MECH0011 Topic Notes

UCL

HD

November 16, 2020

Contents

Part I

Fluids

Chapter 1

Differential equations of mass, momentum and energy

1.1 Differential analysis of fluid flow

What do we want to know? The velocity field, pressures, densities and temperature everywhere and anytime. Hence, these will be a function of (x, y, z, t).

List of variables

Variable	Type	Units
$\overrightarrow{U} = u\hat{i} + v\hat{j} + w\hat{k}$ $\overrightarrow{U} = u_1\hat{i}_1 + u_2\hat{i}_2 + u_3\hat{i}_3$	Velocity/Vector	$\mathrm{m\,s^{-1}}$
p	Pressure/Scalar	$\mathrm{N\,m^{-2}}$
T	Temperature/Scalar	°C
ρ	Density/Scalar	$\mathrm{kg\,m^{-3}}$
$T = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$	Stress Tensor	$\mathrm{N\,m^{-2}}$

Since we have 12 variables, we need 12 equations to describe the fluid!

From last year, we have our conservation of mass equation

$$\frac{\partial}{\partial t}\int_V \rho \,\mathrm{d}V + \oint_S (\rho\overrightarrow{V} \cdot \hat{n})\,\mathrm{d}S = 0$$

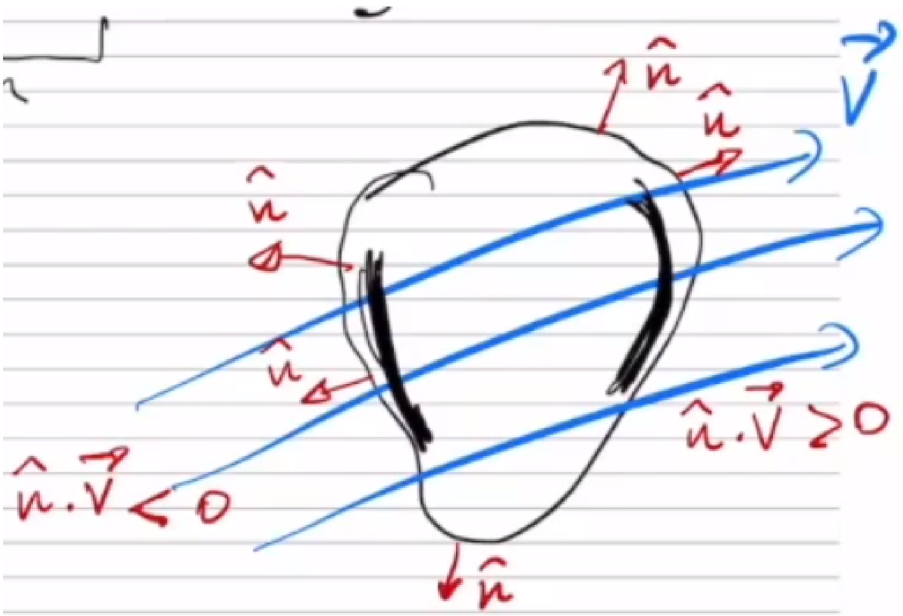


Figure 1.1: Consider $\hat{n}$ to be a vector coming out of the control volume. Depending on where $\hat{n}$ is, our dot product will either be greater than or less than 0.

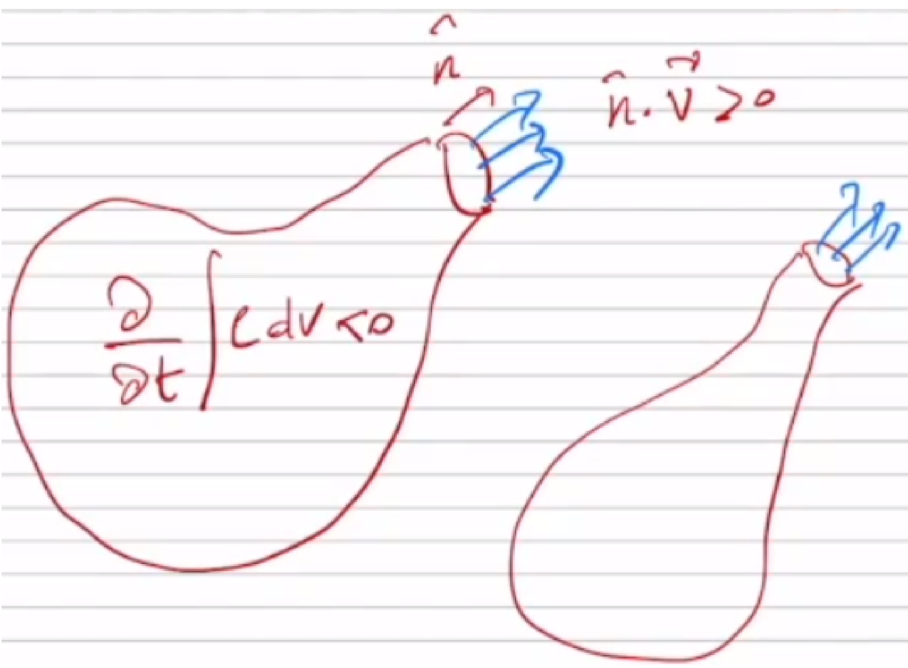
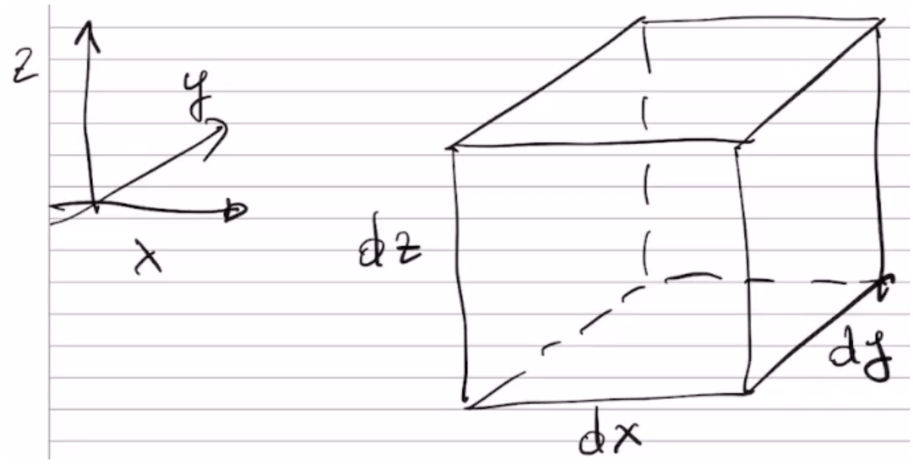


Figure 1.2: There is a velocity exiting the balloon. The amount of mass inside will decrease with time. The volume of the balloon will become smaller. This will be equal to the amount of mass which came out of the control volume (the balloon). If the second term of the continuity equation is positive, the first term must be negative.

1.2 Conservation of mass

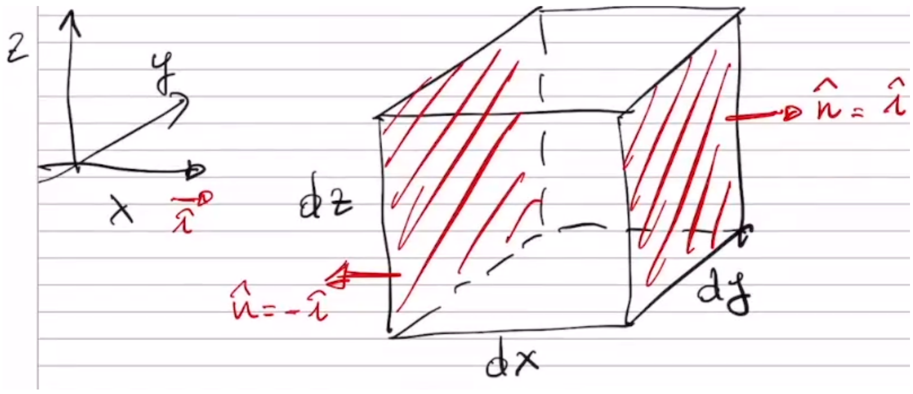
Let us consider an infinitesimally small cube:



Consider term 1 in our continuity equation - the mass variation inside the control volume.

$$\frac{\partial \rho}{\partial t} \cdot dV = \frac{\partial \rho}{\partial t} \cdot \mathrm{d}x \cdot \mathrm{d}y \cdot \mathrm{d}z$$

Consider term 2 - the contribution of mass from the sides of the cube, which are orthogonal to x, shown below.



Left side:

$$\rho \overrightarrow{V} \cdot \hat{n}\, \mathrm{d}S = \rho \overrightarrow{V} \cdot (-\hat{\imath})\, \mathrm{d}z\, \mathrm{d}y \tag{1.1}$$

$$= -\rho u\, \mathrm{d}z\, \mathrm{d}y \tag{1.2}$$

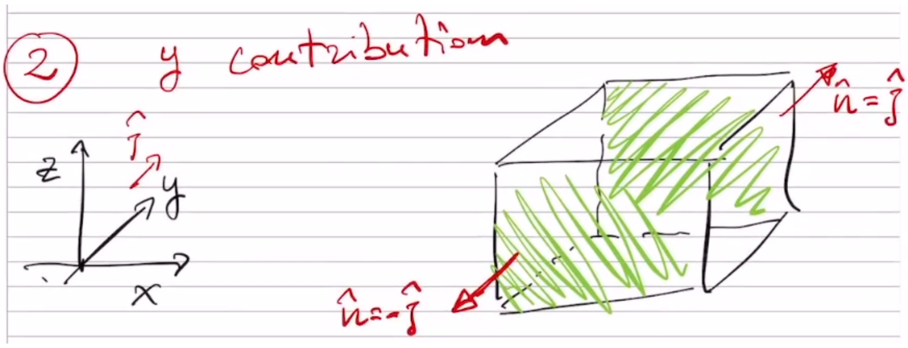
Looking from the right side, we will not have a negative (which is coming from the fact that the normal vector is going in the opposite direction to i)

$$(\rho u + \frac{\partial \rho u}{\partial x} \cdot \mathrm{d}x)\, \mathrm{d}z\, \mathrm{d}y \tag{1.3}$$

The net contribution from the orthogonal x direction is

$$= (\rho u + \frac{\partial \rho u}{\partial x} \cdot \mathrm{d}x)\,\mathrm{d}z \cdot \mathrm{d}y - \rho u\,\mathrm{d}z\,\mathrm{d}y \tag{1.4}$$
$$= \frac{\partial \rho u}{\partial x}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \tag{1.5}$$

y orthogonal contribution



Front side

$$\rho \overrightarrow{V} \cdot \hat{n}\,\mathrm{d}S = \rho \overrightarrow{V} \cdot (-\hat{j})\,\mathrm{d}x\,\mathrm{d}z \tag{1.6}$$
$$= -\rho v\,\mathrm{d}x\,\mathrm{d}z \tag{1.7}$$

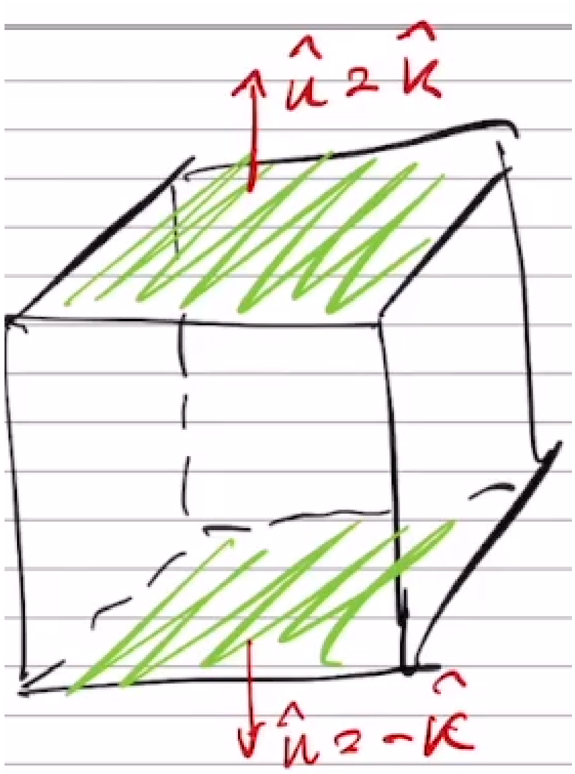
Back side

$$(\rho v + \frac{\partial \rho v}{\partial y} \cdot \mathrm{d}y)\,\mathrm{d}z\,\mathrm{d}x \tag{1.8}$$

Final contribution

$$\frac{\partial \rho v}{\partial y}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x \tag{1.9}$$

z orthogonal contribution



Bottom side

$$\rho\overrightarrow{V} \cdot \hat{n}\,\mathrm{d}S = \rho\overrightarrow{V} \cdot (-\hat{k})\,\mathrm{d}x\,\mathrm{d}y \tag{1.10}$$
$$= -\rho w\,\mathrm{d}x\,\mathrm{d}y \tag{1.11}$$

Top side

$$(\rho w + \frac{\partial \rho w}{\partial z} \cdot \mathrm{d}z)\,\mathrm{d}x\,\mathrm{d}y \tag{1.12}$$

Final contribution

$$\frac{\partial \rho w}{\partial z}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \tag{1.13}$$

If we add up all of the contributions above, we get the conservation of mass for an infinitesimal volume.

$$\frac{\partial \rho}{\partial t} \cdot \mathrm{d}x \cdot \mathrm{d}y \cdot \mathrm{d}z + \frac{\partial \rho u}{\partial x}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y + \frac{\partial \rho v}{\partial y}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x + \frac{\partial \rho w}{\partial z}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{1.14}$$

This simplifies to:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{1.15}$$

We can simplify this a bit more by introducing a term called the divergence.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\overrightarrow{V}) = 0 \tag{1.16}$$

Where $\nabla \cdot (\rho\overrightarrow{V})$ is the divergence of the vector $\rho\overrightarrow{V}$. It is a scalar.

$$\frac{\partial \rho}{\partial t} + \overrightarrow{V} \cdot \nabla\rho + \rho\nabla \cdot \overrightarrow{V} = 0 \tag{1.17}$$

$\nabla \cdot \overrightarrow{V}$ is the divergence of the vector $\overrightarrow{V}$ and it is a scalar. $\nabla \rho$ is the gradient of the density ρ and is a vector. It can be expanded as:

$$\frac{\partial \rho}{\partial x}\hat{i} + \frac{\partial \rho}{\partial y}\hat{j} + \frac{\partial \rho}{\partial z}\hat{k} \tag{1.18}$$

$$\overrightarrow{V} \cdot \rho\nabla = (u\hat{i} + v\hat{j} + w\hat{k}) \cdot \left(\frac{\partial \rho}{\partial x}\hat{i} + \frac{\partial \rho}{\partial y}\hat{j} + \frac{\partial \rho}{\partial z}\hat{k}\right) = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \tag{1.19}$$

$$\rho\nabla \cdot \overrightarrow{V} = \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{1.20}$$

For steady flow:

$$\frac{\partial \rho}{\partial t} = 0 \rightarrow \nabla \cdot (\rho\overrightarrow{V}) = 0 \tag{1.21}$$

[H] For incompressible flow, the density is constant. This means all derivatives of ρ are 0. Hence, our equation reduces to:

$$\rho = \text{const} \rightarrow \nabla \cdot \overrightarrow{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1.22}$$

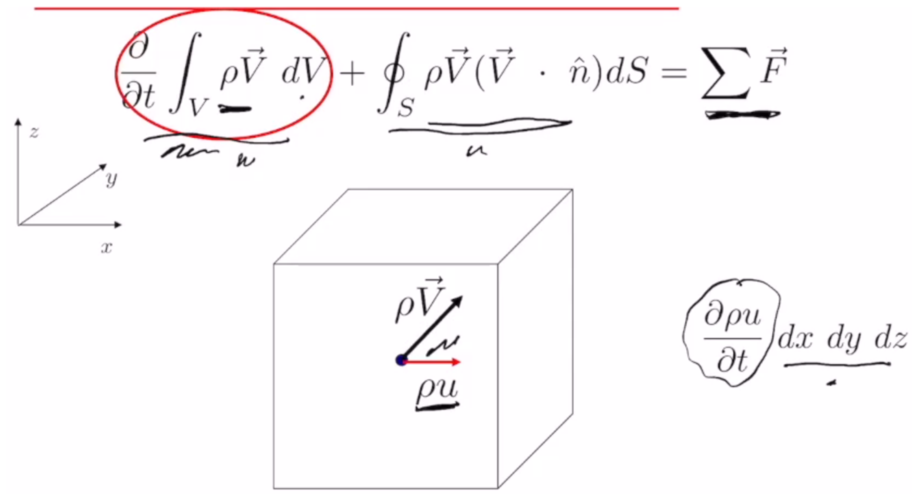
Each of these derivatives represent the stretch or compression of the fluid particle in the orthogonal direction. When these are all added up, it gives the variation in volume. If this is positive, it shows that the volume has increased with time. If ρ is constant, then the volume cannot change. Which is why equation (1.22) must equal 0.

1.3 Conservation of momentum

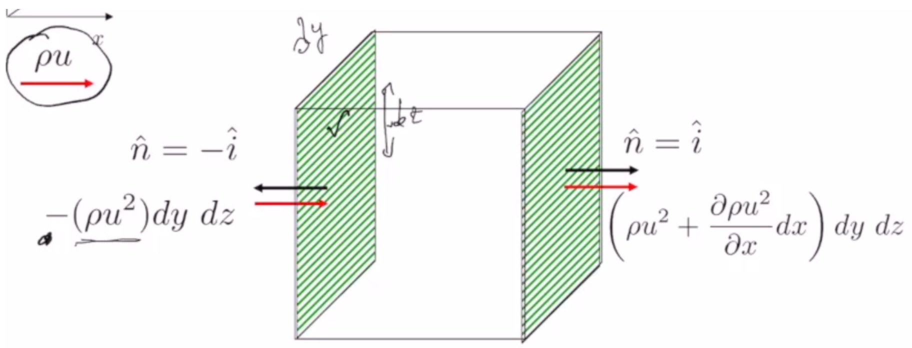
$$\frac{\partial}{\partial t}\int_V \rho\overrightarrow{V}\,\mathrm{d}V + \oint_S \rho\overrightarrow{V}(\overrightarrow{V} \cdot \hat{n})\,\mathrm{d}S = \sum \overrightarrow{F} \tag{1.23}$$

We have two types of external force that can act on our infinitesimal fluid element, **volumetric** forces (e.g. gravity) and **surface** forces (shear, pressure).

Momentum is a vector as we have to take into account momentum in 3D (first term). We want to know how these change with time. The second term looks at the flux of momentum through the sides of the control volume. In this example, we are only looking at the momentum in the x direction.

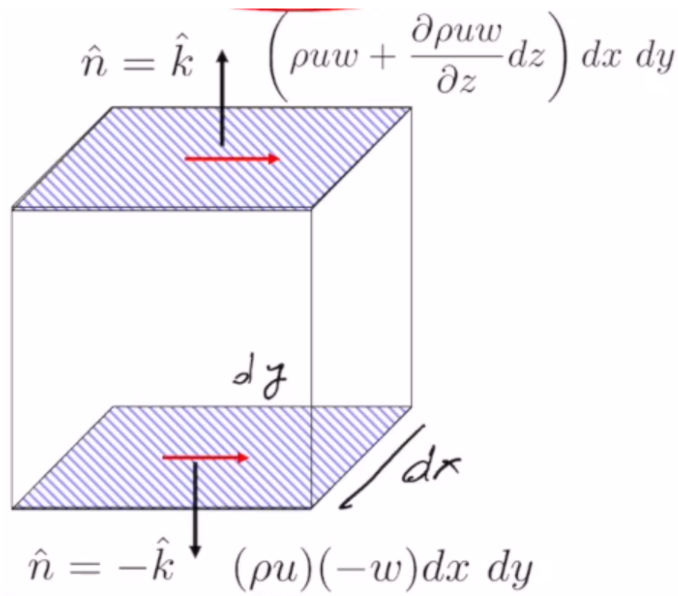


Where $\frac{\partial \rho u}{\partial t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is the momentum in x.



Adding the left and the right sides, we get

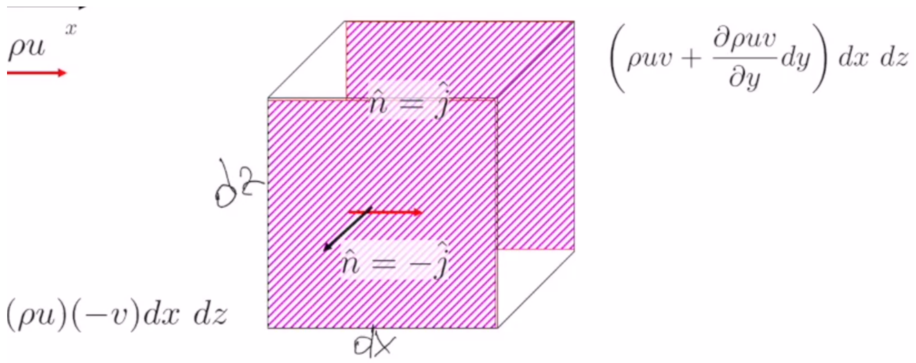
$$\rho u(\overrightarrow{V}\cdot\hat{n})\,\mathrm{d}S = \frac{\partial(pu^2)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.24}$$



Adding the net contribution to our equation:

$$\rho u(\overrightarrow{V}\cdot\hat{n})\,\mathrm{d}S = \frac{\partial(\rho u^2)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \frac{\partial(\rho u w)}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.25}$$

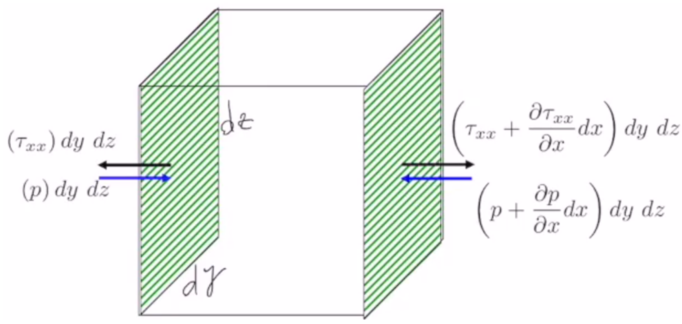
y orthogonal



Adding the net contribution to our equation:

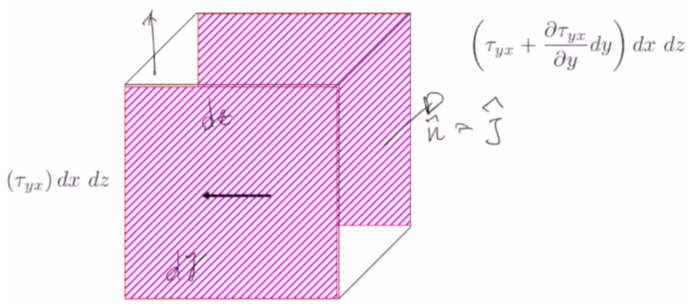
$$\rho u(\overrightarrow{V}\cdot\hat{n})\,\mathrm{d}S = \frac{\partial(\rho u^2)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \frac{\partial(\rho u w)}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \frac{\partial \rho u v}{\partial y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.26}$$

Let us look at the $\sum \overrightarrow{F}$ term. Pressure is always exerted orthogonal to a face. We also have our τ stresses acting orthogonally



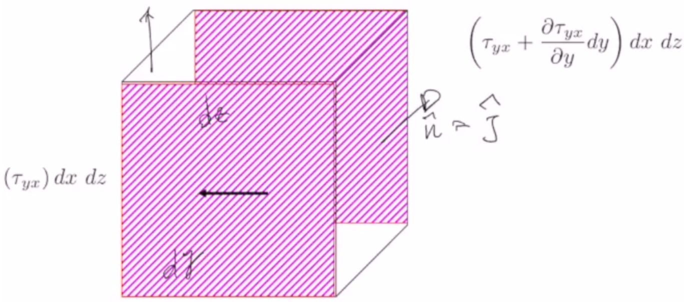
$$\sum F_x = -\left(\frac{\partial p}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \left(\frac{\partial \tau_{xx}}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.27}$$

z orthogonal shear force.



$$\sum F_x = -\left(\frac{\partial p}{\partial x}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial \tau_{xx}}{\partial x}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial \tau_{zx}}{\partial z}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{1.28}$$

y orthogonal shear force z orthogonal shear force.



$$\sum F_x = -\left(\frac{\partial p}{\partial x}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial \tau_{xx}}{\partial x}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial \tau_{zx}}{\partial z}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial \tau_{yx}}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{1.29}$$

Substituting this back into our original equation, the conservation of momentum in the x direction is:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{1.30}$$

y direction:

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho vu) + \frac{\partial}{\partial y}(\rho vv) + \frac{\partial}{\partial z}(\rho vw) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{1.31}$$

z direction (we add ρg here due to the gravitational force acting downwards).

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho wu) + \frac{\partial}{\partial y}(\rho wv) + \frac{\partial}{\partial z}(\rho ww) = -\frac{\partial p}{\partial x} - \rho g + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \tag{1.32}$$

These can be added to find that we arrive with two terms, one being the continuity equation, which must equal 0. To summarise, we have our continuity equation and momentum equations below.

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overrightarrow{V}) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{1.33}$$

x direction momentum

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{1.34}$$

y direction momentum

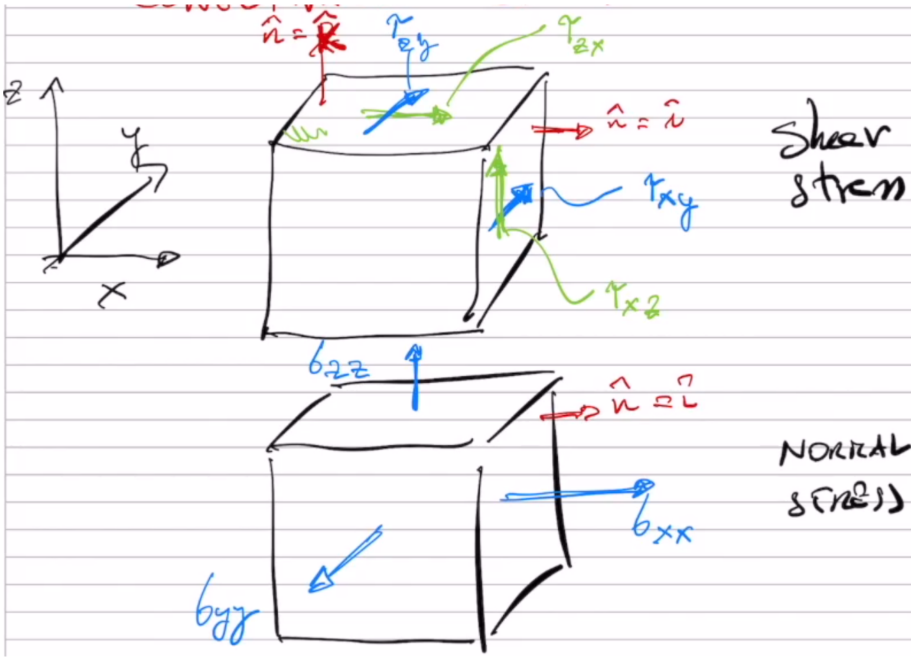
$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{1.35}$$

z direction momentum

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \tag{1.36}$$

1.4 Stress tensor notation

To identify a stress component we use a double subscript notation (tensor notation). The first subscript indicates the direction of the normal to the plane on which stress acts. Second subscript indicates the direction of the stress. Thus, the symbol τ_{ij} denotes a stress in j direction on a face normal to the i-axis.



The normal stresses two contributions are pressure p and viscous stress τ. Pressure is always negative due to it acting against the surface (if we take the arrow coming out of the surface as positive). τ accounts for the extra stress coming from viscosity.

$$\sigma_{xx} = -p + \tau_{xx} \tag{1.37}$$

$$\sigma_{yy} = -p + \tau_{yy} \tag{1.38}$$

$$\sigma_{zz} = -p + \tau_{zz} \tag{1.39}$$

Parts on the opposite sides of a stress tensor are equal.

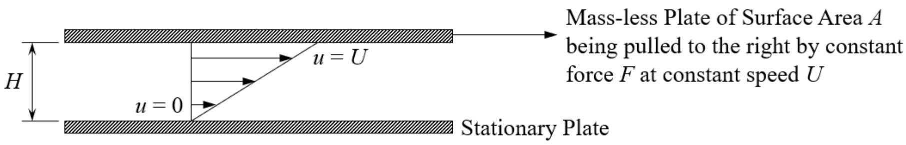
$$\tau_{xy} = \tau_{yx}, \;\; \tau_{xz} = \tau_{zx}, \;\; \tau_{yz} = \tau_{zy}$$

Chapter 2

Navier-Stokes equations

2.1 Constitutive equations

We want to find a way to link the stress tensor τ with the velocity field i.e. $\tau = f(u,\ v,\ w)$.



The angle of deformation $\Delta\theta$ can be used to derive the following:

$$\tan\Delta\theta = \frac{u \cdot \Delta T}{H} \tag{2.1}$$

$$\tan\Delta\theta = \mathrm{d}\theta = \frac{u \cdot t}{H} \rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{u}{H} \tag{2.2}$$

$$\tau = \frac{F}{A} \propto \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{u}{H} \tag{2.3}$$

$$\tau = \mu\frac{\mathrm{d}\theta}{\mathrm{d}t} = \mu\frac{u}{H} \tag{2.4}$$

$$\tau = \mu\frac{\mathrm{d}u}{\mathrm{d}y} \tag{2.5}$$

- τ is the shear stress
- $\frac{\mathrm{d}u}{\mathrm{d}y}$ is the shear rate
- μ is the dynamic viscosity and has units $\mathrm{N\,s\,m^{-2}}$
- $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity and has units $\mathrm{m^2\,s^{-1}}$

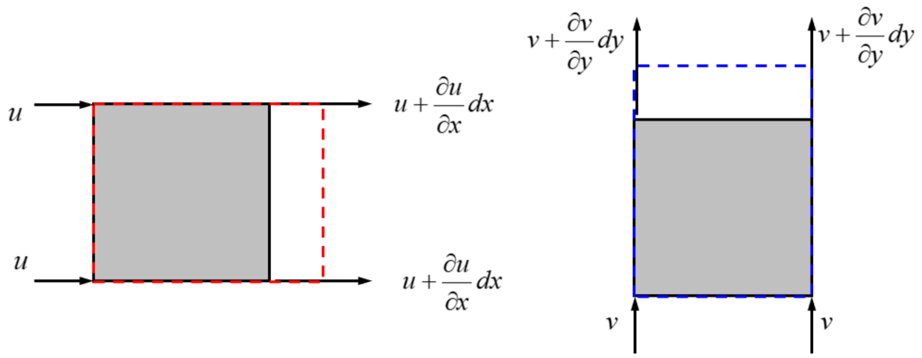
For Newtonian fluids, μ is constant. In the case above, our stress tensor is τ_{yx}, hence:

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} \tag{2.6}$$

Our velocity gradient can be defined as:

$$\nabla \vec{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{2.7}$$

The left diagonal components are the normal deformation, orthogonal to the surface.



A simplified way of writing these left diagonal terms is

$$\nabla \cdot \vec{V} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{2.8}$$

The repeated index i means sum in the x, y and z directions.

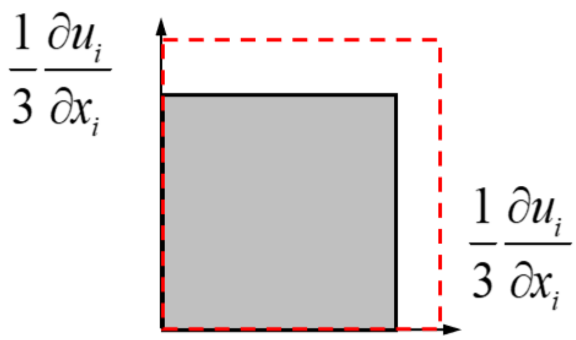
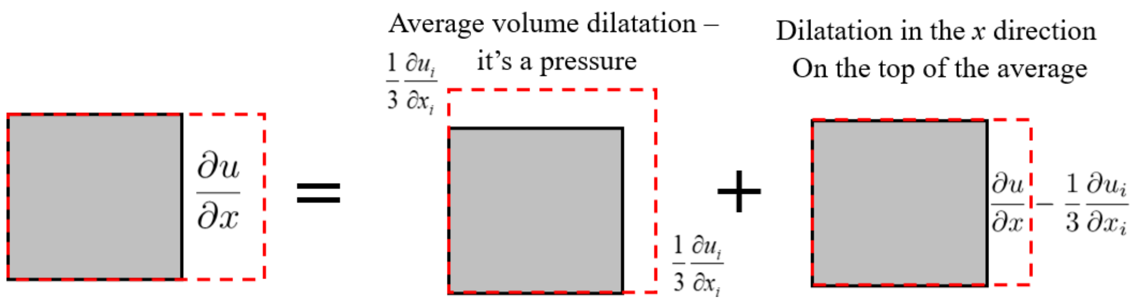


Figure 2.1: 1/3 symbolises the average deformation in x, y and z.

To find $\frac{\partial u}{\partial x}$, we can do the following

$$\frac{\partial u}{\partial x} = \frac{1}{3}\frac{\partial u_i}{\partial x_i} + \left(\frac{\partial u}{\partial x} - \frac{1}{3}\frac{\partial u_i}{\partial x_i} \right) \tag{2.9}$$



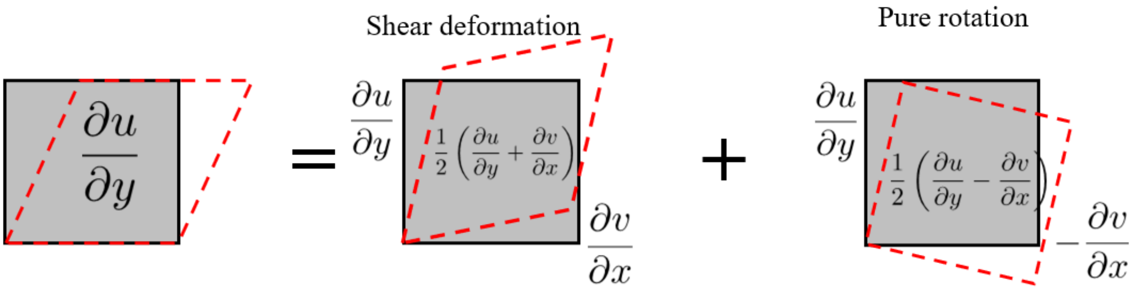
This can be also done for the other two orthogonal directions

$$\frac{\partial v}{\partial y} = \frac{1}{3}\frac{\partial u_i}{\partial x_i} + \left(\frac{\partial v}{\partial y} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\right) \tag{2.10}$$

$$\frac{\partial w}{\partial z} = \frac{1}{3}\frac{\partial u_i}{\partial x_i} + \left(\frac{\partial w}{\partial z} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\right) \tag{2.11}$$

Let us consider another term, such as $\frac{\partial u}{\partial y}$. We can define this as a component of deformation and rotation of the fluid particle.

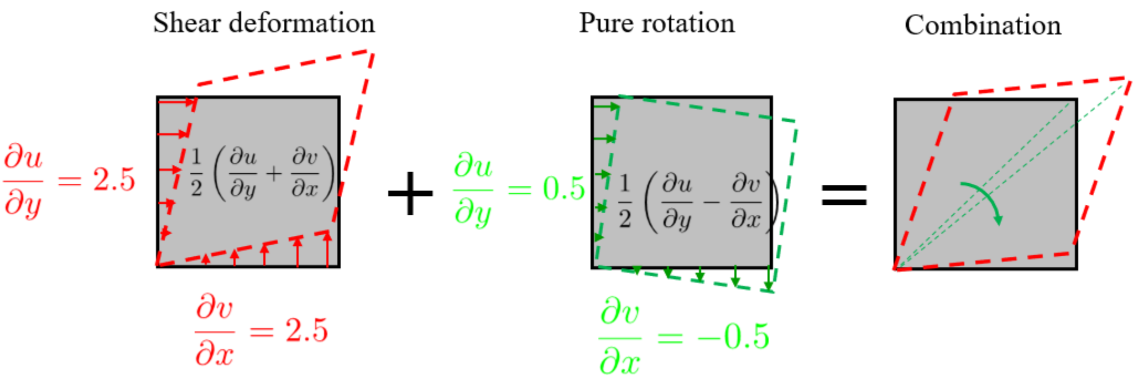
$$\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \tag{2.12}$$

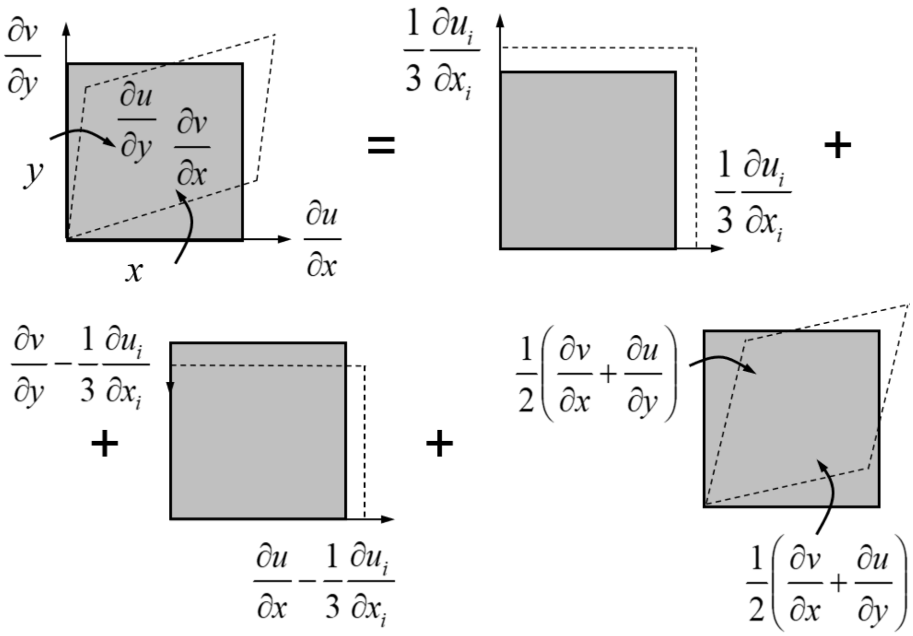


Example

$$\frac{\partial u}{\partial y} = 3 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 2.5 + 0.5 \tag{2.13}$$

$$\frac{\partial v}{\partial x} = 2 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 2.5 - 0.5 \tag{2.14}$$





2.1.1 Strain rate tensor

$$s = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial x}\right] & \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \\ \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] & \frac{\partial v}{\partial y} & \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] \\ \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] & \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] & \frac{\partial w}{\partial z} \end{bmatrix} = \tag{2.15}$$

$$\begin{bmatrix} \frac{1}{3}\nabla\cdot\vec{V} & 0 & 0 \\ 0 & \frac{1}{3}\nabla\cdot\vec{V} & 0 \\ 0 & 0 & \frac{1}{3}\nabla\cdot\vec{V} \end{bmatrix} + \begin{bmatrix} \left[\frac{\partial u}{\partial x} - \frac{1}{3}\nabla\cdot\vec{V}\right] & \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial x}\right] & \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \\ \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] & \left[\frac{\partial v}{\partial y} - \frac{1}{3}\nabla\cdot\vec{V}\right] & \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] \\ \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] & \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] & \left[\frac{\partial w}{\partial z} - \frac{1}{3}\nabla\cdot\vec{V}\right] \end{bmatrix} \tag{2.16}$$

$$\text{Deformation part which goes in pressure } \rho + \text{Deformation part which goes in the stress tensor } T \tag{2.17}$$

Compact notation of the strain rate tensor, indices $i,\ j = 1,\ 2,\ 3$

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.18}$$

2.1.2 Stress tensor

$$T = \begin{bmatrix} \tau_x x & \tau_y x & \tau_z x \\ \tau_x y & \tau_y y & \tau_z y \\ \tau_x z & \tau_y z & \tau_z z \end{bmatrix} = \tag{2.19}$$

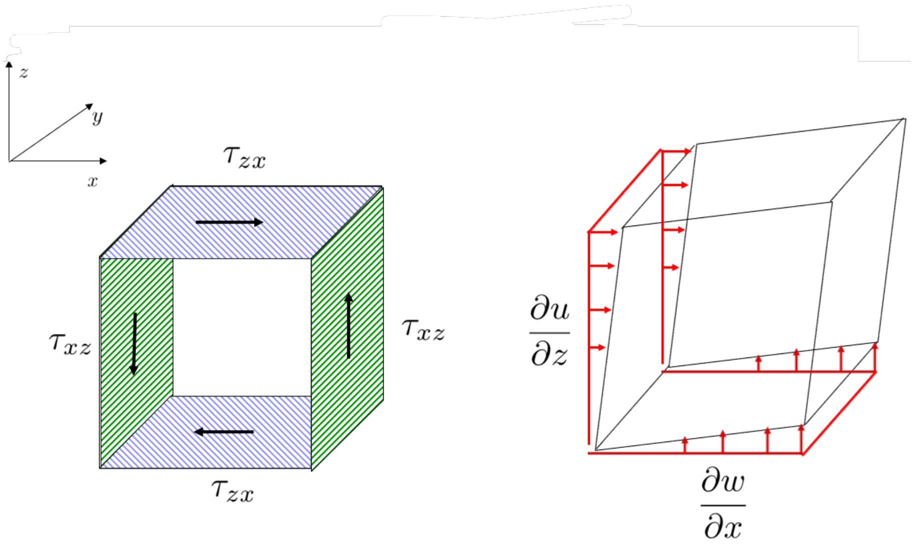
$$\begin{bmatrix} \mu\left[2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V}\right] & \mu\left[\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial x}\right] & \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \\ \mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] & \mu\left[2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\vec{V}\right] & \mu\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] \\ \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] & \mu\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] & \mu\left[2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\vec{V}\right] \end{bmatrix} \tag{2.20}$$

Compact notation for constitutive equation:

$$\tau_{ij} = 2\mu\left[s_{ij} - \frac{1}{3}(\nabla\cdot\vec{V})\delta_{ij}\right] = \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}(\nabla\cdot\vec{V})\delta_{ij}\right] \tag{2.21}$$

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{2.22}$$

The stress tensor is always symmetric along the left diagonal.



We now have 10/12 equations to describe the fluid. The final two relate to the temperature and the energy of the fluid. We will not be considering the energies of the fluid in this course. Our state equation can be $p = \rho RT$, when the fluid is compressible and if our fluid is incompressible we take ρ as constant. All in all, 11 variables and 11 equations to describe the fluid

2.2 Navier-Stokes Equations

Navier-Stokes equations are a system of equations that can be used to describe the behaviour of a fluid. They can be obtained through inserting the Constitutive Equations into the Conservation of Momentum Equations, rearranging, and simplifying them. The Navier-Stokes Equations for an incompressible fluid in 3D are as follows:

Conservation of Mass:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.23}$$

Conservation of Momentum (x, y, z)**:**

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{2.24}$$

$$\rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \tag{2.25}$$

$$\rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \rho g \tag{2.26}$$

The Navier-Stokes Equations can be further simplified if the following occur:

- Constant Density
- Assume Steady Flow (No Time-Dependent Terms)
- Assume No External Forces
- Assume Fluid is Incompressible

The Navier-Stokes Equations in 2D, with the above assumptions applied are below:

Conservation of Mass (2D):

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.27}$$

Conservation of Momentum (x, y)**:**

$$\rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{2.28}$$

$$\rho \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{2.29}$$

2.3 Lagrangian vs. Eulerian

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right) \tag{2.30}$$

- $\frac{\mathrm{D}}{\mathrm{D}t}$ - Lagrangian/Material Derivative: Variation in time of a property (for example temperature, density or velocity component) of a fluid particle. The reference system is moving with the fluid particle.
- $\frac{\partial}{\partial t}$ - Eulerian Derivative: Variation in time of a property (for example temperature, density etc..) in a fixed point in space (x, y , z). Reference system fixed in space.
- $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ - Convection Terms in the x, y and z: Variation of a property due to how the particle is moving in space.

Chapter 3

Inviscid and Irrotational flow

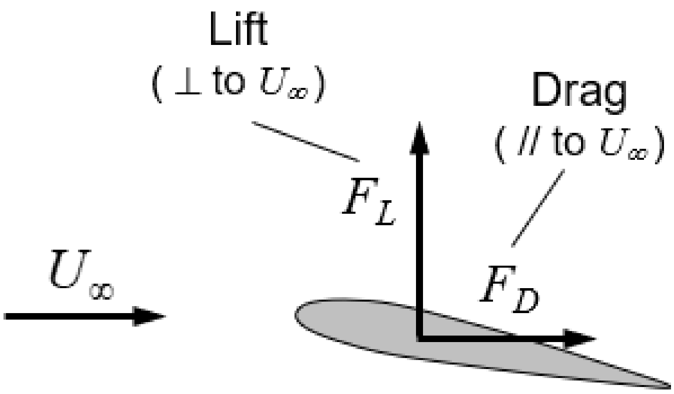
3.1 Lift and drag

Typical forces of interest for bodies in a flow are **drag** and **lift**. We can represent these in dimensionless form:

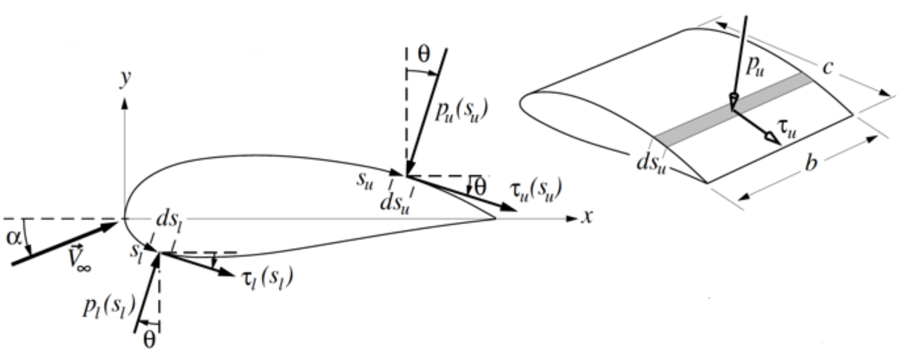
$$\text{Drag coefficient: } c_D = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 S} \tag{3.1}$$

$$\text{Lift coefficient: } c_L = \frac{F_L}{\frac{1}{2}\rho U_\infty^2 S} \tag{3.2}$$

Where S is a representative area for the body, determined by convention.



3.2 Pressure and frictional force distribution



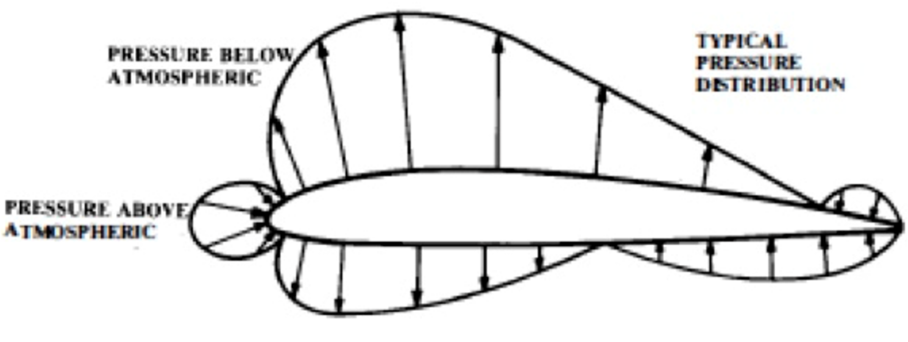
$$L = -\int_S \left(p(\hat{n}\cdot\hat{j})\right)\,\mathrm{d}S + \int^S \left(\vec{\tau}\cdot\hat{j}\right)\,\mathrm{d}S \tag{3.3}$$

$$D = -\int \left(p(\hat{n}\cdot\hat{j})\right)\,\mathrm{d}S + \int \left(\vec{\tau}\cdot\hat{i}\right)\,\mathrm{d}S \tag{3.4}$$

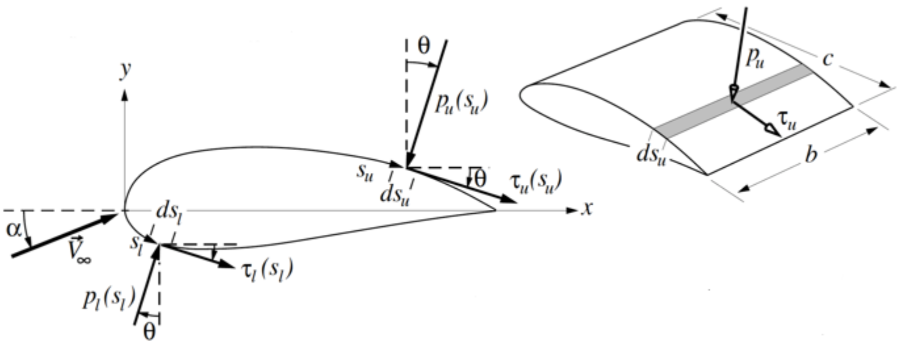
To determine the lift and drag coefficients c_L and c_D, we are interested in the pressure distribution over the airfoil, or more specifically in the local pressure difference from the stream pressure p_∞.

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} \tag{3.5}$$

Free stream pressure and velocity are p_∞ and V_∞.



- Local suction (depression): $c_p < 0$ Vectors point away from the airfoil surface
- Local pushing: $c_p > 0$ Vectors point towards the airfoil surface



$$L = -\int_S \left(p\hat{n}\cdot\hat{j}\right)\,\mathrm{d}S = \tag{3.6}$$

$$c_L = -\frac{1}{S}\int_S \left(\frac{p-p_\infty}{\frac{1}{2}\rho V_\infty^2}\hat{n}\cdot\hat{j}\right)\,\mathrm{d}S = -\frac{1}{S}\int_S \left(c_p\hat{n}\cdot\hat{j}\right)\,\mathrm{d}S \tag{3.7}$$

The lift coefficient per unit of span-wise length is:

$$c'_L - \frac{1}{c}\int_c^0 \left(c_p\hat{n}\cdot\hat{j}\right)\,\mathrm{d}x \tag{3.8}$$

3.3 Rearrangement of momentum equation - x direction

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.9}$$

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + v\frac{\partial u}{\partial y} - v\frac{\partial u}{\partial y} \tag{3.10}$$

$$= -v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial u^2}{\partial x} + \frac{\partial v^2}{\partial x}\right) \tag{3.11}$$

$$= -v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2) \tag{3.12}$$

(u^2+v^2) is the total kinetic energy of the fluid particle. The derivative is the element that takes into the account the variation of this kinetic energy. $\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ relates to the rotation of the particle. This rotation is related to the difference of velocity gradient.

$$\rho\left[-v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2)\right] = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.13}$$

$$-v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.14}$$

Our Bernoulli term in the above equation is $\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right)$, gravitational energy is negligible. $\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ is an anti-clockwise rotation. Hence, the vorticity component in the z direction is:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{3.15}$$

Our final momentum equations in x and y are:

$$-v\omega_z = -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.16}$$

$$u\omega_z = -\frac{\partial}{\partial y}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3.17}$$

We can make some assumptions:

- Inviscid flow - $\nu = 0$ (this may be realistic in some parts of a fluid domain but in real life, inviscid fluids do not exist)
- Irrotational flow - $\omega_z = 0$

This reduces our equations to:

$$0 = -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + 0 \tag{3.18}$$

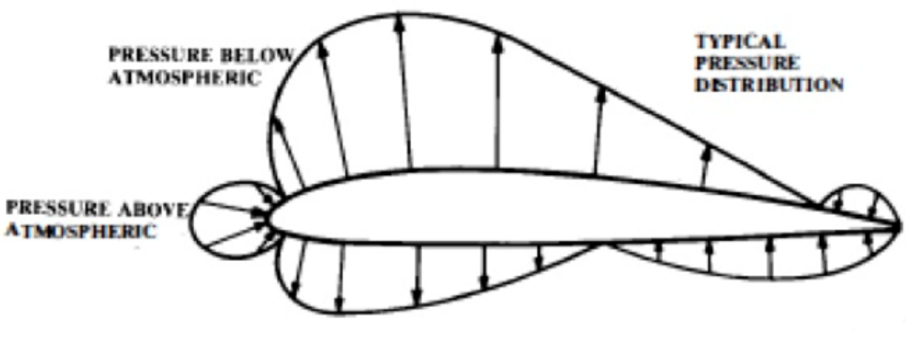
$$0 = -\frac{\partial}{\partial y}\left(\frac{p}{\rho} + \frac{u^2+v^2}{2}\right) + 0 \tag{3.19}$$

3.4 Application of Bernoulli

$$p_\infty + \frac{1}{2}\rho V_\infty^2 = p + \frac{1}{2}\rho(u^2+v^2) = \text{constant} \tag{3.20}$$

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{u^2+v^2}{V_\infty^2} = 1 - \frac{||V||^2}{V_\infty^2} \tag{3.21}$$

If $c_p < 0$, $||V|| > V_\infty$ and vice versa. If a fluid particle enters a region where c_p is negative it is accelerated and when c_P is positive it will lose velocity relative to the free stream.



Extending this to 3D, we can derive the 3D vorticity equation. We sum the momentum equations with the assumptions above and take the ijk components as so:

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} \tag{3.22}$$

If $\vec{\omega} = 0$ then a potential function, $\phi(x, y, z)$ exists such:

$$\begin{cases} u = \frac{\partial\phi(x,y,z)}{\partial x} \\ v = \frac{\partial\phi(x,y,z)}{\partial y} \\ w = \frac{\partial\phi(x,y,z)}{\partial z} \end{cases} \tag{3.23}$$

By plugging in these equations into our continuity equation, we get:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y} + \frac{\partial^2 \phi}{\partial z} \tag{3.24}$$

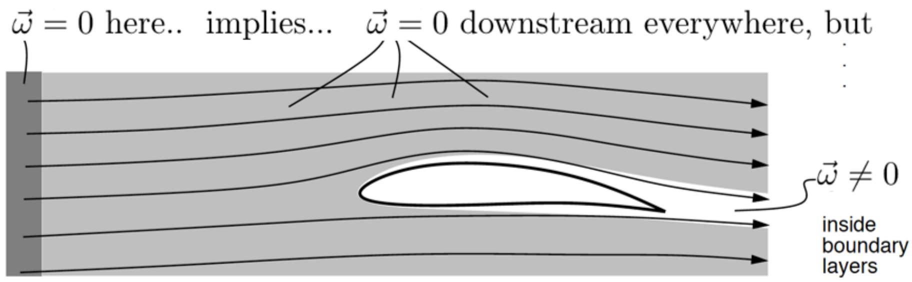
Conservation of momentum (Bernoulli term) is simply:

$$p + \frac{1}{2}\rho(u^2 + v^2) = \text{constant} \tag{3.25}$$

3.5 Applicability of irrotational flow

The flow domain can be subdivided into two parts.

- **Irrotational flow region**, outside of the boundary layer, Bernoulli equation, potential flow and stream function apply.
- **Boundary layer**, layer where all vorticity is confined. The friction shear the airfoil surface acts as a source of vorticity.



In the case where we consider our fluid inviscid, there is no shear stress being applied on the fluid by the airfoil. We need to understand how the no-slip condition changes. We still need a boundary condition to know the value of ϕ in our flow domain.

3.6 No-Slip condition for inviscid flow

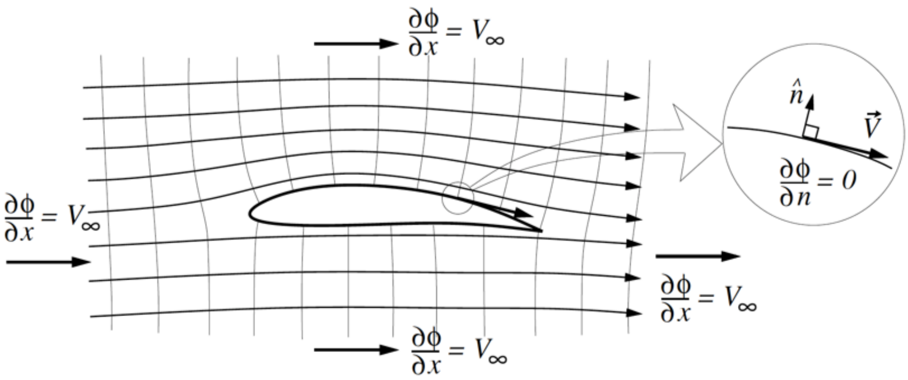
Viscous fluid

$$\nu \neq 0 \rightarrow \vec{V} = 0 \tag{3.26}$$

Inviscid flow

$$\nu = 0 \rightarrow V_n = \vec{V} \cdot \hat{n} = \frac{\partial \phi}{\partial n} = 0 \tag{3.27}$$

In essence, with inviscid flow, we are accepting that there is some movement on the boundary, however this is only parallel to the surface. n is the direction orthogonal to the boundary.



3.7 Stream function

In a 2D flow a stream function, $\psi(x, y)$, can be defined which is always aligned/parallel with the local velocity vector and visualise a streamline. Different streamlines are identified with different values of $\psi(x, y)$

$$\begin{cases} u = \frac{\partial\psi(x,y)}{\partial x} \\ v = \frac{\partial\psi(x,y)}{\partial y} \end{cases} \tag{3.28}$$

Iso-potential lines and streamlines are orthogonal to each other. Streamlines visualise the trajectory of a particle in the field.

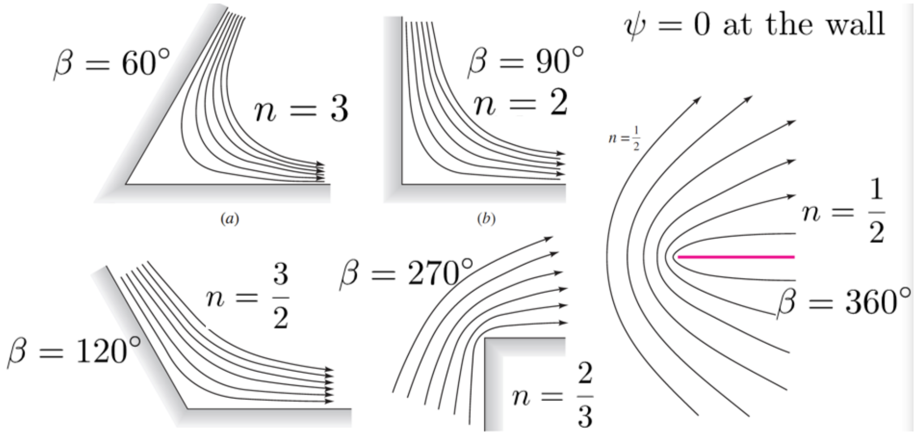
3.8 Potential flow past bodies

Flow fields for which an incompressible fluid is assumed to be frictionless and the motion to be irrotational are commonly referred to as **potential** flows. Paradoxically, potential flows can be simulated by a slow moving, viscous flow between closely spaced parallel plates.

3.9 Flow around a corner of arbitrary angle, β

Considering a radial coordinate system:

$$\phi = Ar^n \cos(n\theta) \tag{3.29}$$
$$\psi = Ar^n \sin(n\theta) \tag{3.30}$$



3.10 Cylindrical coordinates

3D vorticity equation:

$$\vec{\omega} = \omega_r \hat{i}_r + \omega_\theta \hat{i}_\theta + \omega_z \hat{i}_z \tag{3.31}$$

$$\vec{\omega} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{i}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{i}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\hat{i}_z \tag{3.32}$$

Potential flow function and stream function:

$$\text{Potential flow}\begin{cases} u_r = \frac{\partial \phi}{\partial r} \\ u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \\ u_z = \frac{\partial \phi}{\partial z} \end{cases} \quad \text{Stream function}\begin{cases} u_\theta = -\frac{\partial \psi}{\partial r} \\ u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \end{cases} \tag{3.33}$$

Conservation of mass (continuity equation):

$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{3.34}$$

Chapter 4

Modelling the flow around a bluff body

27/10/2020

4.1 Uniform Flow

Cartesian Coordinates:

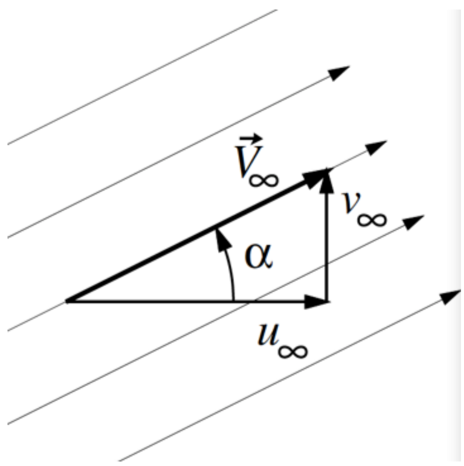
$$\phi = V_\infty \left[x\cos(\alpha) + y\sin(\alpha)\right] \tag{4.1}$$
$$\psi = V_\infty \left[y\cos(\alpha) - x\sin(\alpha)\right] \tag{4.2}$$

The conservation of mass is balanced:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.3}$$

The flow is irrotational:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{4.4}$$

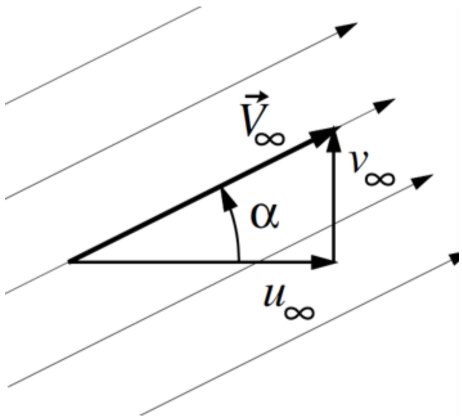


Cylindrical Coordinates:

$$\phi(r,\theta) = V_\infty r\cos(\theta-\alpha) \tag{4.5}$$
$$\psi(r,\theta) = V_\infty r\sin(\theta-\alpha) \tag{4.6}$$

The conservation of mass is satisfied for cylindrical coordinates:

$$\nabla\cdot\overrightarrow{V} = \frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} \tag{4.7}$$
$$= \frac{\partial r(v_\infty\cos(\theta-\alpha))}{\partial r} - \frac{\partial v_\infty\sin(\theta-\alpha)}{\partial\theta} \tag{4.8}$$
$$v_\infty\cos(\theta-\alpha) - v_\infty\cos(\theta-\alpha) = 0 \tag{4.9}$$



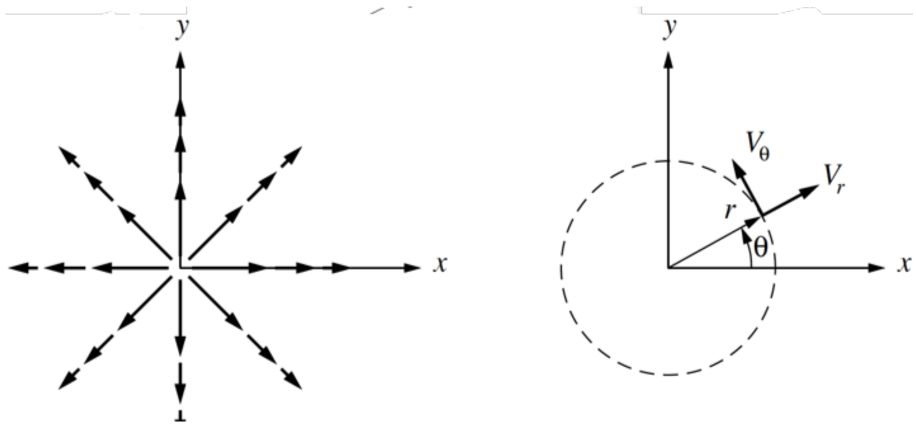
4.2 Source/Sink Flow

Cartesian Coordinates:

$$\phi = \frac{\Lambda}{2\pi}\ln(\sqrt{x^2+y^2}) \tag{4.10}$$
$$\psi = \frac{\Lambda}{2\pi}\arctan\left(\frac{y}{x}\right) \tag{4.11}$$

Cylindrical Coordinates:

$$\phi = \frac{\Lambda}{2\pi}\ln(r) \tag{4.12}$$
$$\psi = \frac{\Lambda}{2\pi}\theta \tag{4.13}$$



In cylindrical coordinates, we do not have a θ component as it is moving radially outwards from a source.

$$u_r = \frac{\partial \phi}{\partial r} = \frac{\Lambda}{2\pi r} \tag{4.14}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \tag{4.15}$$

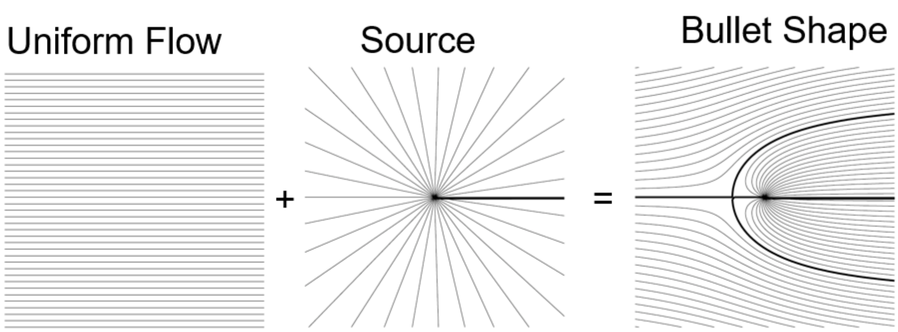
Here we can see the magnitude of the velocity is dependent on $\frac{1}{r}$. If $\Lambda > 0$, we have a source and if $\Lambda < 0$, we have a sink.

4.3 Uniform Flow + Source

$$\phi(r,\theta) = \frac{\Lambda}{2\pi}\ln(r) + V_\infty r\cos\theta \tag{4.16}$$

$$\psi(r,\theta) = \frac{\Lambda}{2\pi}\theta + V_\infty r\sin\theta \tag{4.17}$$

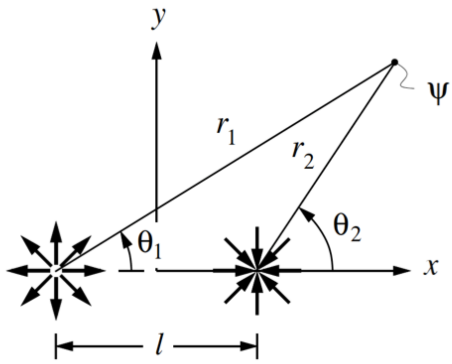
Stream function $\psi(r,\theta)$ of:

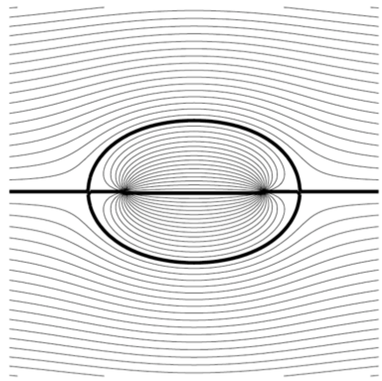


4.4 Uniform Flow + Source + Sink

$$\phi - V_\infty r \cos\theta + \frac{\Lambda}{2\pi}(\ln(r_1) - \ln(r_2)) \tag{4.18}$$

$$\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) \tag{4.19}$$



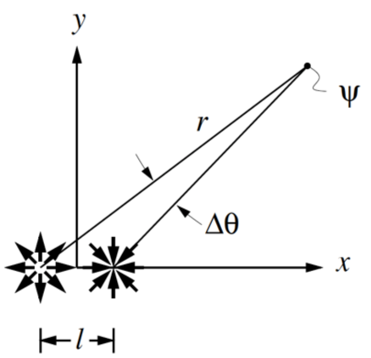


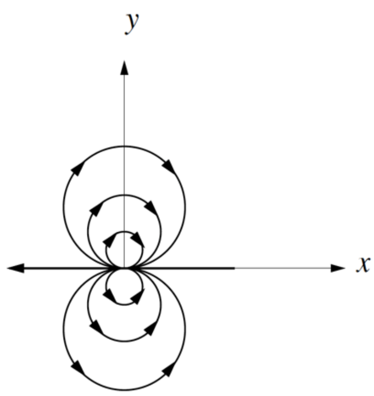
4.5 Doublet

Consider a pair of source and sink of $\pm\Lambda$ who are l apart and $l \times \Lambda =$ constant.

$$\psi = \lim_{l\to 0} \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) = -\frac{k}{2\pi}\frac{\sin\theta}{r} \tag{4.20}$$

$$\phi = \frac{k}{2\pi}\frac{\cos\theta}{r} \tag{4.21}$$

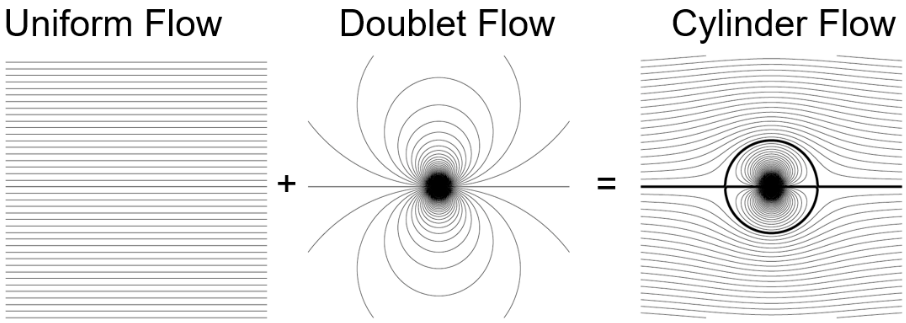




4.6 Cylinder (Uniform Flow + Doublet)

$$\phi = V_\infty r\cos\theta + \frac{k}{2\pi}\frac{\cos\theta}{r} \tag{4.22}$$

$$\psi = V_\infty r\sin\theta - \frac{k}{2\pi}\frac{\sin\theta}{r} \tag{4.23}$$



The radius of the cylinder can be derived as so:

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty\cos\theta - \frac{k}{2\pi}\frac{\cos\theta}{r^2} \tag{4.24}$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -\left(V_\infty\sin\theta + \frac{k}{2\pi}\frac{\sin\theta}{r^2}\right) \tag{4.25}$$

On the cylinder, $\vec{u}\cdot\hat{n} = 0$

$$\hat{n} = \hat{i}_r \rightarrow u_r(R) = 0 \tag{4.26}$$

$$V_\infty\cos\theta - \frac{k}{2\pi}\frac{\cos\theta}{R^2} = 0 \tag{4.27}$$

$$R = \sqrt{\frac{k}{2\pi V_\infty}} \tag{4.28}$$

We can rewrite ϕ and ψ

$$\phi = V_\infty r\cos\theta\left(1 + \frac{R^2}{r^2}\right) \tag{4.29}$$

$$\psi = V_\infty r\sin\theta\left(1 - \frac{R^2}{r^2}\right) \tag{4.30}$$

On the cylinder surface, $r = R$ and inputting this into ψ:

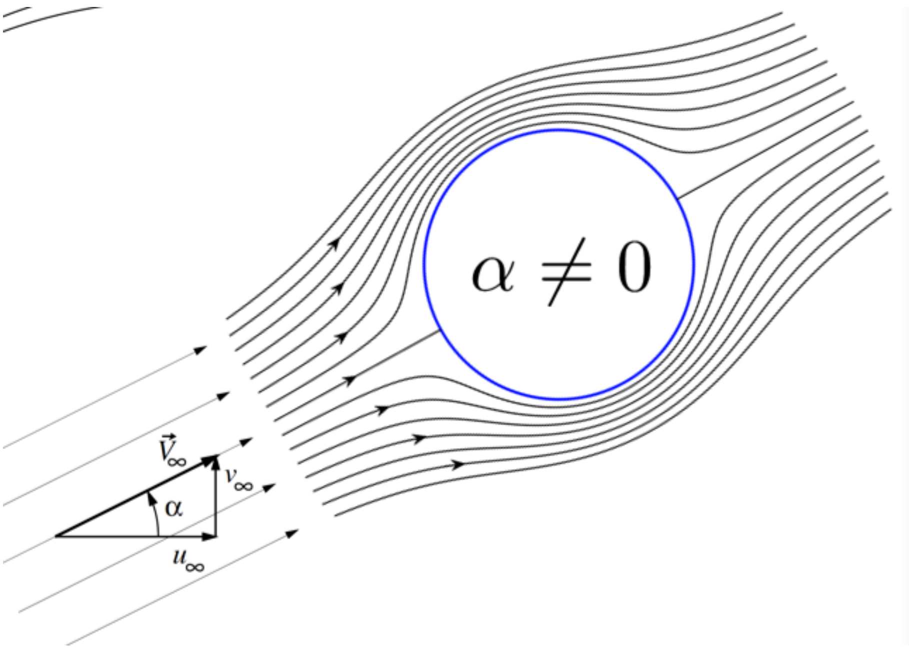
$$\psi = 0 \tag{4.31}$$

4.7 Uniform Stream with Varying Direction

All we need to do to generalise our equations a bit more is to rewrite our equations with an extra angular term, α:

$$\phi = V_\infty r\cos(\theta-\alpha)\left(1+\frac{R^2}{r^2}\right) \tag{4.32}$$

$$\psi = V_\infty r\sin(\theta-\alpha)\left(1-\frac{R^2}{r^2}\right) \tag{4.33}$$



4.8 Adding Circulation with a Vortex Flow

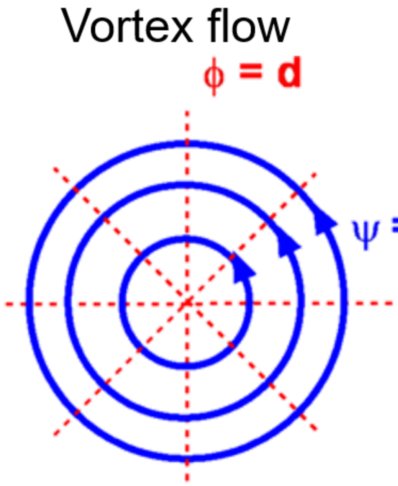
$$\phi = -\frac{\Gamma}{2\pi}\theta \tag{4.34}$$

$$\psi = \frac{\Gamma}{2\pi}\ln\left(\frac{r}{R}\right) \tag{4.35}$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0 \tag{4.36}$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -\frac{\Gamma}{2\pi r} \tag{4.37}$$

Where $\Gamma < 0$ is anti-clockwise motion and $\Gamma > 0$ is clockwise motion.



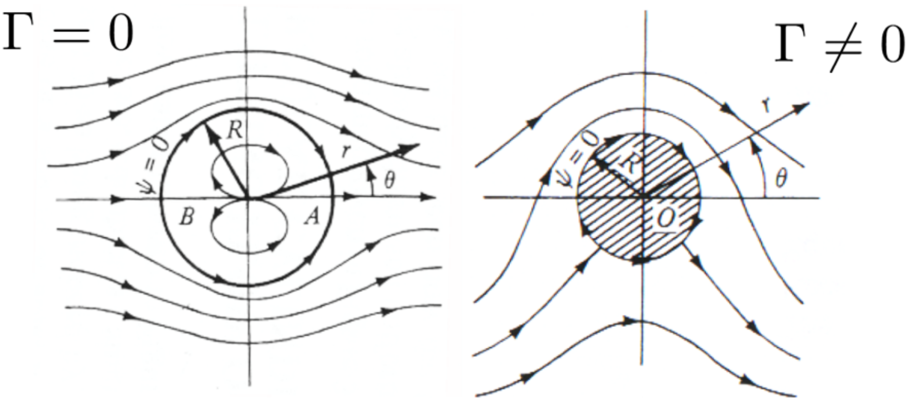
4.9 Cylinder with a Vortex Flow

$$\psi = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi} \ln\left(\frac{r}{R}\right) \tag{4.38}$$

$$\phi = V_\infty r \cos\theta \left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi}\theta \tag{4.39}$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty \cos\theta \left(1 - \frac{R^2}{r^2}\right) \tag{4.40}$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -V_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right) \tag{4.41}$$



4.10 Lift and Drag of a Cylinder with Circulation

Apply Bernoulli:

$$p_\infty + \frac{1}{2}\rho V_\infty^2 = p(r,\theta) + \frac{1}{2}\rho(u_r^2 + u_\theta^2) \tag{4.42}$$

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{u_r^2 + u_\theta^2}{V_\infty^2} \tag{4.43}$$

On the cylinder surface: $u_r = 0$

$$c_p(R,\theta) = 1 - \frac{u_\theta^2}{V_\infty^2} = 1 - \frac{(2V_\infty \sin\theta + \frac{\Gamma}{2\pi R})^2}{V_\infty^2} \tag{4.44}$$

$$= 1 - \left(4\sin^2(\theta) + \frac{\Gamma^2}{4\pi^2 V_\infty^2 R^2} + \frac{2\Gamma\sin(\theta)}{V_\infty \pi R}\right) \tag{4.45}$$

4.11 Lift of the Cylinder

We need to calculate c_p on the surface of the cylinder.

$$L = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(c_p \cdot \hat{n} \cdot \hat{j} R\right) \mathrm{d}\theta \tag{4.46}$$

$$= -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(c_p \sin\theta R\right)\, \mathrm{d}\theta \tag{4.47}$$

$$L = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(1 - \frac{(2V_\infty \sin\theta + \frac{\Gamma}{2\pi R})^2}{V_\infty^2}\right) \sin\theta R \,\mathrm{d}\theta \tag{4.48}$$

Expanding the integral out, we arrive at:

$$L = -\frac{1}{2}\rho V_\infty^2 \left[\int_0^{2\pi} \left(1 - \frac{\Gamma^2}{4\pi^2 R^2 V_\infty^2}\right) \cdot \sin\theta \cdot R\,\mathrm{d}\theta + \int_0^{2\pi} -4\sin\theta^3 \cdot R\,\mathrm{d}\theta + \right.$$

$$\left. \int_0^{2\pi} -\frac{2\Gamma}{\pi R V_\infty} \cdot \sin\theta^2 \cdot R\,\mathrm{d}\theta \right] \tag{4.49}$$

Because the first term and the second term have an odd power of sin, when we integrate these, they will have negligible outcome on the lift of the cylinder. We can reduce our equation to:

$$L = -\frac{1}{2}\rho V_\infty^2 \left[\int_0^{2\pi} -\frac{2\Gamma}{\pi R V_\infty} \cdot \sin\theta^2 \cdot R\,\mathrm{d}\theta\right] \tag{4.50}$$

$$= \frac{\rho V_\infty \Gamma}{\pi} \int_0^{2\pi} \sin\theta^2 \,\mathrm{d}\theta \tag{4.51}$$

$$= \frac{\rho V_\infty \Gamma}{\pi} \left[\frac{1}{2}\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} \tag{4.52}$$

$$L = \rho V_\infty \Gamma \tag{4.53}$$

We can see here that our vortex factor Γ has a proportional effect on our lift force. We can see an example of this in real life when a football is kicked. When the ball is kicked in such a way that it has a spin, we see the ball curves in certain directions.

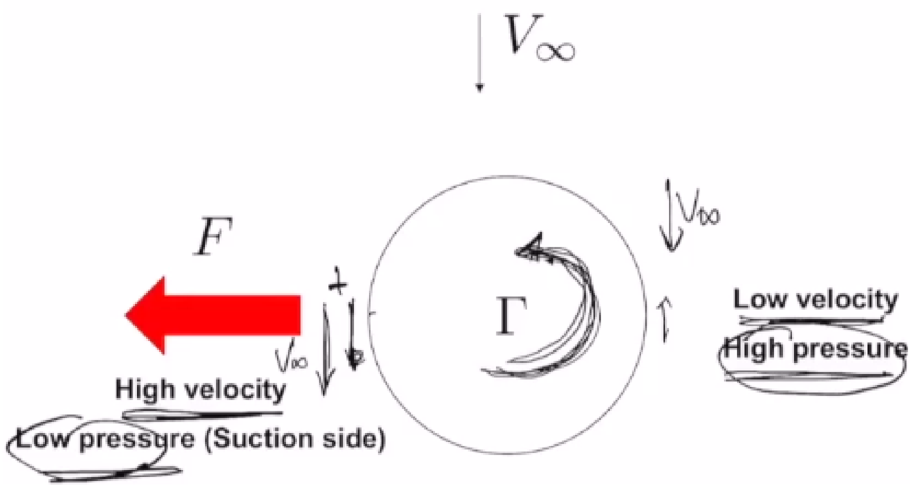
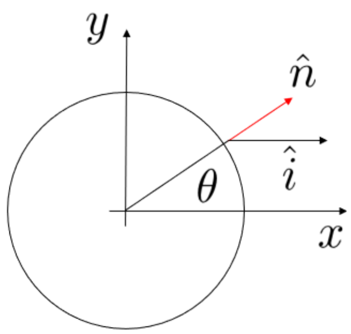


Figure 4.1: We can see that when the ball spins, the velocity from the free stream and the vortex combine to create regions of low and high pressure. This creates a net force, leading to a suction effect.

4.12 Drag of a cylinder



$$D = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} c_p \vec{n} \cdot \hat{i} R \, \mathrm{d}\theta \tag{4.54}$$

$$= -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} c_p \cos\theta R \, \mathrm{d}\theta \tag{4.55}$$

$$D = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left(1 - \frac{\left(2V_\infty \sin\theta + \frac{\Gamma}{2\pi R}\right)^2}{V_\infty^2}\right) \cos\theta R \, \mathrm{d}\theta \tag{4.56}$$

The first term has an odd power of cosine, and so is negligible. The second term of the drag integral is:

$$\int_0^{2\pi} -4\sin\theta^2 \cos\theta R \, \mathrm{d}\theta = -4R\left(\frac{1}{3}\sin\theta^3\right)_p^{2\pi} = 0 \tag{4.57}$$

The third term of the drag integral is:

$$\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} -\frac{2\Gamma}{\pi R V_\infty} \sin\theta \cos\theta R \,\mathrm{d}\theta = -\frac{\rho V_\infty \Gamma}{\pi}\left(-\frac{1}{2}\cos 2\theta\right)_0^{2\pi} = 0 \tag{4.58}$$

Therefore, we see that our drag is in fact:

$$D = 0 \tag{4.59}$$

This is due to our assumption that our flow is inviscid and that the pressure forces are symmetrical to the left and right sides to the y axis. Therefore the net forces acting on the x direction is zero. This is where our model starts failing and is called the D'Alambert Paradox.

4.13 Inviscid and Viscous Flow past a body

High Re implies that the magnitude of the inertia forces are much greater than the magnitudes of the viscous forces in a system. This might imply that the effects of viscosity are insignificant compared with the inertia forces, but this would be a dangerous conclusion. Compare theory for zero viscosity with experiment for high Re flow past a cylinder:

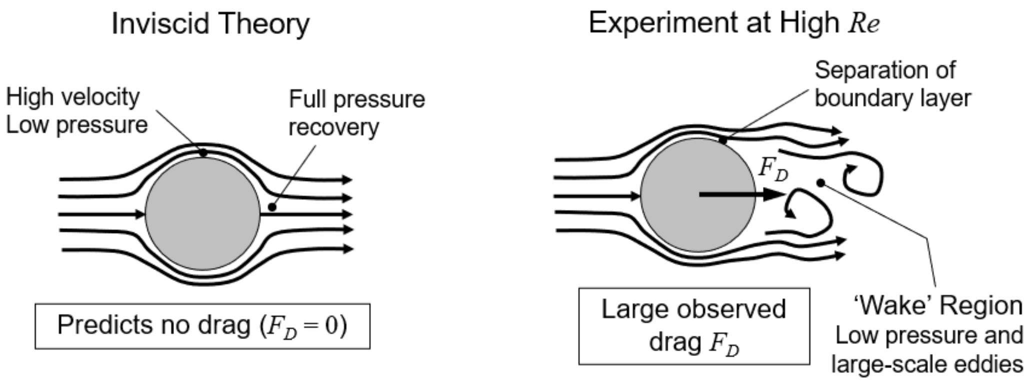
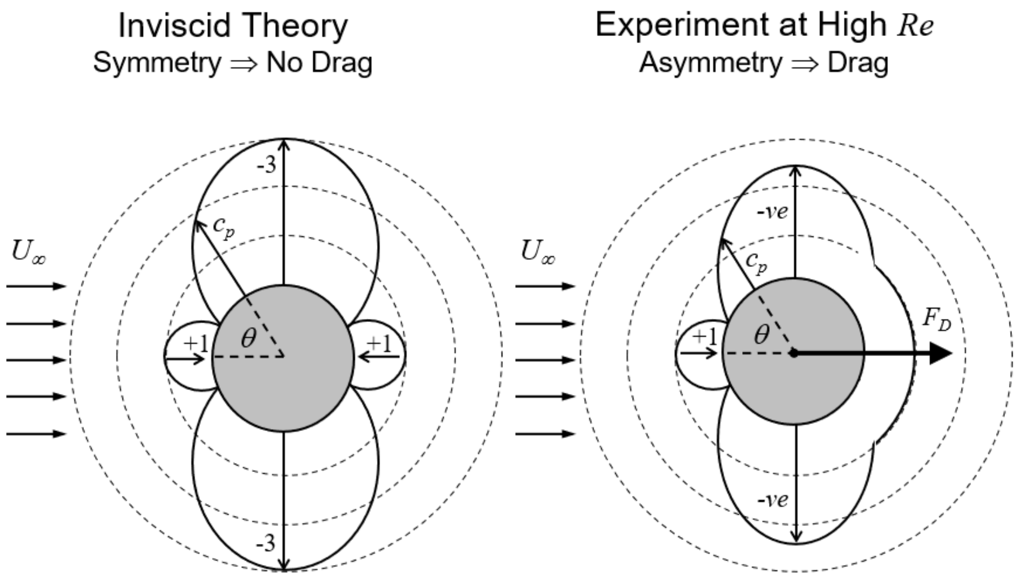
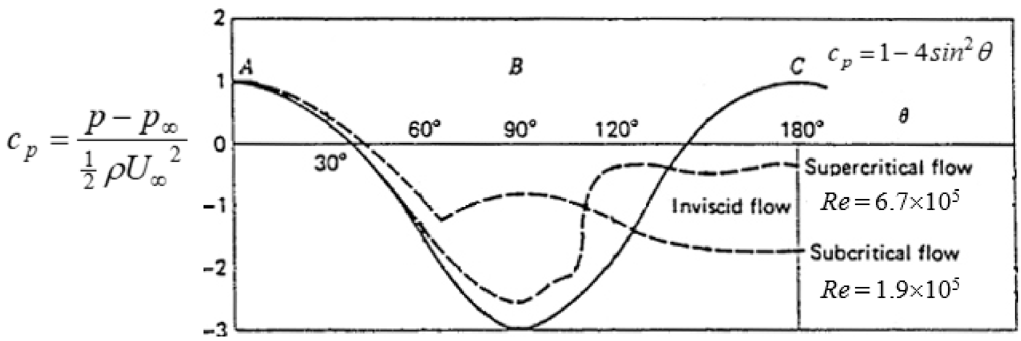


Figure 4.2: We have a net force in our experiment due to the fact that a net force is created from the low pressure wake region and the high pressure front side of the cylinder.

4.14 Flow past a cylinder - pressure coefficient



We can plot the pressure coefficient and see the difference between the inviscid theory and an experiment at high Re.



For a long circular cylinder, the lift coefficient c_L and the form drag coefficient to c_D are related to c_p by:

$$c_L = \frac{1}{2}\int_0^{2\pi} c_p \sin\theta \,\mathrm{d}\theta \tag{4.60}$$

$$c_D = \frac{1}{2}\int_0^{2\pi} c_p \cos\theta \,\mathrm{d}\theta \tag{4.61}$$

Some examples of viscous flow past bodies: