MECH0013 Topic Notes

UCL

HD

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Chapter 1

Basic Concepts of Structure Mechanics

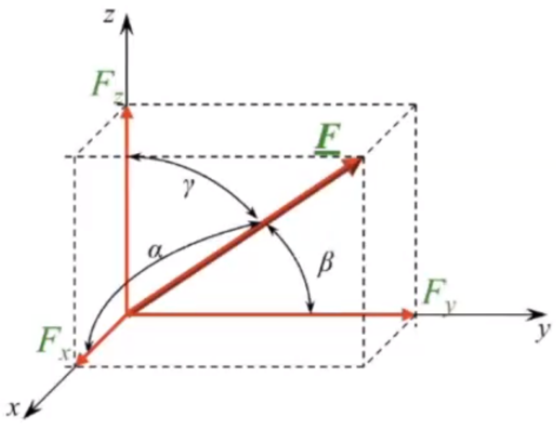
09/10/2020

1.1 Actions and Deformations

1.1.1 Vector Quantities

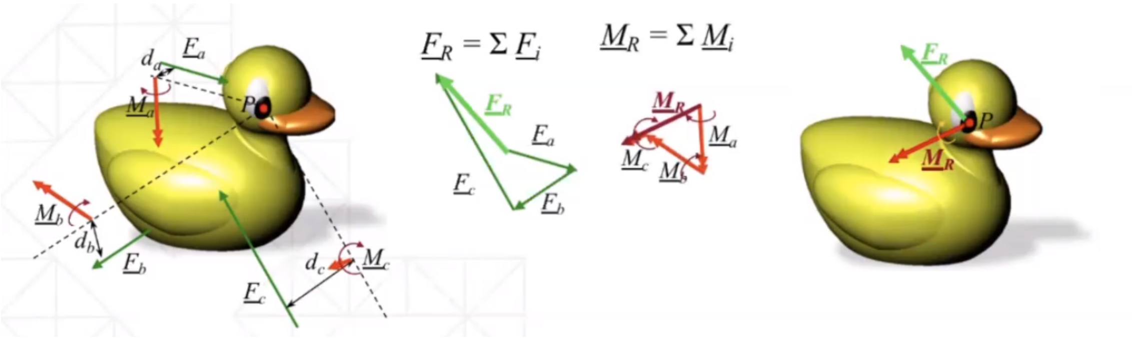
A vector is a quantity defined by **magnitude** and **direction**. Mechanical actions (forces and moments) can be represented as **vectors**.

Vector quantities can be decomposed in components, that can be conveniently oriented with the Cartesian reference system.



$$F_x = F \cdot \cos\alpha \qquad M_x = M \cdot \cos\alpha \tag{1.1}$$
$$F_y = F \cdot \cos\beta \qquad M_y = M \cdot \cos\beta \tag{1.2}$$
$$F_z = F \cdot \cos\gamma \qquad M_z = M \cdot \cos\gamma \tag{1.3}$$

On the other hand, a set of vector forces can be composed in a resultant force applied to any point P, and the moment they produce about P.



$$\vec{F_R} = \sum \vec{F_i} \qquad \vec{M_R} = \sum \vec{F_i} \tag{1.4}$$
$$F_R = \sqrt{(F_x)^2 + (F_y)^2 + (F_z)^2} \tag{1.5}$$
$$M_R = \sqrt{(M_x)^2 + (M_y)^2 + (M_z)^2} \tag{1.6}$$

1.1.2 Equilibrium State

If a configuration is in equilibrium, the resultant of all external forces and moments is zero. This can be expressed mathematically in the following 6 equations:

$$\begin{array}{ccc} \sum F_x = 0 & \sum F_y = 0 & \sum F_z = 0 \\ \sum M_x = 0 & \sum M_y = 0 & \sum M_z = 0 \end{array}$$

These equations have to be valid for the entire body, and for any of its portions.

1.1.3 Deformations

Mechanical actions produce **deformations** in the body. These can be:

- Tension
- Compression
- Bending

- Twisting

These deformations translate into local strains and are opposed and balanced by internal reaction forces (and stresses), that guarantee the structural congruence of the body.

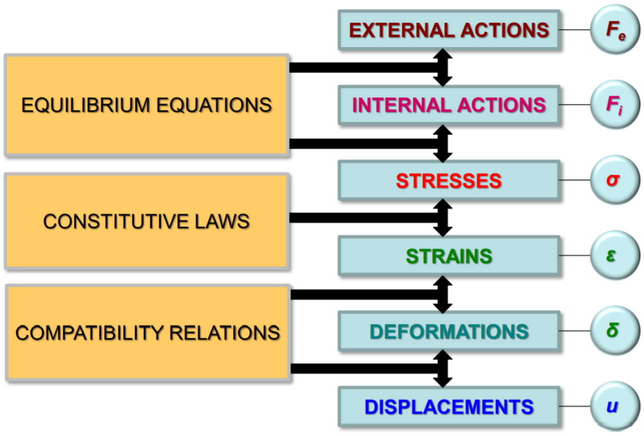
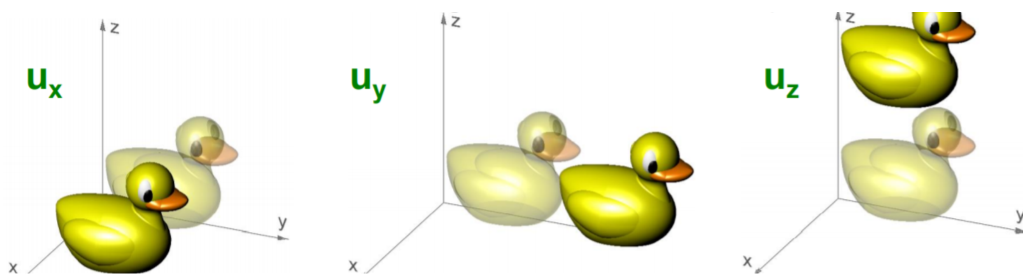


Figure 1.1: Solid Mechanics Equation: When dealing with mechanical action problems, the actions listed in the flowchart above occur, starting with external/internal forces and ending with displacements/deformations

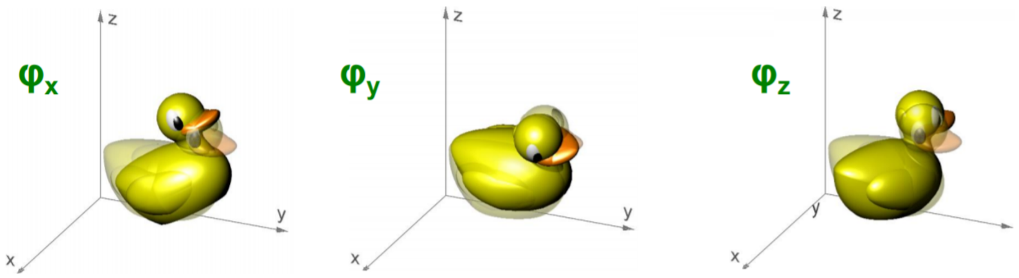
1.2 Degree of Freedom and Supports

We define **degree of freedom** of a system as all the basic kinematical parameters (or all the forms of movement) allowed. A rigid body in the space, in a coordinate system, has 6 degrees of freedom:

3 translations along the coordinate axes x, y and z



3 rotations about the coordinate axes x, y and z



The total translational and rotational movement of an object can be shown with the following expression:

$$\vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \\ \phi_x \\ \phi_y \\ \phi_z \end{bmatrix}$$

In a 2D plane, the degree of freedom reduces to pnly 3 variables:

$$\vec{u} = \begin{bmatrix} u_x \\ u_y \\ \phi_z \end{bmatrix}$$

1.2.1 Constraint

We define **constraint** as a limitation of the degree of freedom of the system. The most common constraints are:

- Supports providing the required reacting forces to maintain overall equilibrium
- Connections providing reaction forces between two components of the system

The table below summarizes the different types of supports (constraints) that will be used throughout the course:

Fixed	Rotating	Roller	Sliding
			
$\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\vec{u} = \begin{bmatrix} 0 \\ 0 \\ \phi \end{bmatrix}$	$\vec{u} = \begin{bmatrix} u_x \\ 0 \\ \phi \end{bmatrix}$	$\vec{u} = \begin{bmatrix} u_x \\ 0 \\ 0 \end{bmatrix}$
$\vec{R} = \begin{bmatrix} R_x \\ R_y \\ M_z \end{bmatrix}$	$\vec{R} = \begin{bmatrix} R_x \\ R_y \\ 0 \end{bmatrix}$	$\vec{R} = \begin{bmatrix} 0 \\ R_y \\ 0 \end{bmatrix}$	$\vec{R} = \begin{bmatrix} 0 \\ R_y \\ M \end{bmatrix}$

1.3 Beams and Sign Conventions

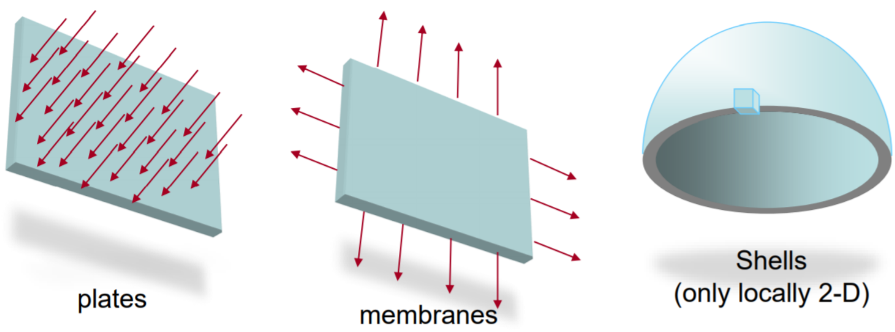
Structures are sets of solid bodies components with the function of carrying loads. All solid bodies are 3-dimensional, however, often it is possible to identify some dimension that is more relevant. Many structures can be analysed as bi-dimensional (2D) or mono-dimensional (1D).

1.3.1 Types of Structures

Bi-dimensional Structures

If one of the dimensions is negligible compared to the other two, the structure can be studied as bi-dimensional. Some examples are:

- Plates
- Membranes
- Shells (Only locally 2D)

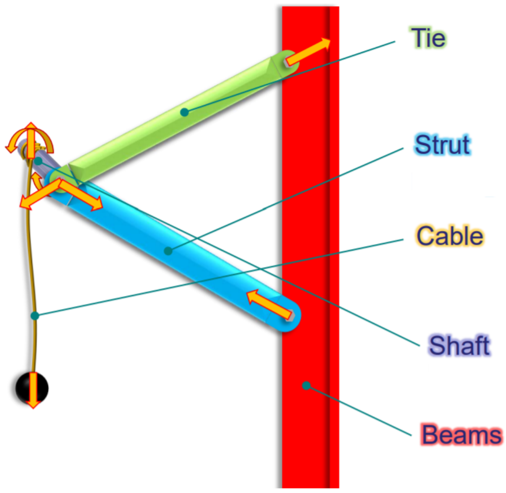


For shells to be considered bi-dimensional, they need to be looked at very closely where they can resemble a plate. The difference between plates and membranes is the bending rigidity; force is required to bend plates while membranes are really floppy.

Mono-dimensional Structures

If two of the dimensions are negligible compared to the other one, the structure can be studied as mono-dimensional. Some examples are:

- Tie - Prevents two parts of the structure from moving away
- Strut - Prevents two parts of the structure from moving forward
- Cable - Flexible string that stands only tensile loads
- Shaft - Is used for the transmission of torque
- Beams - Can carry also transverse loads



1.3.2 Beams

The generic mono-dimensional components of structures, able to carry also transverse loads are called **beams**. Beams are between the most common and important components in structures. In order to study beams as mono-dimensional structures, all mechanical actions have to act on the **centre of gravity (CG)** of the beam section. If there is a case where a force isn't acting on the CG, it will be converted to act on it, so that the beam can be analysed in a simple manner.

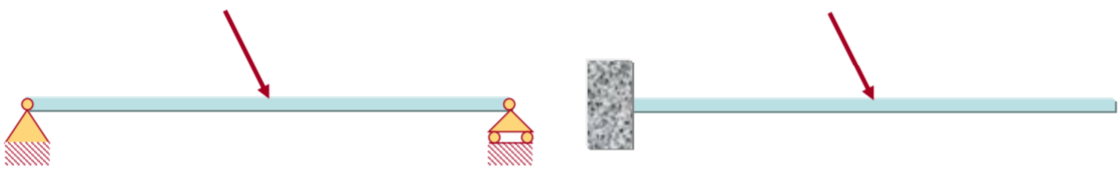
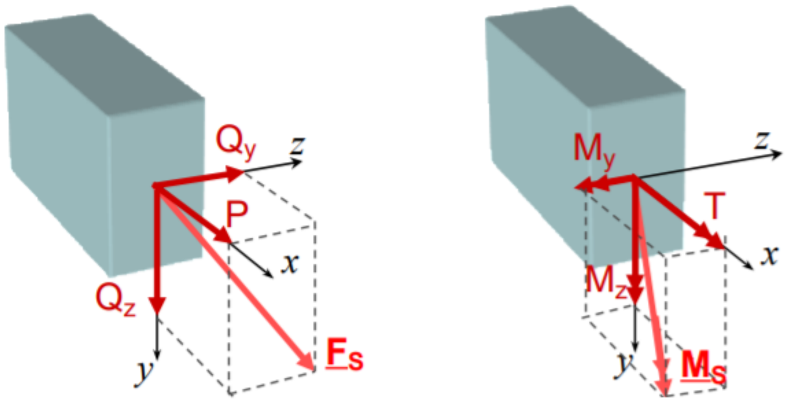


Figure 1.2: On the left: Simply supported beam (rotating support + roller support), On the right: Cantilever beam (with a fixed end)

1.3.3 Internal Forces

Each point of the beam is characterised by a specific set of internal forces. We consider a cross section of the beam to investigate these forces.

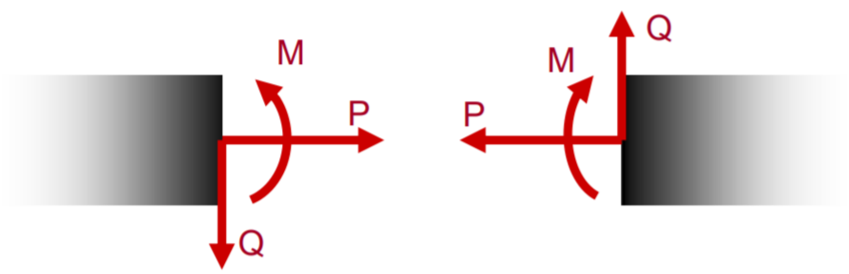


- P - Longitudinal force
- Q_y - y shear forces
- Q_z - z shear forces
- T - Torque
- M_y - y bending moment
- M_z - z bending moment

The internal forces at every point of the beam can be characterised with the following expression:

$$\vec{F} = \begin{bmatrix} P \\ Q_y \\ Q_z \\ \\ T \\ M_y \\ M_z \end{bmatrix}$$

In a 2D plane, it simplifies to:



- P - Longitudinal force

- Q - Shear forces
- M - Bending moment

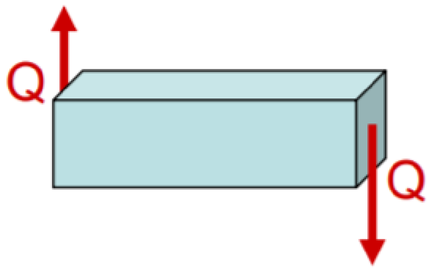
$$\vec{F} = \begin{bmatrix} P \\ Q \\ M \end{bmatrix}$$

1.3.4 Sign Conventions

Longitudinal Force

The direction of pulling is considered to be positive.
The direction of compression is considered to be negative.

Shear Force

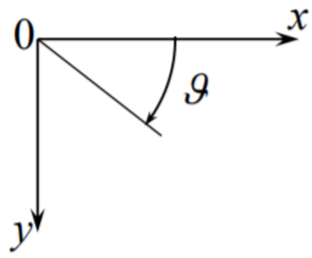


The left side pointing upwards and right side pointing downwards are taken as positive.
The left side pointing downwards and right side pointing upwards are taken as negative.

Bending Moment

If the bending moment is making the beam a upwards concave shape (like U), it is considered to be positive.
If the bending moment is making the beam a downwards concave shape, it is considered to be negative.

Reference System



x axis (horizontal direction) to the right is taken as positive.
y axis (vertical direction) downwards is taken as positive.

Chapter 2

Bending of Beams

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2.1 Internal Forces and Diagrams

2.1.1 Bending Moment

The **bending moment** is by far the most relevant of the internal forces, since it produces the largest levels of deformations and stress into the beam. Therefore, it is essential to be able to determine the distribution of the bending moment along the members, in order to assess the mechanical and functional safety of the structure.

2.1.2 Diagrams and Determination of Internal Forces

To determine the internal forces of a body, and draw the relevant diagrams, the following steps are followed:

Step 1

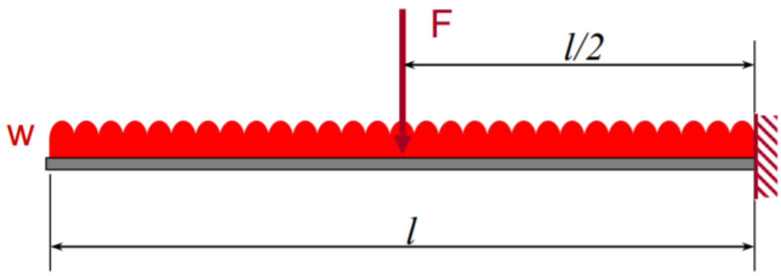
Apply to the body the force and moment equilibrium equations to find support reactions (it is possible only if the system is statically determinate)

Step 2

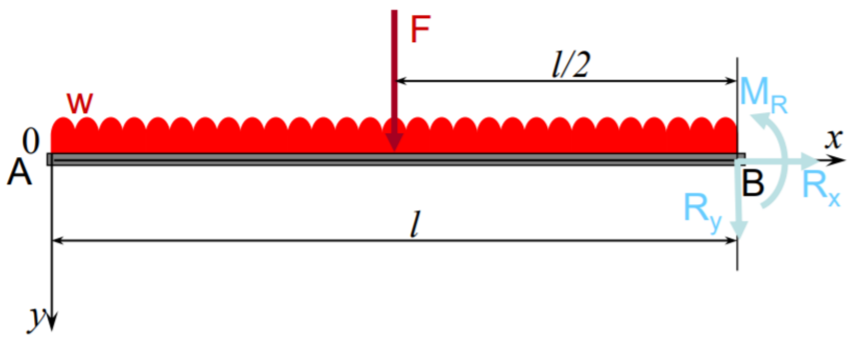
Imagine to cut the beam at a section at a distance x from the beam extreme. Balance the forces and moments of the rigid body by applying the required forces and moment at the cut section

- The **Shear Force** on any given section of a structural member is the algebraic sum of the forces to **one side only** of the section considered.
- The **Bending Moment** on any given section of a structural member is the algebraic sum of the moments of all the forces to **one side only** of the section, about the section
- The maximum value of bending moment occurs at the point where the Shear Force is zero

Example: Cantilever beam having combined concentrated and distributive loads



Determination of Support Reactions:



$$\vec{R_B} = \begin{bmatrix} R_x \\ R_y \\ M \end{bmatrix}$$

$$\sum F_x : R_x = 0 \tag{2.1}$$

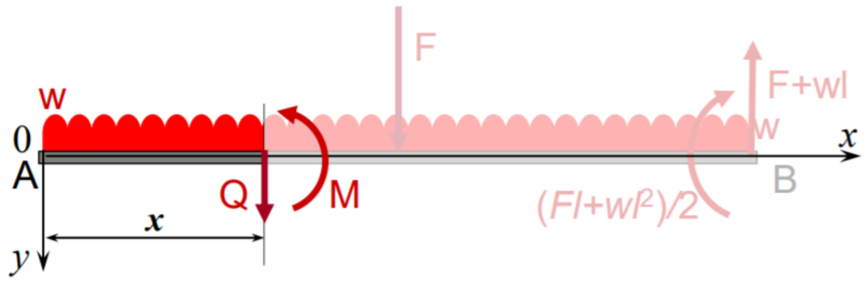
$$\sum F_y : R_y + F + wl = 0 \tag{2.2}$$

$$R_y = -(F + wl) \tag{2.3}$$

$$\sum M : M - F\frac{l}{2} - wl\frac{l}{2} = 0 \tag{2.4}$$

$$M = F\frac{l}{2} + wl\frac{l}{2} = \frac{Fl + wl^2}{2} \tag{2.5}$$

Determination of internal forces (from $x = 0$ to $x = \frac{l}{2}$)



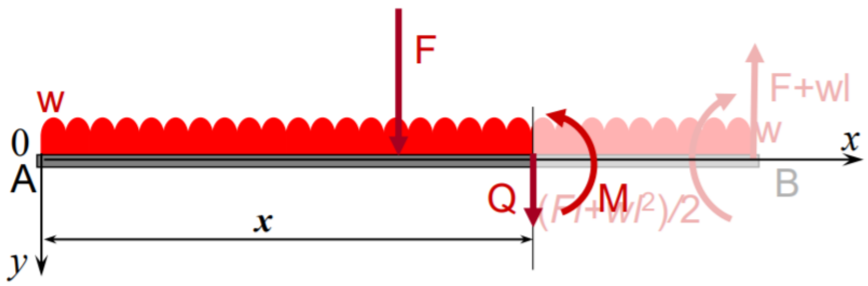
$$\sum F_y : Q + wx = 0 \tag{2.6}$$
$$Q = -wx \tag{2.7}$$

Q varies linearly: it is zero at $x = 0$ and is $\frac{-wl}{2}$ at $x = \frac{l}{2}$.

$$\sum M : M_x + wx\frac{x}{2} = 0 \tag{2.8}$$
$$M_x = \frac{-wx^2}{2} \tag{2.9}$$

M varies parabolically: it is zero at $x = 0$ and $\frac{-wl^2}{8}$ at $x = \frac{l}{2}$.

Determination of internal forces (from $x = \frac{l}{2}$ to $x = l$)



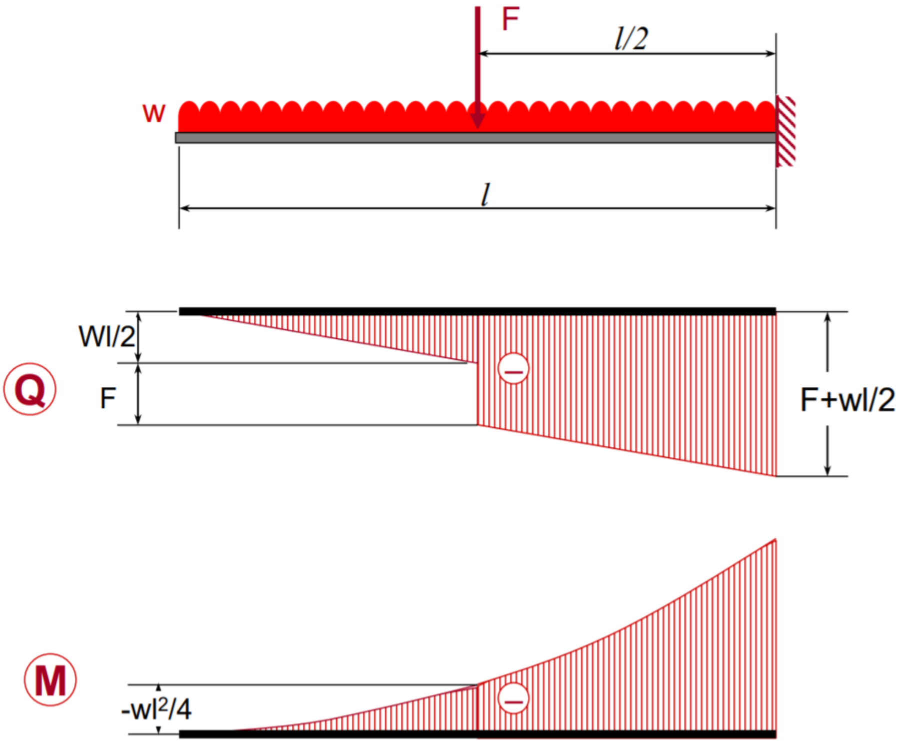
$$\sum F_y : Q + wx + F = 0 \tag{2.10}$$
$$Q = -wx - F \tag{2.11}$$

Q varies linearly between $x = \frac{l}{2}$ (where it is $-(\frac{wl}{2} + F)$) and $x = l$ (where it is $-(F + wl)$)

$$\sum M : M_x + wx\frac{x}{2} + F(x - \frac{l}{2}) = 0 \tag{2.12}$$
$$M_x = \frac{-wx^2}{2} - F(x - \frac{l}{2}) \tag{2.13}$$

At $x = \frac{l}{2}$, M distribution changes into a parabola with a steeper slope

Diagram of Internal Forces



2.2 Derivation of Bending Equations

2.2.1 Equilibrium – Beam Bending Equations

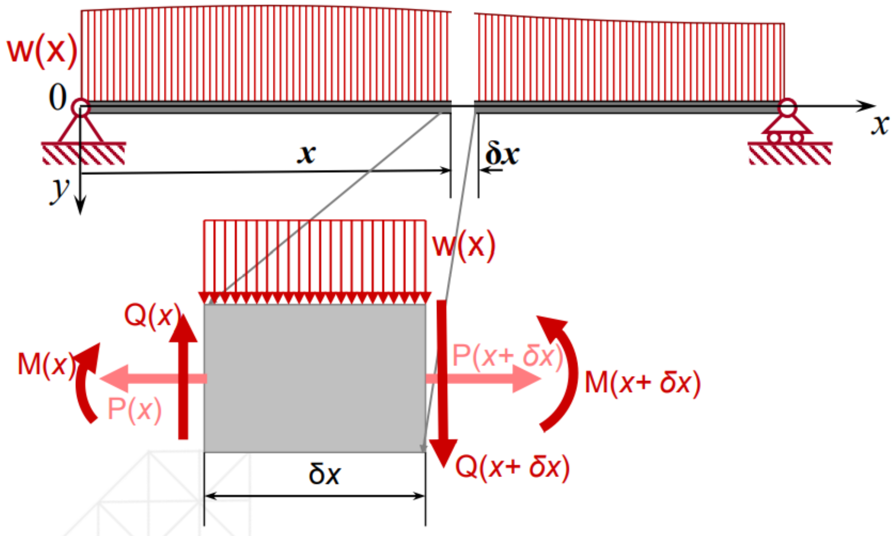


Figure 2.1: Horizontal beam with vertical distributed load. We focus a section of the beam with the very small length δx.

y-direction

$$Q(x+\delta x) - Q(x) + w(x)\delta x = 0 \tag{2.14}$$

x-direction

$$M(x+\delta x) - M(x) - Q(x)\delta x + w(x)\cdot\delta x\cdot\frac{\delta x}{2} = 0 \tag{2.15}$$

We assume that δx is so small that no matter the distribution of the load on the beam, $w(x)$ is constant along the investigated beam section.

Rearranging the terms in the y direction yields:

$$\frac{Q(x+\delta x) - Q(x)}{\delta x} = -w(x) \tag{2.16}$$

Taking δx to the smallest limit (0), the above equation can be written as:

$$\delta x = 0 \tag{2.17}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -w(x) \tag{2.18}$$

$$w(x) = -\frac{\mathrm{d}Q}{\mathrm{d}x} \tag{2.19}$$

The load w in the y direction is the derivative of the shear force Q. Integrating equation (**??**):

$$Q = -\int w\,\mathrm{d}x \tag{2.20}$$

We consider the bending moment at the distance δx (right-side for the example). Rearranging the terms in the x direction yields:

$$\frac{M(x+\delta x)-M(x)}{\delta x}=Q(x)-\frac{1}{2}w(x)\delta x \tag{2.21}$$

Taking δx to the smallest limit (0), the above equation can be written as:

$$\delta x = 0 \tag{2.22}$$

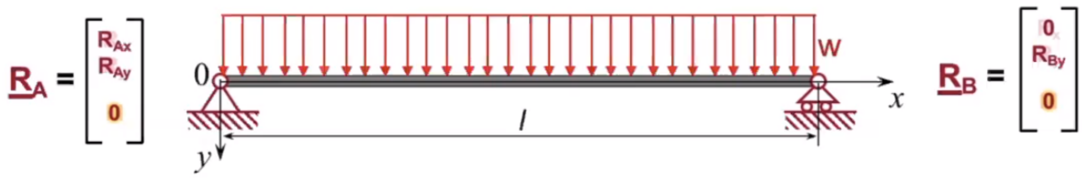
$$Q(x)=\frac{\mathrm{d}M}{\mathrm{d}x} \tag{2.23}$$

$$\tag{2.24}$$

The shear force Q is the derivative of the bending Moment M. Integrating equation (??):

$$M=\int Q\,\mathrm{d}x \tag{2.25}$$

Example:



$$Q=-\int w\,\mathrm{d}x \tag{2.26}$$

$$=-wx+Q_0 \tag{2.27}$$

$$\tag{2.28}$$

$$M=\int Q\,\mathrm{d}x \tag{2.29}$$

$$=\int(-wx+Q_0)\,\mathrm{d}x \tag{2.30}$$

$$=-\frac{1}{2}wx^2+Q_0\cdot x+M_0 \tag{2.31}$$

Applying the boundary conditions:

$$M(0)=0\rightarrow-\frac{1}{2}w\cdot 0^2+Q_0\cdot 0+M_0=0 \tag{2.32}$$

$$M_0=0 \tag{2.33}$$

$$\tag{2.34}$$

$$M(l)=0\rightarrow-\frac{1}{2}wl^2+Q_0\cdot l+0=0 \tag{2.35}$$

$$Q_0=\frac{1}{2}wl \tag{2.36}$$

Overall:

$$Q = -wx + \frac{1}{2}wl \tag{2.37}$$

$$M = -\frac{1}{2}wx^2 + \frac{1}{2}wlx \tag{2.38}$$

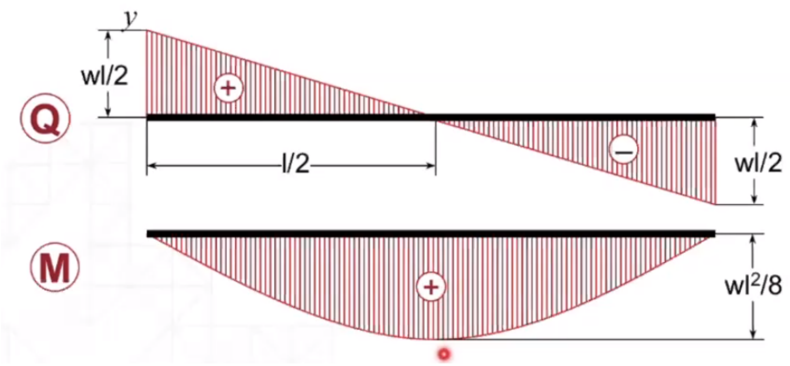


Figure 2.2: The shear force and bending moment varying along x

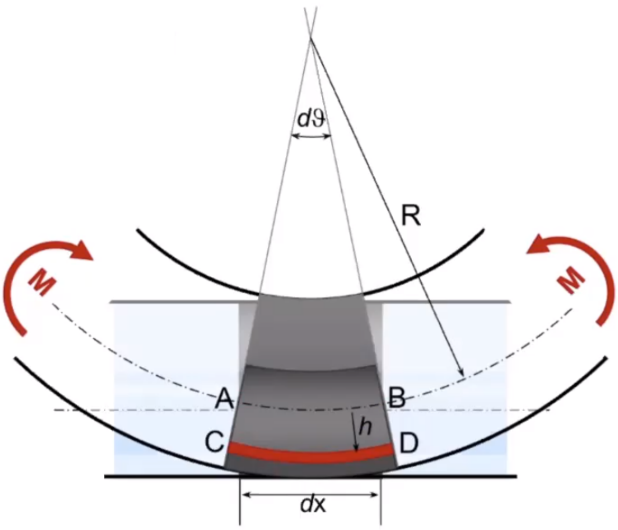
2.3 Differential Equations for Deflection

2.3.1 Theory of Pure Bending

- The beam is initially straight and unstressed;
- The beam material is perfectly homogeneous and isotropic;
- Plane cross-sections remain plane before and after bending;
- Every cross-section in the beam is symmetrical about the plane of bending;
- There is no resultant force perpendicular to any cross-section.
- The elastic limit is nowhere exceeded;
- Young's Modulus for the material is the same in tension and compression;

2.3.2 Compatibility - Strains in Pure Bending

Consider a portion of the length dx from the beam subject to uniform bending.



Lower fibres stretch and upper fibres shorten. Hence, since cross-sections remain plane, there must be a plane where the fibre elongation is zero. This is called **neutral plane** (it is a plane because section is symmetrical) and the intersection with the plane of bending is called **neutral axis**.

If we indicate with R the radius of curvature of the neutral axis, along the neutral plane:

$$\overline{AB} = \widehat{AB} \tag{2.39}$$

$$\widehat{AB} = \mathrm{d}x = R\,\mathrm{d}\theta \tag{2.40}$$

For a generic plane CD, distant h from N.A.:

$$\widehat{CD} = (R + h)\,\mathrm{d}\theta \tag{2.41}$$

The longitudinal strain of CD is given by:

$$\epsilon(h) = \frac{elongation}{initial\ length} = \frac{\widehat{CD} - \overline{CD}}{\overline{CD}} \tag{2.42}$$

$$\tag{2.43}$$

$$= \frac{(R+h)\cdot \mathrm{d}\theta - R\cdot \mathrm{d}\theta}{R\cdot \mathrm{d}\theta} = \frac{h}{R} \tag{2.44}$$

$$\tag{2.45}$$

$$\epsilon(h) = \frac{h}{R} \tag{2.46}$$

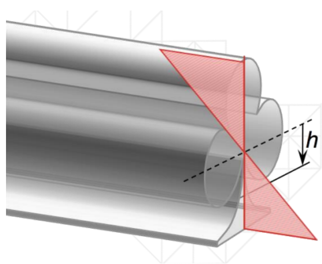


Figure 2.3: Strains are distributed linearly across the section

Stresses and strains can be associated with each other. For most materials, under small deformation:

$$\sigma = E \cdot \epsilon \tag{2.47}$$

Where:

- σ is the stress
- E is the Young Modulus
- ϵ is the strain

Therefore, from equations (**??**) and (**??**):

$$\sigma_x(h) = E\frac{h}{R} \tag{2.48}$$

2.3.3 Constitutive - Stress-Curvature Relation

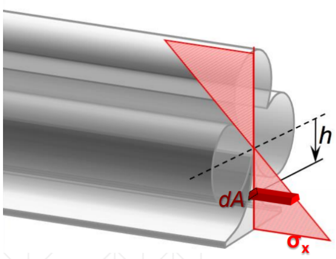
Stress is also distributed linearly across the section, being 0 at the neutral plane and maximum (in tension and compression) at the outer surfaces, where the distance from the neutral plane is maximum.

$$\sigma_{min} = E\frac{h_{min}}{R} \tag{2.49}$$

$$\sigma_{max} = E\frac{h_{max}}{R} \tag{2.50}$$

The minimum value is at the compression and the maximum value is at the tension. This just follows the convention; the tension is taken as positive and compression is taken as negative, in terms of the stress direction.

2.3.4 Equilibrium - Force-Stress Relation



Considering an elemental area dA, the force associated with bending stress is:

$$\mathrm{d}F_x = \sigma_x \cdot \mathrm{d}A \tag{2.51}$$

$$= E\frac{h}{R} \cdot \mathrm{d}A \tag{2.52}$$

For the force equilibrium of the entire section:

$$\int_A \mathrm{d}F_x = \int_A E\frac{h}{R} \cdot \mathrm{d}A = 0 \tag{2.53}$$

$$\tag{2.54}$$

E and R are constants, so they don't affect the integral and can be taken out. Hence the first moment of area is:

$$\int_A h \cdot \mathrm{d}A = 0 \tag{2.55}$$

The first moment of area of a section is zero if it is calculated about the centroid. **The neutral axis corresponds to the centroid of the section.**

2.3.5 Equilibrium - Bending-Stress Relation

Considering an elemental area $\mathrm{d}A$, the internal moment produced by the bending stress is:

$$\mathrm{d}M = \sigma_x \cdot h \cdot \mathrm{d}A \tag{2.56}$$

$$= \mathrm{d}F_x \cdot h \tag{2.57}$$

$$= E\frac{h}{R} \cdot h \cdot \mathrm{d}A \tag{2.58}$$

Moment of the entire section:

$$M = \int_A \mathrm{d}M = \int_A E\frac{h^2}{R} \cdot \mathrm{d}A = \frac{E}{R}\int_A h^2 \cdot \mathrm{d}A \tag{2.59}$$

The second moment of area is:

$$\int_A h^2 \cdot \mathrm{d}A = I \tag{2.60}$$

The second moment of area represents how easy or difficult it is to bend a beam, depending on the shape of the cross section. The overall relation is:

$$M = \frac{EI}{R} \tag{2.61}$$

This defines how the cross section, elastic modulus, curvature, and bending moment are related to each other.

2.3.6 Solid Mechanics Equations

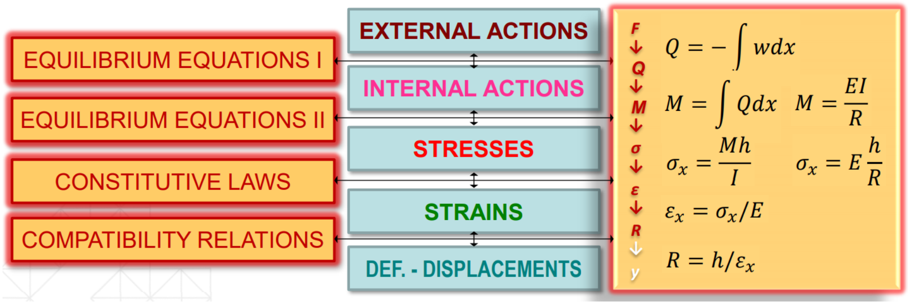


Figure 2.4: The relationships between Force, Shear Force, Bending Moment, Stress, Strain, Curvature

2.3.7 Geometric - Slope-Deflection Relation

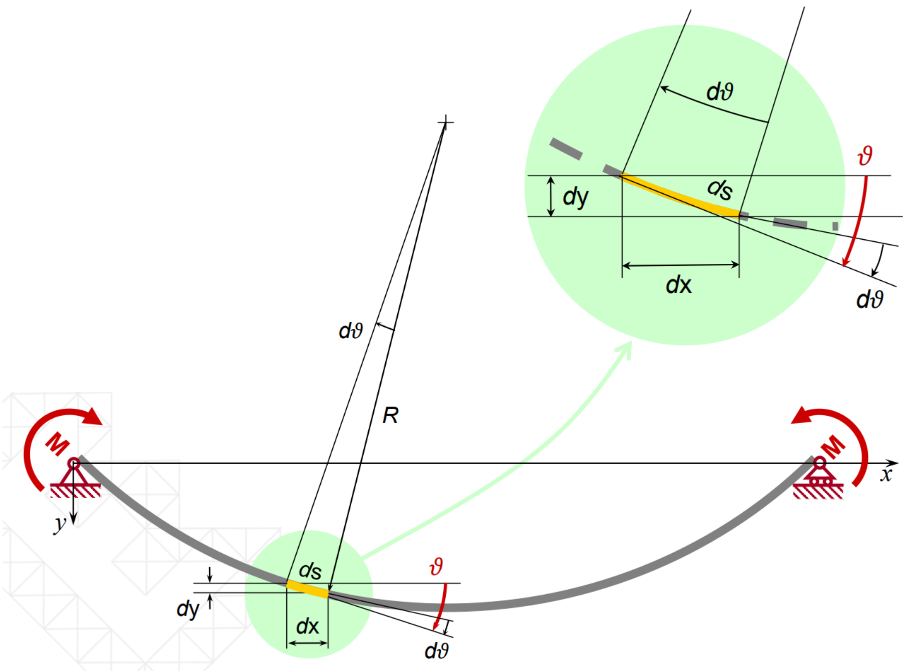


Figure 2.5: The geometry of the beam under bending. The direction of the angle θ is based on the "right-hand rule".

For infinitesimal deformations:

$$\mathrm{d}x \approx \mathrm{d}s = -R \cdot \mathrm{d}\theta \tag{2.62}$$

$$\frac{1}{R} = -\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{2.63}$$

The negative sign comes from the situation that the x axis direction is towards right while the angle (θ) direction is to the left. Hence, they are opposite. Since:

$$M = \frac{EI}{R} \tag{2.64}$$

$$\rightarrow M = -EI\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{2.65}$$

Assuming anlge θ is small:

$$\theta \approx \tan(\theta) = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{2.66}$$

- The load w in the y direction is the derivative of the shear force Q.
- The shear force Q is the derivative of the bending moment M.
- The bending moment M is proportional to the derivative of the slope θ.
- The slope θ is the derivative of the deflection y : $\theta = \frac{\mathrm{d}y}{\mathrm{d}x}$

2.3.8 Stresses Due to Shear Force

The shear force also produces stresses into the section (shear stresses). However:

- They are much lower than bending stresses
- They are zero at surfaces, where bending stresses are maximum
- Their effect on the deformation is negligible compared to bending stresses

In general, neglecting the shear stresses due to the shear force is a good approximation for both the calculation of failure and deflections.

2.3.9 Summary of Beam Bending Equations

Deflection:

$$y \tag{2.67}$$

Slope:

$$\theta = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{2.68}$$

Bending Moment:

$$M = -EI\frac{\mathrm{d}\theta}{\mathrm{d}x} = -EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \tag{2.69}$$

Shear Force:

$$Q = \frac{\mathrm{d}M}{\mathrm{d}x} = -EI\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} \tag{2.70}$$

Load Distribution:

$$w = -\frac{\mathrm{d}Q}{\mathrm{d}x} = EI\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} \tag{2.71}$$

The equations above give a scenario of being given deflection y, and eventually finding load distribution w. However, the inverse can also occur where load distribution w is given, and the y can be found through integration:

Load Distribution

$$w \tag{2.72}$$

Shear Force:

$$Q = -\int w \cdot \mathrm{d}x \tag{2.73}$$

Bending Moment:

$$M = \int Q \cdot \mathrm{d}x = -\int\int w \cdot \mathrm{d}x\,\mathrm{d}x \tag{2.74}$$

Slope:

$$\theta = -\frac{1}{EI}\int M \cdot \mathrm{d}x = \frac{1}{EI}\int\int\int w \cdot \mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x \tag{2.75}$$

Deflection:

$$y = \int \theta \cdot \mathrm{d}x = \frac{1}{EI}\int\int\int\int w \cdot \mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x \tag{2.76}$$

2.3.10 Direct Integration Method

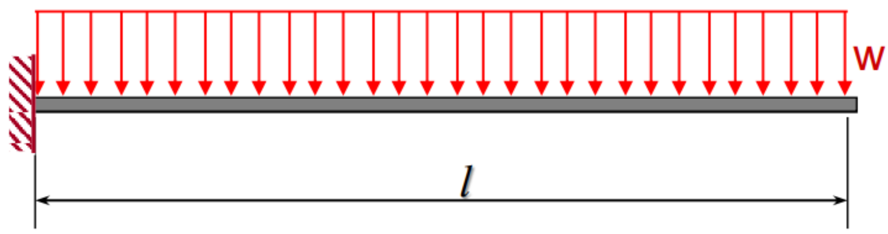
STEP 1: Determination of Support Reactions

Apply to the body the force and moment equilibrium equations to find support reactions (it is possible only if the system is statically determinate).

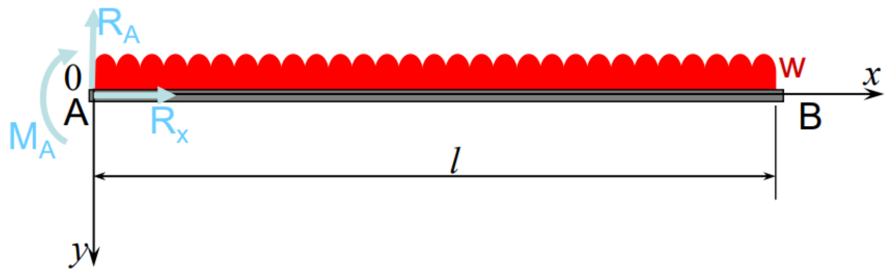
STEP 2: Determination of Deflection

1. Write bending moment expression
2. Use double integration on bending moment expression. This would result in 2 constants of integration.
3. Use boundary conditions to determinate constant of integration.

Example: Uniformly Distributed Load on a Cantilever Beam



Determination of Support Reactions:



$$\vec{R_A} = \begin{bmatrix} R_x \\ R_y \\ M_A \end{bmatrix}$$

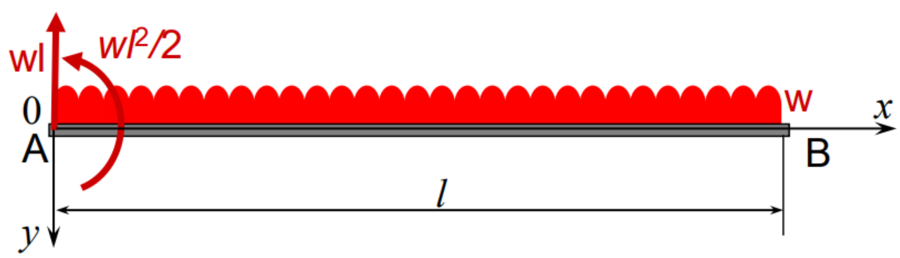
$$\sum F_x : R_x = 0 \tag{2.77}$$

$$\sum F_y : R_y = wl \tag{2.78}$$

$$\sum M : M_A + wl\frac{l}{2} = 0 \tag{2.79}$$

$$M_A = -\frac{wl^2}{2} \tag{2.80}$$

Determination of Deflection:



Direct Integration:

$$Q = -\int w \cdot \mathrm{d}x = -wx + Q_0 \tag{2.81}$$

$$\tag{2.82}$$

$$M = \int Q \cdot \mathrm{d}x \tag{2.83}$$

$$= \int (-wx + Q_0)\,\mathrm{d}x \tag{2.84}$$

$$= -\frac{1}{2}wx^2 + Q_0 x + M_0 \tag{2.85}$$

Boundary Conditions:

$$Q(0) = R_y = wl \tag{2.86}$$

$$\rightarrow Q_0 = wl \tag{2.87}$$

$$M(0) = M_A = -\frac{1}{2}wl^2 \tag{2.88}$$

$$\rightarrow M_0 = -\frac{1}{2}wl^2 \tag{2.89}$$

Therefore:

$$Q = -w(x + l) \tag{2.90}$$

$$M = -\frac{1}{2}wx^2 + wlx - \frac{1}{2}wl^2 \tag{2.91}$$

Relating Deflection to the Bending Moment:

$$M = -\frac{1}{2}wx^2 + wlx - \frac{1}{2}wl^2 \tag{2.92}$$

$$\theta = -\frac{1}{EI}\int M \cdot \mathrm{d}x = -\frac{1}{EI}\left(-\frac{1}{6}wx^3 + \frac{1}{2}wlx^2 - \frac{1}{2}wl^2x\right) + \theta_0 \tag{2.93}$$

$$y = \int \theta \cdot \mathrm{d}x = -\frac{1}{EI}\left(-\frac{1}{24}wx^4 + \frac{1}{6}wlx^3 - \frac{1}{4}wl^2x^2\right) + \theta_0 x + y_0 \tag{2.94}$$

$$\tag{2.95}$$

Boundary Conditions:

$$\theta(0) = 0 \rightarrow \theta_0 = 0 \tag{2.96}$$

$$y(0) = 0 \rightarrow y_0 = 0 \tag{2.97}$$

Therefore:

$$y = \frac{1}{EI}\left(\frac{1}{24}wx^4 - \frac{1}{6}wlx^3 + \frac{1}{4}wl^2x^2\right) \tag{2.98}$$

$$y_{max} = y(l) = \frac{wl^4}{8EI} \tag{2.99}$$

2.3.11 Statically Determinate Beams

23/10/2020

Statically determinate (or isostatic) beams are characterised by the fact that all unknown supports reactions can be determined from the forces and equilibrium equations: the number of supports reactions is equal to the degree of freedom.

A beam in 2D plane has 3 DOF. If only lateral loads and moments act (horizontal actions and reactions = 0), the beam has only 2 degree of freedom.

Beams subjected to lateral loads are statically determinate if they have **only 2 unknown support reactions:** the two equilibrium equation for lateral forces and moments are sufficient to solve the system. If the supports produce **three or more unknown reactions**, the beams are **statically indeterminate**.

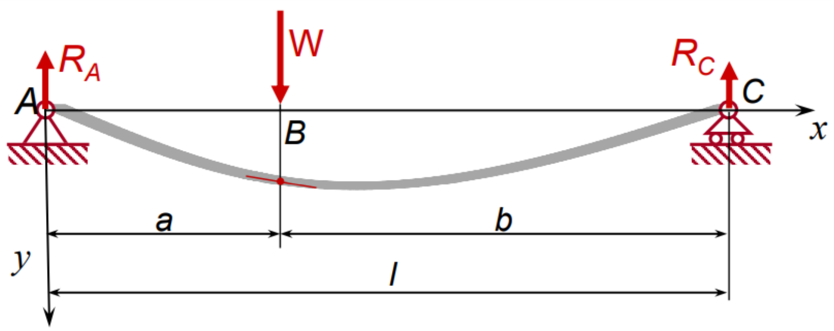
2.3.12 Double Integration Method

In more complex cases, such as discontinuous loads, the multiple integration method can still be used:

1. Write one bending moment expression for each part of the beam (e.g. AB and BC).
2. Use double integration on each bending moment expression. This would result in 2 constants of integration for each part of the beam (in this case, 4 constants).
3. Use boundary conditions to determine constant of integration.

The boundary conditions at the supports are generally not sufficient to obtain all the constants of integration (in our case we need 4 and they are 2) $\therefore$ We need other boundary conditions.

Example: Concentrated Load on a Simply Supported Beam



We know that, tough the load is discontinuous; the slope and deflection of the beam must be continuous from one section to the next. $\therefore$ at C:

$$\theta_{AC} = \theta_{CB} \tag{2.100}$$
$$y_{AC} = y_{CB} \tag{2.101}$$

The 1st equation refers to continuous slope, while the 2nd equation refers to continuous deflection. Continuity of slope and deflection of the beam from adjacent sections provides the supplementary equations that allow us to determine all constant of integration. By solving the system of equations, the following can be obtained:

$$y_a = -\frac{x}{6EIl}[R_A l(x^2 - l^2) + W(l-a)^3] \tag{2.102}$$
$$y_b = -\frac{1}{6EIl}[R_A lx(x^2 - l^2) - Wl(x-a)^3 + Wx(l-a)^3] \tag{2.103}$$

Memorising or fully understanding these equations are not that important; it is more crucial to understand the whole process of double integration method.

If a system is composed of n parts:

- Determine n equations of the bending moment
- Solve $2n$ integrals
- Solve a system of $2n$ equations in $2n$ unknowns

2.4 Macaulay's Method

2.4.1 Step Function

We define the step function:

$$\langle x-a\rangle^n = \begin{cases} (x-a)^n & x-a \geq 0 \\ 0 & x-a < 0 \end{cases}$$

The step function can easily describe the common forms of bending moment distribution. The expression of the bending moment can be obtained, moving from the left extreme of the beam to the right, adding every time we find a new load the quantities:

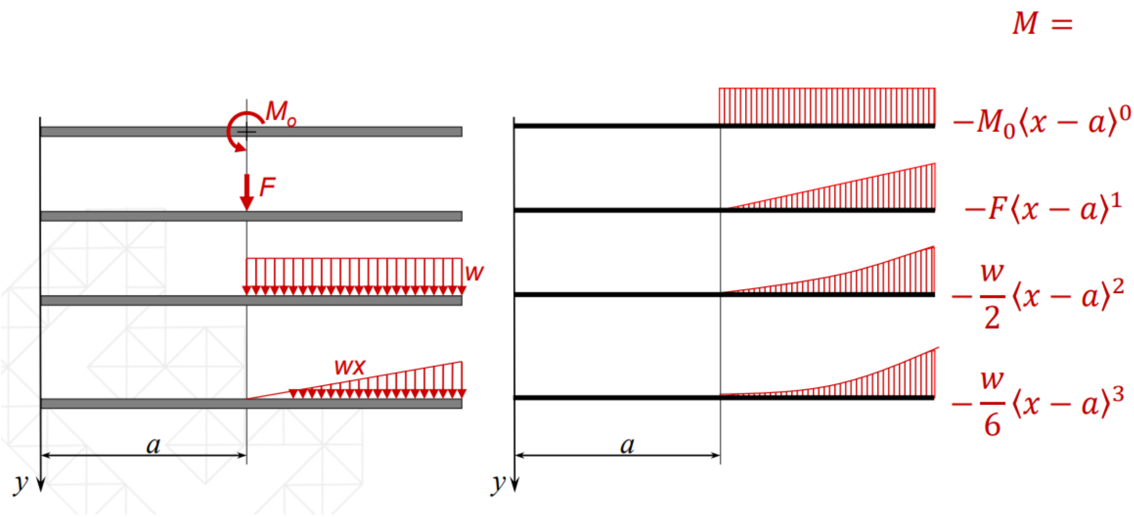


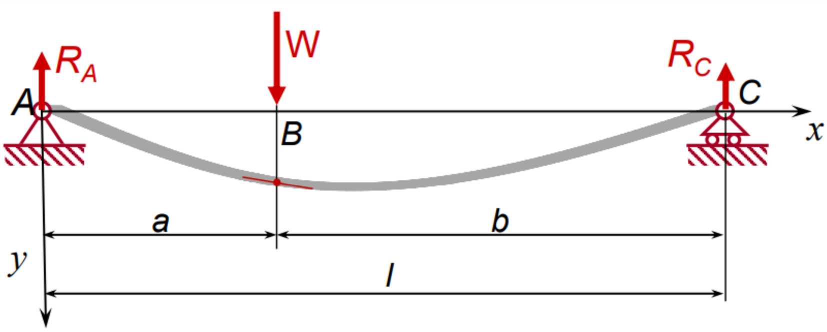
Figure 2.6: Different loads/moments applied to beams, their respective moment distributions and step functions

The crucial property of the step function is that it can be integrated just like an ordinary polynomial expression:

$$\int \langle x-a\rangle^n \,\mathrm{d}x = \frac{\langle x-a\rangle^{n+1}}{n+1} + C \tag{2.104}$$

Using the step function, the expressions of the different parts of a beam (under discontinuous load) can be put together into one single expression.

Example: Simply Supported Beam with a Concentrated Load



The bending distribution in AB and BC respectivelty is:

$$M_a = R_A x \tag{2.105}$$
$$M_b = R_A x - W(x-a) \tag{2.106}$$

Using the step function, the bending distribution in the entire beam is:

$$M = R_A x - W\langle x - a\rangle \tag{2.107}$$

$$\theta = -\frac{1}{EI}\int M\,\mathrm{d}x = -\frac{1}{EI}\left[R_A + \int x\,\mathrm{d}x - W\int \langle x-a\rangle\,\mathrm{d}x\right] \tag{2.108}$$

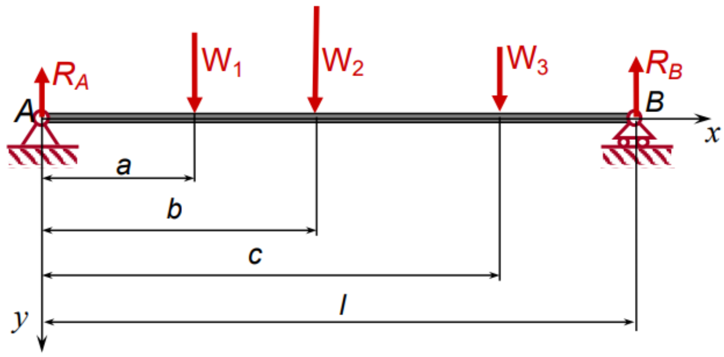
$$= -\frac{1}{EI}\left(\frac{R_A x^2}{2} - \frac{W\langle x-a\rangle^2}{2}\right) + \theta_0 \tag{2.109}$$

$$y = \int \theta\,\mathrm{d}x = -\frac{1}{EI}\left[\frac{R_A x^3}{6} - \frac{W\langle x-a\rangle^3}{6}\right] + \theta_0 x + y_0 \tag{2.110}$$

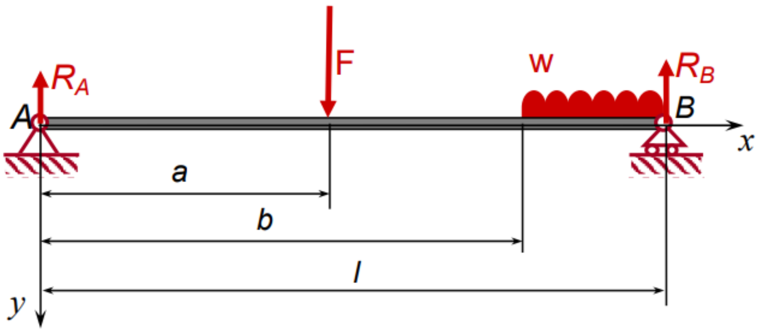
There are only 2 integration constants $\therefore$ 2 boundary conditions are sufficient to solve the equation.

2.4.2 Examples

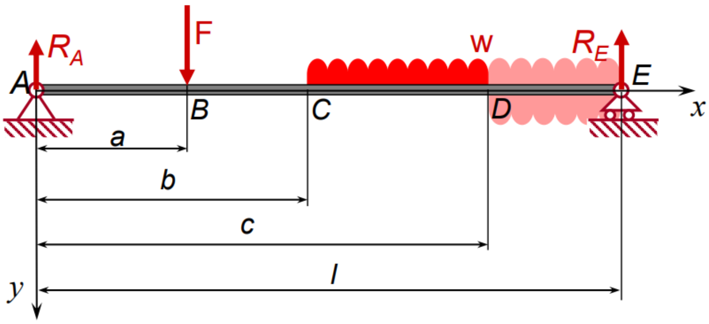
The following figures and equations are just examples of generic loads on simply supported beams, and what their step functions are. To find the deflection, the functions need to be integrated twice, and the boundary conditions must be substituted to find the integration constants.



$$M = R_A x - W_1\langle x-a\rangle - W_2\langle x-b\rangle - W_3\langle x-c\rangle \tag{2.111}$$



$$M = R_A x - F\langle x-a\rangle - \frac{w}{2}\langle x-b\rangle^2 \tag{2.112}$$



In this example, the distributed load is not acting all the way to the end point E $\therefore$ A counteracting distributed load is added to cancel the contribution of the distributed load between DE.

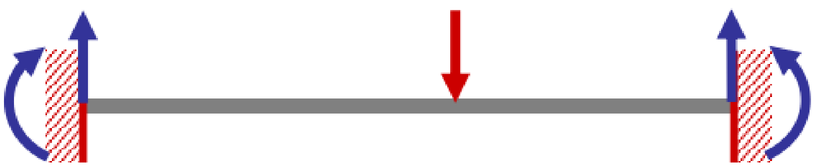
$$M = R_A x - F\langle x-a\rangle - \frac{w}{2}\langle x-b\rangle^2 + \frac{w}{2}\langle x-c\rangle^2 \tag{2.113}$$

2.5 Statically Indeterminate Beams

30/10/2020

So far, we have analysed statically-determinate beams. These are characterised by only **two unknown support reactions**, that can be determined from the force and moment equilibrium equations.

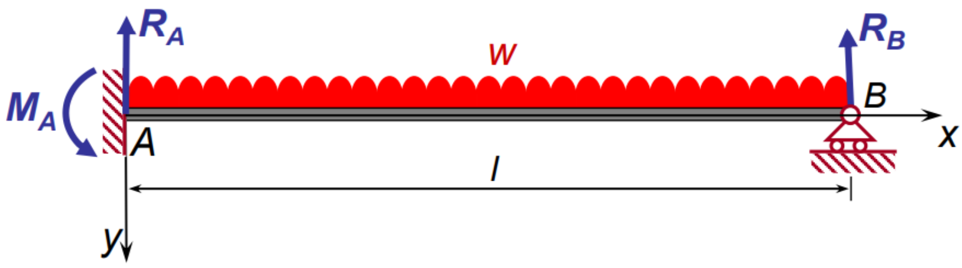
If the supports produce **three or more unknown reactions**, the beam is **statically indeterminate**.



The analysis of statically indeterminate structures requires the calculation of displacements to find a sufficient number of equations to solve for all unknown reactions.

The double integration method or the Macaulay's method can be used to find the support reactions and the solution in statically indeterminate problems.

Example: Propped Cantilever with Uniformly Distributed Load (Double Integration Method)



The equilibrium equations:

$$R_A - wl + R_B = 0 \tag{2.114}$$

$$R_A l - M_A - \frac{1}{2}wl^2 = 0 \tag{2.115}$$

Moment at the point distance x away from the left end:

$$M(x) = -M_A - \frac{1}{2}wx^2 + R_A x \tag{2.116}$$

Integrating the moment equation:

$$\theta(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{EI}\int\left(-M_A - \frac{1}{2}wx^2 + R_A x\right)\mathrm{d}x \tag{2.117}$$

$$= -\frac{1}{EI}\left(-M_A x - \frac{1}{6}wx^3 + \frac{1}{2}R_A x^2\right) + \theta_0 \tag{2.118}$$

Boundary Condition: $\theta(0) = 0 \longrightarrow \theta_0 = 0$

$$\therefore \theta(x) = -\frac{1}{EI}\left(-M_A x - \frac{1}{6}wx^3 + \frac{1}{2}R_A x^2\right) \tag{2.119}$$

Integrating further:

$$y(x) = -\frac{1}{EI}\int\left(-M_A x - \frac{1}{6}wx^3 + \frac{1}{2}R_A x^2\right)\mathrm{d}x \tag{2.120}$$

$$= -\frac{1}{EI}\left(-\frac{1}{2}M_A x^2 - \frac{1}{24}wx^4 + \frac{1}{6}R_A x^3\right) + y_0 \tag{2.121}$$

Boundary Condition: $y(0) = 0 \longrightarrow y_0 = 0$

$$\therefore y(x) = -\frac{1}{EI}\left(-\frac{1}{2}M_A x^2 - \frac{1}{24}wx^4 + \frac{1}{6}R_A x^3\right) \tag{2.122}$$

Applying another boundary condition $y(l) = 0$ (to consider the other end of the beam):

$$y(l) = -\frac{1}{EI}\left(-\frac{1}{2}M_A l^2 - \frac{1}{24}wl^4 + \frac{1}{6}R_A l^3\right) = 0 \tag{2.123}$$

$$\longrightarrow \frac{1}{2}M_A l^2 + \frac{1}{24}wl^4 - \frac{1}{6}R_A l^3 = 0 \tag{2.124}$$

$$\longrightarrow 12M_A + wl^2 - 4R_A l = 0 \tag{2.125}$$

The R_A, R_B, and M_A are still **unknowns** at this point. Combining the equilibrium conditions with the double integration (equation (**??**)):

$$R_A - wl + R_B = 0 \tag{2.126}$$

$$R_A l - M_A - \frac{1}{2}wl^2 = 0 \tag{2.127}$$

$$4R_A l - 12M_A - wl^2 = 0 \tag{2.128}$$

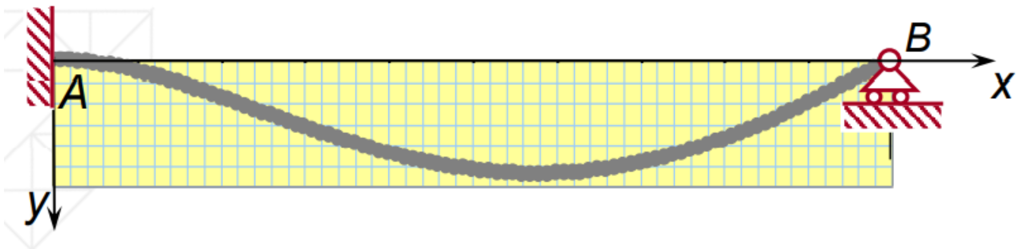
Solving for the above system of equations yields:

$$R_A = \frac{5}{8}wl \tag{2.129}$$

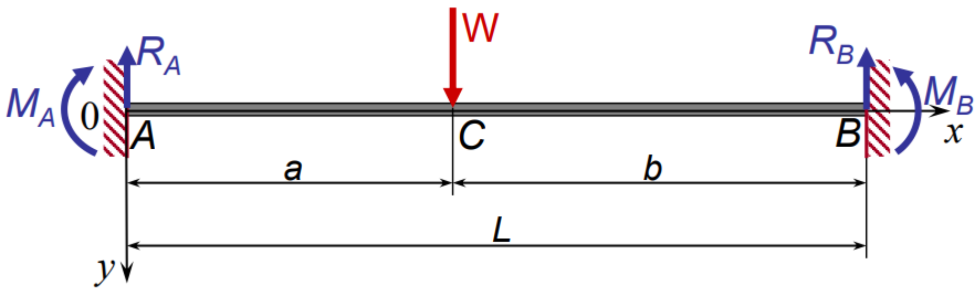
$$R_B = \frac{3}{8}wl \tag{2.130}$$

$$M_A = \frac{wl^2}{8} \tag{2.131}$$

Using these values in all previous equations, all quantities can be determined.



Example: Fixed Beam with Concentrated Load (Macaulay's Method)



The equilibrium conditions:

$$R_A - W + R_B = 0 \tag{2.132}$$

$$M_A - Wb + R_A L - M_B = 0 \tag{2.133}$$

Moment at the point distance x away from the left end:

$$M(x) = M_A + R_A x - W\langle x - a\rangle \tag{2.134}$$

Integrating the moment equation:

$$\theta(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{EI}\int \big(M_A + R_A x - W\langle x-a\rangle\big)\,\mathrm{d}x \tag{2.135}$$

$$= -\frac{1}{EI}\left(M_A x + \frac{1}{2}R_A x^2 - \frac{1}{2}W\langle x-a\rangle^2\right) + \theta_0 \tag{2.136}$$

Boundary Condition: $\theta(0) = 0 \longrightarrow \theta_0 = 0$

$$\therefore \theta(x) = -\frac{1}{EI}\left(M_A x + \frac{1}{2}R_A x^2 - \frac{1}{2}W\langle x-a\rangle^2\right) \tag{2.137}$$

Integrating further:

$$y(x) = -\frac{1}{EI}\int \left(M_A x + \frac{1}{2}R_A x^2 - \frac{1}{2}W\langle x-a\rangle^2\right)\mathrm{d}x \tag{2.138}$$

$$= -\frac{1}{EI}\left(\frac{1}{2}M_A x^2 + \frac{1}{6}R_A x^3 - \frac{1}{6}W\langle x-a\rangle^3\right) + y_0 \tag{2.139}$$

Boundary Condition: $y(0) = 0 \longrightarrow y_0 = 0$

$$\therefore y(x) = -\frac{1}{EI}\left(\frac{1}{2}M_A x^2 + \frac{1}{6}R_A x^3 - \frac{1}{6}W\langle x-a\rangle^3\right) \tag{2.140}$$

Applying other boundary conditions $\theta(L) = 0$, $y(L) = 0$ (to consider the other end of the beam):

$$\theta(L) = -\frac{1}{EI}\left(M_A L + \frac{1}{2}R_A L^2 - \frac{1}{2}W(L-a)^2\right) = 0 \tag{2.141}$$

$$y(L) = -\frac{1}{EI}\left(\frac{1}{2}M_A L^2 + \frac{1}{6}R_A L^3 - \frac{1}{6}W(L-a)^3\right) = 0 \tag{2.142}$$

$$\text{Overall:} \begin{cases} R_A - W + R_B = 0 \\ M_A - Wb + R_A L - M_B = 0 \\ 2M_A L + R_A L^2 - W(L-a)^2 = 2M_A L + R_A L^2 - Wb^2 = 0 \\ 3M_A L^2 + R_A L^3 - W(L-a)^3 = 3M_A L^2 + R_A L^3 - Wb^3 = 0 \end{cases} \tag{2.143}$$

The R_A, R_B, M_A, and M_B are the 4 **unknowns** with 4 equations. Hence, they can be solved:

$$R_A = \frac{Wb^2}{L^3}(3a+b) \tag{2.144}$$

$$M_A = -\frac{Wab^2}{L^2} \tag{2.145}$$

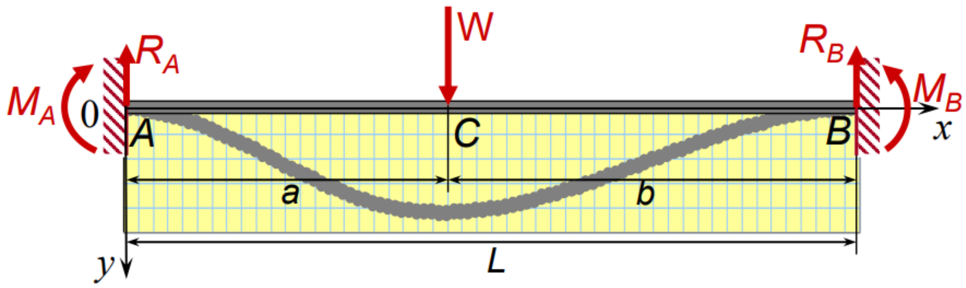
$$R_B = \frac{Wa^2}{L^3}(3b+a) \tag{2.146}$$

$$M_B = -\frac{Wa^2 b}{L^2} \tag{2.147}$$

The equation of deflection:

$$y = -\frac{W}{2EI}\left[\frac{b^2(3a+b)x^3}{3L^3} - \frac{ab^2x^2}{L^2} - \frac{\langle x-a\rangle^3}{3}\right] \tag{2.148}$$

$$y(a) = \frac{Wa^3b^3}{3EIL^3} \tag{2.149}$$



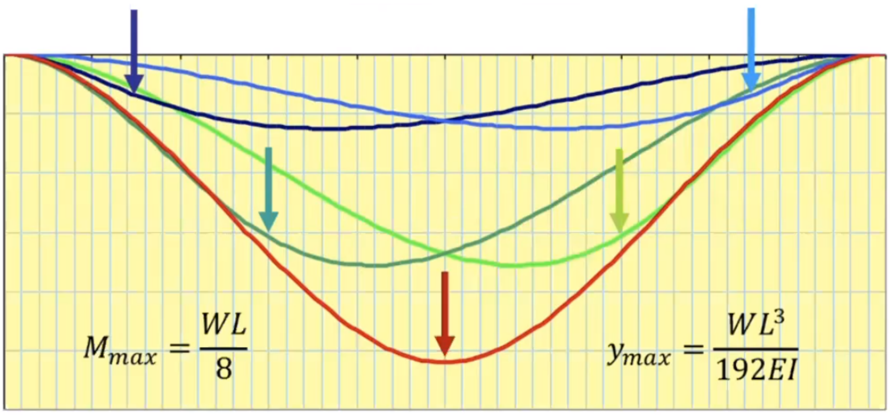


Figure 2.7: Depending on the loading, the profile of the deflection will be different. Maximum diflection occurs when $a = b = \frac{L}{2}$

Statically indeterminate structures produce **lower displacement and stress levels**, resulting in savings in material. **Failure** of a member in a statically determinate structure **leads to collapse**. In case of failure of a member, statically indeterminate structures can find **alternative load paths**, at least temporarily.

Chapter 3

Plastic collapse

20/11/2020

What you already know on bending is sufficient to enable the basic engineering design of static beam structures. Basic engineering design is primarily concerned with:

- Controlling elastic deformations
- Maintaining structures within their elastic range

However, for safety reasons, it is essental for engineers to be able to predict how a structure would fail if unexpected loading condiitons occur.

3.1 Introduction

3.1.1 Elastic and Plastic Regimes

Stressing within the **elastic regime**:

- Material returns to original state upon removal of external actions
- Deformation depends solely upon stress and not upon load history

Exceeding the elastic regime, **plastic regime** is reached:

- Permanent distortions take place in the material
- Deformation depends upon stress and load history

Equations of solid mechanics apply to both cases.

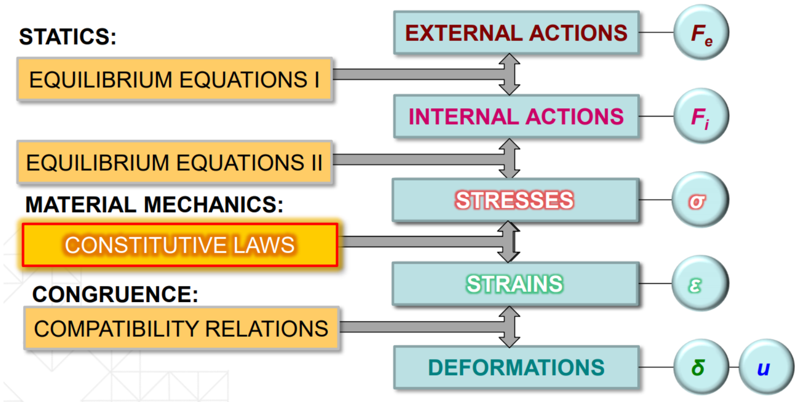


Figure 3.1: Solid Mechanics Equations: The relationship between actions, stresses, strains and deformations

3.1.2 Stress-Strain Plastic Relationship

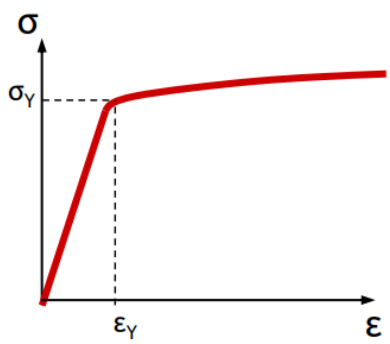


Figure 3.2: Typical elasto-plastic material

Typical elasto-plastic material:

- Region 1 - linear elastic behaviour up to yield stress σ_y
- Region 2 – non linear development with strain-hardening

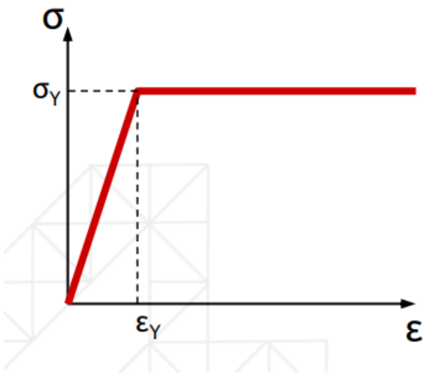


Figure 3.3: Perfectly elasto-plastic material

Perfectly elasto-plastic material:

- Region 1 - linear elastic behaviour up to yield stress σ_y
- Region 2 – strain increases at constant stress

Structural steels are elastoplastic materials and can be modelled as perfectly elastoplastic (neglecting the strain-hardening is conservative for safety).

3.2 Plastic Theory of Collapse

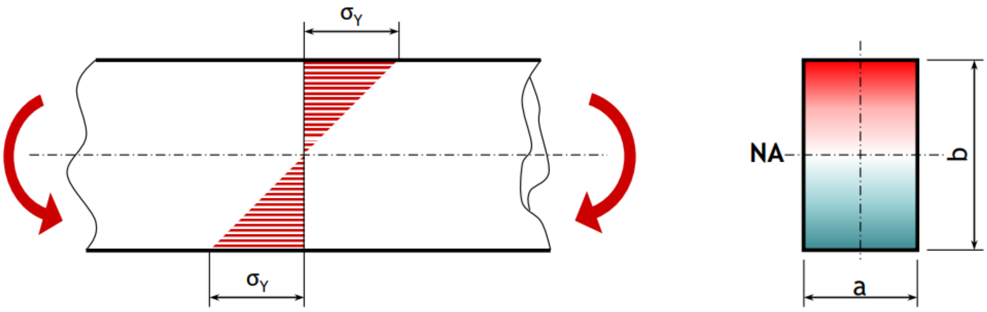
3.2.1 Bending and Plastic Collapse

Bending moment is by far the most relevant of the internal forces, since it produces the largest levels of deformations and stress into the beam. Therefore, **plastic collapse** of beam structures is commonly associated with **plastic bending**. Assumptions:

- Material is perfectly elasto-plastic. (in the plastic region stress will be constant)
- Yield stress is the same in tension and compression
- Transverse cross-sections remain plane (strain is proportional to the distance from the NA)
- When a cross-section is fully plastic (plastic hinge), its resisting moment remains constant until collapse of the whole structure
- Loads increase monotonically

3.2.2 Elastic Bending Moment

Consider a beam of rectangular cross-section, subjected to pure bending.



Maximum stress reaches the elastic limit σ_y when the bending moment is equal to:

$$M = \frac{EI}{R} \tag{3.1}$$

$$\sigma = \frac{Eb}{2R} \tag{3.2}$$

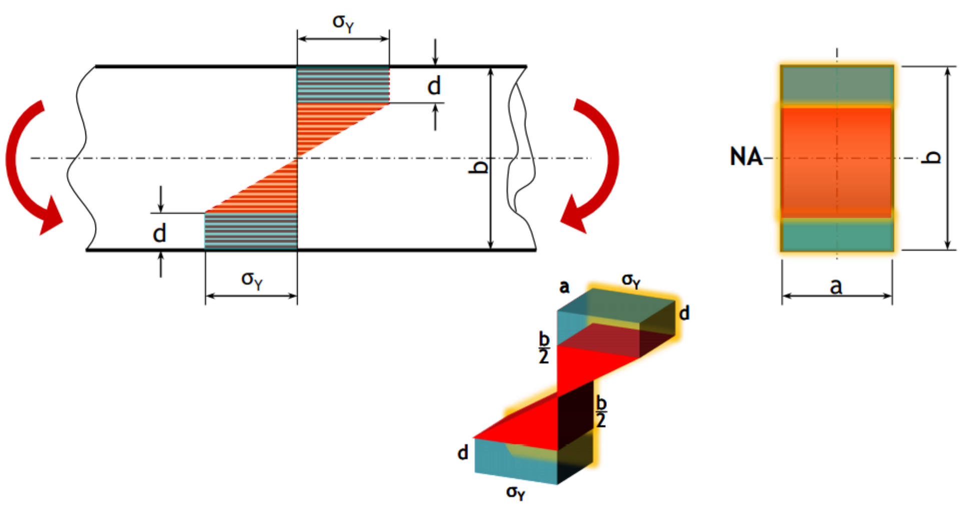
$$I = \frac{ab^3}{12} \tag{3.3}$$

$$\therefore \text{ The yield bending moment : } M_Y = \sigma_Y \frac{ab^2}{6} \tag{3.4}$$

At the elastic moment, all fibres are still in the elastic condition. If moment increases further, external fibres exceed the elastic limit and yield: deformation increases but stress keeps constant and equal to σ_y. With increasing bending moment, the plastic region penetrates deeper toward the Neutral Axis (NA).

3.2.3 Elasto-Plastic Bending Moment

Consider a beam of rectangular cross-section, subjected to pure bending.



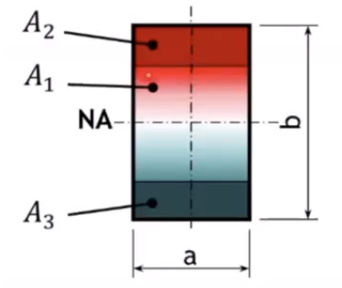
The moment is given by:

$$M = \frac{\sigma I}{h} \tag{3.5}$$

$$I = ? \tag{3.6}$$

$$\therefore \text{Elasto-plastic bending moment} : M = \frac{\sigma_Y a b^2}{6}\left[1 + 2\frac{d}{b}\left(1 - \frac{d}{b}\right)\right] \tag{3.7}$$

The derivation of the bending moment:



The cross-section is divided into 3 parts:

1. Elastic part in the middle (A_1)
2. Plastic part at the top (A_2)
3. Plastic part at the bottom (A_3)

$$M = \int_A \mathrm{d}M = \int_A h \cdot \sigma \, \mathrm{d}A \tag{3.8}$$

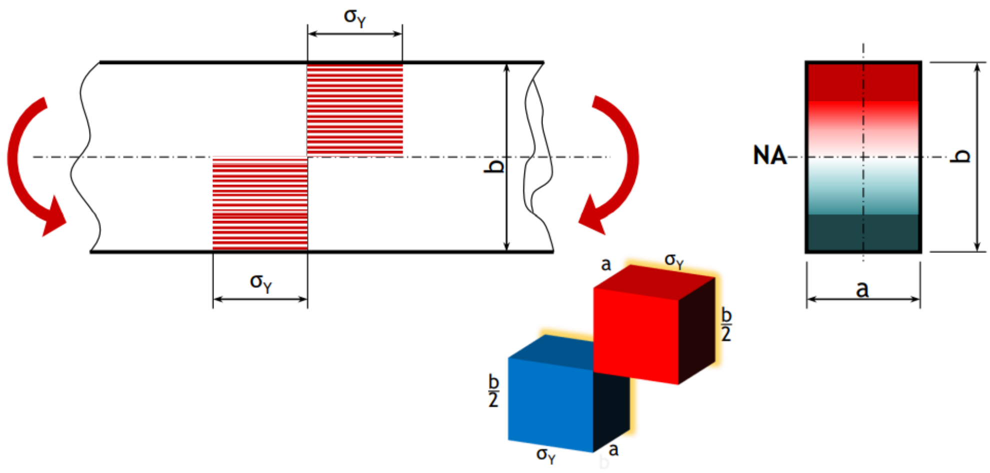
$$= \int_{A_1} \frac{\sigma_Y h}{b/2 - d} h \, \mathrm{d}A + \int_{A_2} \sigma_Y h \, \mathrm{d}A + \int_{A_3} \sigma_Y h \, \mathrm{d}A \tag{3.9}$$

$$= \int_{-b/2+d}^{b/2-d} \frac{\sigma_Y a}{b/2 - d} h^2 \, \mathrm{d}h + \int_{b/2-d}^{b/2} \sigma_Y a h \, \mathrm{d}h + \int_{-b/2}^{-b/2+d} \sigma_Y a h \, \mathrm{d}h \tag{3.10}$$

$$= \frac{\sigma_Y a b^2}{6}\left[1 + 2\frac{d}{b}\left(1 - \frac{d}{b}\right)\right] \tag{3.11}$$

3.2.4 Plastic Bending Moment

Consider a beam of rectangular cross-section, subjected to pure bending.



If bending moment keeps increasing, the plastic region propagate to the NA. In this case the moment is equal to:

$$\text{Plastic bending moment} : M_P = \sigma_Y \frac{ab^2}{4} \tag{3.12}$$

In practice, the plastic moment is given by the product of the yield stress by (the 1st moment of area (of the cross section) above the plastic NA + the 1st moment of area below the plastics NA).

3.2.5 Shape Factor

The yielding and plastic bending moments are different. In this case (rectangular cross-section) they are:

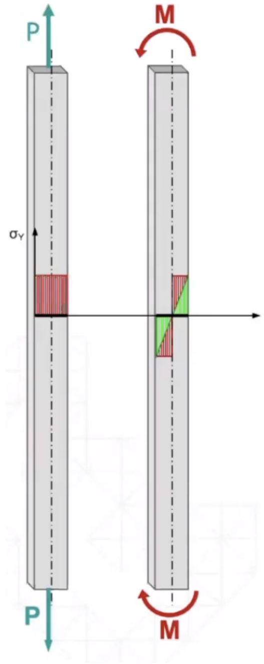
$$M_Y = \sigma_Y \frac{ab^2}{6} \qquad M_P = \sigma_Y \frac{ab^2}{4} \tag{3.13}$$

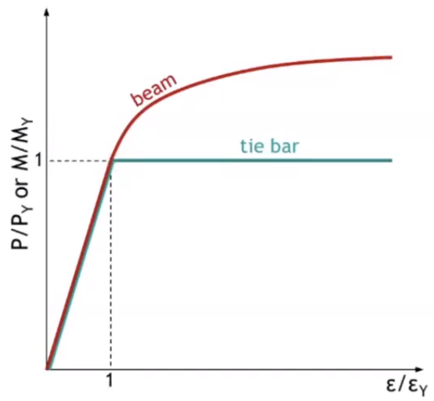
The ratio $f = \frac{M_P}{M_Y}$ between the yielding bending moment and the plastic bending moment is solely a function of the shape of the cross-section, and it is called **shape factor**. In this case (rectangular cross-section) it is:

$$f = \frac{M_P}{M_Y} = \frac{\sigma_Y ab^2}{4} \frac{6}{\sigma_Y ab^2} = 1.5 \tag{3.14}$$

The shape factor is an indicator of the reserve strength that the beam can offer after yielding first begins.

3.2.6 Plastic Collapse Under Bending



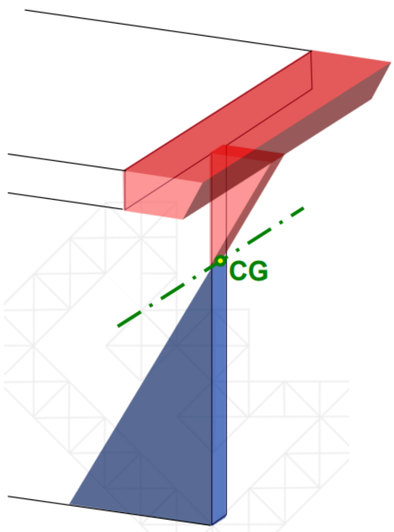


Contrary to the case of normal load, the plastic collapse of a beam under pure bending occurs at a moment greater than the yield moment.

For a rectangular cross section, it will occur at a moment 50% higher than the one that initiate yielding.

3.2.7 Neutral Axis with Asymmetry

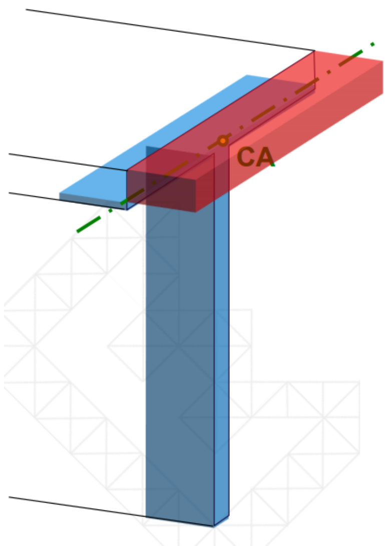
In the case of asymmetrical sections (though still singly symmetric), the analysis becomes more complex:



Equilibrium of forces in the ELASTIC case:

$$F = \int_A \sigma \cdot \mathrm{d}A = \int_A \frac{M}{I} h \cdot \mathrm{d}A = \frac{M}{I} \int_A h \cdot \mathrm{d}A = 0 \tag{3.15}$$

The $\int_A h \cdot \mathrm{d}A$ is the 1st moment of area. The **elastic neutral axis** corresponds to the centroid of the section (centre of gravity).



Equilibrium of forces in the PLASTIC case:

$$F = \int_A \sigma_Y \cdot \mathrm{d}A = \int_{A_T} \sigma_Y \cdot \mathrm{d}A - \int_{A_C} \sigma_Y \cdot \mathrm{d}A = 0 \tag{3.16}$$

$$\therefore \sigma_Y A_T = \sigma_Y A_C \tag{3.17}$$

$$\therefore A_T = A_C \tag{3.18}$$

The **plastic neutral axis** corresponds to the centre of area (belongs to the line that divides the section into 2 equal areas).

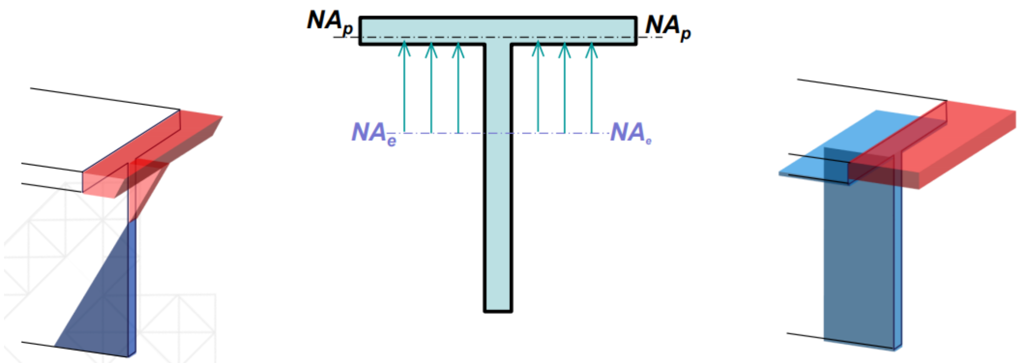
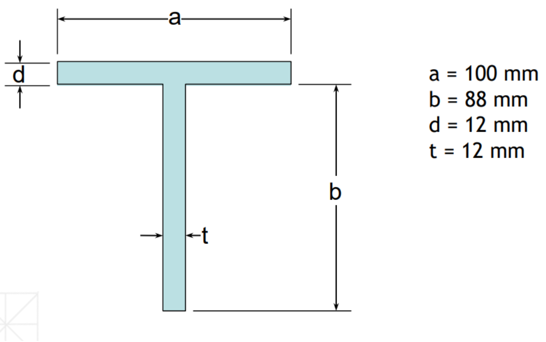


Figure 3.4: In the elasto-plastic phase, the neutral axis shifts from the centroid (centre of gravity) to the centre of area of the section

3.2.8 Example: Shape factor for T-beam

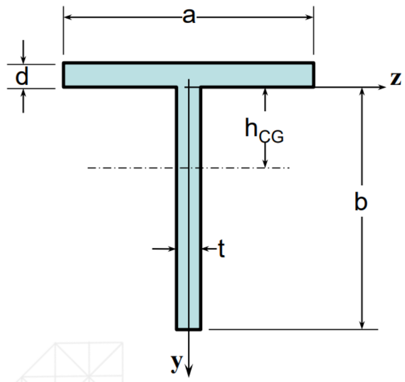


Yield moment M_Y and plastic moment M_P need to be calculated:

$$M_Y = \frac{\sigma_Y \cdot I}{h_{max}} \tag{3.19}$$

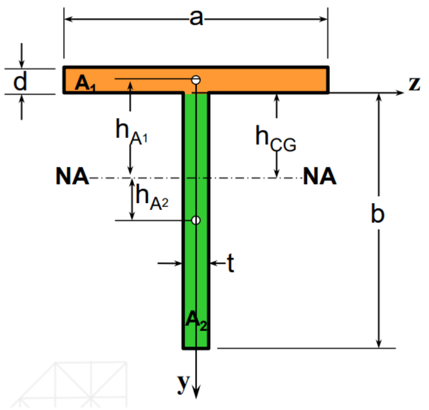
$$M_P = \sigma_Y (Q_1 + Q_2) \tag{3.20}$$

Elastic Neutral Axis:



$$\text{Centre of Gravity} = h_{CG} = \frac{\int_A h \cdot \mathrm{d}A}{A} \tag{3.21}$$

2nd Moment of Area:



$$h_{A1} = h_{CG} + \frac{d}{2} = 23.4\text{mm} \tag{3.22}$$

$$h_{A2} = \frac{b}{2} - h_{CG} = -26.6\text{mm} \tag{3.23}$$

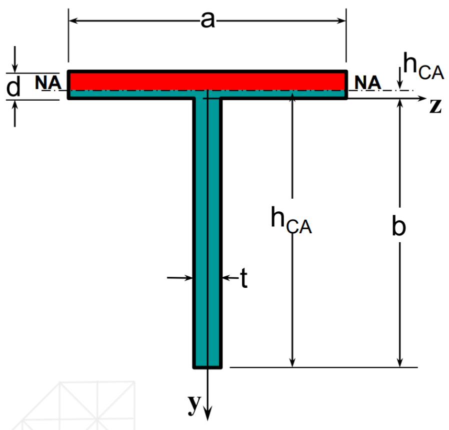
2nd moment of area (parallel axis theorem):

$$I_{zz} = (I_{A_1} + A_1 \cdot h_{A_1}^2) + (I_{A_2} + A_2 \cdot h_{A_2}^2) = 2.10 \cdot 10^6\text{mm}^4 \tag{3.24}$$

Yield Moment:

$$M_Y = \frac{\sigma_Y \cdot I_{zz}}{h_{CG}} = 29745 \cdot \sigma_Y \tag{3.25}$$

Plastic Neutral Axis:

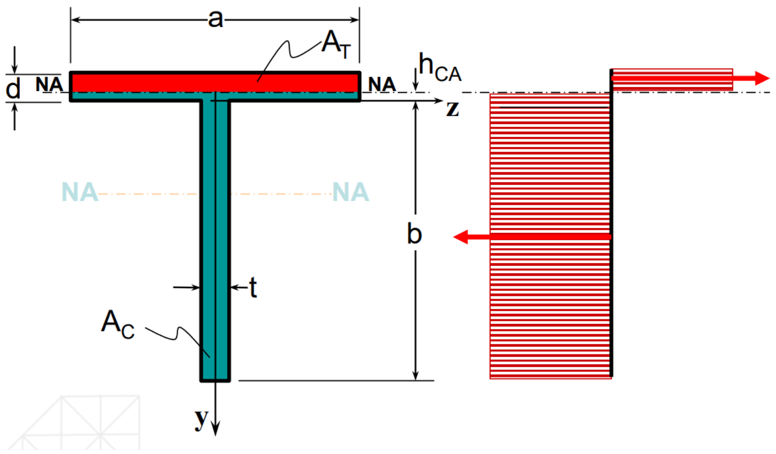


$$A_T = A_C = \frac{A_{total}}{2} \tag{3.26}$$

$$\text{Centre of Area} = h_{CA} = 0.72\text{mm} \tag{3.27}$$

Where A_T and A_C represent the areas of the top and bottom parts.

1st Moment of Area:



1st moment of the areas in tension & compression:

$$Q_{A_T} = \int_{A_T} h \cdot \mathrm{d}A = 6362\mathrm{mm}^3 \tag{3.28}$$

$$Q_{A_C} = \int_{A_C} h \cdot \mathrm{d}A = 46846\mathrm{mm}^3 \tag{3.29}$$

Plastic Moment:

$$M_P = \sigma_Y(Q_{A_T} + Q_{A_C}) = \sigma_Y \cdot 6362 + \sigma_Y \cdot 46846 = 53208 \cdot \sigma_Y \tag{3.30}$$

Shape Factor:

Calculated yield and plastic moments:

$$M_Y = \frac{\sigma_Y \cdot I_{zz}}{y_{CG}} = 29745 \cdot \sigma_Y \tag{3.31}$$

$$M_P = \sigma_Y(Q_{A_T} + Q_{A_C}) = 53208 \cdot \sigma_Y \tag{3.32}$$

$$\therefore \text{Shape Factor:} f = \frac{M_P}{M_Y} = 1.79 \tag{3.33}$$

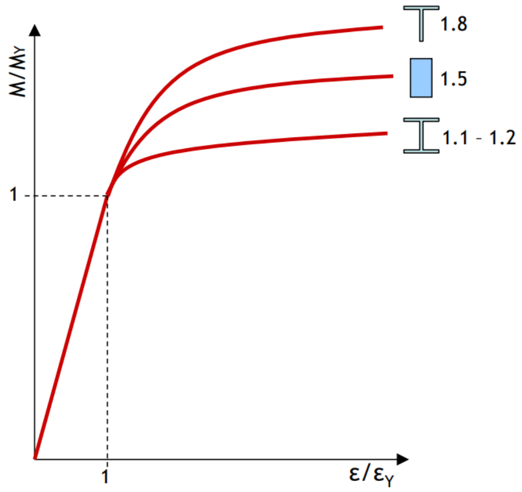


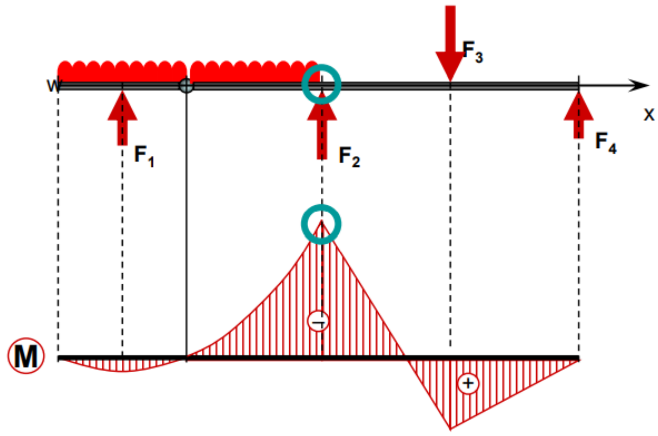
Figure 3.5: The shape factors of Rectangular, T-shaped and I-shaped cross sections

27/11/2020

3.3 Plastic Hinges

So far we have considered the case of pure bending: predicted propagation of plastic region occurs at all sections in same way. However, pure bending rarely occurs. Distribution of bending moment is commonly variable along beam length.

In this case, the section with highest bending moment would reach fully plastic state first. The fully plastic section can not react to further increase of moment: **it loses its capability of reacting to rotations and its curvature can increase indefinitely**. The section behaves like a hinge and is called **PLASTIC HINGE**.



A plastic hinge is defined as a hinge which can undergo rotation when the bending moment reaches the plastic moment M_P.

- for $M < M_P \longrightarrow$ No Rotation

- for $M = M_P \longrightarrow$ Rotation

  - M_P is the limiting value for the bending moment

The effect of the plastic hinge is equivalent to introducing in the section an additional pin joint and a concentrated bending moment equal to M_P.

Simply Supported Beam with Central Load

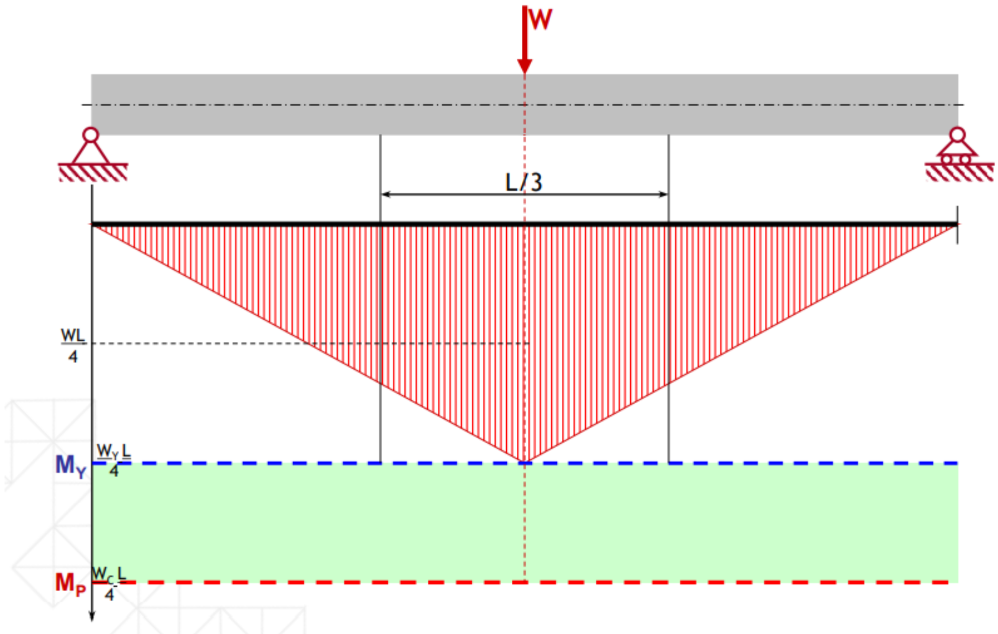


Figure 3.6: As W increases the bending moment raise until the yielding moment is reached at midspan. Yielding starts at midspan, at the top and bottom edges of the section, and spreads inwards to the NA and at either sides.

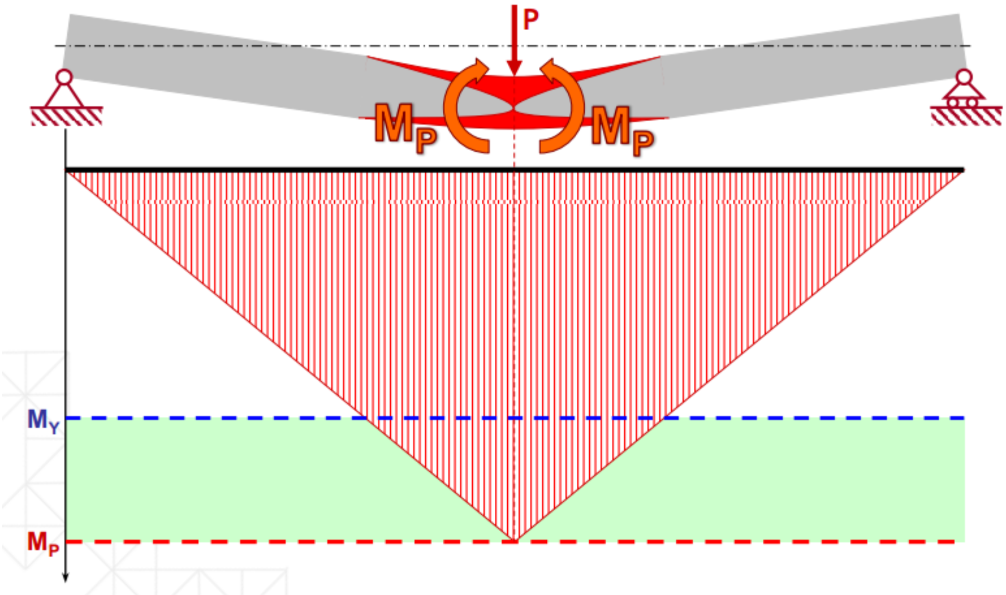


Figure 3.7: By the time plastic hinge forms at the midspan, yielding is quite widespread in the beam.

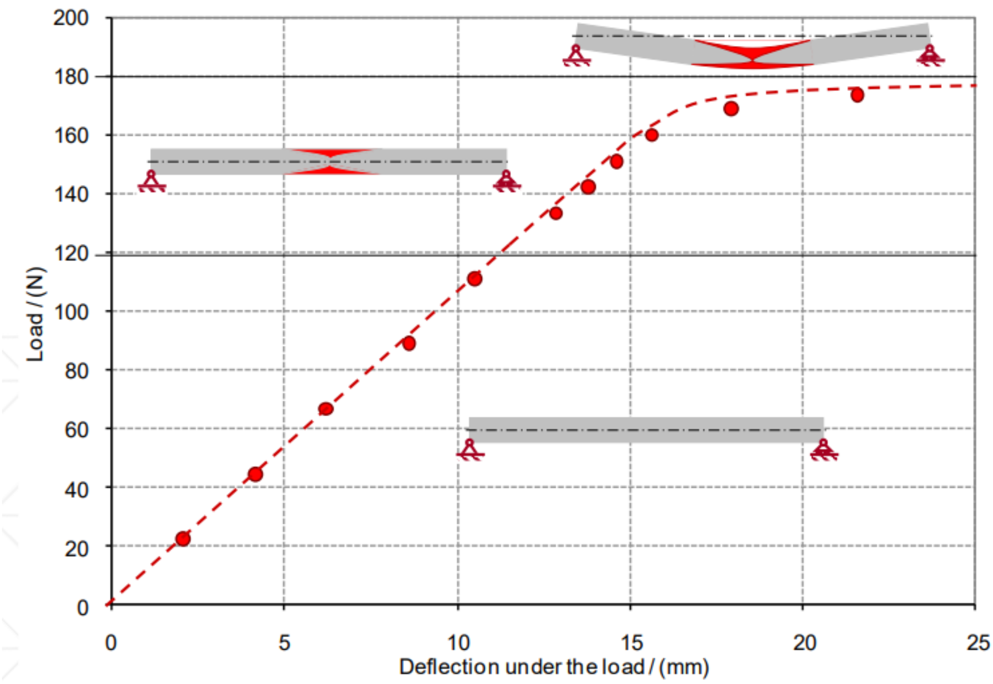


Figure 3.8: Experimental Load-Deflection Characteristic

Simply Supported Beam with Uniform Load

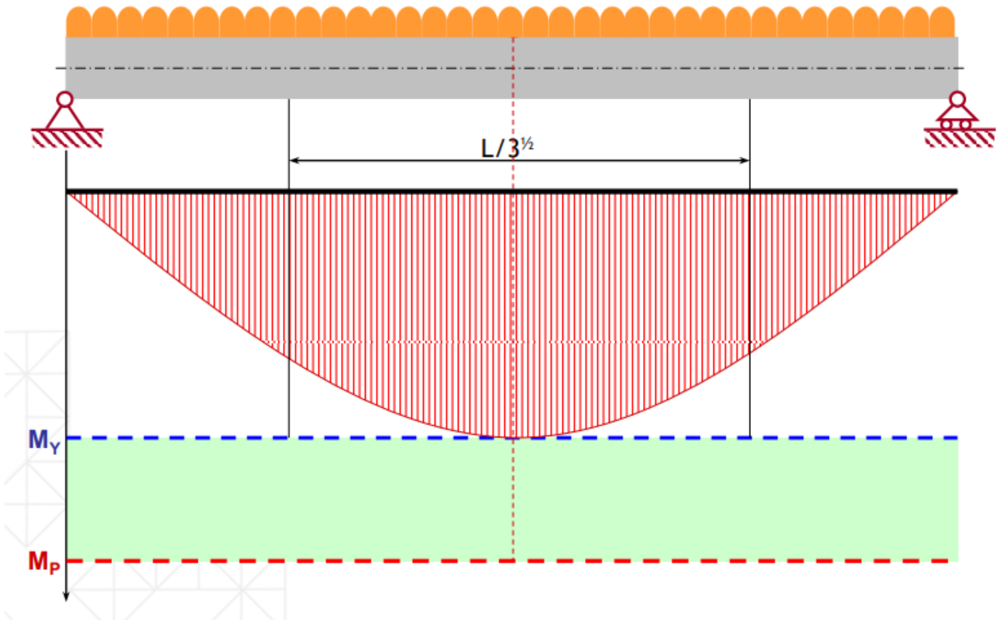


Figure 3.9: The moment increases as the load increases while keeping profile the same, reaching M_Y at a certain point.

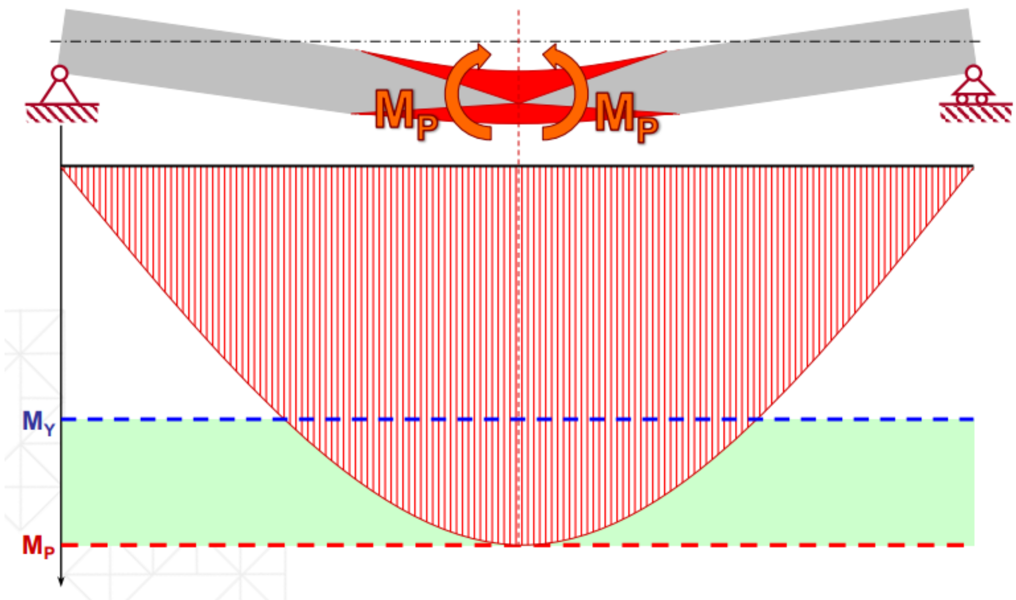


Figure 3.10: As the load is further increased, M_P is reached. The spread of the plastic region occurs on both the cross-section and to the sides. Plastic hinge formations occurs.

3.4 Collapse Mechanism

In the following discussion we will also assume that:

- Cross sections behave elastically up to the plastic moment
- Plastic hinges affect only the section of maximum bending (partial yielding at points away from the plastic hinge is neglected)
- Failure modes other than plastic collapse are prevented
- Effect of normal and shear forces can be neglected

3.4.1 Statically Determinate Beam

Collapse occurs when a sufficient number of hinges have developed to reduce a structure to a mechanism. If the structure is **statically determinate** the development of a **single hinge** leads to collapse.

Case 1: load factor for a beam with a concentrated central point load on a simple support

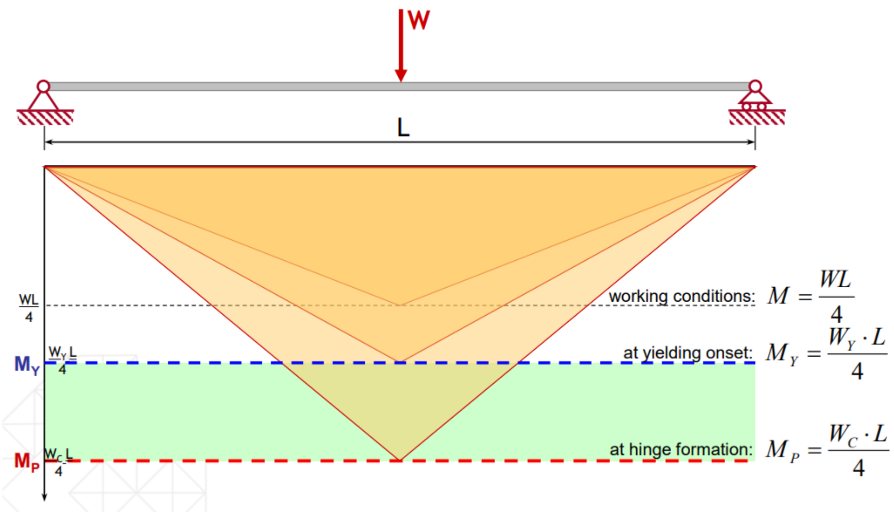


Figure 3.11: The beam under working condition, at yielding onset, and at hinge formation. The moment should be kept at/below the working condition for safety reasons.

The load that initiates yielding is:

$$W_Y = \frac{4M_Y}{L} \tag{3.34}$$

Collapse occurs at load:

$$W_C = \frac{4M_P}{L} \tag{3.35}$$

Load Factor and Safety Factor

The safety factor K_S, used when conventional elastic approach is adopted, is defined as the **ratio between the load that initiate yielding** W_Y **and the maximum working load** W_{max}:

$$K_S = \frac{W_Y}{W_{max}} \longrightarrow \text{This is what you decide} \tag{3.36}$$

Similarly, the load factor K_L is defined as the **ratio between the critical load** W_C **that produces the structure's collapse and the maximum working load** W_{max}:

$$K_L = \frac{W_C}{W_{max}} \tag{3.37}$$

For a statically determinate structure, the load factor K_L corresponds to the product between the shape factor f and the safety factor K_S:

$$K_L = f \cdot K_S \tag{3.38}$$

In the case of statically indeterminate structures, more than one hinge is required, and therefore:

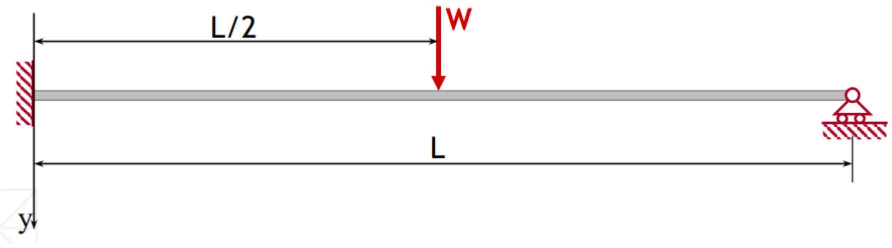
$$K_L \geq f \cdot K_S \tag{3.39}$$

3.4.2 Statically Indeterminate Beam

Collapse occurs when a sufficient number of hinges have developed to reduce a structure to a mechanism. If the structure is **statically determinate** the development of a **single hinge** leads to **collapse**.

If the structure is **statically indeterminate** the development of a **single hinge** leads to an **alternative structure, with higher degree of freedom**. In this case, before structure collapses, a sufficient number of hinges have to develop, in order to make the structure labile.

Case 2: load factor for a propped cantilever with concentrated central load



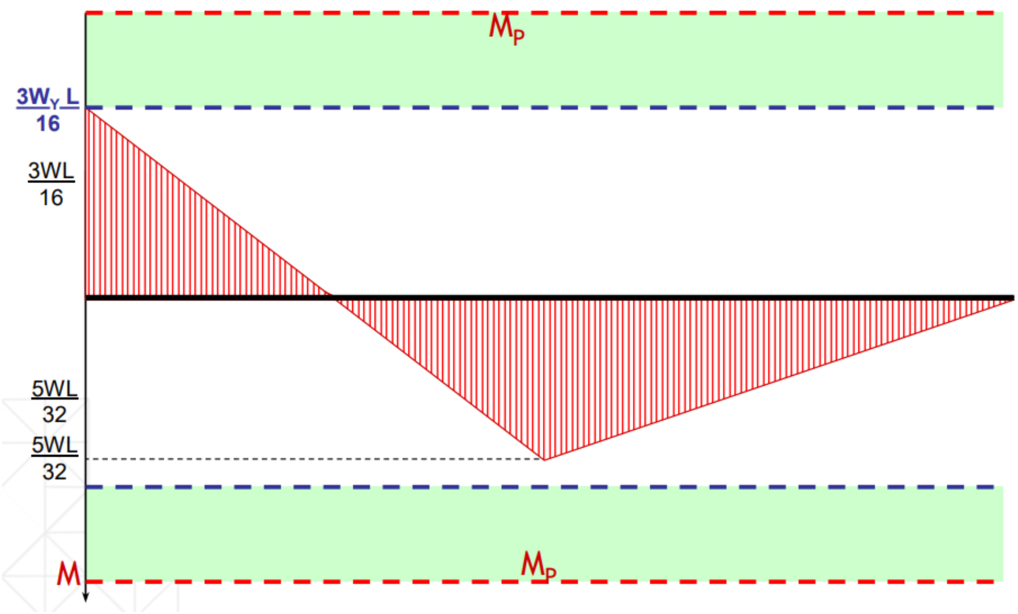


Figure 3.12: The moment diagrams when the load is increased to the yielding onset/moment

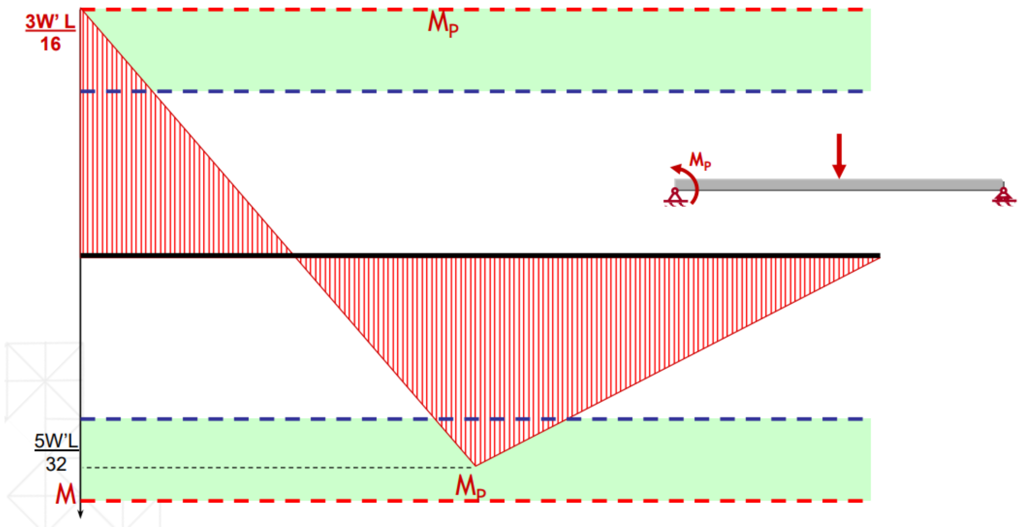


Figure 3.13: The moment diagrams when the highest moment (on the left) reaches plastic moment, and the moment at the midpoint surpasses the yielding moment. At this point, a plastic hinge is formed at the left end of this beam, hence the boundary condition changes. However, the plastic moment M_P to resist the bending still exists.

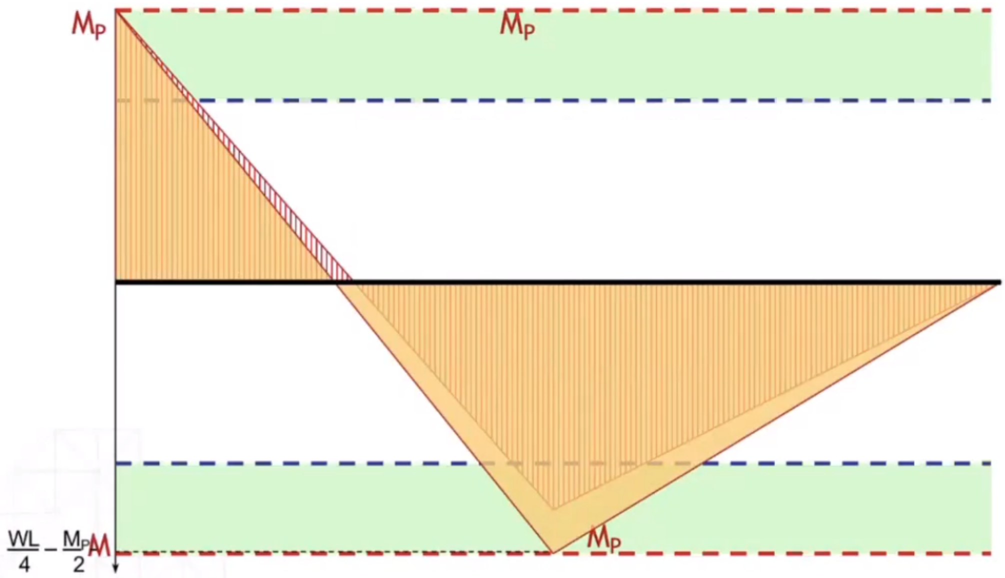


Figure 3.14: The additional load W'' will further increase the bending moment at the midspan, that will eventually reach the plastic moment and form another hinge: the structure will become labile and collapse.

The load that initiates yielding is:

$$W_Y = \frac{16M_Y}{3L} \tag{3.40}$$

First hinges formation occurs at load:

$$W' = \frac{16M_P}{3L} \tag{3.41}$$

Collapse occurs at load:

$$W_C = \frac{6M_P}{L} \tag{3.42}$$

The Load Factor is:

$$K_L = \frac{9}{8} f \cdot K_S \tag{3.43}$$

$$K_L = \frac{W_C}{W_Y} K_S = \frac{6M_P/L}{16M_Y/3L} K_S = \frac{9}{8} f \cdot K_S \tag{3.44}$$

Where:

- $\frac{9}{8}$ depends on the beam cross section
- f depends on loading and boundary conditions

Load Deflection Curve

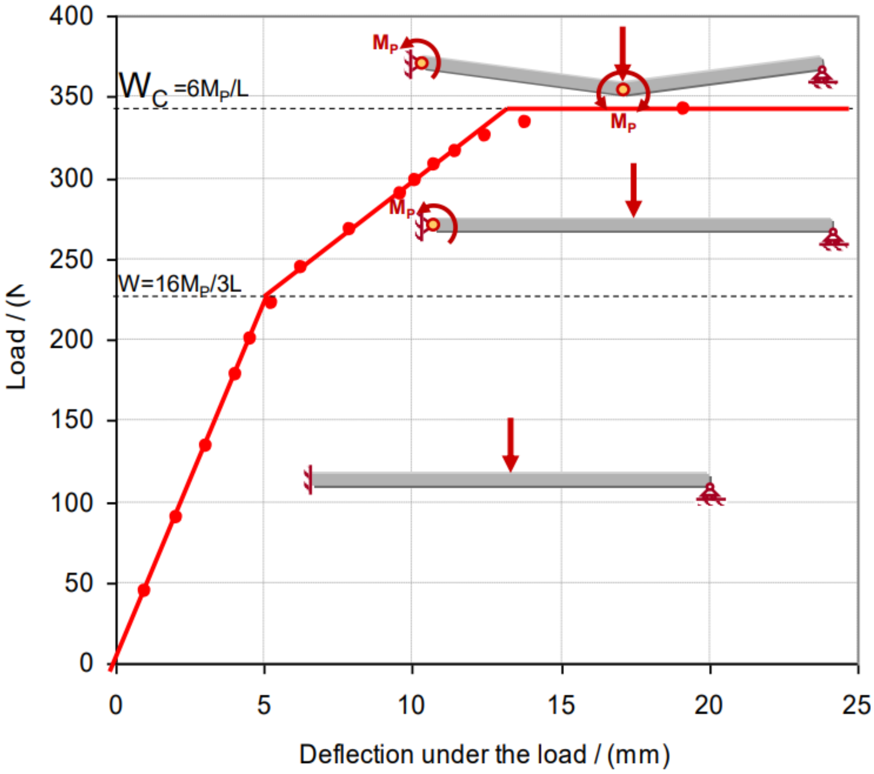


Figure 3.15: Experimental Load-Deflection Characteristic

3.5 Collapse Loads for Portal Frames

Portal frames are rigid structures designed to offer rigidity and stability in their plane.

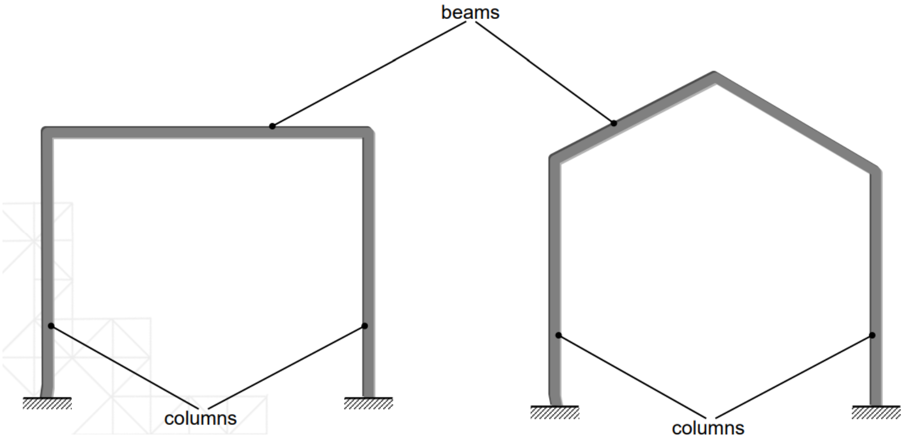
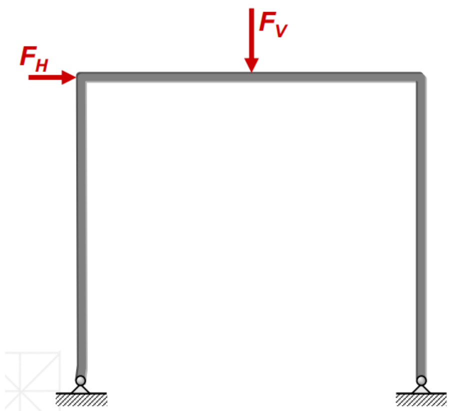


Figure 3.16: Left - rectangular portal frame, Right - pitched-roof portal frame

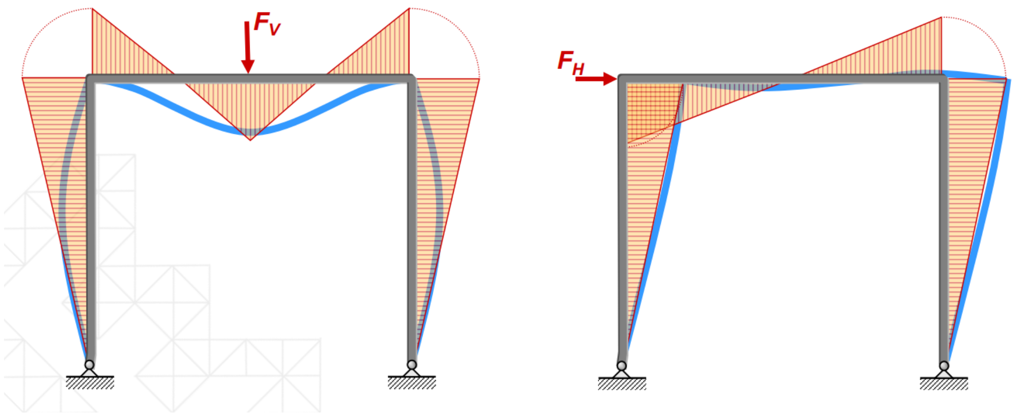
3.6 Portal Frames with Hinged Bases

Consider a flat-roofed portal frame with pinned feet, carrying a vertical concentrated load F_V at the centre of the beam and an horizontal concentrated load F_H at the top of one column.



3.6.1 Deformations and Bending Moments

In this case, the deformations and bending moment distribution produced by the horizontal and vertical loads considered independently are respectively:



Vertical Loads

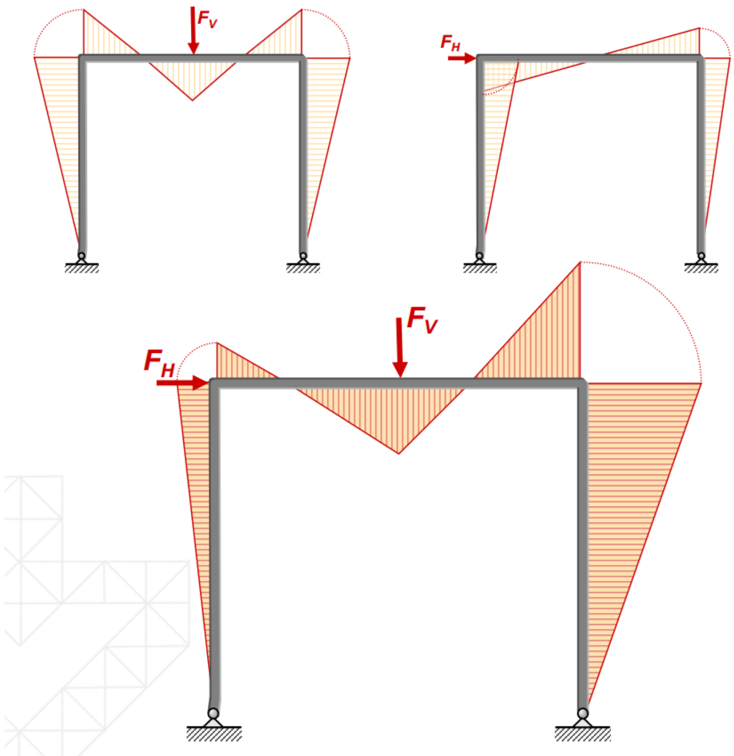
- The mid span off the horizontal beam deforms downwards
- The corners are rigid joints
- The corner points are rotated (from the vertical column's POV)
  - Right Corner - CCW
  - Left Corner - CW
- The bending moment is highest at the mid-span
- The magnitude of the bending moments at the corner (left & right) are the same
  - The arc indicates that they are the same

Horizontal Loads

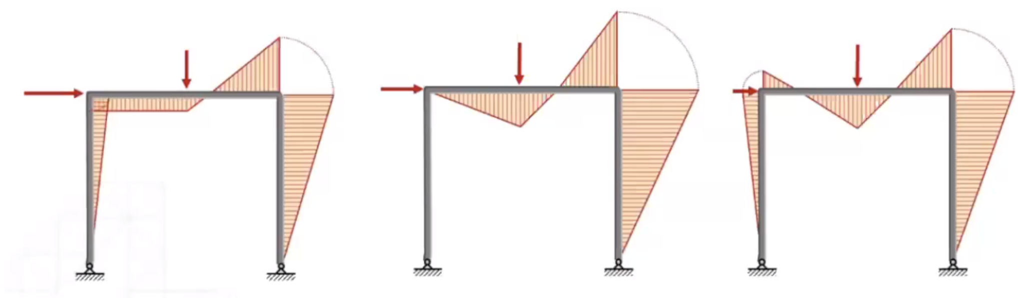
- The frame is displaced to the side
- The corners are rigid joints
- Due to the rigidity of the beam, the top corners are rotated
- The magnitude of the bending moments at the corner (right) are the same

3.6.2 Bending Moment Distribution

By adding the contributions together, we obtain the bending moment distribution below:



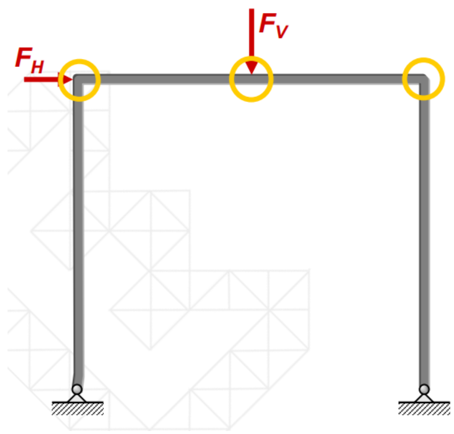
Of course, the exact distribution will depend on the relative magnitude of the two acting forces, and may vary significantly:



However, for any distribution, the points of maximum bending moment are located at the joints or/and at the points of application of the load.

3.6.3 Plastic Hinge Sections

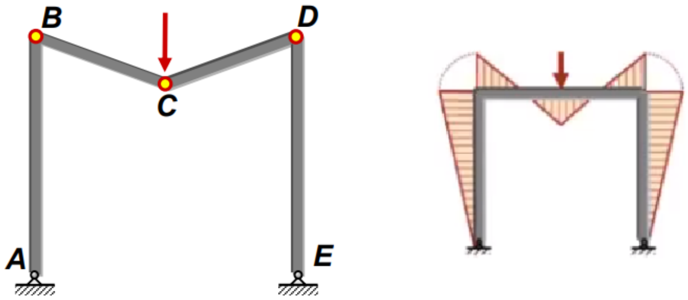
In a framework with rigid joints, the points of local maximum bending moment will occur at the joints as well as under any applied loads.



As a consequence, if plastic collapse is reached, the plastic hinges develop in some of these sections

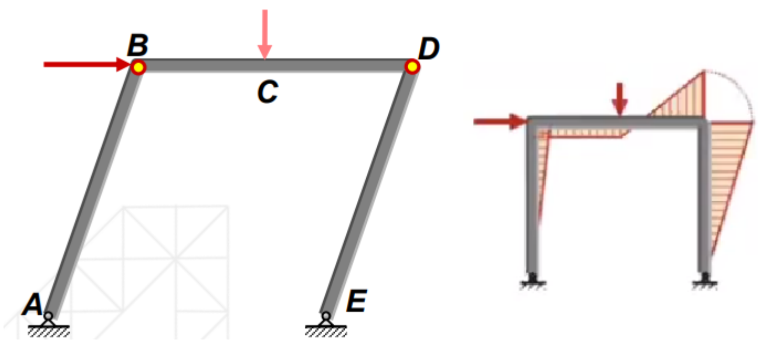
3.6.4 Possible Plastic Collapse Mechanisms

Beam Collapse:



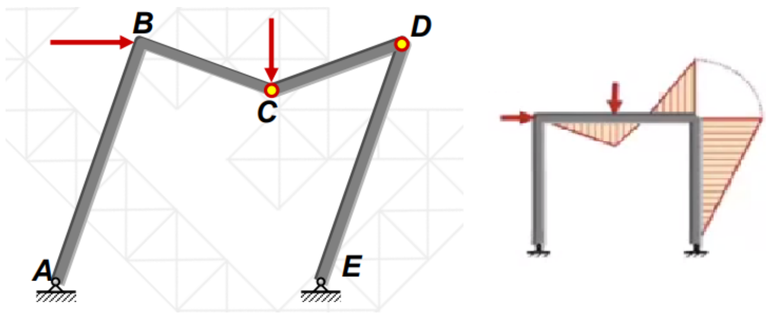
Similar mechanism to fixed-ended beams – hinges form at sections B, C and D. (vertical load much larger than horizontal)

Sway Collapse:



Occurs by overall sway of the frame – hinges form at sections B and D. (horizontal load much larger than vertical)

Combined Beam and Sway Collapse:



Combined beam and sway collapse – hinges form at sections C and D and section B keeps elastic. (horizontal and vertical loads comparable)

3.7 Principle of Virtual Works

3.7.1 Energy

Energy of a system is the ability of the system to do work. Energy may exist in many forms, and can be transformed from one for to another. All forms of energy can be put into two main categories:

1. Kinetic Energy - Motion (of waves, electrons, atoms, molecules, substances and objects)
2. Potential Energy - Stored energy and energy of position (gravitational)

3.7.2 Principle of Virtual Works

If the possible collapse mechanisms are easy to recognise (as in this case), the **principle of virtual works** can be very useful to analyse the collapse mechanisms.

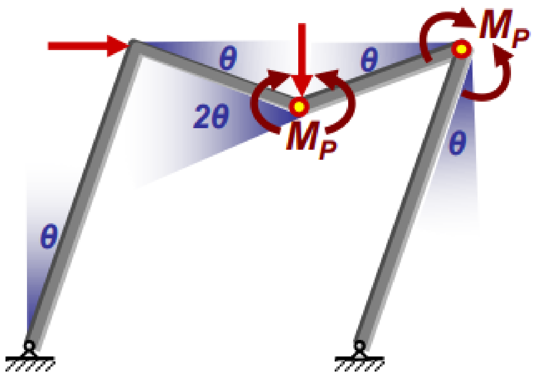
Consider a structure subjected to a set of external mechanical actions. Assume for the structure a set of virtual (imaginary) displacements. The principle of virtual works states that:

The virtual work done by the real external actions as effect of the virtual displacements is equal to the virtual work done by the real internal reactions as effect of the virtual deformations.

3.7.3 Applications to Portal Frames

In the case of portal frames:

- The set of external mechanical actions corresponds to the real forces applied to the structure;
- The set of internal reactions are the internal forces and moments reacting to the forces.
  - Shear force
  - Bending moment
- It is convenient to chose the possible collapse mechanisms as set virtual displacements (only one of them will occur as effect of the applied load).
- In this case the deformations are concentrated at plastic hinges, and are the rotations produced by the internal plastic moments.

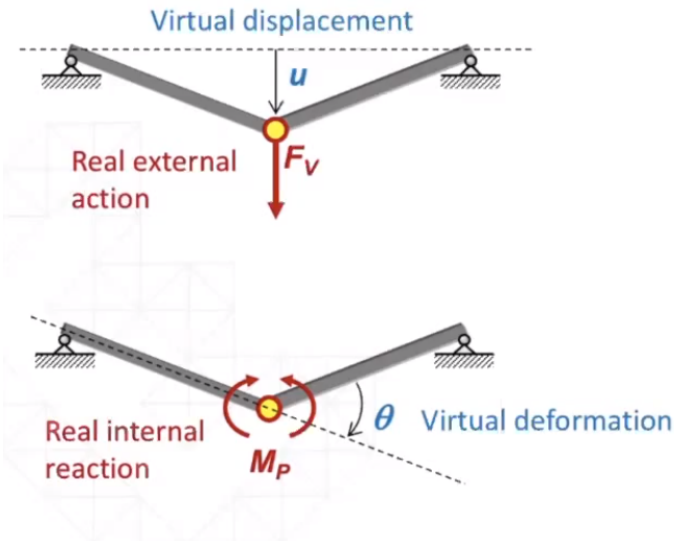


Then, the virtual work principle can be written as:

$$\sum_{i=1}^{n} F_i \cdot u_i = \sum_{j=1}^{m} M_{Pj} \cdot \theta_j \tag{3.45}$$

Where:

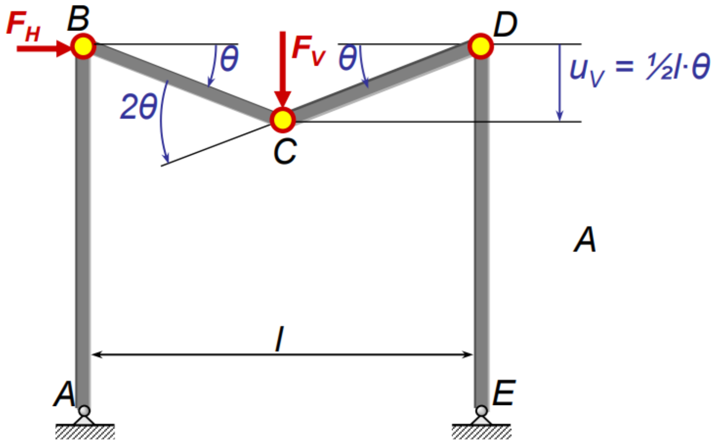
- F_i are the applied forces
- M_{Pj} are the plastic moments at the hinges
- u_i is the relative linear displacement
- θ_i is the relative angular displacement



We do this to find out the level of F_V such that the frame is collapse:

$$F_V = f(M_P) \tag{3.46}$$

Beam Collapse:



- F_H is ignored in this example as it is not causing any sort of displacement
- F_i is the F_V in this example, causing a vertical displacement at the middle
- The linear displacement u_i is considered for point C
- The angular displacement θ is assumed to be very small $\therefore |CD| = \frac{l}{2}$

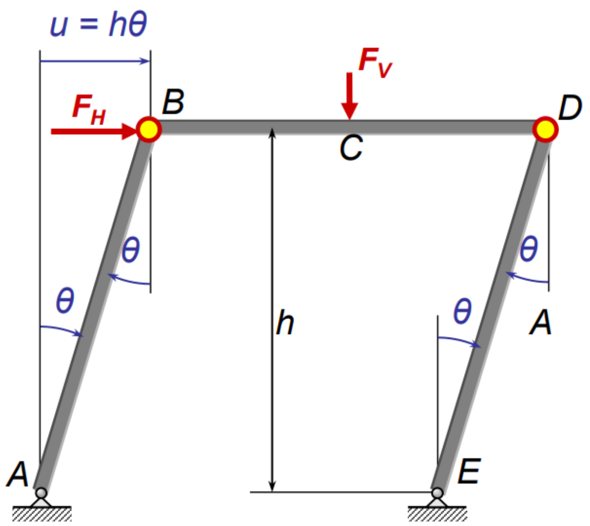
$$F_V \cdot \frac{l}{2}\theta = M_P \cdot \theta + M_P \cdot 2\theta + M_P \cdot \theta \tag{3.47}$$

The RHS is considered for points B,C,D respectively.

$$F_V \cdot \frac{l}{2}\theta = 4M_P \cdot \theta \tag{3.48}$$

$$F_V = 8\frac{M_P}{l} \tag{3.49}$$

Sway Collapse:



- F_V is ignored in this example as it is not causing any sort of displacement
- $F_i = F_H$, causing a horizontal displacement
- $u_i = h\theta$
- $\theta \approxeq 0 \therefore |CD| = \frac{l}{2}$

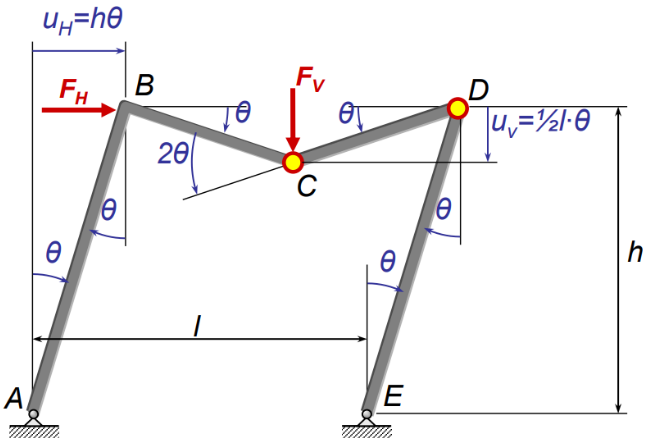
$$F_H \cdot h\theta = M_P \cdot \theta + M_P \cdot \theta \tag{3.50}$$

The RHS is considered for points B and D.

$$F_H \cdot h\theta = 2M_P \cdot \theta \tag{3.51}$$

$$F_H = 2\frac{M_P}{h} \tag{3.52}$$

Combined Collapse:



- Only points C and D are under collapse
- $F_i = F_H$ and F_V
- $u_h = h\theta$
- $u_v = \frac{l}{2}\theta$
- $\theta \approxeq 0 \therefore |CD| = \frac{l}{2}$

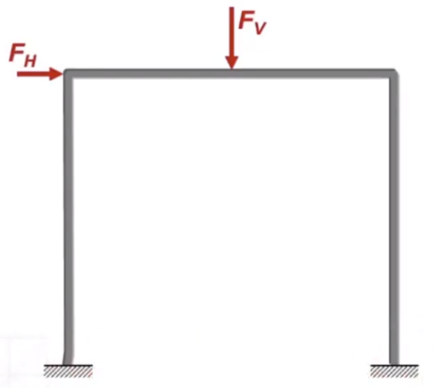
$$F_H h + F_V \frac{l}{2} = 4M_P \tag{3.53}$$

3.7.4 Collapse Mechanism

The collapse mechanism of the portal frame, under the effect of the external forces considered is **the one that requires lower critical forces** (the one that is reached first).

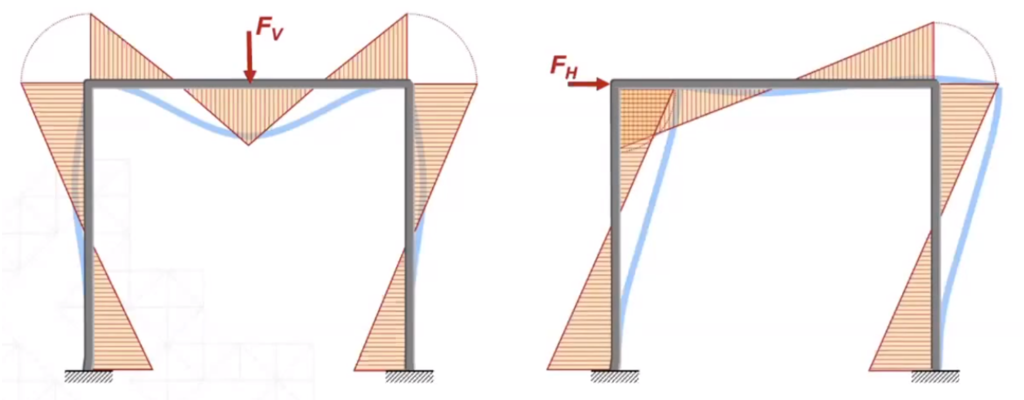
3.8 Portal Frames with Fixed Bases

Consider a flat-roofed portal frame with pinned feet, carrying a vertical concentrated load F_V at the centre of the beam and an horizontal concentrated load F_H at the top of one column.



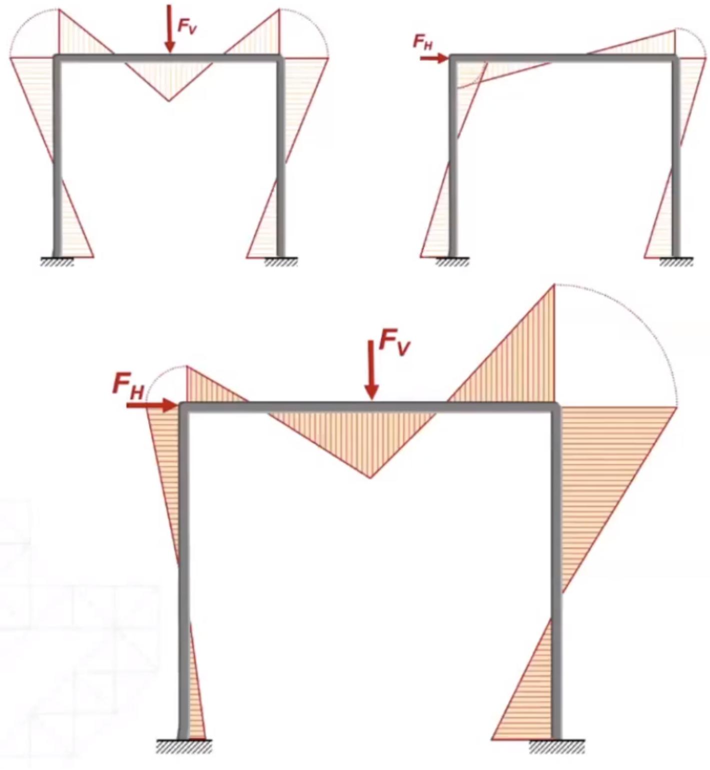
3.8.1 Deformations and Bending Moments

In this case, the deformations and bending moment distribution produced by the horizontal and vertical loads considered independently are respectively:

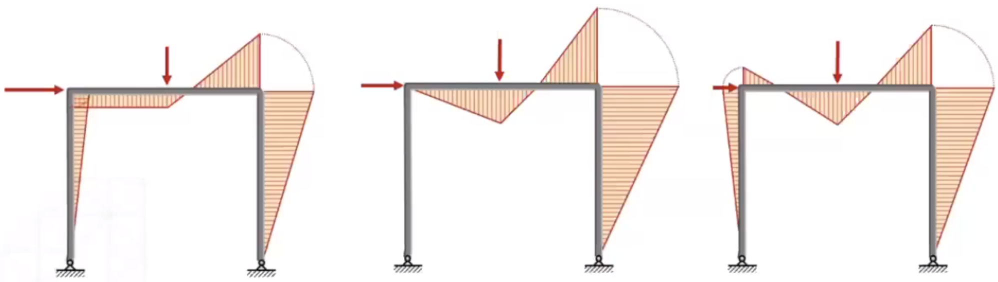


3.8.2 Bending Moment Distribution

By adding the contributions together, we obtain the bending moment distribution below:



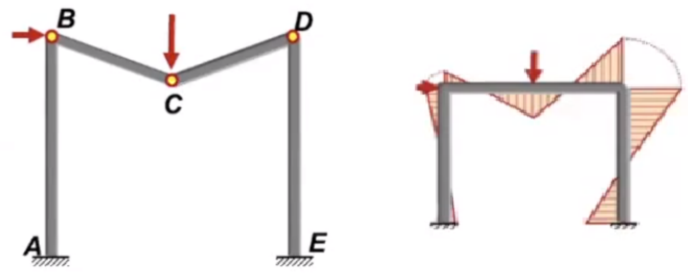
Again, the exact distribution will depend on the relative magnitude of the two acting forces, and may vary significantly (with one column left unstressed in one of the possible configurations):



It is confirmed again that the sections of maximum bending moment are located at the joints or/and at the points of application of the load. Therefore, the plastic hinges develop in some of these sections.

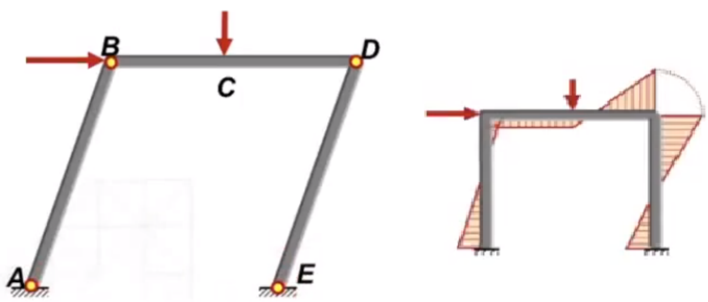
3.8.3 Possible Plastic Collapse Mechanisms

Beam Collapse:



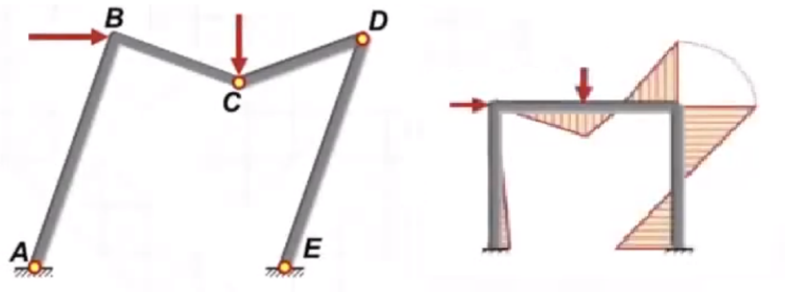
Similar mechanism to fixed-ended beams – hinges form at sections B, C and D. (occurs when vertical load is dominant)

Sway Collapse:



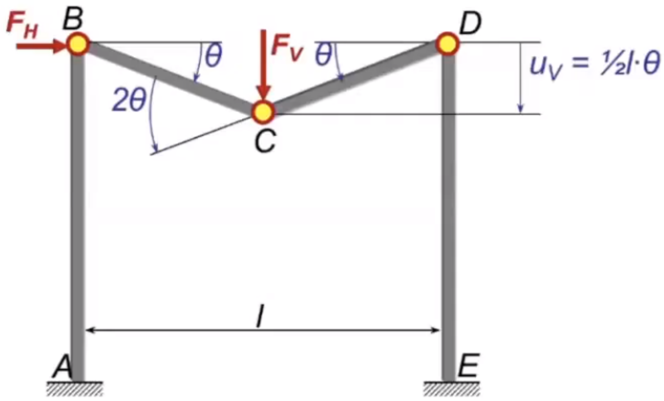
Occurs by overall sway of the frame – hinges form at sections A, B, D and E. (due to a dominant horizontal load)

Combined Beam and Sway Collapse:



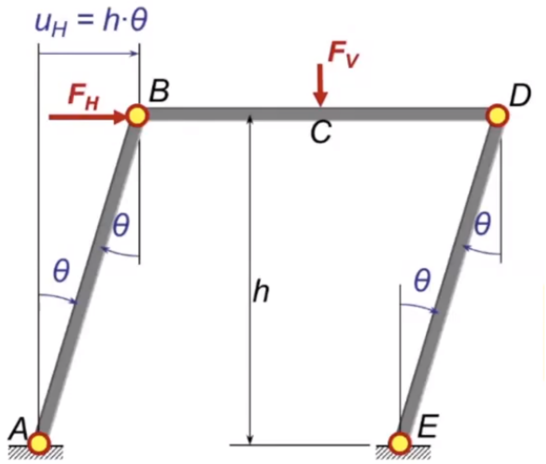
Combined beam and sway collapse – hinges form at sections A, C, D and E and section B keeps elastic. (horizontal and vertical loads comparable)

Beam Collapse:



$$F_V = 8\frac{M_P}{l} \tag{3.54}$$

Sway Collapse:

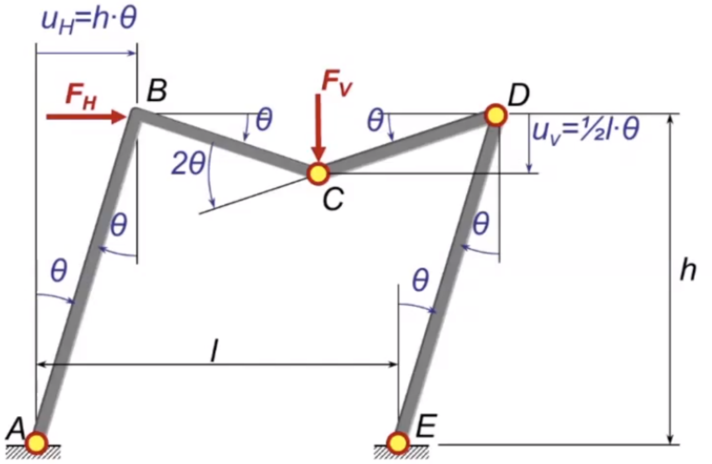


$$F_H \cdot h\theta = M_P\theta + M_P\theta + M_P\theta + M_P\theta \tag{3.55}$$

$$F_H \cdot h\theta 4M_P\theta \tag{3.56}$$

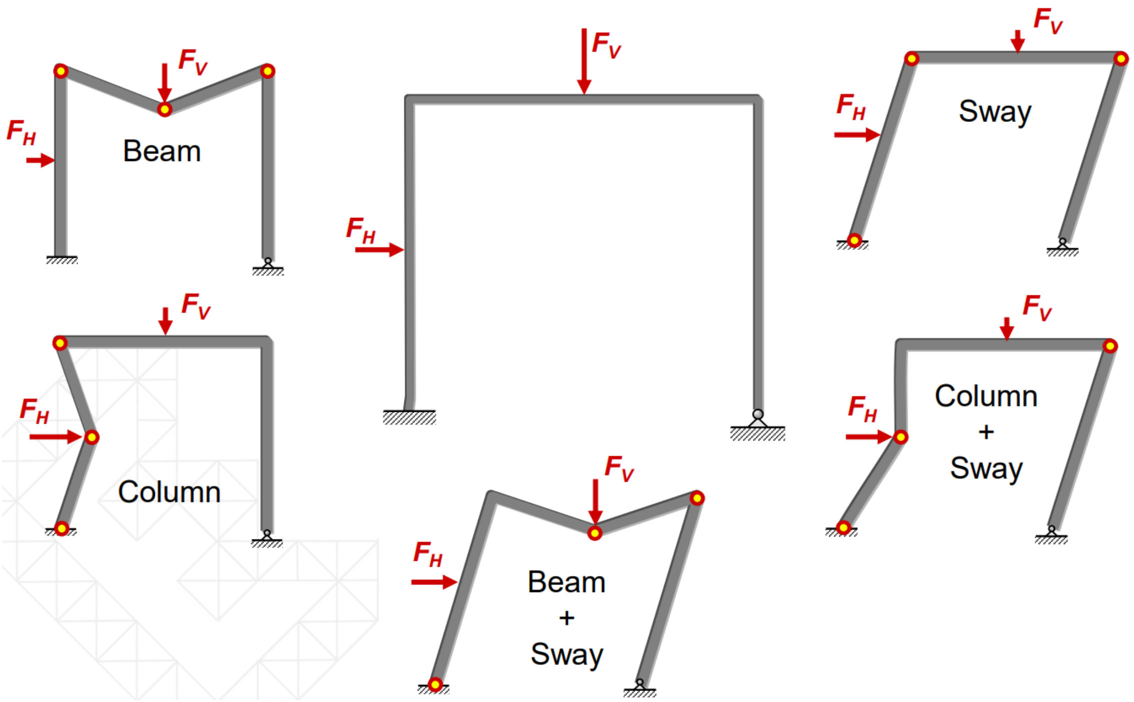
$$F_H = 4\frac{M_P}{h} \tag{3.57}$$

Combined Collapse:

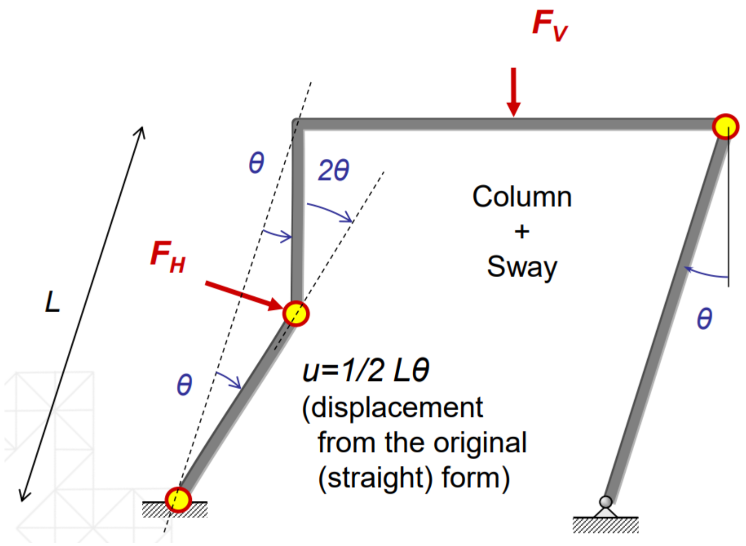


$$F_H h + F_V \frac{l}{2} = 6M_P \tag{3.58}$$

Example



These are all possible modes of collapse. Find out which mode is the easiest to occur and under what level of load.



Other modes are the same as previous examples.

Chapter 4

Energy methods

4.1 Strain energy

The energy of a system is the ability of the system to do work.

Energy may exist in many forms and can be transformed from one for to another. All forms of energy can be put into two main categories:

- Kinetic energy - motion (of waves, electrons, atoms, molecules, substances and objects)
- Potential energy - stored energy

Strain energy is a form of potential energy that is stored within a material which has been subjected to strain (deformation). If the material remains **elastic** whilst the deformation is applied, then it is capable of doing an amount of work equal to the stored strain energy when returning to its original dimensions.

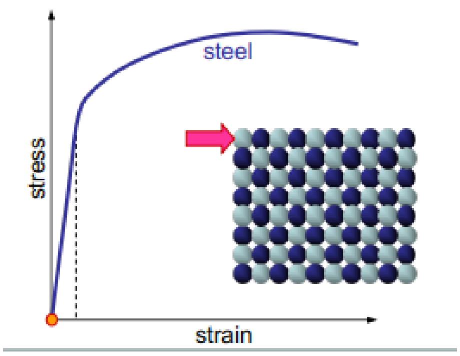


Figure 4.1:

The work done W by the external actions on the structure as effect of the displacements, is equal to the strain energy U stored by internal reaction forces as effect of the deformations within the structure:

$$W = U \tag{4.1}$$

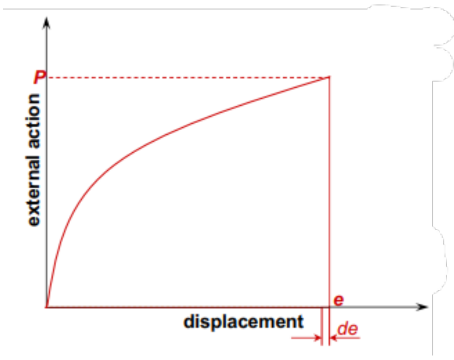


Figure 4.2:

For a gradually applied load, the work done by a force acting on the structure, is given by the area under the force - displacement curve:

$$W = \int_0^e P \text{ de} \tag{4.2}$$

If displacement is proportional to load (Hookean behaviour)

$$W = \frac{1}{2}Pe \tag{4.3}$$

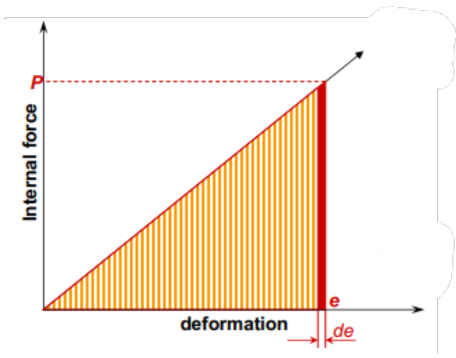


Figure 4.3:

For a gradually applied load, the energy stored by the internal actions into the structure, is given by the area under the internal force - deformation curve:

$$U = \int_0^e P \text{ de} \tag{4.4}$$

If deformation is proportional to load (Hookean behaviour):

$$U = \frac{1}{2}Pe \tag{4.5}$$

4.1.1 Bending moment

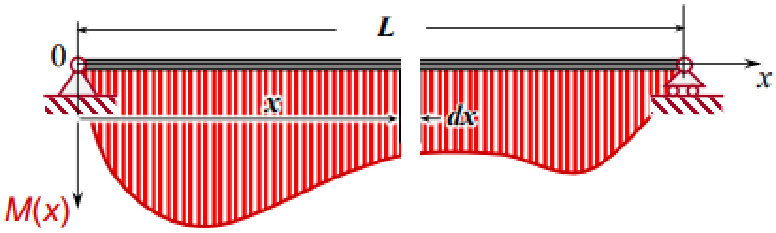


Figure 4.4:

We can determine through various methodologies the distribution of the bending moment. If we want to determine the strain energy at a particular point along the beam, we can extract an elemental portion of the beam with length $\mathrm{d}x$ from a position x in our coordinate system. This is subjected to a certain bending moment and this will produce a $\mathrm{d}\theta$ with radius R.

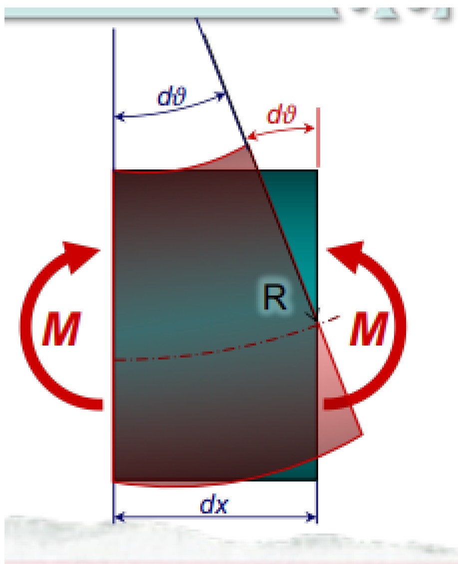


Figure 4.5:

We then know that:

$$\mathrm{d}U_M = \frac{1}{2} M \, \mathrm{d}\theta \tag{4.6}$$

It is inconvenient to use θ in our equations and we would like to substitute this for x. We know that:

$$\mathrm{d}\theta = \frac{\mathrm{d}x}{R} \text{ and } \frac{1}{R} = \frac{M}{EI} \tag{4.7}$$

(from theory of bending). Therefore:

$$\mathrm{d}\theta = \frac{M}{EI} \, \mathrm{d}x \tag{4.8}$$

Substituting:

$$\mathrm{d}U_M = \frac{1}{2}\frac{M^2}{EI}\,\mathrm{d}x \tag{4.9}$$

$$U_M = \int_0^L \frac{M^2}{2EI}\,\mathrm{d}x \tag{4.10}$$

4.2 Castigliano's Theorem

Castigliano stated in his engineering thesis that:

> ... the partial derivative of the strain energy, considered as a function of the applied forces acting o na linear elastic structure, with respect to one of these forces, is equal to the displacement in the direction of the force of its point of application.

This simply means that if we want to know the displacement from an applied force at the point of application, we can simply take the partial derivative of the strain energy.

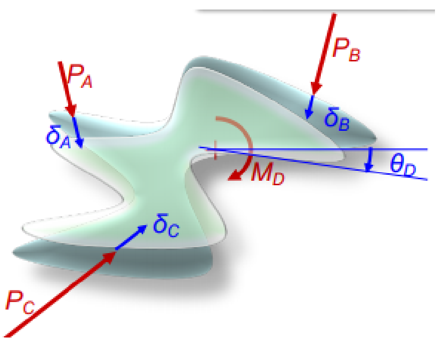


Figure 4.6:

$$\delta_A = \frac{\partial U}{\partial P_A},\ \ \delta_B = \frac{\partial U}{\partial P_B} \tag{4.11}$$

$$\delta_C = \frac{\partial U}{\partial P_C},\ \ \theta_D = \frac{\partial U}{\partial M_D} \tag{4.12}$$

In practice, the displacement δ_i at any load P_i acting upon a system is given by:

$$\delta_i = \frac{\partial U}{\partial P_i} \tag{4.13}$$

where U is the strain energy of the system due to all loads including P_i. The load may be a force, in which case a linear displacement is obtained; or a rotational moment, in which case an angular displacement is obtained. The displacement (linear or angular) will result positive in the same sense as the applied action (force or moment).

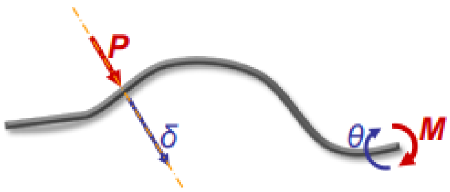


Figure 4.7:

4.2.1 Application of Castigliano's theorem

We know that:

$$\delta_i = \frac{\partial U_M}{\partial P_i} \tag{4.14}$$

We also know that:

$$U_M = \int_0^L \frac{M^2}{EI}\,\mathrm{d}x \tag{4.15}$$

Substituting:

$$\delta_i = \frac{\partial}{\partial P_i}\int_0^L \frac{M^2}{2EI}\,\mathrm{d}x \tag{4.16}$$

$$\delta_i = \int_0^L \frac{\partial}{\partial P_i}\left(\frac{M^2}{2EI}\right)\mathrm{d}x \tag{4.17}$$

An alternative expression using the chain rule can be derived as:

$$\delta_i = \frac{\partial U_M}{\partial P_i} = \frac{\partial U_M}{\partial M}\cdot\frac{\partial M}{\partial P_i} = \int_0^L \frac{\partial}{\partial M}\left(\frac{M^2}{2EI}\right)\frac{\partial M}{\partial P_i}\,\mathrm{d}x \tag{4.18}$$

This leads to:

$$\frac{\partial}{\partial M}\frac{\partial M^2}{2EI} = \frac{M}{EI} \tag{4.19}$$

$$\delta_i = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_i}\,\mathrm{d}x \tag{4.20}$$

4.3 Examples and applications

4.3.1 Dummy loads

Example 1

Determine the deflection of the free end of the beam in the direction of P.

Figure 4.8:

Bending moment:

$$M = -Px \tag{4.21}$$

Castigliano's theorem:

$$\delta = \int_0^L \frac{\partial}{\partial P}\left(\frac{M^2}{2EI}\right) \mathrm{d}x = \int_0^L \frac{\partial}{\partial P}\left(\frac{P^2x^2}{2EI}\right) \mathrm{d}x = \int_0^L \frac{Px^2}{EI} \mathrm{d}x = \frac{PL^3}{3EI} \tag{4.22}$$

$$\delta = \frac{PL^3}{3EI} \tag{4.23}$$

Example 2

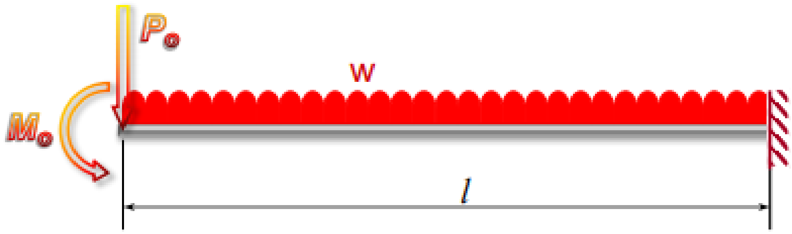


Figure 4.9:

We can utilise dummy loads in order to analyse our system. The bending moment is:

$$M = -\frac{wx^2}{2} - P_0x - M_0 \tag{4.24}$$

Castigliano's theorem (alternative equation):

$$\delta = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_0} \mathrm{d}x \text{ with } \frac{\partial M}{\partial P_0} = \frac{\partial}{\partial P_0}\left(-\frac{wx^2}{2} - P_0x - M_0\right) = -x \tag{4.25}$$

Substituting:

$$\delta = \int_0^L \frac{1}{EI}\left(-\frac{wx^2}{2}\right)(-x) \mathrm{d}x = \frac{w}{2EI}\int_0^L x^3 \mathrm{d}x = \frac{wL^4}{8EI} \tag{4.26}$$

Note: as P_0x and M_0 are dummy loads, they do not need to be included in our final equation for δ.

4.3.2 Curved beams

In many applications beams are curved in the form of circular arcs, rather than straight. Standard formulae used for straight beam deflections cannot be used for curved beams. In this case the application of Castigliano's theorem allows for a solution.

> **Thin curved beams** are characterised by a beam depth which is small compared to the radius of curvature.

As a result:

- curvature effects and stress concentration in the cross-section are negligible: stresses are essentially linear like those of a straight beam
- effects of normal and shear forces are typically negligible

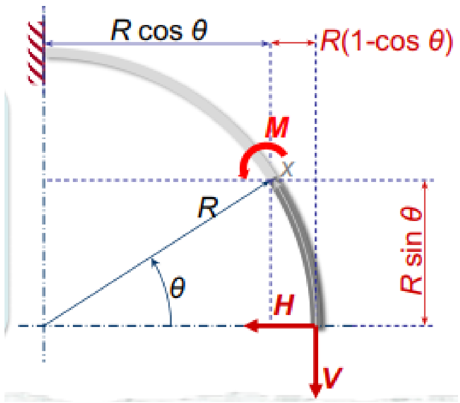


Figure 4.10:

In the case of curved beams, it is more convenient to refer to the angle θ than to to the beam length. The bending moment can be characterised as:

- Contribution to bending moment from horizontal force H: M_H = force · distance = $H \cdot R\sin\theta$
- Contribution to bending moment from vertical force V: M_V = force · distance = $V \cdot R\,(1-\cos\theta)$

$$M(\theta) = HR\sin\theta + VR\,(1-\cos\theta) \tag{4.27}$$

Example 3

Determine the displacement at the free end.

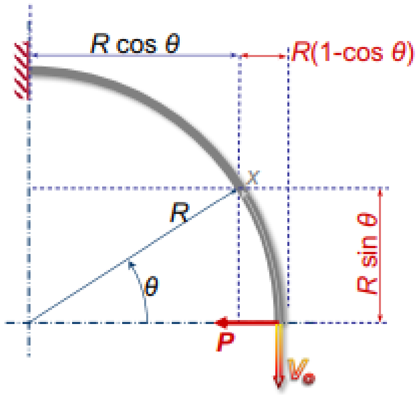


Figure 4.11:

Bending moment:

$$M = PR\sin\theta + V_0 R\,(1-\cos\theta) \tag{4.28}$$

Horizontal displacement:

$$\delta_H = \frac{\partial U}{\partial P} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}\,\mathrm{d}x \tag{4.29}$$

$$\frac{\partial M}{\partial P} = \frac{\partial}{\partial P}\big(PR\sin\theta + V_0 R\,(1-\cos\theta)\big) = R\sin\theta \tag{4.30}$$

Converting reference system to polar ($\mathrm{d}x = R\,\mathrm{d}\theta$):

$$\delta_H = \int_0^{\frac{\pi}{2}} \frac{1}{EI}\big(PR\sin\theta + V_0 R(1-\cos\theta)\big)\,(R\sin\theta)(R)\;\mathrm{d}\theta \tag{4.31}$$

$V_0 R(1-\cos\theta)$ is a dummy load and can be ignored.

$$= \frac{PR^3}{EI}\int_{)}^{\frac{\pi}{2}} \sin^2\theta\;\mathrm{d}\theta \tag{4.32}$$

$$= \frac{PR^3}{EI}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{PR^3}{EI}\left[\frac{\pi}{4} - 0\right] \tag{4.33}$$

$$\delta_H = \frac{\pi PR^3}{4EI} \tag{4.34}$$

Vertical displacement:

$$\delta_V = \frac{\partial U}{\partial V_0} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial V_0}\,\mathrm{d}x \tag{4.35}$$

$$\frac{\partial M}{\partial V_0} = \frac{\partial}{\partial V_0}\big(PR\sin\theta + V_0 R\,(1-\cos\theta)\big) = R\,(1-\cos\theta) \tag{4.36}$$

Converting reference system to polar ($\mathrm{d}x = R\,\mathrm{d}\theta$):

$$\delta_V = \int_0^{\frac{\pi}{2}} \frac{1}{EI} \left(PR\sin\theta + V_0R(1-\cos\theta)\right) \cdot R\,(1-\cos\theta) \tag{4.37}$$

$V_0R(1-\cos\theta)$ is a dummy load and can be ignored.

$$= \frac{PR^3}{EI} \int_0^{\frac{\pi}{2}} (\sin\theta - \sin\theta\cos\theta)\ \mathrm{d}\theta \tag{4.38}$$

$$= \frac{PR^3}{EI} \left[-\cos\theta - \frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}} = \frac{PR^3}{EI}\left[\left(0-\frac{1}{2}\right) - (-1-0)\right] \tag{4.39}$$

$$\delta_v = \frac{PR^3}{2EI} \tag{4.40}$$

Example 4 - proving ring

Determine the maximum diametrical deflection. These setups are often used to calibrate strain gauges as these systems are purely mechanical. We can use two lines of symmetry to simplify our problem. We assume that the bending does not extend past 90° as shown in the middle figure, as this would imply there is a plastic hinge.

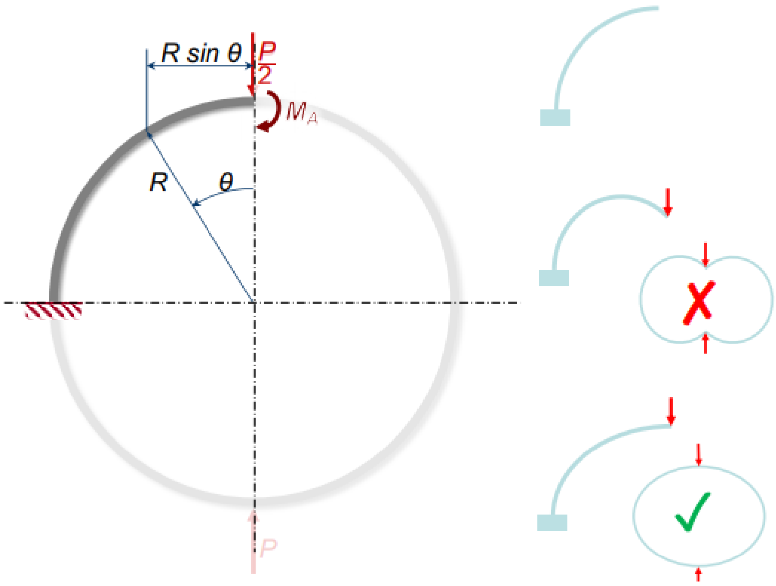


Figure 4.12:

Bending moment:

$$M = M_A + \frac{PR}{2}\sin\theta \tag{4.41}$$

Boundary condition:

$$\varphi_A = \frac{\partial U}{\partial M_A} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_A}\,\mathrm{d}x = 0 \tag{4.42}$$

$$\frac{\partial M}{\partial M_A} = \frac{\partial}{\partial M_A}\left(M_A + \frac{PR}{2}\sin\theta\right) = 1 \tag{4.43}$$

Where φ_A is our rotation. Converting reference system to polar ($\mathrm{d}x = R\,\mathrm{d}\theta$):

$$\varphi_A = \int_0^{\frac{\pi}{2}} \frac{1}{EI}\left(M_A + \frac{PR}{2}\sin\theta\right)(1)(R)\,\mathrm{d}\theta \tag{4.44}$$

$$= \frac{R}{EI}\int_0^{\frac{\pi}{2}}\left(M_A + \frac{PR}{2}\sin\theta\right)\mathrm{d}\theta \tag{4.45}$$

$$= \frac{R}{EI}\left(M_A\frac{\pi}{2} + \frac{PR}{2}\right) = 0 \tag{4.46}$$

$$M_A = -\frac{PR}{\pi} \tag{4.47}$$

Thus, our bending moment becomes:

$$M = -\frac{PR}{\pi} + \frac{PR}{2}\sin\theta = \frac{PR}{2}\left(\sin\theta - \frac{2}{\pi}\right) \tag{4.48}$$

Vertical displacement:

$$\delta = \frac{\partial U}{\partial \frac{P}{2}} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial \frac{P}{2}}\,\mathrm{d}x \tag{4.49}$$

$$\frac{\partial M}{\partial \frac{P}{2}} = \frac{\partial}{\partial \frac{P}{2}}\left(\frac{P}{2}R\left(\sin\theta - \frac{2}{\pi}\right)\right) = R\left(\sin\theta - \frac{2}{\pi}\right) \tag{4.50}$$

Converting reference system to polar ($\mathrm{d}x = R\,\mathrm{d}\theta$):

$$\delta = \int_0^{\frac{\pi}{2}} \frac{1}{EI}\left(\frac{PR}{2}\left(\sin\theta - \frac{2}{\pi}\right)\right)\left(R\left(\sin\theta - \frac{2}{\pi}\right)\right)(R)\,\mathrm{d}\theta \tag{4.51}$$

$$= \frac{PR^3}{2EI}\int_0^{\frac{\pi}{2}}\left(\sin^2\theta - \frac{4}{\pi}\sin\theta + \frac{4}{\pi^2}\right)\mathrm{d}\theta \tag{4.52}$$

$$= \frac{PR^3}{2EI}\left(\frac{\pi}{4} - \frac{2}{\pi}\right) \tag{4.53}$$

Therefore our diametrical deflection is:

$$D = 2\delta = \frac{PR^3}{EI}\left(\frac{\pi}{4} - \frac{2}{\pi}\right) \tag{4.54}$$

Example 5 - split rings

Consider a piston ring. We must be wary that we do not have two axes of symmetry and only one in this case. Hence, we must integrate from $o - \pi$.

Figure 4.13:

Example 6 - discontinuous loads

Develop an expression for the deflection at the free end.

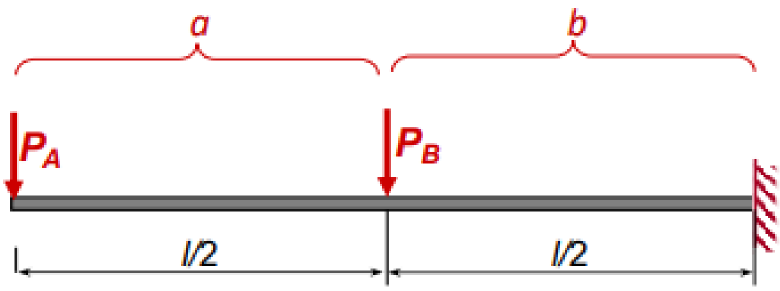


Figure 4.14:

If expressions for the internal forces are discontinuous, the structure is divided into sections (over which the internal forces are continuous) and the intergration processes are summed to give the total strain energy of the structure/member.

$$\text{a. } 0 \leq x \leq \frac{l}{2} \rightarrow M_a = -P_a x \rightarrow \frac{\partial M_a}{\partial P_A} = -x \tag{4.55}$$

$$\text{b. } \frac{l}{2} \leq x \leq l \rightarrow M_b = -P_a x - P_b\left(x - \frac{l}{2}\right) \rightarrow \frac{\partial M_b}{\partial P_A} = -x \tag{4.56}$$

We take the derivative with respect to P_A as we want to find the deflection at that point.

$$\delta_A = \frac{\partial U}{\partial P_A} = \int_0^{\frac{l}{2}} \frac{\partial M_A}{EI} \frac{\partial M_A}{\partial P_A} \, \mathrm{d}x + \int_{\frac{l}{2}}^{l} \frac{M_b}{EI} \frac{\partial M_b}{\partial P_a} \, \mathrm{d}x \tag{4.57}$$

$$= \int_0^{\frac{l}{2}} \frac{Px^2}{EI} \, \mathrm{d}x + \int_{\frac{l}{2}}^{l} \frac{1}{EI} \left(Px^2 + P \left(x^2 - \frac{l}{2}x \right) \right) \, \mathrm{d}x \tag{4.58}$$

$$= \frac{P}{EI} \left(\int_0^{\frac{l}{2}} x^2 \, \mathrm{d}x + \int_{\frac{l}{2}}^{l} \left(2x^2 - \frac{l}{2}x \right) \, \mathrm{d}x \right) \tag{4.59}$$

$$\delta_A = \frac{7}{16} \frac{PL^3}{EI} \tag{4.60}$$

Chapter 5

Beam buckling

Potential energy

We know from everyday life that external loads of tension and compression produce very different responses in beams (not jus a change in the stress sign). This is due to the fact that tension and compression are associated with different behaviour in term of equilibrium.

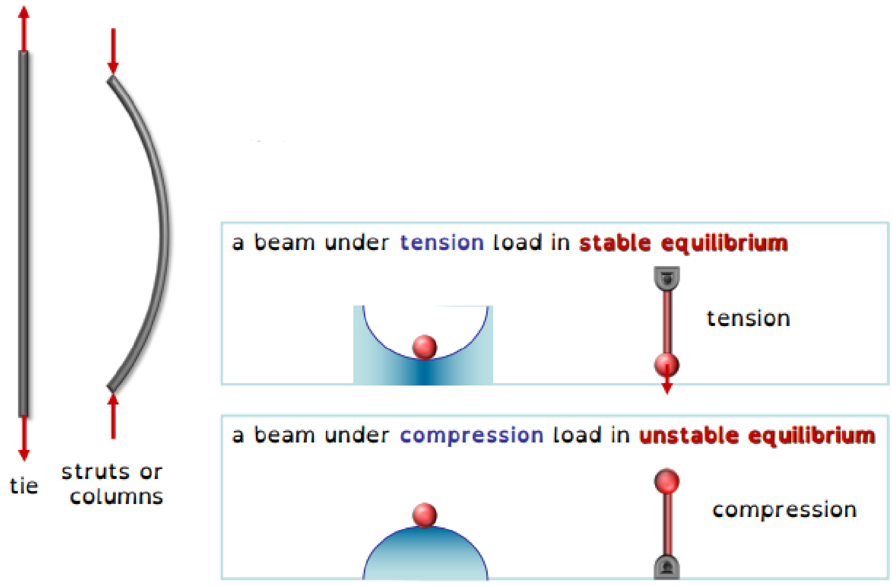


Figure 5.1:

Buckling

This phenomenon, that produce the sudden bow of long struts, is known as instability or buckling, as the member is said to buckle. It is essential that for the design of safe structures we have to be able to predict if a member under compressive load would buckle.

Figure 5.2:

5.1 Euler's analysis of buckling

5.1.1 Euler buckling theory

Assumptions:

- The strut is ideal and perfectly straight when unloaded
- Axial load is applied at the centroid of cross-section
- The dimensions of the section are much smaller than the strut length
- All main assumptions of bending theory apply
- The beam material is perfectly homogeneous and isotropic
- The elastic limit is nowhere exceeded
- Young's Modulus for the material is the same in tension and compression
- Plane cross-sections remain plane before and after bending
- Every cross-section in the beam is symmetrical about the plane of bending

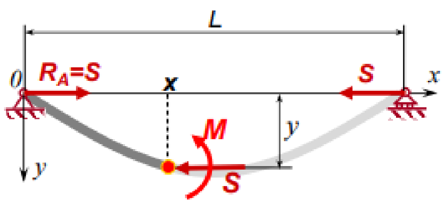


Figure 5.3:

Internal forces:

$$x)\ P = -S \tag{5.1}$$
$$X)\ M = S \cdot y \tag{5.2}$$

From the theory of bending:

$$\frac{d^2 y}{dx^2} = -\frac{1}{EI}M = -\frac{S}{EI}y \tag{5.3}$$

$$\frac{d^2 y}{dx^2} + \frac{S}{EI}y = 0 \tag{5.4}$$

To solve a second order differential equation, we have to:

1. find the complementary function y_{CF}
2. find the particular integral y_{PI}
3. the general solution is given by $y = y_{CF} + y_{PI}$
4. find the particular solution using boundary conditions

For differential equations in the form:

$$\frac{d^2 y}{dx^2} + \alpha^2 y = f(x) \tag{5.5}$$

y_{CF} has standard solution:

$$y_{CF} = A\sin\alpha x + B\cos\alpha x \tag{5.6}$$

Then, using substitution:

$$\alpha^2 = \frac{S}{EI} \tag{5.7}$$

This becomes:

$$\frac{d^2 y}{dy^2} + \alpha^2 y = 0 \tag{5.8}$$

With standard solution:

$$y_{CF} = A\sin\alpha x + B\cos\alpha x \tag{5.9}$$

Since $f(x) = 0$, the particular integral is:

$$y_{PI} = 0 \tag{5.10}$$

Therefore, the complementary function corresponds to the general solution:

$$y = A\sin\alpha x + B\cos\alpha x \tag{5.11}$$

Boundary conditions:

$$\begin{cases} y(0) = 0 \\ y(L) = 0 \end{cases} \tag{5.12}$$

First boundary condition:

$$y(0) = A\sin 0 + B\cos 0 \tag{5.13}$$

$$\therefore B = 0 \tag{5.14}$$

Our equation becomes:

$$y = A \sin \alpha x \tag{5.15}$$

Second boundary condition:

$$y(L) = A \sin \alpha L = 0 \tag{5.16}$$

Equations accepts infinite solutions:

$$\begin{cases} A = 0 \\ \alpha L = n\pi \text{ with } n = 0,\ 1,\ 2\,, \ldots \end{cases} \tag{5.17}$$

Trivial solutions:

$$A = 0 \rightarrow y(x) = 0 \text{ no buckling} \tag{5.18}$$

$$\alpha L = 0 \rightarrow y(x) = 0 \text{ no buckling} \tag{5.19}$$

$$\alpha^2 = \frac{S}{EI} \rightarrow \alpha L = \sqrt{\frac{S}{EI}} \cdot L = 0 \rightarrow \begin{cases} L = 0 \\ S = 0 \\ E = \infty \\ I = \infty \end{cases} \tag{5.20}$$

Solutions of interest:

$$\text{Solution } \alpha L = \pi \ (n = 1) \tag{5.21}$$

$$\rightarrow y(x) = A \sin\left(\frac{\pi}{L} x\right) \tag{5.22}$$

$$\alpha^2 = \frac{S}{EI} \rightarrow \alpha L = \sqrt{\frac{S}{EI}} \cdot L = \pi \tag{5.23}$$

Critical load (mode 1):

$$S_{cr} = \frac{\pi^2 EI}{L^2} \tag{5.24}$$

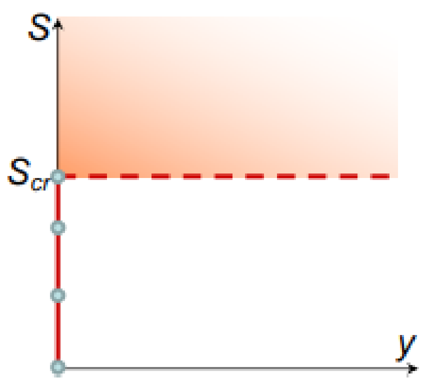


Figure 5.4:

Critical load (higher modes)

$$S_{cr} = \frac{n^2 \pi^2 EI}{L^2} \tag{5.25}$$

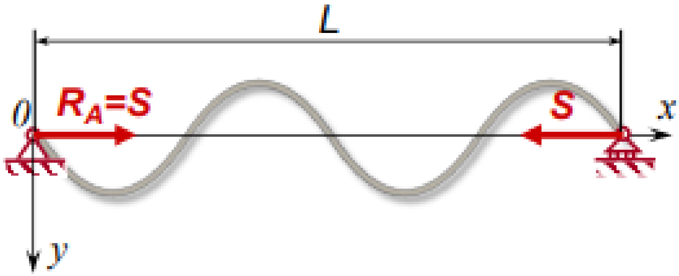


Figure 5.5:

5.1.2 Buckling modes

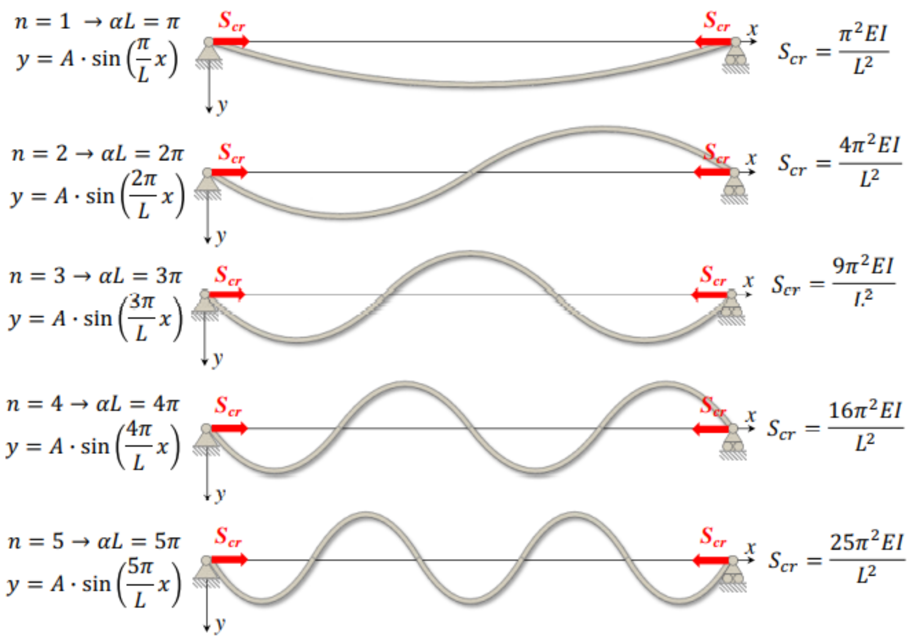


Figure 5.6:

The first buckling mode (n=1) is reached at the lowest critical load, and therefore is the one that occurs in normal conditions. Higher modes can be produced by applying external constrains at the points of contraflexure.

5.1.3 Stress in buckling

For the equilibrium of internal forces, stresses at each section will have to balance:

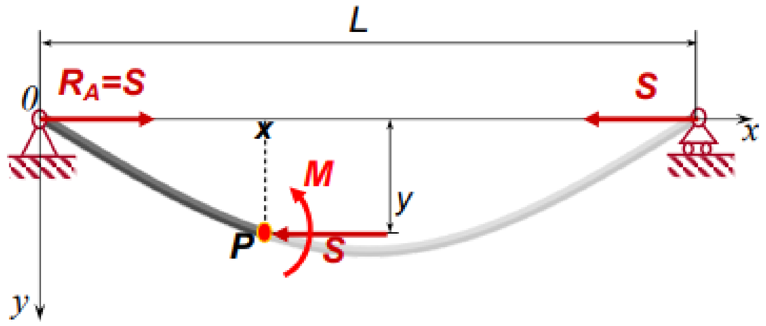


Figure 5.7:

Normal force contribution:

$$\sigma'_x = -\frac{S}{A} \tag{5.26}$$

Where A is the area of the section.

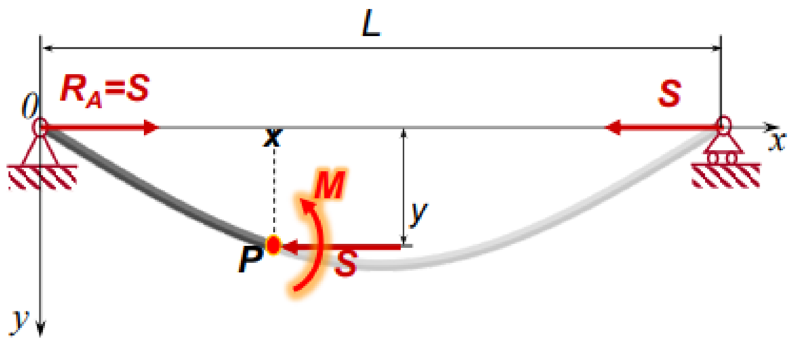


Figure 5.8:

Bending moment contribution:

$$\sigma''_x = \frac{M}{I}h = \frac{Sh}{I}y \tag{5.27}$$

Where I is the area of the section.

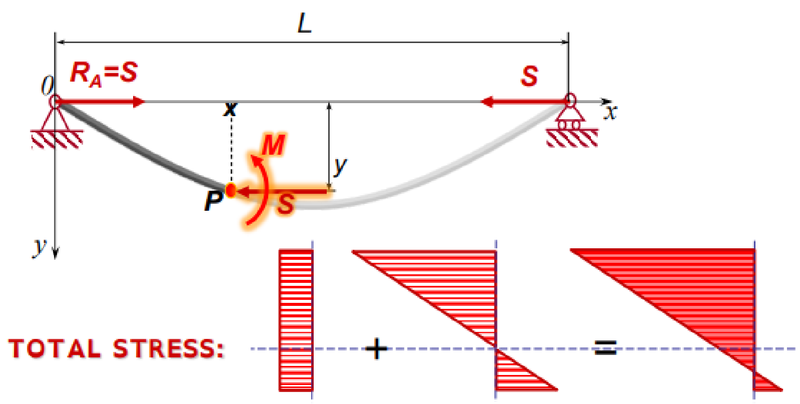


Figure 5.9:

$$\sigma_x = \sigma_x' + \sigma_x'' = -\frac{S}{A} + \frac{Sh}{I}y = S\left(\frac{h}{I}y - \frac{1}{A}\right) \tag{5.28}$$

5.1.4 Struts with other end conditions

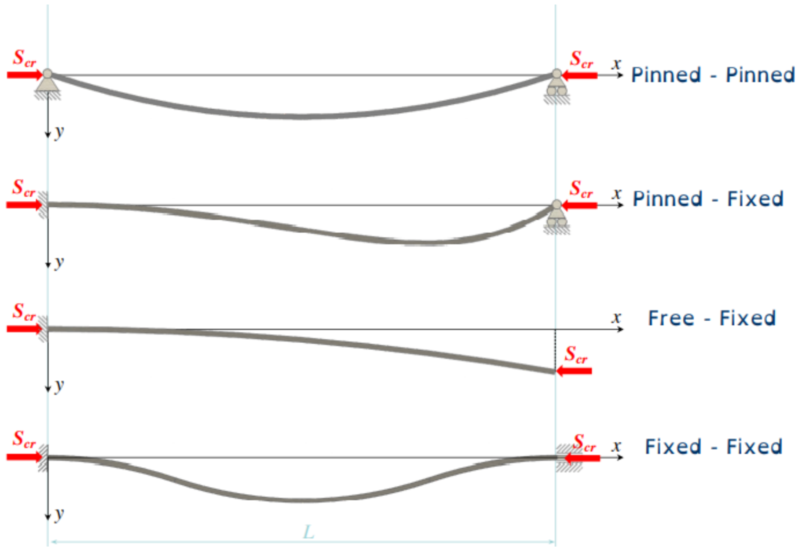


Figure 5.10:

5.1.5 Effective length

Expressions for the critical load for each end condition can be calculated adopting the same approach as for the pinned-pinned case. The form of expression is identical irrespective of the end conditions.

$$S_{cr} = \frac{\pi^2 EI}{L_e^2} \tag{5.29}$$

Where L_e is the **effective length**.

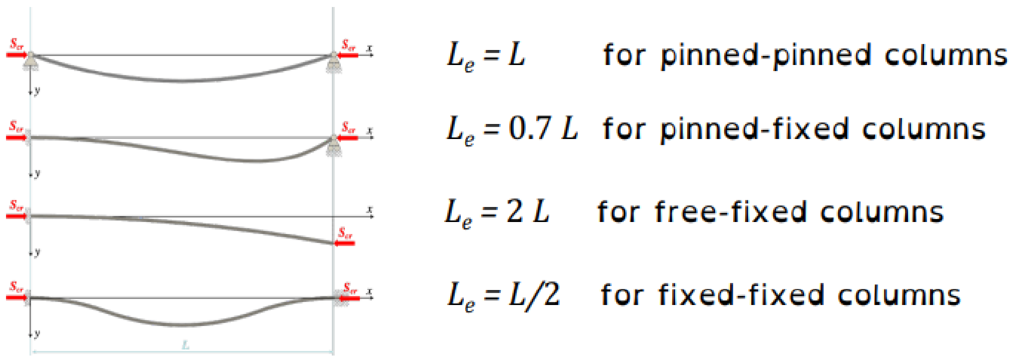


Figure 5.11:

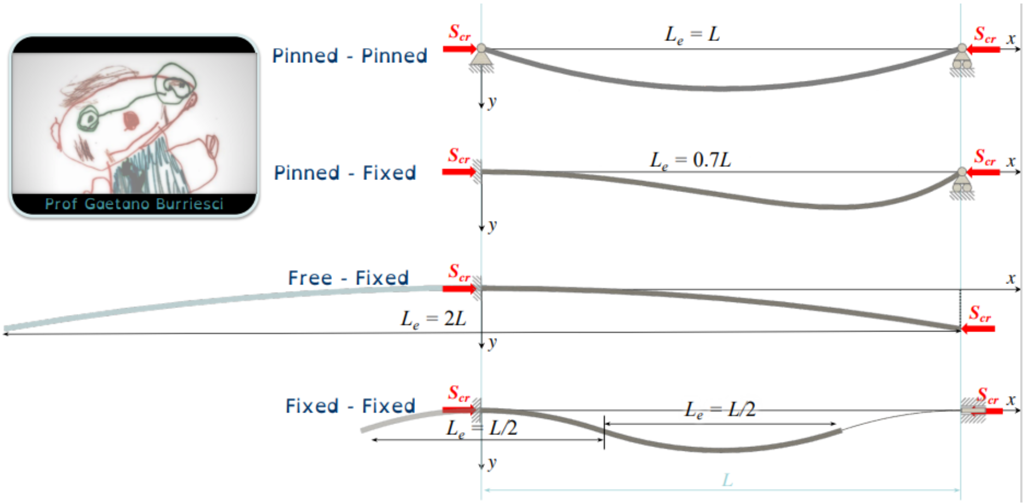


Figure 5.12:

5.1.6 Design against buckling

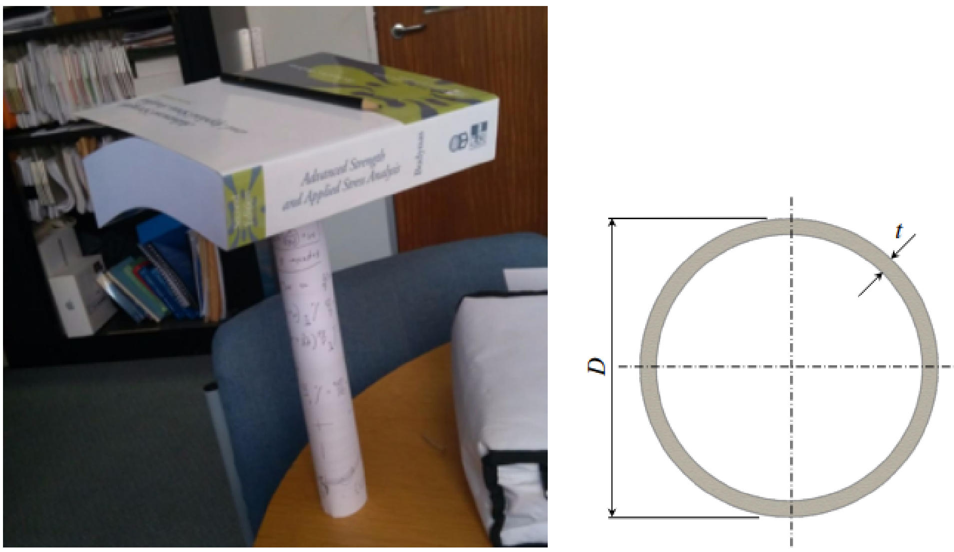


Figure 5.13: Hollow sections provide a greater critical load for a given cross-sectional area.

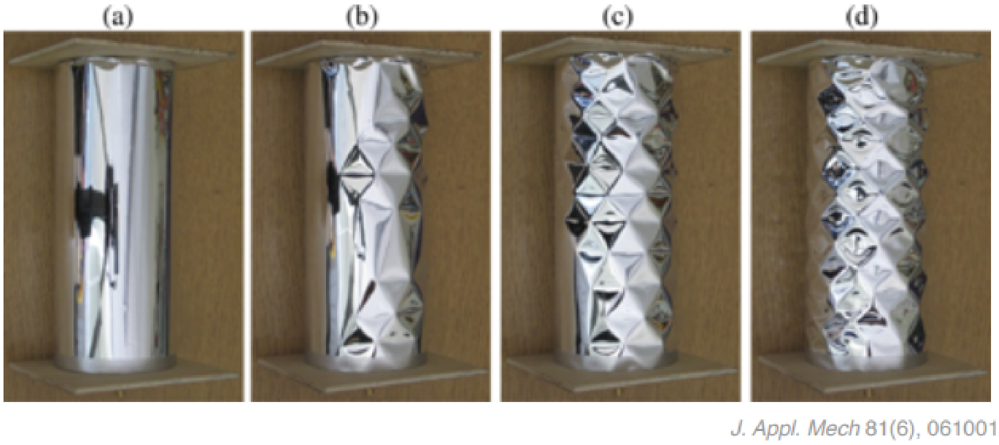


Figure 5.14: If the $\frac{D}{t}$ ration is too high, then local instability in the wall will occur.

Structural sections are often made to have a preferred failure plane: buckling occurs in the plane of least second moment of area of the cross-section.

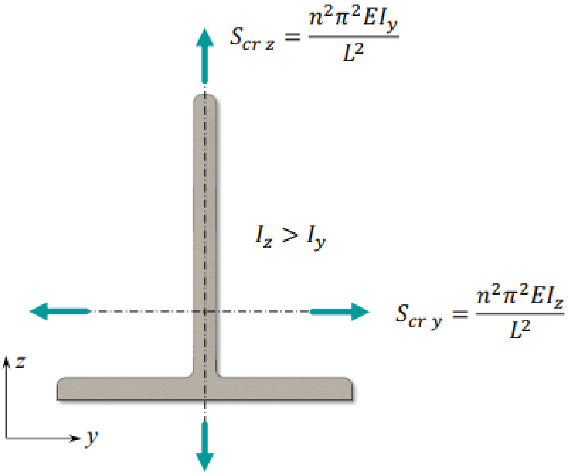


Figure 5.15:

Additional supports shorten the effective length of the column (structure has to buckle following higher critical loads modes).

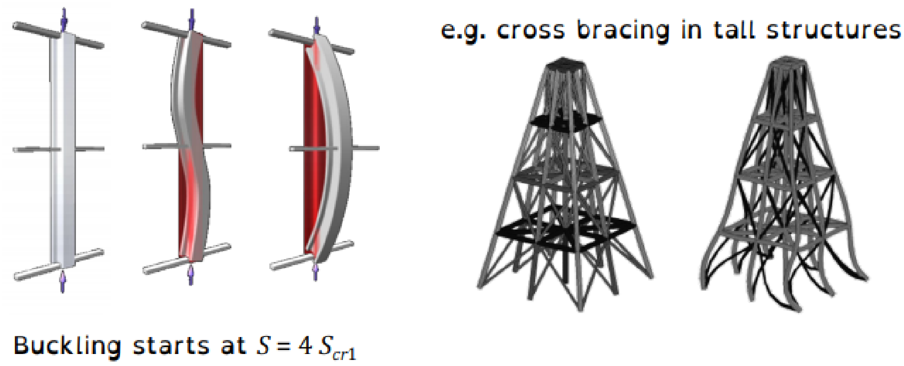

e.g. cross bracing in tall structures
Buckling starts at $S = 4\,S_{cr1}$

Figure 5.16:

Chapter 6

Principle stress and failure criteria

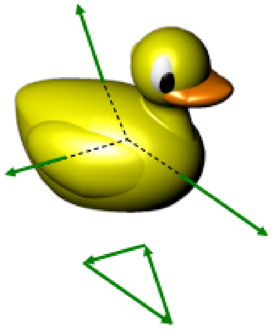
6.1 Plane stress state

6.1.1 State of equilibrium

The basis of structural analysis is the **equilibrium state**:

> If a configuration is in equilibrium, the resultant of all external forces and moments is zero.

This can be expressed mathematically in the following six equations:



$$\begin{cases} \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases} \\ \begin{cases} \sum M_x = 0 \\ \sum M_y = 0 \\ \sum M_z = 0 \end{cases} \end{cases} \tag{6.1}$$

Figure 6.1:

These equations have to be valid for or any portion of the body.

6.1.2 Stress components

The force distribution, in a generic point of a section, will have components in the *normal* and *tangential* direction.
If a Cartesian reference system is fixed, with the x direction normal to the section, the normal and shear stresses can be expressed in the coordinate system.
The stress can be decomposed into *normal component* σ_x; and two *shear components* τ_{xy} and τ_{xz}.

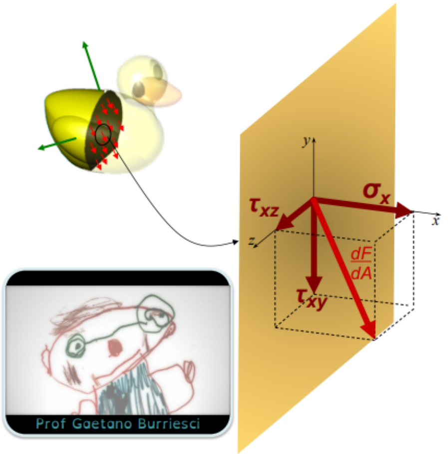


Figure 6.2:

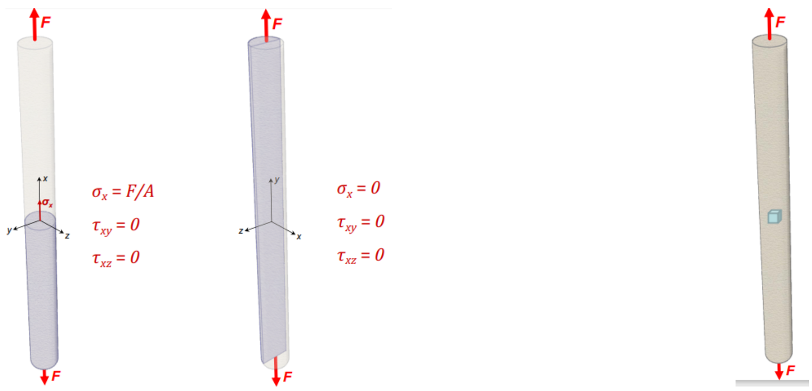


Figure 6.3:

Stress components associated with a specific direction are not sufficient to describe the stress state at one point, as they depend on the selected reference system. To define the stress state at one point, we need to consider all surfaces surrounding the point.

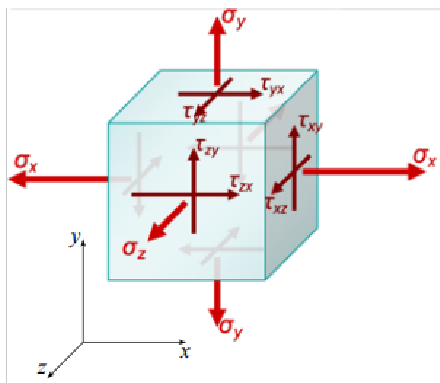


Figure 6.4:

Extract from the body an infinitesimal element, sectioned along the defined Cartesian planes, of dimensions dx, dy, dz. The *stress state* of the point where the element is extracted can be described by the 18 stress components (3 per each face):

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \qquad \begin{aligned} \tau_{xy} &= -\tau_{yx} \\ \tau_{xz} &= -\tau_{zx} \\ \tau_{yz} &= -\tau_{zy} \end{aligned} \tag{6.2}$$

Equilibrium of forces and moments reduces the independent components to six. The complete *state of stresses* at a point is defined by the six stress components:

$$\sigma_x,\ \sigma_y,\ \sigma_z;\ \tau_{xy},\ \tau_{yz},\ \tau_{zx} \tag{6.3}$$

Bidimensional case

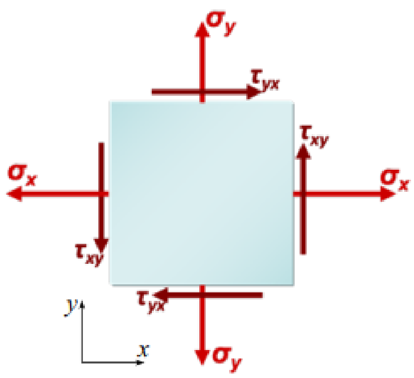


Figure 6.5:

In the common cases of plane state of stress (all stress components lay on a plane), the entire state of stress can be defined by only three stress components:

$$\sigma_x,\ \sigma_y;\ \tau_{xy} \qquad \sigma = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \tag{6.4}$$

6.1.3 Normal and shear stress in a plane σ & $\tau @ \theta$

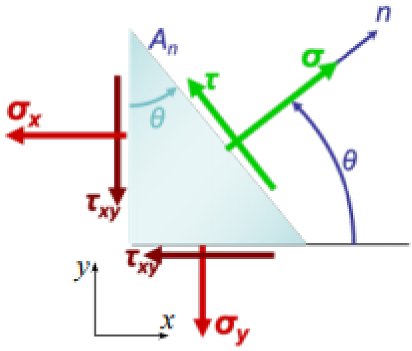


Figure 6.6:

Imagine to cut the infinitesimal element with a a plane parallel to z, with normal n at an arbitrary angle θ from x. Due to equilibrium, the new section of area A_n will be characterised by normal and shear stresses σ and τ.

Normal component of the stress (equilibrium along n)

$$\sigma \cdot A_n \tag{6.5}$$
$$-\sigma_x (A_n \cos\theta)\cos\theta - \tau_{xy}(\cos\theta)\sin\theta \tag{6.6}$$
$$-\sigma_y (A_n \sin\theta)\sin\theta - \tau_{xy}(A_n \sin\theta)\cos\theta \tag{6.7}$$
$$= 0 \tag{6.8}$$
$$\sigma = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta \cdot \cos\theta \tag{6.9}$$

Shear component of the stress (equilbrium along normal to n in the plane)

$$\tau \cdot A_n \tag{6.10}$$
$$+\sigma_x (A_n \cos\theta)\sin\theta - \tau_{xy}(A_n \cos\theta)\cos\theta \tag{6.11}$$
$$-\sigma_y (A_n \sin\theta)\cos\theta + \tau_{xy}(A_n \sin\theta)\sin\theta\sin\theta \tag{6.12}$$
$$= 0 \tag{6.13}$$
$$\tau = -\left(\sigma_x - \sigma_y\right)\sin\theta \cdot \cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right) \tag{6.14}$$

Normal and shear component of the stress

Since:

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \tag{6.15}$$
$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \tag{6.16}$$
$$\sin\theta\cdot\cos\theta = \frac{1}{2}\sin 2\theta \tag{6.17}$$

Expressions of σ and τ become:

$$\sigma = \frac{1}{2}\sigma_x(1+\cos 2\theta) + \frac{1}{2}\sigma_y(1-\cos 2\theta) + \tau_{xy}\sin 2\theta \tag{6.18}$$
$$\tau = -\left(\sigma_x-\sigma_y\right)\left(\frac{1}{2}\sin 2\theta\right) + \frac{1}{2}\tau_{xy}\left[(1+\cos 2\theta)-(1-\cos 2\theta)\right] \tag{6.19}$$

Further simplification:

$$\sigma = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{6.20}$$
$$\tau = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{6.21}$$

Given a specific state of stress, at a point P, these expressions give the normal and tangential stress to any plane passing through the point.

6.2 Principal stress

6.2.1 Maximum and minimum stress

σ and τ vary as the selected plane changes inclination. The maximum and minimum of σ when:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta} = 0 \rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}\theta} = -\left(\sigma_x-\sigma_y\right)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0 \tag{6.22}$$
$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x-\sigma_y} \tag{6.23}$$

The function $\tan 2\theta$ defines 2 orientations in the range 0-360° with 90° inclination with respect to each other. The max and min normal stresses are called **principal stresses** (respectively σ_1 and σ_2) and their planes **principal planes**. The minimum of the tangential shear stresses are zero on the **principal planes**.

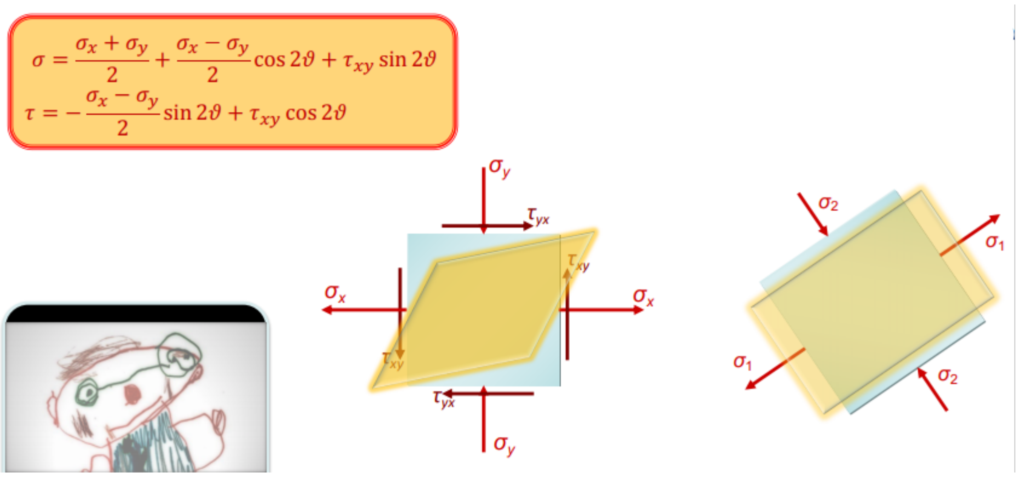


Figure 6.7:

6.3 Mohr's circles

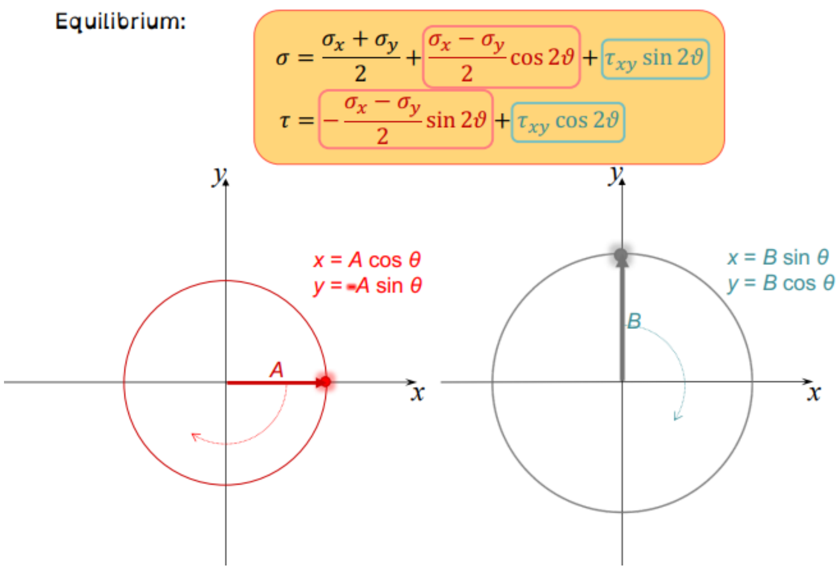


Figure 6.8:

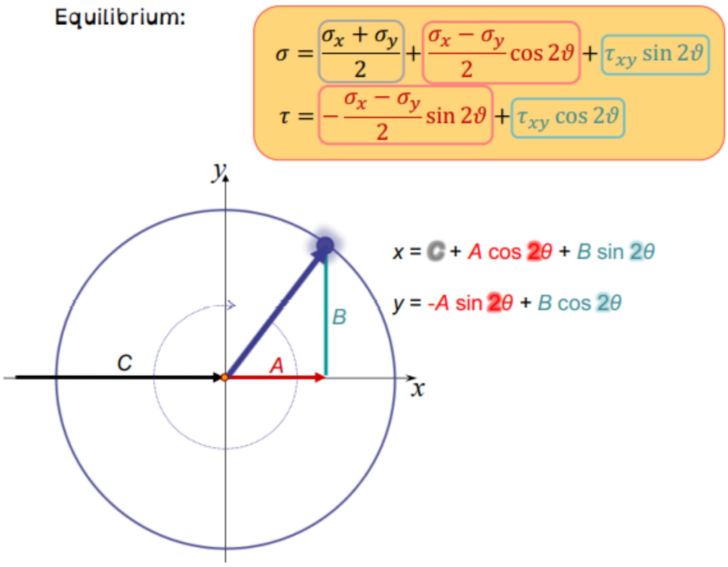


Figure 6.9:

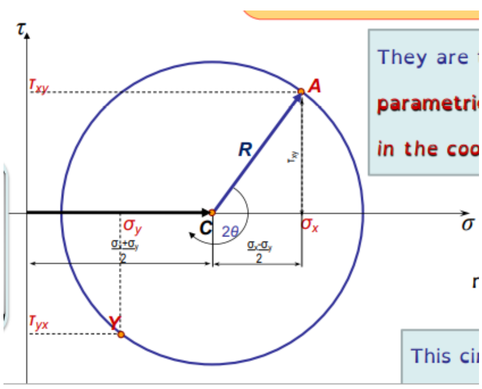


Figure 6.10:

Mohr's circles are the equations of a circle in parametric form (with parameter θ, in the coordinates σ and τ).

$$\text{centre: } C \equiv \left(\frac{\sigma_x + \sigma_y}{2}, 0\right) \tag{6.24}$$

$$\text{radius: } R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{6.25}$$

Mohr's circles give a graphical representation of the state of stress in one point, describing the normal and tangent component of the stress for any plane passing through the point. The points on the Mohr's circles represent combinations of normal and shear stress that exist at all possible orientations.

Note that angles in Mohr's circles are double angles and their direction is opposite to the physical one.

This can be fixed by inverting the τ axis.

6.3.1 Construction of Mohr's circles - A

The knowledge of the tensors, σ_x, σ_y and τ_{xy} relative to any plane is sufficient to build the Mohr's circle:

1. Plot the coordinates σ_x and τ_{xy} (relative to position 'X') in the $\sigma - \tau$ plane. This is along the x-axis and then corresponds to the angle $\theta = 2\theta = 0$.

2. Plot the coordinates σ_y and τ_{yx} (relative to position 'Y') in the $\sigma - \tau$ plane. This is along the y-axis and corresponds to the angle $\theta = 90°$ ($2\theta = 180°$).

3. The intersection of the line joining 'X' and 'Y' with σ-axis identifies the centre C of the circle (C has abscissa equal to the average of σ_x and σ_y).

4. Use 'C' and 'X' (or 'Y') to build the circle.

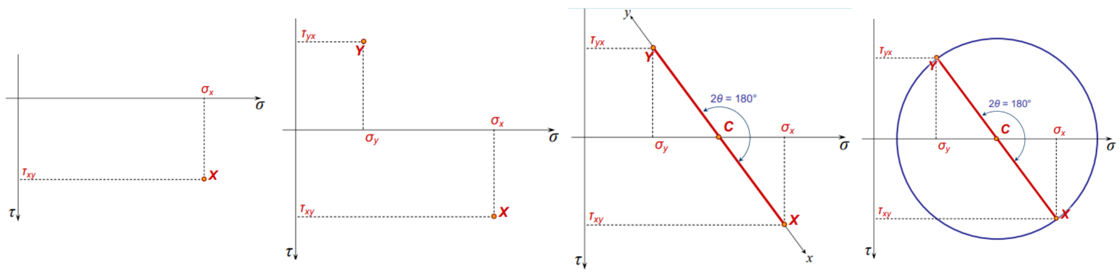


Figure 6.11:

6.3.2 Construction of Mohr's circles - B

The knowledge of the tensors, σ_x, σ_y and τ_{xy} relative to any plane is sufficient to build the Mohr's circle:

1. Determine the centre C of the circle from: $C \equiv \left(\frac{\sigma_x+\sigma_y}{2}, 0\right)$.

2. Determine the radius R of the circle from: $R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$

3. Draw a circle of centre C and radius R.

4. Position on the circle points relative to 'X' (and 'Y') to identify x-axis.

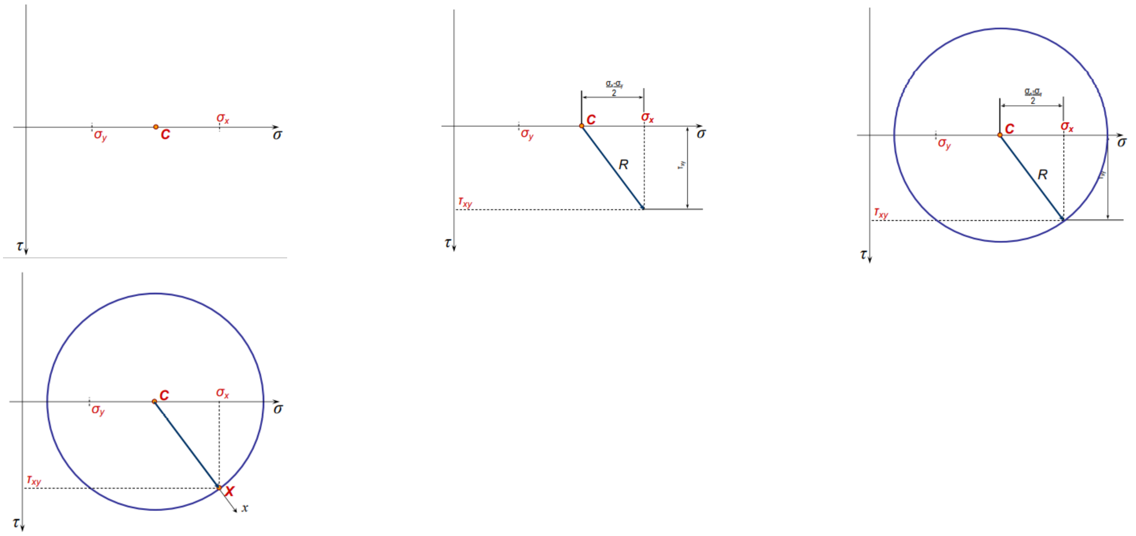


Figure 6.12:

6.4 Application of Mohr's circles

6.4.1 Determination of principal stresses

The most important application of Mohr's circles is the geometrical determination of the value and direction of the principal stress.

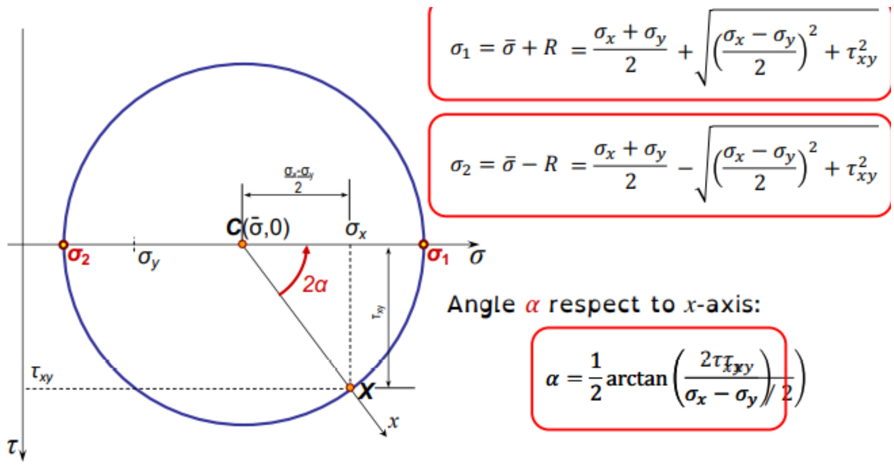


Figure 6.13:

6.4.2 Determination of max shear stress

Another Mohr's circle application is the geometrical determination of the value and direction of the maximum shear stresses. The *maximum shear stress* is always at an angle $\theta = 45^\circ\,(2\theta = 90^\circ)$ with the principal stress directions and has magnitude

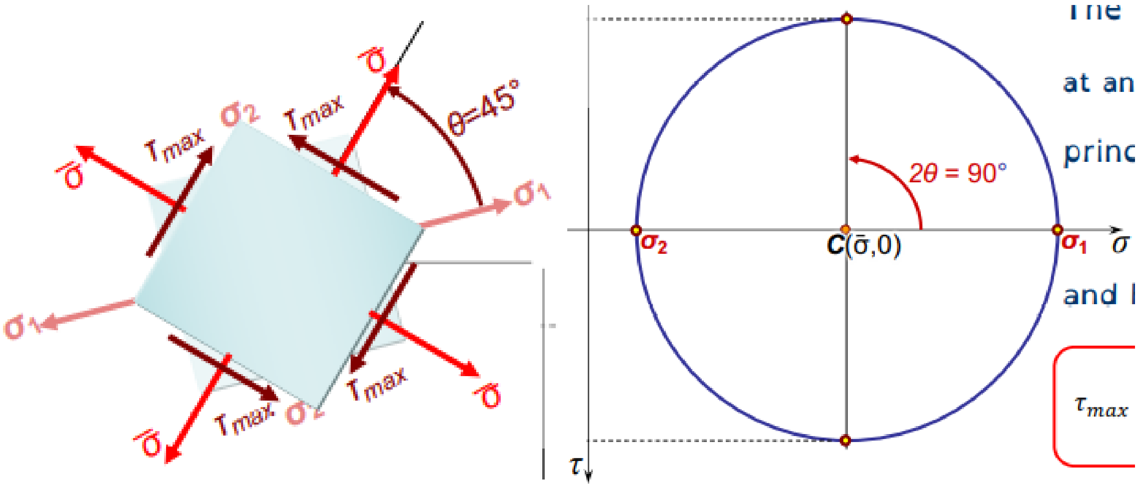


Figure 6.14:

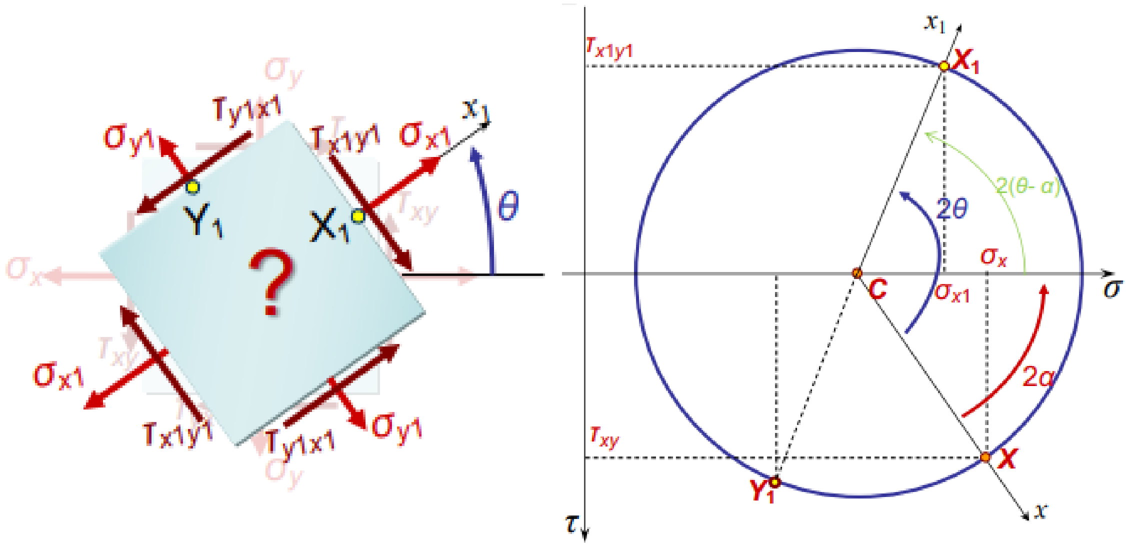


Figure 6.15:

equal to:

$$\tau_{max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_1 - \sigma_2}{2} \tag{6.26}$$

Mohr's circles allow to calculate the stress tensors for any plane.

$$\sigma_{x1} = \bar{\sigma} + R\cos 2(\theta - \alpha) \tag{6.27}$$

$$\sigma_{y1} = \bar{\sigma} - R\cos 2(\theta - \alpha) \tag{6.28}$$

$$\tau_{x1\,y1} = -R\sin 2(\theta - \alpha) \tag{6.29}$$

With $\bar{\sigma} = \frac{\sigma_x+\sigma_y}{2} = \frac{\sigma_1+\sigma_2}{2}$ and $R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_1-\sigma_2}{2}$.

6.5 Tri-axial state of stress

6.5.1 Plane stress state

We have seen that if we only consider two stresses in the plane, the state of stress is defined by 3 stress components (tensors).

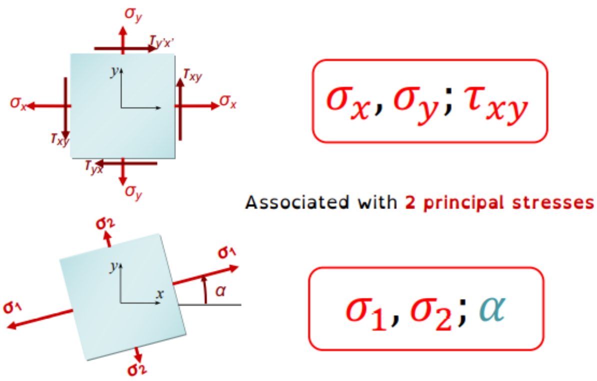


Figure 6.16:

6.5.2 Tri-axial stress state

In the most generic state of stress (tri-axial) is defined by 6 stress components (tensors):

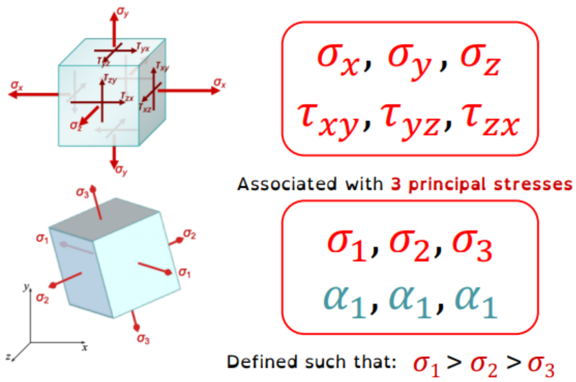


Figure 6.17:

6.5.3 Complete Mohr's circles

Mohr's representation will be characterised by 3 circles, relative to the three orthogonal principal planes:

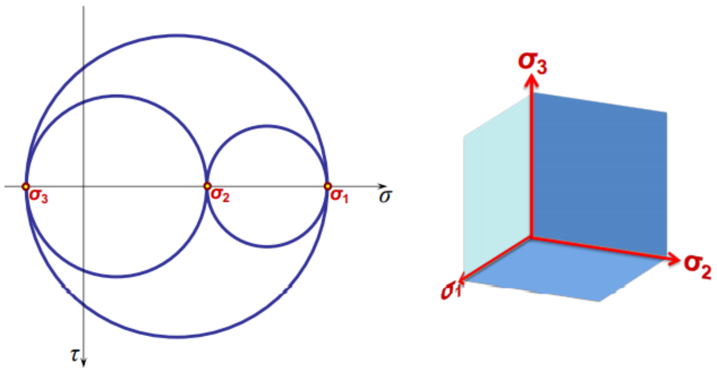


Figure 6.18:

Maximum shear stresses in the three principal planes will be:

$$(\tau_{max})_3 = \frac{|\sigma_1 - \sigma_2|}{2} \qquad (\tau_{max})_1 = \frac{|\sigma_2 - \sigma_3|}{2} \qquad (\tau_{max})_2 = \frac{|\sigma_3 - \sigma_1|}{2} \tag{6.30}$$

Absolute maximum equal to the largest value.

$$\tau_{max} = (\tau_{max})_2 = \frac{\sigma_1 - \sigma_3}{2} \tag{6.31}$$

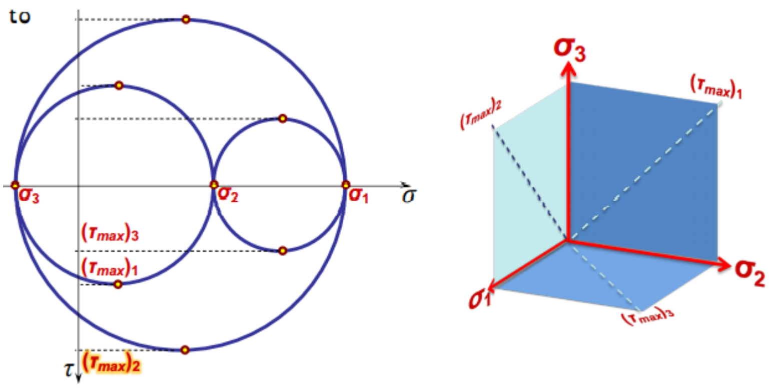


Figure 6.19:

Also a plane state of stress ($\sigma_1 \neq 0$, $\sigma_2 \neq 0$, $\sigma_3 = 0$) is associated with three Mohr's circles. The maximum shear stress is not necessarily in the plane of the stresses that are different from zero!

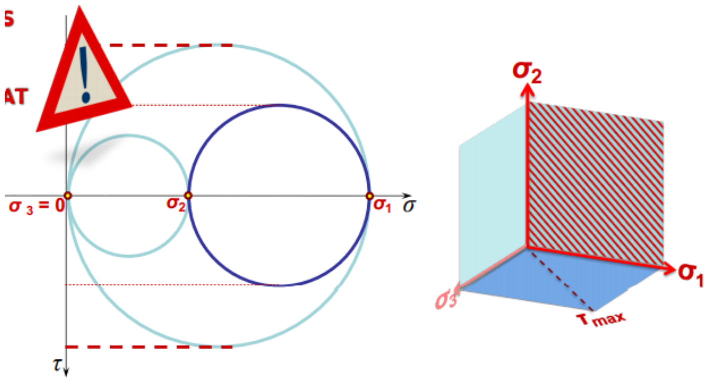


Figure 6.20:

6.6 Stress strain relations

6.6.1 Principal strains

In the case of an isotropic, linear-elastic material, the *principal strains* are defined as the strains in the directions of the principal stresses.

$$\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3 \tag{6.32}$$

Principal stress and principal strains are linked through the Young's Modulus and the Poisson ratio (constitutive relations).

6.6.2 What is the Poisson's ratio? (ν)

The Poisson's Ratio is the negative ratio of the transverse to longitudinal strain:

$$\nu = -\frac{\text{transverse strain}}{\text{longitudinal strain}} = -\frac{\varepsilon_t}{\varepsilon_l} \tag{6.33}$$

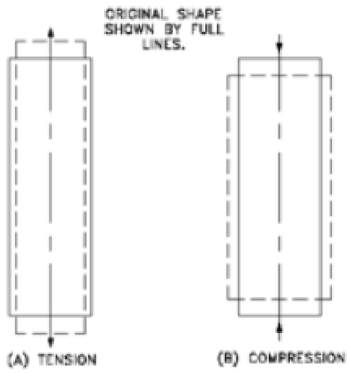


Figure 6.21:

6.6.3 Uni-axial case

$$\begin{array}{ll} \varepsilon_1 = \frac{\sigma_1}{E} & \sigma_1 = E\varepsilon_1 \\ \varepsilon_2 = -\frac{\nu}{E}\sigma_1 & \sigma_2 = 0 \\ \varepsilon_3 = -\frac{\nu}{E}\sigma_1 & \sigma_3 = 0 \end{array} \tag{6.34}$$

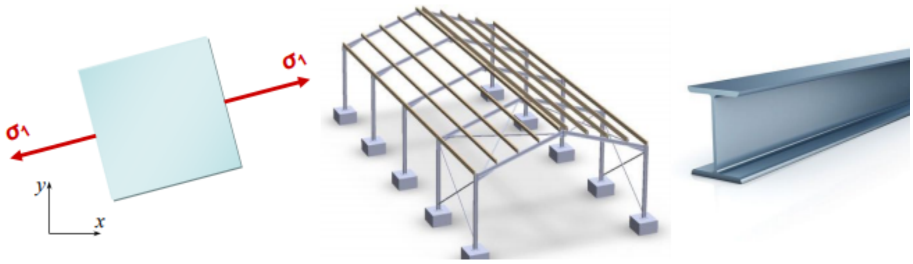


Figure 6.22:

6.6.4 Bi-axial case

$$\begin{array}{ll} \varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\sigma_2\right] & \sigma_1 = \frac{E}{1-\nu^2}\left[\varepsilon_1 + \nu\varepsilon_2\right] \\ \varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu\sigma_1\right] & \sigma_2 = \frac{E}{1-\nu^2}\left[\varepsilon_2 + \nu\varepsilon_1\right] \\ \varepsilon_3 = -\frac{\nu}{E}\left(\sigma_1 + \sigma_2\right) & \sigma_3 = 0 \end{array} \tag{6.35}$$

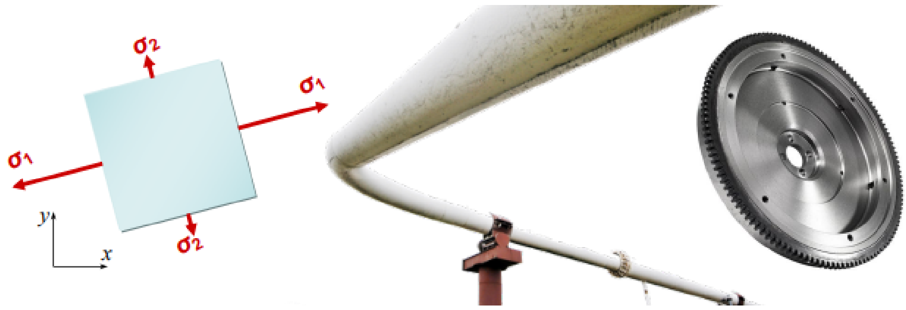


Figure 6.23:

6.6.5 Tri-axial case

$$\begin{array}{ll} \varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right] & \sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\varepsilon_1 + \nu\left(\varepsilon_2 + \varepsilon_3\right)\right] \\ \varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu\left(\sigma_3 + \sigma_1\right)\right] & \sigma_2 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\varepsilon_2 + \nu\left(\varepsilon_3 + \varepsilon_1\right)\right] \\ \varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right)\right] & \sigma_3 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\varepsilon_3 + \nu\left(\varepsilon_1 + \varepsilon_2\right)\right] \end{array} \tag{6.36}$$

Figure 6.24: