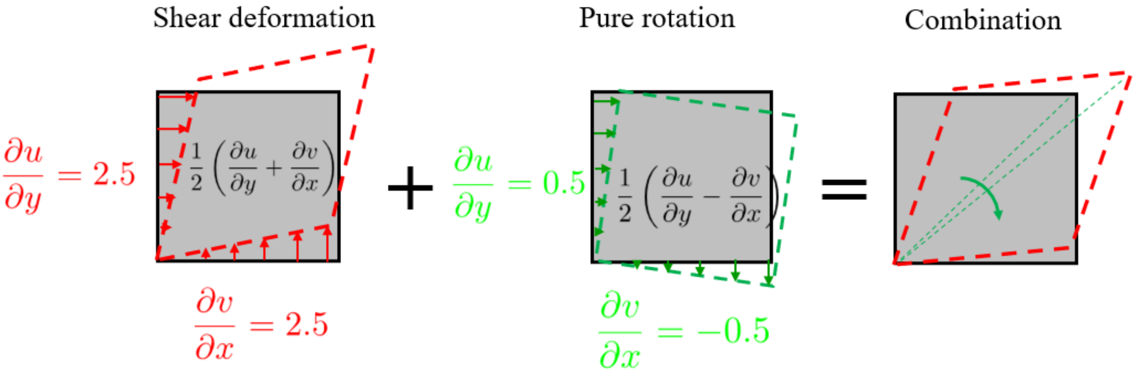
$$\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \tag{2.12}$$

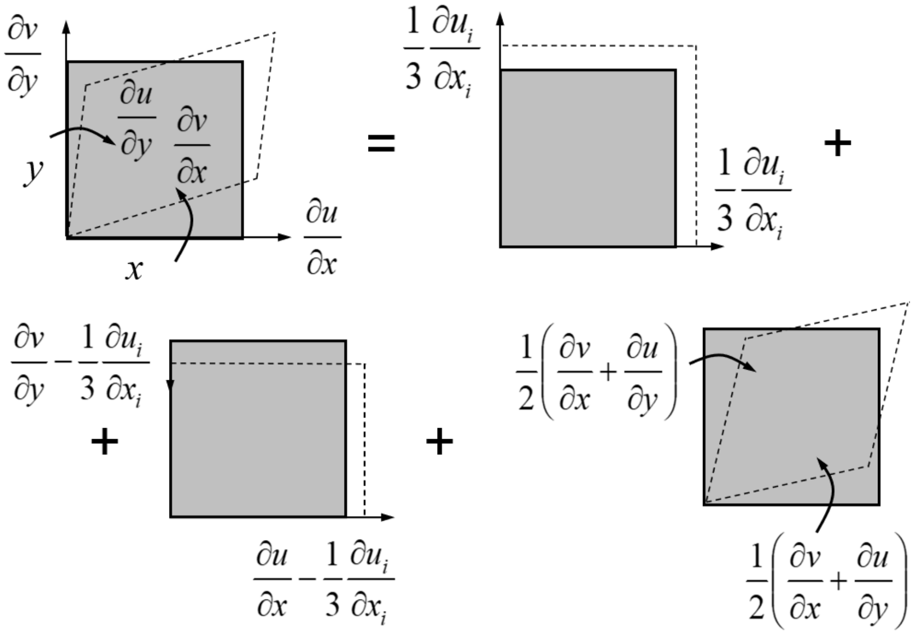


Example

$$\frac{\partial u}{\partial y} = 3 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 2.5 + 0.5 \tag{2.13}$$

$$\frac{\partial v}{\partial x} = 2 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 2.5 - 0.5 \tag{2.14}$$





2.1.1 Strain rate tensor

$$s = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial x}\right] & \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \\ \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] & \frac{\partial v}{\partial y} & \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] \\ \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] & \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] & \frac{\partial w}{\partial z} \end{bmatrix} = \tag{2.15}$$

$$\begin{bmatrix} \frac{1}{3}\nabla\cdot\vec{V} & 0 & 0 \\ 0 & \frac{1}{3}\nabla\cdot\vec{V} & 0 \\ 0 & 0 & \frac{1}{3}\nabla\cdot\vec{V} \end{bmatrix} + \begin{bmatrix} \left[\frac{\partial u}{\partial x} - \frac{1}{3}\nabla\cdot\vec{V}\right] & \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial x}\right] & \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \\ \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] & \left[\frac{\partial v}{\partial y} - \frac{1}{3}\nabla\cdot\vec{V}\right] & \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] \\ \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] & \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] & \left[\frac{\partial w}{\partial z} - \frac{1}{3}\nabla\cdot\vec{V}\right] \end{bmatrix} \tag{2.16}$$

$$\text{Deformation part which goes in pressure } \rho + \text{Deformation part which goes in the stress tensor } T \tag{2.17}$$

Compact notation of the strain rate tensor, indices $i,\ j = 1,\ 2,\ 3$

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.18}$$

2.1.2 Stress tensor

$$T = \begin{bmatrix} \tau_x x & \tau_y x & \tau_z x \\ \tau_x y & \tau_y y & \tau_z y \\ \tau_x z & \tau_y z & \tau_z z \end{bmatrix} = \tag{2.19}$$

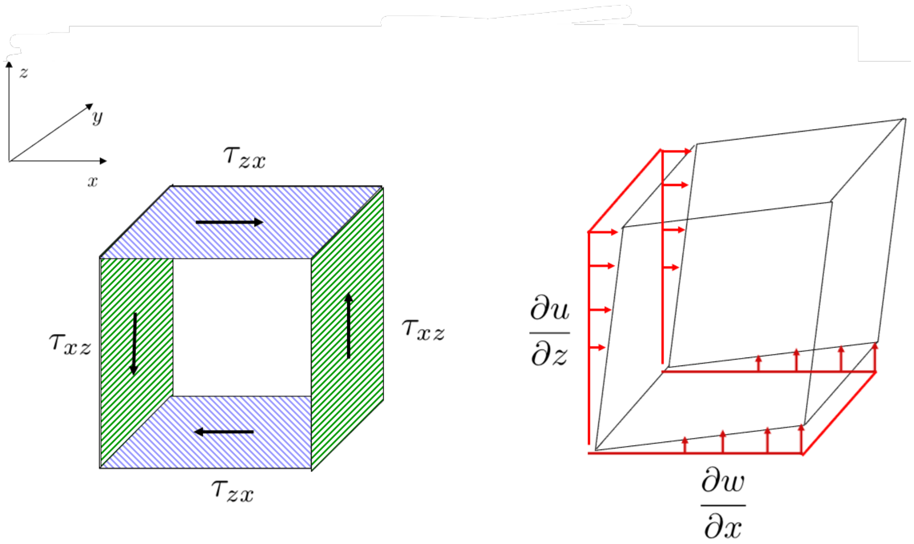
$$\begin{bmatrix} \mu\left[2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V}\right] & \mu\left[\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial x}\right] & \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \\ \mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] & \mu\left[2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\vec{V}\right] & \mu\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] \\ \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] & \mu\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] & \mu\left[2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\vec{V}\right] \end{bmatrix} \tag{2.20}$$

Compact notation for constitutive equation:

$$\tau_{ij} = 2\mu\left[s_{ij} - \frac{1}{3}(\nabla\cdot\vec{V})\delta_{ij}\right] = \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}(\nabla\cdot\vec{V})\delta_{ij}\right] \tag{2.21}$$

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{2.22}$$

The stress tensor is always symmetric along the left diagonal.



We now have 10/12 equations to describe the fluid. The final two relate to the temperature and the energy of the fluid. We will not be considering the energies of the fluid in this course. Our state equation can be $p = \rho RT$, when the fluid is compressible and if our fluid is incompressible we take ρ as constant. All in all, 11 variables and 11 equations to describe the fluid

2.2 Navier-Stokes Equations

Navier-Stokes equations are a system of equations that can be used to describe the behaviour of a fluid. They can be obtained through inserting the Constitutive Equations into the Conservation of Momentum Equations, rearranging, and simplifying them. The Navier-Stokes Equations for an incompressible fluid in 3D are as follows:

Conservation of Mass:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.23}$$

Conservation of Momentum (x, y, z):

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{2.24}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{2.25}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - \rho g \tag{2.26}$$

The Navier-Stokes Equations can be further simplified if the following occur:

- Constant Density
- Assume Steady Flow (No Time-Dependent Terms)
- Assume No External Forces
- Assume Fluid is Incompressible

The Navier-Stokes Equations in 2D, with the above assumptions applied are below:

Conservation of Mass (2D):

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.27}$$

Conservation of Momentum (x, y):

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2.28}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2.29}$$

2.3 Lagrangian vs. Eulerian

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right) \tag{2.30}$$

- $\frac{\mathrm{D}}{\mathrm{D}t}$ - Lagrangian/Material Derivative: Variation in time of a property (for example temperature, density or velocity component) of a fluid particle. The reference system is moving with the fluid particle.
- $\frac{\partial}{\partial t}$ - Eulerian Derivative: Variation in time of a property (for example temperature, density etc..) in a fixed point in space (x, y , z). Reference system fixed in space.
- $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ - Convection Terms in the x, y and z: Variation of a property due to how the particle is moving in space.

Chapter 3

Inviscid and Irrotational flow

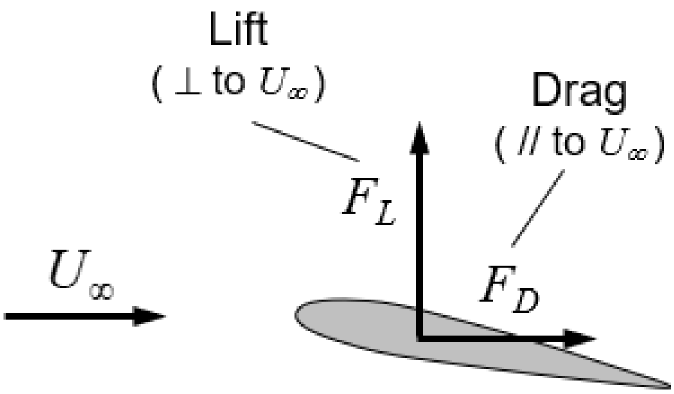
3.1 Lift and drag

Typical forces of interest for bodies in a flow are **drag** and **lift**. We can represent these in dimensionless form:

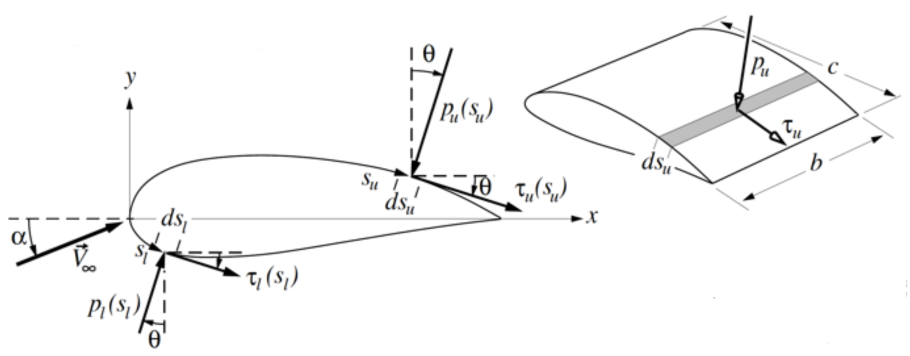
$$\text{Drag coefficient: } c_D = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 S} \tag{3.1}$$

$$\text{Lift coefficient: } c_L = \frac{F_L}{\frac{1}{2}\rho U_\infty^2 S} \tag{3.2}$$

Where S is a representative area for the body, determined by convention.



3.2 Pressure and frictional force distribution



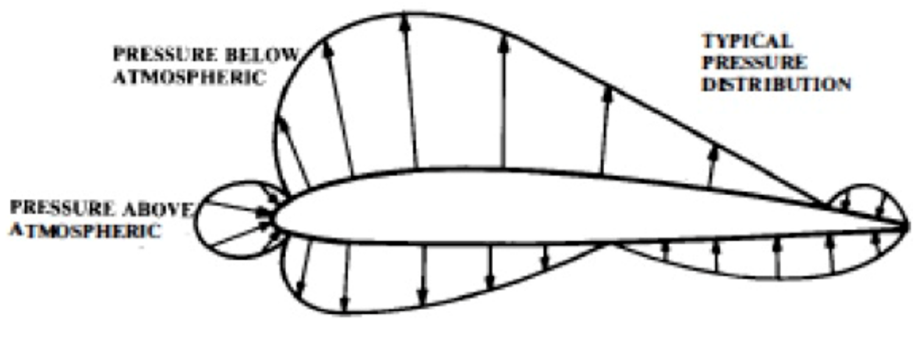
$$L = -\int_S \Big(p(\hat{n}\cdot\hat{j})\Big)\ \mathrm{d}S + \int^S \Big(\vec{\tau}\cdot\hat{j}\Big)\ \mathrm{d}S \tag{3.3}$$

$$D = -\int \Big(p(\hat{n}\cdot\hat{j})\Big)\ \mathrm{d}S + \int \Big(\vec{\tau}\cdot\hat{i}\Big)\ \mathrm{d}S \tag{3.4}$$

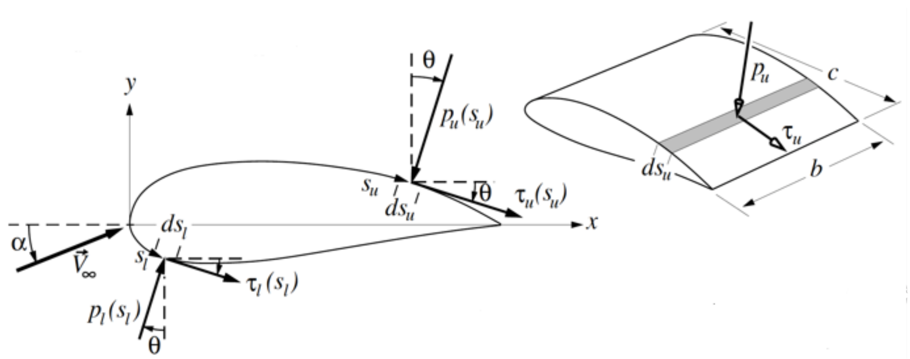
To determine the lift and drag coefficients c_L and c_D, we are interested in the pressure distribution over the airfoil, or more specifically in the local pressure difference from the stream pressure p_∞.

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} \tag{3.5}$$

Free stream pressure and velocity are p_∞ and V_∞.



- Local suction (depression): $c_p < 0$ Vectors point away from the airfoil surface
- Local pushing: $c_p > 0$ Vectors point towards the airfoil surface



$$L = -\int_S \left(p\hat{n}\cdot\hat{j}\right)\,\mathrm{d}S = \tag{3.6}$$

$$c_L = -\frac{1}{S}\int_S \left(\frac{p-p_\infty}{\frac{1}{2}\rho V_\infty^2}\hat{n}\cdot\hat{j}\right)\,\mathrm{d}S = -\frac{1}{S}\int_S \left(c_p\hat{n}\cdot\hat{j}\right)\,\mathrm{d}S \tag{3.7}$$

The lift coefficient per unit of span-wise length is:

$$c_L' - \frac{1}{c}\int_c^0 \left(c_p\hat{n}\cdot\hat{j}\right)\,\mathrm{d}x \tag{3.8}$$

3.3 Rearrangement of momentum equation - x direction

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.9}$$

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + v\frac{\partial u}{\partial y} - v\frac{\partial u}{\partial y} \tag{3.10}$$

$$= -v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial u^2}{\partial x} + \frac{\partial v^2}{\partial x}\right) \tag{3.11}$$

$$= -v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2) \tag{3.12}$$

(u^2+v^2) is the total kinetic energy of the fluid particle. The derivative is the element that takes into the account the variation of this kinetic energy. $\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ relates to the rotation of the particle. This rotation is related to the difference of velocity gradient.

$$\rho\left[-v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2)\right] = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.13}$$

$$-v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.14}$$

Our Bernoulli term in the above equation is $\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right)$, gravitational energy is negligible. $\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ is an anti-clockwise rotation. Hence, the vorticity component in the z direction is:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{3.15}$$

Our final momentum equations in x and y are:

$$-v\omega_z = -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.16}$$

$$u\omega_z = -\frac{\partial}{\partial y}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3.17}$$

We can make some assumptions:

- Inviscid flow - $\nu = 0$ (this may be realistic in some parts of a fluid domain but in real life, inviscid fluids do not exist)
- Irrotational flow - $\omega_z = 0$

This reduces our equations to:

$$0 = -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + 0 \tag{3.18}$$

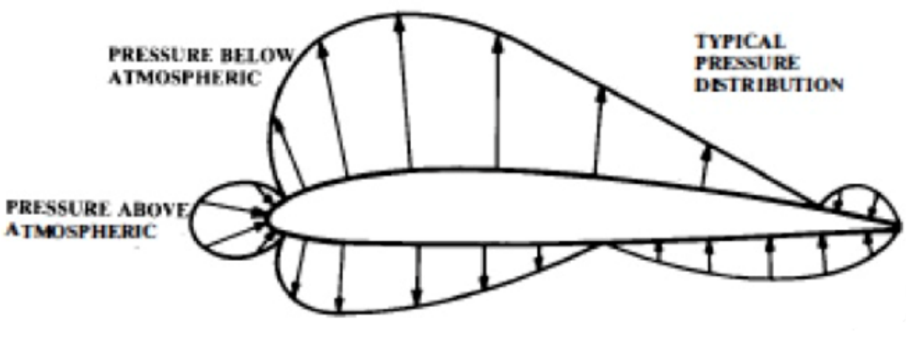
$$0 = -\frac{\partial}{\partial y}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + 0 \tag{3.19}$$

3.4 Application of Bernoulli

$$p_\infty + \frac{1}{2}\rho V_\infty^2 = p + \frac{1}{2}\rho(u^2+v^2) = \text{constant} \tag{3.20}$$

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{u^2+v^2}{V_\infty^2} = 1 - \frac{||V||^2}{V_\infty^2} \tag{3.21}$$

If $c_p < 0$, $||V|| > V_\infty$ and vice versa. If a fluid particle enters a region where c_p is negative it is accelerated and when c_P is positive it will lose velocity relative to the free stream.



Extending this to 3D, we can derive the 3D vorticity equation. We sum the momentum equations with the assumptions above and take the ijk components as so:

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} \tag{3.22}$$

If $\vec{\omega} = 0$ then a potential function, $\phi(x, y, z)$ exists such:

$$\begin{cases} u = \frac{\partial \phi(x,y,z)}{\partial x} \\ v = \frac{\partial \phi(x,y,z)}{\partial y} \\ w = \frac{\partial \phi(x,y,z)}{\partial z} \end{cases} \tag{3.23}$$

By plugging in these equations into our continuity equation, we get:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y} + \frac{\partial^2 \phi}{\partial z} \tag{3.24}$$

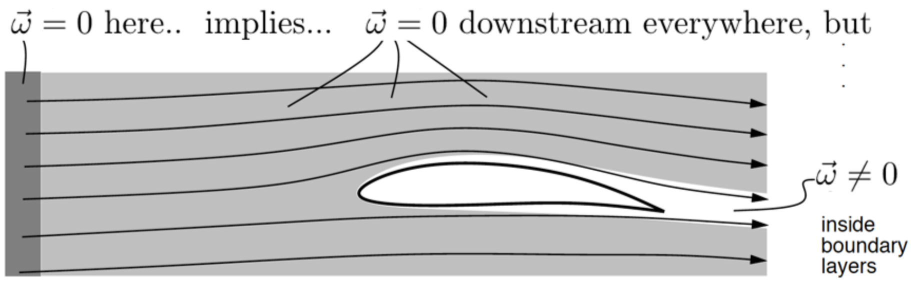
Conservation of momentum (Bernoulli term) is simply:

$$p + \frac{1}{2}\rho(u^2 + v^2) = \text{constant} \tag{3.25}$$

3.5 Applicability of irrotational flow

The flow domain can be subdivided into two parts.

- **Irrotational flow region**, outside of the boundary layer, Bernoulli equation, potential flow and stream function apply.
- **Boundary layer**, layer where all vorticity is confined. The friction shear the airfoil surface acts as a source of vorticity.



In the case where we consider our fluid inviscid, there is no shear stress being applied on the fluid by the airfoil. We need to understand how the no-slip condition changes. We still need a boundary condition to know the value of ϕ in our flow domain.

3.6 No-Slip condition for inviscid flow

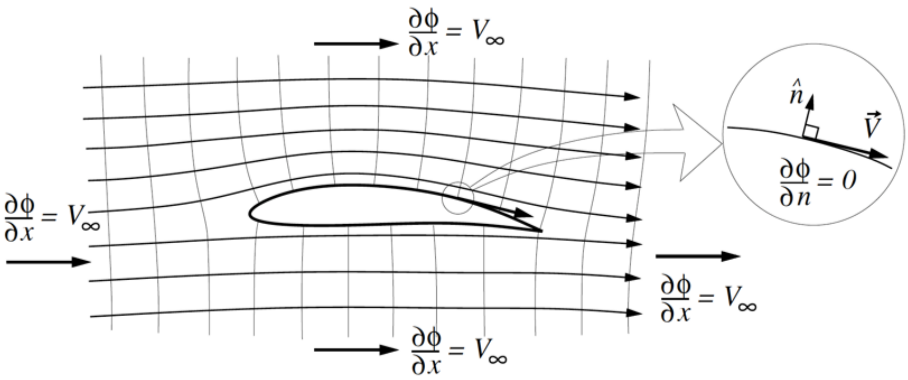
Viscous fluid

$$\nu \neq 0 \to \vec{V} = 0 \tag{3.26}$$

Inviscid flow

$$\nu = 0 \to V_n = \vec{V} \cdot \hat{n} = \frac{\partial \phi}{\partial n} = 0 \tag{3.27}$$

In essence, with inviscid flow, we are accepting that there is some movement on the boundary, however this is only parallel to the surface. n is the direction orthogonal to the boundary.



3.7 Stream function

In a 2D flow a stream function, $\psi(x, y)$, can be defined which is always aligned/parallel with the local velocity vector and visualise a streamline. Different streamlines are identified with different values of $\psi(x, y)$

$$\begin{cases} u = \frac{\partial\psi(x,y)}{\partial x} \\ v = \frac{\partial\psi(x,y)}{\partial y} \end{cases} \tag{3.28}$$

Iso-potential lines and streamlines are orthogonal to each other. Streamlines visualise the trajectory of a particle in the field.

3.8 Potential flow past bodies

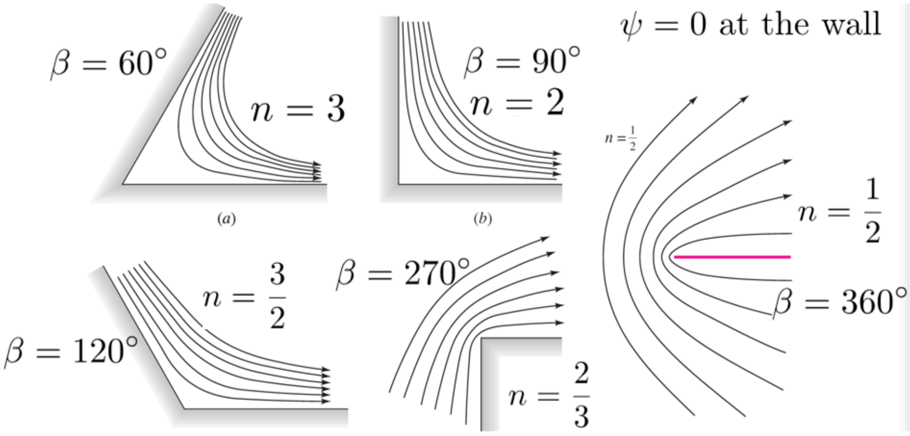
Flow fields for which an incompressible fluid is assumed to be frictionless and the motion to be irrotational are commonly referred to as **potential** flows. Paradoxically, potential flows can be simulated by a slow moving, viscous flow between closely spaced parallel plates.

3.9 Flow around a corner of arbitrary angle, β

Considering a radial coordinate system:

$$\phi = Ar^n \cos(n\theta) \tag{3.29}$$

$$\psi = Ar^n \sin(n\theta) \tag{3.30}$$



3.10 Cylindrical coordinates

3D vorticity equation:

$$\vec{\omega} = \omega_r \hat{i}_r + \omega_\theta \hat{i}_\theta + \omega_z \hat{i}_z \tag{3.31}$$

$$\vec{\omega} = \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \hat{i}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{i}_\theta + \frac{1}{r} \left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right) \hat{i}_z \tag{3.32}$$

Potential flow function and stream function:

$$\text{Potential flow} \begin{cases} u_r = \frac{\partial \phi}{\partial r} \\ u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \\ u_z = \frac{\partial \phi}{\partial z} \end{cases} \quad \text{Stream function} \begin{cases} u_\theta = -\frac{\partial \psi}{\partial r} \\ u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \end{cases} \tag{3.33}$$

Conservation of mass (continuity equation):

$$\nabla \cdot \vec{V} = \frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{3.34}$$

Chapter 4

Modelling the flow around a bluff body

27/10/2020

4.1 Uniform Flow

Cartesian Coordinates:

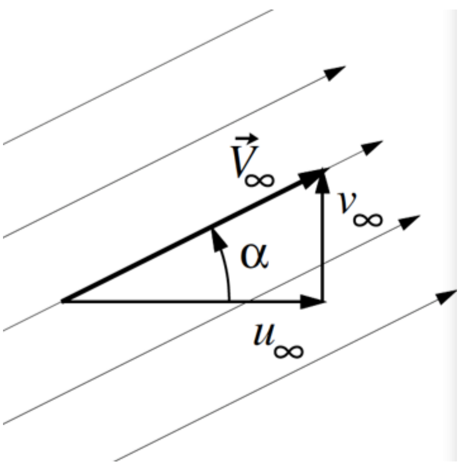
$$\phi = V_\infty \left[x\cos(\alpha) + y\sin(\alpha) \right] \tag{4.1}$$
$$\psi = V_\infty \left[y\cos(\alpha) - x\sin(\alpha) \right] \tag{4.2}$$

The conservation of mass is balanced:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.3}$$

The flow is irrotational:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{4.4}$$

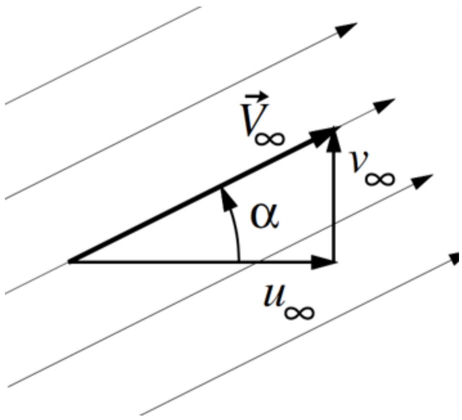


Cylindrical Coordinates:

$$\phi(r,\theta) = V_\infty r\cos(\theta-\alpha) \tag{4.5}$$
$$\psi(r,\theta) = V_\infty r\sin(\theta-\alpha) \tag{4.6}$$

The conservation of mass is satisfied for cylindrical coordinates:

$$\nabla\cdot\overrightarrow{V} = \frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \tag{4.7}$$
$$= \frac{\partial r(v_\infty \cos(\theta-\alpha))}{\partial r} - \frac{\partial v_\infty \sin(\theta-\alpha)}{\partial \theta} \tag{4.8}$$
$$v_\infty\cos(\theta-\alpha) - v_\infty\cos(\theta-\alpha) = 0 \tag{4.9}$$



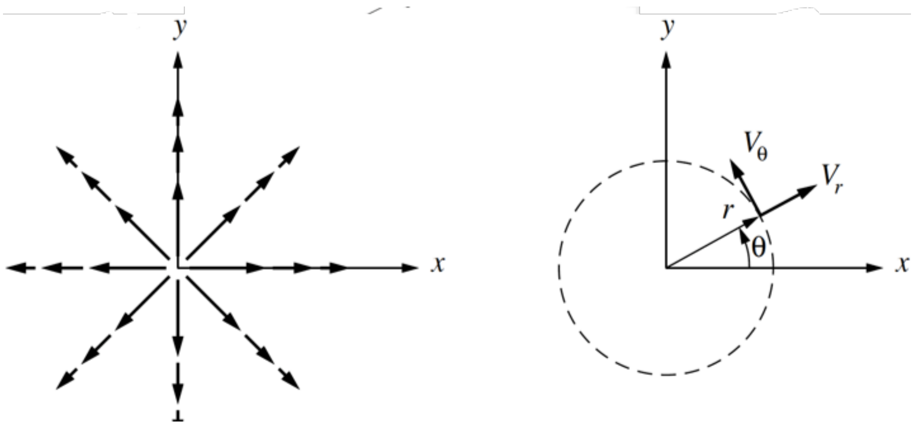
4.2 Source/Sink Flow

Cartesian Coordinates:

$$\phi = \frac{\Lambda}{2\pi}\ln(\sqrt{x^2+y^2}) \tag{4.10}$$
$$\psi = \frac{\Lambda}{2\pi}\arctan\left(\frac{y}{x}\right) \tag{4.11}$$

Cylindrical Coordinates:

$$\phi = \frac{\Lambda}{2\pi}\ln(r) \tag{4.12}$$
$$\psi = \frac{\Lambda}{2\pi}\theta \tag{4.13}$$



In cylindrical coordinates, we do not have a θ component as it is moving radially outwards from a source.

$$u_r = \frac{\partial \phi}{\partial r} = \frac{\Lambda}{2\pi r} \tag{4.14}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \tag{4.15}$$

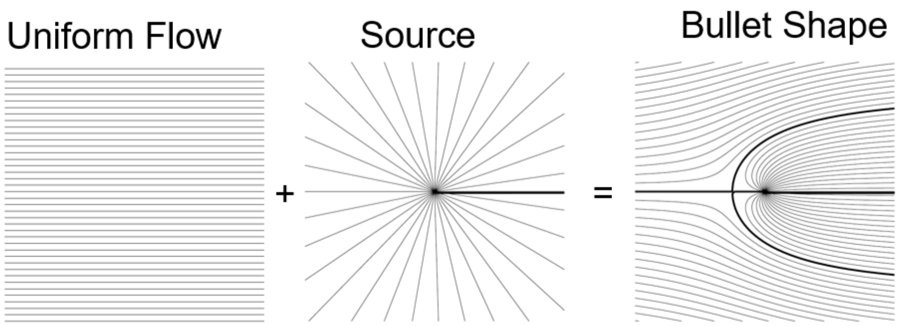
Here we can see the magnitude of the velocity is dependent on $\frac{1}{r}$. If $\Lambda > 0$, we have a source and if $\Lambda < 0$, we have a sink.

4.3 Uniform Flow + Source

$$\phi(r,\theta) = \frac{\Lambda}{2\pi}\ln(r) + V_\infty r\cos\theta \tag{4.16}$$

$$\psi(r,\theta) = \frac{\Lambda}{2\pi}\theta + V_\infty r\sin\theta \tag{4.17}$$

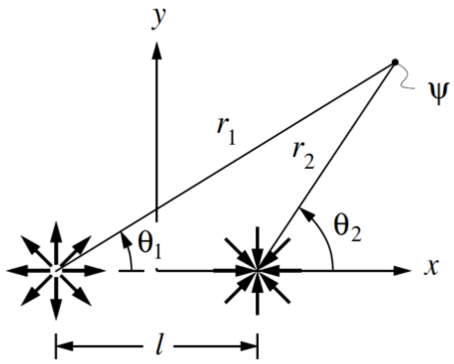
Stream function $\psi(r,\theta)$ of:

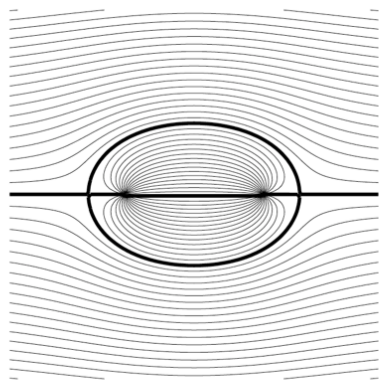


4.4 Uniform Flow + Source + Sink

$$\phi - V_\infty r\cos\theta + \frac{\Lambda}{2\pi}(\ln(r_1) - \ln(r_2)) \tag{4.18}$$

$$\psi = V_\infty r\sin\theta + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) \tag{4.19}$$



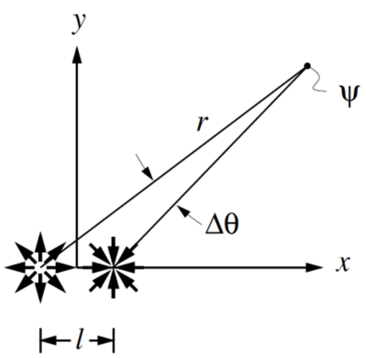


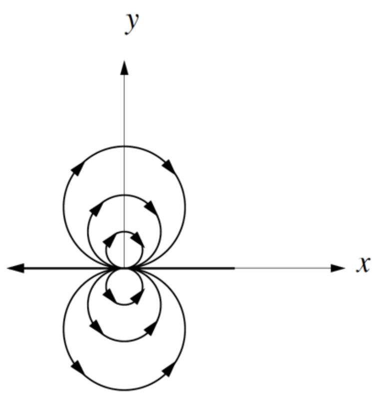
4.5 Doublet

Consider a pair of source and sink of $\pm\Lambda$ who are l apart and $l \times \Lambda =$ constant.

$$\psi = \lim_{l\to 0}\frac{\Lambda}{2\pi}(\theta_1 - \theta_2) = -\frac{k}{2\pi}\frac{\sin\theta}{r} \tag{4.20}$$

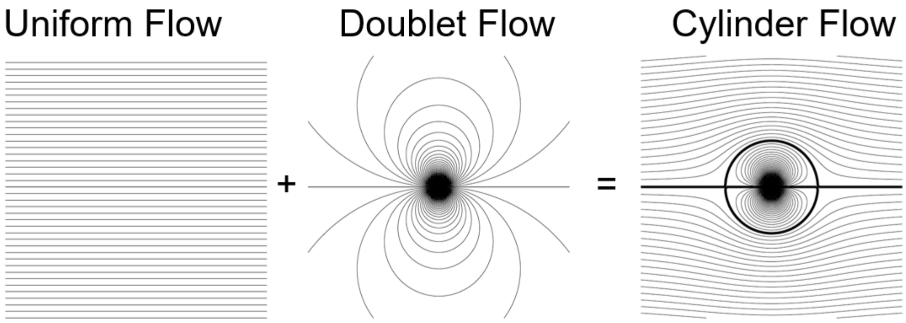
$$\phi = \frac{k}{2\pi}\frac{\cos\theta}{r} \tag{4.21}$$





4.6 Cylinder (Uniform Flow + Doublet)

$$\phi = V_\infty r\cos\theta + \frac{k}{2\pi}\frac{\cos\theta}{r} \tag{4.22}$$
$$\psi = V_\infty r\sin\theta - \frac{k}{2\pi}\frac{\sin\theta}{r} \tag{4.23}$$



The radius of the cylinder can be derived as so:

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty\cos\theta - \frac{k}{2\pi}\frac{\cos\theta}{r^2} \tag{4.24}$$
$$u_\theta = -\frac{\partial\psi}{\partial r} = -\left(V_\infty\sin\theta + \frac{k}{2\pi}\frac{\sin\theta}{r^2}\right) \tag{4.25}$$

On the cylinder, $\vec{u}\cdot\hat{n} = 0$

$$\hat{n} = \hat{i}_r \rightarrow u_r(R) = 0 \tag{4.26}$$
$$V_\infty\cos\theta - \frac{k}{2\pi}\frac{\cos\theta}{R^2} = 0 \tag{4.27}$$
$$R = \sqrt{\frac{k}{2\pi V_\infty}} \tag{4.28}$$

We can rewrite ϕ and ψ

$$\phi = V_\infty r\cos\theta\left(1 + \frac{R^2}{r^2}\right) \tag{4.29}$$
$$\psi = V_\infty r\sin\theta\left(1 - \frac{R^2}{r^2}\right) \tag{4.30}$$

On the cylinder surface, $r = R$ and inputting this into ψ:

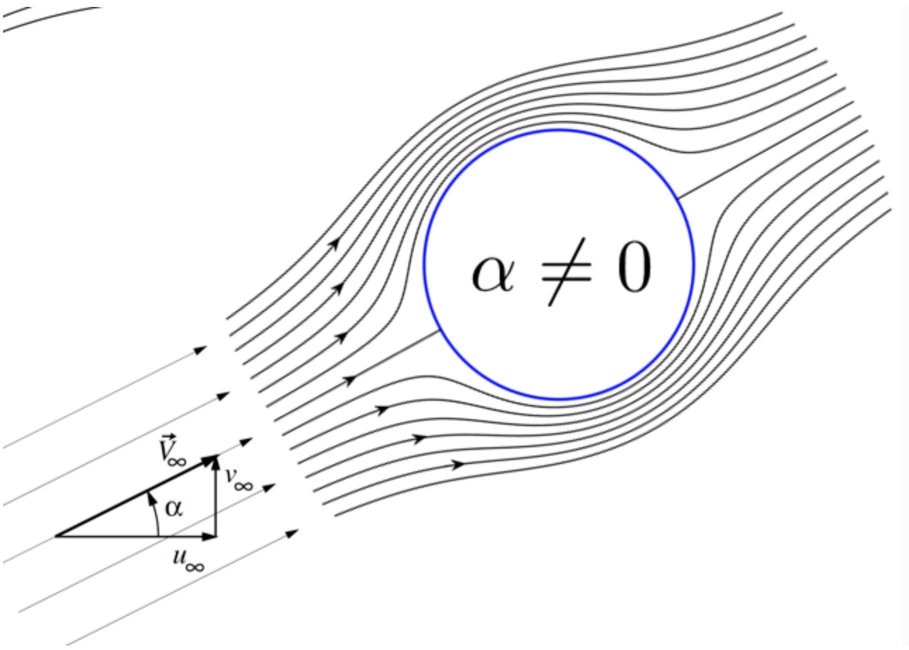
$$\psi = 0 \tag{4.31}$$

4.7 Uniform Stream with Varying Direction

All we need to do to generalise our equations a bit more is to rewrite our equations with an extra angular term, α:

$$\phi = V_\infty r \cos(\theta - \alpha)\left(1 + \frac{R^2}{r^2}\right) \tag{4.32}$$

$$\psi = V_\infty r \sin(\theta - \alpha)\left(1 - \frac{R^2}{r^2}\right) \tag{4.33}$$



4.8 Adding Circulation with a Vortex Flow

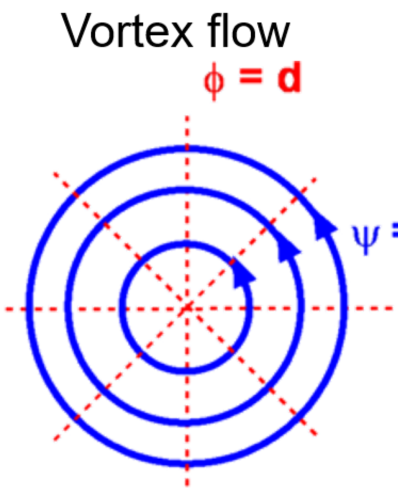
$$\phi = -\frac{\Gamma}{2\pi}\theta \tag{4.34}$$

$$\psi = \frac{\Gamma}{2\pi}\ln\left(\frac{r}{R}\right) \tag{4.35}$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0 \tag{4.36}$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -\frac{\Gamma}{2\pi r} \tag{4.37}$$

Where $\Gamma < 0$ is anti-clockwise motion and $\Gamma > 0$ is clockwise motion.



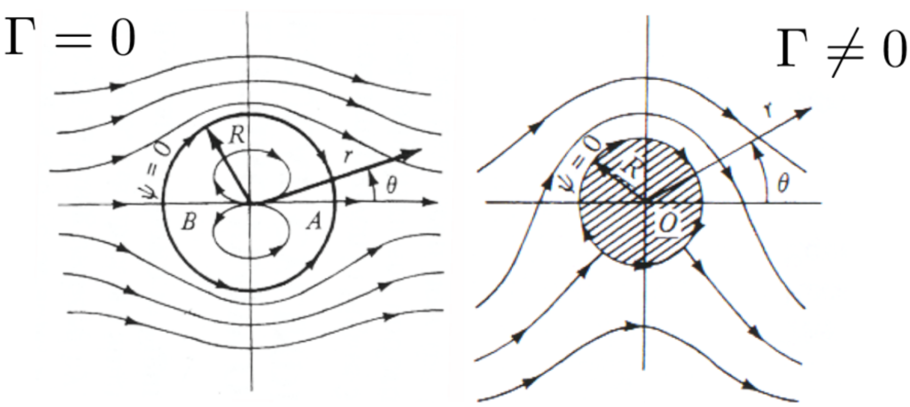
4.9 Cylinder with a Vortex Flow

$$\psi = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi} \ln\left(\frac{r}{R}\right) \tag{4.38}$$

$$\phi = V_\infty r \cos\theta \left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi}\theta \tag{4.39}$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty \cos\theta \left(1 - \frac{R^2}{r^2}\right) \tag{4.40}$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -V_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right) \tag{4.41}$$



4.10 Lift and Drag of a Cylinder with Circulation

Apply Bernoulli:

$$p_\infty + \frac{1}{2}\rho V_\infty^2 = p(r,\theta) + \frac{1}{2}\rho(u_r^2 + u_\theta^2) \tag{4.42}$$

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{u_r^2 + u_\theta^2}{V_\infty^2} \tag{4.43}$$

On the cylinder surface: $u_r = 0$

$$c_p(R,\theta) = 1 - \frac{u_\theta^2}{V_\infty^2} = 1 - \frac{(2V_\infty \sin\theta + \frac{\Gamma}{2\pi R})^2}{V_\infty^2} \tag{4.44}$$

$$= 1 - \left(4\sin^2(\theta) + \frac{\Gamma^2}{4\pi^2 V_\infty^2 R^2} + \frac{2\Gamma\sin(\theta)}{V_\infty \pi R}\right) \tag{4.45}$$

4.11 Lift of the Cylinder

We need to calculate c_p on the surface of the cylinder.

$$L = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(c_p \cdot \hat{n} \cdot \hat{j} R\right) \mathrm{d}\theta \tag{4.46}$$

$$= -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(c_p \sin\theta R\right) \mathrm{d}\theta \tag{4.47}$$

$$L = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(1 - \frac{(2V_\infty \sin\theta + \frac{\Gamma}{2\pi R})^2}{V_\infty^2}\right) \sin\theta R \,\mathrm{d}\theta \tag{4.48}$$

Expanding the integral out, we arrive at:

$$L = -\frac{1}{2}\rho V_\infty^2 \left[\int_0^{2\pi} \left(1 - \frac{\Gamma^2}{4\pi^2 R^2 V_\infty^2}\right) \cdot \sin\theta \cdot R\,\mathrm{d}\theta + \int_0^{2\pi} -4\sin\theta^3 \cdot R\,\mathrm{d}\theta + \int_0^{2\pi} -\frac{2\Gamma}{\pi R V_\infty} \cdot \sin\theta^2 \cdot R\,\mathrm{d}\theta\right] \tag{4.49}$$

Because the first term and the second term have an odd power of sin, when we integrate these, they will have negligible outcome on the lift of the cylinder. We can reduce our equation to:

$$L = -\frac{1}{2}\rho V_\infty^2 \left[\int_0^{2\pi} -\frac{2\Gamma}{\pi R V_\infty} \cdot \sin\theta^2 \cdot R\,\mathrm{d}\theta\right] \tag{4.50}$$

$$= \frac{\rho V_\infty \Gamma}{\pi} \int_0^{2\pi} \sin\theta^2 \,\mathrm{d}\theta \tag{4.51}$$

$$= \frac{\rho V_\infty \Gamma}{\pi} \left[\frac{1}{2}\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} \tag{4.52}$$

$$L = \rho V_\infty \Gamma \tag{4.53}$$

We can see here that our vortex factor Γ has a proportional effect on our lift force. We can see an example of this in real life when a football is kicked. When the ball is kicked in such a way that it has a spin, we see the ball curves in certain directions.

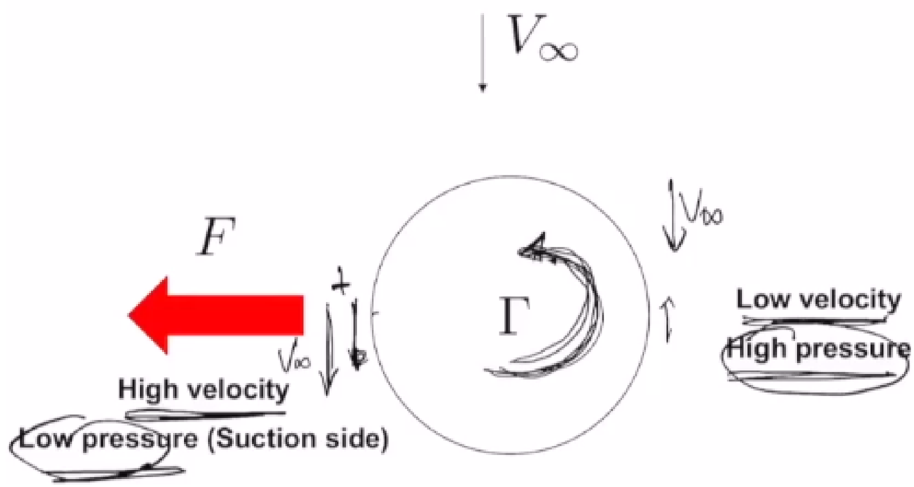
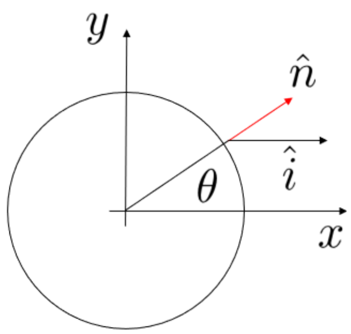


Figure 4.1: We can see that when the ball spins, the velocity from the free stream and the vortex combine to create regions of low and high pressure. This creates a net force, leading to a suction effect.

4.12 Drag of a cylinder



$$D = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} c_p \vec{n} \cdot \hat{i} R \, \mathrm{d}\theta \tag{4.54}$$

$$= -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} c_p \cos\theta R \, \mathrm{d}\theta \tag{4.55}$$

$$D = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(1 - \frac{\left(2V_\infty \sin\theta + \frac{\Gamma}{2\pi R}\right)^2}{V_\infty^2}\right) \cos\theta R \, \mathrm{d}\theta \tag{4.56}$$

The first term has an odd power of cosine, and so is negligible. The second term of the drag integral is:

$$\int_0^{2\pi} -4\sin\theta^2 \cos\theta R \, \mathrm{d}\theta = -4R\left(\frac{1}{3}\sin\theta^3\right)_p^{2\pi} = 0 \tag{4.57}$$

The third term of the drag integral is:

$$\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} -\frac{2\Gamma}{\pi R V_\infty} \sin\theta \cos\theta R \, \mathrm{d}\theta = -\frac{\rho V_\infty \Gamma}{\pi} \left(-\frac{1}{2} \cos 2\theta \right)_0^{2\pi} = 0 \tag{4.58}$$

Therefore, we see that our drag is in fact:

$$D = 0 \tag{4.59}$$

This is due to our assumption that our flow is inviscid and that the pressure forces are symmetrical to the left and right sides to the y axis. Therefore the net forces acting on the x direction is zero. This is where our model starts failing and is called the D'Alambert Paradox.

4.13 Inviscid and Viscous Flow past a body

High Re implies that the magnitude of the inertia forces are much greater than the magnitudes of the viscous forces in a system. This might imply that the effects of viscosity are insignificant compared with the inertia forces, but this would be a dangerous conclusion. Compare theory for zero viscosity with experiment for high Re flow past a cylinder:

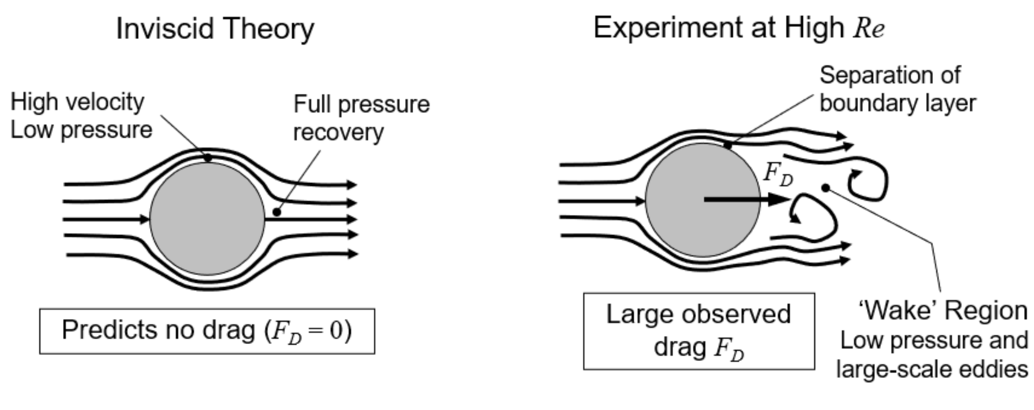
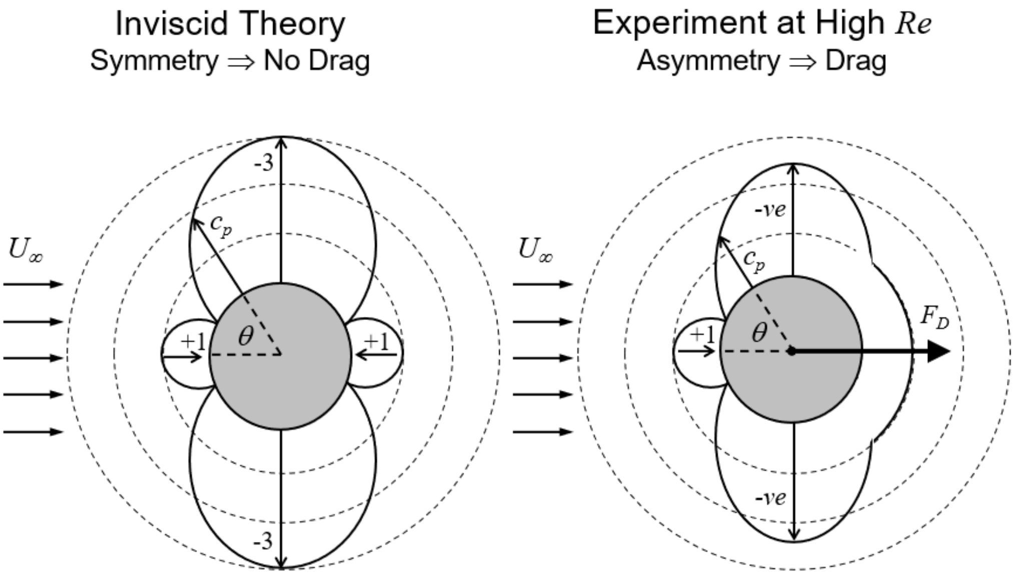


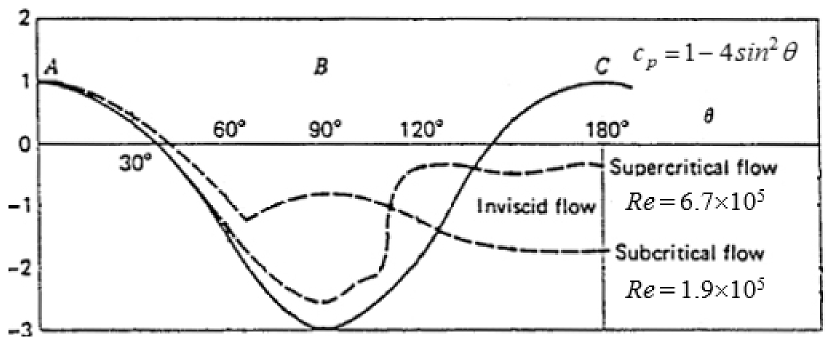
Figure 4.2: We have a net force in our experiment due to the fact that a net force is created from the low pressure wake region and the high pressure front side of the cylinder.

4.14 Flow past a cylinder - pressure coefficient



We can plot the pressure coefficient and see the difference between the inviscid theory and an experiment at high Re.

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^{\ 2}}$$

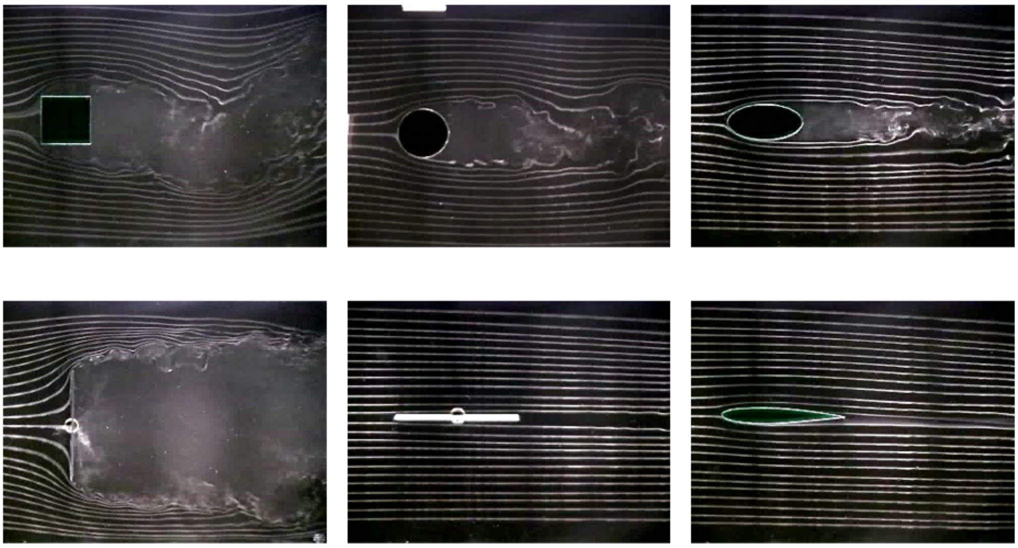


For a long circular cylinder, the lift coefficient c_L and the form drag coefficient to c_D are related to c_p by:

$$c_L = \frac{1}{2}\int_0^{2\pi} c_p \sin\theta \,\mathrm{d}\theta \tag{4.60}$$

$$c_D = \frac{1}{2}\int_0^{2\pi} c_p \cos\theta \,\mathrm{d}\theta \tag{4.61}$$

Some examples of viscous flow past bodies:



Chapter 5

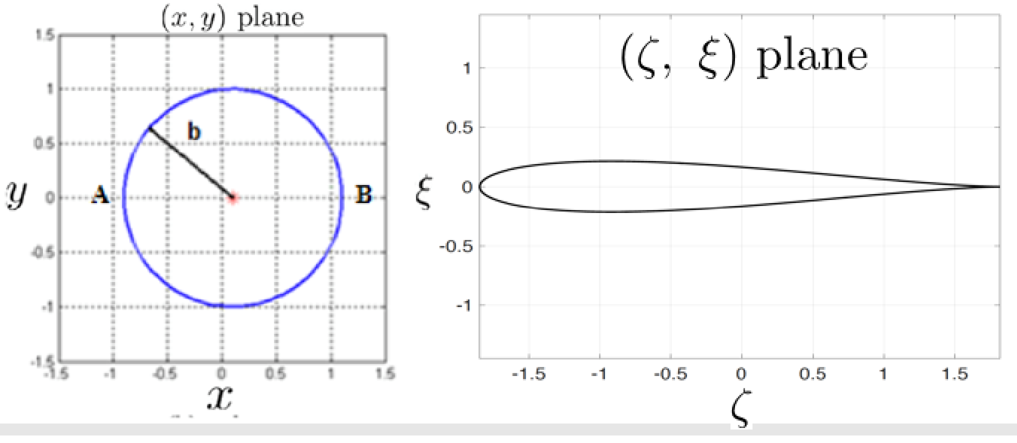
Modelling the flow around a streamlined body

5.1 Conformal mapping

5.1.1 Joukowsky Transformation

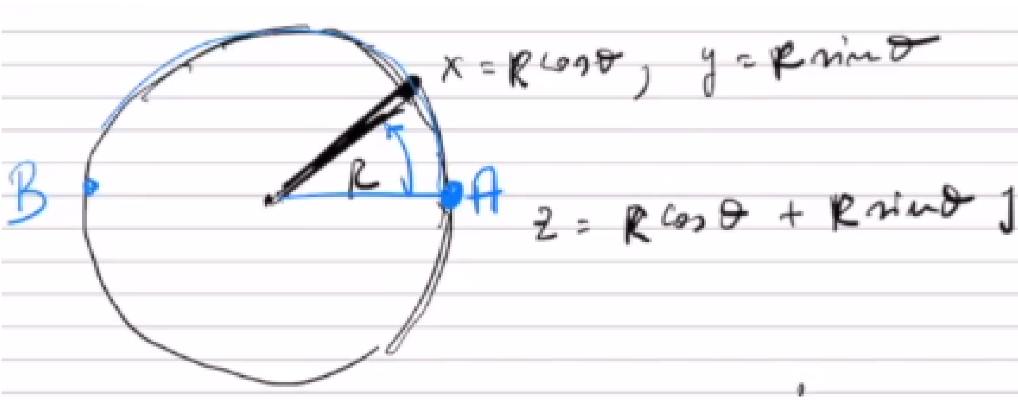
The question is how can we transform our cylinder into something that looks like an airfoil? We achieve this by using conformal mapping which maps each part of our coordinate space to a new one.

$$x = r\cos\theta \tag{5.1}$$
$$y = r\sin\theta \tag{5.2}$$
$$w = G(z) = z + \frac{\lambda^2}{z} \tag{5.3}$$
$$z = re^{i\theta} = x + yi \tag{5.4}$$
$$w = \zeta + \xi i \tag{5.5}$$



5.1.2 Flat plate and Ellipse

Lets consider a cylinder:



$$z = R\cos\theta + Ri\sin\theta \tag{5.6}$$

$$w = G(z) = z + \frac{\lambda^2}{z} \tag{5.7}$$

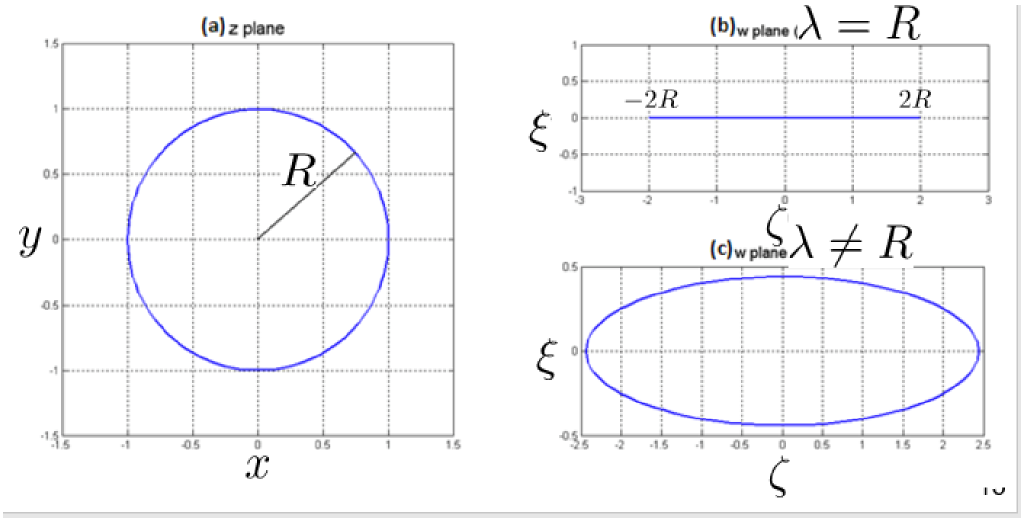
$$= R\cos\theta + Ri\sin\theta + \frac{\lambda^2}{R\cos\theta + Ri\sin\theta} \tag{5.8}$$

$$= R\cos\theta + Ri\sin\theta + \frac{\lambda^2(R\cos\theta - Ri\sin\theta)}{(R\cos\theta + Ri\sin\theta)(R\cos\theta - Ri\sin\theta)} \tag{5.9}$$

$$= R\cos\theta + Ri\sin\theta + \frac{\lambda^2(R\cos\theta - Ri\sin\theta)}{R^2} \tag{5.10}$$

$$w = R\cos\theta\left(1 + \frac{\lambda^2}{R}\right) + Ri\sin\theta\left(1 - \frac{\lambda^2}{R}\right) \tag{5.11}$$

When $\lambda = R$, we achieve a flat plate transformation as our imaginary term is always 0. When $\lambda \neq R$, we achieve an elliptical shape.



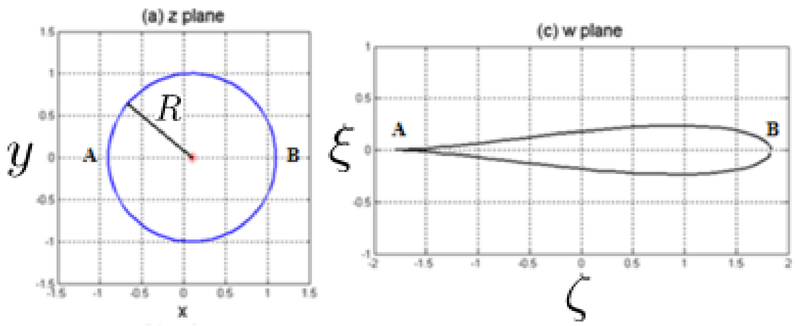
5.1.3 Aerofoils

We can further refine our conformal mapping by adding two elements to our equation, x_c and y_c. These two terms effect the center of the cylinder in x and y.

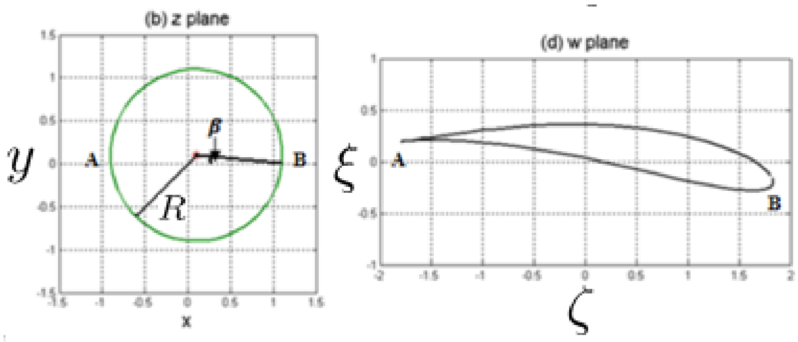
$$w = G(z) = z + \frac{\lambda^2}{z} \tag{5.12}$$
$$\lambda = R - \sqrt{x_c^2 + y_c^2} \tag{5.13}$$
$$z = x + x_c + (y + y_c)i \rightarrow w = \zeta + \xi i \tag{5.14}$$

Mapping these into their respective planes:

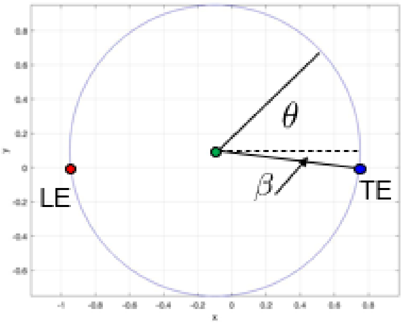
$$x_c \neq 0 \tag{5.15}$$
$$y_c = 0 \tag{5.16}$$



$$x_c \neq 0 \tag{5.17}$$
$$y_c \neq 0 \tag{5.18}$$



5.1.4 Cambered airfoil



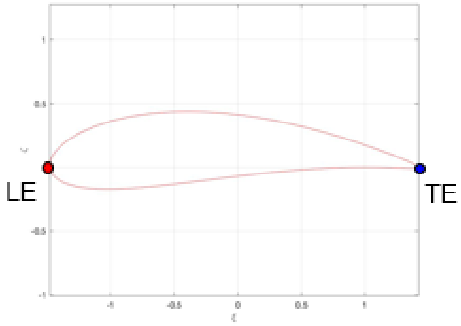


Figure 5.1: $x_c = -0.1$, $y_c = 0.1$, $sin\beta = \frac{y_c}{R}$

5.2 Uniform stream + circulation

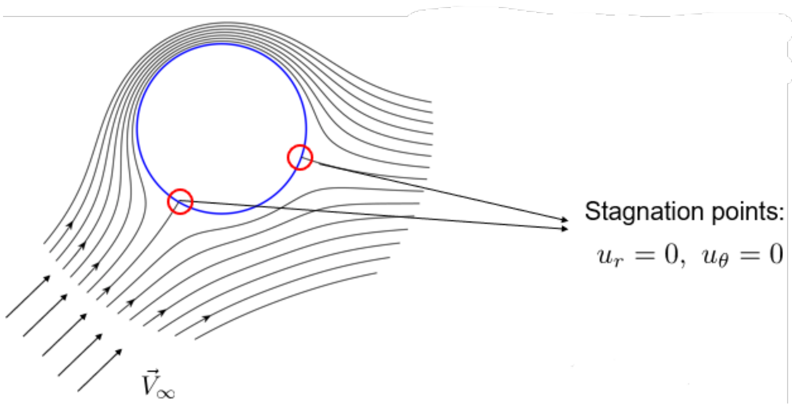
What is the correct relationship between the circulation and the angle of incidence?

$$\Gamma = f(\alpha) \tag{5.19}$$

$$\text{Stagnation points: } u_r = 0,\ u_\theta = 0 \tag{5.20}$$

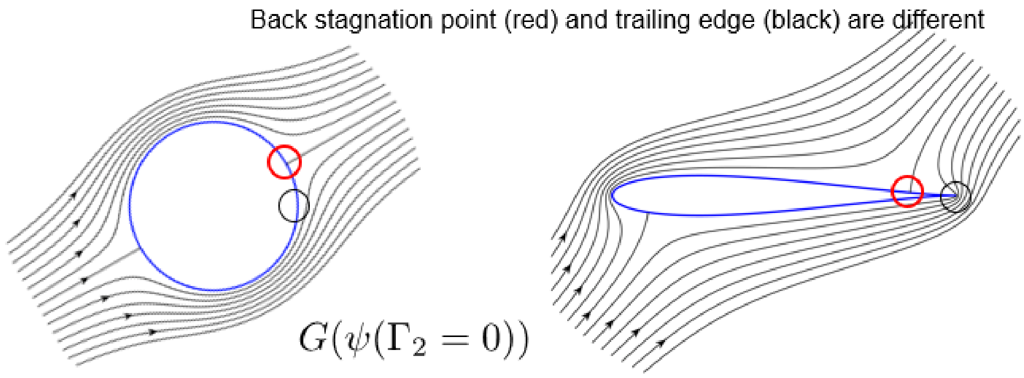
$$\phi = V_\infty r \cos(\theta - \alpha)\left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi}\theta \tag{5.21}$$

$$\psi = V_\infty r \sin(\theta - \alpha)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\log\left(\frac{r}{R}\right) \tag{5.22}$$

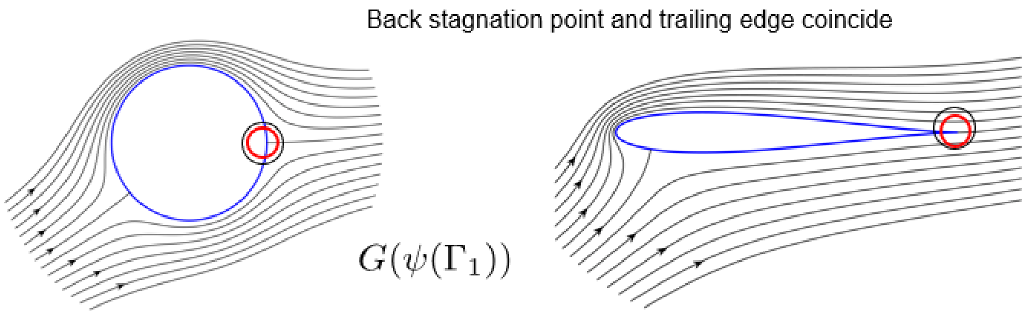


5.3 Kutta condition

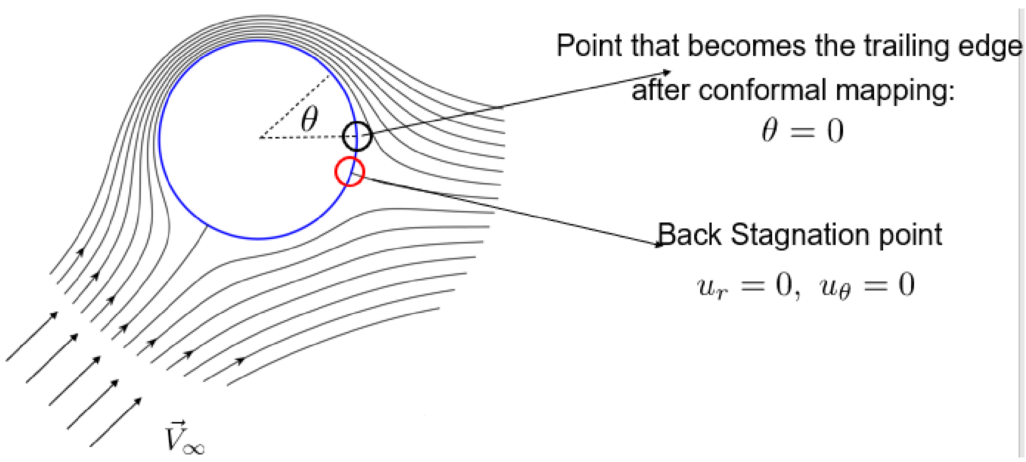
I can apply the Joukowsky Transformation to the streamlines around the cylinder and obtain the streamlines around the airfoil. However I need to carefully select the circulation, Γ.



We can see that this streamline is unrealistic. This is because to curl around the trailing edge cusp, the speed goes to infinity. This is clearly impossible and flow separation would occur instead. This is the wrong circulation value for the given angle of incidence.



This streamline is more realistic as the velocity is parallel to the trailing edge cusp and its component in the vertical direction is zero. This circulation can be used to predict the lift F_L but still predicts no drag ($F_D = 0$). The next question is how do we achieve this mapping?



We can see here that for a given angle of coincidence, the circulation has to be selected by imposing that the back stagnation point (red circle) and the point corresponding to the airfoil trailing edge (black circle) coincide.

$$\phi = V_\infty r \cos(\theta - \alpha)\left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi}\theta \tag{5.23}$$

$$\psi = V_\infty r \sin(\theta - \alpha)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\log\left(\frac{r}{R}\right) \tag{5.24}$$

Velocity components:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_\infty r \cos(\theta - \alpha)\left(1 - \frac{R^2}{r^2}\right) \tag{5.25}$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin(\theta - \alpha)\left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi r} \tag{5.26}$$

On the cylinder $r = R$:

$$u_r = 0 \tag{5.27}$$

$$u_\theta = -2V_\infty \sin(\theta - \alpha) - \frac{\Gamma}{2\pi R} \tag{5.28}$$

Kutta condition:

$$u_\theta(\theta = 0) = 0 \tag{5.29}$$

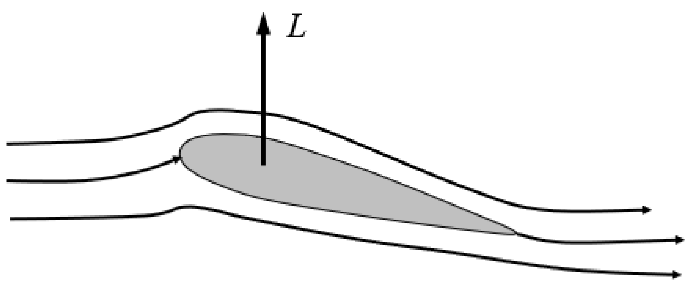
$$\Gamma = -4\pi V_\infty R \sin(-\alpha) = 4\pi V_\infty R \sin\alpha \tag{5.30}$$

For a cambered aerofoil:

$$u_\theta(\theta = -\beta) = 0 \tag{5.31}$$

$$\Gamma = -4\pi V_\infty R \sin(-\beta - \alpha) = 4\pi V_\infty R \sin(\beta + \alpha) \tag{5.32}$$

5.4 Aerofoil lift



The lift is given by:

$$L = \rho V \Gamma = 4\rho\pi V_\infty^2 R \sin\alpha \tag{5.33}$$

The lift coefficient is:

$$c_L = \frac{L}{\frac{1}{2}\rho V_\infty^2 c} = 9\pi \frac{R}{c}\sin\alpha \tag{5.34}$$

Flow is inviscid so zero drag.

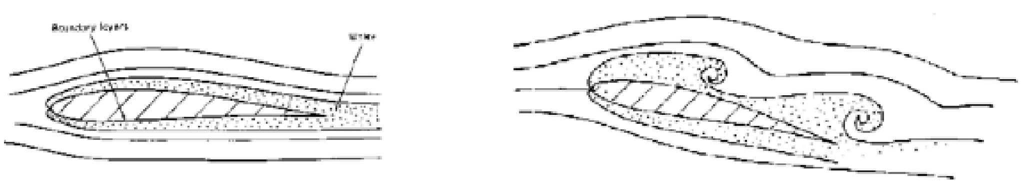
5.5 Flow past an Aerofoil

5.5.1 How was the circulation created?

When an airfoil is still on the ground there is no flow and total circulation is zero. As soon as the airfoil takes off a negative circulation is created at the back oft he airfoil and a positive one is stored within the airfoil boundary layer.

5.5.2 Stalling

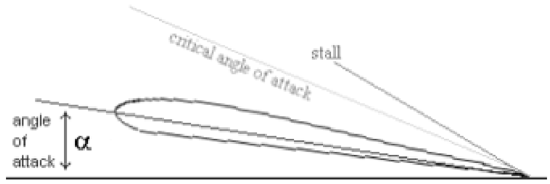
If the angle of attack becomes too great and boundary layer separation occurs on the top of the aerofoil the pressure pattern will change dramatically. In such a case, the lift component is insufficient to overcome the weight of the aircraft and disaster is imminent. This phenomenon is known as stalling. When stalling occurs, all, or most, of the 'suction' pressure is lost and the plane will suddenly drop from the sky. The solution to this is to put the plane into a dive to regain the boundary layer. A transverse lift force is then exerted on the wing which gives the pilot some control and allows the plane to be pulled out of the dive.

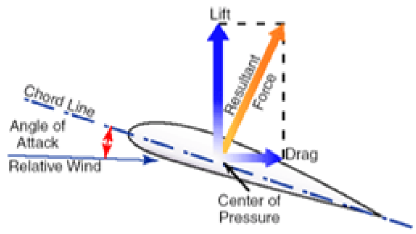


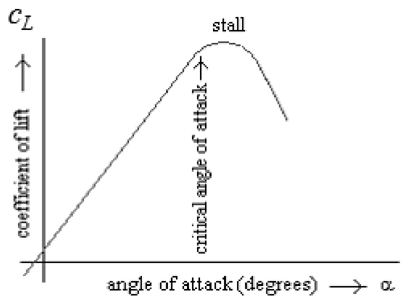
Boundary layer separation occurs on the upper surface (where the adverse pressure gradients are large) amd produces a different c_p distribution from the unseparated case. As skin friction is predominant for the unstalled wing, the profile drag is sensitive to any increase in form drag.

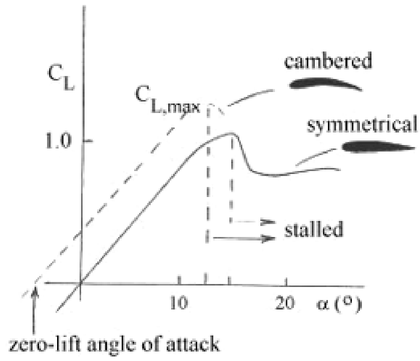
- At a small angle of attack a: separation is close to trailing edge, wake is thin, low form of drag
- As the angle of attack a increases: separation moves along the top of the aerofoil, wake width increases, form drag increases
- The critical angle of attack a_{crit}: value of a form maximum c_L, stall angle
- If the angle of attack $a > a_{crit}$: separation from most of the upper surface, wider wake with turbulence, considerable form drag

The exact value of the critical angle of attack depends on the type and shape of the aerofoil.

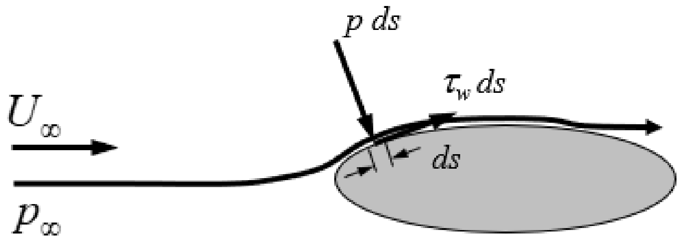








5.6 Form drag and skin friction



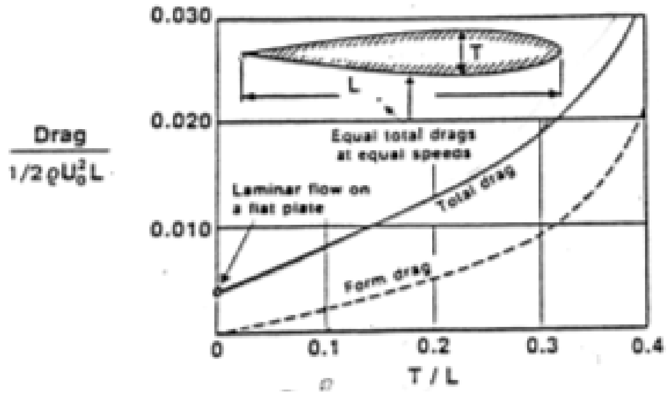
Force due to pressure p on element of surface area ds is $p\,$ds. The force $p\,$ds has a component in the stream direction and if this component is integrated over the whole body surface, it gives the drag due to pressure distribution or the form drag. Form drag is the resultant of the forces normal to the surface (pressure distribution: normal stresses). The same element of area ds experiences a shear stress τ_w due ot the velocity of the gradient normal to the surface and the associated shear force is $\tau_w\,$ds. This also has a component in the stream direction which when integrated over the body surface gives the drag due due to skin friction. Skin friction drag is resultant of forces tangential to the surface (shear stresses).

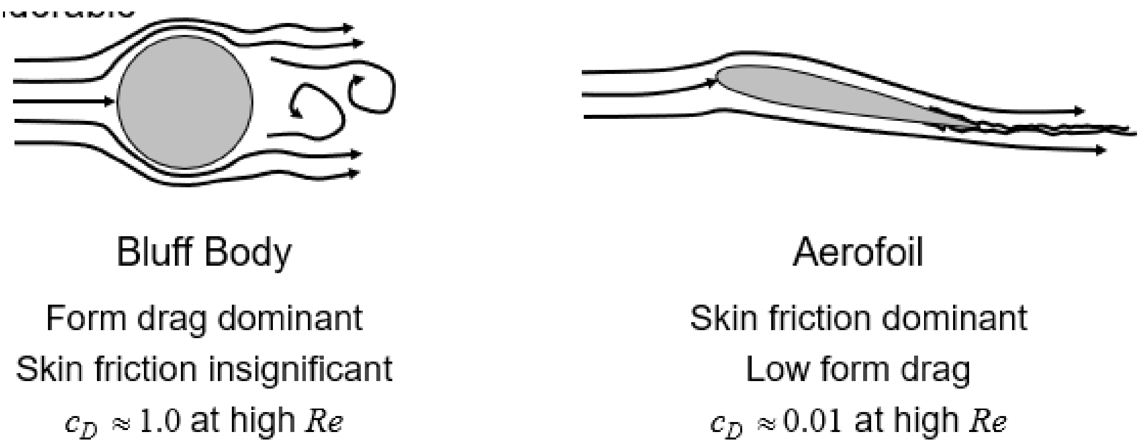
5.7 Total drag

Total drag (also known as profile drag) is given by:

$$\text{Total drag} = \text{Form drag} + \text{Skin friction} \tag{5.35}$$

Separation is what leads to large form drag. The benefits of streamlining can be considerable.





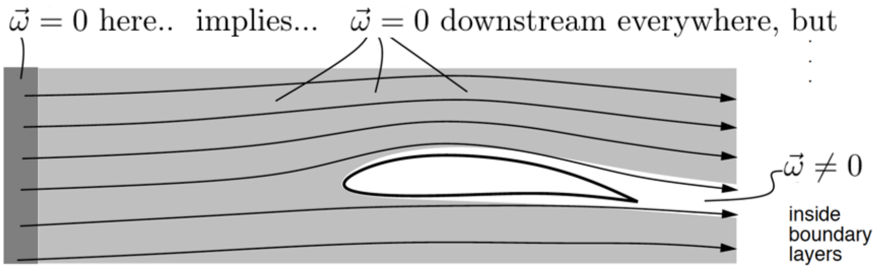
Chapter 6

Boundary layer theory, separation and control

Boundary layer theory (Prandtl)

The flow domain can be subdivided into two parts:

- **Irrotational flow region**, outside of the boundary layer, Bernoulli equation, potential flow and stream function apply
- **Boundary layer** where all vorticity is confined, The friction shear at the airfoil surface acts as a source of vorticity.



6.1 Boundary layer concept

Immediately adjacent to a solid surface, the fluid particles are slowed by the strong shear force between the fluid particles and the surface. This relatively slower moving layer of fluid is the boundary layer.

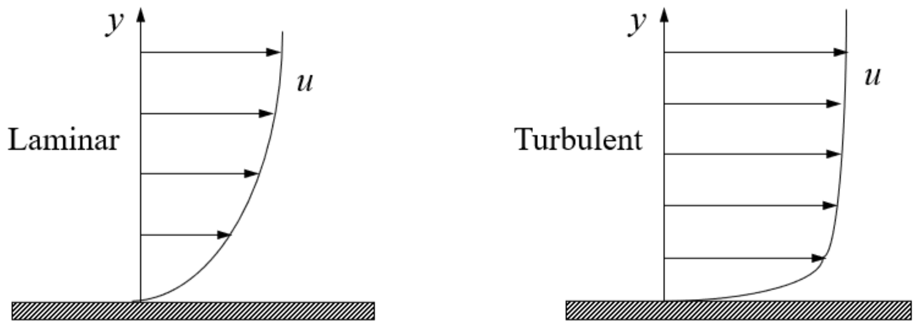
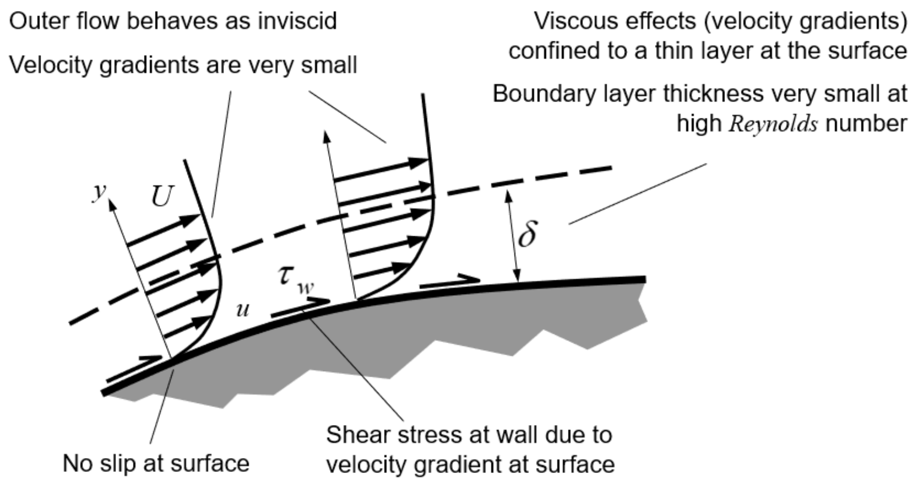


Figure 6.1: In turbulent flow, you have a higher shear stress due to the higher velocity close to the boundary. This produces more frictional drag against the motion of the solid surface. ($\tau = \mu \frac{\partial u}{\partial y}$)

6.2 Boundary layer theory

Prandtl's concept of the boundary layer in high *Re* flow is that although viscous forces can be considered small, their effects are not negligible.



Let us see what happens when we consider the Navier stokes equations for a steady - 2D flow with *Re* >> 1. Continuity equation (conservation of mass):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.1}$$

Conservation of momentum equations:

$$\text{x-direction: } u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{6.2}$$

$$\text{y-direction: } u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{6.3}$$

Remember that:

$$Re = \frac{U_\infty L}{\nu} \tag{6.4}$$

Where L is the length of the airfoil being analysed, U_∞ is the velocity of the free stream and ν the kinematic viscosity. Inside the boundary layer, one thing we can assume that the inertial forces and the viscous forces are comparable, because viscous diffusion is large and of the same order of magnitude of the inertial terms. Therefore:

$$\frac{u\frac{\partial u}{\partial x}}{\nu\frac{\partial^2 u}{\partial y^2}} \approx 1 \tag{6.5}$$

We need to exploit this relationship in order to derive an expression for δ in comparison to L. Some approximations we can make below are:

$$u \approx U_\infty \tag{6.6}$$

$$x \approx L \rightarrow \frac{\partial}{\partial x} \propto \frac{1}{L} \tag{6.7}$$

$$y \approx \delta \rightarrow \frac{\partial}{\partial y} \propto \frac{1}{\delta} \tag{6.8}$$

Combining this with our above assumption:

$$\frac{\frac{U_\infty^2}{L}}{\nu\frac{U_\infty}{\delta^2}} = \frac{\delta^2}{L^2}Re \approx 1 \tag{6.9}$$

$$\delta^2 \approx \frac{L^2}{Re} \tag{6.10}$$

Looking at the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.11}$$

$$\frac{U_\infty}{L} + \frac{v}{\delta} = 0 \tag{6.12}$$

$$v = \frac{\delta}{L}U_\infty \tag{6.13}$$

$$v = \frac{1}{\sqrt{Re}}U_\infty \tag{6.14}$$

(signs were ignored). Looking at the momentum equation in the x-direction:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{6.15}$$

$$\text{First term: } u\frac{\partial u}{\partial x} \propto \frac{U_\infty^2}{L} \tag{6.16}$$

$$\text{Second term: } v\frac{\partial u}{\partial y} \propto \frac{1}{\sqrt{Re}}\frac{U_\infty}{\delta}U_\infty = \frac{U_\infty^2}{\sqrt{Re}}\frac{\sqrt{Re}}{L} = \frac{U_\infty^2}{L} \tag{6.17}$$

$$\text{Fourth term: } \nu\frac{\partial^2 u}{\partial x^2} \propto \nu\frac{U_\infty}{L^2} = \frac{\nu}{LU_\infty}\frac{U_\infty^2}{L} = \frac{1}{Re}\frac{U_\infty^2}{L} \tag{6.18}$$

$$\text{Fifth term: } \nu\frac{\partial^2 u}{\partial y^2} \propto \nu\frac{I_\infty}{\delta^2} = \frac{\nu}{LU_\infty}\frac{U_\infty^2 L}{\delta^2} = \frac{1}{Re}\frac{U_\infty^2 Re}{L} = \frac{U_\infty^2}{L} \tag{6.19}$$

We can see in the fourth term that $Re >> 1$, hence it is neglected i.e. there will not be a viscous term due to the variation of the velocity in the x direction w.r.t x. Looking at the momentum equation in the y-direction:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{6.20}$$

$$\text{First term: } u\frac{\partial v}{\partial x} \propto U_\infty^2\frac{\delta}{L^2} = \frac{1}{\sqrt{Re}}\frac{U_\infty^2}{L} \tag{6.21}$$

$$\text{Second term: } v\frac{\partial v}{\partial y} \propto \frac{U_\infty^2}{L^2}\frac{\delta^2}{\delta} = \frac{1}{\sqrt{Re}}\frac{U_\infty^2}{L} \tag{6.22}$$

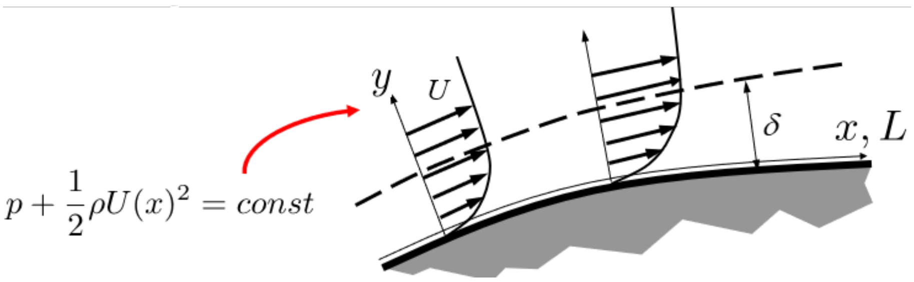
$$\text{Fourth term: } \nu\frac{\partial^2 v}{\partial x^2} \propto \nu\frac{U_\infty\delta}{L}\frac{1}{L^2} = \frac{\nu}{LU_\infty}\frac{U_\infty^2\delta}{L^2} = \frac{1}{Re}\frac{U_\infty^2}{L\sqrt{Re}} = \frac{1}{\sqrt{Re}^3}\frac{U_\infty^2}{L} \tag{6.23}$$

$$\text{Fifth term: } \nu\frac{\partial^2 v}{\partial y^2} \propto \nu\frac{U_\infty\delta}{L}\frac{1}{\delta^2} = \frac{\nu}{LU_\infty}\frac{U_\infty^2}{\delta} = \frac{1}{Re}\frac{U_\infty^2\sqrt{Re}}{L} = \frac{1}{\sqrt{Re}}\frac{U_\infty^2}{L} \tag{6.24}$$

We see that in the first, second, fourth and fifth terms, $Re >> 1$, hence they are neglected. Let us see what this analysis tells us:

$$\text{x-direction: } u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{6.25}$$

$$\text{y-direction: } \frac{\partial p}{\partial y} = 0 \tag{6.26}$$



Pressure does not change along thickness of the boundary layer. Therefore I can estimate the pressure outside of the boundary layer using Bernoulli (inviscid flow theory), and then apply that pressure on the body wall. Outside of the boundary layer, we know:

$$\frac{\partial}{\partial y} = 0 \;\&\; v = 0 \tag{6.27}$$

$$U(x) = \frac{\partial U(x)}{\partial x} = \frac{\partial \frac{1}{2}U(x)^2}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{6.28}$$

$$p + \frac{1}{2}\rho U(x)^2 = \text{const} \tag{6.29}$$

6.3 Flow separation

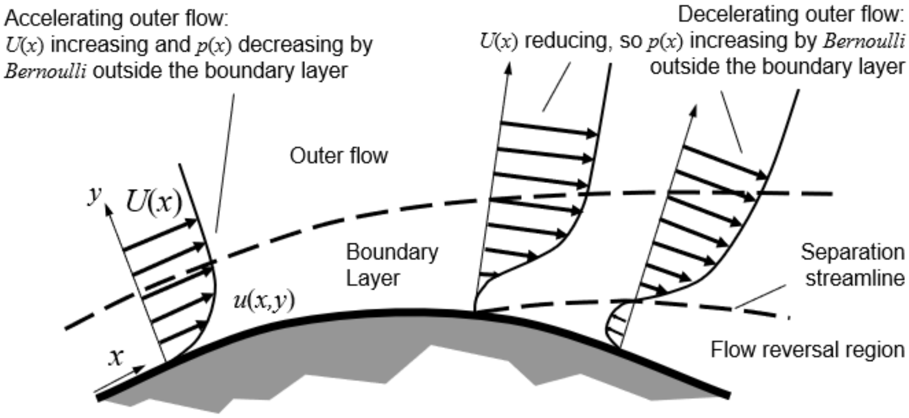


Figure 6.2: Formation of a vortex, due to flow separation and flow reversal.

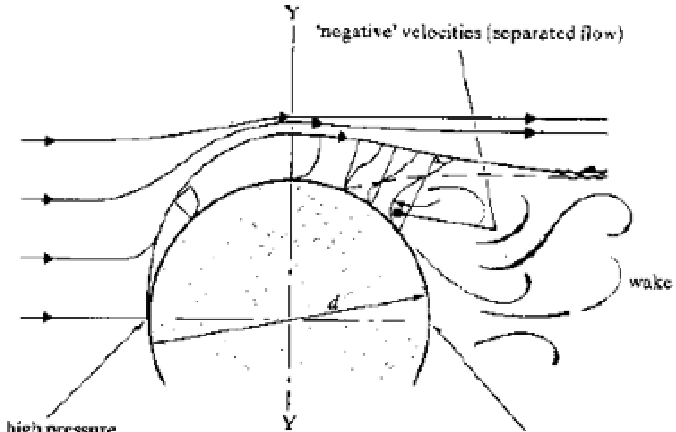
Favourable pressure gradient:

$$\frac{\partial U(x)}{\partial x} > 0, \; \frac{\partial p}{\partial x} < 0 \tag{6.30}$$

Adverse pressure gradient:

$$\frac{\partial U(x)}{\partial x} < 0, \; \frac{\partial p}{\partial x} > 0 \tag{6.31}$$

Adverse pressure gradient produces further retardation of fluid (which is already slowed down close to wall). The fluid near to the wall is eventually brought to rest, flow reverse and separated. Separation occurs at or near the point where $\frac{dp}{dx}$ first becomes zero; at the first point of separation, at $y = 0$, $\frac{du}{du} = 0$. The formation of separation occurs as the fluid accelerates from the centre to get round the cylinder (it must accelerate as it has further to go than the surround fluid). It reaches a maximum at $Y - Y$, where it has also dropped in pressure. The adverse pressure gradient between here and the downstream side of the cylinder will cause separation if the flow is fast enough ($Re > 2$).



6.4 Laminar and turbulent flow separation

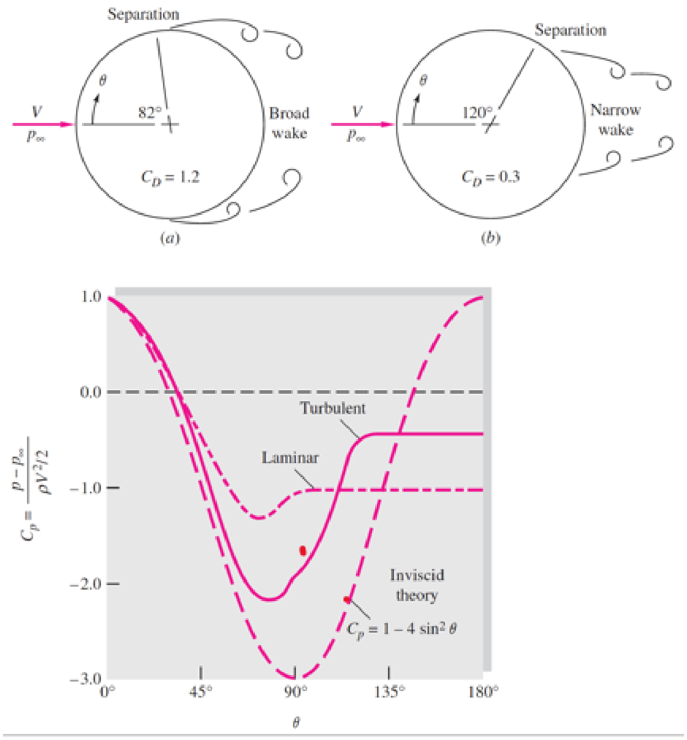
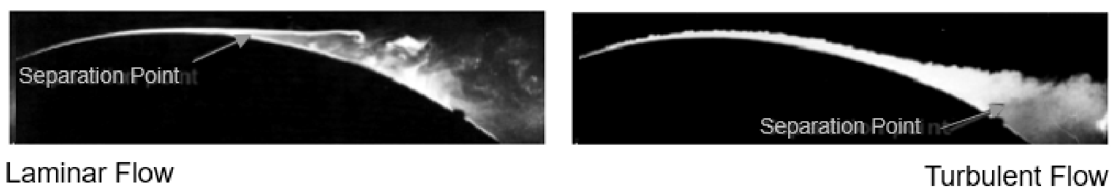
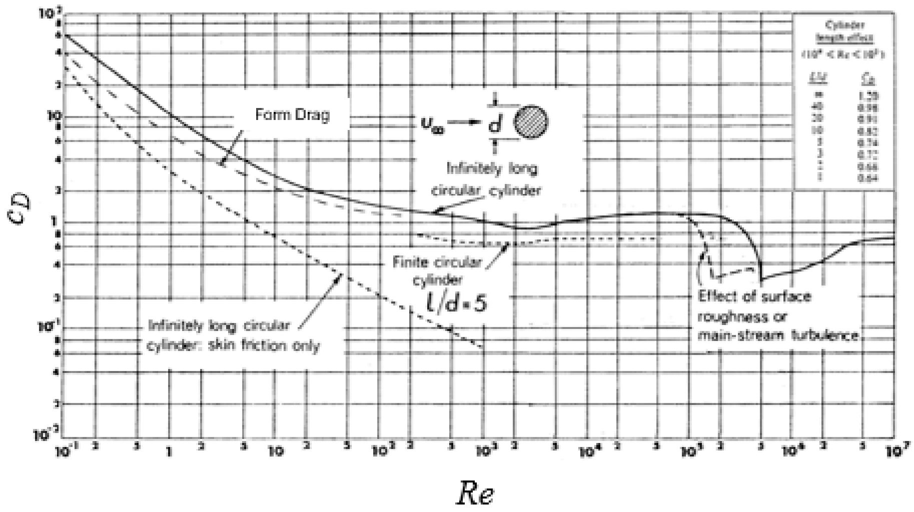


Figure 6.3: Left: laminar, right: turbulent

The boundary layer has a high momentum deficit and in the case of laminar flow, it is unable to adjust to the increasing pressure and separates. A turbulent boundary layer is more resistant to separation than a laminar one. Due to the chaotic nature of turbulence, a turbulent boundary layer has a great capacity for mixing and absorbing energy from the free stream (this is obvious from its profile and its rate of growth compared with a laminar layer). The effect of increased mixing is increased transport of momentum (due to turbulent stresses) between the free-stream and the flow near the wall. This increased transport of momentum from the free-stream to the wall increases the stream-wise momentum in the boundary layer allowing the flow to overcome $\frac{dp}{dx}$ for longer and to separate further downstream (larger velocities near the wall are not so easily halted).



6.5 Flow past a cylinder - Drag coefficient

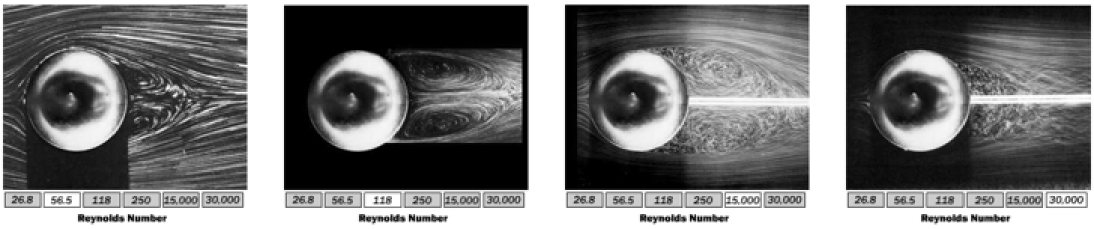


For a cylinder,

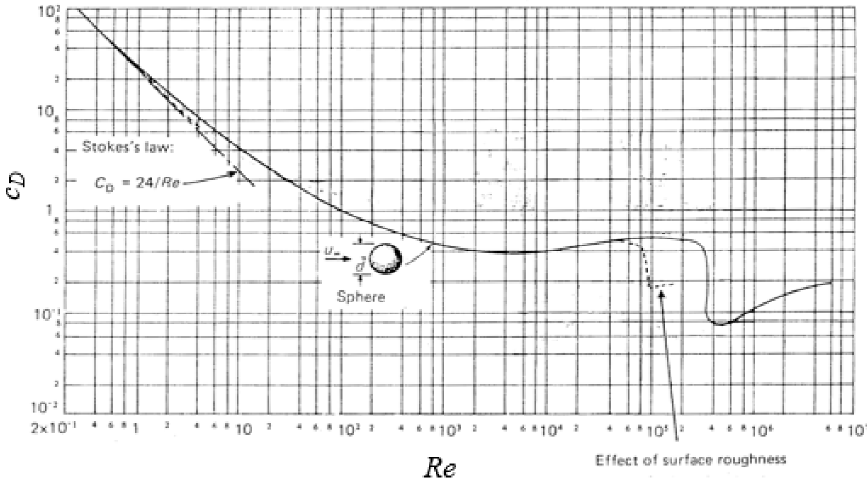
- c_D is nearly constant when $Re = 10^3 - 10^5$, but varies considerably as the boundary layer becomes turbulent between $Re = 10^5 - 10^6$
- At $Re \approx 2000$, c_D reaches minimum of ≈ 0.9
- At $Re \approx 30000$, c_D rises to ≈ 1.2 (partly due to increasing turbulence in the wake)
- At $Re \approx 200000$, c_D drops suddenly to ≈ 0.3
- At $Re > 200000$, c_D increases slowly.
- The form drag is $\approx 50\%$ of the profile drag as $Re \to 0$
- By $Re \approx 200$ the form drag is $\approx 90\%$ of the profile drag
- At $Re \approx 30000$, skin friction is insignificant.

6.6 Flow past a sphere

The flow over a sphere starts to separate from the surface at $Re \approx 20$. As Re increases, a pair of recirculating vortices becomes visible in the wake region. At flows up to $Re \approx 200$ the flow is still steady but beyond this value oscillations start to appear and the separation point moves upstream of the top of the sphere, increasing the drag.



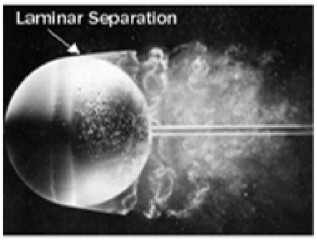
6.6.1 Drag coefficient

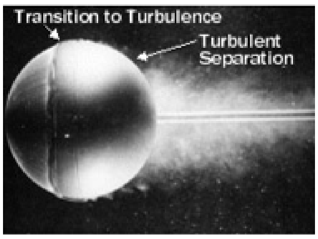


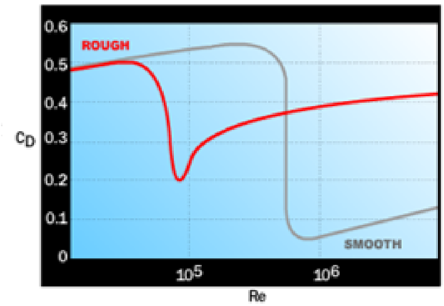
For a sphere, c_D is nearly constant for $Re = 10^3 - 10^5$ but varies considerably as the boundary layer becomes turbulent between $Re = 10^5 - 10^6$ and, in fact, this transition causes c_D to dramatically increase. As $Re \to 0$, c_D is given by Stokes's law ($c_D = \frac{24}{Re}$)

6.6.2 Drag reduction

Deliberately tripping the boundary layer from laminar to turbulent (i.e. by a trip wire) results in an increase in skin drag (due to increased shear stresses), but a decrease in form drag (due to delayed separation point). The reduction in form drag more than compensates for the increased drag due to turbulent shearing, resulting in a greatly reduced overall drag coefficient c_D. Drag reduction can also be accomplished by roughening the surface and, thereby, lowering Re at which transition occurs. However, the drag coefficient for a roughened surface will typically rise faster in comparison to a smooth surface, once the transition point is passed.

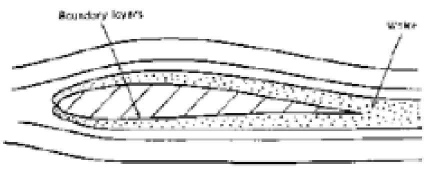


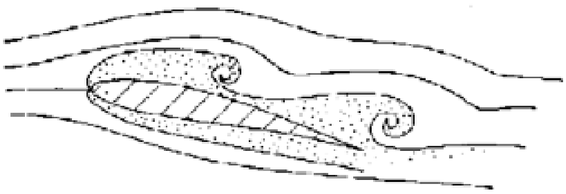


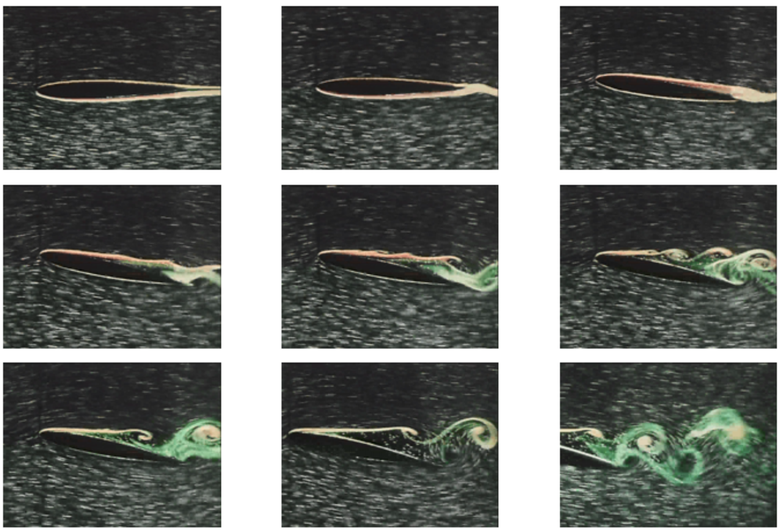


6.7 Flow past an aerofoil - stalling

If the angle of attack becomes too great and boundary layer separation occurs on top of the aerofoil, the pressure pattern will change dramatically. In such a case, the lift component is insufficient to overcome the weight of one aircraft and disaster is imminent. This phenomenon is known as **stalling**. When stalling occurs, all, or most, of the 'suction' pressure is lost, and the plane will suddenly drop from the sky. The solution to this is to put the plane into a dive to regain the boundary layer. A transverse lift force is then exerted on the wing which gives the pilot some control and allows the plane to be pulled out of the dive.





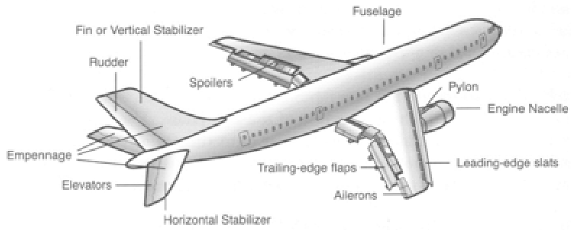


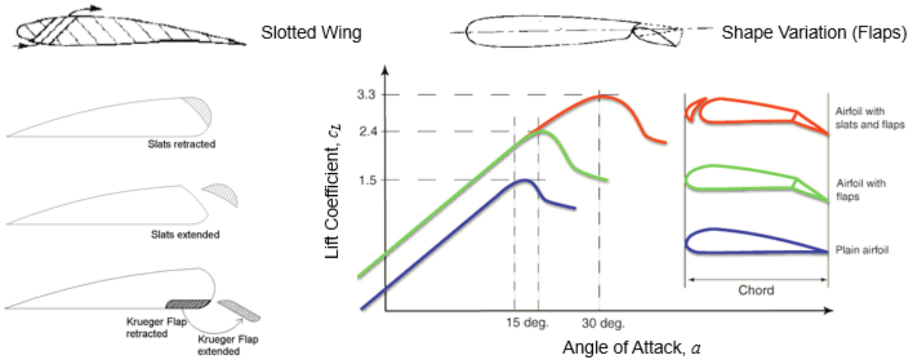
6.8 Boundary layer control

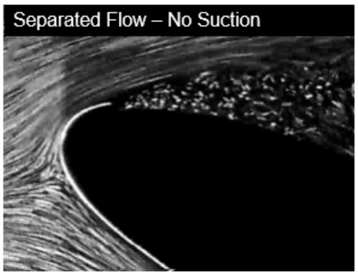
There are mechanisms for preventing the boundary layer from separating in the first place.

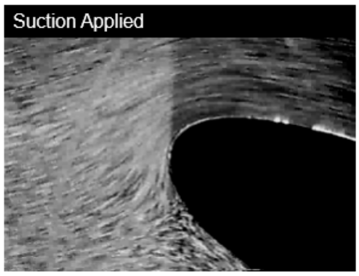
- Surface roughening
  - To initiate turbulent boundary layer, thus delay separation and achieve a narrower wake of higher pressure
- Shape variation
  - Putting a flap at the end of the wing and tilting it before separation occurs increases the velocity over the top of the wing, reduces the pressure and chance of separation
- Fluid injection
  - Accelerate boundary layer in direction of flow (but this will produce very turbulent flow and more skin friction)
- Fluid removal
  - Suction through small holes or a porous surface
  - Downstream of suction position, boundary layer is thinner and faster thus better able to withstand adverse pressure gradient. Suction delays laminar to turbulent transition and leads to less skin friction
- Slotted wing
  - Slower moving air on the upper surface can be accelerated by bringing air from the high-pressure area on the bottom of the wing through slots. Pressure decreases on the top and adverse pressure gradient reduces
- Engine arrangement
  - Engine intakes draw slow air from the boundary layer at the rear of the wing through small holes. This keeps the boundary layer close to the wing and greater pressure gradients can be maintained before separation

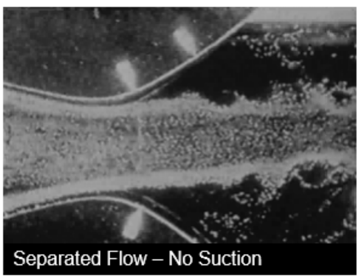
Control systems are often costly and difficult to install. Advantages for some conditions, hindrance of others (advantages maybe offset by the added drag). Planes take advantage of the cambered aerofoil's high lift characteristics by enabling shape variation of the basic wing. Leading edge 'Krüger' flaps or slats and trailing edge 'Fowler' flaps are extended from the wing and change the aerofoil's shape into a cambered form, generating greater lift during low-speed flight.

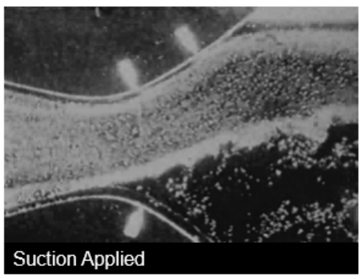

Fuselage
Fin or Vertical Stabilizer
Rudder
Spoilers
Pylon
Engine Nacelle
Empennage
Elevators
Trailing-edge flaps
Leading-edge slats
Ailerons
Horizontal Stabilizer

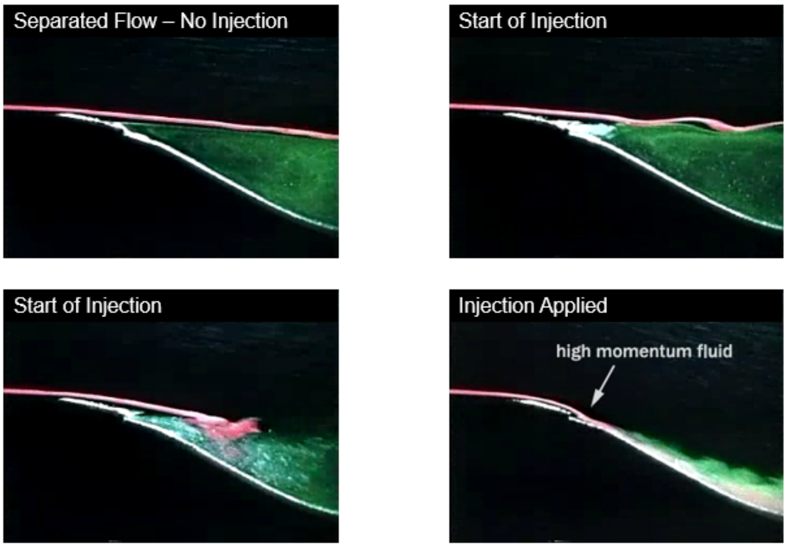

Slotted Wing
Shape Variation (Flaps)
Slats retracted
Slats extended
Krueger Flap retracted
Krueger Flap extended
3.3
2.4
1.5
Lift Coefficient, c_L
15 deg.
30 deg.
Angle of Attack, α
Airfoil with slats and flaps
Airfoil with flaps
Plain airfoil
Chord


Separated Flow – No Suction


Suction Applied


Separated Flow – No Suction


Suction Applied


Separated Flow – No Injection
Start of Injection
Start of Injection
Injection Applied
high momentum fluid

Chapter 7

Boundary layer of a flat plate

7.1 Boundary Layer Theory

x-direction:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{7.1}$$

y-direction:

$$\frac{\partial p}{\partial y} = 0 \tag{7.2}$$

Pressure does no change along the thickness of the boundary layer. Therefore, I can estimate the pressure outside of boundary layer using Bernoulli (inviscid flow theory), and then apply that pressure on the body wall. By definition, outside of the boundary layer:

$$\frac{\partial}{\partial y} = 0 \;\&\; v = 0 \tag{7.3}$$

$$U(x) = \frac{\partial U(x)}{\partial x} = \frac{\partial \frac{1}{2}U(x)^2}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{7.4}$$

$$p + \frac{1}{2}\rho U(x)^2 = \text{const (from } \frac{\partial \frac{1}{2}U(x)^2}{\partial x}) \tag{7.5}$$

7.2 Flat plate boundary layer

Inside the boundary layer:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{7.6}$$

$$\frac{\partial p}{\partial y} = 0 \tag{7.7}$$

Pressure does not change along boundary layer thickness:

$$U_\infty = \frac{\partial U_\infty}{\partial x} = 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{7.8}$$

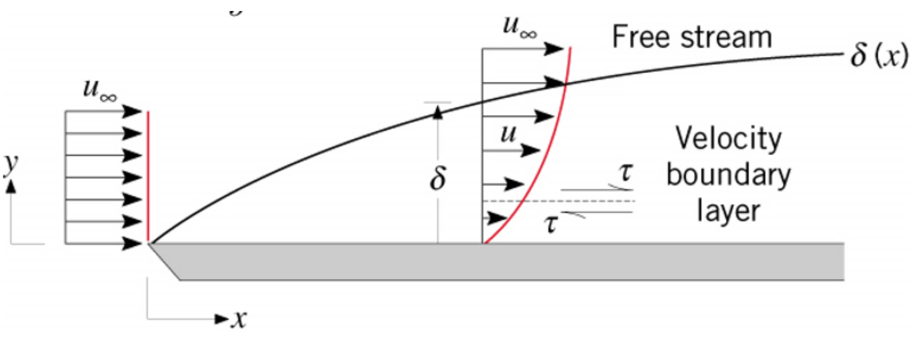


Figure 7.1:

7.2.1 Analysis of order of magnitude

$$v << u \tag{7.9}$$

$$\left|\frac{\partial}{\partial x}\right| \approx \frac{1}{L} \tag{7.10}$$

$$\left|\frac{\partial}{\partial y}\right| \approx \frac{1}{\delta} \tag{7.11}$$

Inside the boundary layer, viscous diffusion is large and of the same order of magnitude of inertial terms, therefore:

$$\frac{u\frac{\partial u}{\partial x}}{\nu\frac{\partial^2 u}{\partial y^2}} \approx 1 \begin{cases} u \approx U_\infty \\ y \approx \delta \\ x \approx L \end{cases} \rightarrow \frac{\frac{U_\infty^2}{L}}{\frac{\nu U_\infty}{\delta^2}} = \frac{\delta^2}{L^2} Re \approx 1 \tag{7.12}$$

7.3 Blasius solution

In 1908, Blasius found the analytical solution to the flat plate boundary layer problem. The velocity profiles are self-similar at different axial distances from the leading edge.

$$\eta = \frac{y}{\delta(x)} = \frac{y\sqrt{Re_x}}{x} = \frac{y}{x}\sqrt{\frac{U_\infty x}{\nu}} = y\sqrt{\frac{U_\infty}{x\nu}} \tag{7.13}$$

$$\frac{u}{U_\infty} = f'(\eta) \tag{7.14}$$

$$\frac{v}{U_\infty} = \frac{1}{2}\left(\frac{\nu}{xU_\infty}\right)^{0.5}\left(\eta f'(\eta) - f(\eta)\right) \tag{7.15}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \rightarrow \frac{1}{2}f(\eta)f''(\eta) + f'''(\eta) = 0 \tag{7.16}$$

Boundary conditions:

$$\begin{cases} u = v = 0 & @\ y = 0 \\ u = U_\infty & @\ y = \infty \end{cases} \rightarrow \begin{cases} f = f' = 0 & @\ \eta = 0 \\ f' = 1 & @\ \eta = \infty \end{cases} \tag{7.17}$$

$$\frac{\delta}{x} = \frac{A}{\sqrt{Re_x}} \rightarrow A = ? \tag{7.18}$$

Blasius solved this problem and derived the analytical solution, which is shown below:

$y[U/(\nu x)]^{1/2}$	u/U	$y[U/(\nu x)]^{1/2}$	u/U
0.0	0.0	2.8	0.81152
0.2	0.06641	3.0	0.84605
0.4	0.13277	3.2	0.87609
0.6	0.19894	3.4	0.90177
0.8	0.26471	3.6	0.92333
1.0	0.32979	3.8	0.94112
1.2	0.39378	4.0	0.95552
1.4	0.45627	4.2	0.96696
1.6	0.51676	4.4	0.97587
1.8	0.57477	4.6	0.98269
2.0	0.62977	4.8	0.98779
2.2	0.68132	5.0	0.99155
2.4	0.72899	∞	1.00000
2.6	0.77246		

Figure 7.2:

The boundary layer corresponds to when we have $0.99U_\infty$, we have $A = 5$, hence:

$$\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \tag{7.19}$$

Let us look at the shear stress and the friction coefficient:

$$\tau(x) = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} \rightarrow c_f = \frac{\tau(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{0.664}{\sqrt{Re_x}} \tag{7.20}$$

How does the shear stress, $\tau(x)$, vary with distance from the leading edge? If we look at our Re_x term, we see that as x increases c_f decreases. Looking at the gradients of $\frac{\partial u}{\partial y}$ along our boundary layer, we see that the gradient is steeper at $y = 0$. We can also plot $\tau(x)$ against x.

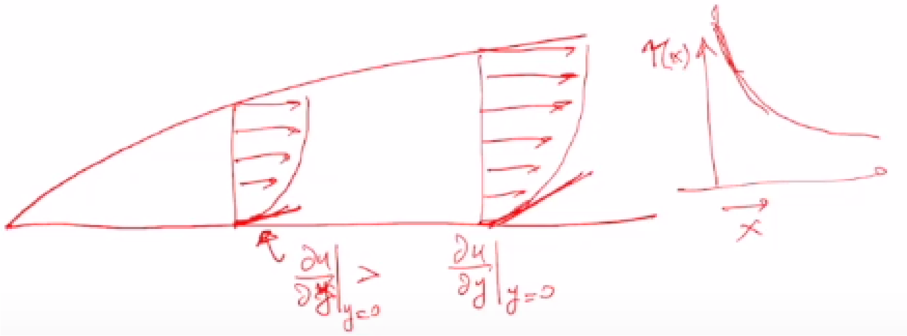


Figure 7.3:

Let us look at the friction drag on the flat plate:

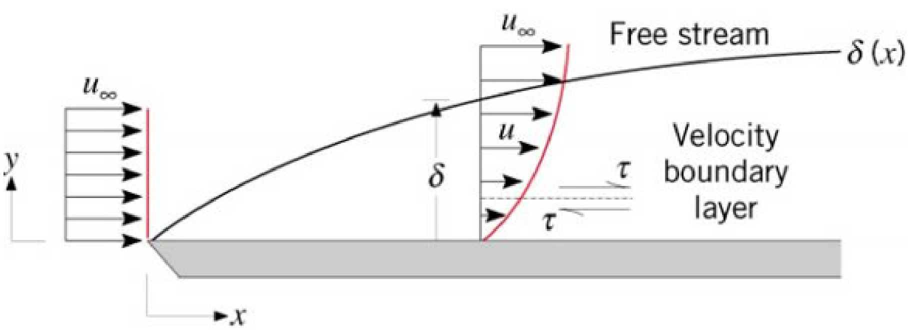


Figure 7.4:

What is the friction drag on the flat plate per unit of span?

$$D = \int_0^L \tau(x)\,\mathrm{d}x = 0.664\rho U_\infty^2 \frac{L}{\sqrt{Re_L}} \tag{7.21}$$

$$c_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 L} = \frac{1.328}{\sqrt{Re_L}} = 2c_f(L) \tag{7.22}$$

7.4 Laminar and turbulent profiles

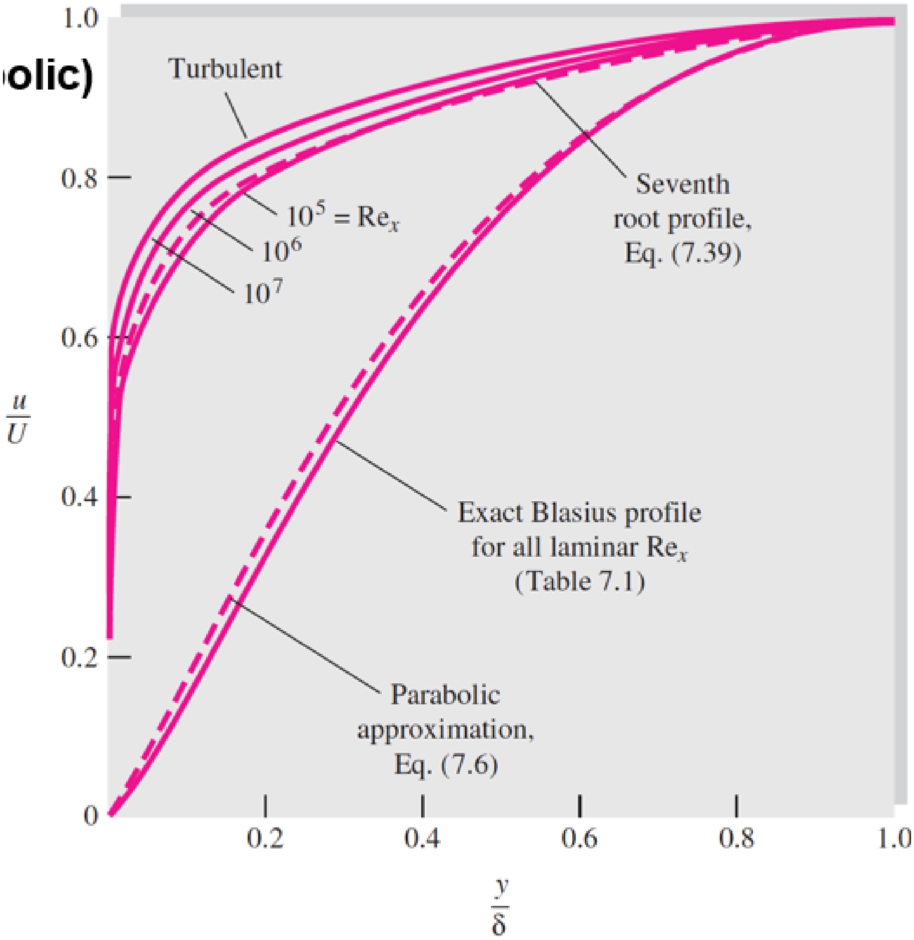


Figure 7.5:

Our parabolic approximation is given by the Karman approximation:

$$\frac{u}{U_\infty} = \left(2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) \tag{7.23}$$

Turbulent flow can be described as:

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{\frac{1}{5}} \tag{7.24}$$

The Prandtl approximation is given as:

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \tag{7.25}$$

In Figure (7.5), we see that with increasing Re, the bulge in our curve also increases. This also means that $\frac{\partial u}{\partial y}$ is also increasing, meaning our shear stress also increases with Re.

7.4.1 Turbulent friction and drag coefficients

For $n = \frac{1}{5}$ (fully turbulent flow):

$$\frac{\delta}{x} = \frac{0.385}{(Re_x)^{\frac{1}{5}}} \tag{7.26}$$

$$c_f = \frac{\tau(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{0.0594}{(Re_x)^{\frac{1}{5}}} \tag{7.27}$$

$$c_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 L} = \frac{0.074}{(Re_L)^{\frac{1}{5}}} = \frac{5}{4}c_f(L) \tag{7.28}$$

For $n = \frac{1}{7}$ (fully turbulent flow):

$$\frac{\delta}{x} = \frac{0.16}{(Re_x)^{\frac{1}{7}}} \tag{7.29}$$

$$c_f = \frac{\tau(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{0.027}{(Re_x)^{\frac{1}{7}}} \tag{7.30}$$

$$c_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 L} = \frac{0.031}{(Re_L)^{\frac{1}{7}}} = \frac{7}{6}c_f(L) \tag{7.31}$$

7.5 Displacement thickness

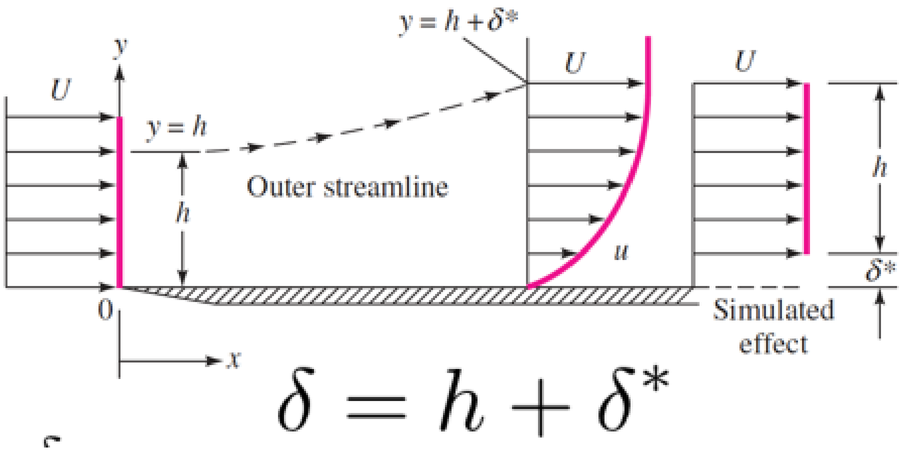


Figure 7.6:

Based on the conservation of mass; the outer streamline must displace outwards to conserve the mass flow rate. The displacement thickness measure the streamline displacement for different x:

$$\rho U h = \int_0^\delta \rho u \,\mathrm{d}y = \int_0^\delta \rho\,(u + U - U)\;\mathrm{d}y \tag{7.32}$$

$$Uh = U(h + \delta^*) + \int_0^\delta (u - U)\;\mathrm{d}y \tag{7.33}$$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right)\;\mathrm{d}y \tag{7.34}$$

7.6 Momentum thickness

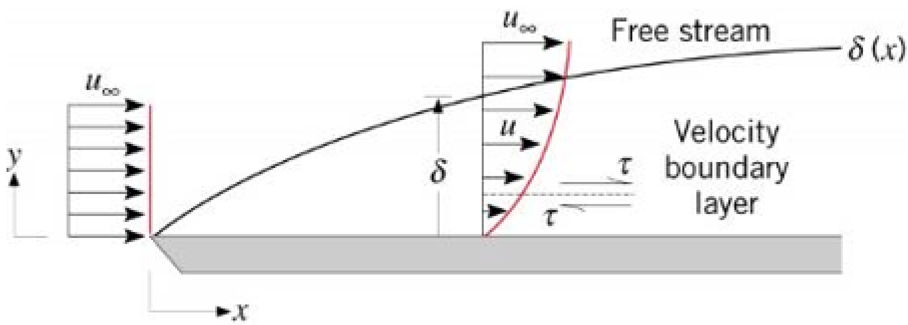


Figure 7.7:

Based on momentum loss; the total loss os momentum is equivalent to the loss of momentum occurring over a distance θ, denoted as the "momentum thickness."

$$\rho U^2 \theta = \int_0^\delta \rho u (U - u)\,\mathrm{d}y \tag{7.35}$$

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)\;\mathrm{d}y \tag{7.36}$$

- The "momentum thickness" measures the deficit in the rate of transport momentum in the layer compared with the rate of transport of momentum in the absence of the layer.

- Alternatively it can be seen as the thickness of a completely stagnant layer of fluid giving the same deficit in momentum as the actual profile.

For a flat plate (zero pressure gradient), the momentum loss is also equal to the friction drag along the flat plate.

$$D = \int_0^{\delta} \rho u(U_\infty - u)\,\mathrm{d}y \tag{7.37}$$

$$D = \int_0^{x} \tau_w\,\mathrm{d}x \tag{7.38}$$

$$\tau_w = \frac{dD}{dx} = \rho U_\infty^2 \frac{d\theta}{dx} \tag{7.39}$$

$$\frac{\tau_w}{\rho U_\infty^2} = \frac{d\theta}{dx} \tag{7.40}$$

The following equations are valid:

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = 2\frac{d\theta}{dx} \tag{7.41}$$

If we know the momentum thickness, I can find the shear stress/friction coefficient on the plate and vice versa.

7.7 General case with pressure gradient & shape factor

The shape factor is the ratio between the displacement thickness and the momentum thickness.

$$H = \frac{\delta^*}{\theta} = \frac{\int_0^{\delta}\left(1 - \frac{u}{U}\right)\,\mathrm{d}y}{\int_0^{\delta}\frac{u}{U}\left(1 - \frac{u}{U}\right)\,\mathrm{d}y} > 1 \tag{7.42}$$

This leads to:

$$\frac{\tau_w}{\rho U_\infty^2} = \frac{d\theta}{dx} + U\frac{dU}{dx}(H+2)\theta \tag{7.43}$$

where $U\frac{dU}{dx}(H+2)\theta$ is an extra term added due to the presence of a pressure gradient.

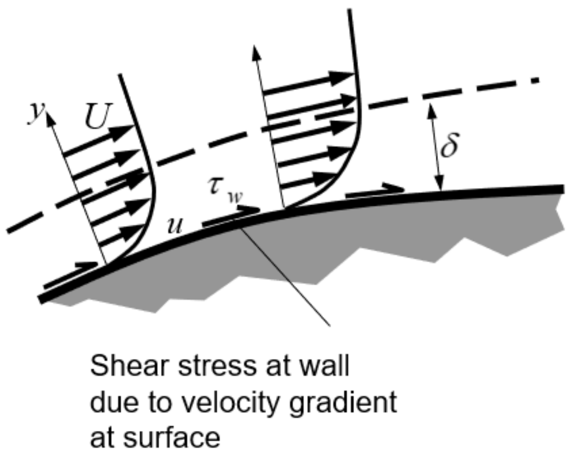


Figure 7.8:

For laminar flow (flat plate)

$$H = 2.59 \tag{7.44}$$

For turbulent flow (flat plate)

$$H = 1.29 \tag{7.45}$$

A large shape factor is indicative of boundary layer separation.

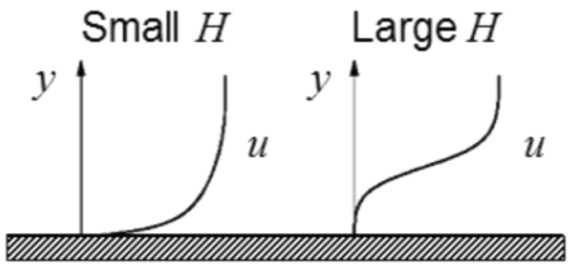


Figure 7.9:

7.8 Laminar - turbulent flow transition

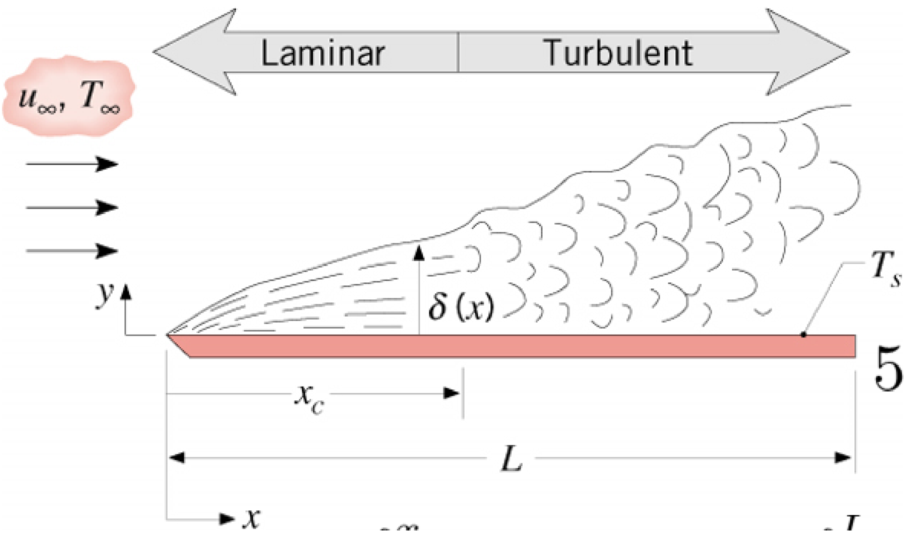


Figure 7.10:

x_c is defined as the point along our plate where our flow turns turbulent.

$$Re = \frac{U_\infty x_c}{\nu} \tag{7.46}$$

$$\tag{7.47}$$

This is normally in the region where $Re = 5 \times 10^5 - 8 \times 10^7$.

$$D = \int_0^{x_c} \tau_{\text{lam}}(x)\, \mathrm{d}x + \int_{x_c}^{L} \tau_{\text{turb}}(x)\, \mathrm{d}x \tag{7.48}$$

$$c_D = \frac{1}{L}\left(\int_0^{x_c} c_{fl}(x)\, \mathrm{d}x + \int_{x_c}^{L} c_{ft}(x)\, \mathrm{d}x\right) \tag{7.49}$$

7.9 Flat plate drag coefficient

$$c_D \approx \left(1.98 + 1.62 \log \frac{L}{\epsilon}\right)$$

ϵ **Roughness**

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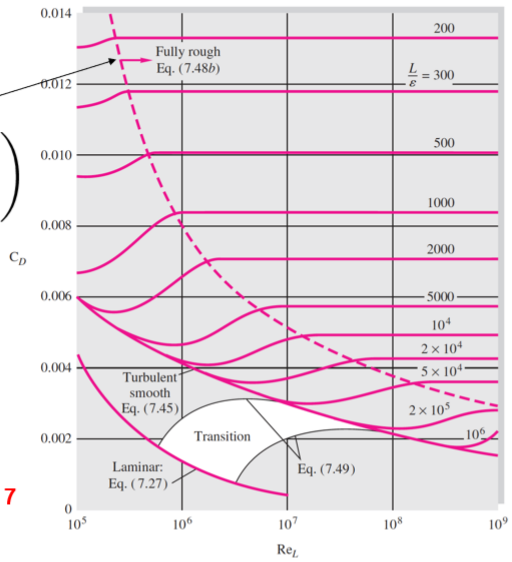

Figure 7.11: