UCL Mechanical Engineering 2021/2022
MECH0026 Coursework One

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Let us analyse a case of uniaxial tension and use the concept of superposition to find our stress distribution.

The biharmonic equation written in polar coordinates is:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right)=0 \tag{2.1}$$

We know our boundary conditions which are:

$$\sigma_r,\, \tau_{r\theta}=0,\, r=a \tag{2.2}$$

$$\frac{\sigma_y}{\sigma_x}\rightarrow 2.5;\, r\rightarrow\infty \tag{2.3}$$

$$\tau_{xy}\rightarrow 0;\, r\rightarrow\infty \tag{2.4}$$

We can use separation of variables to obtain a solution of the form:

$$\phi = R(r)\Theta(\theta) \tag{2.5}$$

One solution is:

$$\phi=\left(Ar^2+Br^4\frac{C}{r^2}+D\right)\cos{(2\theta)} \tag{2.6}$$

Let us also consider:

$$\phi = A \ln r + Cr^2 \tag{2.7}$$

A linear combination of 2.6 and 2.7 yields a stress function that can be shown to represent the stress distribution in a large plate with a circular hole subjected to a uniform tensile stress σ. By inputting our boundary conditions we can come to the following stress function:

$$\phi = \frac{\sigma}{2}\left(2r^2 - a^2 \ln r\right) - \frac{\sigma}{4}\left(r^2 + \frac{a^4}{r^2} - 2a^2\right)\cos\left(2\theta\right) \tag{2.8}$$

Now we can find our stresses:

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{\sigma}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos\left(2\theta\right) \tag{2.9}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos\left(2\theta\right) \tag{2.10}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -\frac{\sigma}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin\left(2\theta\right) \tag{2.11}$$

Superimposing a tensile stress at a 90° angle simply involves adding a $\frac{\pi}{2}$ term to our θ components. Let us denote the stress in the vertical direction as σ_y and the stress in the horizontal direction as σ_x:

$$\begin{aligned}\sigma_r = \frac{\sigma_x}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_x}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos\left(2\theta\right) + \\ \frac{\sigma_y}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_y}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos\left(2\left(\theta + \frac{\pi}{2}\right)\right)\end{aligned} \tag{2.12}$$

$$\begin{aligned}\sigma_\theta = \frac{\sigma_x}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_x}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos\left(2\theta\right) + \\ \frac{\sigma_y}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_y}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos\left(2\left(\theta + \frac{\pi}{2}\right)\right)\end{aligned} \tag{2.13}$$

$$\tau_{r\theta} = -\frac{\sigma_x}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin\left(2\theta\right) - \frac{\sigma_y}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin\left(2\left(\theta + \frac{\pi}{2}\right)\right) \tag{2.14}$$

Simplifying and inputting our boundary conditions:

$$\sigma_r = \frac{\sigma_x + \sigma_y}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_x + \sigma_y}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos\left(2\theta\right) \tag{2.15}$$

$$\sigma_\theta = \frac{\sigma_x - \sigma_y}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_x - \sigma_y}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos\left(2\theta\right) \tag{2.16}$$

$$\tau_{r\theta} = \frac{\sigma_y - \sigma_x}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin\left(2\theta\right) \tag{2.17}$$

At $r = a$:

$$\sigma_r = 0 \tag{2.18}$$

$$\sigma_\theta = \sigma_x + \sigma_y - 2\left(\sigma_x - \sigma_y\right)\cos\left(2\theta\right) \tag{2.19}$$

$$\tau_{r\theta} = 0 \tag{2.20}$$

σ_θ takes a maximum value at $\theta = \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$:

$$\sigma_\theta = 3\left(\sigma_x + \sigma_y\right) \tag{2.21}$$