## 0.1 Comparing Lifting Surfaces

### 0.1.1 Subsonic Aerofoil

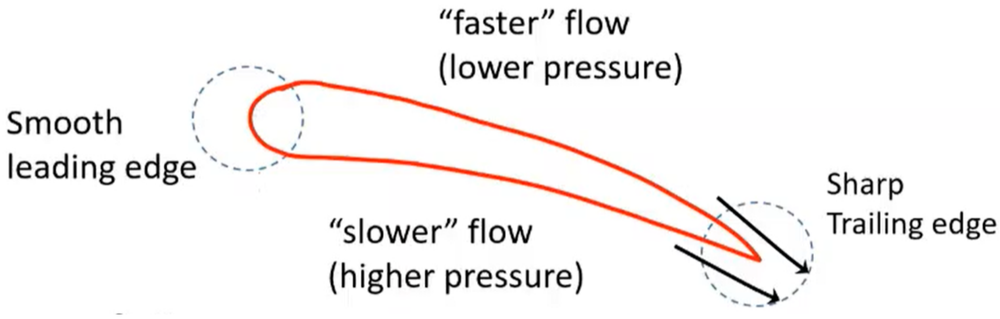


Figure 1: Diagram of a subsonic aerefoil with the relevant pressures around it

For a subsonic aerefoil, the pressure around it varies continuously.

### 0.1.2 Supersonic Aerofoil

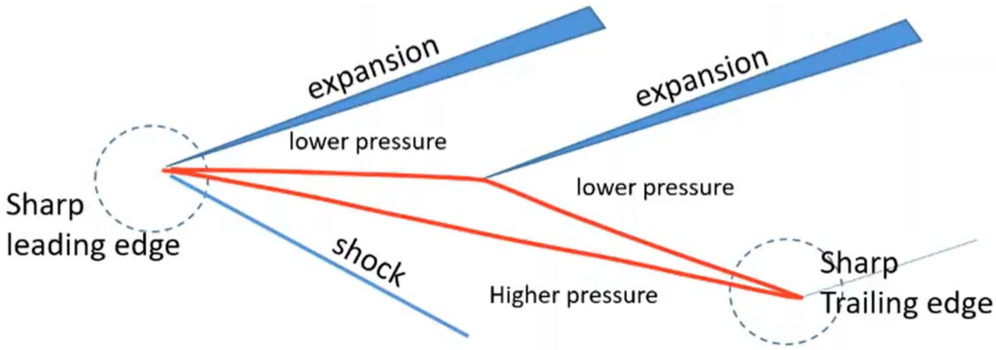


Figure 2: Diagram of a supersonic aerefoil with the relevant pressures around it

For a supersonic aerefoil, the pressure around it is constant in different regions.

## 0.2 Nonlinear Analysis

We consider the case of a flat plate and examine the forces on it.

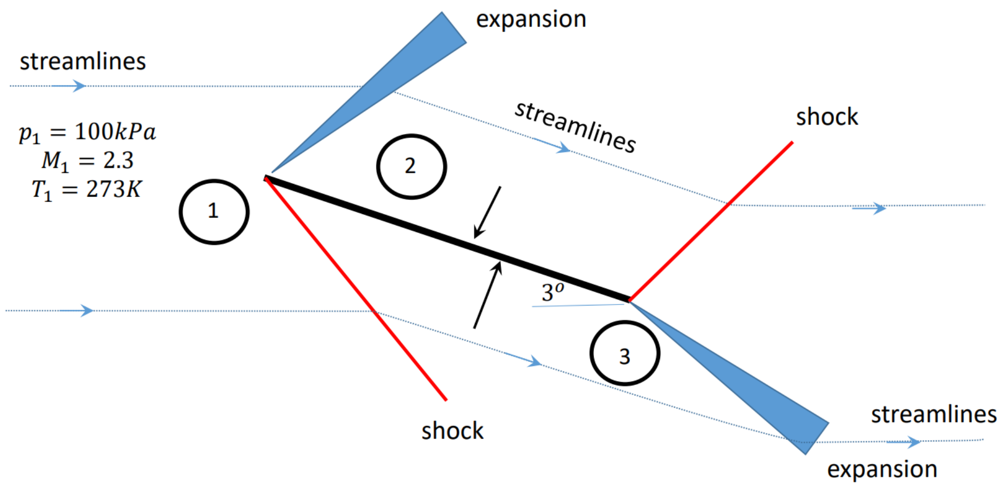

Figure 3: The diagram of a flat plate with the relevant regions indicated

The initial conditions are:

$$p_1 = 100\text{kPa} \tag{1}$$

$$M_1 = 2.3 \tag{2}$$

$$T_1 = 273\text{K} \tag{3}$$

### 0.2.1 Use Shock Charts from Region 1 to Region 3

The following results are obtained:

$$\frac{p_3}{p_1} = 1.2 \tag{4}$$

Hence, the pressure in region 2 is calculated as:

$$p_3 = 1.2 \cdot 100 = 120\text{kPa} \tag{5}$$

### 0.2.2 Use Expansion Charts for Region 1 to Region 2

The following results are obtained:

$$M_2 = 2.4 \tag{6}$$

$$\frac{p_2}{p_0} = 0.068 \tag{7}$$

$$\frac{p_1}{p_0} = 0.079 \tag{8}$$

Hence, the pressure in region 2 is calculated as:

$$\frac{p_2}{p_1} = \frac{p_2}{p_0} \cdot \frac{p_0}{p_1} = 0.861 \tag{9}$$

$$\therefore p_2 = 0.861 \cdot 100 = 86.1\text{kPa} \tag{10}$$

### 0.2.3 Force Calculation

The total force (per unit length) is:

$$F_N = (p_3 - p_2)L \tag{11}$$

$$= ((120 - 86.1) \cdot 10^3\text{Pa}) \cdot 1 \tag{12}$$

$$= (34 \cdot 10^3\text{Pa}) \cdot 1 = 34\text{kN}\,\text{m}^{-1} \tag{13}$$

### 0.2.4 Lift and Drag Coefficients

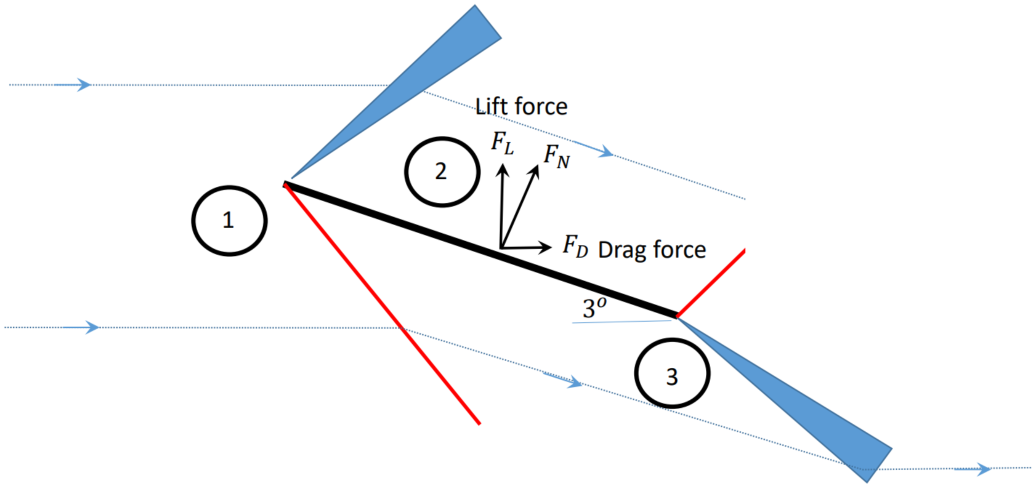


Figure 4: The diagram of a flat plate with the lift and drag coefficients shown

Using:

$$R = 287\,\mathrm{J\,kg^{-1}\,K^{-1}} \qquad \& \qquad \gamma = 1.4 \tag{14}$$

The air speed is calculated as:

$$u_1 = c_1 M_1 \tag{15}$$

$$c_1 = \sqrt{\gamma R T_1} \tag{16}$$

$$u_1 = \sqrt{\gamma R T_1} \cdot M_1 = \sqrt{1.4 \cdot 287 \cdot 273} \cdot 2.3 = 762\,\mathrm{m\,s^{-1}} \tag{17}$$

The drag and lift forces are defined by:

$$F_D = F_N \sin\alpha \tag{18}$$

$$F_L = F_N \cos\alpha \tag{19}$$

The lift and drag coefficients are:

$$c_D = \frac{F_D}{\frac{1}{2}\rho_1 u_1^2 L} \qquad \& \qquad c_L = \frac{F_L}{\frac{1}{2}\rho_1 u_1^2 L} \tag{20}$$

Substituting in the values yields:

$$c_D = 0.0046 \tag{21}$$

$$c_L = 0.088 \tag{22}$$

Two observations can be made here:

1. It is quite hard to calculate
2. $\frac{c_L}{c_D} >> 1$

## 0.3 Linear Methods for Calculating Forces

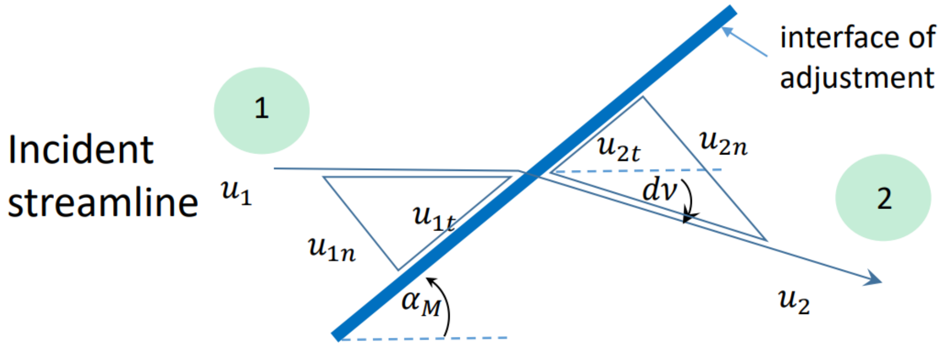


Figure 5

Conservation of momentum parallel to interface:

$$u_1 \cos \alpha_M = u_2 \cos(\alpha_M + dv) \tag{23}$$

From the double angle formula:

$$\cos(\alpha_M + dv) \approx \cos \alpha_M - dv \sin \alpha_M \tag{24}$$

Rearranging gives:

$$\frac{\mathrm{d}u}{u} = \frac{u_2 - u_1}{u_2} = \frac{\sin \alpha_M}{\cos \alpha_M} dv \tag{25}$$

We are trying to relate changes in pressure to the deflection angle. All angles are relative to incident streamline and small so that $|dv| << 1$, where:

- $\mathrm{d}v$ - deflection or wedge angle
- $\alpha_M$ - shock wave angle
- $\sin \alpha_M = \dfrac{1}{M_1}$

To this linear approximation, a shock = - expansion.

From the normal momentum equation:

$$\rho_1 u_{1n}(u_{2n} - u_{1n}) = -(p_2 - p_1) \tag{26}$$

or

$$\rho_1 u_1 \sin \alpha_M (u_2 \sin(\alpha_M + \mathrm{d}v) - u_1 \sin \alpha_M) = -(p_2 - p_1) \tag{27}$$

Rearranging gives:

$$\frac{p_2 - p_1}{\frac{1}{2}\rho_1 u_1^2} = -\frac{2 \sin \alpha_M}{u_1}(u_2(\sin \alpha_M + \cos \alpha_M \,\mathrm{d}v) - u_1 \sin \alpha_M) \tag{28}$$

$$= -\frac{2 \sin \alpha_M}{u_1}\left(u_2 \,\mathrm{d}v \frac{\sin^2 \alpha_M}{\cos \alpha_M} + u_2 \cos \alpha_M \,\mathrm{d}v\right) \tag{29}$$

$$\therefore \frac{p_2 - p_1}{\frac{1}{2}\rho_1 u_1^2} = -\frac{2 \sin \alpha_M}{\cos \alpha_M} \mathrm{d}v \frac{u_2}{u_1} \approx -\frac{2 \,\mathrm{d}v}{\sqrt{M_1^2 - 1}} \tag{30}$$

This is known as the linear model for small changes where $u_1$ and $M_1$ do not change.

## 0.4 Example: Nonlinear Calculation

We consider the case of a flat plate and examine the forces on it:

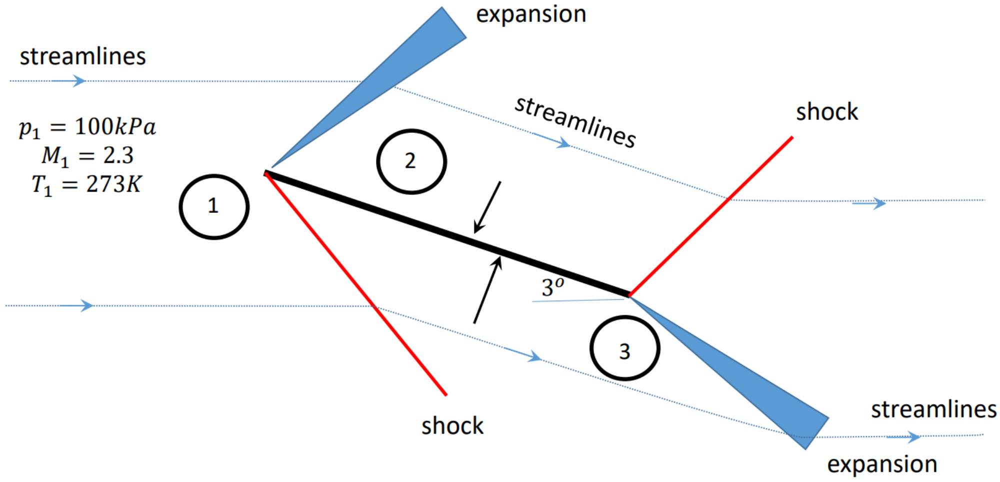


Figure 6: The diagram of a flat plate with the relevant regions indicated

In applying this technique we need to make sure that we understand when a shock or expansion occurs. This tells us the sign of the pressure change.

$$\frac{p_2 - p_1}{\frac{1}{2}\rho_1 u_1^2} \approx -\frac{2\alpha}{\sqrt{M_1^2 - 1}} \tag{31}$$

$$\frac{p_3 - p_1}{\frac{1}{2}\rho_1 u_1^2} \approx -\frac{2\alpha}{\sqrt{M_1^2 - 1}} \tag{32}$$

### 0.4.1 Lift and Drag Coefficients

The normal force per unit length is calculated as:

$$F_N = (p_3 - p_2)L \tag{33}$$

$$= \frac{\frac{1}{2}\rho_1 u_1^2 \big((2\alpha) - (-2\alpha)\big) L}{\sqrt{M_1^2 - 1}} \tag{34}$$

The lift and drag coefficients are:

$$c_D = \frac{F_N \sin\alpha}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{\frac{\frac{1}{2}\rho_1 u_1^2\big((2\alpha)-(-2\alpha)\big)L}{\sqrt{M_1^2-1}} \sin\alpha}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{4\alpha \sin\alpha}{\sqrt{M_1^2 - 1}} = 0.0059 \text{ (Nonlinear 0.0048)} \tag{35}$$

$$c_L = \frac{F_N \cos\alpha}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{\frac{\frac{1}{2}\rho_1 u_1^2\big((2\alpha)-(-2\alpha)\big)L}{\sqrt{M_1^2-1}} \cos\alpha}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{4\alpha \cos\alpha}{\sqrt{M_1^2 - 1}} = 0.1 \text{ (Nonlinear 0.092)} \tag{36}$$

## 0.5 Linear Calculation for More Complicated Geometry

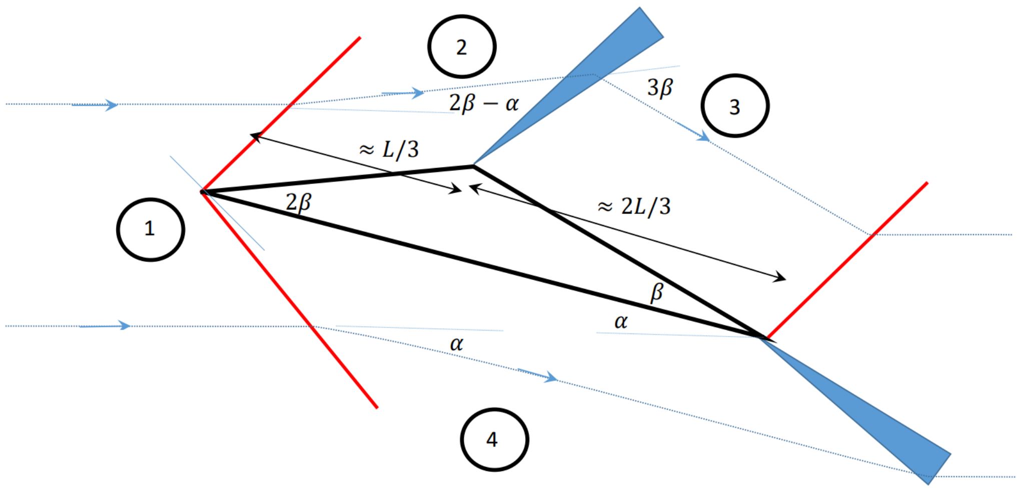


Figure 7: The diagram for a more complicated geometry

To simplify the analysis, define:

$$A = \frac{\frac{1}{2}\rho_1 u_1^2}{\sqrt{M_1^2 - 1}} \tag{37}$$

Techniques for solution:

1. Draw a large figure
2. Calculate the deflection angles
3. Calculate the pressure differences between the different regions

Pressures in each region:

$$p_2 - p_1 = 2(2\beta - \alpha)A \tag{38}$$

$$p_3 - p_2 = -6\beta A \tag{39}$$

$$p_4 - p_1 = 2\alpha A \tag{40}$$

The second equation can be written as:

$$p_3 - p_1 = -(2\alpha + 2\beta) \tag{41}$$

The lift force is calculated as:

$$F_L = p_4 L \cos\alpha - p_2 \frac{L}{3}\cos(2\beta - \alpha) - p_3 \frac{2L}{3}\cos(\alpha + \beta) \tag{42}$$

Through small angle approximation:

$$F_L = p_4 L - p_2 \frac{L}{3} - p_3 \frac{2L}{3} \tag{43}$$

$$\approx \left(2\alpha - \frac{2}{3}(2\beta - \alpha) + \frac{2}{3}(2\alpha + 2\beta)\right) AL \tag{44}$$

The drag force is calculated as:

$$F_D = p_4 L \sin\alpha + p_2 \frac{L}{3}\sin(2\beta-\alpha) - p_3\frac{2L}{3}\sin(\alpha+\beta) \tag{45}$$

Through small angle approximation:

$$F_D = p_4 L\alpha + p_2\frac{L}{3}(2\beta-\alpha) - p_3\frac{2L}{3}(\alpha+\beta) \tag{46}$$

$$\approx \left(2(\alpha)(\alpha) + \frac{2}{3}(2\beta-\alpha)(2\beta-\alpha) + \frac{2}{3}(2\alpha+2\beta)(\alpha+\beta)\right)AL \tag{47}$$

$$= \left(2\alpha^2 + \frac{2}{3}(4\beta^2 - 4\beta\alpha + \alpha^2) + \frac{4}{3}(\alpha^2 + 2\beta\alpha + \beta^2)\right)AL \tag{48}$$

$$= (4\alpha^2 + 4\beta^2)AL \tag{49}$$

The lift and drag coefficients are:

$$c_L = \frac{F_L}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{4\alpha}{\sqrt{M_1^2-1}} \tag{50}$$

$$c_D = \frac{F_D}{\frac{1}{2}\rho_1 u_1^2 L} = \frac{4(\alpha^2+\beta^2)}{\sqrt{M_1^2-1}} \tag{51}$$

The solution reduces to flat plate case when $\beta = 0$.

## 0.6 Lift Coefficient Variation with Mach Number

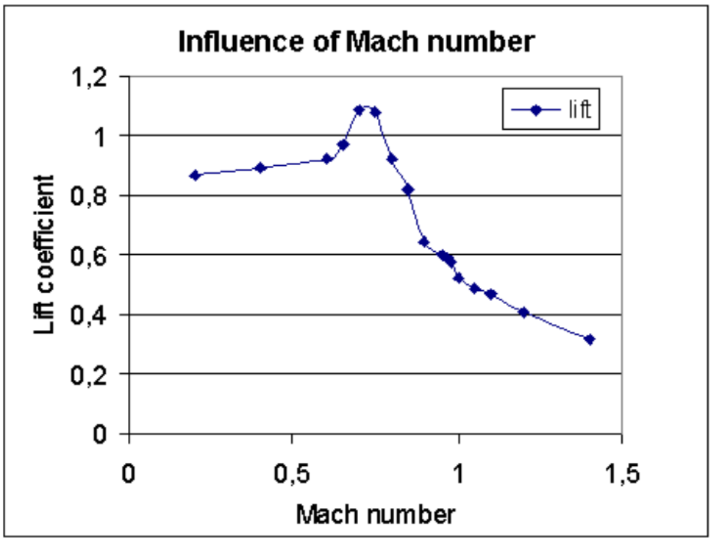


Figure 8: Lift coefficient variation with Mach number

For low Mach number:

$$c_D \sim constant \tag{52}$$

$$c_L \sim 2\pi\alpha \tag{53}$$

For high Mach number:

$$c_D \sim \frac{4\alpha^2}{\sqrt{M_1^2 - 1}} \tag{54}$$

$$c_L \sim \frac{4\alpha}{\sqrt{M_1^2 - 1}} \tag{55}$$