MECH0024 Topic Notes

UCL

HD

November 14, 2021

Contents

Chapter 1

Introduction and Normal Shocks

1.1 Revision of Fundamental Concepts

1.1.1 Dimensionless Measures

Reynolds number:

$$Re = \frac{Ud}{v} \tag{1.1}$$

Mach number:

$$M = \frac{q}{c} = \frac{q}{(\gamma RT)^{\frac{1}{2}}} \tag{1.2}$$

Where:

- q = local flow speed
- U = characteristics flow speed
- d = characteristic lengthscale
- c = either local or characteristic speed of sound

The difference between a characteristic and a local measure is important, especially for compressible flows.

1.1.2 Classical Thermodynamics

First Law of Thermodynamics (change in internal energy, E):

$$\Delta E = Q - W \tag{1.3}$$

Second Law of Thermodynamics (entropy cannot decrease):

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} \tag{1.4}$$

1.1.3 Equation of State

The relationships for a perfect gas are:

$$p = \rho R T \tag{1.5}$$

$$c_p - c_v = R \tag{1.6}$$

$$\mathrm{d}U = c_p \, \mathrm{d}T \tag{1.7}$$

$$\mathrm{d}E = c_v \, \mathrm{d}T \tag{1.8}$$

Where E is the internal energy, U is the enthalpy. The isentropic index is:

$$\gamma = \frac{c_p}{c_v} \tag{1.9}$$

Gas	$R(\mathrm{Km^2s^{-2}})$	$\rho(kgm^{-3})$	γ
H_2	4124	0.822	1.41
He	2077	1.63	1.66
Dry air	287	1.18	1.40
N_2	297	1.14	1.40

Figure 1.1

1.1.4 Terminology

Adiabatic - no heat in / work done (strong changes)
Isentropic – no change in entropy (weak changes)

1.1.5 Conservation Principles

There are two frameworks to analyse fluid and solid mechanics.

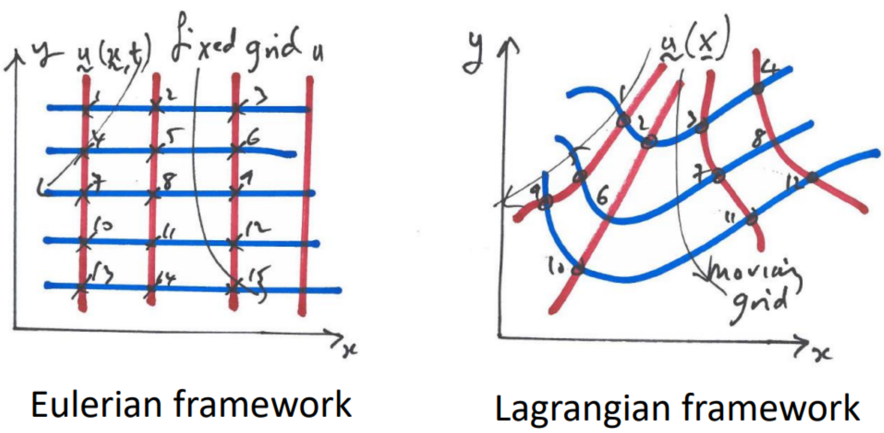


Figure 1.2

- Eulerian – information at fixed points
- Lagrangian – information at points that move with fluid or solid
- They both have advantages and disadvantages.

1.1.6 Conservation of Mass

Integral form of conservation law for a Lagrangian control volume:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_L} \rho \,\mathrm{d}V = 0 \tag{1.10}$$

Differential form for the conservation of mass:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho(\nabla \cdot u) \tag{1.11}$$

For an incompressible fluid:

$$\nabla \cdot u = 0 \tag{1.12}$$

1.1.7 Conservation of Linear Momentum

Integral form of the conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_L} \rho u \,\mathrm{d}V = \int_{S_L} \sigma \cdot \hat{n} \,\mathrm{d}S + \int_{V_L} F \,\mathrm{d}V \tag{1.13}$$

Where:

- σ is the stress tensor
- p is the pressure
- τ is the viscous stress tensor

$$\sigma = -pI + \tau \tag{1.14}$$

Differential form of the conservation of momentum:

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = \nabla \cdot \sigma + F \tag{1.15}$$

$$\rho(x,t)\left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right) = -\nabla p \tag{1.16}$$

This is Euler's equation for an inviscid fluid. The flow is compressible (explicitly stated).

1.1.8 Conservation of Angular Momentum

Integral form of the conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_L} \rho x \times u \,\mathrm{d}V = \int_{S_L} x \times \sigma \cdot \hat{n} \,\mathrm{d}S + \int_{V_L} x \times F \,\mathrm{d}V \tag{1.17}$$

The differential form of the conservation law:

$$x \times \left(\rho \frac{\mathrm{D}u}{\mathrm{D}t} - \nabla \cdot \sigma\right) = \epsilon_{ilk}\sigma_{kl} \tag{1.18}$$

Consequence is that the stress tensor is symmetric:

$$\sigma_{ij} = \sigma_{ji} \tag{1.19}$$

1.1.9 Conservation of Energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_L} \rho\left(E + \frac{1}{2}q^2\right)\mathrm{d}V = -\int_{S_L} u \cdot \sigma \cdot \hat{n}\,\mathrm{d}S + \int_{S_L} k\nabla T \cdot \hat{n}\,\mathrm{d}S + \int_{V_L} u \cdot F\,\mathrm{d}V \tag{1.20}$$

$$E_T = E + \frac{1}{2}q^2 \tag{1.21}$$

Where $q = |u|$ is the fluid speed. The differential form of the energy equation is:

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(E + \frac{1}{2}q^2\right) = -\nabla \cdot (u \cdot \sigma) + \nabla \cdot (k\nabla T) + u \cdot F \tag{1.22}$$

The continuum form on the conservation of energy says:

$$\frac{\mathrm{D}E}{\mathrm{D}t} = -\frac{p}{\rho}(\nabla \cdot u) + \frac{1}{\rho}(\phi + \nabla \cdot (k\nabla T)) \tag{1.23}$$

$$\frac{\mathrm{D}E}{\mathrm{D}t} = -\frac{pD\left(\frac{1}{\rho}\right)}{\mathrm{D}t} + \frac{\phi + \nabla \cdot (k\nabla T)}{\rho} \tag{1.24}$$

The dissipation is:

$$\phi = \nabla \cdot (u\sigma) - u \cdot \nabla\sigma \tag{1.25}$$

Compare to the differential form that you have met before:

$$\mathrm{d}E = -p\,\mathrm{d}\left(\frac{1}{\rho}\right) + \mathrm{d}Q \tag{1.26}$$

For an inviscid fluid (with no viscous dissipation $\sigma = -pI$) and no diffusion of heat:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(E + \frac{1}{2}q^2\right) = -\frac{1}{\rho}\nabla \cdot (pu) \tag{1.27}$$

1.1.10 Bernoulli's Equation

Form	Equation	Conservation Law
Isothermal	$\frac{p}{\rho} + \frac{1}{2}q^2 = const$	Mechanical energy
Isothermal	$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}q^2 + \Upsilon = const$	Mechanical energy
Adiabatic	$E + \frac{p}{\rho} + \frac{1}{2}q^2 = const$	1st law of thermodynamics
Adiabatic & perfect gas	$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}q^2 = const$	1st law of thermodynamics & perfect gas law

Figure 1.3

1.1.11 Reference State (Stagnation)

There are two possible reference states.

(a) Stagnation flow condition

$$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}q^2 = const \tag{1.28}$$

The speed of sound is c where:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \gamma RT = \frac{\gamma p}{\rho} \tag{1.29}$$

$$\frac{p}{\rho}\left(1 + \frac{1}{2}(\gamma - 1)M^2\right) = const \tag{1.30}$$

$$T\left(1 + \frac{1}{2}(\gamma - 1)M^2\right) = const \tag{1.31}$$

We can set the reference constant to be when the flow is a rest or stagnant. For flows where changes are significant, this state cannot be realised without other processes occurring, so that:

$$1 + \frac{1}{2}(\gamma - 1)M^2 = \frac{T_0}{T} \tag{1.32}$$

This tells:

$$1 + \frac{1}{2}(\gamma - 1)M^2 = \frac{p_0\rho}{p\rho_0} \tag{1.33}$$

This relationship is not useful, unless combined with the isentropic relationship $\dfrac{p}{\rho^\gamma} = const$:

$$\frac{p}{p_0} = \left(1 + \frac{1}{2}(\gamma - 1)M^2\right)^{-\dfrac{\gamma}{\gamma - 1}} \tag{1.34}$$

$$\frac{\rho}{\rho_0} = \left(1 + \frac{1}{2}(\gamma - 1)M^2\right)^{-\dfrac{1}{\gamma - 1}} \tag{1.35}$$

(b) Sonic flow condition

$$\frac{p_*}{p_0} = \left(\frac{2}{\gamma + 1}\right)^{\dfrac{\gamma}{\gamma - 1}} = 0.5283 \tag{1.36}$$

$$\frac{\rho_*}{\rho_0} = \left(\frac{2}{\gamma + 1}\right)^{\dfrac{1}{\gamma - 1}} = 0.63 \tag{1.37}$$

$$\frac{T_*}{T_0} = \frac{2}{\gamma + 1} = 0.8333 \tag{1.38}$$

1.2 Normal Shocks

1.2.1 Assumptions

The flow adjusts over a short distance from one region to another and the streamlines are parallel and not deflected.

This is called a normal shock. The distance can be very short (comparable with the mean-free path, $10\mu m$) so that the thickness of the wave may be ignored. Although viscous effects may be important within the wave, an inviscid analysis can be applied to understand these processes. We consider the flow across a shock wave and denote the flow properties with 1 upstream and 2 downstream.

1.2.2 Conservation Principles

Conservation of mass:

$$\nabla \cdot (\rho u) = 0 \tag{1.39}$$

Conservation of momentum:

$$\rho u \cdot \nabla u + \nabla p = 0 \tag{1.40}$$

Conservation of energy:

$$\rho u \cdot \nabla \left(E + \frac{1}{2} q^2 \right) + \nabla \cdot (pu) = 0 \tag{1.41}$$

To apply these relationships we have to integrate them across the shock. Remember, the gradient of these variables is zero on each side because the changes are confined to the thin shock.

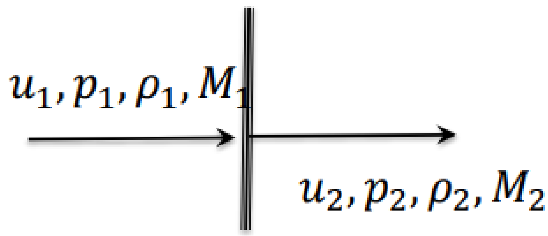


Figure 1.4

$$\rho_2 u_2 = \rho_1 u_1 \tag{1.42}$$

$$\rho_2 u_2^2 + p_2 = \rho_1 u_1^2 + p_1 \tag{1.43}$$

$$\frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} + \frac{1}{2} u_2^2 = \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} + \frac{1}{2} u_1^2 \tag{1.44}$$

$$\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2} \tag{1.45}$$

Strength of shock is:

$$\frac{p_2}{p_1} \tag{1.46}$$

1.2.3 Frame of Reference

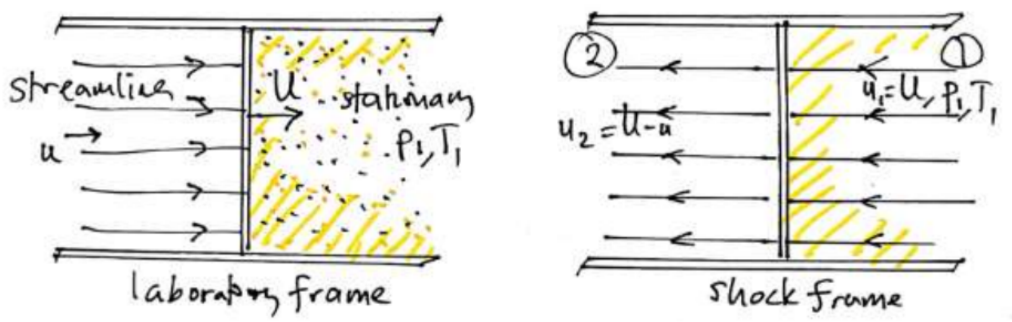


Figure 1.5: Diagram showing the frame of reference for an example question

When we consider scenarios such as shocks propagating in stationary flows, like a bomb or a pressure pulse is moving into a stationary region, we don't investigate it in this complex form. We hop on a frame of reference of the shock so the flow tends to be steady, and then we can conduct an analysis on this framce of reference.

1.2.4 Solution Technique

Aim of the calculation is to relate the flow upstream of the shock to downstream of the shock. There are 4 equations: mass, momentum, energy and state. One useful way to solve this is to use 3 of them at a time. The systems are solved in pairs:

(a) mass, momentum and state (Rayleigh flow) - This occurs in when heat is added to a flow
(b) mass, energy and state (Fanno flow) - This occurs in when pipe friction is important

1.2.5 Algebraic Manipulation

From the momentum equation:

$$p_1 - p_2 = \rho_2 u_2^2 - \rho_1 u_1^2 = \rho_1 u_1 (u_2 - u_1) \tag{1.47}$$

Rearranging gives:

$$u_2^2 - u_1^2 = \frac{(p_1 - p_2)(u_1 + u_2)}{\rho_1 u_1} = (p_1 - p_2)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \tag{1.48}$$

$$\frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right) = (p_1 - p_2)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \tag{1.49}$$

Rankine-Hugoniot relation:

$$\frac{\rho_2}{\rho_1} = \frac{\left(\dfrac{\gamma+1}{\gamma-1}\right)\dfrac{p_2}{p_1} + 1}{\dfrac{p_2}{p_1} + \dfrac{\gamma+1}{\gamma-1}} \tag{1.50}$$

The strength of the shock is:

$$\frac{p_2}{p_1} = \frac{\left(\dfrac{\gamma+1}{\gamma-1}\right)\dfrac{\rho_2}{\rho_1} - 1}{\dfrac{\gamma+1}{\gamma-1} - \dfrac{\rho_2}{\rho_1}} \tag{1.51}$$

This relationship is valid of shocks and highly non-linear behaviour. The first point in the discussion is what happens when the shock is weak.

1.2.6 Weak Shocks

This can be demonstrated analytically by considering changes of pressure and density across the shock. To show this, let $p_2 = p_1 + \Delta p$ and $\rho_2 = \rho_1 + \Delta\rho$, then substituting into the Rankine-Hugoniot gives:

$$\frac{\Delta p}{p_1} = \frac{p_2 - p_1}{p_1} = \frac{\dfrac{2\gamma(\rho_2-\rho_1)}{(\gamma-1)\rho_1}}{\dfrac{2}{\gamma-1}} = \frac{\gamma\Delta\rho}{\rho_1} \tag{1.52}$$

which is the same as the isentropic approximation obtained by taking the differential of $\frac{p}{\rho^\gamma} = const.$

We can write ρu^2 as $\rho c^2 M^2 = \gamma p M^2$. Thus the momentum equation may be written as:

$$p_1 - p_2 = \gamma p_2 M_2^2 - \gamma p_1 M_1^2 \tag{1.53}$$

or

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \tag{1.54}$$

The density ratio is:

$$\frac{\rho_1}{\rho_2} = \frac{p_2}{p_1}\left(\frac{M_2}{M_1}\right)^2 \tag{1.55}$$

This is known as the **Rayleigh line**.

Since there is no change in stagnation temperature across the shock:

$$\frac{T_2}{T_1} = \frac{1+\frac{1}{2}(\gamma-1)M_1^2}{1+\frac{1}{2}(\gamma-1)M_2^2} \tag{1.56}$$

From the equations of continuity and state:

$$\frac{T_2}{T_1} = \frac{p_2\rho_1}{p_1\rho_2} = \frac{p_2 u_2}{p_1 u_1} = \frac{p_2 M_2}{p_1 M_1}\left(\frac{T_2}{T_1}\right)^{\frac{1}{2}} \tag{1.57}$$

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\frac{M_2}{M_1}\right)^2 \tag{1.58}$$

Substituting into the equation of state gives:

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{1+\frac{1}{2}(\gamma-1)M_1^2}{1+\frac{1}{2}(\gamma-1)M_2^2}\right)^{\frac{1}{2}} \tag{1.59}$$

This is known as the **Fanno line**.

Matching the solutions to the Fanno and Rayleigh lines gives the combined solution to the mass, momentum, energy conservation equations and the equation of state:

$$\frac{M_1\left(1+\frac{1}{2}(\gamma-1)M_1^2\right)^{\frac{1}{2}}}{1+\gamma M_1^2} = \frac{M_2\left(1+\frac{1}{2}(\gamma-1)M_2^2\right)^{\frac{1}{2}}}{1+\gamma M_2^2} \tag{1.60}$$

The solutions to this are:

$$M_1 = M_2 \tag{1.61}$$

$$M_2^2 = \frac{1+\frac{1}{2}(\gamma-1)M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma-1)} \tag{1.62}$$

Note that for $M_1 = M_2 = 1$ there is no shock. For upstream supersonic flows $M_1 > 1$, the downstream flow is subsonic $M_2 < 1$.

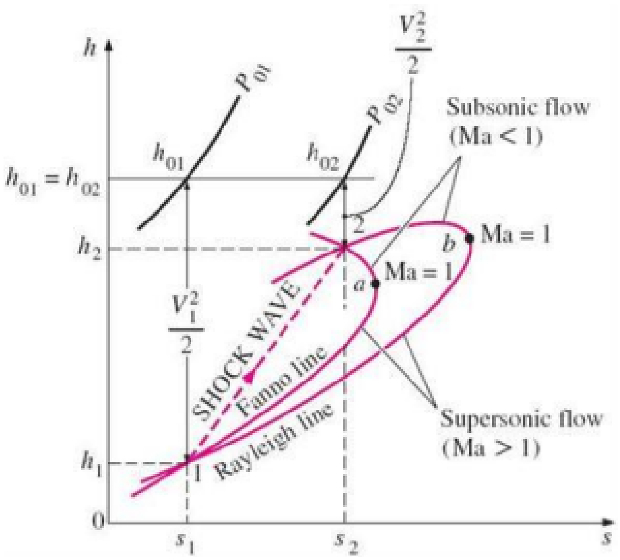


Figure 1.6

The pressure and density ratios are important:

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \tag{1.63}$$

$$\frac{\rho_2}{\rho_1} = \frac{\frac{1}{2}(\gamma+1)M_1^2}{1+\frac{1}{2}(\gamma-1)M_1^2} \tag{1.64}$$

Stagnation pressure values can be calculated from:

$$\frac{p_{20}}{p_{10}} = \frac{p_{20}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{10}} \tag{1.65}$$

where the relationship been static and stagnation pressures have been defined.

1.2.7 Entropy Considerations

From the conservation of energy:

$$\mathrm{d}Q = \mathrm{d}E + \mathrm{d}W = c_v\,\mathrm{d}T + pd\left(\frac{1}{\rho}\right) \tag{1.66}$$

Since:

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} = \frac{c_v\,\mathrm{d}T}{T} + \frac{p}{T}d\left(\frac{1}{\rho}\right) = \frac{c_v + R}{T}\,\mathrm{d}T - R\frac{\mathrm{d}p}{p} \tag{1.67}$$

Integrating gives an entropy change of:

$$\Delta s = \int_1^2 \mathrm{d}s = c_p \log\left(\frac{T_2}{T_1}\right) - R\log\left(\frac{p_2}{p_1}\right) = c_p \log\left(\frac{T_2}{T_1}\left(\frac{p_1}{p_2}\right)^{\dfrac{\gamma-1}{\gamma}}\right) \tag{1.68}$$

Since $c_p = \dfrac{\gamma R}{\gamma - 1}$, we can rearrange as:

$$\frac{\Delta s}{R} = \frac{\gamma}{\gamma - 1}\left(\log\left(\frac{\rho_1}{\rho_2}\right) + \frac{1}{\gamma}\log\left(\frac{p_2}{p_1}\right)\right) \tag{1.69}$$

Specifically for a shock:

$$\frac{\Delta s}{R} = \frac{\gamma}{\gamma - 1}\log\left(\frac{2}{(\gamma+1)M_1^2} + \frac{\gamma-1}{\gamma+1}\right) + \frac{1}{\gamma-1}\log\left(\frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\right) \tag{1.70}$$

When $M_1 < 1$, $M_2 > 1$ and $\Delta s < 0$. This is unphysical and is ignored. But when $M_1 > 1$, $M_2 < 1$ and $\Delta s > 0$.

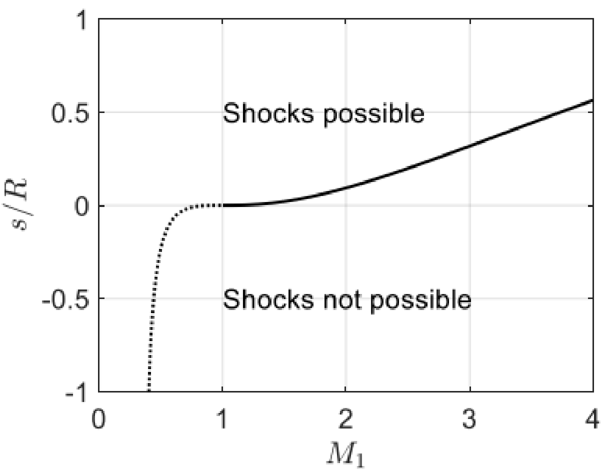


Figure 1.7

1.2.8 $T-s$ and $p-1/\rho$ Diagrams

We use these types of figures to analyse systems.

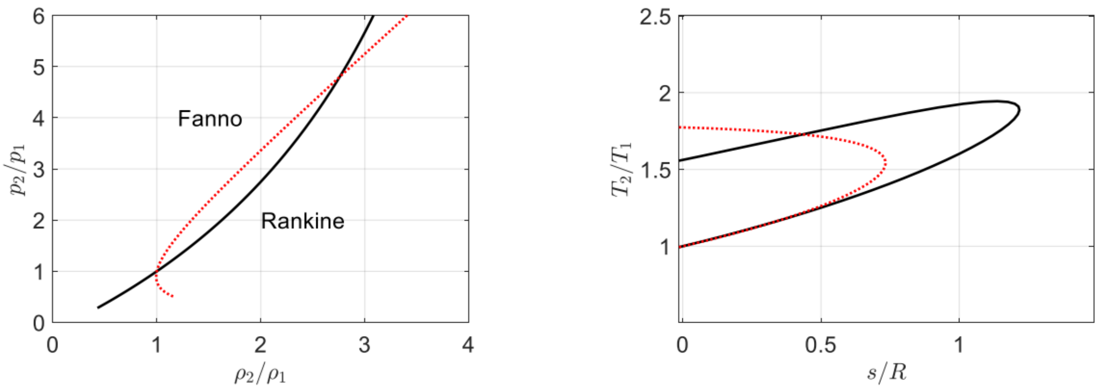


Figure 1.8

$\frac{p_{02}}{p_{01}} < 1$	Loss in stagnation pressure
$\frac{p_2}{p_1} > 1$	Increase in static pressure
$\frac{T_2}{T_1} > 1$	Increase in static temperature
$\frac{\rho_2}{\rho_1} > 1$	Increase in density
$\frac{u_2}{u_1} < 1$	Decrease in velocity
$M_2 < 1$	Subsonic flow behind shock
$\frac{T_{02}}{T_{01}} = 1$	No change in stagnation temperature

Figure 1.9

Chapter 2

Oblique Shocks

2.1 Recap

2.1.1 Analogy with hydraulic jumps

"Supersonic flows" in the kitchen. In free-surface flows, we have the Froude number

$$Fr = \frac{u}{c} \tag{2.1}$$

- $Fr < 1$: subcritical
- $Fr > 1$: supercritical
- u: (depth averaged) flow speed
- c: wave speed

Understanding these flows are important for looking at forces on buildings/bridges. The same conservation principles apply (except energy; which is not conserved here) and is equivalent to $\delta = 2$. Used by Ernest Mach to illustrate shock collision with walls.

2.1.2 Oblique shocks and terminology

The Mach number varies with position in space. Shock represents:

1. change in stagnation pressure

and depends on

2. stagnation pressure unchanged

3. rapid changes over a short distance

- Oblique shock - flow is deflected and squashed.
- Weak oblique shocks (Mach number is supersonic on both sides).
- Strong oblique shock (Mach number is subsonic after shock)

2.1.3 Streamlines

The local **instantaneous** velocity $u(x, t)$ is tangential to the local streamline. Streamlines have directions and need arrows!

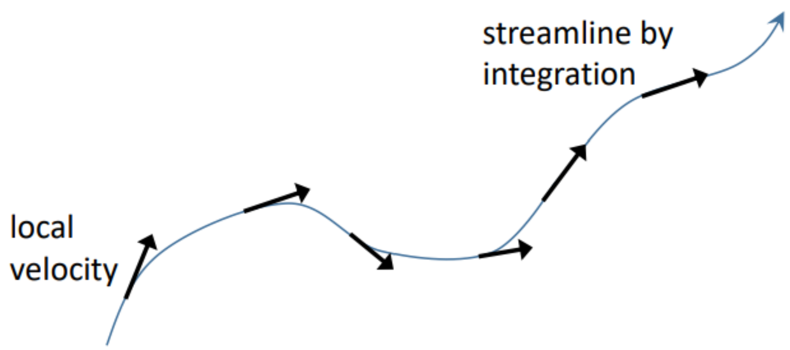


Figure 2.1: Streamline.

(note the difference from pathlines and streaklines).

- Streamlines do not cross (except at point source and sinks, or where mass is created or lost)
- They cannot meet at rigid bodies except at a special point (the stagnation point)
- THe surface of a body is a streamline

2.2 Oblique shocks

2.2.1 Two types of oblique shock waves

Geometry controlled shocks

This is where the deflection angle of the flow is specified.

Pressure controlled shocks

THis is where the pressure after the shock is known. Occurs in supersonic jets.

2.2.2 Schematic and notation

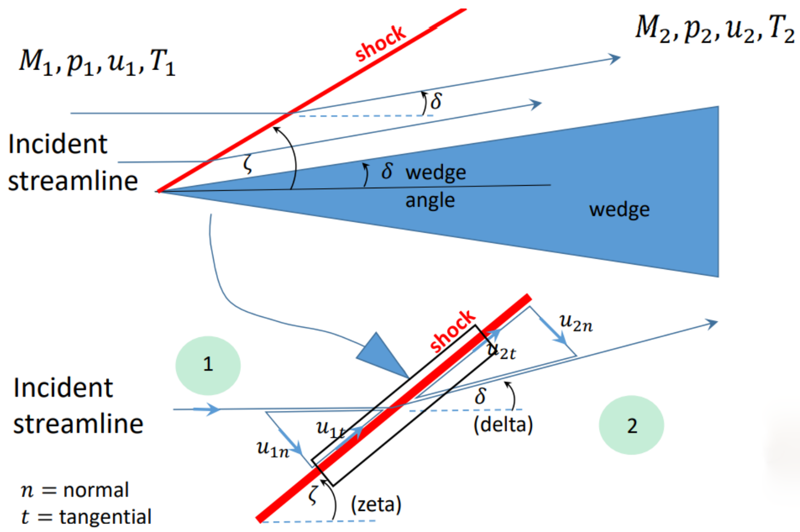


Figure 2.2: Schematic and notation.

Geometry controlled case. All angles are relative to incident streamline.

$$\delta \text{ - deflection or wedge angle} \tag{2.2}$$
$$\zeta \text{ - shock angle} \tag{2.3}$$

The aim is to relate δ, ζ, M and $\frac{p_2}{p_1}$. We need to know M_1 and δ - geometrically constrained or M_1 and $\frac{p_2}{p_1}$ pressure constrained.

2.2.3 Conservation equations

The differential form of the conservation laws are mass:

$$\underline{\nabla} \cdot (\rho \underline{u}) = 0 \text{ (scalar)} \tag{2.4}$$

Momentum:

$$\rho \underline{u} \cdot \underline{\nabla u} + \underline{\nabla} p = 0 \text{ (vector)} \tag{2.5}$$

Energy:

$$\rho \underline{u} \cdot \underline{\nabla} \left(E + \frac{1}{2} q^2 \right) + \underline{\nabla} \cdot (p \underline{u}) = 0 \tag{2.6}$$

The word equation form is sometimes more useful:

1. Mass: mass flux is conserved
2. Momentum: increase in momentum flux is balanced by a decrease in pressure
3. Energy: sum of internal energy and kinetic energy is equal to the work done by press.

2.2.4 Normal shock relations

$$\rho_2 u_{2n} = \rho_1 u_{1n} \tag{2.7}$$
$$\rho_2 u_{2n}^2 + p_2 = \rho_1 u_{1n}^2 + p_1 \tag{2.8}$$
$$\frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}q_2^2 = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}q_1^2 \tag{2.9}$$

The conservation of energy is expressed in terms of the (specific) kinetic energy that depends on the gas speed $q^2 = u^2 + v^2$. As we are going to see, these relationships are identical to the oblique shock analysis.

2.2.5 Oblique shock relations

Integrating the conservation laws:

$$\rho_2 u_{2n} = \rho_1 u_{1n} \tag{2.10}$$
$$\rho_2 u_{2n}^2 + p_2 = \rho_1 u_{1n}^2 + p_1 \tag{2.11}$$
$$\frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}\left(u_{2n}^2 - \cancel{u_{2t}^2}\right) = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}\left(u_{1n}^2 - \cancel{u_{1t}^2}\right) \tag{2.12}$$

The conservation of momentum tangential to the shock is:

$$\rho \underline{u} \cdot \underline{\nabla} u_t = \underline{u} p \cdot \underline{\hat{t}} \tag{2.13}$$

Where $\underline{\hat{t}}$ is the unit vector tangential to the shock. Since the pressure is constant on both sides of the shock, the RHS is zero. This means that:

$$\rho_1 u_{1n} u_{1t} = \rho_2 u_{2n} u_{2t} \text{ or } u_{1t} = u_{2t} \tag{2.14}$$

From the inclination of the shock and deflection angle:

$$u_{1n} = u_1 \sin\zeta \tag{2.15}$$
$$u_{2n} = u_2 \sin(\zeta - \delta) \tag{2.16}$$
$$u_{1t} = u_1 \cos\zeta \tag{2.17}$$
$$u_{2t} = u_2 \cos(\zeta - \delta) \tag{2.18}$$

Normal shock relationship:

$$M_{2n}^2 = \frac{1 + \frac{1}{2}(\gamma - 1) M_{1n}^2}{\gamma M_{1n}^2 - \frac{1}{2}(\gamma - 1s)} \tag{2.19}$$

Oblique shock relationship:

$$M_2^2 \sin^2(\zeta - \delta) = \frac{1 + \frac{1}{2}(\gamma - 1) M_1^2 \sin^2\zeta}{\gamma M_1^2 \sin^2\zeta - \frac{1}{2}(\gamma - 1)} \tag{2.20}$$

This gives a relationship between 3 variables:

$$\tan\delta = \frac{\cot\left(\zeta\left(M_1^2 \sin^2(\zeta - 1)\right)\right)}{\frac{1}{2}(\gamma + 1) M_1^2 - M_1^2 \sin^2(\zeta) + 1} \tag{2.21}$$

2.2.6 Relationship between M_1, ζ and δ

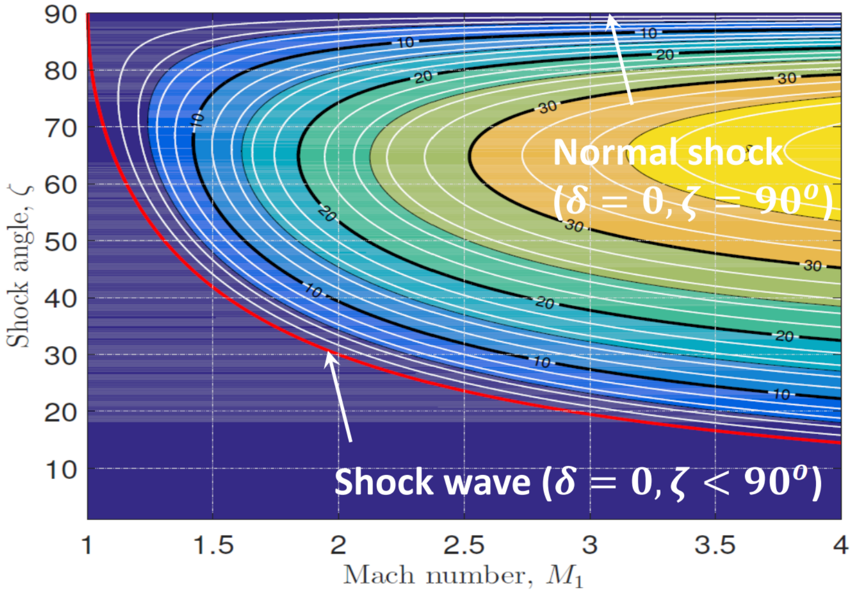


Figure 2.3: Shock angle vs Mach number.

We deconstruct the solution into parts that we can understand before looking at more specifically the solutions. For $\delta = 0$ (no streamline deflection), we have:

- $\zeta = 90°$: normal shock
- $\zeta < 90°$: shock wave

2.2.7 Sonic waves

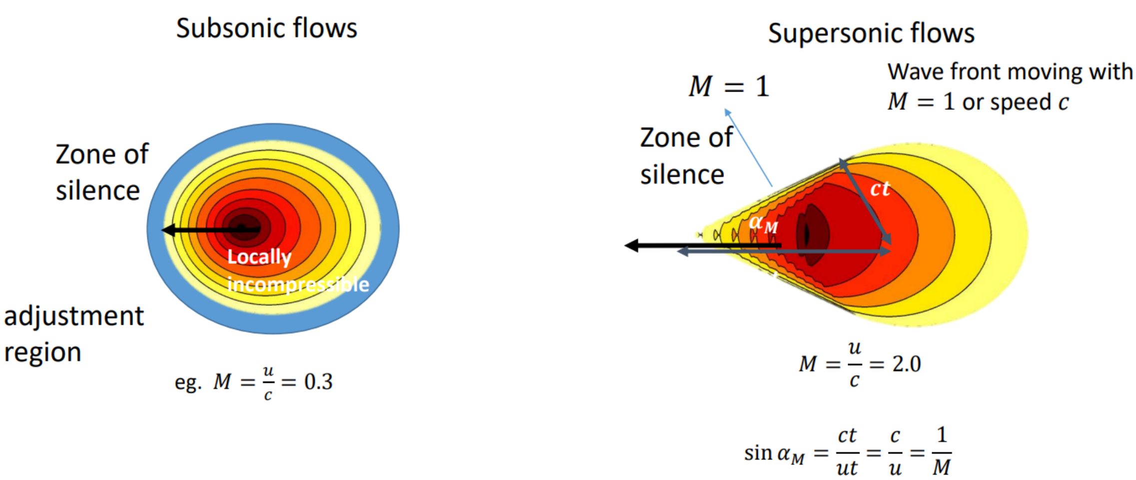


Figure 2.4: Subsonic vs supersonic flows.

2.2.8 Multiple solutions (δ, ζ, M_1)

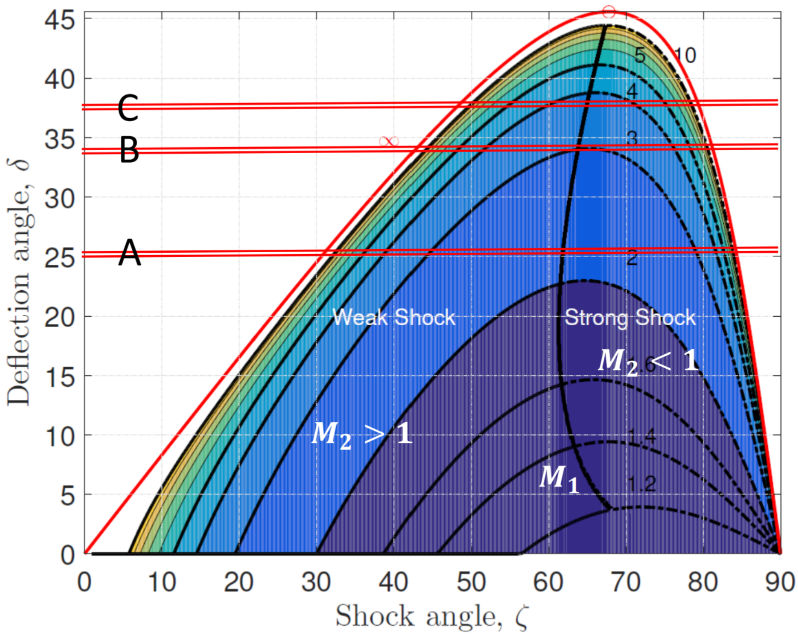


Figure 2.5: Deflection angle vs shock angle.

Look at the case where $M_1 = 3$ with three examples of $\delta = 25°$, $34°$ and $38°$.

- (A) There are two solutions here, a weak shock where $\zeta = 45°$ and $78°$
- (B) There is one unique solution $\zeta = 68°$ for $\delta = 34°$. At wedge angles greater than this, we have case (C)
- (C) There is no solution here. This is because there is no way for the flow to be able to adjust by an oblique shock. Something else happens.

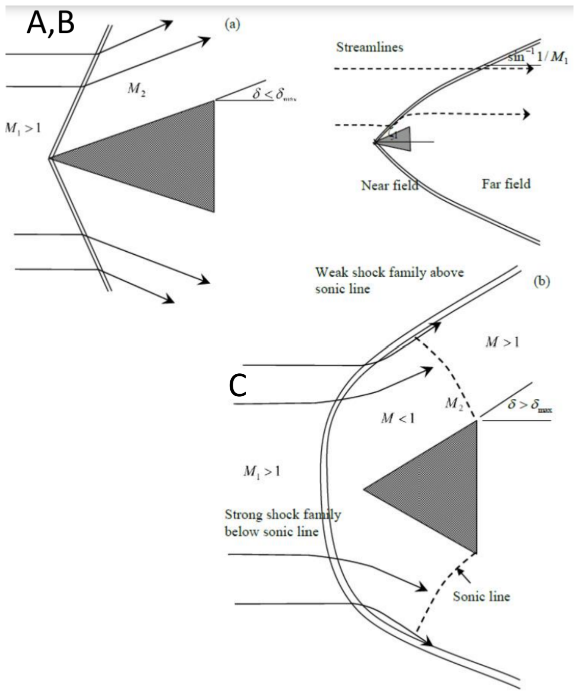


Figure 2.6: The shock angles changes from the near field to the far field.

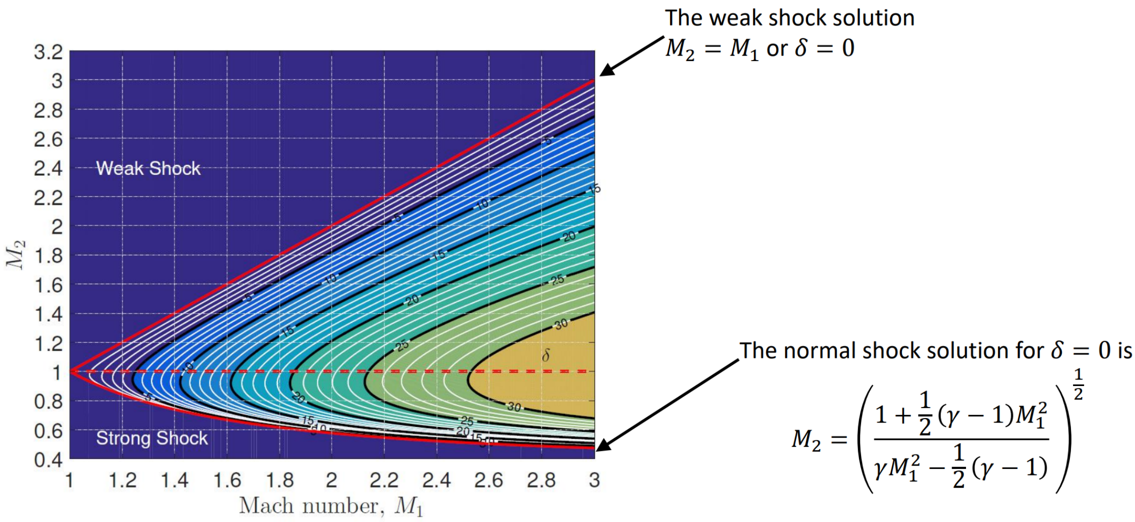


Figure 2.7: M_2 vs Mach number (M_1).

2.2.9 Pressure ratio $\frac{p_2}{p_1}$

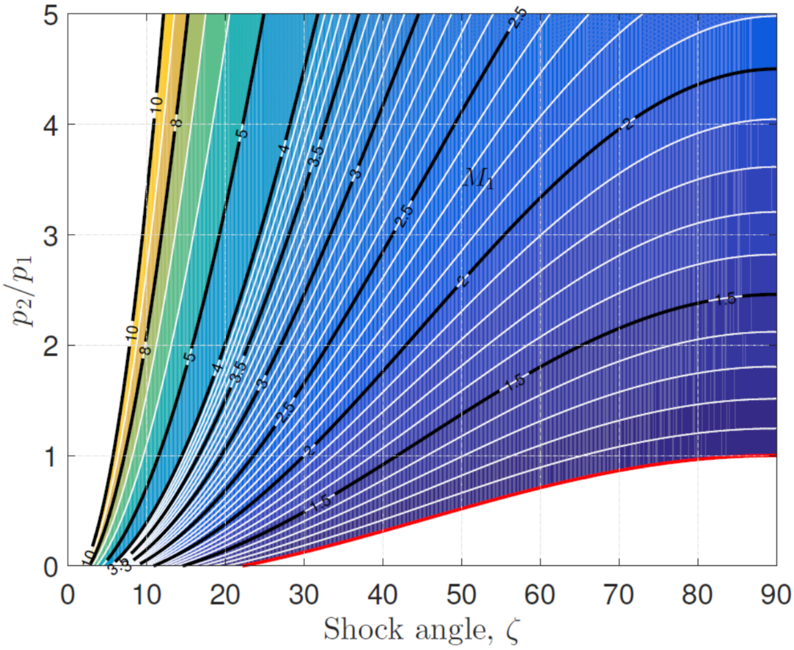


Figure 2.8: Pressure ratio vs shock angle.

The progression of the pressure across the shock is important and is calculated graphically here. Notice that in all cases, $\frac{p_2}{p_1} > 1$. In the examples we have seen, the flow is controlled by the bounding geometry. In many other cases however, it is controlled by pressure, for instance, at the edge of a rocket exit.

2.2.10 Summary

Two types of shocks: geometry controlled or pressure controlled. Two types of shocks: weak shocks and strong shocks.

Chapter 3

Prandtl-Meyer Expansion Fans

3.1 Prandtl-Meyer Expansion Fans

3.1.1 Examples of supersonic expansion

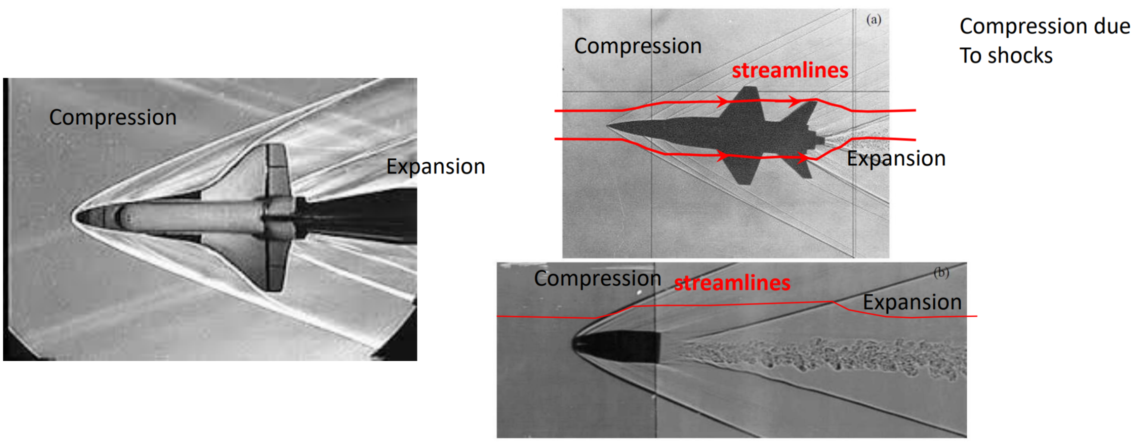


Figure 3.1: Examples of supersonic expansion.

3.1.2 Sonic waves

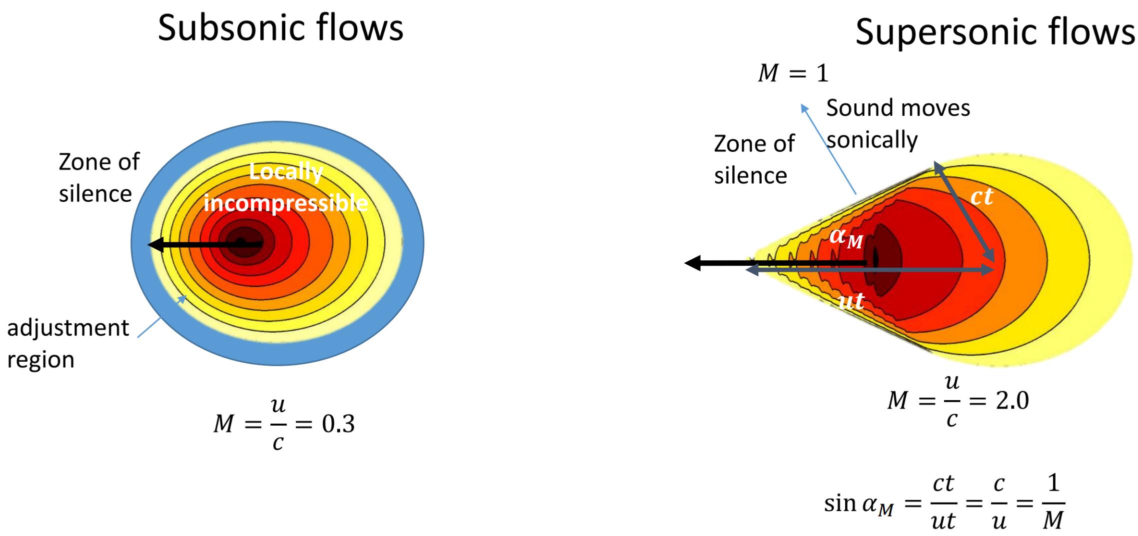


Figure 3.2: Sonic waves.

3.1.3 Turning a corner

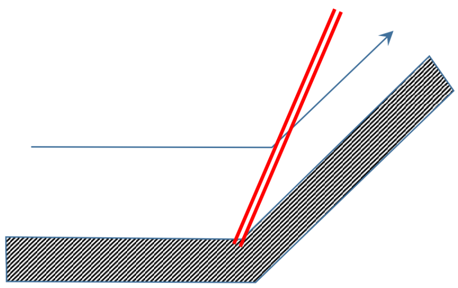


Figure 3.3: Shock is formed. Entropy decreases.

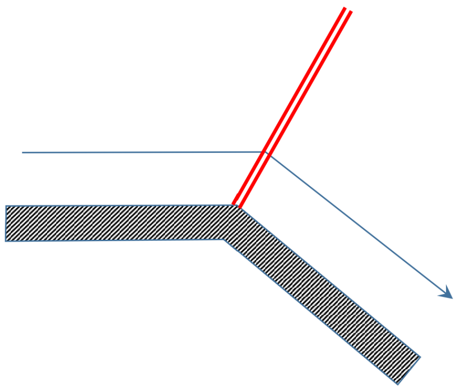


Figure 3.4: If a single shock is formed, entropy increases and this is unphysical.

A we have seen in the previous section, as the flow is turned by a wedge, the flow is compressed and the Mach number decreases. Whether the flow downstream of an oblique shock is supersonic depends on the deflection angle.

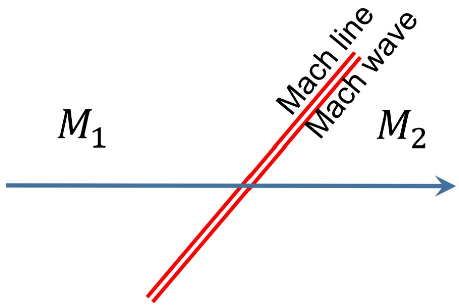


Figure 3.5

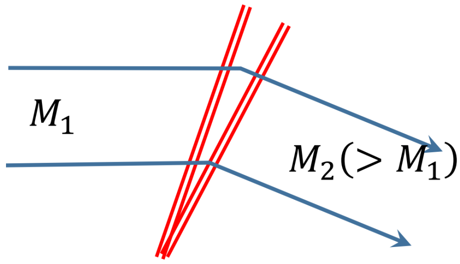


Figure 3.6

A Mach line or waves is a very weak shock comprised of an isentropic process. We can study the relationship between the turning angle and the change in Mach number by studying the deflection process in more detail.

Here we consider the case where the flow expands and the process of adjustment. The Mach number of the flow will increase due to expansion, but the pressure wil decrease. The process of adjustment takes place by shockwaves being emitted. The first result we need to consider is the angle ν that the shock waves make to the incident flow.

3.1.4 Schematic and notation

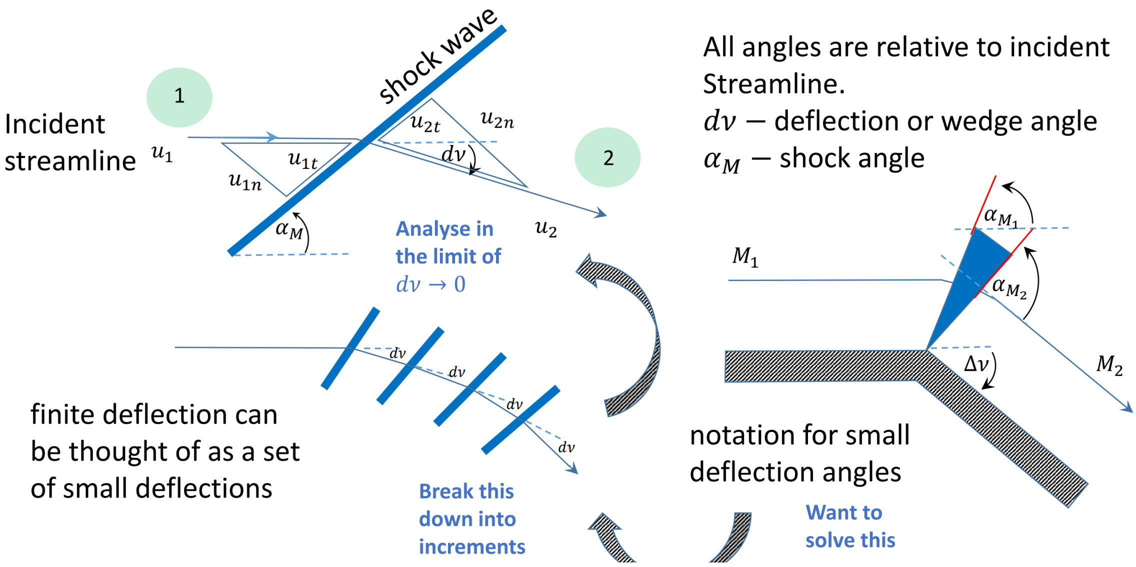


Figure 3.7: Schematic and notation.

3.1.5 Geometrical characteristics

The slip velocity is continuous so that:

$$u_1 \cos \alpha_M = u_2 \cos (\alpha_M + \mathrm{d}v) \tag{3.1}$$

From the double angle formula:

$$\cos\left(\alpha_M + \mathrm{d}v\right) \approx \cos\alpha_M - \mathrm{d}v\sin\alpha_M \tag{3.2}$$

Rearranging gives:

$$\frac{\mathrm{d}u}{u} = \frac{u_2 - u_1}{u_2} = \frac{\sin\alpha_M}{\cos\alpha_M}\,\mathrm{d}v \tag{3.3}$$

Since the Mach wave angle α_m is related to the Mach number by:

$$\sin\alpha_M = \frac{1}{M} \tag{3.4}$$

Combining and using $\cos^2\alpha_M = 1 - \sin^2\alpha_M$, gives:

$$\frac{\mathrm{d}u}{u} = \frac{\sin\alpha_M}{\left(1 - \sin^2\alpha_M\right)^{\frac{1}{2}}}\,\mathrm{d}v = \frac{\mathrm{d}v}{\left(M^2 - 1\right)^{\frac{1}{2}}} \tag{3.5}$$

We can write:

$$\frac{u^2}{M^2} = \frac{u^2}{\left(\dfrac{u^2}{\gamma RT}\right)} = \gamma RT \tag{3.6}$$

From the Conservation of Energy,

$$T\left(1 + \frac{1}{2}\left(\gamma - 1\right)M^2\right) = \text{const} \tag{3.7}$$

Taking the square root gives:

$$\frac{u}{M}\left(1 + \frac{1}{2}\left(\gamma - 1\right)M^2\right)^{\frac{1}{2}} = \text{const} \tag{3.8}$$

Taking the logarithm:

$$\log u - \log M + \frac{1}{2}\log\left(1 + \frac{1}{2}\left(\gamma - 1\right)M^2\right) = \text{const} \tag{3.9}$$

To find the differential change we take the differential to give us:

$$\frac{\mathrm{d}u}{u} - \frac{\mathrm{d}M}{M} + \frac{\frac{1}{2}\left(\gamma - 1\right)\frac{1}{2}2M}{1 + \frac{1}{2}\left(\gamma - 1\right)M^2}\,\mathrm{d}M = 0 \tag{3.10}$$

Rearranging gives:

$$\frac{\mathrm{d}u}{u} = \left(\frac{1}{M} - \frac{\frac{1}{2}\left(\gamma - 1\right)\frac{1}{2}2M}{1 + \frac{1}{2}\left(\gamma - 1\right)M^2}\right)\frac{\mathrm{d}M}{M} \tag{3.11}$$

$$= \frac{1}{1 + \frac{1}{2}\left(\gamma - 1\right)M^2}\frac{\mathrm{d}M}{M} \tag{3.12}$$

Combining the above equations:

$$\frac{\mathrm{d}v}{\left(M^2 - 1\right)^{\frac{1}{2}}} = \frac{\mathrm{d}u}{u} = \frac{\mathrm{d}M}{M\left(1 + \frac{1}{2}\left(\gamma - 1\right)M^2\right)} \tag{3.13}$$

In total,

$$\Delta v = \int_0^{\Delta v} = \int_{M_1}^{M_2}\left(\frac{\left(M^2 - 1\right)^{\frac{1}{2}}}{M\left(1 + \frac{1}{2}\left(\gamma - 1\right)M^2\right)}\right)\mathrm{d}M \tag{3.14}$$

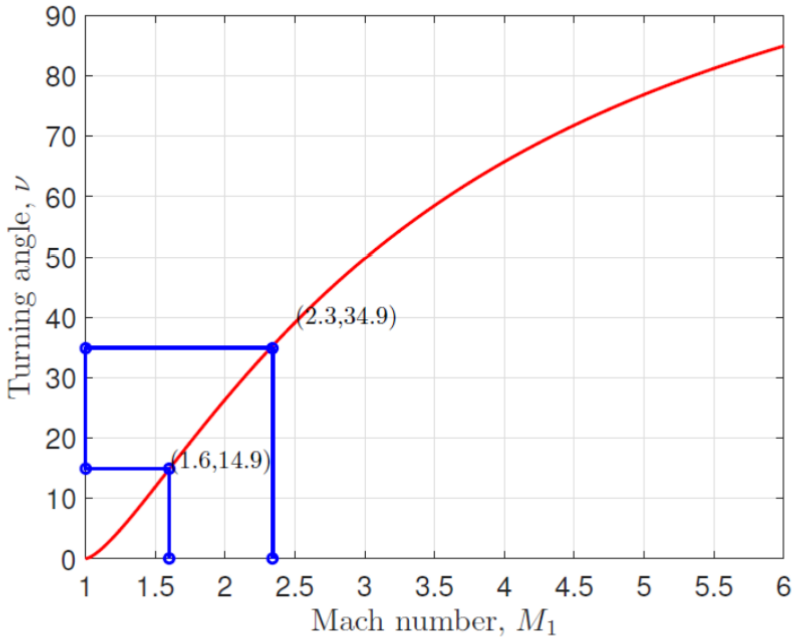
3.1.6 Relationship between turning angle and M

$$\Delta v = \int_0^{\Delta v} = \int_{M_1}^{M_2} \left(\frac{\left(M^2-1\right)^{\frac{1}{2}}}{M\left(1+\frac{1}{2}\left(\gamma-1\right)M^2\right)} \right) \mathrm{d}M \tag{3.15}$$

This is quite difficult to use since it would require tabulating Δv against M_1 and M_2. Instead, we use the sonic reference condition and set $M_1 = 1$ and $M_2 = M$. This gives:

$$v = \left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}} \arctan\left(\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}} \left(M^2-1\right)^{\frac{1}{2}}\right) - \arctan\left(M^2-1\right)^{\frac{1}{2}} \tag{3.16}$$

This formula is plotted in the Prandtl-Meyer charts below.



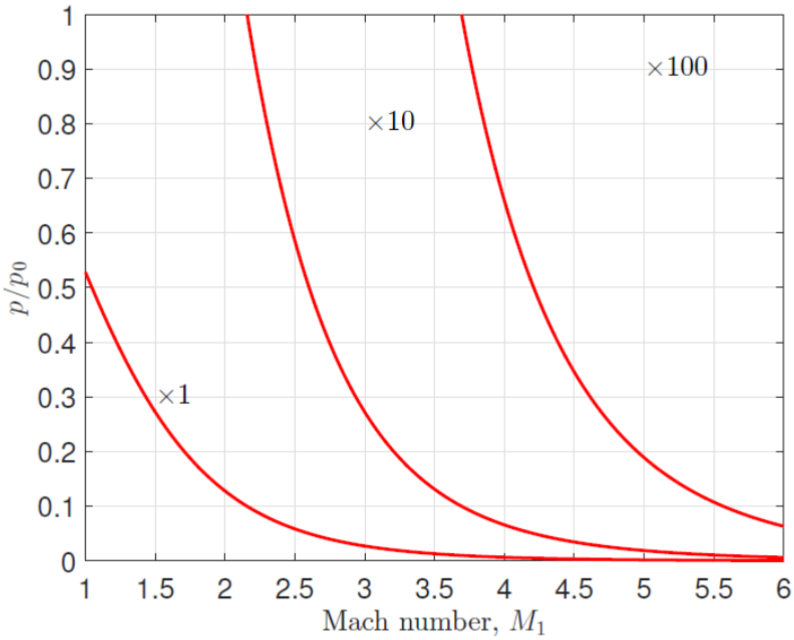


Figure 3.8: Prandtl-Meyer charts.

3.1.7 How to use expansion charts

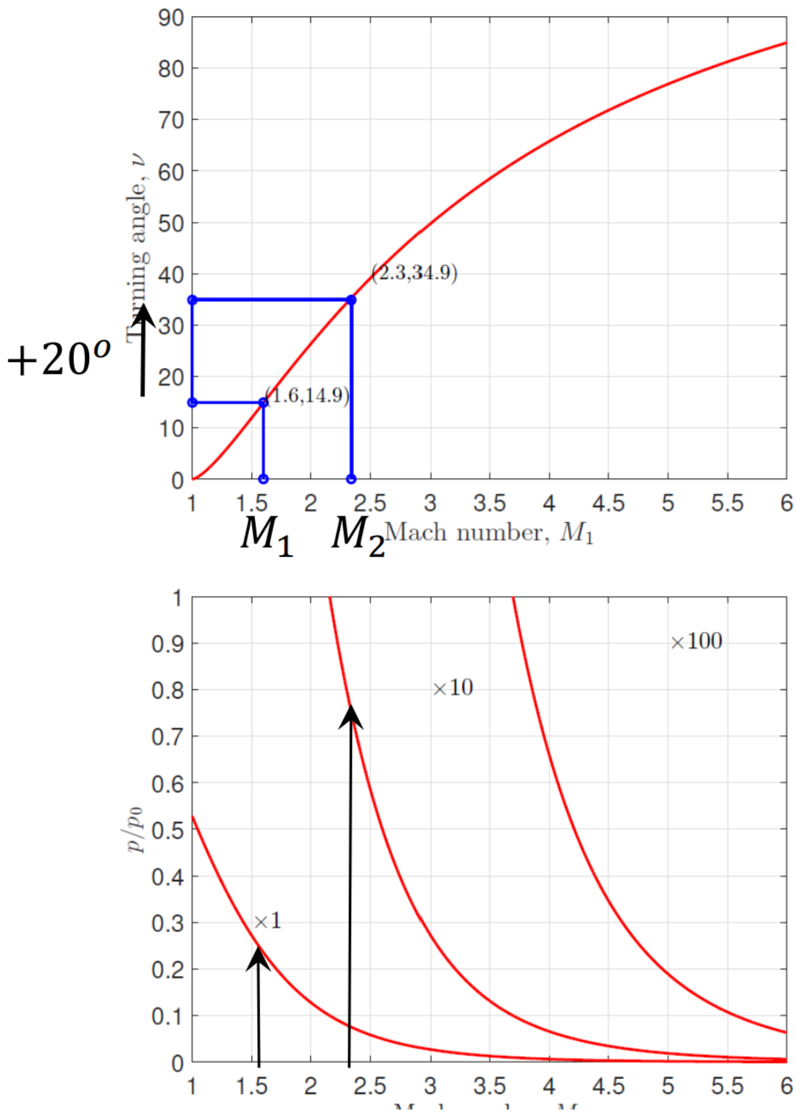


Figure 3.9: Prandtl-Meyer charts.

To illustrate how the charts are used, consider the finite deflection of a flow $M_1 = 1.6$ and the flow turns an angle $\Delta v = 20°$. We have to determine the turning angle from $M = 1$ to $M_1 = 1.6$. From the chart above this is $V_1 = 14.9°$. In total,

$$v_2 = v_1 + \Delta v = 14.9° + 20° = 34.9° \tag{3.17}$$

The corresponding Mach number is:

$$M_2 = 2.31 \tag{3.18}$$

The fan angles are:

$$\alpha_{M_1} = \arcsin \frac{1}{M_1} = 38.7° \tag{3.19}$$

$$\alpha_{M_2} = \arcsin \frac{1}{M_2} = 25.7° \tag{3.20}$$

For an expansion wave, the stagnation pressure is conserved.

$$p_0 = p_1 \left(1 + \frac{1}{2}(\gamma - 1) M_1^2\right)^{\frac{\gamma}{\gamma-1}} \tag{3.21}$$

$$= p_2 \left(1 + \frac{1}{2}(\gamma - 1) M_2^2\right)^{\frac{\gamma}{\gamma-1}} \tag{3.22}$$

This enable the pressure after the expansion fan to be estimated. The stagnation pressure p_0 is constant across the expansion fan.

3.1.8 Maximum turning angle

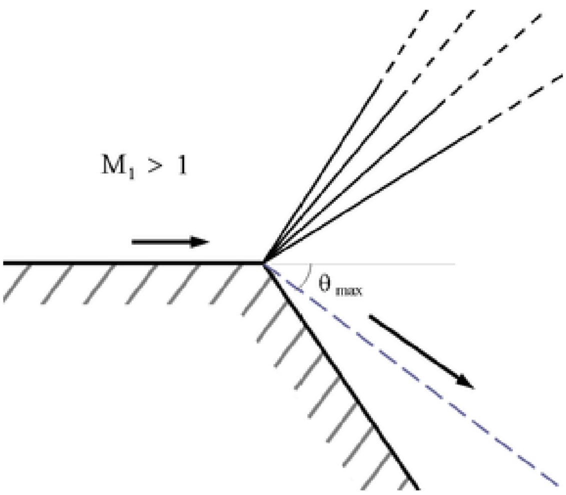


Figure 3.10: Prandtl-Meyer charts.

From the formula, we find that:

$$v_{max} = \frac{\pi}{2}\left(\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}} - 1\right) \tag{3.23}$$

This places a limit on the maximum turning angle that can be supported. When the deflection angle is greater than:

$$\theta_{max} = v_{max} - v\left(M_1\right) \tag{3.24}$$

a slip plane is generated.

3.1.9 Turning a corner

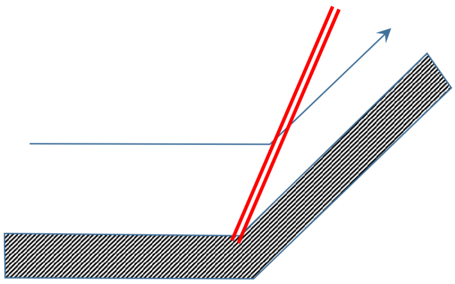


Figure 3.11: Shock is formed.

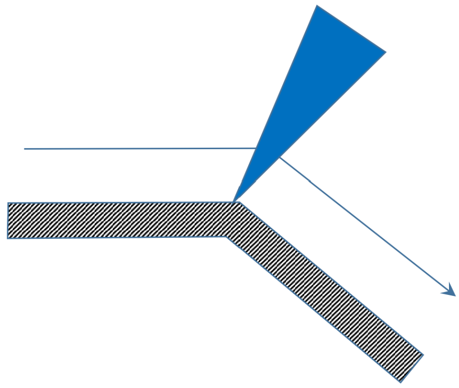


Figure 3.12: Expansion fan.

As we have seen in the previous section, as the flow is turned by a wedge, the flow is compressed and the Mach number decreases. Whether the flow downstream of an oblique shock is supersonic depends on the deflection angle.

Chapter 4

Non-linear and Linear Aerofoil Theory

4.1 Comparing Lifting Surfaces

4.1.1 Subsonic Aerofoil

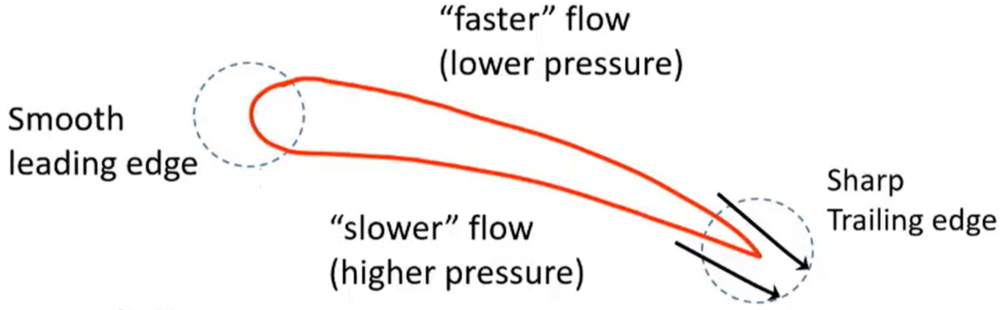


Figure 4.1: Diagram of a subsonic aerefoil with the relevant pressures around it

For a subsonic aerefoil, the pressure around it varies continuously.

4.1.2 Supersonic Aerofoil

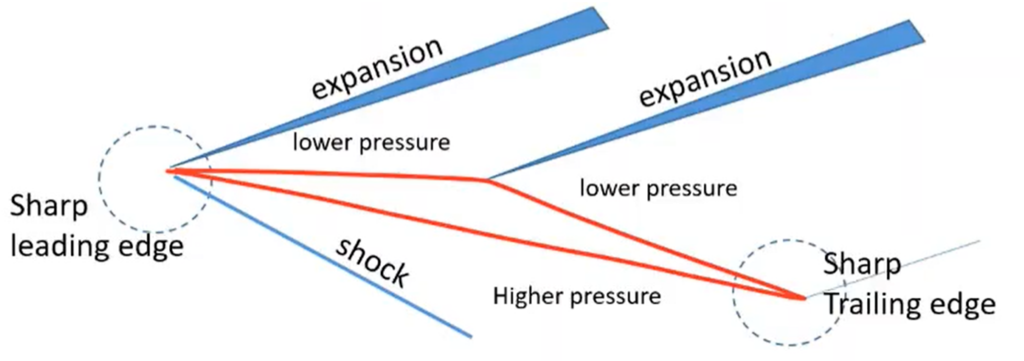


Figure 4.2: Diagram of a supersonic aerefoil with the relevant pressures around it

For a supersonic aerefoil, the pressure around it is constant in different regions.

4.2 Nonlinear Analysis

We consider the case of a flat plate and examine the forces on it.

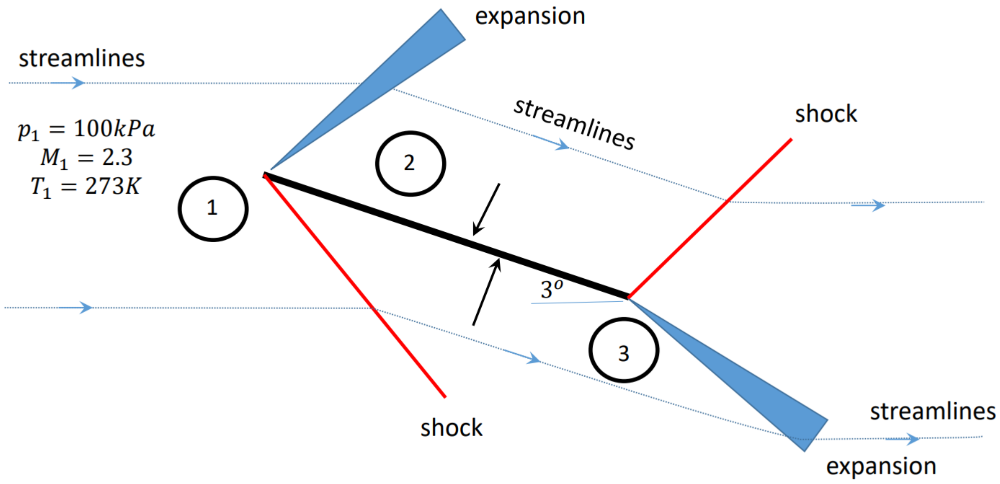


Figure 4.3: The diagram of a flat plate with the relevant regions indicated

The initial conditions are:

$$p_1 = 100\text{kPa} \tag{4.1}$$

$$M_1 = 2.3 \tag{4.2}$$

$$T_1 = 273\text{K} \tag{4.3}$$

4.2.1 Use Shock Charts from Region 1 to Region 3

The following results are obtained:

$$\frac{p_3}{p_1} = 1.2 \tag{4.4}$$

Hence, the pressure in region 2 is calculated as:

$$p_3 = 1.2 \cdot 100 = 120\text{kPa} \tag{4.5}$$

4.2.2 Use Expansion Charts for Region 1 to Region 2

The following results are obtained:

$$M_2 = 2.4 \tag{4.6}$$

$$\frac{p_2}{p_0} = 0.068 \tag{4.7}$$

$$\frac{p_1}{p_0} = 0.079 \tag{4.8}$$

Hence, the pressure in region 2 is calculated as:

$$\frac{p_2}{p_1} = \frac{p_2}{p_0} \cdot \frac{p_0}{p_1} = 0.861 \tag{4.9}$$

$$\therefore p_2 = 0.861 \cdot 100 = 86.1\text{kPa} \tag{4.10}$$

4.2.3 Force Calculation

The total force (per unit length) is:

$$F_N = (p_3 - p_2)L \tag{4.11}$$

$$= ((120 - 86.1) \cdot 10^3\text{Pa}) \cdot 1 \tag{4.12}$$

$$= (34 \cdot 10^3\text{Pa}) \cdot 1 = 34\text{kN}\,\text{m}^{-1} \tag{4.13}$$

4.2.4 Lift and Drag Coefficients

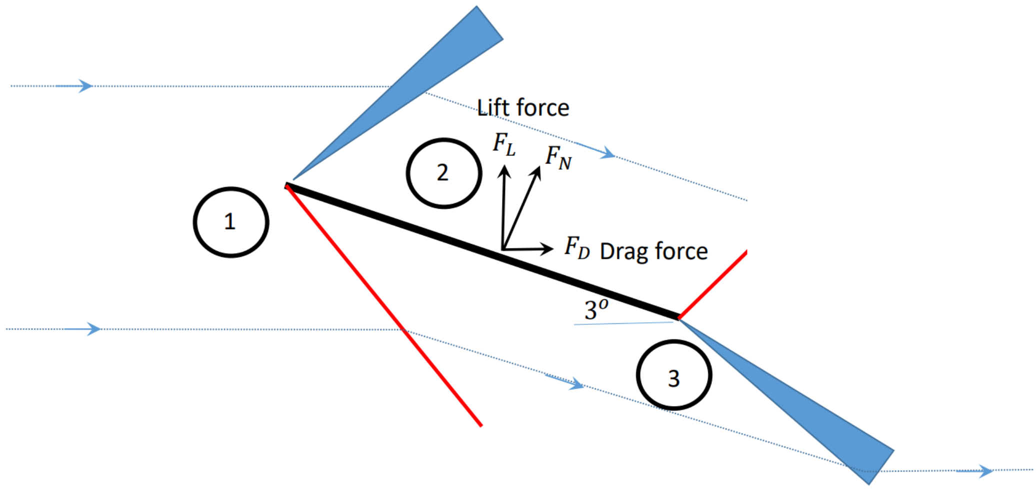


Figure 4.4: The diagram of a flat plate with the lift and drag coefficients shown

Using:

$$R = 287\mathrm{J\,kg^{-1}\,K^{-1}} \qquad \& \qquad \gamma = 1.4 \tag{4.14}$$

The air speed is calculated as:

$$u_1 = c_1 M_1 \tag{4.15}$$

$$c_1 = \sqrt{\gamma R T_1} \tag{4.16}$$

$$u_1 = \sqrt{\gamma R T_1} \cdot M_1 = \sqrt{1.4 \cdot 287 \cdot 273} \cdot 2.3 = 762\mathrm{m\,s^{-1}} \tag{4.17}$$

The drag and lift forces are defined by:

$$F_D = F_N \sin\alpha \tag{4.18}$$

$$F_L = F_N \cos\alpha \tag{4.19}$$

The lift and drag coefficients are:

$$c_D = \frac{F_D}{\frac{1}{2}\rho_1 u_1^2 L} \qquad \& \qquad c_L = \frac{F_L}{\frac{1}{2}\rho_1 u_1^2 L} \tag{4.20}$$

Substituting in the values yields:

$$c_D = 0.0046 \tag{4.21}$$

$$c_L = 0.088 \tag{4.22}$$

Two observations can be made here:

1. It is quite hard to calculate
2. $\frac{c_L}{c_D} >> 1$

4.3 Linear Methods for Calculating Forces

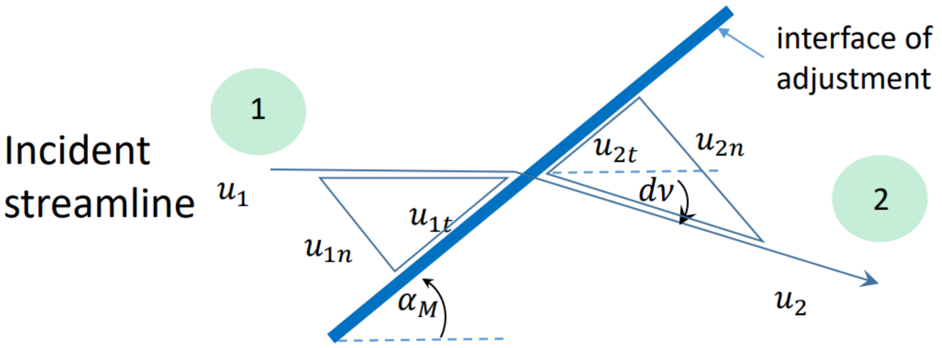


Figure 4.5

Conservation of momentum parallel to interface:

$$u_1 \cos \alpha_M = u_2 \cos(\alpha_M + dv) \tag{4.23}$$

From the double angle formula:

$$\cos(\alpha_M + dv) \approx \cos \alpha_M - dv \sin \alpha_M \tag{4.24}$$

Rearranging gives:

$$\frac{\mathrm{d}u}{u} = \frac{u_2 - u_1}{u_2} = \frac{\sin \alpha_M}{\cos \alpha_M} dv \tag{4.25}$$

We are trying to relate changes in pressure to the deflection angle. All angles are relative to incident streamline and small so that $|dv| << 1$, where:

- dv - deflection or wedge angle
- α_M - shock wave angle
- $\sin \alpha_M = \dfrac{1}{M_1}$

To this linear approximation, a shock = - expansion.

From the normal momentum equation:

$$\rho_1 u_{1n}(u_{2n} - u_{1n}) = -(p_2 - p_1) \tag{4.26}$$

or

$$\rho_1 u_1 \sin \alpha_M (u_2 \sin(\alpha_M + \mathrm{d}v) - u_1 \sin \alpha_M) = -(p_2 - p_1) \tag{4.27}$$

Rearranging gives:

$$\frac{p_2 - p_1}{\frac{1}{2}\rho_1 u_1^2} = -\frac{2\sin\alpha_M}{u_1}(u_2(\sin\alpha_M + \cos\alpha_M\,\mathrm{d}v) - u_1\sin\alpha_M) \tag{4.28}$$

$$= -\frac{2\sin\alpha_M}{u_1}\left(u_2\,\mathrm{d}v\frac{\sin^2\alpha_M}{\cos\alpha_M} + u_2\cos\alpha_M\,\mathrm{d}v\right) \tag{4.29}$$

$$\therefore \frac{p_2 - p_1}{\frac{1}{2}\rho_1 u_1^2} = -\frac{2\sin\alpha_M}{\cos\alpha_M}\mathrm{d}v\frac{u_2}{u_1} \approx -\frac{2\,\mathrm{d}v}{\sqrt{M_1^2 - 1}} \tag{4.30}$$

This is known as the linear model for small changes where u_1 and M_1 do not change.

4.4 Example: Nonlinear Calculation

We consider the case of a flat plate and examine the forces on it:

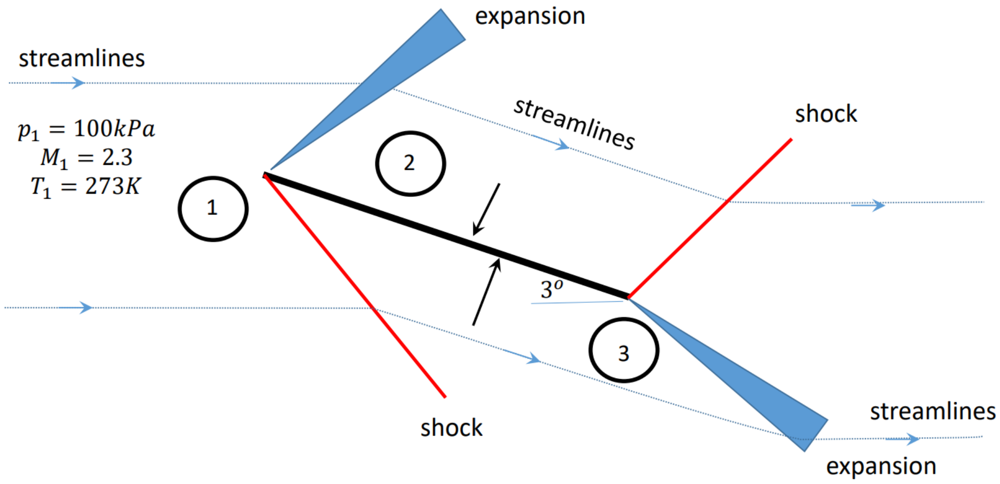


Figure 4.6: The diagram of a flat plate with the relevant regions indicated

In applying this technique we need to make sure that we understand when a shock or expansion occurs. This tells us the sign of the pressure change.

$$\frac{p_2 - p_1}{\frac{1}{2}\rho_1 u_1^2} \approx -\frac{2\alpha}{\sqrt{M_1^2 - 1}} \tag{4.31}$$

$$\frac{p_3 - p_1}{\frac{1}{2}\rho_1 u_1^2} \approx -\frac{2\alpha}{\sqrt{M_1^2 - 1}} \tag{4.32}$$

4.4.1 Lift and Drag Coefficients

The normal force per unit length is calculated as:

$$F_N = (p_3 - p_2)L \tag{4.33}$$

$$= \frac{\frac{1}{2}\rho_1 u_1^2 \big((2\alpha) - (-2\alpha)\big) L}{\sqrt{M_1^2 - 1}} \tag{4.34}$$

The lift and drag coefficients are:

$$c_D = \frac{F_N \sin\alpha}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{\frac{\frac{1}{2}\rho_1 u_1^2\big((2\alpha)-(-2\alpha)\big)L}{\sqrt{M_1^2-1}}\sin\alpha}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{4\alpha\sin\alpha}{\sqrt{M_1^2-1}} = 0.0059 \text{ (Nonlinear 0.0048)} \tag{4.35}$$

$$c_L = \frac{F_N \cos\alpha}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{\frac{\frac{1}{2}\rho_1 u_1^2\big((2\alpha)-(-2\alpha)\big)L}{\sqrt{M_1^2-1}}\cos\alpha}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{4\alpha\cos\alpha}{\sqrt{M_1^2-1}} = 0.1 \text{ (Nonlinear 0.092)} \tag{4.36}$$

4.5 Linear Calculation for More Complicated Geometry

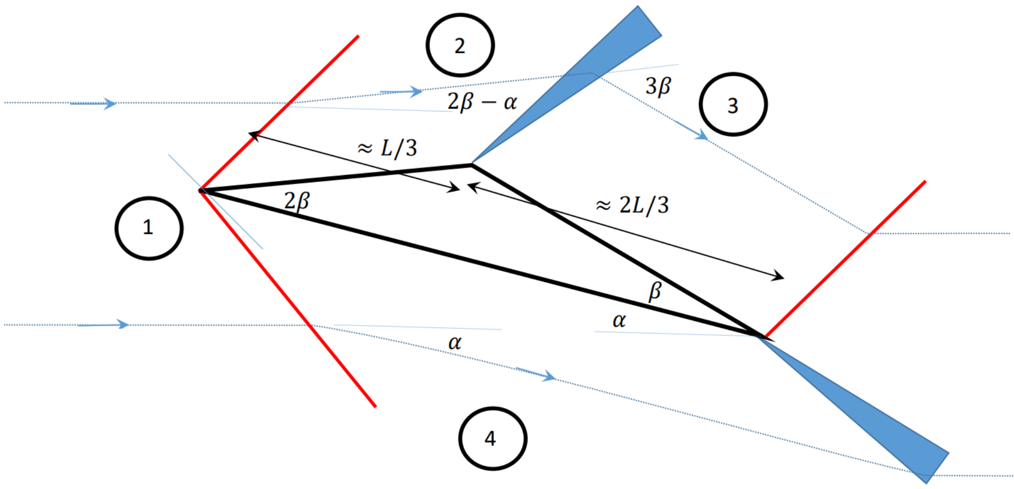


Figure 4.7: The diagram for a more complicated geometry

To simplify the analysis, define:

$$A = \frac{\frac{1}{2}\rho_1 u_1^2}{\sqrt{M_1^2 - 1}} \tag{4.37}$$

Techniques for solution:

1. Draw a large figure
2. Calculate the deflection angles
3. Calculate the pressure differences between the different regions

Pressures in each region:

$$p_2 - p_1 = 2(2\beta - \alpha)A \tag{4.38}$$

$$p_3 - p_2 = -6\beta A \tag{4.39}$$

$$p_4 - p_1 = 2\alpha A \tag{4.40}$$

The second equation can be written as:

$$p_3 - p_1 = -(2\alpha + 2\beta) \tag{4.41}$$

The lift force is calculated as:

$$F_L = p_4 L\cos\alpha - p_2\frac{L}{3}\cos(2\beta - \alpha) - p_3\frac{2L}{3}\cos(\alpha + \beta) \tag{4.42}$$

Through small angle approximation:

$$F_L = p_4 L - p_2\frac{L}{3} - p_3\frac{2L}{3} \tag{4.43}$$

$$\approx \left(2\alpha - \frac{2}{3}(2\beta - \alpha) + \frac{2}{3}(2\alpha + 2\beta)\right) AL \tag{4.44}$$

The drag force is calculated as:

$$F_D = p_4 L\sin\alpha + p_2\frac{L}{3}\sin(2\beta - \alpha) - p_3\frac{2L}{3}\sin(\alpha + \beta) \tag{4.45}$$

Through small angle approximation:

$$F_D = p_4 L\alpha + p_2\frac{L}{3}(2\beta - \alpha) - p_3\frac{2L}{3}(\alpha + \beta) \tag{4.46}$$

$$\approx \left(2(\alpha)(\alpha) + \frac{2}{3}(2\beta - \alpha)(2\beta - \alpha) + \frac{2}{3}(2\alpha + 2\beta)(\alpha + \beta)\right) AL \tag{4.47}$$

$$= \left(2\alpha^2 + \frac{2}{3}(4\beta^2 - 4\beta\alpha + \alpha^2) + \frac{4}{3}(\alpha^2 + 2\beta\alpha + \beta^2)\right) AL \tag{4.48}$$

$$= (4\alpha^2 + 4\beta^2)AL \tag{4.49}$$

The lift and drag coefficients are:

$$c_L = \frac{F_L}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{4\alpha}{\sqrt{M_1^2 - 1}} \tag{4.50}$$

$$c_D = \frac{F_D}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{4(\alpha^2 + \beta^2)}{\sqrt{M_1^2 - 1}} \tag{4.51}$$

The solution reduces to flat plate case when $\beta = 0$.

4.6 Lift Coefficient Variation with Mach Number

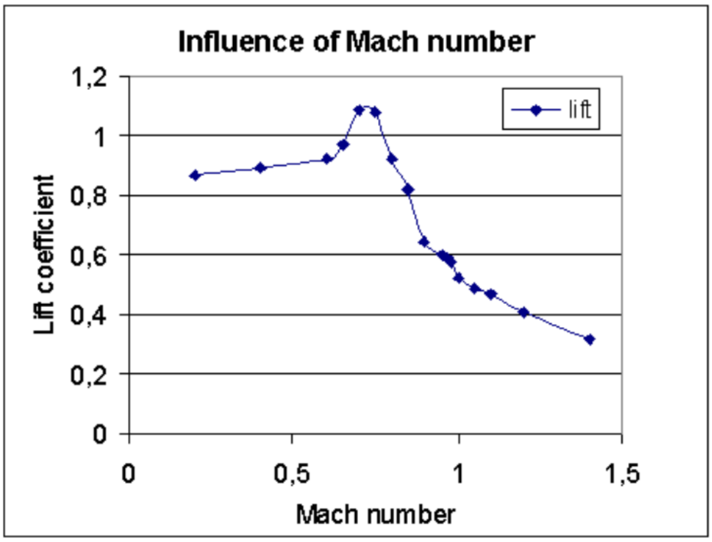


Figure 4.8: Lift coefficient variation with Mach number

For low Mach number:

$$c_D \sim constant \tag{4.52}$$

$$c_L \sim 2\pi\alpha \tag{4.53}$$

For high Mach number:

$$c_D \sim \frac{4\alpha^2}{\sqrt{M_1^2 - 1}} \tag{4.54}$$

$$c_L \sim \frac{4\alpha}{\sqrt{M_1^2 - 1}} \tag{4.55}$$

Chapter 5

Inlets

5.1 Inlets

5.1.1 Characteristics of a jet engine

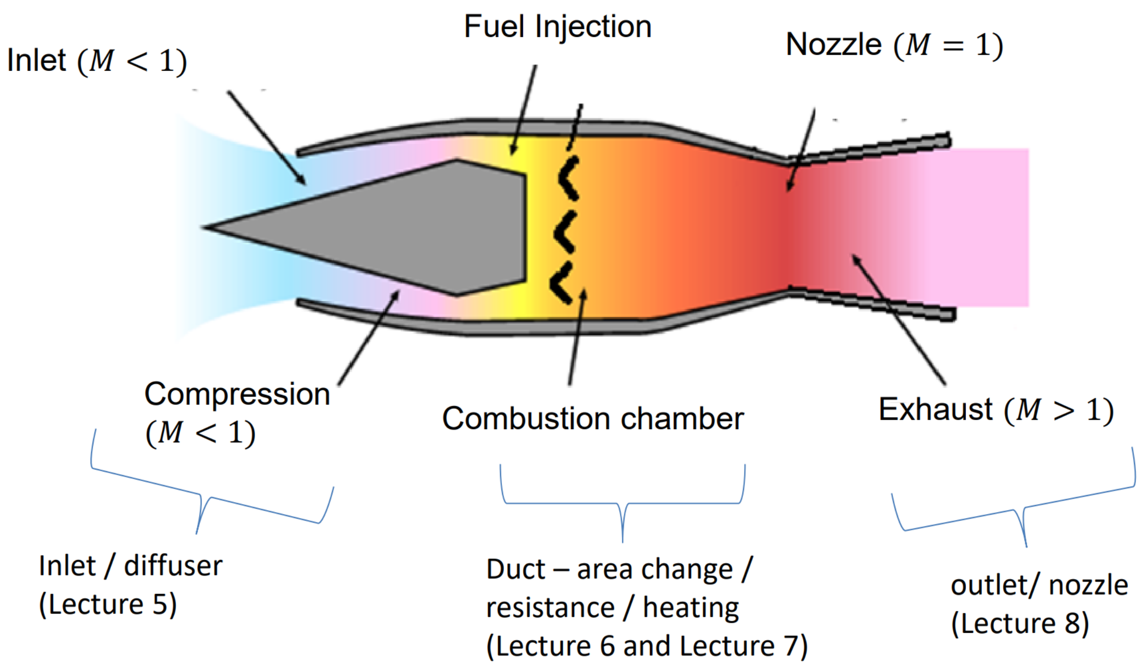


Figure 5.1: Components of a jet engine.

5.1.2 Thrust from a jet/turbine

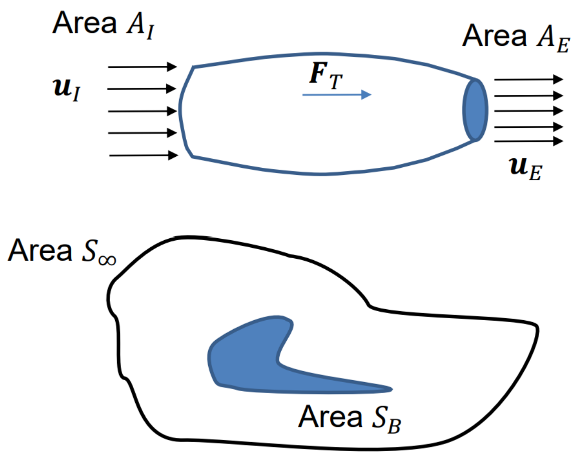


Figure 5.2: Inlet areas.

The thrust force on a body is determined by the integrated (pressure) force over its surface S_B:

$$\underline{F}_T = \int_{S_B} (p\hat{\underline{n}})\,\mathrm{d}S \tag{5.1}$$

From the conservation of momentum, the force can be expressed in terms of integrals over S_∞ and S_B:

$$\underline{F}_T = -\int_{S_\infty} (p\hat{\underline{n}})\,\mathrm{d}S + \int_{S_B+S_\infty} \left(\rho\left(\underline{u}\cdot\hat{\underline{n}}\right)\underline{u}\right)\mathrm{d}S \tag{5.2}$$

Taking a control surface that envelopes the body:

$$\underline{F}_T\cdot\hat{\underline{x}} = A_E p_E - A_1 p_1 + \dot{m}\left(u_E - u_1\right) \tag{5.3}$$

5.1.3 Purpose of an inlet

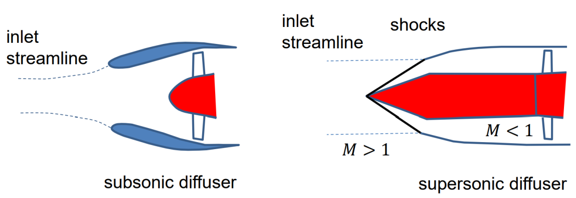


Figure 5.3: Inlet streamlines with various diffusers.

There are two requirements:

1. Efficiency of engine depends on reducing losses (such as loss of stagnation pressure)
2. Provide the required inlet mass flow (there are constraints to this). This is limited by choking of the inlet.

5.1.4 What do they look like?

Figure 5.4: Subsonic inlet.

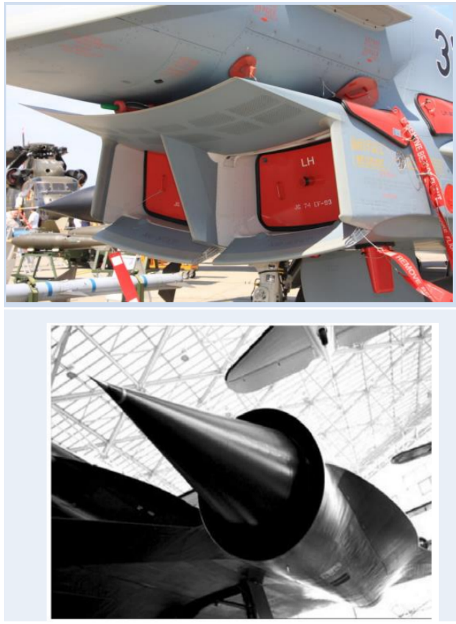


Figure 5.5: Supersonic inlets.

Modern jets are turbofans. This consists of a low speed fan and an inner high speed compressor. This is preferred because of the lower exhaust speed that is more efficient and gives rise to less noise.

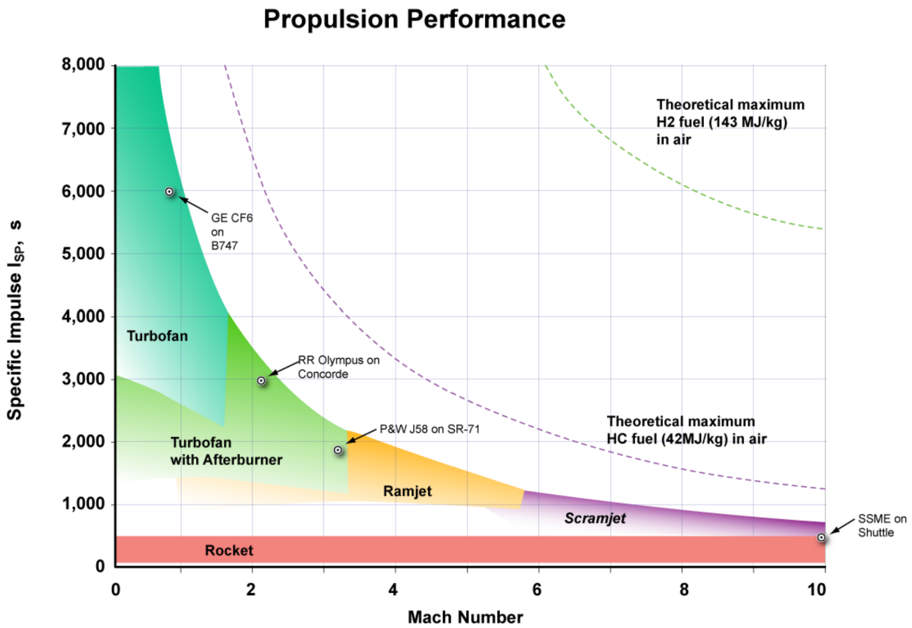


Figure 5.6: Graph to show propulsion performance for different inlet types.

Thrust divided by rate at which mass of propellant is consumed $\dfrac{F_T}{\dot{m}}$.

5.2 Jet engines

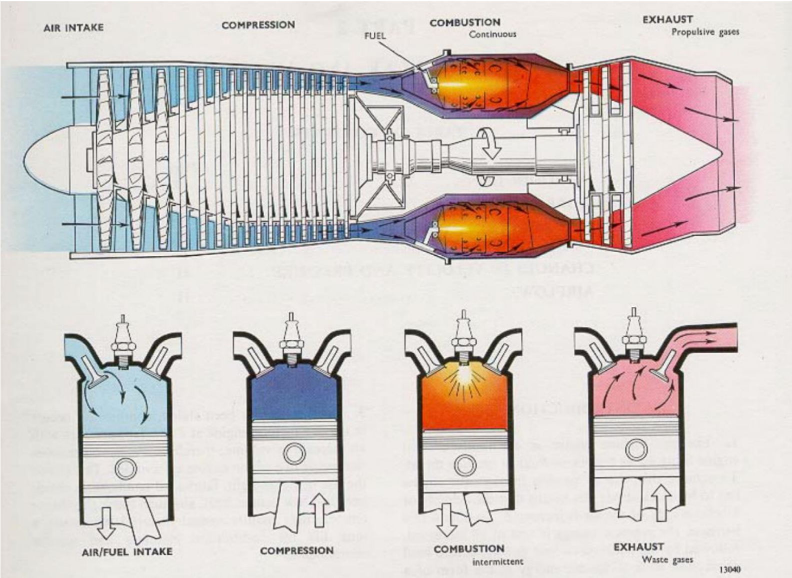


Figure 5.7: Jet engine.

In the first half - need high pressure, in the second half - need high velocity. There are four main components to a jet engine:

1. Inlet
2. Compression
3. Combustion
4. Exhaust

Thrust:

1. Exhaust gas
2. Pushing the flow

5.2.1 Ramjet

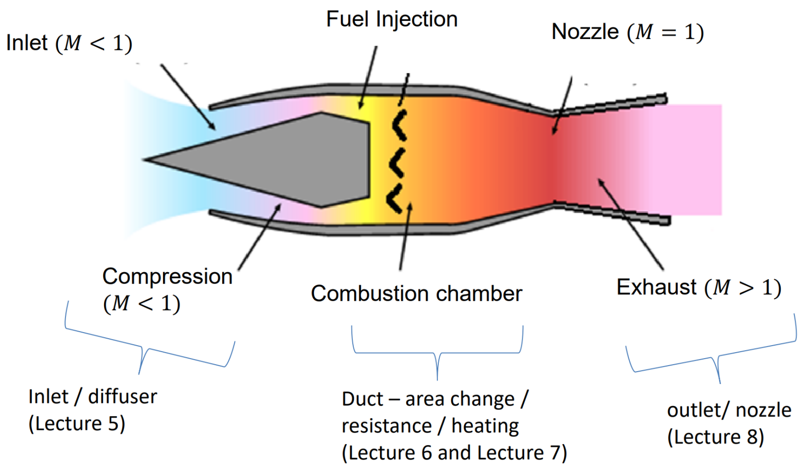


Figure 5.8: Ramjet.

This is a form of airbreathing jet engine that uses the engine's forward motion to compress incoming air without an axial compressor. Ramjets work most efficiently at supersonic speeds, around Mach 3 and can operate up to speeds of Mach 6 ($7350\,\mathrm{km\,h^{-1}}$).

5.2.2 Scramjet

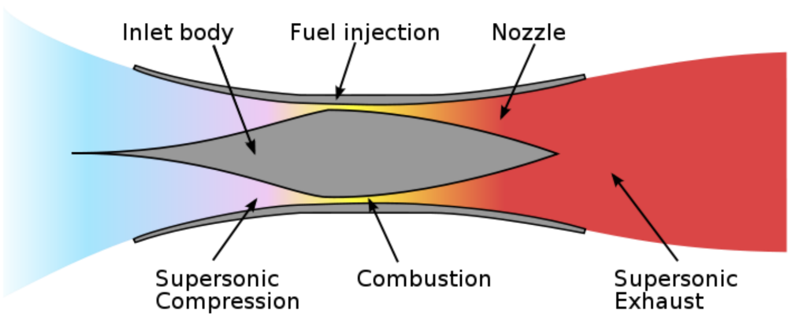


Figure 5.9: Scramjet.

This consists of a converging inlet, where incoming air is compressed; a combustor, where gaseous fuel is burned with atmospheric oxygen to produce heat. There is a diverging nozzle, where the heated air is accelerated to produce thrust. The flow is supersonic through the entire flow. The thrust is:

$$F = \dot{m}\,(V_2 - V_1) \tag{5.4}$$

Increasing pressure of airflow increases the potential energy. Increasing velocity increases kinetic energy. The purpose of the inlet, compressor and exhaust is to convert potential energy to kinetic energy.

5.3 Inlet types

5.3.1 Subsonic

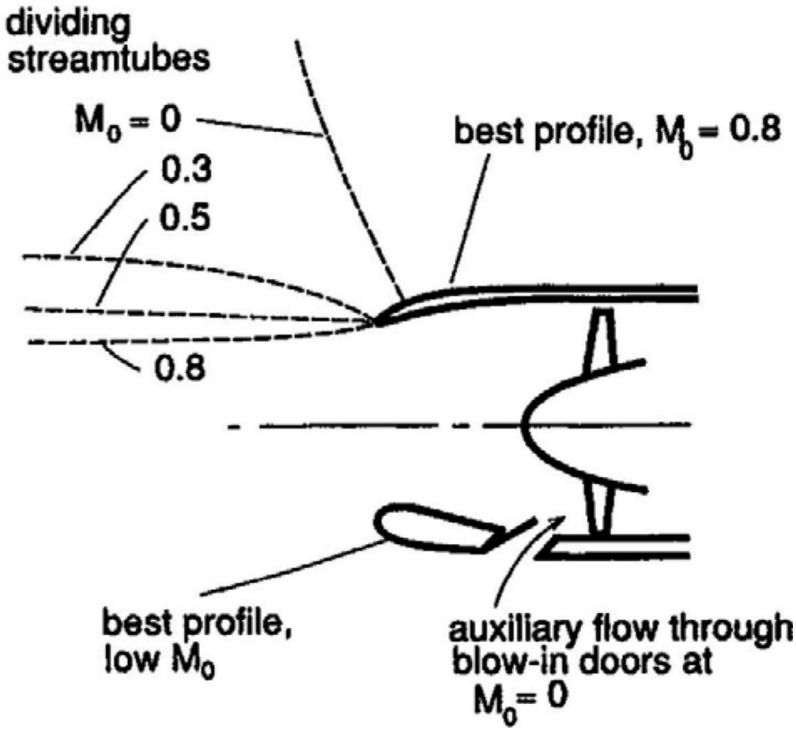


Figure 5.10: Subsonic inlet profile.

5.3.2 Supersonic inlets

An inlet for a supersonic aircraft, on the other hand, has a relatively sharp lip. The inlet lip is sharpened to minimise the performance losses from shock waves that occur during supersonic flight. For a supersonic aircraft, the inlet must slow the flow down to **subsonic** speeds before the air reaches the compressor.

5.4 Simple normal shock inflow

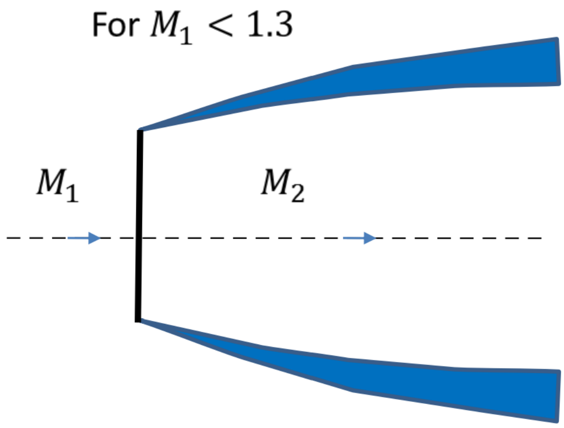


Figure 5.11: Simple normal shock inflow.

Relationship between flow properties upstream and downstream of shock:

$$M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1) M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)} \tag{5.5}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right) \tag{5.6}$$

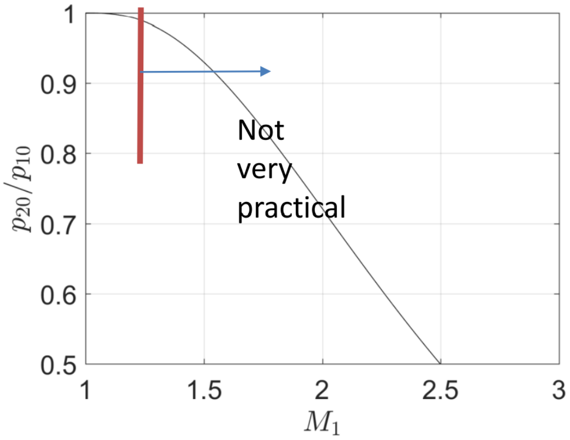


Figure 5.12: Graph to show ratio of stagnation pressure vs Mach number.

Ratio of stagnation pressure downstream to upstream of normal shock:

$$\frac{p_{10}}{p_{20}} = \left(\frac{2\gamma M_1^2}{\gamma + 1} - \frac{\gamma - 1}{\gamma + 1}\right)^{\frac{1}{\gamma - 1}} \left(\frac{(\gamma - 1) M_1^2 + 2}{(\gamma + 1) M_1^2}\right)^{\frac{\gamma}{\gamma - 1}} \tag{5.7}$$

5.5 Basic concept - reflection of shock at wall

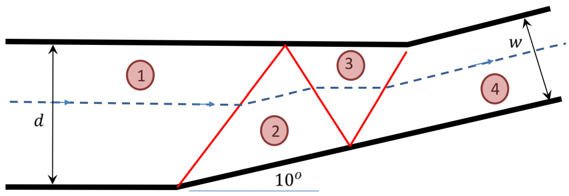


Figure 5.13: Diagram of shock at wall.

Example from Problem Sheet 2. The inlet Mach number is $M_1 = 2.8$. For a turning angle of 10°, calculate the width of the duct (w) in terms of the inlet width d, given that the shock cancels out at the corner.

Region 1	Region 2	Region 3	Region 4
$M_1 = 2.8$	$M_2 = 2.34$	$M_3 = 1.94$	$M_4 = 1.58$
	$\zeta_2 = *$	$\zeta_3 = *$	$\zeta_4 = *$
	$\dfrac{p_2}{p_1} = 2.0$	$\dfrac{p_3}{p_2} = 1.82$	$\dfrac{p_4}{p_3} = 1.*$

Table 5.1: Mach numbers in different regions.

5.6 External vs internal compression

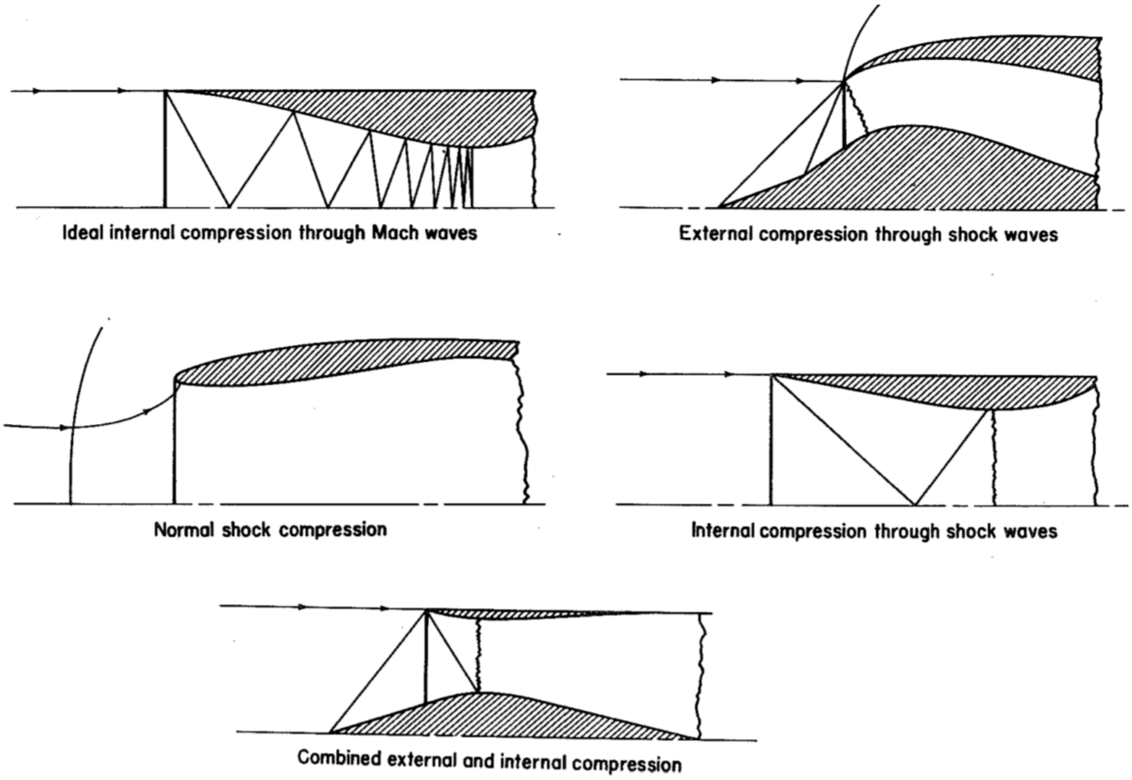


Figure 5.14: External vs internal compression.

5.7 Strategy for increasing stagnation pressure

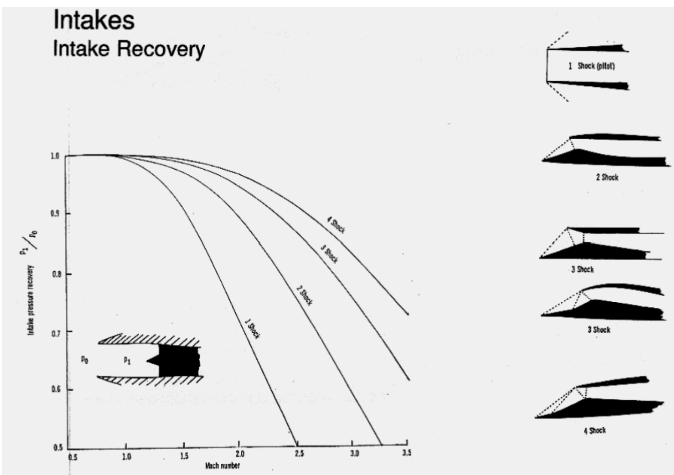


Figure 5.15: Intake recovery graph.

To increase the stagnation pressure after the compression stage, it is useful to compress the flow with a series of external and internal shocks. This shows the stagnation pressure ratio due to a series of shocks which are of equal strength. Equal strength means that:

$$\frac{p_2}{p_1} = \frac{p_3}{p_2} = \frac{p_4}{p_3} = ... \tag{5.8}$$

5.7.1 Optimal choice

The shocks are set up so that they tend to meet at the edge of the cowl. This is to:

1. minimise spillage
2. minimise buzz - oscillations in the inlet
3. minimise loss of stagnation pressure

We tend to ensure that the ratio of stagnation pressure across the shocks is the same (i.e. they have the same strength).

5.8 Oblique diffusers

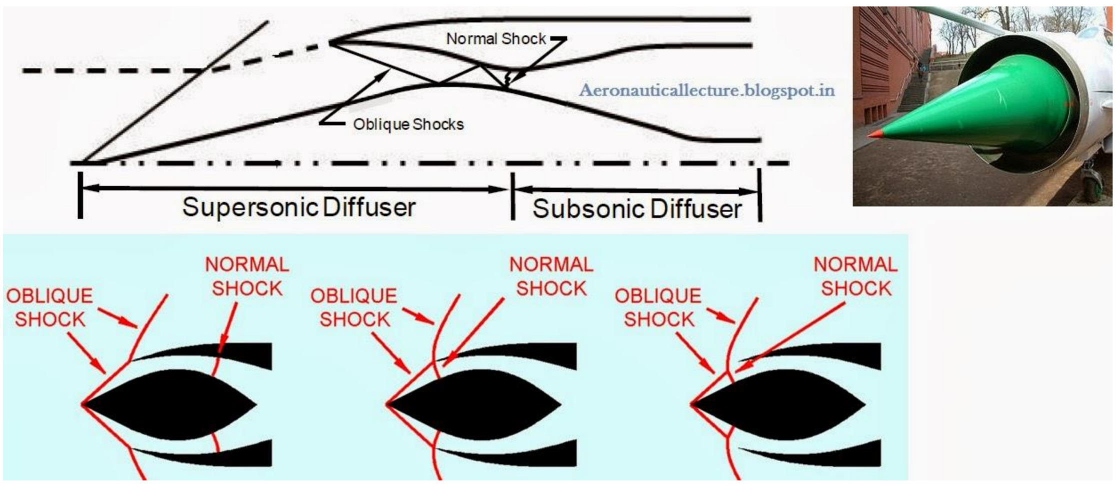


Figure 5.16: Oblique diffusers.

5.8.1 Adaptation

A movable cone called a 'spike' is used. For subsonic flow, the spike is pushed forward. When the aircraft accelerated past Mach 1.6, an internal jackscrew moved the spike up 66 cm inwards, directed by an analog air inlet computer that took into account pitot-static system, pitch, roll, yaw and angle of attack. Moving the spike tip drew the shock wave riding on it closer to the inlet cowling until it touched just slightly inside the cowling.

Chapter 6

Compressible Duct Flows

6.1 Compressible duct flows

6.1.1 Problem types

We are concerned with flow in ducts or 'long' pipes whose cross-section area is simple (rectangular / circular / conical). There are three types of problems where:

1. changes in cross-sectional area are important ($dA \neq 0$).
2. frictional forces are important (**momentum is not conserved**) - *Fanno flow* (mass, energy, state).
3. heating and cooling are important (**energy is not conserved**) - *Rayleigh flow* (mass, momentum, state).

We study these effects separately and then consider them combined.

6.1.2 Influence of changes in cross-section area

To understand the physics, we simplify the analysis to gas moving without frictional forces and the addition of heat. There are two approaches to analyse the problem:

1. Differential approach - how flow properties vary along pipe (Rayleigh flow, Laval nozzle) - sometimes called a 1D model.
2. Integral approach - relationship between two flow states (Fanno flow, isothermal examples) - sometimes called a 0D model.

We have met these two approaches before:

1. Navier-Stokes equation.

$$\rho\frac{D\underline{u}}{Dt} = -\underline{\nabla}p + \mu\underline{\nabla}^2\underline{u}$$

2. Momentum-integral approach.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V (\rho u)\,\mathrm{d}V = \int_S (-p\underline{I} + \underline{\tau})\cdot\hat{\underline{n}}\,\mathrm{d}S$$

Where the integral is taken over a volume V bounded by a surface S.

6.1.3 Recap of reference states

The stagnation values are valid for isentropic flows:

$$\frac{p}{p_0} = \left(1 + \frac{1}{2}(\gamma - 1)M^2\right)^{-\frac{\gamma}{\gamma-1}} \tag{6.1}$$

$$\frac{\rho}{\rho_0} = \left(1 + \frac{1}{2}(\gamma - 1)M^2\right)^{-\frac{\gamma}{\gamma-1}} \tag{6.2}$$

and for adiabatic flows:

$$\frac{T}{T_0} = \left(1 + \frac{1}{2}(\gamma - 1)M^2\right)^{-1} \tag{6.3}$$

It is important to be aware of what is conserved for adiabatic or isentropic conditions. The reference state of $M = 1$ is useful for flow in pipes where the sonic condition is common. The sonic reference condition is usually referred to as the critical condition and denoted with a '*'. Therefore:

$$\frac{p_*}{p_0} = \left(\frac{1}{2}(\gamma + 1)(M = 1)^2\right)^{-\frac{\gamma}{\gamma-1}} \approx 0.528 \tag{6.4}$$

$$\frac{\rho_*}{\rho_0} = \left(\frac{1}{2}(\gamma + 1)(M = 1)^2\right)^{-\frac{1}{\gamma-1}} \approx 0.91 \tag{6.5}$$

$$\frac{T_*}{T_0} = \left(\frac{1}{2}(\gamma + 1)(M = 1)^2\right)^{-1} \approx 0.833 \tag{6.6}$$

In duct flows, the critical state is used. p_0, ρ_0 are constant when there is no shock. T_0 is constant even with a shock.

6.1.4 Conservation of mass

The purpose is to relate A and M. The conservation of mass requires (integral or 0D approach and assuming no shocks at the moment):

$$\dot{m} = \rho u A = \text{const} = \rho^* u^* A^* \tag{6.7}$$

or

$$\frac{\rho}{\rho^*}\frac{u}{u^*}\frac{A}{A^*} \tag{6.8}$$

Substituting for our isentropic and adiabatic parts:

$$\frac{\rho}{\rho^*} = \left(\frac{1 + \frac{1}{2}(\gamma - 1) M^2}{\frac{1}{2}(\gamma + 1)} \right)^{-\frac{1}{\gamma - 1}} \tag{6.9}$$

$$\frac{u}{u^*} = \frac{M}{M^*}\frac{c}{c^*} = \frac{M}{M^*}\left(\frac{T}{T^*}\right)^{\frac{1}{2}} = M \left(\frac{\frac{1}{2}(\gamma + 1)}{1 + \frac{1}{2}(\gamma - 1) M^2} \right)^{\frac{1}{2}} \tag{6.10}$$

The right-hand side is a constant. It is a state that may, or may not, be realised

6.1.5 Relationship between flow state and area

We are to explore some of the important relationships between the state of the flow and the cross-sectional area.

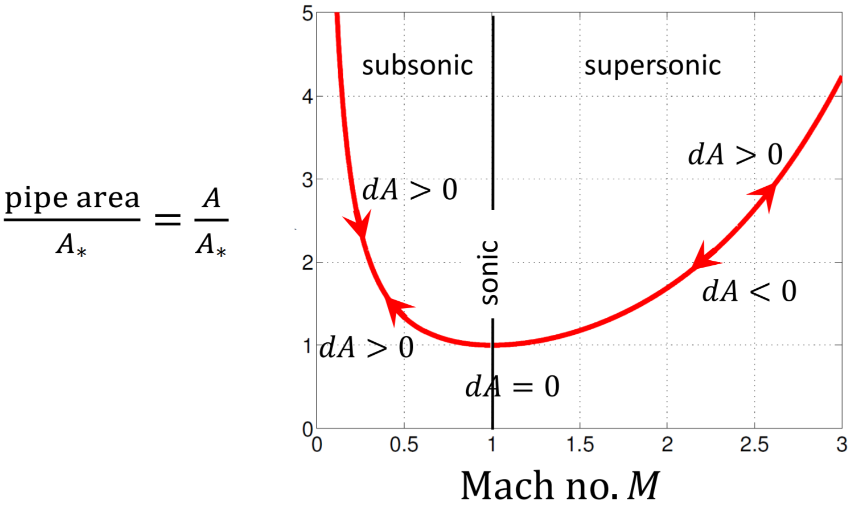


Figure 6.1: Graph to show pipe area vs Mach number.

$\mathrm{d}A > 0$	Subsonic	$\mathrm{d}M < 0$
$\mathrm{d}A < 0$	Subsonic	$\mathrm{d}M > 0$
$\mathrm{d}A > 0$	Supersonic	$\mathrm{d}M > 0$
$\mathrm{d}A < 0$	Supersonic	$\mathrm{d}M < 0$

Table 6.1: Table to show pipe area conditions and Mach number

Profound elements - 2 solutions are possible!

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1+\frac{1}{2}(\gamma-1)M^2}{\frac{1}{2}(\gamma+1)}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{6.11}$$

$$M = 1 \text{ at } \mathrm{d}A = 0 \text{ (point of inflection)} \tag{6.12}$$

6.1.6 Creating supersonic flows

Figure 6.2: Supersonic flow generator.

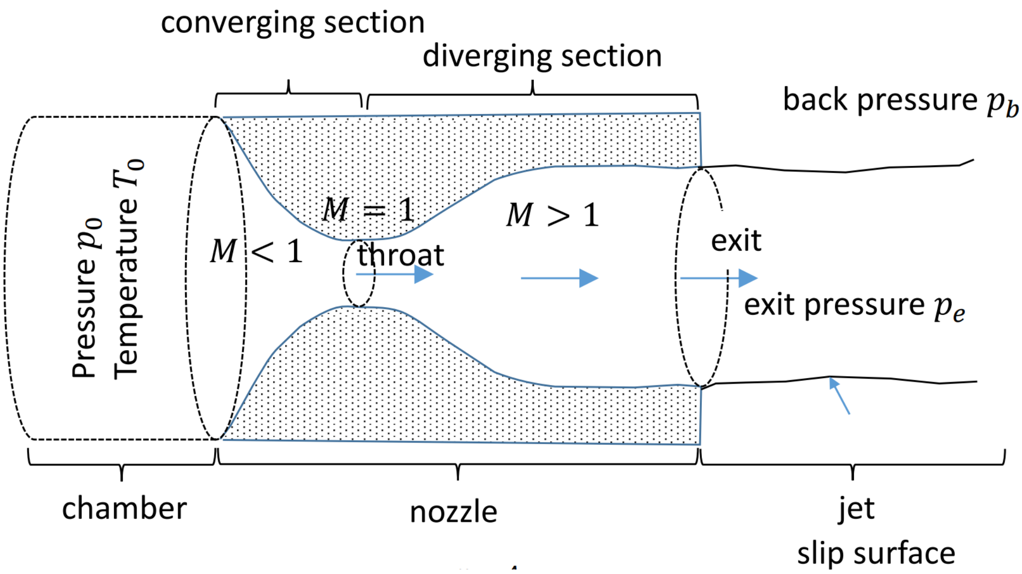


Figure 6.3: Geometry of supersonic flow generator.

This requires a converging, throat and diverging sections. The flow is determined by $\frac{p_b}{p_0}$, $\frac{A_e}{A_t}$.

6.1.7 Differential approach to derivation

Conservation of mass shows:

$$\rho u A = \text{const} \tag{6.13}$$

This can be converted to a differential form:

$$\log(\rho u A) = \log\rho + \log u + \log A = \text{const} \tag{6.14}$$

Taking the differential gives:

$$\frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}A}{A} = 0 \tag{6.15}$$

Conservation of momentum gives:

$$\rho u A\,\mathrm{d}U = -A\,\mathrm{d}p \tag{6.16}$$

For an isentropic flow:

$$\frac{\mathrm{d}p}{p} = -\gamma\frac{\mathrm{d}\rho}{\rho} \tag{6.17}$$

From the conservation of energy:

$$\frac{T}{T_0} = \left(1 + \frac{1}{2}(\gamma-1)M^2\right)^{-1} \tag{6.18}$$

$$\frac{\mathrm{d}T}{T} = \frac{\frac{1}{2}(\gamma-1)\,\mathrm{d}M^2}{1 + \frac{1}{2}(\gamma-1)M^2} \tag{6.19}$$

THis gives:

$$\frac{\mathrm{d}M^2}{M^2} = -\frac{2\left(1 + \frac{1}{2}(\gamma-1)M^2\right)}{1-M^2}\frac{\mathrm{d}A}{A} \tag{6.20}$$

Integrate:

$$\log\frac{A_2}{A_1} = \int_{A_1}^{A_2}\left(\frac{1}{A}\right)\mathrm{d}A = \int_{M_1^2}^{M_2^2}\left(\frac{1}{M^2}\right)\mathrm{d}M^2 \tag{6.21}$$

This can be manipulated to the formula on the crib sheet.

6.1.8 Mass flux relationship

The mass flux (per unit area) is $\frac{m}{A} = \rho u$. Its derivative w.r.t. M is:

$$\frac{\mathrm{d}}{\mathrm{d}M}\left(\frac{\rho u}{\rho_* u_*}\right) = \left(1-M^2\right)\left(1 + \frac{1}{2}(\gamma-1)M^2\right)^{-\frac{\gamma-3}{2(\gamma-1)}}\left(\frac{1}{2}(\gamma+1)\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{6.22}$$

The maximum occurs when $M = 1$, when the flow is choked. The maximum mass flux is:

$$\dot{m}_{max} = \rho^* u^* A^* = \left(\frac{1}{\frac{1}{2}(\gamma+1)}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \left(\frac{\gamma}{RT_0}\right)^{\frac{1}{2}} A^* p_0 \qquad (6.23)$$

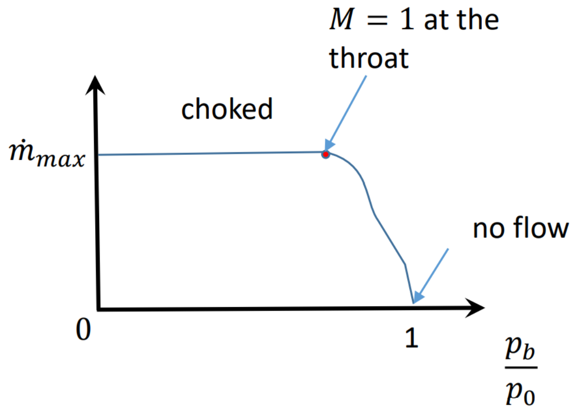


Figure 6.4

where p_b is the back pressure and p_0 is the reservoir pressure.

6.1.9 Types of solution available

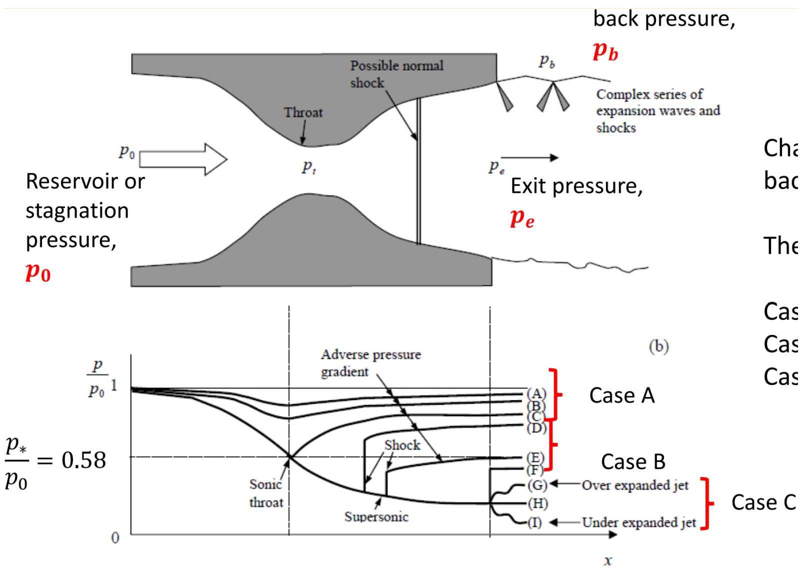


Figure 6.5: Overview of problem.

Characteristics depend on the value of the back pressure. There are three types of solution:

- Case A: Subsonic in whole flow ($p_e = p_b$)
- Case B: Choked but subsonic outlet ($p_e = p_b$)
- Case C: Supersonic outlet ($p_e \lesseqgtr p_b$)

Case A: Subsonic flow

For subsonic flow condition, $M \leq 1$ everywhere and the flow is isentropic because no shocks form. The pressure at the outlet of the duct, p_e, is the same as the back pressure, i.e:

$$p_e = p_b \tag{6.24}$$

The exit Mach number, M_e, is determined from the isentropic relationship:

$$M_e^2 = \frac{2}{\gamma - 1}\left(\left(\frac{p_0}{p_b}\right)^{\frac{\gamma-1}{\gamma}} - 1\right) \tag{6.25}$$

The critical area A^* can be determined from the crib sheet as:

$$A_* = A_e M_e \left(\frac{1 + \frac{1}{2}(\gamma - 1) M_e^2}{\frac{1}{2}(\gamma + 1)}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \tag{6.26}$$

The mass flux can be determined at any point along the duct. Choosing the outlet plane, then:

$$\dot{m} = \rho_e u_e A_e = \rho_0 \sqrt{\gamma R T_0} M_e A_e \left(1 + \frac{1}{2}(\gamma - 1) M_e^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \tag{6.27}$$

This has a maximum value when $\frac{p_b}{p_0} = \frac{p_*}{p_0}$. For a moderate drop in p_b from states A and B, the throat is still subsonic. For curve C, the area ratio is:

$$\frac{A_e}{A_t} = \frac{A_e}{A_*} \tag{6.28}$$

and the flow is sonic at the throat.

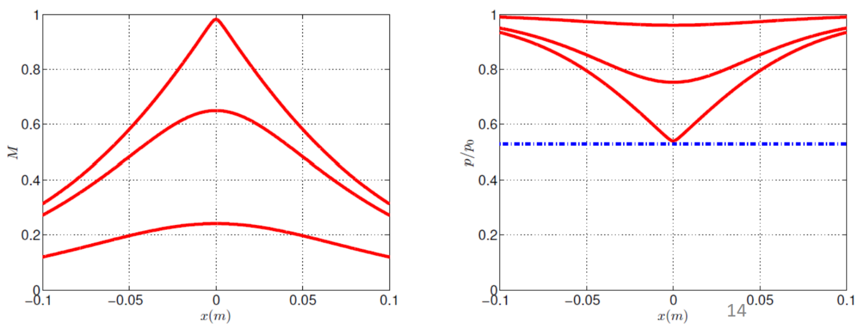


Figure 6.6: Mach number in the supersonic generator, case A.

Case B: Choked flow with subsonic outlet

When p_b decreases, the only way for the flow to adapt is for a shock to be created between the nozzle and the outlet. Isentropic model still applies but supplement by normal shock relationship across the shock. If $p_b < p_*$, the nozzle cannot respond because it is choked at its maximum throat mass flow. The flow switches to the supersonic condition after the throat since the flow is mass constrained. The flow passes smoothly through this transition. The only way of the pressure recovering to match the back pressure is to have a normal shock at some location in the nozzle. The exit's jets is then subsonic and is able to match the back pressure.

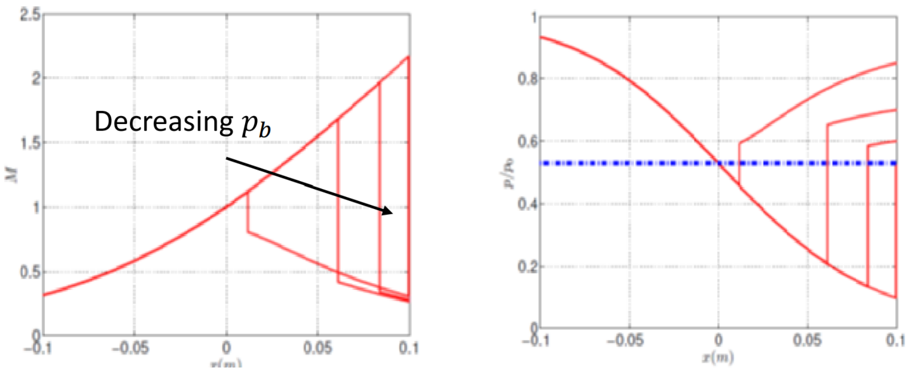


Figure 6.7: Mach number in supersonic generator, case B.

6.7 with subsonic outlet with choked flow and $\frac{p_b}{p_0} = 0.53, 0.60, 0.70$ and 0.85. Note that $\frac{p_*}{p_0} = 0.528$. After the shock A_* is different before and after the shock ($A_* = A_t$ before the shock.)

The flow characteristics are separated into three parts:

1. Choked flow to normal shock.

   In this region, since the flow is choked, $A_* = A_t$. The change in Mach number and pressure are:

$$\frac{A}{A_t} = \frac{1}{M}\left(\frac{1+\frac{1}{2}(\gamma-1)M^2}{\frac{1}{2}(\gamma+1)}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{6.29}$$

$$\frac{p}{p_0} = \left(1+\frac{1}{2}(\gamma-1)M^2\right)^{-\frac{\gamma}{\gamma-1}} \tag{6.30}$$

2. Normal Shock relationship

   If the normal shock occurs at $x = x_s$ where $M_1 = M(x_s)$ and $p_1 = p(x_s)$, the

Mach number and pressure after the shock are:

$$M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1) M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)} \tag{6.31}$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma + 1} M_1^2 - \frac{\gamma - 1}{\gamma + 1} \tag{6.32}$$

The stagnation pressure decreases after the shock and needs to be recalculated:

$$p_{20} = p_2 \left(1 + \frac{1}{2}(\gamma - 1) M_2^2\right)^{\frac{\gamma}{\gamma - 1}} \tag{6.33}$$

The Mach number at the exit is:

$$M_e^2 = \frac{2}{\gamma - 1} \left(\left(\frac{p_{20}}{p_e} \right)^{\frac{\gamma - 1}{\gamma}} - 1 \right) \tag{6.34}$$

F is required normal shock in the duct exit. At the back pressure G no single normal shock can be the job and so the flow compresses outside the exit in a complex series of oblique shocks until it matches p_b. In state F:

$$\frac{A_e}{A_t} = \frac{1}{M_e} \left(\frac{1 + \frac{1}{2}(\gamma - 1) M_e^2}{\frac{1}{2}(\gamma + 1)} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \tag{6.35}$$

$$p_b = p_2 = p_1 \left(\frac{2\gamma}{\gamma + 1} M_e^2 - \frac{\gamma - 1}{\gamma + 1} \right) \tag{6.36}$$

$$p_1 = p_e = p_0 \left(1 + \frac{1}{2}(\gamma - 1) M_e\right)^{-\frac{\gamma}{\gamma - 1}} \tag{6.37}$$

Case C: Supersonic outlet

The final state does not depend on the shape of the throat:

$$\frac{A_e}{A_t} = \frac{1}{M_e} \left(\frac{1 + \frac{1}{2}(\gamma - 1) M_e^2}{\frac{1}{2}(\gamma + 1)} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \tag{6.38}$$

$$\frac{p_e}{p_0} = \left(1 + \frac{1}{2}(\gamma - 1) M_e^2\right)^{-\frac{\gamma}{\gamma - 1}} \tag{6.39}$$

$p_e = p_b$	Design pressure	
$p_e > p_b$	Under-expanded flow	Exit pressure higher than back pressure, need expansion fans to match flow.
$p_e < p_b$	Over-expanded flow	Exit pressure less than back pressure so need shocks to increase pressure.