

# Chapter 1

## Finite Element Method

### 1.1 Introduction to the method

- Discretisation of the model to elements
- Governing equations for each element
- Assembled to give system equations
- $[k]\{U\} = \{F\}$
- $[k]$  is a square matrix, stiffness matrix  $\{U\}$  is the vector of unknown nodal displacements or temperatures and  $\{F\}$  is the vector of applied nodal forces

### 1.2 1D element - the pin jointed bar

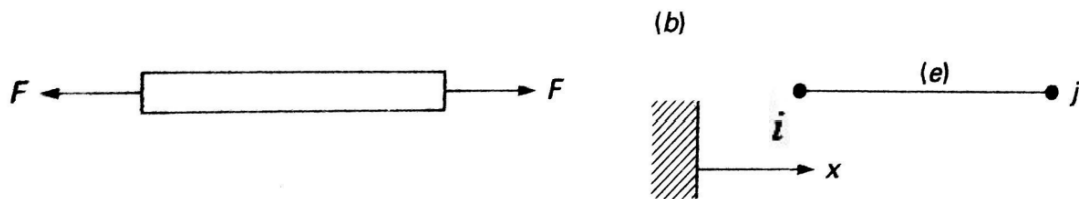


Figure 1.1: Pin jointed bar.

$$u = a + bx \quad (1.1)$$

$$u_i = a + bx_i \quad (1.2)$$

$$u_j = a + bx_j \quad (1.3)$$

where  $u_{i,j}$  are unknown nodal displacements and  $x_{i,j}$  are known nodal coordinates.

This leads to:

$$a = \frac{(u_i x_j - u_j x_i)}{L} \quad (1.4)$$

$$b = \frac{(u_j - u_i)}{L} \quad (1.5)$$

$$u = \frac{x_j - x}{L} u_i + \frac{x - x_i}{L} u_j \quad (1.6)$$

$$u = N_i u_i + N_j u_j \quad (1.7)$$

where  $L = x_j - x_i$ .  $N_i$  and  $N_j$  are called shape functions. When a structure is loaded and reaches an equilibrium its potential energy must be minimum.

$$\Pi = \Lambda - W \quad (1.8)$$

where  $\Lambda$  is strain energy and  $W$  is work done by external loads (pressure load, body force, nodal forces).

$$W = u_i F_i + u_j F_j = \{U\}^T \{F\} \quad (1.9)$$

where:

$$\{U\}^T = \begin{bmatrix} u_i & u_j \end{bmatrix} \quad (1.10)$$

$$\{F\} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix} \quad (1.11)$$

$$\Lambda = \int_{x_i}^{x_j} \left( \frac{1}{2} \sigma \varepsilon A \right) dx = \frac{AE}{2} \int_{x_i}^{x_j} (\varepsilon^2) dx \quad (1.12)$$

where  $\sigma = E\varepsilon$  and is the stress,  $\Lambda$  is strain energy density,  $A$  is surface area and  $x$  is length. Using the definition of strain:

$$\varepsilon = \frac{du}{dx} \quad (1.13)$$

We can differentiate our shape function:

$$\varepsilon = \frac{(-u_i + u_j)}{L} \quad (1.14)$$

Leading to:

$$\Lambda = \frac{AE}{2L} (-u_i + u_j)^2 \quad (1.15)$$

$$\Lambda = \frac{AE}{2L} \begin{bmatrix} u_i & u_j \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \quad (1.16)$$

$$= \frac{1}{2} \{U\}^T [k] \{U\} \quad (1.17)$$

where the stiffness matrix is:

$$[k] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (1.18)$$

Potential energy must be minimised

$$\Pi = \frac{1}{2} \{U\}^T [k] \{U\} - \{U\}^T \{F\} \quad (1.19)$$

$$\frac{\partial \Pi}{\partial u_i} = \frac{\partial \Pi}{\partial u_j} = 0 \text{ or } \frac{\partial \Pi}{\partial \{U\}} = 0 \quad (1.20)$$

$$\frac{\partial \Pi}{\partial \{U\}} = [k] \{U\} - \{F\} = 0 \quad (1.21)$$

$$\begin{Bmatrix} F_i \\ F_j \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \quad (1.22)$$

$$\Pi = \sum_{e=1}^E (\Lambda^{(e)} - W) \quad (1.23)$$

Leading to:

$$\frac{\partial \Pi}{\partial \{U\}} = \left( \sum_{e=1}^E [k^{(e)}] \right) \{U\} - \{F\} = 0 \quad (1.24)$$

### 1.2.1 Exercise

Calculate the displacement and stress using the stiffness matrix.

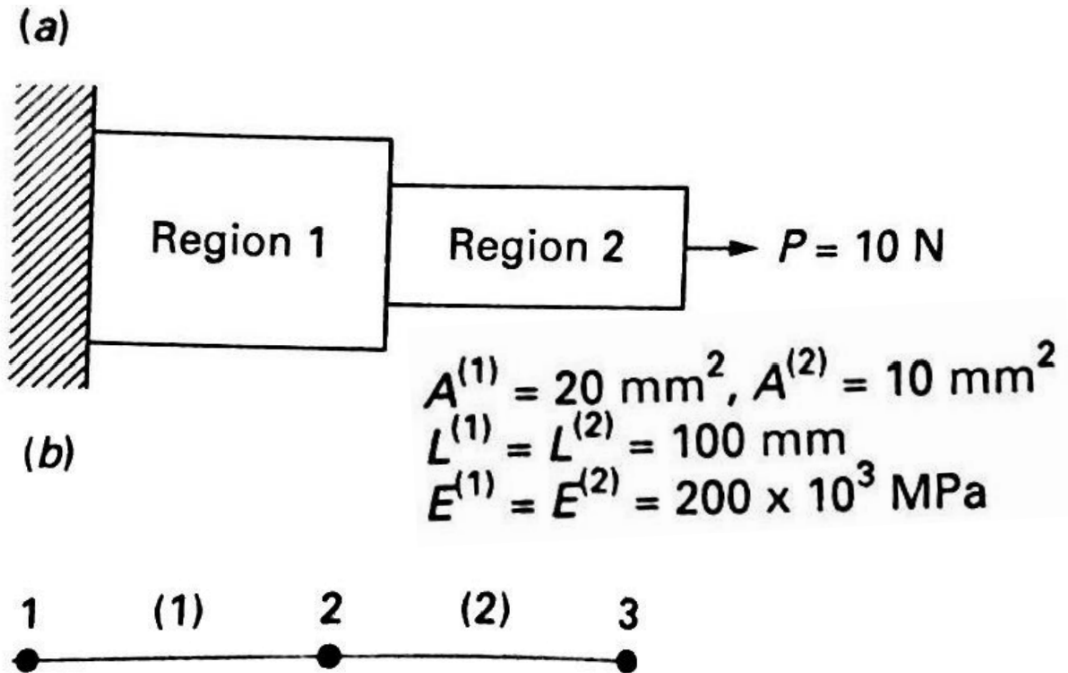


Figure 1.2: Exercise, 3 node 1D problem.

$$[k_1] = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \times 10^4 \text{ N mm}^{-1} \quad (1.25)$$

$$[k_2] = \frac{A_2 E_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \times 10^4 \text{ N mm}^{-1} \quad (1.26)$$

$$[k] = \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \times 10^4 \text{ N mm}^{-1} \quad (1.27)$$

$$\{F\}^T = \begin{bmatrix} 0 & 0 & 10 \end{bmatrix} 10^4 \times \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 + R1 \\ 0 \\ 10 \end{Bmatrix} \quad (1.28)$$

Leading to (commented in Latex):