MECH0024 Topic Notes

UCL

HD

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Contents

Chapter 1

Introduction and Normal Shocks

1.1 Revision of Fundamental Concepts

1.1.1 Dimensionless Measures

Reynolds number:

$$Re = \frac{Ud}{v} \tag{1.1}$$

Mach number:

$$M = \frac{q}{c} = \frac{q}{(\gamma RT)^{\frac{1}{2}}} \tag{1.2}$$

Where:

- q = local flow speed
- U = characteristics flow speed
- d = characteristic lengthscale
- c = either local or characteristic speed of sound

The difference between a characteristic and a local measure is important, especially for compressible flows.

1.1.2 Classical Thermodynamics

First Law of Thermodynamics (change in internal energy, E):

$$\Delta E = Q - W \tag{1.3}$$

Second Law of Thermodynamics (entropy cannot decrease):

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} \tag{1.4}$$

1.1.3 Equation of State

The relationships for a perfect gas are:

$$p = \rho R T \tag{1.5}$$

$$c_p - c_v = R \tag{1.6}$$

$$\mathrm{d}U = c_p \, \mathrm{d}T \tag{1.7}$$

$$\mathrm{d}E = c_v \, \mathrm{d}T \tag{1.8}$$

Where E is the internal energy, U is the enthalpy. The isentropic index is:

$$\gamma = \frac{c_p}{c_v} \tag{1.9}$$

Gas	$R(\mathrm{Km^2s^{-2}})$	$\rho(kgm^{-3})$	γ
H_2	4124	0.822	1.41
He	2077	1.63	1.66
Dry air	287	1.18	1.40
N_2	297	1.14	1.40

Figure 1.1

1.1.4 Terminology

Adiabatic - no heat in / work done (strong changes)
Isentropic – no change in entropy (weak changes)

1.1.5 Conservation Principles

There are two frameworks to analyse fluid and solid mechanics.

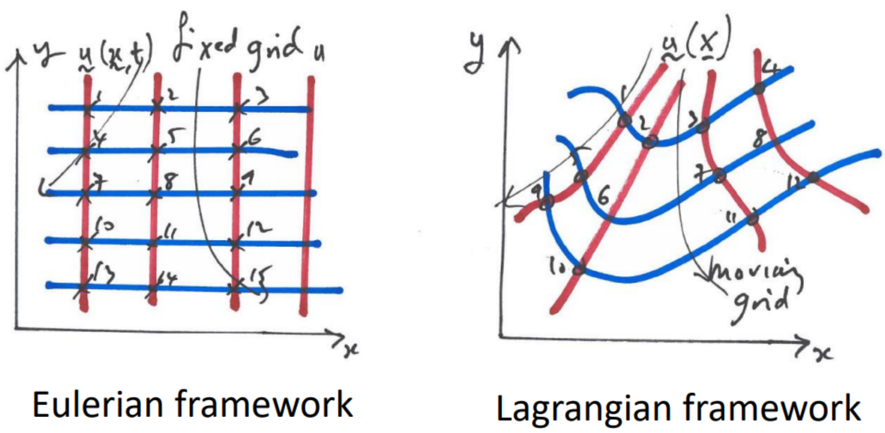

Figure 1.2

- Eulerian – information at fixed points
- Lagrangian – information at points that move with fluid or solid
- They both have advantages and disadvantages.

1.1.6 Conservation of Mass

Integral form of conservation law for a Lagrangian control volume:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_L} \rho \, \mathrm{d}V = 0 \tag{1.10}$$

Differential form for the conservation of mass:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho(\nabla \cdot u) \tag{1.11}$$

For an incompressible fluid:

$$\nabla \cdot u = 0 \tag{1.12}$$

1.1.7 Conservation of Linear Momentum

Integral form of the conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_L} \rho u \, \mathrm{d}V = \int_{S_L} \sigma \cdot \hat{n} \, \mathrm{d}S + \int_{V_L} F \, \mathrm{d}V \tag{1.13}$$

Where:

- σ is the stress tensor
- p is the pressure
- τ is the viscous stress tensor

$$\sigma = -pI + \tau \tag{1.14}$$

Differential form of the conservation of momentum:

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = \nabla \cdot \sigma + F \tag{1.15}$$

$$\rho(x,t)\left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right) = -\nabla p \tag{1.16}$$

This is Euler's equation for an inviscid fluid. The flow is compressible (explicitly stated).

1.1.8 Conservation of Angular Momentum

Integral form of the conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_L} \rho x \times u \,\mathrm{d}V = \int_{S_L} x \times \sigma \cdot \hat{n}\,\mathrm{d}S + \int_{V_L} x \times F\,\mathrm{d}V \tag{1.17}$$

The differential form of the conservation law:

$$x \times \left(\rho \frac{\mathrm{D}u}{\mathrm{D}t} - \nabla \cdot \sigma\right) = \epsilon_{ilk}\sigma_{kl} \tag{1.18}$$

Consequence is that the stress tensor is symmetric:

$$\sigma_{ij} = \sigma_{ji} \tag{1.19}$$

1.1.9 Conservation of Energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_L} \rho\left(E + \frac{1}{2}q^2\right)\mathrm{d}V = -\int_{S_L} u \cdot \sigma \cdot \hat{n}\,\mathrm{d}S + \int_{S_L} k\nabla T \cdot \hat{n}\,\mathrm{d}S + \int_{V_L} u \cdot F\,\mathrm{d}V \tag{1.20}$$

$$E_T = E + \frac{1}{2}q^2 \tag{1.21}$$

Where $q = |u|$ is the fluid speed. The differential form of the energy equation is:

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(E + \frac{1}{2}q^2\right) = -\nabla \cdot (u \cdot \sigma) + \nabla \cdot (k\nabla T) + u \cdot F \tag{1.22}$$

The continuum form on the conservation of energy says:

$$\frac{\mathrm{D}E}{\mathrm{D}t} = -\frac{p}{\rho}(\nabla \cdot u) + \frac{1}{\rho}(\phi + \nabla \cdot (k\nabla T)) \tag{1.23}$$

$$\frac{\mathrm{D}E}{\mathrm{D}t} = -\frac{pD\left(\frac{1}{\rho}\right)}{\mathrm{D}t} + \frac{\phi + \nabla \cdot (k\nabla T)}{\rho} \tag{1.24}$$

The dissipation is:

$$\phi = \nabla \cdot (u\sigma) - u \cdot \nabla\sigma \tag{1.25}$$

Compare to the differential form that you have met before:

$$\mathrm{d}E = -p\,\mathrm{d}\left(\frac{1}{\rho}\right) + \mathrm{d}Q \tag{1.26}$$

For an inviscid fluid (with no viscous dissipation $\sigma = -pI$) and no diffusion of heat:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(E + \frac{1}{2}q^2\right) = -\frac{1}{\rho}\nabla \cdot (pu) \tag{1.27}$$

1.1.10 Bernoulli's Equation

Form	Equation	Conservation Law
Isothermal	$\frac{p}{\rho}+\frac{1}{2}q^2 = const$	Mechanical energy
Isothermal	$\frac{\partial\phi}{\partial t}+\frac{p}{\rho}+\frac{1}{2}q^2+\Upsilon = const$	Mechanical energy
Adiabatic	$E+\frac{p}{\rho}+\frac{1}{2}q^2 = const$	1st law of thermodynamics
Adiabatic & perfect gas	$\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\frac{1}{2}q^2 = const$	1st law of thermodynamics & perfect gas law

Figure 1.3

1.1.11 Reference State (Stagnation)

There are two possible reference states.

(a) Stagnation flow condition

$$\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\frac{1}{2}q^2 = const \tag{1.28}$$

The speed of sound is c where:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \gamma RT = \frac{\gamma p}{\rho} \tag{1.29}$$

$$\frac{p}{\rho}\left(1+\frac{1}{2}(\gamma-1)M^2\right) = const \tag{1.30}$$

$$T\left(1+\frac{1}{2}(\gamma-1)M^2\right) = const \tag{1.31}$$

We can set the reference constant to be when the flow is a rest or stagnant. For flows where changes are significant, this state cannot be realised without other processes occurring, so that:

$$1+\frac{1}{2}(\gamma-1)M^2 = \frac{T_0}{T} \tag{1.32}$$

This tells:

$$1+\frac{1}{2}(\gamma-1)M^2 = \frac{p_0\rho}{p\rho_0} \tag{1.33}$$

This relationship is not useful, unless combined with the isentropic relationship $\frac{p}{\rho^\gamma} = const$:

$$\frac{p}{p_0} = \left(1 + \frac{1}{2}(\gamma - 1)M^2\right)^{-\frac{\gamma}{\gamma - 1}} \tag{1.34}$$

$$\frac{\rho}{\rho_0} = \left(1 + \frac{1}{2}(\gamma - 1)M^2\right)^{-\frac{1}{\gamma - 1}} \tag{1.35}$$

(b) Sonic flow condition

$$\frac{p_*}{p_0} = \left(\frac{2}{\gamma + 1}\right)^{\frac{-\gamma}{\gamma - 1}} = 0.5283 \tag{1.36}$$

$$\frac{\rho_*}{\rho_0} = \left(\frac{2}{\gamma + 1}\right)^{-\frac{1}{\gamma - 1}} = 0.63 \tag{1.37}$$

$$\frac{T_*}{T_0} = \frac{2}{\gamma - 1} = 0.8333 \tag{1.38}$$

1.2 Normal Shocks

1.2.1 Assumptions

The flow adjusts over a short distance from one region to another and the streamlines are parallel and not deflected.

This is called a normal shock. The distance can be very short (comparable with the mean-free path, $10\mu m$) so that the thickness of the wave may be ignored. Although viscous effects may be important within the wave, an inviscid analysis can be applied to understand these processes. We consider the flow across a shock wave and denote the flow properties with 1 upstream and 2 downstream.

1.2.2 Conservation Principles

Conservation of mass:

$$\nabla \cdot (\rho u) = 0 \tag{1.39}$$

Conservation of momentum:

$$\rho u \cdot \nabla u + \nabla p = 0 \tag{1.40}$$

Conservation of energy:

$$\rho u \cdot \nabla \left(E + \frac{1}{2} q^2 \right) + \nabla \cdot (pu) = 0 \tag{1.41}$$

To apply these relationships we have to integrate them across the shock. Remember, the gradient of these variables is zero on each side because the changes are confined to the thin shock.

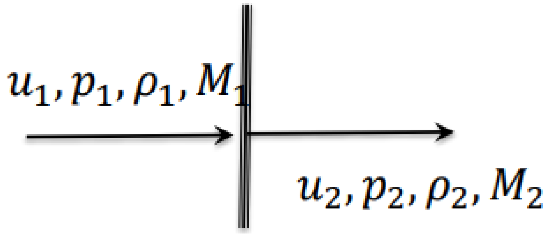

Figure 1.4

$$\rho_2 u_2 = \rho_1 u_1 \tag{1.42}$$

$$\rho_2 u_2^2 + p_2 = \rho_1 u_1^2 + p_1 \tag{1.43}$$

$$\frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} + \frac{1}{2} u_2^2 = \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} + \frac{1}{2} u_1^2 \tag{1.44}$$

$$\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2} \tag{1.45}$$

Strength of shock is:

$$\frac{p_2}{p_1} \tag{1.46}$$

1.2.3 Frame of Reference

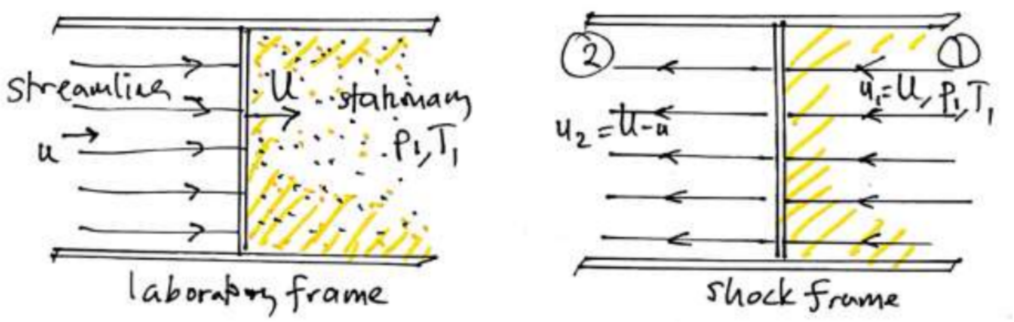

Figure 1.5: Diagram showing the frame of reference for an example question

When we consider scenarios such as shocks propagating in stationary flows, like a bomb or a pressure pulse is moving into a stationary region, we don't investigate it in this complex form. We hop on a frame of reference of the shock so the flow tends to be steady, and then we can conduct an analysis on this framce of reference.

1.2.4 Solution Technique

Aim of the calculation is to relate the flow upstream of the shock to downstream of the shock. There are 4 equations: mass, momentum, energy and state. One useful way to solve this is to use 3 of them at a time. The systems are solved in pairs:

(a) mass, momentum and state (Rayleigh flow) - This occurs in when heat is added to a flow
(b) mass, energy and state (Fanno flow) - This occurs in when pipe friction is important

1.2.5 Algebraic Manipulation

From the momentum equation:

$$p_1 - p_2 = \rho_2 u_2^2 - \rho_1 u_1^2 = \rho_1 u_1 (u_2 - u_1) \tag{1.47}$$

Rearranging gives:

$$u_2^2 - u_1^2 = \frac{(p_1 - p_2)(u_1 + u_2)}{\rho_1 u_1} = (p_1 - p_2)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \tag{1.48}$$

$$\frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right) = (p_1 - p_2)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \tag{1.49}$$

Rankine-Hugoniot relation:

$$\frac{\rho_2}{\rho_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{p_2}{p_1} + 1}{\frac{p_2}{p_1} + \frac{\gamma+1}{\gamma-1}} \tag{1.50}$$

The strength of the shock is:

$$\frac{p_2}{p_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\rho_2}{\rho_1} - 1}{\frac{\gamma+1}{\gamma-1} - \frac{\rho_2}{\rho_1}} \tag{1.51}$$

This relationship is valid of shocks and highly non-linear behaviour. The first point in the discussion is what happens when the shock is weak.

1.2.6 Weak Shocks

This can be demonstrated analytically by considering changes of pressure and density across the shock. To show this, let $p_2 = p_1 + \Delta p$ and $\rho_2 = \rho_1 + \Delta\rho$, then substituting into the Rankine-Hugoniot gives:

$$\frac{\Delta p}{p_1} = \frac{p_2 - p_1}{p_1} = \frac{\dfrac{2\gamma(\rho_2 - \rho_1)}{(\gamma-1)\rho_1}}{\dfrac{2}{\gamma-1}} = \frac{\gamma\Delta\rho}{\rho_1} \tag{1.52}$$

which is the same as the isentropic approximation obtained by taking the differential of $\frac{p}{\rho^\gamma} = const.$

We can write ρu^2 as $\rho c^2 M^2 = \gamma p M^2$. Thus the momentum equation may be written as:

$$p_1 - p_2 = \gamma p_2 M_2^2 - \gamma p_1 M_1^2 \tag{1.53}$$

or

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \tag{1.54}$$

The density ratio is:

$$\frac{\rho_1}{\rho_2} = \frac{p_2}{p_1}\left(\frac{M_2}{M_1}\right)^2 \tag{1.55}$$

This is known as the **Rayleigh line**.

Since there is no change in stagnation temperature across the shock:

$$\frac{T_2}{T_1} = \frac{1 + \frac{1}{2}(\gamma-1)M_1^2}{1 + \frac{1}{2}(\gamma-1)M_2^2} \tag{1.56}$$

From the equations of continuity and state:

$$\frac{T_2}{T_1} = \frac{p_2\rho_1}{p_1\rho_2} = \frac{p_2 u_2}{p_1 u_1} = \frac{p_2 M_2}{p_1 M_1}\left(\frac{T_2}{T_1}\right)^{\frac{1}{2}} \tag{1.57}$$

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\frac{M_2}{M_1}\right)^2 \tag{1.58}$$

Substituting into the equation of state gives:

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{1 + \frac{1}{2}(\gamma-1)M_1^2}{1 + \frac{1}{2}(\gamma-1)M_2^2}\right)^{\frac{1}{2}} \tag{1.59}$$

This is known as the **Fanno line**.

Matching the solutions to the Fanno and Rayleigh lines gives the combined solution to the mass, momentum, energy conservation equations and the equation of state:

$$\frac{M_1\left(1+\frac{1}{2}(\gamma-1)M_1^2\right)^{\frac{1}{2}}}{1+\gamma M_1^2}=\frac{M_2\left(1+\frac{1}{2}(\gamma-1)M_2^2\right)^{\frac{1}{2}}}{1+\gamma M_2^2} \tag{1.60}$$

The solutions to this are:

$$M_1 = M_2 \tag{1.61}$$

$$M_2^2=\frac{1+\frac{1}{2}(\gamma-1)M_1^2}{\gamma M_1^2-\frac{1}{2}(\gamma-1)} \tag{1.62}$$

Note that for $M_1 = M_2 = 1$ there is no shock. For upstream supersonic flows $M_1 > 1$, the downstream flow is subsonic $M_2 < 1$.

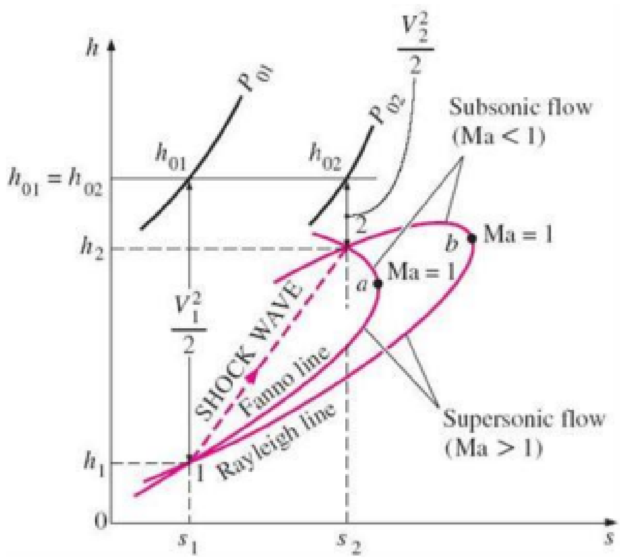

Figure 1.6

The pressure and density ratios are important:

$$\frac{p_2}{p_1}=\frac{2\gamma M_1^2}{\gamma+1}-\frac{\gamma-1}{\gamma+1} \tag{1.63}$$

$$\frac{\rho_2}{\rho_1}=\frac{\frac{1}{2}(\gamma+1)M_1^2}{1+\frac{1}{2}(\gamma-1)M_1^2} \tag{1.64}$$

Stagnation pressure values can be calculated from:

$$\frac{p_{20}}{p_{10}}=\frac{p_{20}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{10}} \tag{1.65}$$

where the relationship been static and stagnation pressures have been defined.

1.2.7 Entropy Considerations

From the conservation of energy:

$$\mathrm{d}Q = \mathrm{d}E + \mathrm{d}W = c_v\,\mathrm{d}T + pd\left(\frac{1}{\rho}\right) \tag{1.66}$$

Since:

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} = \frac{c_v\,\mathrm{d}T}{T} + \frac{p}{T}d\left(\frac{1}{\rho}\right) = \frac{c_v + R}{T}\,\mathrm{d}T - R\frac{\mathrm{d}p}{p} \tag{1.67}$$

Integrating gives an entropy change of:

$$\Delta s = \int_1^2 \mathrm{d}s = c_p \log\left(\frac{T_2}{T_1}\right) - R\log\left(\frac{p_2}{p_1}\right) = c_p \log\left(\frac{T_2}{T_1}\left(\frac{p_1}{p_2}\right)^{\dfrac{\gamma-1}{\gamma}}\right) \tag{1.68}$$

Since $c_p = \dfrac{\gamma R}{\gamma - 1}$, we can rearrange as:

$$\frac{\Delta s}{R} = \frac{\gamma}{\gamma-1}\left(\log\left(\frac{\rho_1}{\rho_2}\right) + \frac{1}{\gamma}\log\left(\frac{p_2}{p_1}\right)\right) \tag{1.69}$$

Specifically for a shock:

$$\frac{\Delta s}{R} = \frac{\gamma}{\gamma-1}\log\left(\frac{2}{(\gamma+1)M_1^2} + \frac{\gamma-1}{\gamma+1}\right) + \frac{1}{\gamma-1}\log\left(\frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\right) \tag{1.70}$$

When $M_1 < 1$, $M_2 > 1$ and $\Delta s < 0$. This is unphysical and is ignored. But when $M_1 > 1$, $M_2 < 1$ and $\Delta s > 0$.

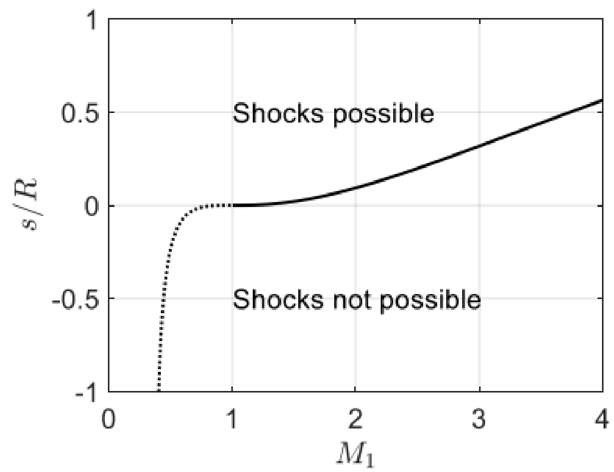

Figure 1.7

1.2.8 $T - s$ and $p - 1/\rho$ Diagrams

We use these types of figures to analyse systems.

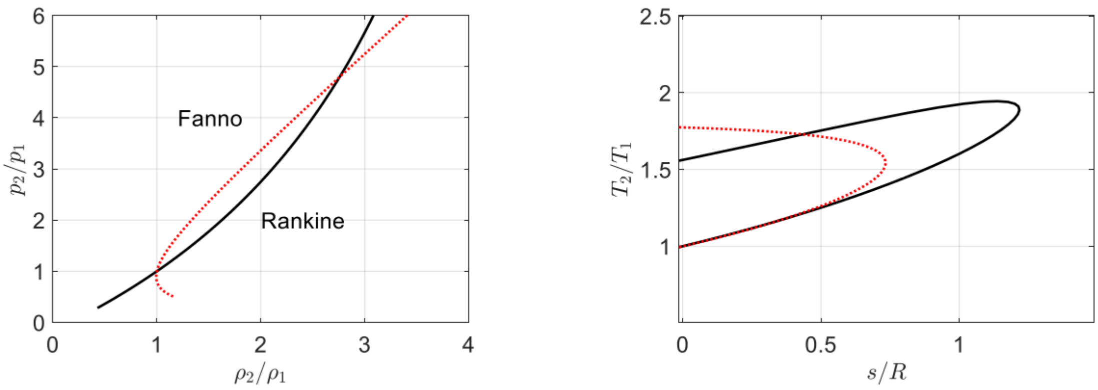

Figure 1.8

$\frac{p_{02}}{p_{01}} < 1$	Loss in stagnation pressure
$\frac{p_2}{p_1} > 1$	Increase in static pressure
$\frac{T_2}{T_1} > 1$	Increase in static temperature
$\frac{\rho_2}{\rho_1} > 1$	Increase in density
$\frac{u_2}{u_1} < 1$	Decrease in velocity
$M_2 < 1$	Subsonic flow behind shock
$\frac{T_{02}}{T_{01}} = 1$	No change in stagnation temperature

Figure 1.9

Chapter 2

Oblique Shocks

2.1 Recap

2.1.1 Analogy with hydraulic jumps

”Supersonic flows” in the kitchen. In free-surface flows, we have the Froude number

$$Fr = \frac{u}{c} \tag{2.1}$$

- $Fr < 1$: subcritical
- $Fr > 1$: supercritical
- u: (depth averaged) flow speed
- c: wave speed

Understanding these flows are important for looking at forces on buildings/bridges. The same conservation principles apply (except energy; which is not conserved here) and is equivalent to $\delta = 2$. Used by Ernest Mach to illustrate shock collision with walls.

2.1.2 Oblique shocks and terminology

The Mach number varies with position in space. Shock represents:

1. change in stagnation pressure

and depends on

2. stagnation pressure unchanged

3. rapid changes over a short distance

- Oblique shock - flow is deflected and squashed.
- Weak oblique shocks (Mach number is supersonic on both sides).
- Strong oblique shock (Mach number is subsonic after shock)

2.1.3 Streamlines

The local **instantaneous** velocity $u(x,t)$ is tangential to the local streamline. Streamlines have directions and need arrows!

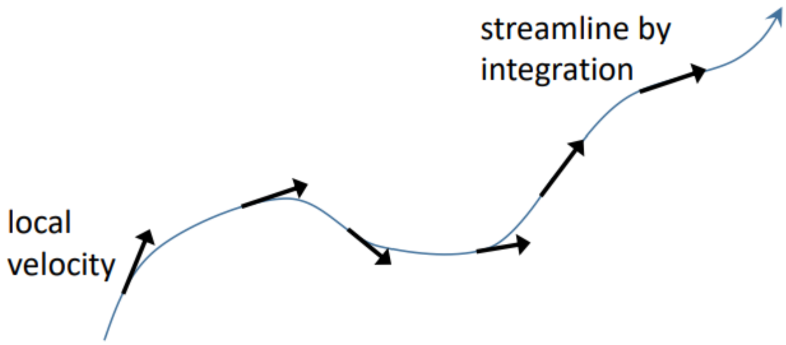

Figure 2.1: Streamline.

(note the difference from pathlines and streaklines).

- Streamlines do not cross (except at point source and sinks, or where mass is created or lost)
- They cannot meet at rigid bodies except at a special point (the stagnation point)
- THe surface of a body is a streamline

2.2 Oblique shocks

2.2.1 Two types of oblique shock waves

Geometry controlled shocks

This is where the deflection angle of the flow is specified.

Pressure controlled shocks

THis is where the pressure after the shock is known. Occurs in supersonic jets.

2.2.2 Schematic and notation

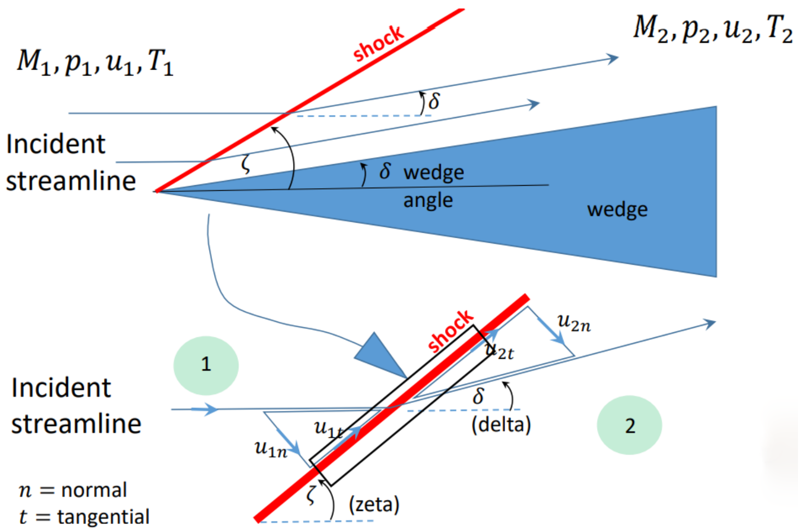

Figure 2.2: Schematic and notation.

Geometry controlled case. All angles are relative to incident streamline.

$$\delta \text{ - deflection or wedge angle} \tag{2.2}$$
$$\zeta \text{ - shock angle} \tag{2.3}$$

The aim is to relate δ, ζ, M and $\frac{p_2}{p_1}$. We need to know M_1 and δ - geometrically constrained or M_1 and $\frac{p_2}{p_1}$ pressure constrained.

2.2.3 Conservation equations

The differential form of the conservation laws are mass:

$$\underline{\nabla} \cdot (\rho \underline{u}) = 0 \text{ (scalar)} \tag{2.4}$$

Momentum:

$$\rho \underline{u} \cdot \underline{\nabla u} + \underline{\nabla} p = 0 \text{ (vector)} \tag{2.5}$$

Energy:

$$\rho \underline{u} \cdot \underline{\nabla} \left(E + \frac{1}{2} q^2 \right) + \underline{\nabla} \cdot (p \underline{u}) = 0 \tag{2.6}$$

The word equation form is sometimes more useful:

1. Mass: mass flux is conserved
2. Momentum: increase in momentum flux is balanced by a decrease in pressure
3. Energy: sum of internal energy and kinetic energy is equal to the work done by press.

2.2.4 Normal shock relations

$$\rho_2 u_{2n} = \rho_1 u_{1n} \tag{2.7}$$
$$\rho_2 u_{2n}^2 + p_2 = \rho_1 u_{1n}^2 + p_1 \tag{2.8}$$
$$\frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{1}{2}q_2^2 = \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}q_1^2 \tag{2.9}$$

The conservation of energy is expressed in terms of the (specific) kinetic energy that depends on the gas speed $q^2 = u^2 + v^2$. As we are going to see, these relationships are identical to the oblique shock analysis.

2.2.5 Oblique shock relations

Integrating the conservation laws:

$$\rho_2 u_{2n} = \rho_1 u_{1n} \tag{2.10}$$
$$\rho_2 u_{2n}^2 + p_2 = \rho_1 u_{1n}^2 + p_1 \tag{2.11}$$
$$\frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{1}{2}\left(u_{2n}^2 - \cancel{u_{2t}^2}\right) = \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}\left(u_{1n}^2 - \cancel{u_{1t}^2}\right) \tag{2.12}$$

The conservation of momentum tangential to the shock is:

$$\rho \underline{u} \cdot \underline{\nabla} u_t = \underline{u} p \cdot \hat{\underline{t}} \tag{2.13}$$

Where $\hat{\underline{t}}$ is the unit vector tangential to the shock. Since the pressure is constant on both sides of the shock, the RHS is zero. This means that:

$$\rho_1 u_{1n} u_{1t} = \rho_2 u_{2n} u_{2t} \text{ or } u_{1t} = u_{2t} \tag{2.14}$$

From the inclination of the shock and deflection angle:

$$u_{1n} = u_1 \sin\zeta \tag{2.15}$$
$$u_{2n} = u_2 \sin(\zeta - \delta) \tag{2.16}$$
$$u_{1t} = u_1 \cos\zeta \tag{2.17}$$
$$u_{2t} = u_2 \cos(\zeta - \delta) \tag{2.18}$$

Normal shock relationship:

$$M_{2n}^2 = \frac{1 + \frac{1}{2}(\gamma - 1) M_{1n}^2}{\gamma M_{1n}^2 - \frac{1}{2}(\gamma - 1s)} \tag{2.19}$$

Oblique shock relationship:

$$M_2^2 \sin^2(\zeta - \delta) = \frac{1 + \frac{1}{2}(\gamma - 1) M_1^2 \sin^2\zeta}{\gamma M_1^2 \sin^2\zeta - \frac{1}{2}(\gamma - 1)} \tag{2.20}$$

This gives a relationship between 3 variables:

$$\tan\delta = \frac{\cot\left(\zeta\left(M_1^2 \sin^2(\zeta - 1)\right)\right)}{\frac{1}{2}(\gamma + 1) M_1^2 - M_1^2 \sin^2(\zeta) + 1} \tag{2.21}$$

2.2.6 Relationship between M_1, ζ and δ

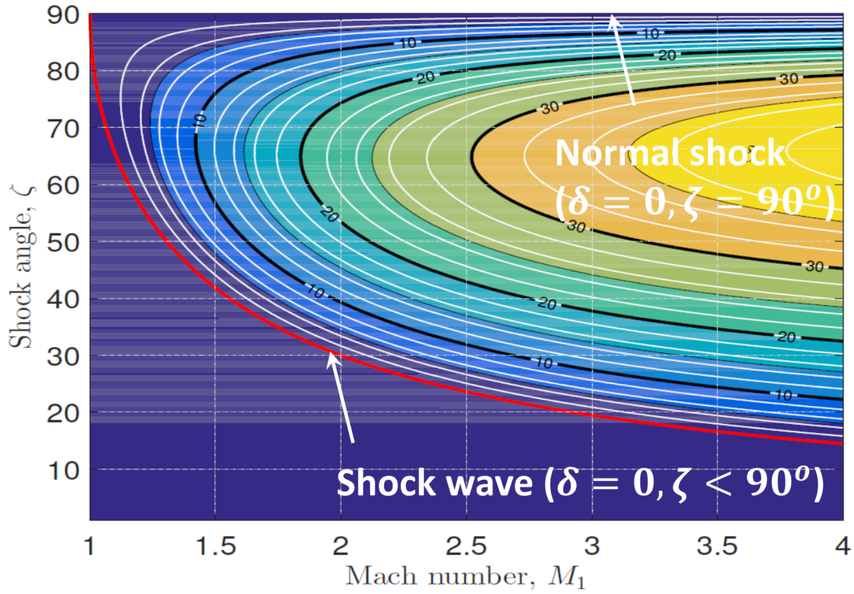

Figure 2.3: Shock angle vs Mach number.

We deconstruct the solution into parts that we can understand before looking at more specifically the solutions. For $\delta = 0$ (no streamline deflection), we have:

- $\zeta = 90°$: normal shock
- $\zeta < 90°$: shock wave

2.2.7 Sonic waves

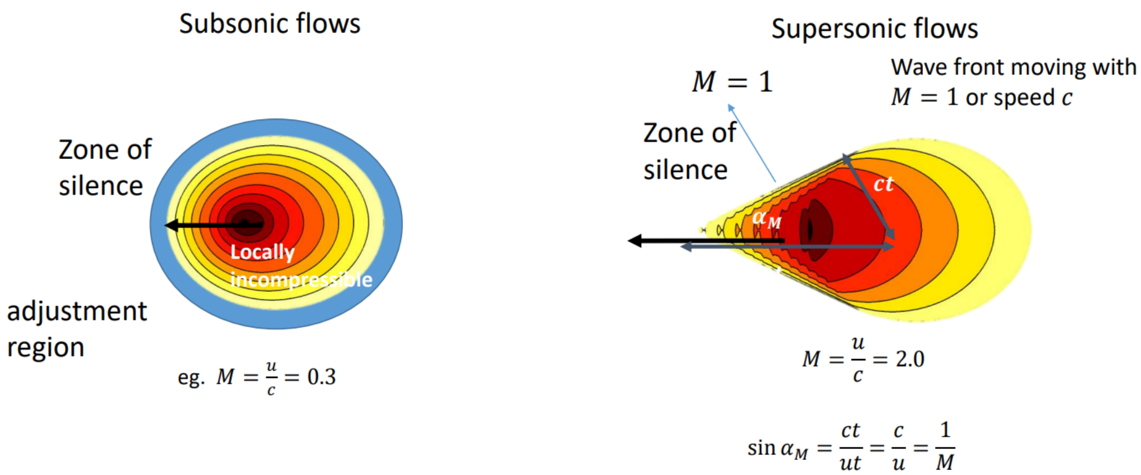

Figure 2.4: Subsonic vs supersonic flows.

2.2.8 Multiple solutions (δ, ζ, M_1)

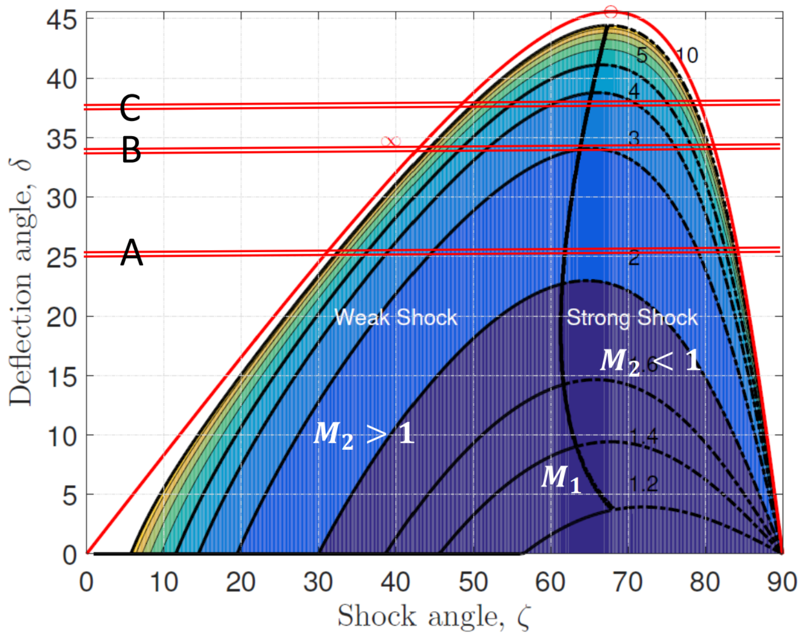

Figure 2.5: Deflection angle vs shock angle.

Look at the case where $M_1 = 3$ with three examples of $\delta = 25°$, $34°$ and $38°$.

- (A) There are two solutions here, a weak shock where $\zeta = 45°$ and $78°$
- (B) There is one unique solution $\zeta = 68°$ for $\delta = 34°$. At wedge angles greater than this, we have case (C)
- (C) There is no solution here. This is because there is no way for the flow to be able to adjust by an oblique shock. Something else happens.

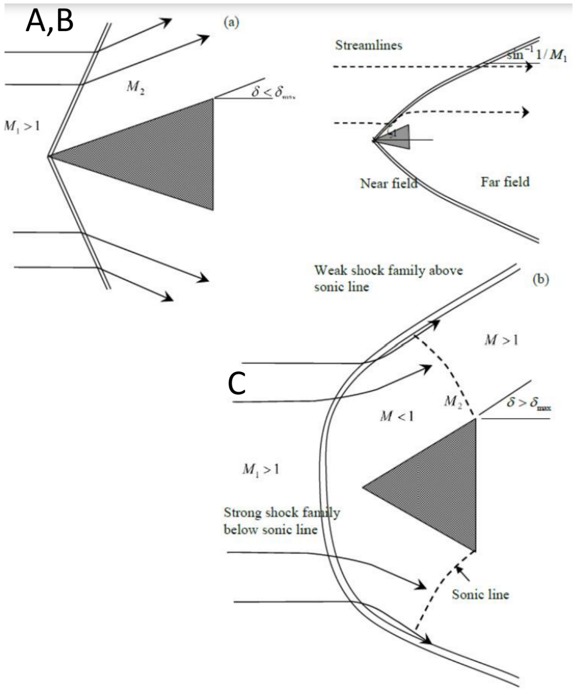

Figure 2.6: The shock angles changes from the near field to the far field.

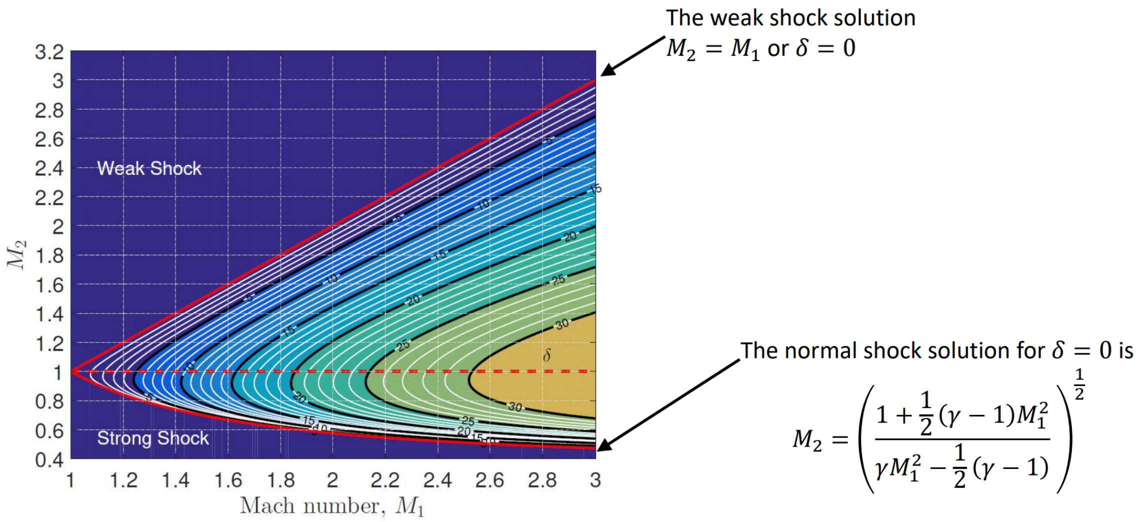

Figure 2.7: M_2 vs Mach number (M_1).

2.2.9 Pressure ratio $\frac{p_2}{p_1}$

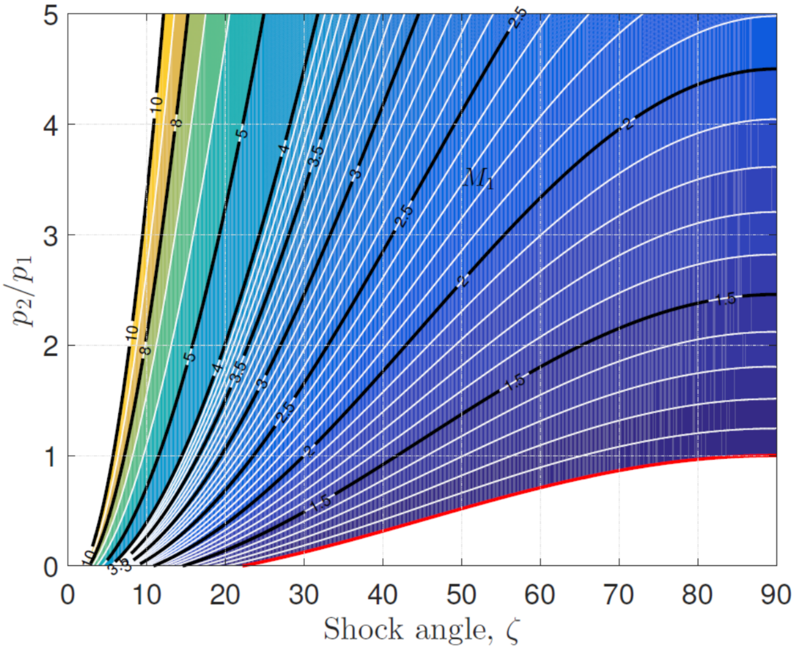

Figure 2.8: Pressure ratio vs shock angle.

The progression of the pressure across the shock is important and is calculated graphically here. Notice that in all cases, $\frac{p_2}{p_1} > 1$. In the examples we have seen, the flow is controlled by the bounding geometry. In many other cases however, it is controlled by pressure, for instance, at the edge of a rocket exit.

2.2.10 Summary

Two types of shocks: geometry controlled or pressure controlled. Two types of shocks: weak shocks and strong shocks.

Chapter 3

Prandtl-Meyer Expansion Fans

3.1 Prandtl-Meyer Expansion Fans

3.1.1 Examples of supersonic expansion

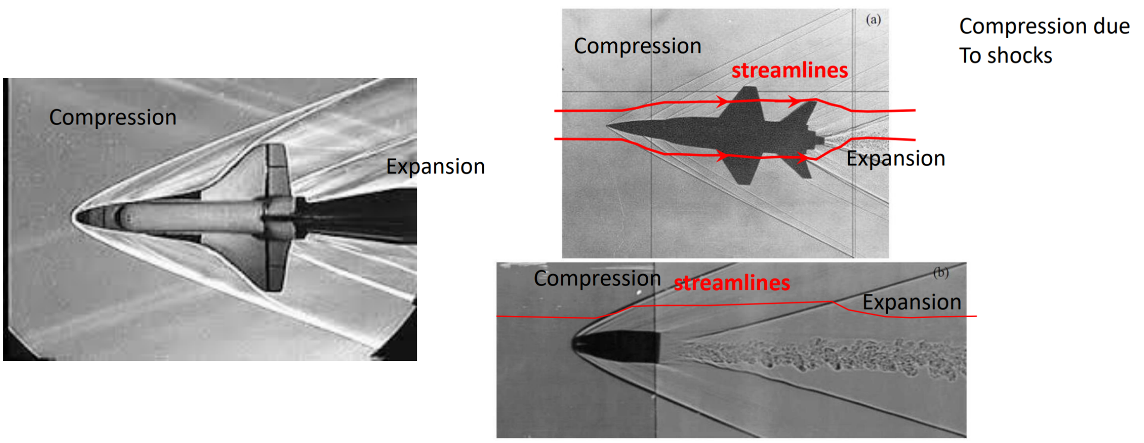

Figure 3.1: Examples of supersonic expansion.

3.1.2 Sonic waves

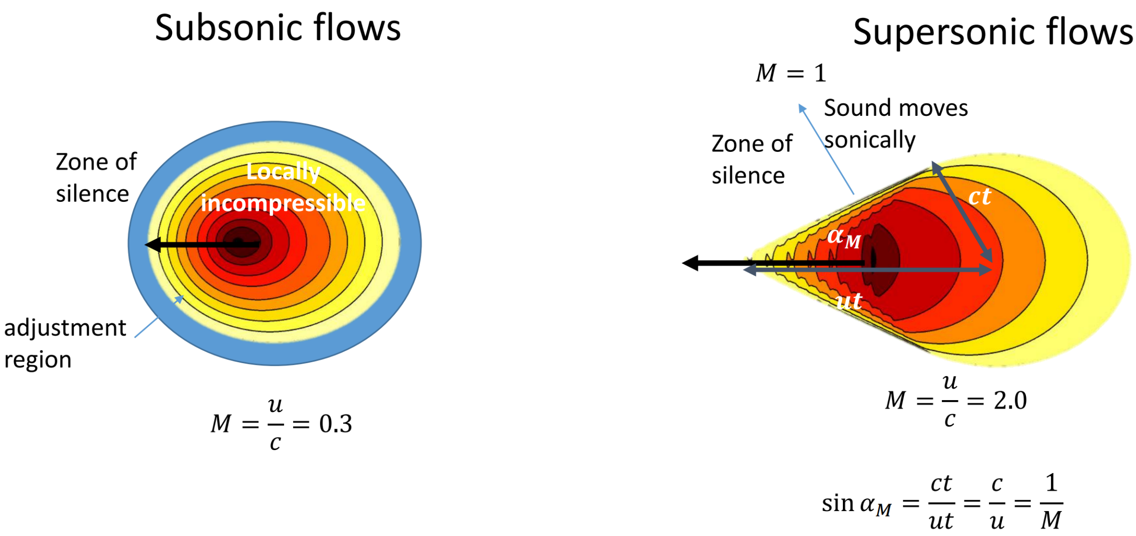

Figure 3.2: Sonic waves.

3.1.3 Turning a corner

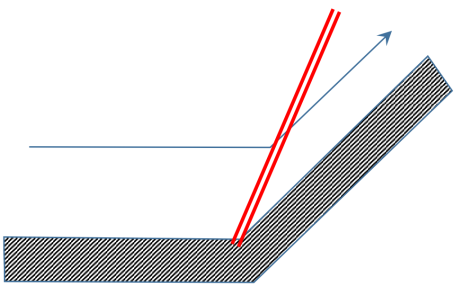

Figure 3.3: Shock is formed. Entropy decreases.

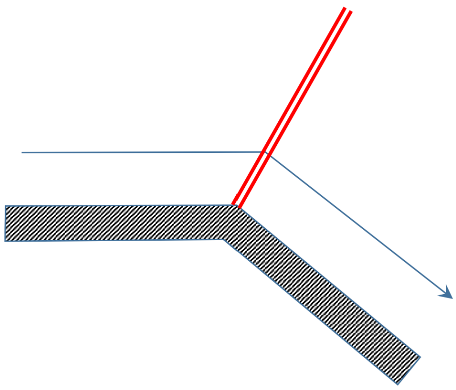

Figure 3.4: If a single shock is formed, entropy increases and this is unphysical.

A we have seen in the previous section, as the flow is turned by a wedge, the flow is compressed and the Mach number decreases. Whether the flow downstream of an oblique shock is supersonic depends on the deflection angle.

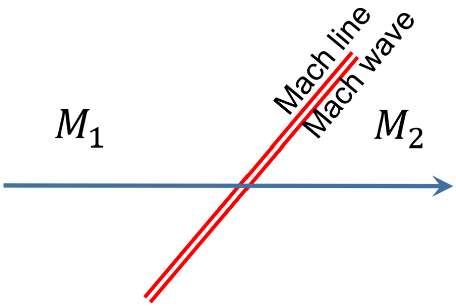

Figure 3.5

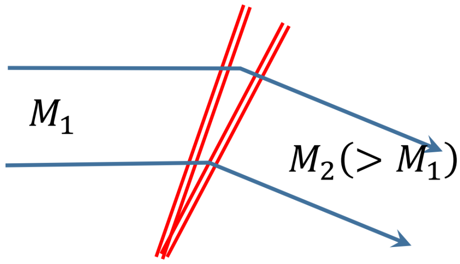

Figure 3.6

A Mach line or waves is a very weak shock comprised of an isentropic process. We can study the relationship between the turning angle and the change in Mach number by studying the deflection process in more detail.

Here we consider the case where the flow expands and the process of adjustment. The Mach number of the flow will increase due to expansion, but the pressure wil decrease. The process of adjustment takes place by shockwaves being emitted. The first result we need to consider is the angle ν that the shock waves make to the incident flow.

3.1.4 Schematic and notation

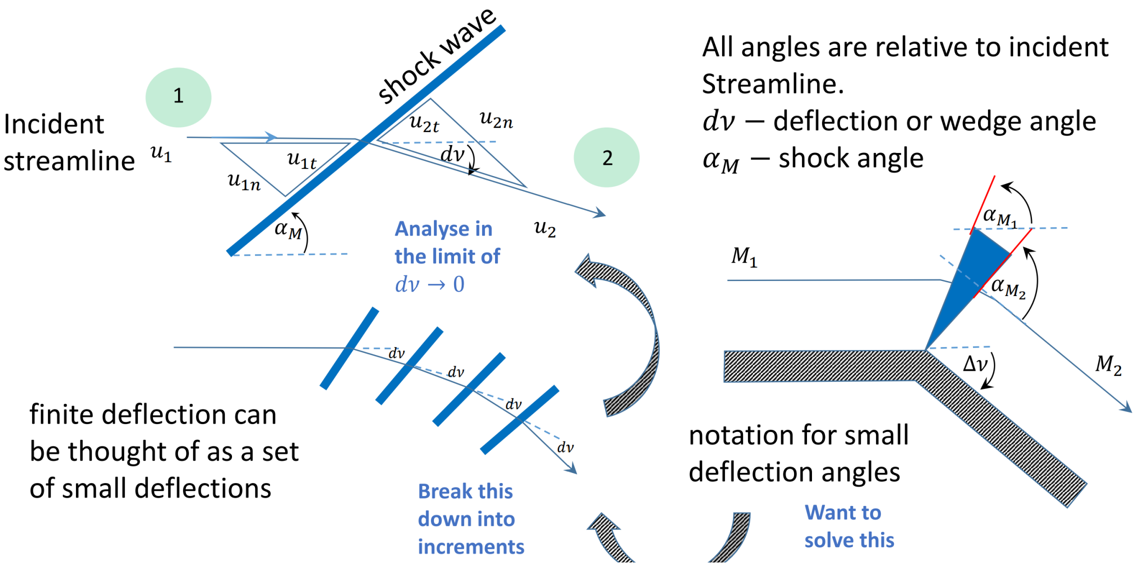

Figure 3.7: Schematic and notation.

3.1.5 Geometrical characteristics

The slip velocity is continuous so that:

$$u_1 \cos \alpha_M = u_2 \cos (\alpha_M + \mathrm{d}v) \tag{3.1}$$

From the double angle formula:

$$\cos\left(\alpha_M + \mathrm{d}v\right) \approx \cos\alpha_M - \mathrm{d}v \sin\alpha_M \tag{3.2}$$

Rearranging gives:

$$\frac{\mathrm{d}u}{u} = \frac{u_2 - u_1}{u_2} = \frac{\sin\alpha_M}{\cos\alpha_M}\,\mathrm{d}v \tag{3.3}$$

Since the Mach wave angle α_m is related to the Mach number by:

$$\sin\alpha_M = \frac{1}{M} \tag{3.4}$$

Combining and using $\cos^2\alpha_M = 1 - \sin^2\alpha_M$, gives:

$$\frac{\mathrm{d}u}{u} = \frac{\sin\alpha_M}{\left(1 - \sin^2\alpha_M\right)^{\frac{1}{2}}}\,\mathrm{d}v = \frac{\mathrm{d}v}{\left(M^2 - 1\right)^{\frac{1}{2}}} \tag{3.5}$$

We can write:

$$\frac{u^2}{M^2} = \frac{u^2}{\left(\dfrac{u^2}{\gamma RT}\right)} = \gamma RT \tag{3.6}$$

From the Conservation of Energy,

$$T\left(1 + \frac{1}{2}\left(\gamma - 1\right)M^2\right) = \mathrm{const} \tag{3.7}$$

Taking the square root gives:

$$\frac{u}{M}\left(1 + \frac{1}{2}\left(\gamma - 1\right)M^2\right)^{\frac{1}{2}} = \mathrm{const} \tag{3.8}$$

Taking the logarithm:

$$\log u - \log M + \frac{1}{2}\log\left(1 + \frac{1}{2}\left(\gamma - 1\right)M^2\right) = \mathrm{const} \tag{3.9}$$

To find the differential change we take the differential to give us:

$$\frac{\mathrm{d}u}{u} - \frac{\mathrm{d}M}{M} + \frac{\frac{1}{2}\left(\gamma - 1\right)\frac{1}{2}2M}{1 + \frac{1}{2}\left(\gamma - 1\right)M^2}\,\mathrm{d}M = 0 \tag{3.10}$$

Rearranging gives:

$$\frac{\mathrm{d}u}{u} = \left(\frac{1}{M} - \frac{\frac{1}{2}\left(\gamma - 1\right)\frac{1}{2}2M}{1 + \frac{1}{2}\left(\gamma - 1\right)M^2}\right)\frac{\mathrm{d}M}{M} \tag{3.11}$$

$$= \frac{1}{1 + \frac{1}{2}\left(\gamma - 1\right)M^2}\frac{\mathrm{d}M}{M} \tag{3.12}$$

Combining the above equations:

$$\frac{\mathrm{d}v}{\left(M^2 - 1\right)^{\frac{1}{2}}} = \frac{\mathrm{d}u}{u} = \frac{\mathrm{d}M}{M\left(1 + \frac{1}{2}\left(\gamma - 1\right)M^2\right)} \tag{3.13}$$

In total,

$$\Delta v = \int_0^{\Delta v} = \int_{M_1}^{M_2}\left(\frac{\left(M^2 - 1\right)^{\frac{1}{2}}}{M\left(1 + \frac{1}{2}\left(\gamma - 1\right)M^2\right)}\right)\mathrm{d}M \tag{3.14}$$

3.1.6 Relationship between turning angle and M

$$\Delta v = \int_0^{\Delta v} = \int_{M_1}^{M_2} \left(\frac{\left(M^2 - 1\right)^{\frac{1}{2}}}{M\left(1 + \frac{1}{2}\left(\gamma - 1\right) M^2\right)} \right) \mathrm{d}M \tag{3.15}$$

This is quite difficult to use since it would require tabulating Δv against M_1 and M_2. Instead, we use the sonic reference condition and set $M_1 = 1$ and $M_2 = M$. This gives:

$$v = \left(\frac{\gamma + 1}{\gamma - 1}\right)^{\frac{1}{2}} \arctan\left(\left(\frac{\gamma + 1}{\gamma - 1}\right)^{\frac{1}{2}} \left(M^2 - 1\right)^{\frac{1}{2}}\right) - \arctan\left(M^2 - 1\right)^{\frac{1}{2}} \tag{3.16}$$

This formula is plotted in the Prandtl-Meyer charts below.

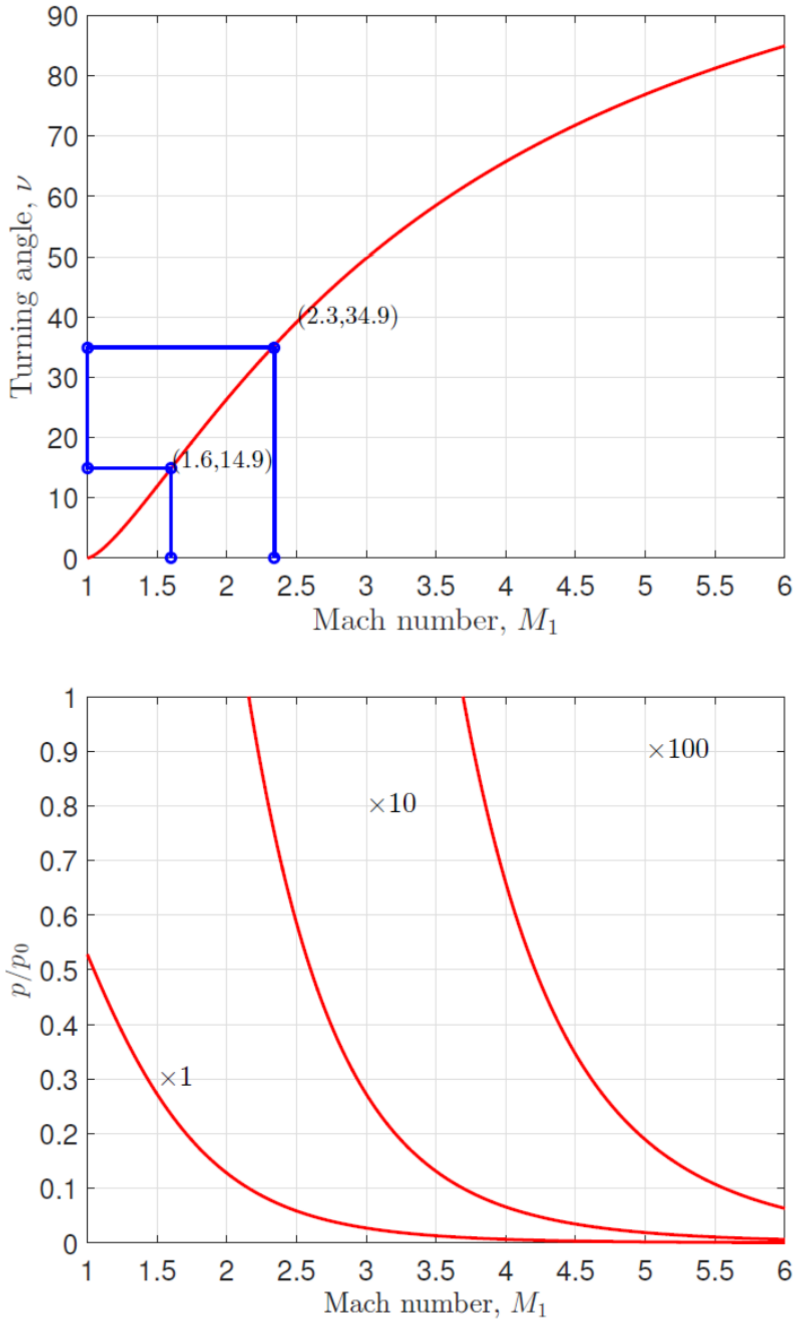

Figure 3.8: Prandtl-Meyer charts.

3.1.7 How to use expansion charts

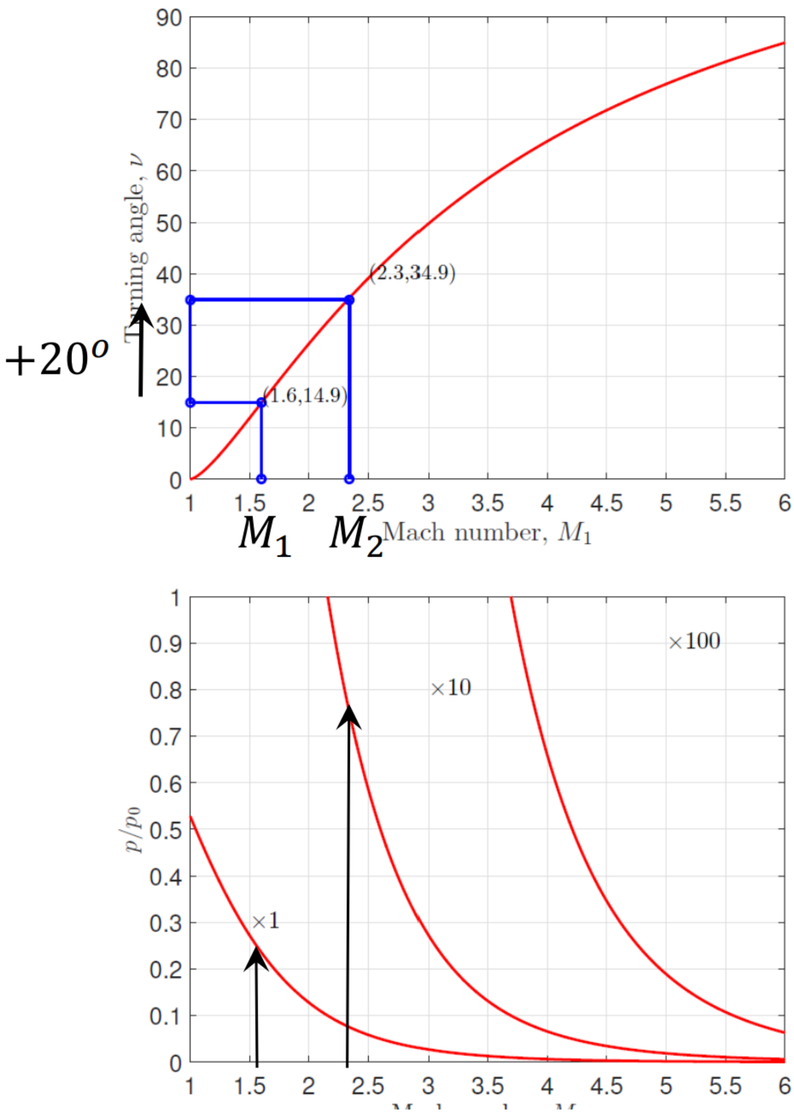

Figure 3.9: Prandtl-Meyer charts.

To illustrate how the charts are used, consider the finite deflection of a flow $M_1 = 1.6$ and the flow turns an angle $\Delta v = 20°$. We have to determine the turning angle from $M = 1$ to $M_1 = 1.6$. From the chart above this is $V_1 = 14.9°$. In total,

$$v_2 = v_1 + \Delta v = 14.9° + 20° = 34.9° \tag{3.17}$$

The corresponding Mach number is:

$$M_2 = 2.31 \tag{3.18}$$

The fan angles are:

$$\alpha_{M_1} = \arcsin \frac{1}{M_1} = 38.7° \tag{3.19}$$

$$\alpha_{M_2} = \arcsin \frac{1}{M_2} = 25.7° \tag{3.20}$$

For an expansion wave, the stagnation pressure is conserved.

$$p_0 = p_1 \left(1 + \frac{1}{2}(\gamma - 1) M_1^2\right)^{\frac{\gamma}{\gamma-1}} \tag{3.21}$$

$$= p_2 \left(1 + \frac{1}{2}(\gamma - 1) M_2^2\right)^{\frac{\gamma}{\gamma-1}} \tag{3.22}$$

This enable the pressure after the expansion fan to be estimated. The stagnation pressure p_0 is constant across the expansion fan.

3.1.8 Maximum turning angle

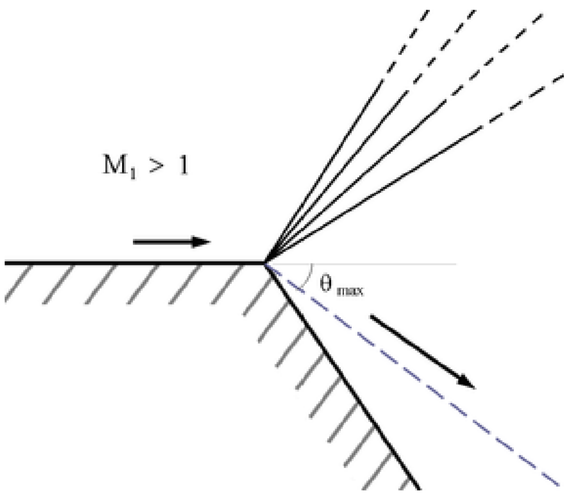

Figure 3.10: Prandtl-Meyer charts.

From the formula, we find that:

$$v_{max} = \frac{\pi}{2}\left(\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}} - 1\right) \tag{3.23}$$

This places a limit on the maximum turning angle that can be supported. When the deflection angle is greater than:

$$\theta_{max} = v_{max} - v\left(M_1\right) \tag{3.24}$$

a slip plane is generated.

3.1.9 Turning a corner

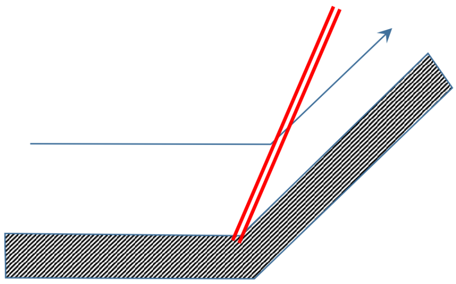

Figure 3.11: Shock is formed.

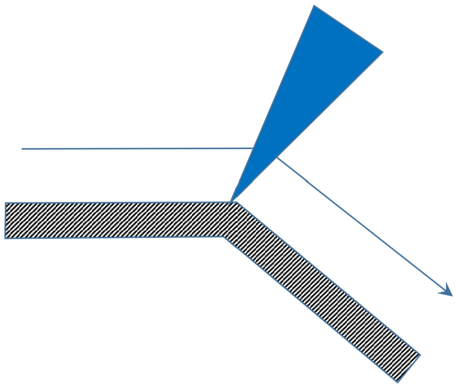

Figure 3.12: Expansion fan.

As we have seen in the previous section, as the flow is turned by a wedge, the flow is compressed and the Mach number decreases. Whether the flow downstream of an oblique shock is supersonic depends on the deflection angle.