Chapter 1

Finite Element Method

1.1 Introduction to the method

- Discretisation of the model to elements
- Governing equations for each element
- Assembled to give system equations
- $[k]\{U\} = \{F\}$
- $[k]$ is a square matrix, stiffness matrix $\{U\}$ is the vector of unknown nodal displacements or temperatures and $\{F\}$ is the vector of applied nodal forces

1.2 1D element - the pin jointed bar

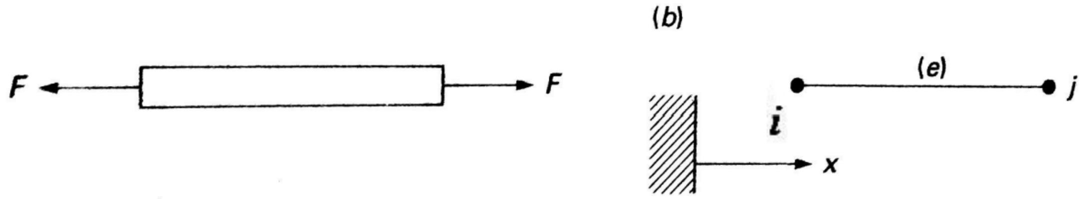

Figure 1.1: Pin jointed bar.

$$u = a + bx \tag{1.1}$$
$$u_i = a + bx_i \tag{1.2}$$
$$u_j = a + bx_j \tag{1.3}$$

where $u_{i,j}$ are unknown nodal displacements and $x_{i,j}$ are known nodal coordinates.

This leads to:

$$a = \frac{\left(u_i x_j - u_j x_i\right)}{L} \tag{1.4}$$

$$b = \frac{\left(u_j - u_i\right)}{L} \tag{1.5}$$

$$u = \frac{x_j - x}{L} u_i + \frac{x - x_i}{L} u_j \tag{1.6}$$

$$u = N_i u_i + N_j u_j \tag{1.7}$$

where $L = x_j - x_i$. N_i and N_j are called shape functions. When a structure is loaded and reaches an equilbirium its potential energy must be minimum.

$$\Pi = \Lambda - W \tag{1.8}$$

where Λ is strain energy and W is work done by external loads (pressure load, body force, nodal forces).

$$W = u_i F_i + u_j F_j = \{U\}^T \{F\} \tag{1.9}$$

where:

$$\{U\}^T = \begin{bmatrix} u_i & u_j \end{bmatrix} \tag{1.10}$$

$$\{F\} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix} \tag{1.11}$$

$$\Lambda = \int_{x_i}^{x_j} \left(\frac{1}{2}\sigma\varepsilon A\right) \mathrm{d}x \Lambda = \frac{AE}{2} \int_{x_i}^{x_j} \left(\varepsilon^2\right) \mathrm{d}x \tag{1.12}$$

where $\sigma = E\varepsilon$ and is the stress, Λ is strain energy density, A is surface area and x is length. Using the definition of strain:

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} \tag{1.13}$$

We can differentiate our shape function:

$$\varepsilon = \frac{\left(-u_i + u_j\right)}{L} \tag{1.14}$$

Leading to:

$$\Lambda = \frac{AE}{2L} \left(-u_i + u_j\right)^2 \tag{1.15}$$

$$\Lambda = \frac{AE}{2L} \begin{bmatrix} u_i & u_j \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{1.16}$$

$$= \frac{1}{2} \{U\}^T [k] \{U\} \tag{1.17}$$

where the stiffness matrix is:

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.18}$$

Potential energy must be minimised

$$\Pi = \frac{1}{2}\{U\}^T[k]\{U\} - \{U\}^T\{F\} \tag{1.19}$$

$$\frac{\partial \Pi}{\partial u_i} = \frac{\partial \Pi}{\partial u_j} = 0 \text{ or } \frac{\partial \Pi}{\partial \{U\}} = 0 \tag{1.20}$$

$$\frac{\partial \Pi}{\partial \{U\}} = [k]\{U\} - \{F\} = 0 \tag{1.21}$$

$$\begin{Bmatrix} F_i \\ F_j \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{1.22}$$

$$\Pi = \sum_{e=1}^{E}\left(\Lambda^{(e)} - W\right) \tag{1.23}$$

Leading to:

$$\frac{\partial \Pi}{\partial \{U\}} = \left(\sum_{e=1}^{E}\left[k^{(e)}\right]\right)\{U\} - \{F\} = 0 \tag{1.24}$$

1.2.1 Exercise

Calculate the displacement and stress using the stiffness matrix.

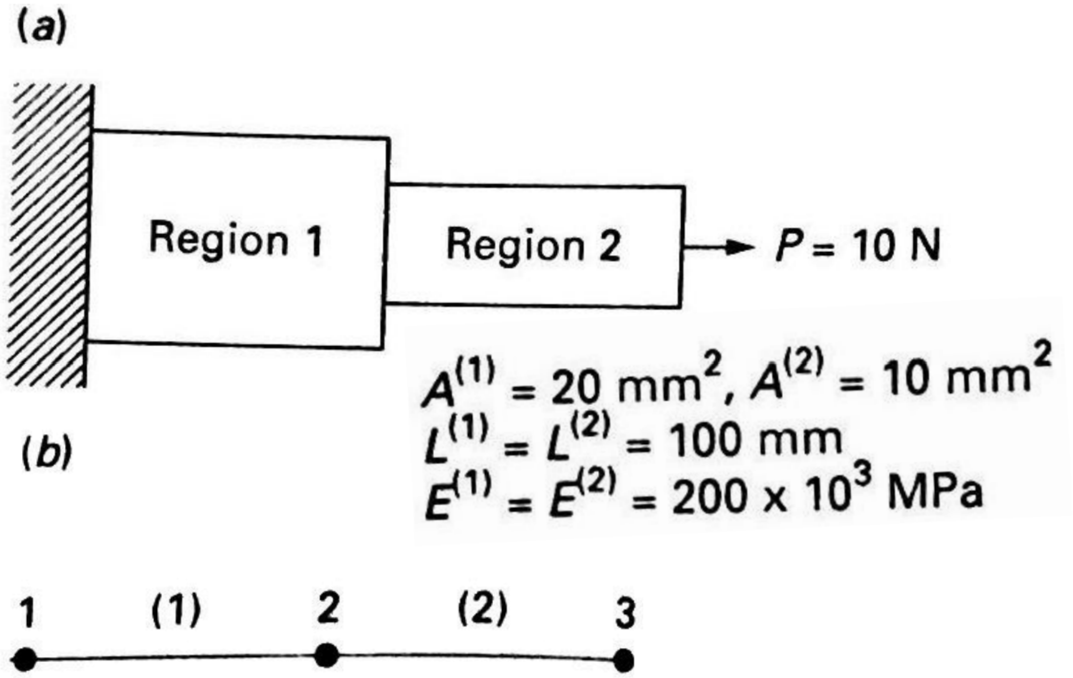

Figure 1.2: Exercise, 3 node 1D problem.

$$[k_1] = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \times 10^4 \,\mathrm{N\,mm^{-1}} \tag{1.25}$$

$$[k_2] = \frac{A_2 E_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \times 10^4 \,\mathrm{N\,mm^{-1}} \tag{1.26}$$

$$[k] = \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \times 10^4 \,\mathrm{N\,mm^{-1}} \tag{1.27}$$

$$\{F\}^T = \begin{bmatrix} 0 & 0 & 10 \end{bmatrix} 10^4 \times \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 + R1 \\ 0 \\ 10 \end{Bmatrix} \tag{1.28}$$

Leading to (commented in Latex):

$$u_1 = 0 \tag{1.29}$$
$$u_2 = 0.25 \times 10^{-3} \,\mathrm{mm} \tag{1.30}$$
$$u_3 = 0.75 \times 10^{-3} \,\mathrm{mm} \tag{1.31}$$
$$R_1 = -10 \,\mathrm{N} \tag{1.32}$$
$$\varepsilon_1 = \frac{(-u_1 + u_2)}{L} = 2.5 \times 10^{-6} \tag{1.33}$$
$$\varepsilon_2 = \frac{(-u_2 + u_3)}{L} = 5 \times 10^{-6} \tag{1.34}$$
$$\sigma_1 = E\varepsilon_1 = 0.5 \,\mathrm{N\,mm^{-1}} \tag{1.35}$$
$$\sigma_2 = E\varepsilon_2 = 1 \,\mathrm{N\,mm^{-1}} \tag{1.36}$$

1.3 1D element - the spring

Consider the same 1D linear spring with stiffness k independent of deflection:

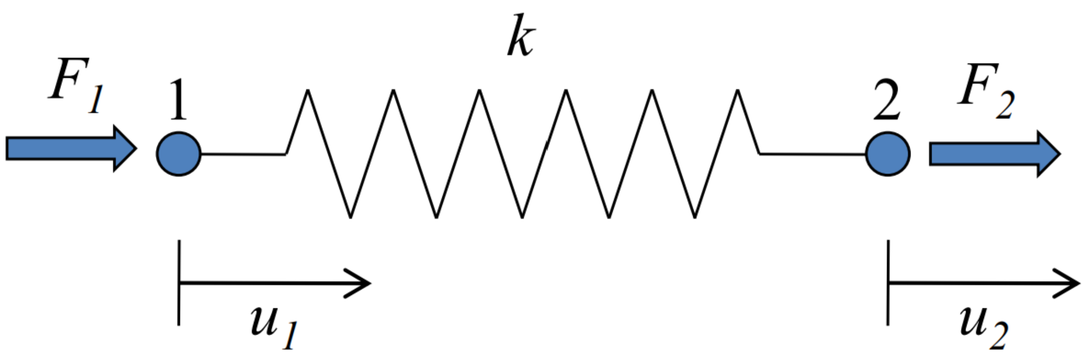

Figure 1.3: Spring system.

$$W_{ext} = \int_0^{u_1} \big(F_1(u)\big)\,\mathrm{d}u + \int_0^{u_2} \big(F_2(u)\big)\,\mathrm{d}u \tag{1.37}$$

$$\delta W_{ext} = F_1 \delta u_1 + F_2 \delta u_2 = \begin{pmatrix} \delta u_1 & \delta u_2 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \tag{1.38}$$

where $F_1 = F_1\left(u_1\right)$ and $F_2 = F_2\left(u_2\right)$.

$$W_{int} = \frac{1}{2}k\left(u_2 - u_1\right)^2 = \frac{1}{2}\begin{pmatrix} u_1 & u_2 \end{pmatrix}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{1.39}$$

$$\delta W_{int} = k\left(u_2 - u_1\right)\left(\delta u_2 - \delta u_1\right) = \begin{pmatrix} \delta u_1 & \delta u_2 \end{pmatrix}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{1.40}$$

Applying the principle of virtual work gives:

$$\delta W_{ext} = \delta W_{int} \rightarrow \delta u_1\left[-k\left(u_2 - u_1\right) - F_1\right] + \delta u_2\left[k\left(u_2 - u_1\right) - F_2\right] = 0 \tag{1.41}$$

and since δu_1 and δu_2 are arbitrary, one obtains:

$$F_1 = ku_1 - ku_2 \text{ and } F_2 = -ku_1 + ku_2 \tag{1.42}$$

Or, in matrix form:

$$\underline{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \underline{Ku} \tag{1.43}$$

Chapter 2

Computational Fluid Dynamics