

# MECH0011: Intermediate Thermodynamics and Fluid Mechanics

Topic Notes

UCL

HD

March 9, 2022

# **Contents**

# Course Outline

## Contact Information

Dr Andrea Ducci	424 Roberts Building	a.ducci@ucl.ac.uk
Prof. Kai Luo	506 Malet Place	k.luo@ucl.ac.uk

## Methods of Assessment

- 50% Final Coursework - Design Exercise (Term 3)
- 10% Quiz - Fluid Mechanics (Term 1, Weekly (5/8))
- 10% Quiz - Thermodynamics (Term 2, Weekly (3/5))
- 15% Fluid Mechanics Lab (Term 1, 9-13 of November, Groups of 4-5)
- 15% Thermodynamics Lab (Term 2)

## Topics to be Covered

### Fluid Mechanics

- Differential equations of mass, momentum and energy balances (F.M. White, Chapters 4 and 8)
- Flow Past a Body (Lift and Drag) - (F.M. White, Chapters 7)
- Flow in Pipe (F.M. White, Sections 6.4-6.7)
- Boundary-layer theory (F.M. White, Sections 7.2-7.4)

### Thermodynamics

- Mixtures of ideal gases
- Basic psychrometric processes
- Elementary combustion
- Steam turbine cycles
- Gas turbine cycles
- Thermodynamics in Renewable energy

# **Part I**

# **Fluids**

# Chapter 1

## Differential Analysis

06/10/2020

In order to understand a fluid flow completely, we want to know:

- Velocity Field
- Pressure
- Density
- Temperature

at any time ( $t$ ) and any place ( $x, y, z$ ). All the variables mentioned are going to be a function of space and time. A mathematical model, based on the mass-conservation equation and momentum-conservation equation, will be formulated to obtain the flow around an object.

### List of Variables:

Name	Variable	Type	Unit	Variables
Velocity	$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$	Vector	( $\text{m s}^{-1}$ )	3
Pressure	$p$	Scalar	( $\text{N m}^{-2}$ )	1
Temperature	$T$	Scalar	( $^{\circ}\text{C}$ )	1
Density	$\rho$	Scalar	( $\text{kg m}^{-3}$ )	1
Stress Tensor	$T = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$		( $\text{N m}^{-2}$ )	6

For most of this course, the fluid will be assumed to be incompressible. Hence, the  $\rho$  will stay constant most of the time. The stress tensor takes into account of the forces which are exerted on the surface of an infinitesimal particle. It has 6 variables; those who are on the opposite side of the diagonal (such as  $\tau_{xy}$  and  $\tau_{yx}$ ) are the same. Overall, there are 12 variables; hence, 12 equations are needed to fully understand a specific flow.

### 1.1 Conservation of Mass - Continuity Equation

Control Volume Analysis can be done with the following equation:

$$\frac{\partial}{\partial t} \int_V \rho dV + \oint_S \rho \vec{V} \cdot \hat{n} dS = 0 \quad (1.1.1)$$

The  $\frac{\partial}{\partial t} \int_V \rho dV$  part of the equation describes the mass in the control volume and how it changes with time.

The  $\oint_S \rho \vec{V} \cdot \hat{n} dS$  part of the equation takes in the account the mass going in and out of the control volume (crossing the control surface).

The equation is equal to 0 as the mass has to be conserved.

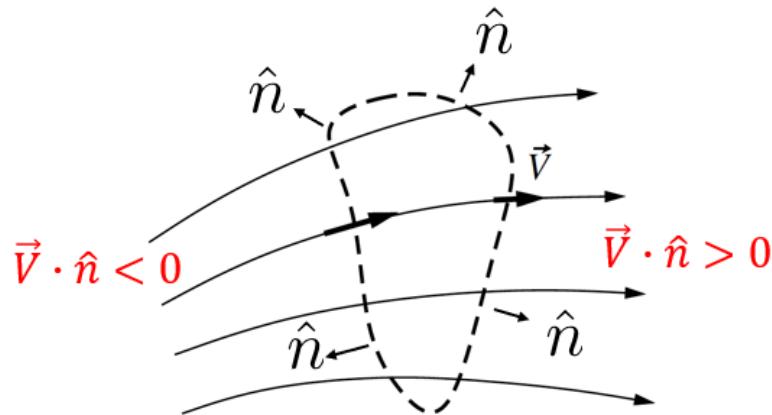
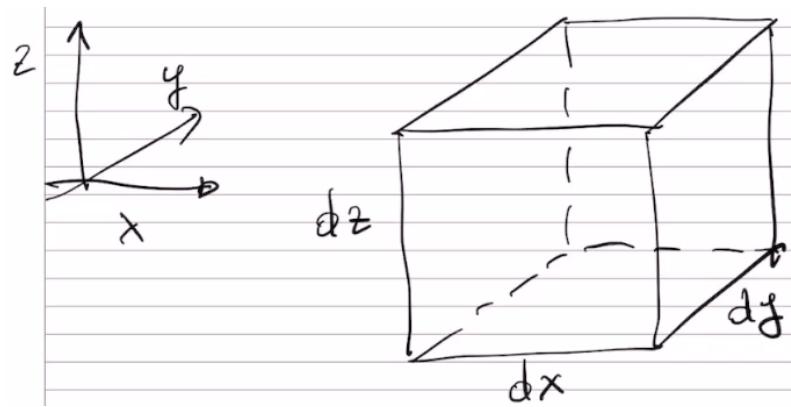


Figure 1.1: A control volume with a flow, velocity field  $\vec{V}$  and the normal vectors  $\hat{n}$

On the right side of the control volume, the angle between  $\vec{V}$  and  $\hat{n}$  is greater than  $90^\circ$ . Therefore,  $\vec{V} \cdot \hat{n} > 0$ , meaning a net mass is going out of the control volume. On the left side of the control volume, the angle between  $\vec{V}$  and  $\hat{n}$  is less than  $90^\circ$ . Therefore,  $\vec{V} \cdot \hat{n} < 0$ , meaning a net mass is going into the control volume.

## 1.2 Differential Form of Continuity Equation

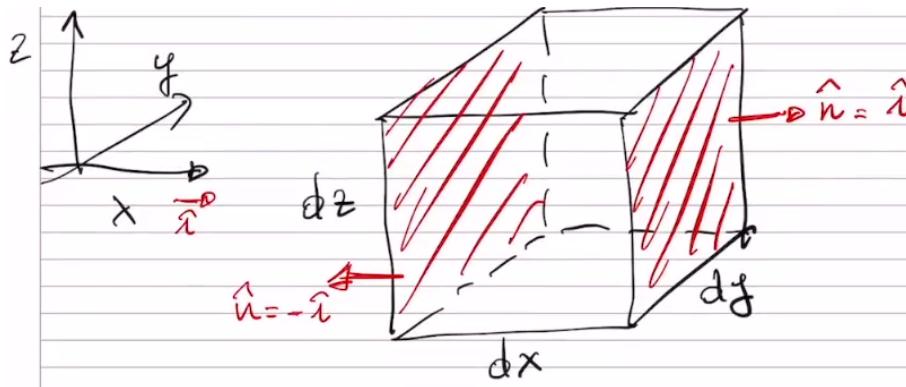
Let us consider an infinitesimally small cube:



Consider Term 1 - the mass variation inside the control volume

$$\frac{\partial \rho}{\partial t} \cdot dV = \frac{\partial \rho}{\partial t} \cdot dx \cdot dy \cdot dz \quad (1.2.1)$$

Consider Term 2 - the contribution of mass from the sides of the cube

**x Orthogonal Contribution:**

Left Side:

$$\rho \vec{V} \cdot \hat{n} dS = \rho \vec{V} \cdot (-\hat{i}) dz dy \quad (1.2.2)$$

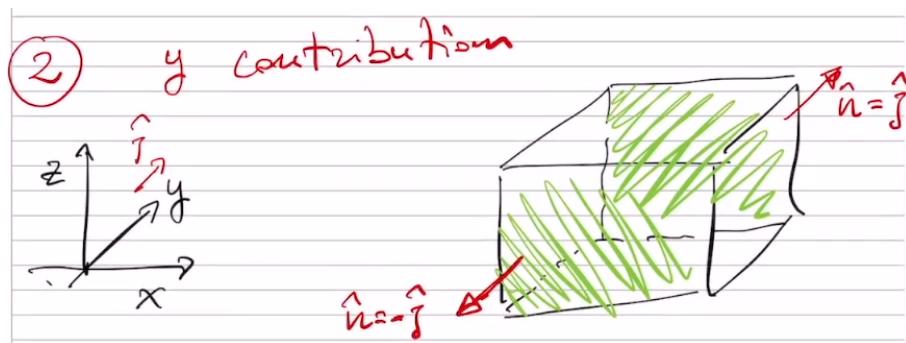
$$= -\rho u dz dy \quad (1.2.3)$$

Right Side:

$$\left( \rho u + \frac{\partial \rho u}{\partial x} \cdot dx \right) dz dy \quad (1.2.4)$$

There won't be a negative (-) sign as the normal vector and velocity field are moving in the same direction. The  $\frac{\partial \rho u}{\partial x} \cdot dx$  part comes from the incremental change in mass while moving in the  $x$  direction. The net contribution from the orthogonal  $x$  direction is

$$= \frac{\partial \rho u}{\partial x} dx dz dy \quad (1.2.5)$$

**y Orthogonal Contribution:**

Front Side:

$$\rho \vec{V} \cdot \hat{n} dS = \rho \vec{V} \cdot (-\hat{j}) dx dz \quad (1.2.6)$$

$$= -\rho v dx dz \quad (1.2.7)$$

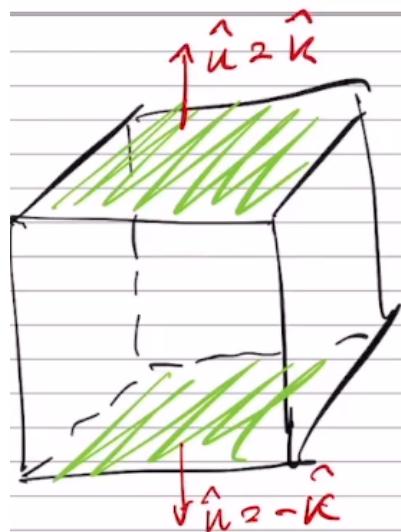
Back Side:

$$\left( \rho v + \frac{\partial \rho v}{\partial y} \cdot dy \right) dz dx \quad (1.2.8)$$

Final Contribution:

$$\frac{\partial \rho v}{\partial y} dy dz dx \quad (1.2.9)$$

*z Orthogonal Contribution:*



Bottom Side:

$$\rho \vec{V} \cdot \hat{n} dS = \rho \vec{V} \cdot (-\hat{k}) dx dy \quad (1.2.10)$$

$$= -\rho w dx dy \quad (1.2.11)$$

Top Side:

$$\left( \rho w + \frac{\partial \rho w}{\partial z} \cdot dz \right) dx dy \quad (1.2.12)$$

Final Contribution:

$$\frac{\partial \rho w}{\partial z} dz dx dy \quad (1.2.13)$$

### Conservation of Mass For an Infinitesimal Volume:

All of the contributions above (??, ??, ??, ??) are added up and the conservation of mass equation (??) becomes:

$$\frac{\partial \rho}{\partial t} \cdot dx \cdot dy \cdot dz + \frac{\partial \rho u}{\partial x} dx dz dy + \frac{\partial \rho v}{\partial y} dy dz dx + \frac{\partial \rho w}{\partial z} dz dx dy = 0 \quad (1.2.14)$$

This simplifies to:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \quad (1.2.15)$$

We can simplify this a bit more by introducing a term called the **divergence**.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \quad (1.2.16)$$

Where  $\nabla \cdot (\rho \vec{V})$  is the divergence of the vector  $\rho \vec{V}$ . It is a scalar.

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} = 0 \quad (1.2.17)$$

$\nabla \cdot \vec{V}$  is the divergence of the vector  $\vec{V}$  and it is a scalar.  $\nabla \rho$  is the gradient of the density  $\rho$  and is a vector. It can be expanded as:

$$\nabla \rho = \frac{\partial \rho}{\partial x} \hat{i} + \frac{\partial \rho}{\partial y} \hat{j} + \frac{\partial \rho}{\partial z} \hat{k} \quad (1.2.18)$$

$$\vec{V} \cdot \rho \nabla = (u \hat{i} + v \hat{j} + w \hat{k}) \cdot \left( \frac{\partial \rho}{\partial x} \hat{i} + \frac{\partial \rho}{\partial y} \hat{j} + \frac{\partial \rho}{\partial z} \hat{k} \right) = u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \quad (1.2.19)$$

$$\rho \nabla \cdot \vec{V} = \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \quad (1.2.20)$$

For steady flow:

$$\frac{\partial \rho}{\partial t} = 0 \rightarrow \nabla \cdot (\rho \vec{V}) = 0 \quad (1.2.21)$$

For incompressible flow, the density is constant. This means all derivatives of  $\rho$  are 0. Hence, our equation reduces to:

$$\rho = \text{const} \rightarrow \nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1.2.22)$$

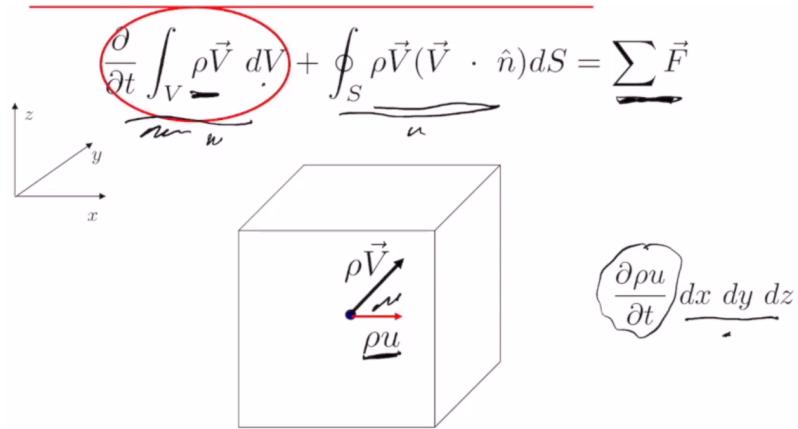
Each of these derivatives represent the stretch or compression of the fluid particle in the orthogonal direction. When these are all added up, it gives the variation in volume. If this is positive, it shows that the volume has increased with time. If  $\rho$  is constant, then the volume cannot change. Which is why equation (??) must equal 0.

### 1.3 Conservation of Momentum

$$\frac{\partial}{\partial t} \int_V \rho \vec{V} dV + \oint_S \rho \vec{V} (\vec{V} \cdot \hat{n}) dS = \sum \vec{F} \quad (1.3.1)$$

We have two types of external force that can act on our infinitesimal fluid element, **volumetric** forces (e.g. gravity) and **surface** forces (shear, pressure). The 2 terms of the conservation of momentum equation are investigated. In the example below, only the momentum in the  $x$  direction is being looked into.

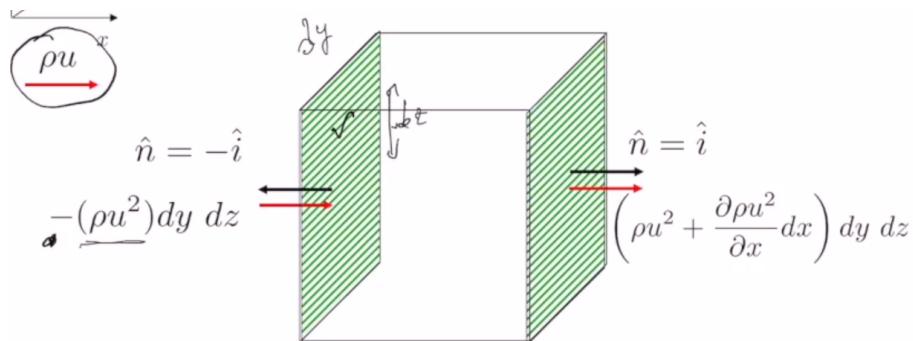
Consider Term 1 - the momentum in 3D changing with time



$$\frac{\partial}{\partial t} \int_V \rho \vec{V} dV = \frac{\partial \rho u}{\partial t} dx dy dz \quad (1.3.2)$$

Consider Term 2 - the flux of momentum through the sides of the control volume

**Momentum Contribution in  $x$  Entering from  $x$  Orthogonal:**



Left Side:

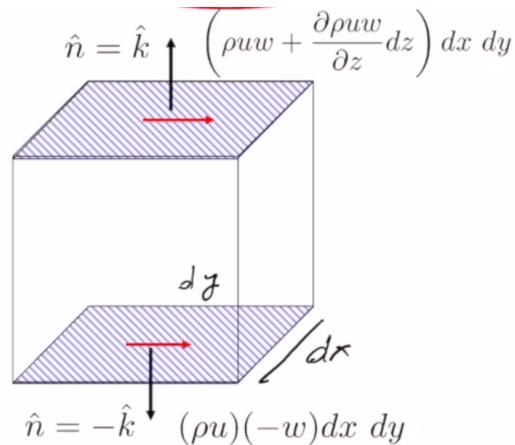
$$-(\rho u^2) dy dz \quad (1.3.3)$$

Right Side:

$$\left( \rho u^2 + \frac{\partial \rho u^2}{\partial x} dx \right) dy dz \quad (1.3.4)$$

Final Contribution:

$$\frac{\partial(\rho u^2)}{\partial x} dx dy dz \quad (1.3.5)$$

**Momentum Contribution in  $x$  Entering from  $z$  Orthogonal:**

Bottom Side:

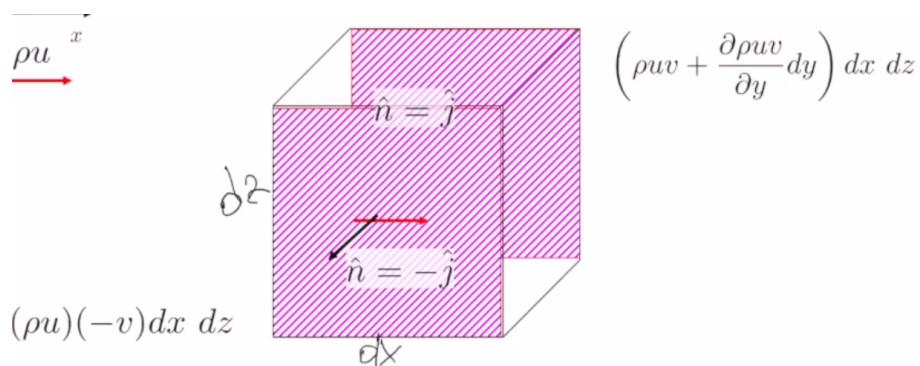
$$-(\rho uw) dx dy \quad (1.3.6)$$

Top Side:

$$\left( \rho uw + \frac{\partial \rho uw}{\partial z} dz \right) dx dy \quad (1.3.7)$$

Final Contribution:

$$\frac{\partial(\rho uw)}{\partial z} dx dy dz \quad (1.3.8)$$

**Momentum Contribution in  $x$  Entering from  $y$  Orthogonal:**

Front Side:

$$-(\rho uv) dx dz \quad (1.3.9)$$

Back Side:

$$\left( \rho uv + \frac{\partial \rho uv}{\partial y} dy \right) dx dz \quad (1.3.10)$$

Final Contribution:

$$\frac{\partial \rho u v}{\partial y} dx dy dz \quad (1.3.11)$$

### All Momentum Contributions:

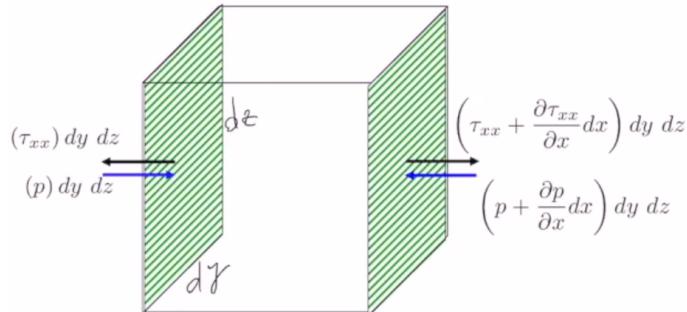
All of the contributions above (??, ??, ??, ??) are added up and the conservation of momentum equation (??) becomes:

$$\frac{\partial \rho u}{\partial t} dx dy dz + \frac{\partial (\rho u^2)}{\partial x} dx dy dz + \frac{\partial (\rho u w)}{\partial z} dx dy dz + \frac{\partial \rho u v}{\partial y} dx dy dz = \sum \vec{F} \quad (1.3.12)$$

Consider Term 3 -  $\sum \vec{F}$  term.

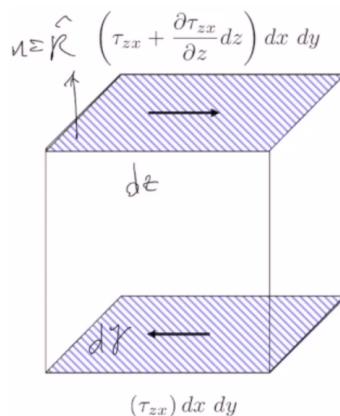
Pressure is always exerted orthogonal to a face. We also have our  $\tau$  stresses acting orthogonally. In the example below, some steps of calculations are skipped (can still be viewed on the figures); only the final contributions on each direction are shown.

### x Orthogonal Shear Force



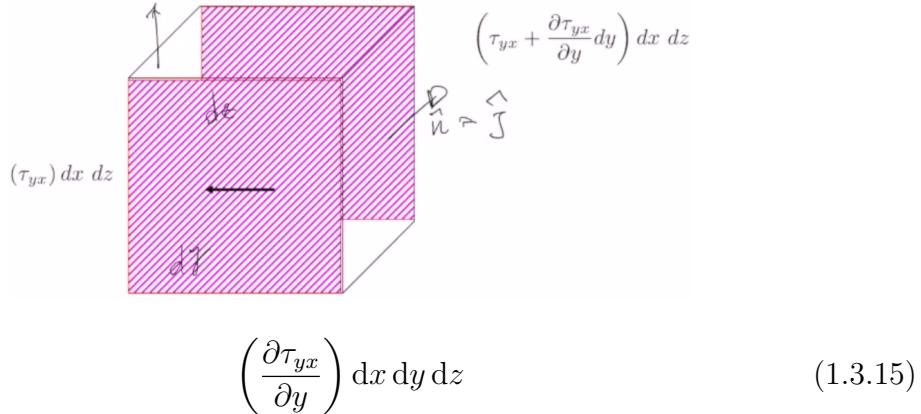
$$-\left(\frac{\partial p}{\partial x}\right) dx dy dz + \left(\frac{\partial \tau_{xx}}{\partial x}\right) dx dy dz \quad (1.3.13)$$

### z Orthogonal Shear Force



$$\left(\frac{\partial \tau_{zx}}{\partial z}\right) dx dy dz \quad (1.3.14)$$

## y Orthogonal Shear Force



## Sum of the Forces $\sum \vec{F}$

$$\begin{aligned} \sum F_x = - \left( \frac{\partial p}{\partial x} \right) dx\,dy\,dz + \left( \frac{\partial \tau_{xx}}{\partial x} \right) dx\,dy\,dz + \\ \left( \frac{\partial \tau_{zx}}{\partial z} \right) dx\,dy\,dz + \left( \frac{\partial \tau_{yx}}{\partial y} \right) dx\,dy\,dz \end{aligned} \quad (1.3.16)$$

## Conservation of Momentum Equations in All Directions

Combining Term 1, Term 2 and Term 3 from above yields the following equations for the conservation of momentum in 3D.

### x Direction:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \quad (1.3.17)$$

### y Direction:

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho vu) + \frac{\partial}{\partial y}(\rho vv) + \frac{\partial}{\partial z}(\rho vw) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \quad (1.3.18)$$

### z Direction:

$(\rho g$  is added here due to the gravitational force acting downwards)

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rhowu) + \frac{\partial}{\partial y}(\rho wv) + \frac{\partial}{\partial z}(\rho ww) = -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \quad (1.3.19)$$

These can be added to find that we arrive with two terms, one being the continuity equation, which must equal 0. To summarise, we have our continuity equation and momentum equations below.

## Conservation of Mass (Continuity Equation)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \quad (1.3.20)$$

***x* Direction Momentum**

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \quad (1.3.21)$$

***y* Direction Momentum**

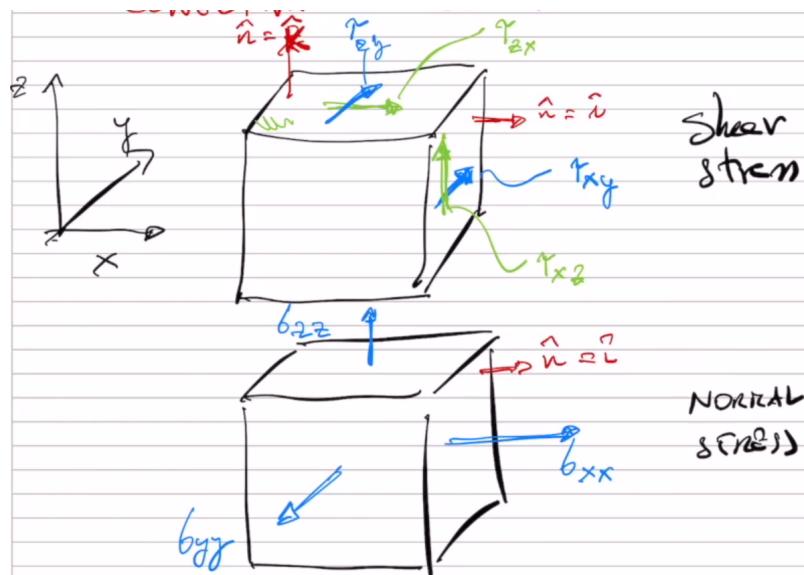
$$\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \quad (1.3.22)$$

***z* Direction Momentum**

$$\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = - \frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \quad (1.3.23)$$

## 1.4 Stress Tensor Notation

To identify a stress component we use a double subscript notation (tensor notation). The first subscript indicates the direction of the normal to the plane on which stress acts. Second subscript indicates the direction of the stress. Thus, the symbol  $\tau_{ij}$  denotes a stress in  $j$  direction on a face normal to the  $i$ -axis.



The normal stresses two contributions are pressure  $p$  and viscous stress  $\tau$ . Pressure is always negative due to it acting against the surface (if we take the arrow coming out of the surface as positive).  $\tau$  accounts for the extra stress coming from viscosity.

$$\sigma_{xx} = -p + \tau_{xx} \quad (1.4.1)$$

$$\sigma_{yy} = -p + \tau_{yy} \quad (1.4.2)$$

$$\sigma_{zz} = -p + \tau_{zz} \quad (1.4.3)$$

Parts on the opposite sides of a stress tensor are equal.

$$\tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}$$

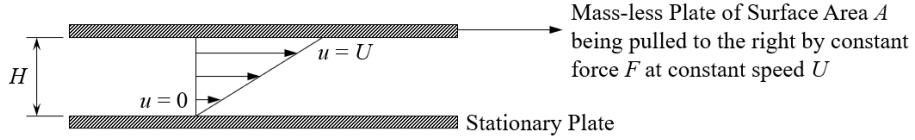
# Chapter 2

## Navier-Stokes Equations

13/10/2020

### 2.1 Constitutive Equations

We want to find a way to link the stress tensor  $\tau$  with the velocity field i.e.  $\tau = f(u, v, w)$ .



The angle of deformation  $\Delta\theta$  can be used to derive the following:

$$\tan \Delta\theta = \frac{u \cdot \Delta T}{H} \quad (2.1.1)$$

$$\tan \Delta\theta = d\theta = \frac{u \cdot t}{H} \rightarrow \frac{d\theta}{dt} = \frac{u}{H} \quad (2.1.2)$$

$$\tau = \frac{F}{A} \propto \frac{d\theta}{dt} = \frac{u}{H} \quad (2.1.3)$$

$$\tau = \mu \frac{d\theta}{dt} = \mu \frac{u}{H} \quad (2.1.4)$$

$$\tau = \mu \frac{du}{dy} \quad (2.1.5)$$

- $\tau$  is the shear stress
- $\frac{du}{dy}$  is the shear rate
- $\mu$  is the dynamic viscosity and has units  $\text{Ns m}^{-2}$
- $\nu = \frac{\mu}{\rho}$  is the kinematic viscosity and has units  $\text{m}^2 \text{s}^{-1}$

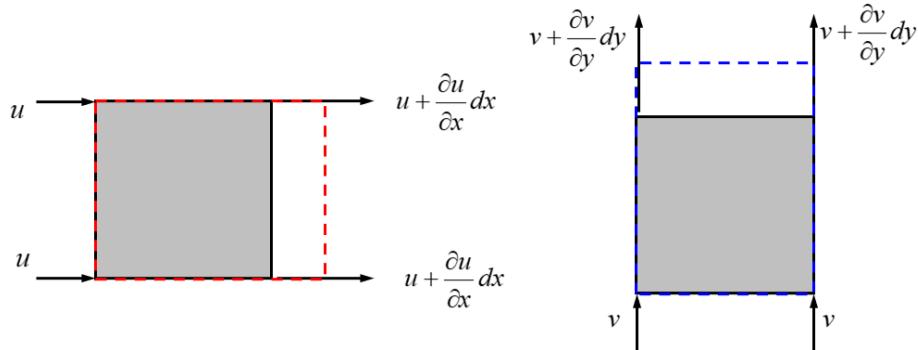
For Newtonian fluids,  $\mu$  is constant. In the case above, our stress tensor is  $\tau_{yx}$ , hence:

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} \quad (2.1.6)$$

Our velocity gradient can be defined as:

$$\nabla \vec{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \quad (2.1.7)$$

The left diagonal components are the normal deformation, orthogonal to the surface.



A simplified way of writing these left diagonal terms is

$$\nabla \cdot \vec{V} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad (2.1.8)$$

The repeated index  $i$  means sum in the x, y and z directions.

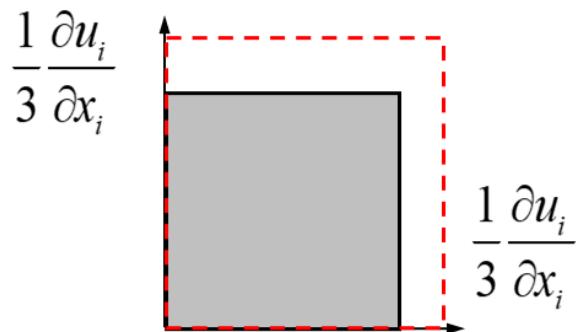
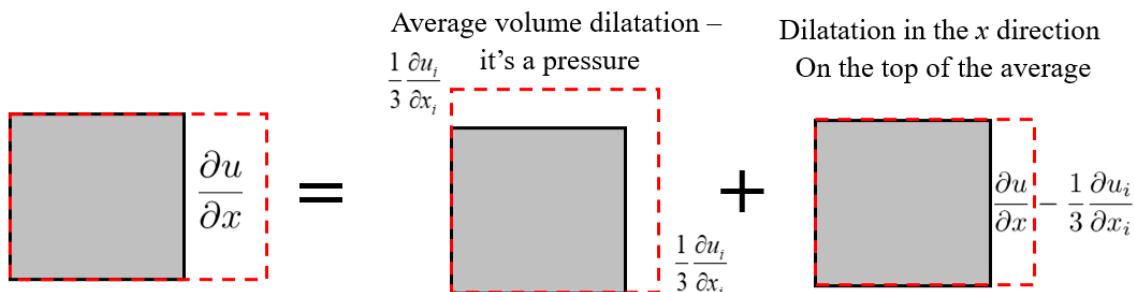


Figure 2.1: 1/3 symbolises the average deformation in x, y and z.

To find  $\frac{\partial u}{\partial x}$ , we can do the following

$$\frac{\partial u}{\partial x} = \frac{1}{3} \frac{\partial u_i}{\partial x_i} + \left( \frac{\partial u}{\partial x} - \frac{1}{3} \frac{\partial u_i}{\partial x_i} \right) \quad (2.1.9)$$



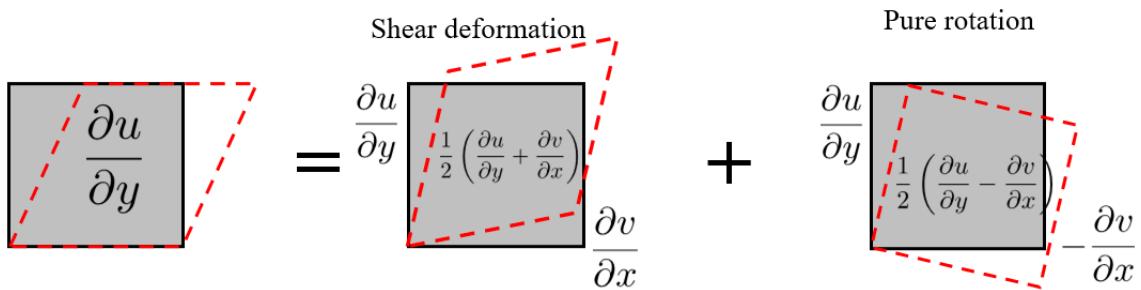
This can be also done for the other two orthogonal directions

$$\frac{\partial v}{\partial y} = \frac{1}{3} \frac{\partial u_i}{\partial x_i} + \left( \frac{\partial v}{\partial y} - \frac{1}{3} \frac{\partial u_i}{\partial x_i} \right) \quad (2.1.10)$$

$$\frac{\partial w}{\partial z} = \frac{1}{3} \frac{\partial u_i}{\partial x_i} + \left( \frac{\partial w}{\partial z} - \frac{1}{3} \frac{\partial u_i}{\partial x_i} \right) \quad (2.1.11)$$

Let us consider another term, such as  $\frac{\partial u}{\partial y}$ . We can define this as a component of deformation and rotation of the fluid particle.

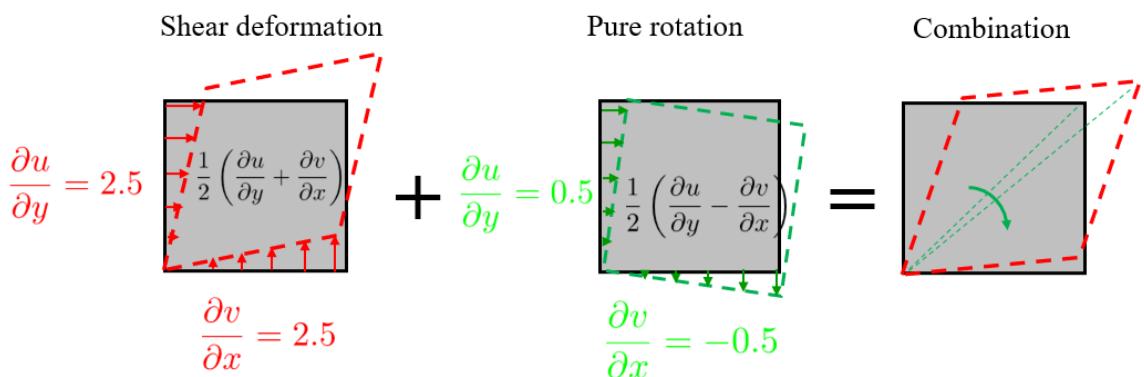
$$\frac{\partial u}{\partial y} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \quad (2.1.12)$$



### Example

$$\frac{\partial u}{\partial y} = 3 = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = 2.5 + 0.5 \quad (2.1.13)$$

$$\frac{\partial v}{\partial x} = 2 = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = 2.5 - 0.5 \quad (2.1.14)$$



### 2.1.1 Strain Rate Tensor

$$s = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left[ \frac{\partial u}{\partial y} + \frac{1}{2} \frac{\partial v}{\partial x} \right] & \left[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right] \\ \frac{1}{2} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] & \frac{\partial v}{\partial y} & \frac{1}{2} \left[ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right] \\ \frac{1}{2} \left[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right] & \frac{1}{2} \left[ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right] & \frac{\partial w}{\partial z} \end{bmatrix} = \quad (2.1.15)$$

$$\begin{bmatrix} \frac{1}{3} \nabla \cdot \vec{V} & 0 & 0 \\ 0 & \frac{1}{3} \nabla \cdot \vec{V} & 0 \\ 0 & 0 & \frac{1}{3} \nabla \cdot \vec{V} \end{bmatrix} + \begin{bmatrix} \left[ \frac{\partial u}{\partial x} - \frac{1}{3} \nabla \cdot \vec{V} \right] & \frac{1}{2} \left[ \frac{\partial u}{\partial y} + \frac{1}{2} \frac{\partial v}{\partial x} \right] & \frac{1}{2} \left[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right] \\ \frac{1}{2} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] & \left[ \frac{\partial v}{\partial y} - \frac{1}{3} \nabla \cdot \vec{V} \right] & \frac{1}{2} \left[ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right] \\ \frac{1}{2} \left[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right] & \frac{1}{2} \left[ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right] & \left[ \frac{\partial w}{\partial z} - \frac{1}{3} \nabla \cdot \vec{V} \right] \end{bmatrix} \quad (2.1.16)$$

Deformation part which goes in pressure  $\rho$  +

Deformation part which goes in the stress tensor  $T$  (2.1.17)

Compact notation of the strain rate tensor, indices  $i, j = 1, 2, 3$

$$s_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2.1.18)$$

## 2.1.2 Stress Tensor

$$T = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \quad (2.1.19)$$

$$\begin{bmatrix} \mu \left[ 2\frac{\partial u}{\partial x} - \frac{2}{3} \nabla \cdot \vec{V} \right] & \mu \left[ \frac{\partial u}{\partial y} + \frac{1}{2} \frac{\partial v}{\partial x} \right] & \mu \left[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right] \\ \mu \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] & \mu \left[ 2\frac{\partial v}{\partial y} - \frac{2}{3} \nabla \cdot \vec{V} \right] & \mu \left[ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right] \\ \mu \left[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right] & \mu \left[ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right] & \mu \left[ 2\frac{\partial w}{\partial z} - \frac{2}{3} \nabla \cdot \vec{V} \right] \end{bmatrix} \quad (2.1.20)$$

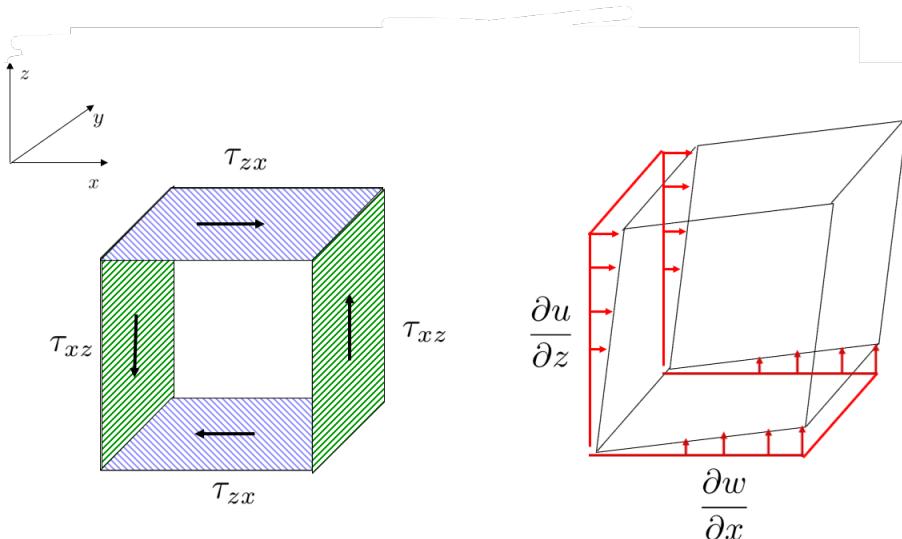
Compact notation for constitutive equation:

$$\tau_{ij} = 2\mu \left[ s_{ij} - \frac{1}{3}(\nabla \cdot \vec{V})\delta_{ij} \right] = \mu \left[ \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3}(\nabla \cdot \vec{V})\delta_{ij} \right] \quad (2.1.21)$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (2.1.22)$$

The stress tensor is always symmetric along the left diagonal.

$$\tau_{xz} = \tau_{zx} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \quad (2.1.23)$$



We now have 10/12 equations to describe the fluid. The final two relate to the temperature and the energy of the fluid. We will not be considering the energies of the fluid in this course. Our state equation can be  $p = \rho RT$ , when the fluid is compressible and if our fluid is incompressible we take  $\rho$  as constant. All in all, 11 variables and 11 equations to describe the fluid

## 2.2 Navier-Stokes Equations

Navier-Stokes equations are a system of equations that can be used to describe the behaviour of a fluid. They can be obtained through inserting the Constitutive Equations into the Conservation of Momentum Equations, rearranging, and simplifying them. The Navier-Stokes Equations for an incompressible fluid in 3D are as follows:

**Conservation of Mass:**

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (2.2.1)$$

**Conservation of Momentum** ( $x, y, z$ ):

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (2.2.2)$$

$$\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \quad (2.2.3)$$

$$\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \rho g \quad (2.2.4)$$

The Navier-Stokes Equations can be further simplified if the following occur:

- Constant Density
- Assume Steady Flow (No Time-Dependent Terms)
- Assume No External Forces
- Assume Fluid is Incompressible

The Navier-Stokes Equations in 2D, with the above assumptions applied are below:

**Conservation of Mass (2D):**

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.2.5)$$

**Conservation of Momentum** ( $x, y$ ):

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2.2.6)$$

$$\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (2.2.7)$$

## 2.3 Lagrangian vs. Eulerian

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \quad (2.3.1)$$

- $\frac{D}{Dt}$  - Lagrangian/Material Derivative: Variation in time of a property (for example temperature, density or velocity component) of a fluid particle. The reference system is moving with the fluid particle.
- $\frac{\partial}{\partial t}$  - Eulerian Derivative: Variation in time of a property (for example temperature, density etc..) in a fixed point in space (x, y , z). Reference system fixed in space.
- $u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$  - Convection Terms in the x, y and z: Variation of a property due to how the particle is moving in space.

# Chapter 3

## Inviscid and Irrotational Flow

20/10/2020

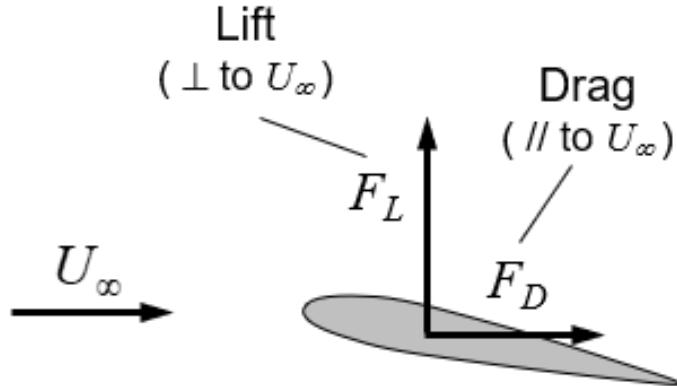
### 3.1 Lift and drag

Typical forces of interest for bodies in a flow are **drag** and **lift**. We can represent these in dimensionless form:

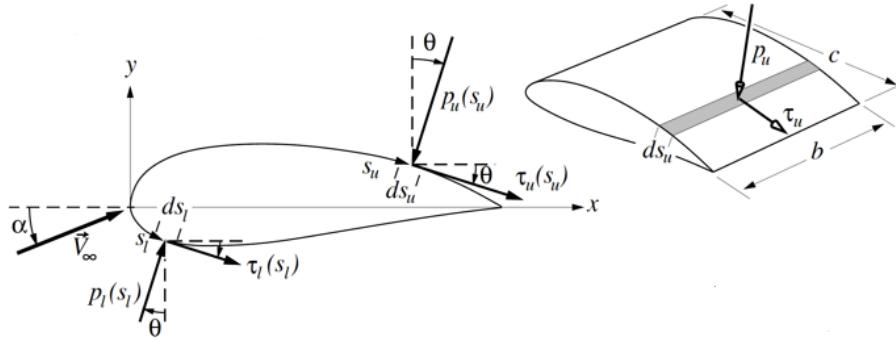
$$\text{Drag coefficient: } c_D = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 S} \quad (3.1.1)$$

$$\text{Lift coefficient: } c_L = \frac{F_L}{\frac{1}{2}\rho U_\infty^2 S} \quad (3.1.2)$$

Where  $S$  is a representative area for the body, determined by convention.



### 3.2 Pressure and frictional force distribution



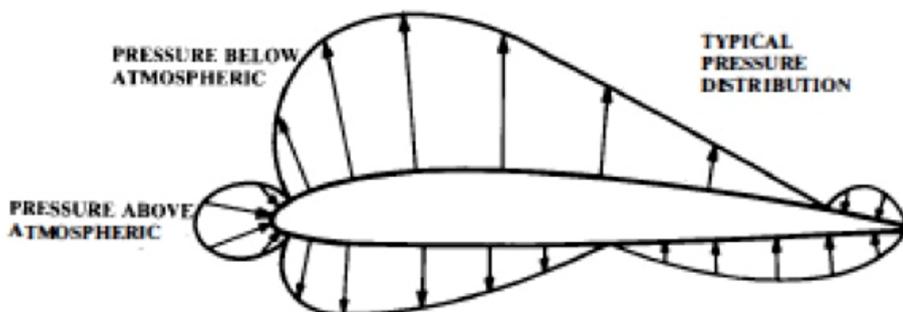
$$L = - \int_S (p(\hat{n} \cdot \hat{j})) \, dS + \int_S (\vec{\tau} \cdot \hat{j}) \, dS \quad (3.2.1)$$

$$D = - \int (p(\hat{n} \cdot \hat{j})) \, dS + \int (\vec{\tau} \cdot \hat{i}) \, dS \quad (3.2.2)$$

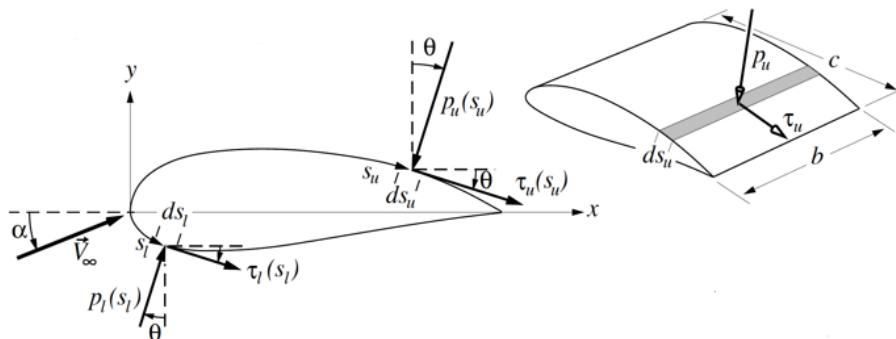
To determine the lift and drag coefficients  $c_L$  and  $c_D$ , we are interested in the pressure distribution over the airfoil, or more specifically in the local pressure difference from the stream pressure  $p_\infty$ .

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} \quad (3.2.3)$$

Free stream pressure and velocity are  $p_\infty$  and  $V_\infty$ .



- Local suction (depression):  $c_p < 0$  Vectors point away from the airfoil surface
- Local pushing:  $c_p > 0$  Vectors point towards the airfoil surface



$$L = - \int_S (p\hat{n} \cdot \hat{j}) \, dS = \quad (3.2.4)$$

$$c_L = -\frac{1}{S} \int_S \left( \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} \hat{n} \cdot \hat{j} \right) \, dS = -\frac{1}{S} \int_S (c_p \hat{n} \cdot \hat{j}) \, dS \quad (3.2.5)$$

The lift coefficient per unit of span-wise length is:

$$c'_L = -\frac{1}{c} \int_c^0 (c_p \hat{n} \cdot \hat{j}) \, dx \quad (3.2.6)$$

### 3.3 Rearrangement of momentum equation - x direction

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (3.3.1)$$

$$\left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + v \frac{\partial u}{\partial y} - v \frac{\partial u}{\partial y} \quad (3.3.2)$$

$$= -v \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{1}{2} \left( \frac{\partial u^2}{\partial x} + \frac{\partial v^2}{\partial x} \right) \quad (3.3.3)$$

$$= -v \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{1}{2} \frac{\partial}{\partial x} (u^2 + v^2) \quad (3.3.4)$$

$(u^2 + v^2)$  is the total kinetic energy of the fluid particle. The derivative is the element that takes into the account the variation of this kinetic energy.  $\left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$  relates to the rotation of the particle. This rotation is related to the difference of velocity gradient.

$$\rho \left[ -v \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{1}{2} \frac{\partial}{\partial x} (u^2 + v^2) \right] = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (3.3.5)$$

$$-v \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = -\frac{\partial}{\partial x} \left( \frac{p}{\rho} + \frac{u^2 + v^2}{2} \right) + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (3.3.6)$$

Our Bernoulli term in the above equation is  $\left( \frac{p}{\rho} + \frac{u^2 + v^2}{2} \right)$ , gravitational energy is negligible.  $\left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$  is an anti-clockwise rotation. Hence, the vorticity component in the z direction is:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (3.3.7)$$

Our final momentum equations in  $x$  and  $y$  are:

$$-v\omega_z = -\frac{\partial}{\partial x} \left( \frac{p}{\rho} + \frac{u^2 + v^2}{2} \right) + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (3.3.8)$$

$$u\omega_z = -\frac{\partial}{\partial y} \left( \frac{p}{\rho} + \frac{u^2 + v^2}{2} \right) + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (3.3.9)$$

We can make some assumptions:

- Inviscid flow -  $\nu = 0$  (this may be realistic in some parts of a fluid domain but in real life, inviscid fluids do not exist)
- Irrotational flow -  $\omega_z = 0$

This reduces our equations to:

$$0 = -\frac{\partial}{\partial x} \left( \frac{p}{\rho} + \frac{u^2 + v^2}{2} \right) + 0 \quad (3.3.10)$$

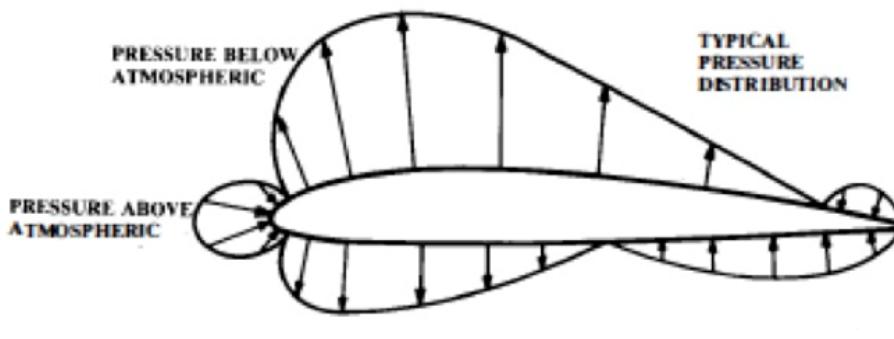
$$0 = -\frac{\partial}{\partial y} \left( \frac{p}{\rho} + \frac{u^2 + v^2}{2} \right) + 0 \quad (3.3.11)$$

## 3.4 Application of Bernoulli

$$p_\infty + \frac{1}{2}\rho V_\infty^2 = p + \frac{1}{2}\rho(u^2 + v^2) = \text{constant} \quad (3.4.1)$$

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{u^2 + v^2}{V_\infty^2} = 1 - \frac{\|V\|^2}{V_\infty^2} \quad (3.4.2)$$

If  $c_p < 0$ ,  $\|V\| > V_\infty$  and vice versa. If a fluid particle enters a region where  $c_p$  is negative it is accelerated and when  $c_p$  is positive it will lose velocity relative to the free stream.



Extending this to 3D, we can derive the 3D vorticity equation. We sum the momentum equations with the assumptions above and take the  $ijk$  components as so:

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} = \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k} \quad (3.4.3)$$

If  $\vec{\omega} = 0$  then a potential function,  $\phi(x, y, z)$  exists such:

$$\begin{cases} u = \frac{\partial \phi(x,y,z)}{\partial x} \\ v = \frac{\partial \phi(x,y,z)}{\partial y} \\ w = \frac{\partial \phi(x,y,z)}{\partial z} \end{cases} \quad (3.4.4)$$

By plugging in these equations into our continuity equation, we get:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \quad (3.4.5)$$

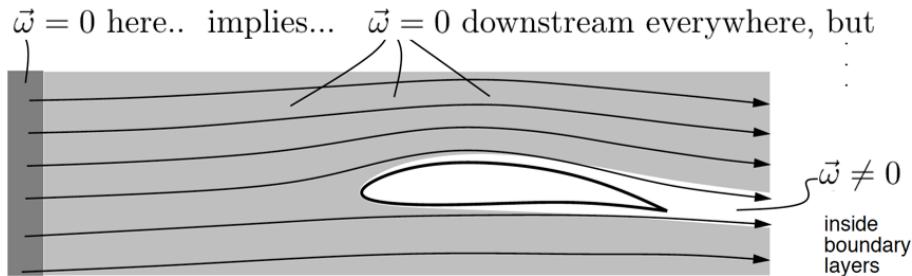
Conservation of momentum (Bernoulli term) is simply:

$$p + \frac{1}{2}\rho(u^2 + v^2) = \text{constant} \quad (3.4.6)$$

## 3.5 Applicability of irrotational flow

The flow domain can be subdivided into two parts.

- **Irrational flow region**, outside of the boundary layer, Bernoulli equation, potential flow and stream function apply.
- **Boundary layer**, layer where all vorticity is confined. The friction shear the airfoil surface acts as a source of vorticity.



In the case where we consider our fluid inviscid, there is no shear stress being applied on the fluid by the airfoil. We need to understand how the no-slip condition changes. We still need a boundary condition to know the value of  $\phi$  in our flow domain.

## 3.6 No-Slip condition for inviscid flow

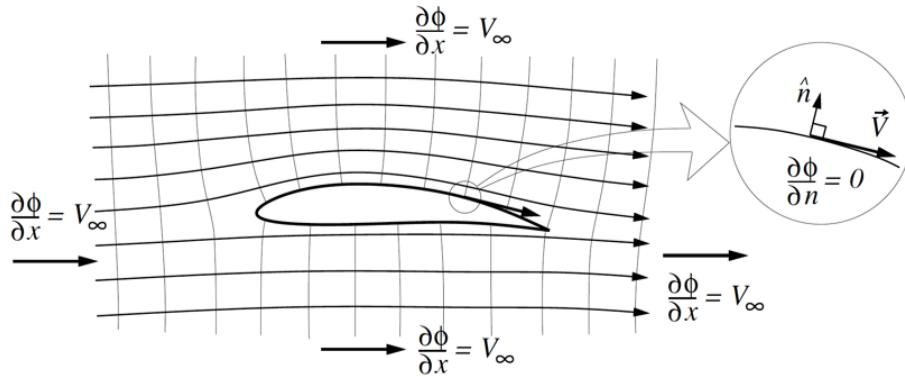
Viscous fluid

$$\nu \neq 0 \rightarrow \vec{V} = 0 \quad (3.6.1)$$

Inviscid flow

$$\nu = 0 \rightarrow V_n = \vec{V} \cdot \hat{n} = \frac{\partial \phi}{\partial n} = 0 \quad (3.6.2)$$

In essence, with inviscid flow, we are accepting that there is some movement on the boundary, however this is only parallel to the surface.  $n$  is the direction orthogonal to the boundary.



### 3.7 Stream function

In a 2D flow a stream function,  $\psi(x, y)$ , can be defined which is always aligned/parallel with the local velocity vector and visualise a streamline. Different streamlines are identified with different values of  $\psi(x, y)$

$$\begin{cases} u = \frac{\partial \psi(x,y)}{\partial x} \\ v = \frac{\partial \psi(x,y)}{\partial y} \end{cases} \quad (3.7.1)$$

Iso-potential lines and streamlines are orthogonal to each other. Streamlines visualise the trajectory of a particle in the field.

### 3.8 Potential flow past bodies

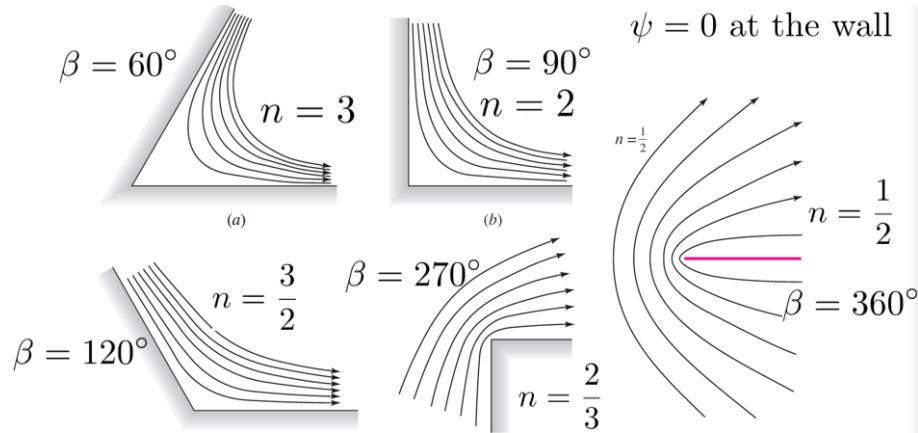
Flow fields for which an incompressible fluid is assumed to be frictionless and the motion to be irrotational are commonly referred to as **potential** flows. Paradoxically, potential flows can be simulated by a slow moving, viscous flow between closely spaced parallel plates.

### 3.9 Flow around a corner of arbitrary angle, $\beta$

Considering a radial coordinate system:

$$\phi = Ar^n \cos(n\theta) \quad (3.9.1)$$

$$\psi = Ar^n \sin(n\theta) \quad (3.9.2)$$



### 3.10 Cylindrical coordinates

3D vorticity equation:

$$\vec{\omega} = \omega_r \hat{i}_r + \omega_\theta \hat{i}_\theta + \omega_z \hat{i}_z \quad (3.10.1)$$

$$\vec{\omega} = \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \hat{i}_r + \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{i}_\theta + \frac{1}{r} \left( \frac{\partial (ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right) \hat{i}_z \quad (3.10.2)$$

Potential flow function and stream function:

$$\text{Potential flow } \begin{cases} u_r = \frac{\partial \phi}{\partial r} \\ u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \\ u_z = \frac{\partial \phi}{\partial z} \end{cases} \quad \text{Stream function } \begin{cases} u_\theta = -\frac{\partial \psi}{\partial r} \\ u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \end{cases} \quad (3.10.3)$$

Conservation of mass (continuity equation):

$$\nabla \cdot \vec{V} = \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \quad (3.10.4)$$

# Chapter 4

## Modelling the Flow Around a Bluff Body

27/10/2020

### 4.1 Uniform Flow

Cartesian Coordinates:

$$\phi = V_\infty [x \cos(\alpha) + y \sin(\alpha)] \quad (4.1.1)$$

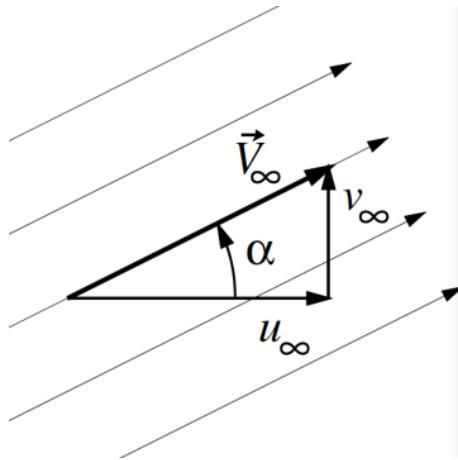
$$\psi = V_\infty [y \cos(\alpha) - x \sin(\alpha)] \quad (4.1.2)$$

The conservation of mass is balanced:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.1.3)$$

The flow is irrotational:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (4.1.4)$$



Cylindrical Coordinates:

$$\phi(r, \theta) = V_\infty r \cos(\theta - \alpha) \quad (4.1.5)$$

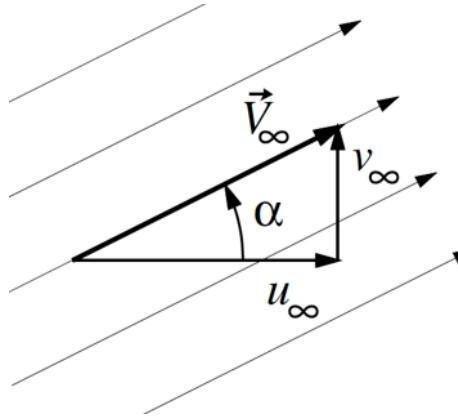
$$\psi(r, \theta) = V_\infty r \sin(\theta - \alpha) \quad (4.1.6)$$

The conservation of mass is satisfied for cylindrical coordinates:

$$\nabla \cdot \vec{V} = \frac{1}{r} \frac{\partial r u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \quad (4.1.7)$$

$$= \frac{\partial r(v_\infty \cos(\theta - \alpha))}{\partial r} - \frac{\partial v_\infty \sin(\theta - \alpha)}{\partial \theta} \quad (4.1.8)$$

$$v_\infty \cos(\theta - \alpha) - v_\infty \cos(\theta - \alpha) = 0 \quad (4.1.9)$$



## 4.2 Source/Sink Flow

Cartesian Coordinates:

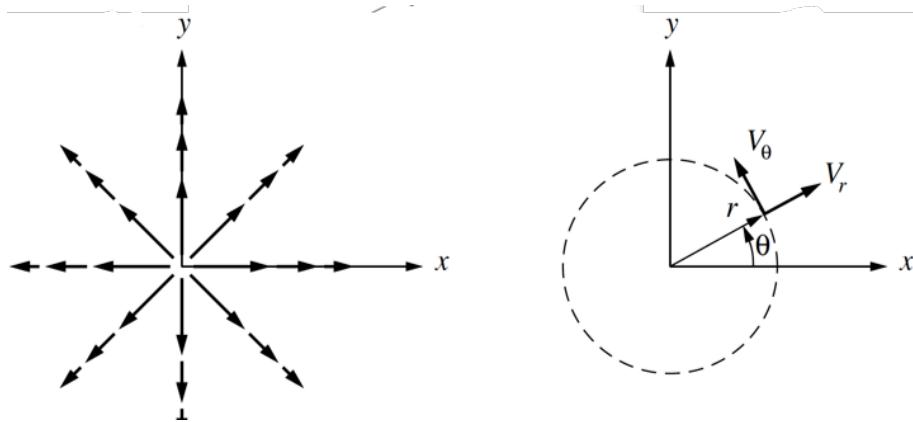
$$\phi = \frac{\Lambda}{2\pi} \ln(\sqrt{x^2 + y^2}) \quad (4.2.1)$$

$$\psi = \frac{\Lambda}{2\pi} \arctan\left(\frac{y}{x}\right) \quad (4.2.2)$$

Cylindrical Coordinates:

$$\phi = \frac{\Lambda}{2\pi} \ln(r) \quad (4.2.3)$$

$$\psi = \frac{\Lambda}{2\pi} \theta \quad (4.2.4)$$



In cylindrical coordinates, we do not have a  $\theta$  component as it is moving radially outwards from a source.

$$u_r = \frac{\partial \phi}{\partial r} = \frac{\Lambda}{2\pi r} \quad (4.2.5)$$

$$u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0 \quad (4.2.6)$$

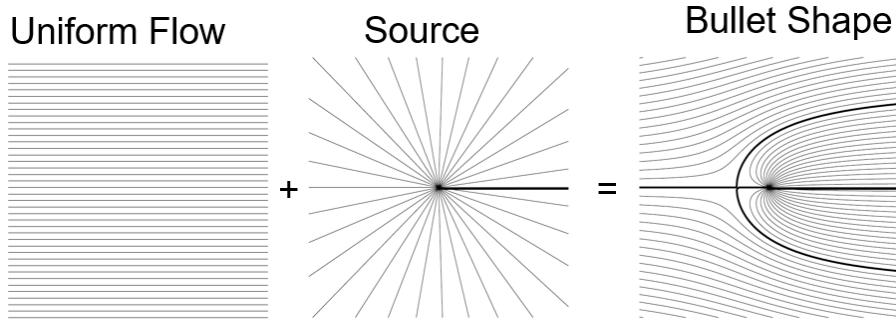
Here we can see the magnitude of the velocity is dependent on  $\frac{1}{r}$ . If  $\Lambda > 0$ , we have a source and if  $\Lambda < 0$ , we have a sink.

### 4.3 Uniform Flow + Source

$$\phi(r, \theta) = \frac{\Lambda}{2\pi} \ln(r) + V_\infty r \cos \theta \quad (4.3.1)$$

$$\psi(r, \theta) = \frac{\Lambda}{2\pi} \theta + V_\infty r \sin \theta \quad (4.3.2)$$

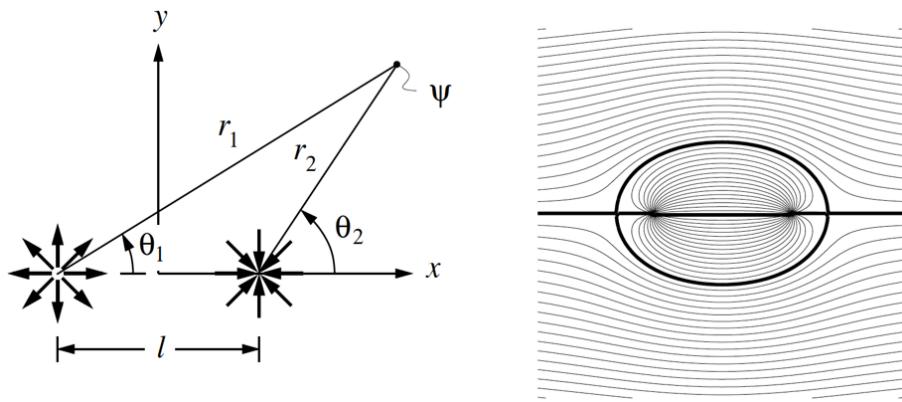
Stream function  $\psi(r, \theta)$  of:



### 4.4 Uniform Flow + Source + Sink

$$\phi = V_\infty r \cos \theta + \frac{\Lambda}{2\pi} (\ln(r_1) - \ln(r_2)) \quad (4.4.1)$$

$$\psi = V_\infty r \sin \theta + \frac{\Lambda}{2\pi} (\theta_1 - \theta_2) \quad (4.4.2)$$

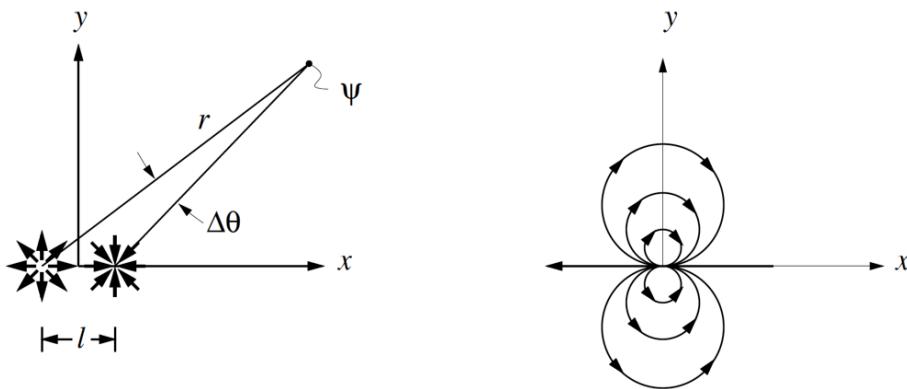


## 4.5 Doublet

Consider a pair of source and sink of  $\pm\Lambda$  who are  $l$  apart and  $l \times \Lambda = \text{constant}$ .

$$\psi = \lim_{l \rightarrow 0} \frac{\Lambda}{2\pi} (\theta_1 - \theta_2) = -\frac{k}{2\pi} \frac{\sin \theta}{r} \quad (4.5.1)$$

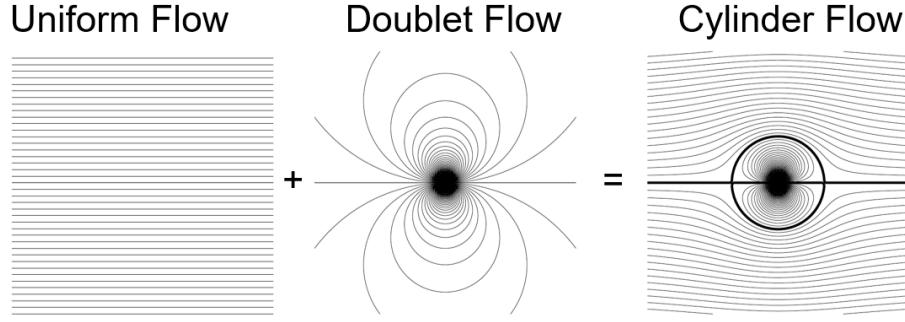
$$\phi = \frac{k}{2\pi} \frac{\cos \theta}{r} \quad (4.5.2)$$



## 4.6 Cylinder (Uniform Flow + Doublet)

$$\phi = V_\infty r \cos \theta + \frac{k}{2\pi} \frac{\cos \theta}{r} \quad (4.6.1)$$

$$\psi = V_\infty r \sin \theta - \frac{k}{2\pi} \frac{\sin \theta}{r} \quad (4.6.2)$$



The radius of the cylinder can be derived as so:

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = V_\infty \cos \theta - \frac{k}{2\pi} \frac{\cos \theta}{r^2} \quad (4.6.3)$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -\left( V_\infty \sin \theta + \frac{k}{2\pi} \frac{\sin \theta}{r^2} \right) \quad (4.6.4)$$

On the cylinder,  $\vec{u} \cdot \hat{n} = 0$

$$\hat{n} = \hat{i}_r \rightarrow u_r(R) = 0 \quad (4.6.5)$$

$$V_\infty \cos \theta - \frac{k}{2\pi} \frac{\cos \theta}{R^2} = 0 \quad (4.6.6)$$

$$R = \sqrt{\frac{k}{2\pi V_\infty}} \quad (4.6.7)$$

We can rewrite  $\phi$  and  $\psi$

$$\phi = V_\infty r \cos \theta \left( 1 + \frac{R^2}{r^2} \right) \quad (4.6.8)$$

$$\psi = V_\infty r \sin \theta \left( 1 - \frac{R^2}{r^2} \right) \quad (4.6.9)$$

On the cylinder surface,  $r = R$  and inputting this into  $\psi$ :

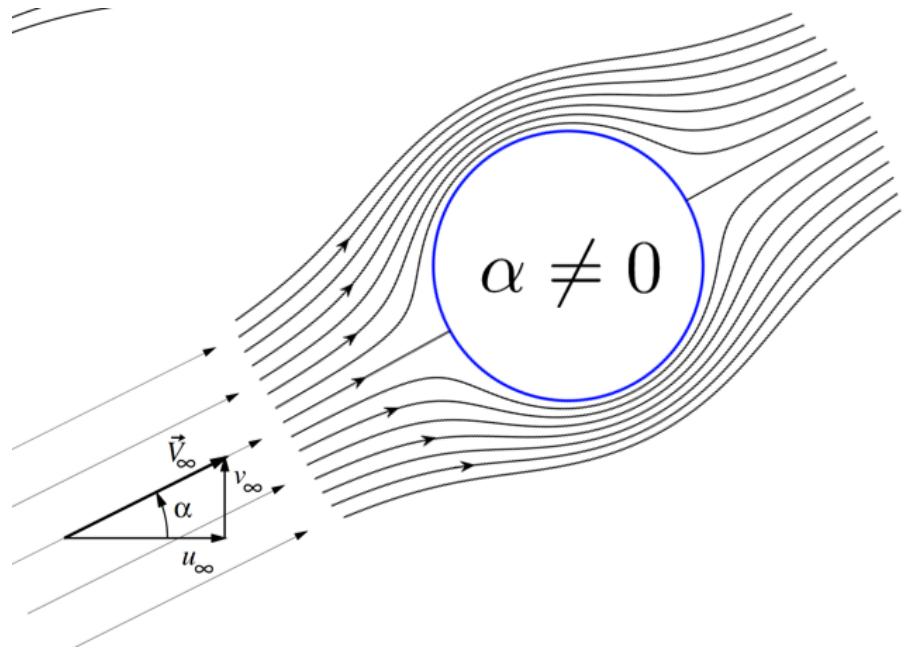
$$\psi = 0 \quad (4.6.10)$$

## 4.7 Uniform Stream with Varying Direction

All we need to do to generalise our equations a bit more is to rewrite our equations with an extra angular term,  $\alpha$ :

$$\phi = V_\infty r \cos(\theta - \alpha) \left( 1 + \frac{R^2}{r^2} \right) \quad (4.7.1)$$

$$\psi = V_\infty r \sin(\theta - \alpha) \left( 1 - \frac{R^2}{r^2} \right) \quad (4.7.2)$$



## 4.8 Adding Circulation with a Vortex Flow

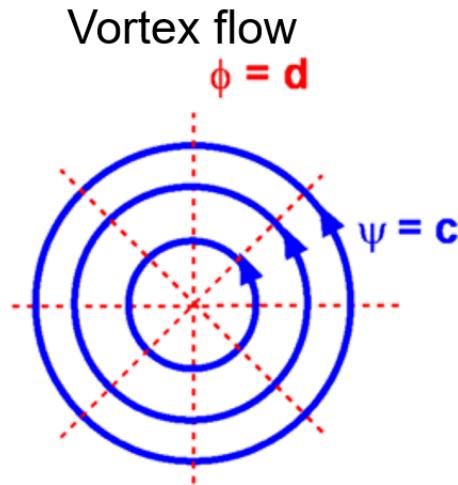
$$\phi = -\frac{\Gamma}{2\pi}\theta \quad (4.8.1)$$

$$\psi = \frac{\Gamma}{2\pi} \ln \left( \frac{r}{R} \right) \quad (4.8.2)$$

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = 0 \quad (4.8.3)$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -\frac{\Gamma}{2\pi r} \quad (4.8.4)$$

Where  $\Gamma < 0$  is anti-clockwise motion and  $\Gamma > 0$  is clockwise motion.



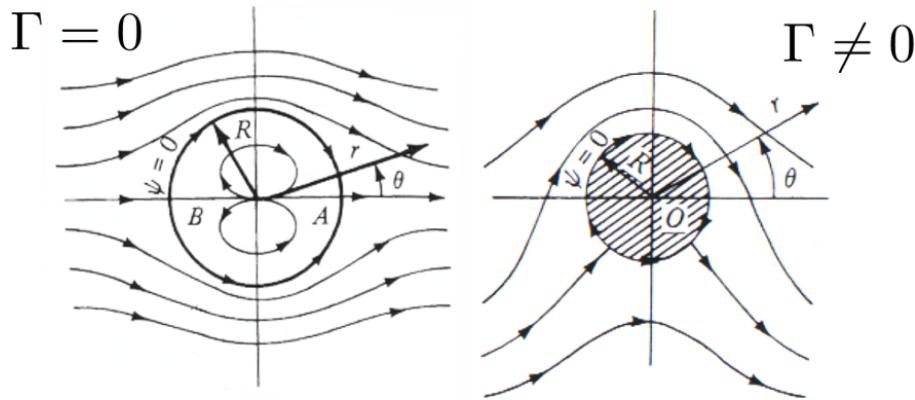
## 4.9 Cylinder with a Vortex Flow

$$\psi = V_\infty r \sin \theta \left( 1 - \frac{R^2}{r^2} \right) + \frac{\Gamma}{2\pi} \ln \left( \frac{r}{R} \right) \quad (4.9.1)$$

$$\phi = V_\infty r \cos \theta \left( 1 + \frac{R^2}{r^2} \right) - \frac{\Gamma}{2\pi} \theta \quad (4.9.2)$$

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = V_\infty \cos \theta \left( 1 - \frac{R^2}{r^2} \right) \quad (4.9.3)$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin \theta \left( 1 + \frac{R^2}{r^2} \right) - \frac{\Gamma}{2\pi r} \quad (4.9.4)$$



## 4.10 Lift and Drag of a Cylinder with Circulation

Apply Bernoulli:

$$p_\infty + \frac{1}{2}\rho V_\infty^2 = p(r, \theta) + \frac{1}{2}\rho(u_r^2 + u_\theta^2) \quad (4.10.1)$$

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{u_r^2 + u_\theta^2}{V_\infty^2} \quad (4.10.2)$$

On the cylinder surface:  $u_r = 0$

$$c_p(R, \theta) = 1 - \frac{u_\theta^2}{V_\infty^2} = 1 - \frac{(2V_\infty \sin \theta + \frac{\Gamma}{2\pi R})^2}{V_\infty^2} \quad (4.10.3)$$

$$= 1 - \left( 4 \sin^2(\theta) + \frac{\Gamma^2}{4\pi^2 V_\infty^2 R^2} + \frac{2\Gamma \sin(\theta)}{V_\infty \pi R} \right) \quad (4.10.4)$$

## 4.11 Lift of the Cylinder

We need to calculate  $c_p$  on the surface of the cylinder.

$$L = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} (c_p \cdot \hat{n} \cdot \hat{j}R) d\theta \quad (4.11.1)$$

$$= -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} (c_p \sin \theta R) d\theta \quad (4.11.2)$$

$$L = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left( 1 - \frac{(2V_\infty \sin \theta + \frac{\Gamma}{2\pi R})^2}{V_\infty^2} \right) \sin \theta R d\theta \quad (4.11.3)$$

Expanding the integral out, we arrive at:

$$\begin{aligned} L &= -\frac{1}{2}\rho V_\infty^2 \left[ \int_0^{2\pi} \left( 1 - \frac{\Gamma^2}{4\pi^2 R^2 V_\infty^2} \right) \cdot \sin \theta \cdot R d\theta + \int_0^{2\pi} -4 \sin \theta^3 \cdot R d\theta + \right. \\ &\quad \left. \int_0^{2\pi} -\frac{2\Gamma}{\pi RV_\infty} \cdot \sin \theta^2 \cdot R d\theta \right] \end{aligned} \quad (4.11.4)$$

Because the first term and the second term have an odd power of  $\sin \theta$ , when we integrate these, they will have negligible outcome on the lift of the cylinder. We can reduce our equation to:

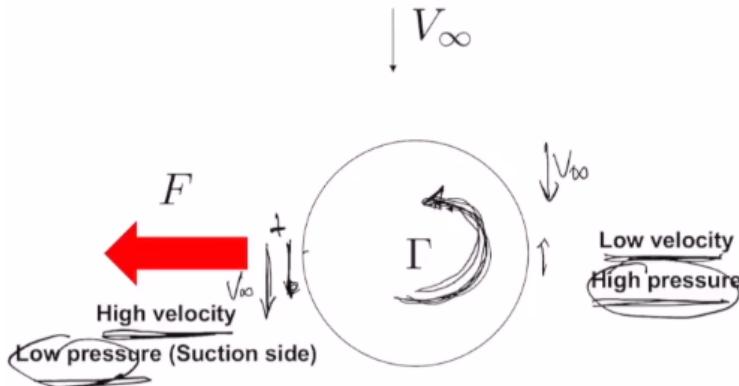
$$L = -\frac{1}{2}\rho V_\infty^2 \left[ \int_0^{2\pi} -\frac{2\Gamma}{\pi RV_\infty} \cdot \sin \theta^2 \cdot R d\theta \right] \quad (4.11.5)$$

$$= \frac{\rho V_\infty \Gamma}{\pi} \int_0^{2\pi} \sin \theta^2 d\theta \quad (4.11.6)$$

$$= \frac{\rho V_\infty \Gamma}{\pi} \left[ \frac{1}{2}\theta - \frac{1}{2}\sin 2\theta \right]_0^{2\pi} \quad (4.11.7)$$

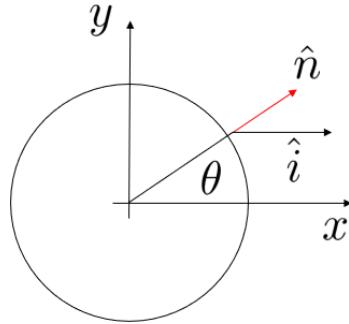
$$L = \rho V_\infty \Gamma \quad (4.11.8)$$

We can see here that our vortex factor  $\Gamma$  has a proportional effect on our lift force. We can see an example of this in real life when a football is kicked. When the ball is kicked in such a way that it has a spin, we see the ball curves in certain directions.



We can see that when the ball spins, the velocity from the free stream and the vortex combine to create regions of low and high pressure. This creates a net force, leading to a suction effect.

## 4.12 Drag of a cylinder



$$D = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} c_p \vec{n} \cdot \hat{i} R d\theta \quad (4.12.1)$$

$$= -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} c_p \cos \theta R d\theta \quad (4.12.2)$$

$$D = -\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} \left( 1 - \frac{(2V_\infty \sin \theta + \frac{\Gamma}{2\pi R})^2}{V_\infty^2} \right) \cos \theta R d\theta \quad (4.12.3)$$

The first term has an odd power of cosine, and so is negligible. The second term of the drag integral is:

$$\int_0^{2\pi} -4 \sin \theta^2 \cos \theta R d\theta = -4R \left( \frac{1}{3} \sin \theta^3 \right)_p^{2\pi} = 0 \quad (4.12.4)$$

The third term of the drag integral is:

$$\frac{1}{2}\rho V_\infty^2 \int_0^{2\pi} -\frac{2\Gamma}{\pi RV_\infty} \sin \theta \cos \theta R d\theta = -\frac{\rho V_\infty \Gamma}{\pi} \left( -\frac{1}{2} \cos 2\theta \right)_0^{2\pi} = 0 \quad (4.12.5)$$

Therefore, we see that our drag is in fact:

$$D = 0 \quad (4.12.6)$$

This is due to our assumption that our flow is inviscid and that the pressure forces are symmetrical to the left and right sides to the y axis. Therefore the net forces acting on the x direction is zero. This is where our model starts failing and is called the D'Alambert Paradox.

## 4.13 Inviscid and Viscous Flow past a body

High  $Re$  implies that the magnitude of the inertia forces are much greater than the magnitudes of the viscous forces in a system. This might imply that the effects of viscosity are insignificant compared with the inertia forces, but this would be a dangerous conclusion. Compare theory for zero viscosity with experiment for high  $Re$  flow past a cylinder:

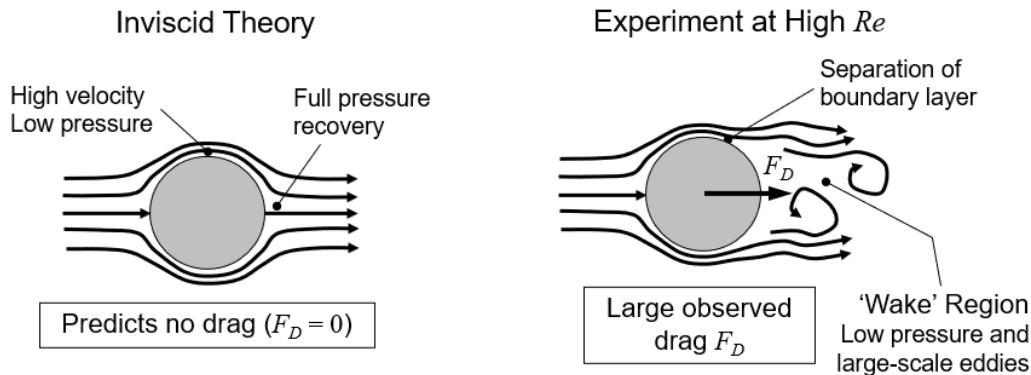
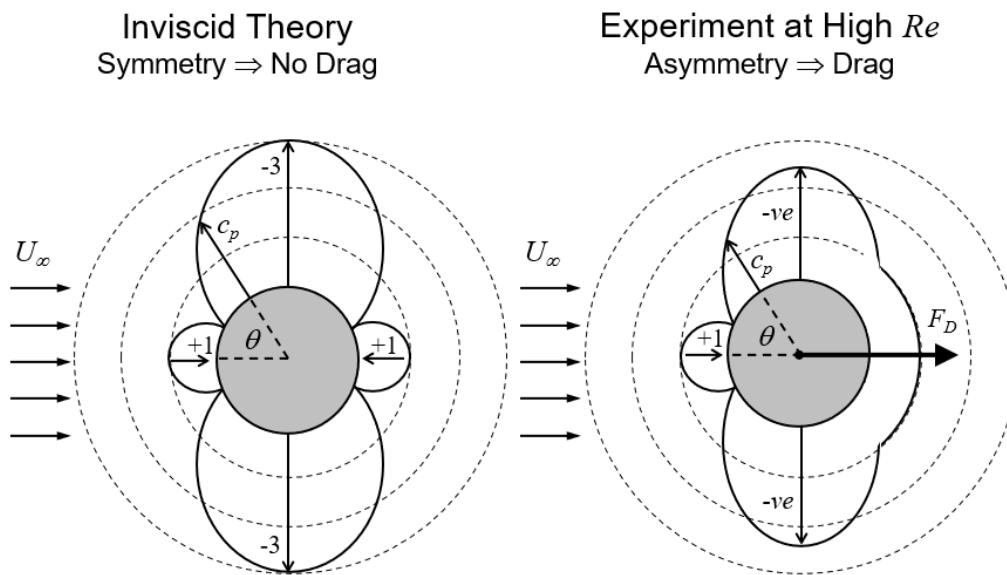
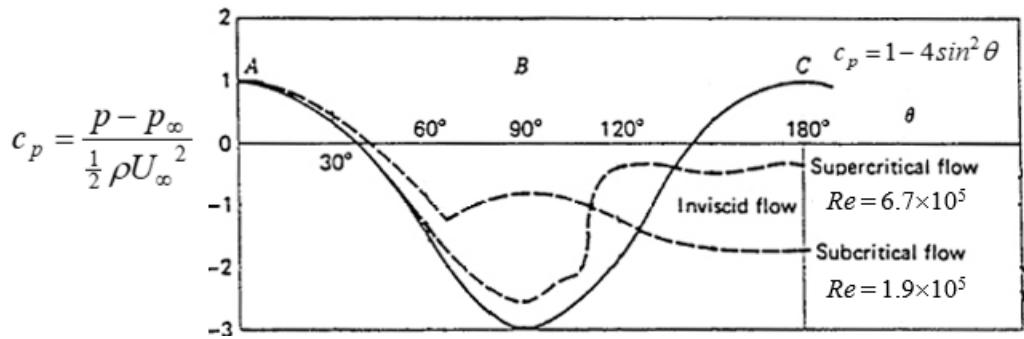


Figure 4.1: We have a net force in our experiment due to the fact that a net force is created from the low pressure wake region and the high pressure front side of the cylinder.

## 4.14 Flow past a cylinder - pressure coefficient



We can plot the pressure coefficient and see the difference between the inviscid theory and an experiment at high  $Re$ .

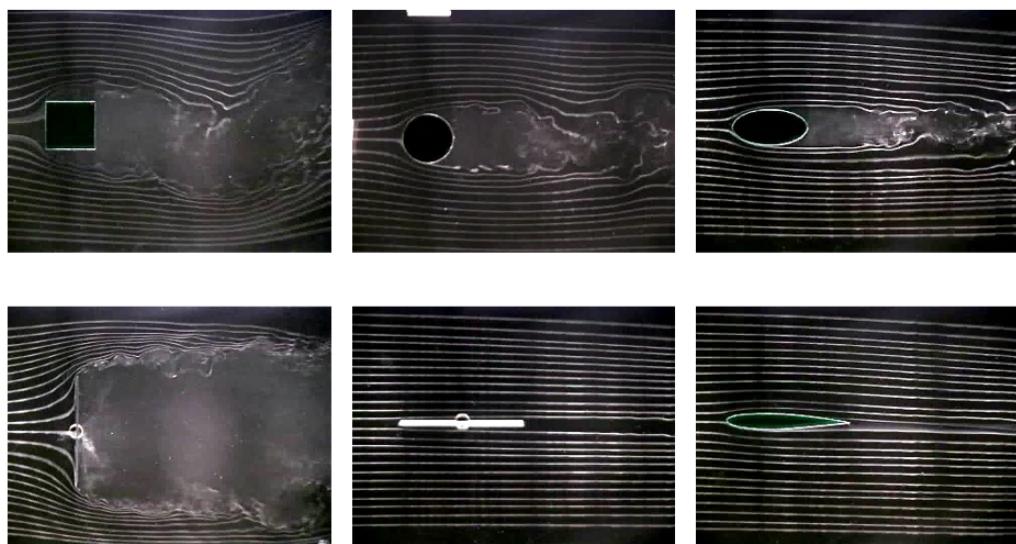


For a long circular cylinder, the lift coefficient  $c_L$  and the form drag coefficient to  $c_D$  are related to  $c_p$  by:

$$c_L = \frac{1}{2} \int_0^{2\pi} c_p \sin \theta \, d\theta \quad (4.14.1)$$

$$c_D = \frac{1}{2} \int_0^{2\pi} c_p \cos \theta \, d\theta \quad (4.14.2)$$

Some examples of viscous flow past bodies:



# Chapter 5

## Modelling the Flow Around Streamlined Bodies

17/11/2020

### 5.1 Conformal mapping

#### 5.1.1 Joukowsky Transformation

The question is how can we transform our cylinder into something that looks like an airfoil? We achieve this by using conformal mapping which maps each part of our coordinate space to a new one.

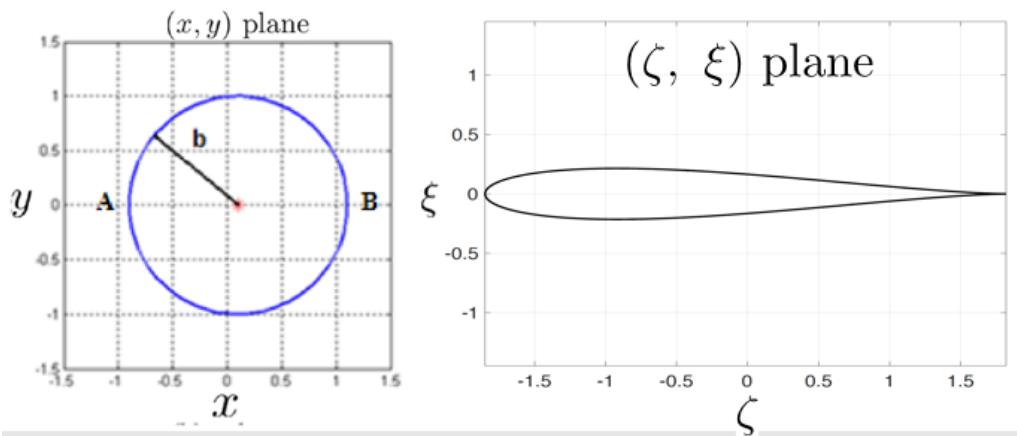
$$x = r \cos \theta \quad (5.1.1)$$

$$y = r \sin \theta \quad (5.1.2)$$

$$w = G(z) = z + \frac{\lambda^2}{z} \quad (5.1.3)$$

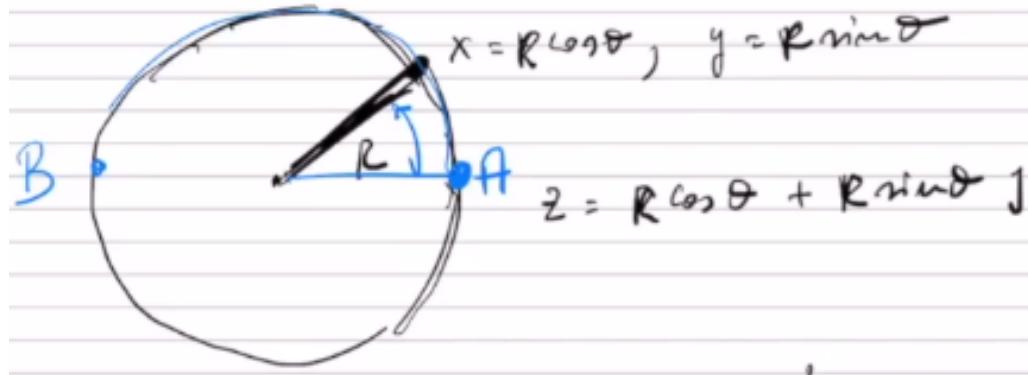
$$z = re^{i\theta} = x + yi \quad (5.1.4)$$

$$w = \zeta + \xi i \quad (5.1.5)$$



### 5.1.2 Flat plate and Ellipse

Lets consider a cylinder:



$$z = R \cos \theta + Ri \sin \theta \quad (5.1.6)$$

$$w = G(z) = z + \frac{\lambda^2}{z} \quad (5.1.7)$$

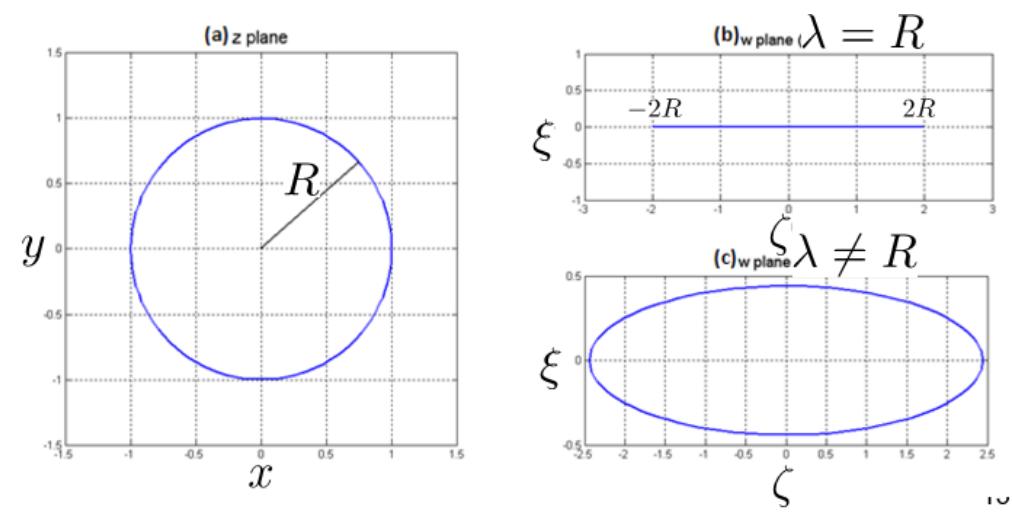
$$= R \cos \theta + Ri \sin \theta + \frac{\lambda^2}{R \cos \theta + Ri \sin \theta} \quad (5.1.8)$$

$$= R \cos \theta + Ri \sin \theta + \frac{\lambda^2(R \cos \theta - Ri \sin \theta)}{(R \cos \theta + Ri \sin \theta)(R \cos \theta - Ri \sin \theta)} \quad (5.1.9)$$

$$= R \cos \theta + Ri \sin \theta + \frac{\lambda^2(R \cos \theta - Ri \sin \theta)}{R^2} \quad (5.1.10)$$

$$w = R \cos \theta \left( 1 + \frac{\lambda^2}{R} \right) + Ri \sin \theta \left( 1 - \frac{\lambda^2}{R} \right) \quad (5.1.11)$$

When  $\lambda = R$ , we achieve a flat plate transformation as our imaginary term is always 0. When  $\lambda \neq R$ , we achieve an elliptical shape.



### 5.1.3 Aerofoils

We can further refine our conformal mapping by adding two elements to our equation,  $x_c$  and  $y_c$ . These two terms effect the center of the cylinder in  $x$  and  $y$ .

$$w = G(z) = z + \frac{\lambda^2}{z} \quad (5.1.12)$$

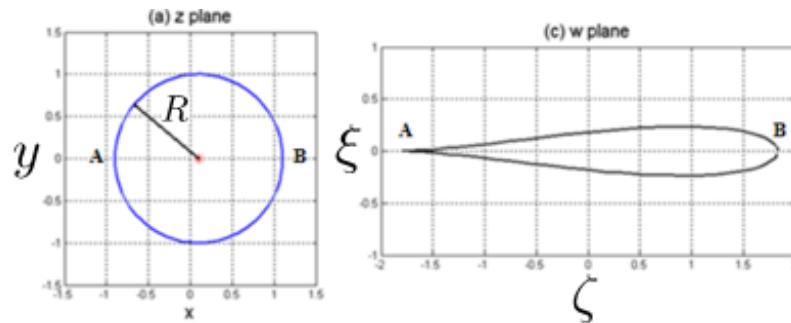
$$\lambda = R - \sqrt{x_c^2 + y_c^2} \quad (5.1.13)$$

$$z = x + x_c + (y + y_c)i \rightarrow w = \zeta + \xi i \quad (5.1.14)$$

Mapping these into their respective planes:

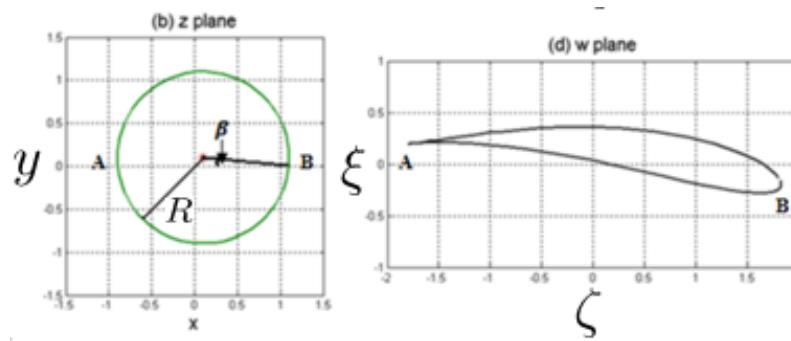
$$x_c \neq 0 \quad (5.1.15)$$

$$y_c = 0 \quad (5.1.16)$$



$$x_c \neq 0 \quad (5.1.17)$$

$$y_c \neq 0 \quad (5.1.18)$$



### 5.1.4 Cambered airfoil

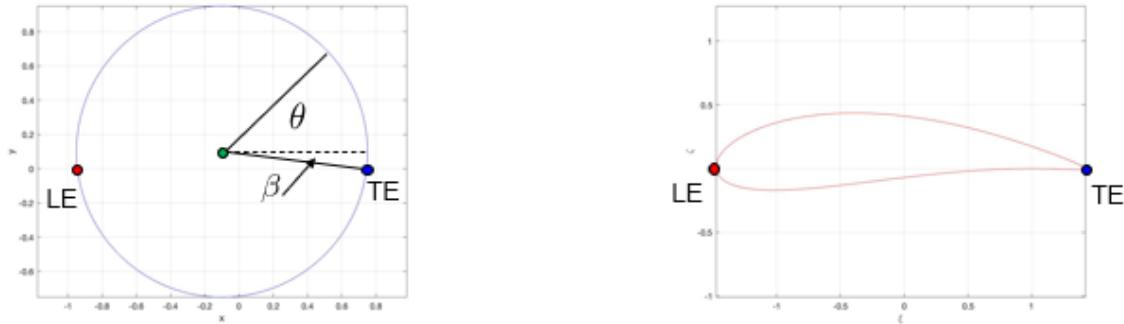


Figure 5.1:  $x_c = -0.1$ ,  $y_c = 0.1$ ,  $\sin\beta = \frac{y_c}{R}$

## 5.2 Uniform stream + circulation

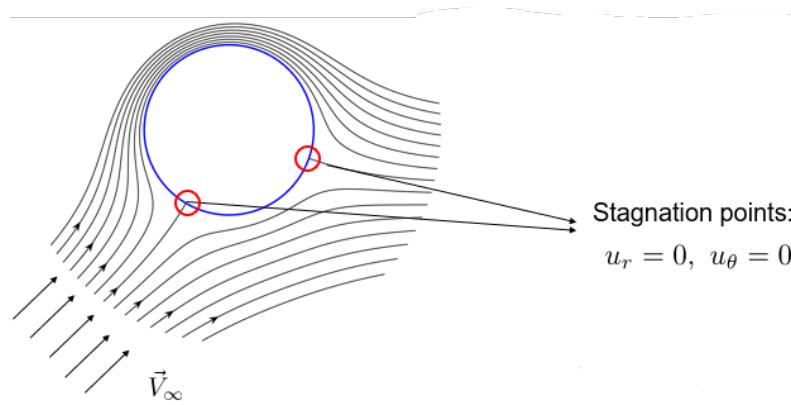
What is the correct relationship between the circulation and the angle of incidence?

$$\Gamma = f(\alpha) \quad (5.2.1)$$

$$\text{Stagnation points: } u_r = 0, u_\theta = 0 \quad (5.2.2)$$

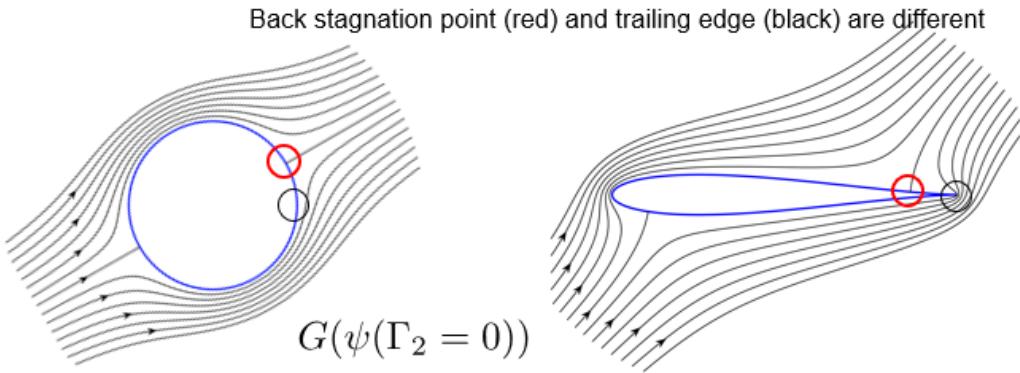
$$\phi = V_\infty r \cos(\theta - \alpha) \left( 1 + \frac{R^2}{r^2} \right) - \frac{\Gamma}{2\pi} \theta \quad (5.2.3)$$

$$\psi = V_\infty r \sin(\theta - \alpha) \left( 1 - \frac{R^2}{r^2} \right) + \frac{\Gamma}{2\pi} \log \left( \frac{r}{R} \right) \quad (5.2.4)$$

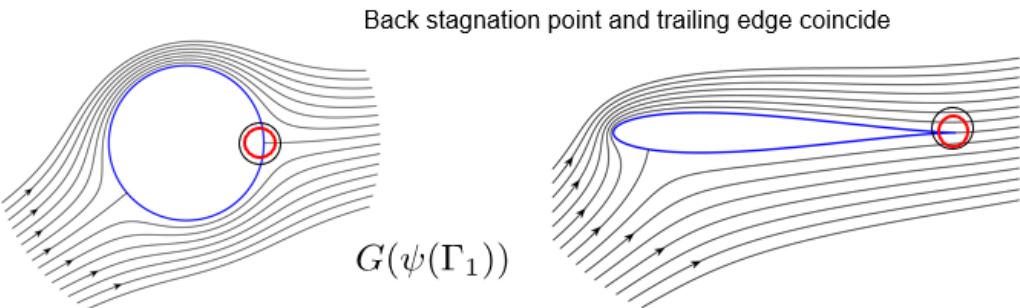


## 5.3 Kutta condition

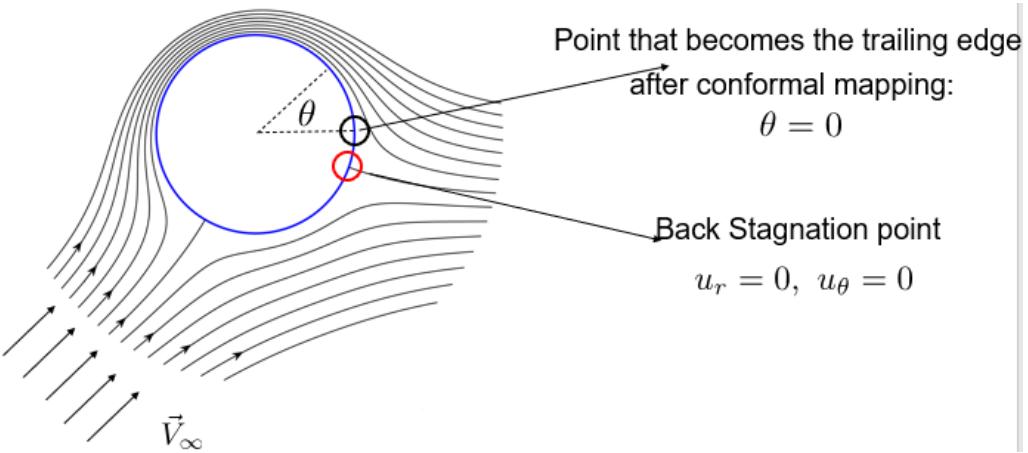
I can apply the Joukowsky Transformation to the streamlines around the cylinder and obtain the streamlines around the airfoil. However I need to carefully select the circulation,  $\Gamma$ .



We can see that this streamline is unrealistic. This is because to curl around the trailing edge cusp, the speed goes to infinity. This is clearly impossible and flow separation would occur instead. This is the wrong circulation value for the given angle of incidence.



This streamline is more realistic as the velocity is parallel to the trailing edge cusp and its component in the vertical direction is zero. This circulation can be used to predict the lift  $F_L$  but still predicts no drag ( $F_D = 0$ ). The next question is how do we achieve this mapping?



We can see here that for a given angle of incidence, the circulation has to be selected by imposing that the back stagnation point (red circle) and the point corresponding to the airfoil trailing edge (black circle) coincide.

$$\phi = V_\infty r \cos(\theta - \alpha) \left( 1 + \frac{R^2}{r^2} \right) - \frac{\Gamma}{2\pi} \theta \quad (5.3.1)$$

$$\psi = V_\infty r \sin(\theta - \alpha) \left( 1 - \frac{R^2}{r^2} \right) + \frac{\Gamma}{2\pi} \log \left( \frac{r}{R} \right) \quad (5.3.2)$$

Velocity components:

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = V_\infty r \cos(\theta - \alpha) \left( 1 - \frac{R^2}{r^2} \right) \quad (5.3.3)$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin(\theta - \alpha) \left( 1 + \frac{R^2}{r^2} \right) - \frac{\Gamma}{2\pi r} \quad (5.3.4)$$

On the cylinder  $r = R$ :

$$u_r = 0 \quad (5.3.5)$$

$$u_\theta = -2V_\infty \sin(\theta - \alpha) - \frac{\Gamma}{2\pi R} \quad (5.3.6)$$

Kutta condition:

$$u_\theta(\theta = 0) = 0 \quad (5.3.7)$$

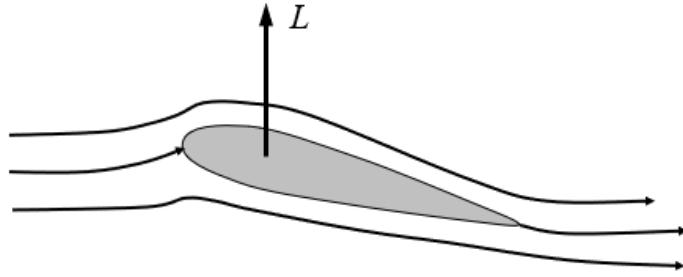
$$\Gamma = -4\pi V_\infty R \sin(-\alpha) = 4\pi V_\infty R \sin \alpha \quad (5.3.8)$$

For a cambered aerofoil:

$$u_\theta(\theta = -\beta) = 0 \quad (5.3.9)$$

$$\Gamma = -4\pi V_\infty R \sin(-\beta - \alpha) = 4\pi V_\infty R \sin(\beta + \alpha) \quad (5.3.10)$$

## 5.4 Aerofoil lift



The lift is given by:

$$L = \rho V \Gamma = 4\rho\pi V_\infty^2 R \sin \alpha \quad (5.4.1)$$

The lift coefficient is:

$$c_L = \frac{L}{\frac{1}{2}\rho V_\infty^2 c} = 9\pi \frac{R}{c} \sin \alpha \quad (5.4.2)$$

Flow is inviscid so zero drag.

## 5.5 Flow past an Aerofoil

### 5.5.1 How was the circulation created?

When an airfoil is still on the ground there is no flow and total circulation is zero. As soon as the airfoil takes off a negative circulation is created at the back of the airfoil and a positive one is stored within the airfoil boundary layer.

### 5.5.2 Stalling

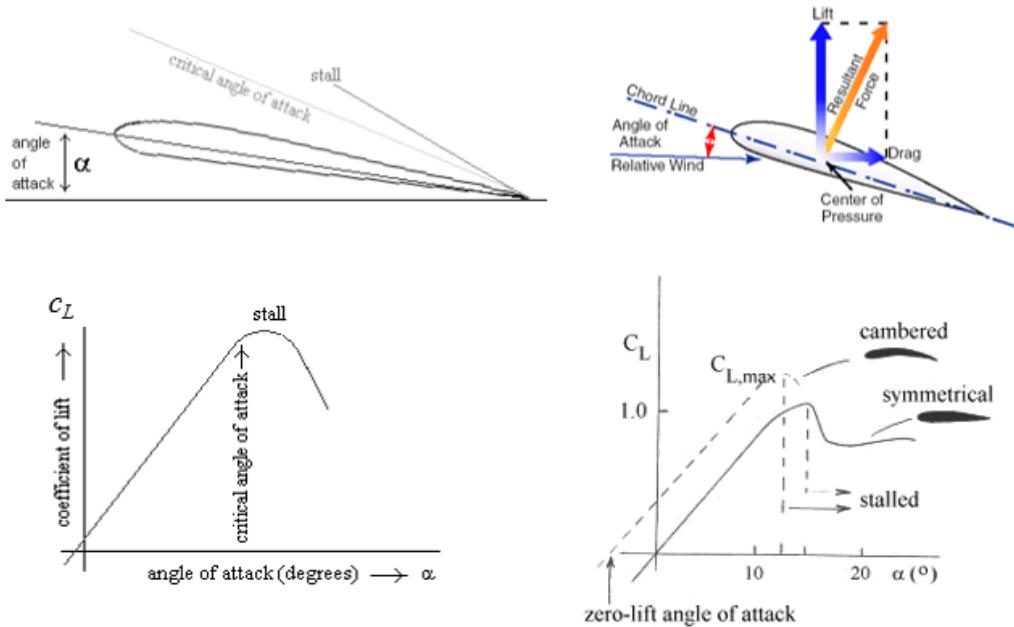
If the angle of attack becomes too great and boundary layer separation occurs on the top of the aerofoil the pressure pattern will change dramatically. In such a case, the lift component is insufficient to overcome the weight of the aircraft and disaster is imminent. This phenomenon is known as stalling. When stalling occurs, all, or most, of the 'suction' pressure is lost and the plane will suddenly drop from the sky. The solution to this is to put the plane into a dive to regain the boundary layer. A transverse lift force is then exerted on the wing which gives the pilot some control and allows the plane to be pulled out of the dive.



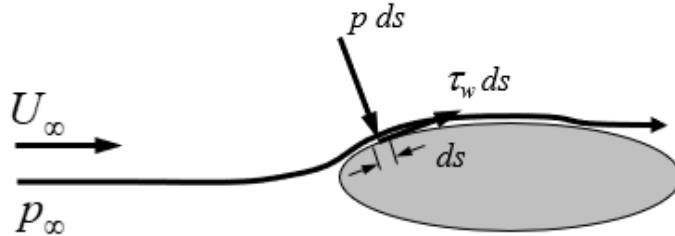
Boundary layer separation occurs on the upper surface (where the adverse pressure gradients are large) and produces a different  $c_p$  distribution from the unseparated case. As skin friction is predominant for the stalled wing, the profile drag is sensitive to any increase in form drag.

- At a small angle of attack  $\alpha$ : separation is close to trailing edge, wake is thin, low form of drag
- As the angle of attack  $\alpha$  increases: separation moves along the top of the aerofoil, wake width increases, form drag increases
- The critical angle of attack  $\alpha_{crit}$ : value of  $\alpha$  for maximum  $c_L$ , stall angle
- If the angle of attack  $\alpha > \alpha_{crit}$ : separation from most of the upper surface, wider wake with turbulence, considerable form drag

The exact value of the critical angle of attack depends on the type and shape of the aerofoil.



## 5.6 Form drag and skin friction



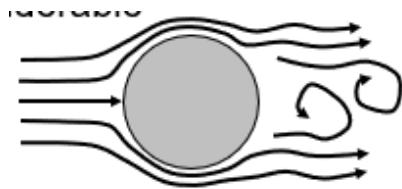
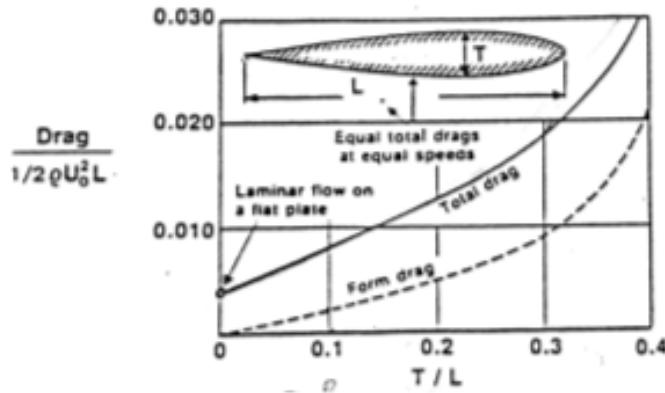
Force due to pressure  $p$  on element of surface area  $ds$  is  $p ds$ . The force  $p ds$  has a component in the stream direction and if this component is integrated over the whole body surface, it gives the drag due to pressure distribution or the form drag. Form drag is the resultant of the forces normal to the surface (pressure distribution: normal stresses). The same element of area  $ds$  experiences a shear stress  $\tau_w$  due to the velocity of the gradient normal to the surface and the associated shear force is  $\tau_w ds$ . This also has a component in the stream direction which when integrated over the body surface gives the drag due to skin friction. Skin friction drag is resultant of forces tangential to the surface (shear stresses).

## 5.7 Total drag

Total drag (also known as profile drag) is given by:

$$\text{Total drag} = \text{Form drag} + \text{Skin friction} \quad (5.7.1)$$

Separation is what leads to large form drag. The benefits of streamlining can be considerable.



Bluff Body

Form drag dominant  
Skin friction insignificant  
 $c_D \approx 1.0$  at high  $Re$



Aerofoil

Skin friction dominant  
Low form drag  
 $c_D \approx 0.01$  at high  $Re$

# Chapter 6

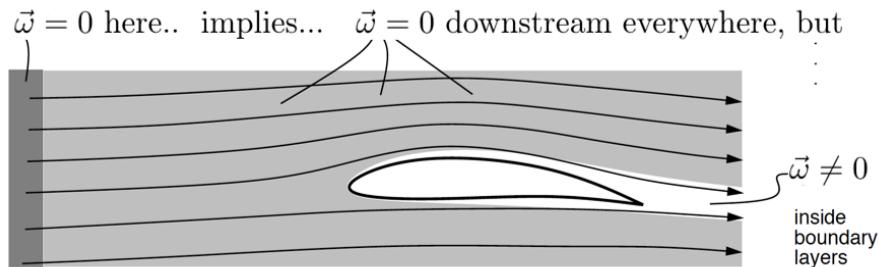
## Boundary Layer Theory, Separation and Control

24/11/2020

### Boundary layer theory (Prandtl)

The flow domain can be subdivided into two parts:

- **Irrational flow region**, outside of the boundary layer, Bernoulli equation, potential flow and stream function apply
- **Boundary layer** where all vorticity is confined, The friction shear at the airfoil surface acts as a source of vorticity.



### 6.1 Boundary layer concept

Immediately adjacent to a solid surface, the fluid particles are slowed by the strong shear force between the fluid particles and the surface. This relatively slower moving layer of fluid is the boundary layer.

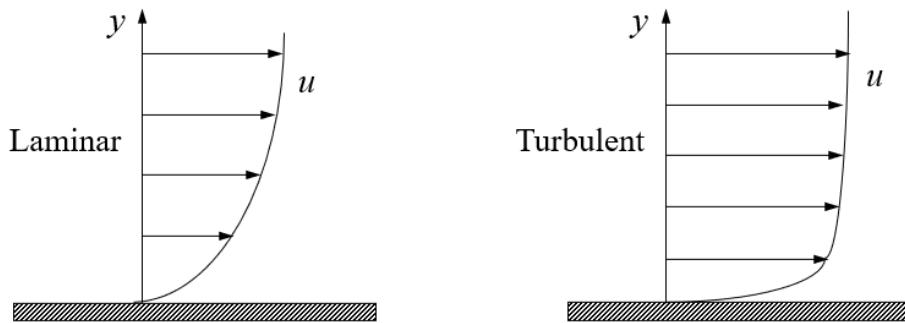
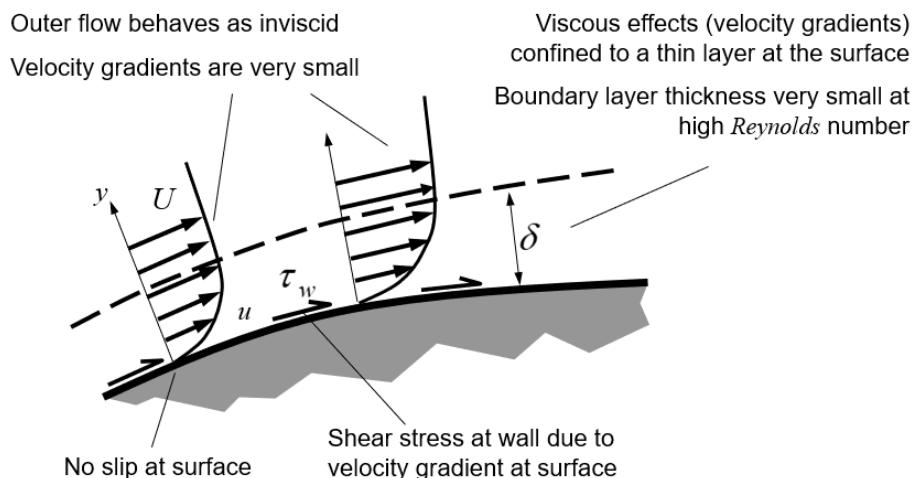


Figure 6.1: In turbulent flow, you have a higher shear stress due to the higher velocity close to the boundary. This produces more frictional drag against the motion of the solid surface. ( $\tau = \mu \frac{\partial u}{\partial y}$ )

## 6.2 Boundary layer theory

Prandtl's concept of the boundary layer in high  $Re$  flow is that although viscous forces can be considered small, their effects are not negligible.



Let us see what happens when we consider the Navier Stokes equations for a steady - 2D flow with  $Re \gg 1$ . Continuity equation (conservation of mass):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6.2.1)$$

Conservation of momentum equations:

$$\text{x-direction: } u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (6.2.2)$$

$$\text{y-direction: } u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (6.2.3)$$

Remember that:

$$Re = \frac{U_\infty L}{\nu} \quad (6.2.4)$$

Where  $L$  is the length of the airfoil being analysed,  $U_\infty$  is the velocity of the free stream and  $\nu$  the kinematic viscosity. Inside the boundary layer, one thing we can assume is that the inertial forces and the viscous forces are comparable, because viscous diffusion is large and of the same order of magnitude of the inertial terms. Therefore:

$$\frac{u \frac{\partial u}{\partial x}}{\nu \frac{\partial^2 u}{\partial y^2}} \approx 1 \quad (6.2.5)$$

We need to exploit this relationship in order to derive an expression for  $\delta$  in comparison to  $L$ . Some approximations we can make below are:

$$u \approx U_\infty \quad (6.2.6)$$

$$x \approx L \rightarrow \frac{\partial}{\partial x} \propto \frac{1}{L} \quad (6.2.7)$$

$$y \approx \delta \rightarrow \frac{\partial}{\partial y} \propto \frac{1}{\delta} \quad (6.2.8)$$

Combining this with our above assumption:

$$\frac{\frac{U_\infty^2}{L}}{\nu \frac{U_\infty}{\delta^2}} = \frac{\delta^2}{L^2} Re \approx 1 \quad (6.2.9)$$

$$\delta^2 \approx \frac{L^2}{Re} \quad (6.2.10)$$

Looking at the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6.2.11)$$

$$\frac{U_\infty}{L} + \frac{v}{\delta} = 0 \quad (6.2.12)$$

$$v = \frac{\delta}{L} U_\infty \quad (6.2.13)$$

$$v = \frac{1}{\sqrt{Re}} U_\infty \quad (6.2.14)$$

(signs were ignored). Looking at the momentum equation in the x-direction:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (6.2.15)$$

$$\text{First term: } u \frac{\partial u}{\partial x} \propto \frac{U_\infty^2}{L} \quad (6.2.16)$$

$$\text{Second term: } v \frac{\partial u}{\partial y} \propto \frac{1}{\sqrt{Re}} \frac{U_\infty}{\delta} U_\infty = \frac{U_\infty^2}{\sqrt{Re}} \frac{\sqrt{Re}}{L} = \frac{U_\infty^2}{L} \quad (6.2.17)$$

$$\text{Fourth term: } \nu \frac{\partial^2 u}{\partial x^2} \propto \nu \frac{U_\infty}{L^2} = \frac{\nu}{LU_\infty} \frac{U_\infty^2}{L} = \frac{1}{Re} \frac{U_\infty^2}{L} \quad (6.2.18)$$

$$\text{Fifth term: } \nu \frac{\partial^2 u}{\partial y^2} \propto \nu \frac{I_\infty}{\delta^2} = \frac{\nu}{LU_\infty} \frac{U_\infty^2 L}{\delta^2} = \frac{1}{Re} \frac{U_\infty^2 Re}{L} = \frac{U_\infty^2}{L} \quad (6.2.19)$$

We can see in the fourth term that  $Re \gg 1$ , hence it is neglected i.e. there will not be a viscous term due to the variation of the velocity in the x direction w.r.t x. Looking at the momentum equation in the y-direction:

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (6.2.20)$$

$$\text{First term: } u \frac{\partial v}{\partial x} \propto U_\infty^2 \frac{\delta}{L^2} = \frac{1}{\sqrt{Re}} \frac{U_\infty^2}{L} \quad (6.2.21)$$

$$\text{Second term: } v \frac{\partial v}{\partial y} \propto \frac{U_\infty^2}{L^2} \frac{\delta^2}{\delta} = \frac{1}{\sqrt{Re}} \frac{U_\infty^2}{L} \quad (6.2.22)$$

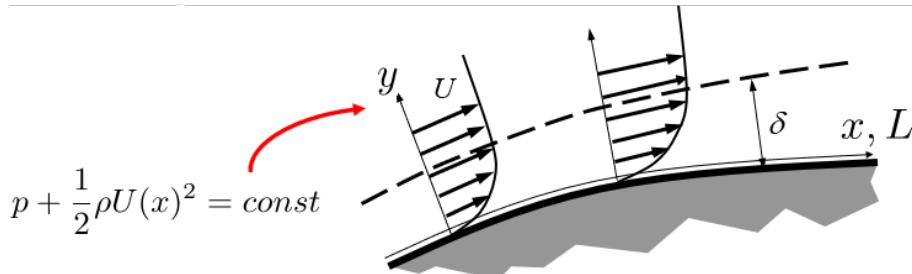
$$\text{Fourth term: } \nu \frac{\partial^2 v}{\partial x^2} \propto \nu \frac{U_\infty \delta}{L} \frac{1}{L^2} = \frac{\nu}{LU_\infty} \frac{U_\infty^2 \delta}{L^2} = \frac{1}{Re} \frac{U_\infty^2}{L \sqrt{Re}} = \frac{1}{\sqrt{Re}^3} \frac{U_\infty^2}{L} \quad (6.2.23)$$

$$\text{Fifth term: } \nu \frac{\partial^2 v}{\partial y^2} \propto \nu \frac{U_\infty \delta}{L} \frac{1}{\delta^2} = \frac{\nu}{LU_\infty} \frac{U_\infty^2}{\delta} = \frac{1}{Re} \frac{U_\infty^2 \sqrt{Re}}{L} = \frac{1}{\sqrt{Re}} \frac{U_\infty^2}{L} \quad (6.2.24)$$

We see that in the first, second, fourth and fifth terms,  $Re \gg 1$ , hence they are neglected. Let us see what this analysis tells us:

$$\text{x-direction: } u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad (6.2.25)$$

$$\text{y-direction: } \frac{\partial p}{\partial y} = 0 \quad (6.2.26)$$



Pressure does not change along thickness of the boundary layer. Therefore I can estimate the pressure outside of the boundary layer using Bernoulli (inviscid flow theory), and then apply that pressure on the body wall. Outside of the boundary layer, we know:

$$\frac{\partial}{\partial y} = 0 \quad \& \quad v = 0 \quad (6.2.27)$$

$$U(x) = \frac{\partial U(x)}{\partial x} = \frac{\partial \frac{1}{2} U(x)^2}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (6.2.28)$$

$$p + \frac{1}{2} \rho U(x)^2 = \text{const} \quad (6.2.29)$$

## 6.3 Flow separation

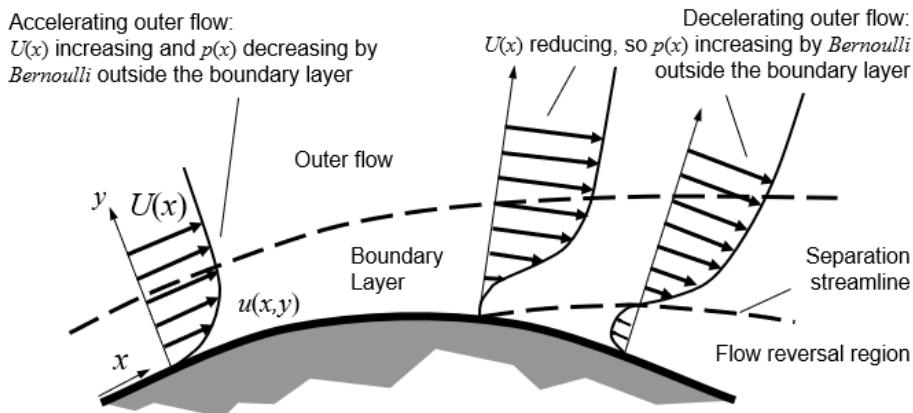


Figure 6.2: Formation of a vortex, due to flow separation and flow reversal.

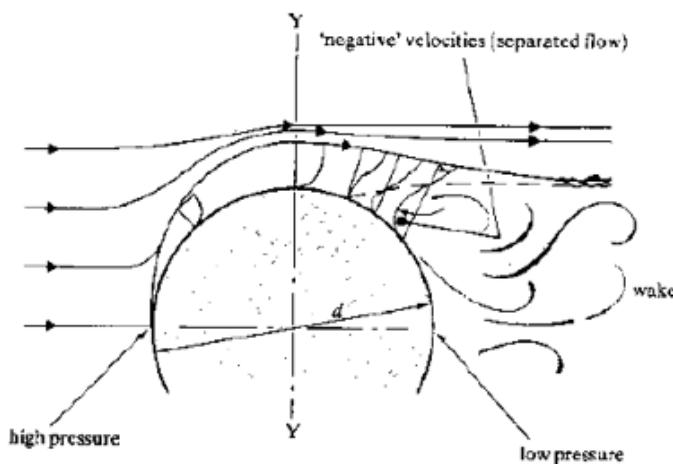
Favourable pressure gradient:

$$\frac{\partial U(x)}{\partial x} > 0, \quad \frac{\partial p}{\partial x} < 0 \quad (6.3.1)$$

Adverse pressure gradient:

$$\frac{\partial U(x)}{\partial x} < 0, \quad \frac{\partial p}{\partial x} > 0 \quad (6.3.2)$$

Adverse pressure gradient produces further retardation of fluid (which is already slowed down close to wall). The fluid near to the wall is eventually brought to rest, flow reverse and separated. Separation occurs at or near the point where  $\frac{dp}{dx}$  first becomes zero; at the first point of separation, at  $y = 0$ ,  $\frac{du}{dy} = 0$ . The formation of separation occurs as the fluid accelerates from the centre to get round the cylinder (it must accelerate as it has further to go than the surround fluid). It reaches a maximum at  $Y - Y$ , where it has also dropped in pressure. The adverse pressure gradient between here and the downstream side of the cylinder will cause separation if the flow is fast enough ( $Re > 2$ ).



## 6.4 Laminar and turbulent flow separation

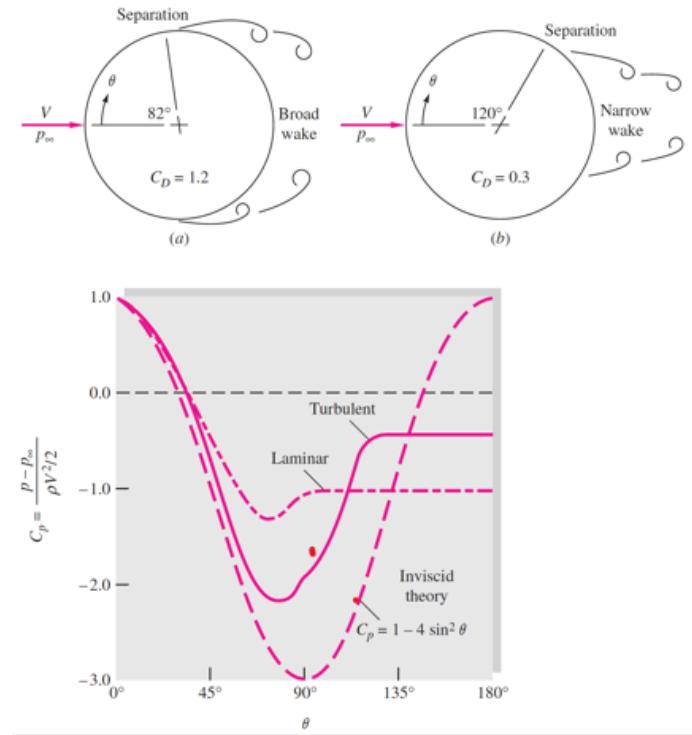
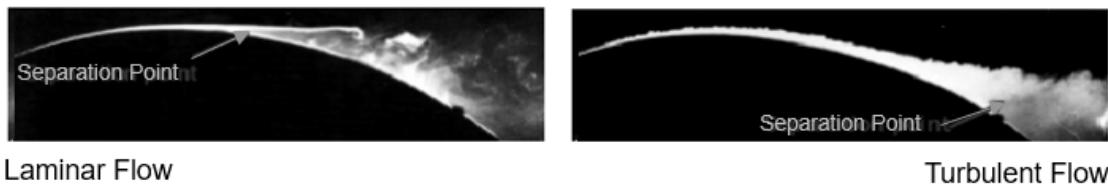
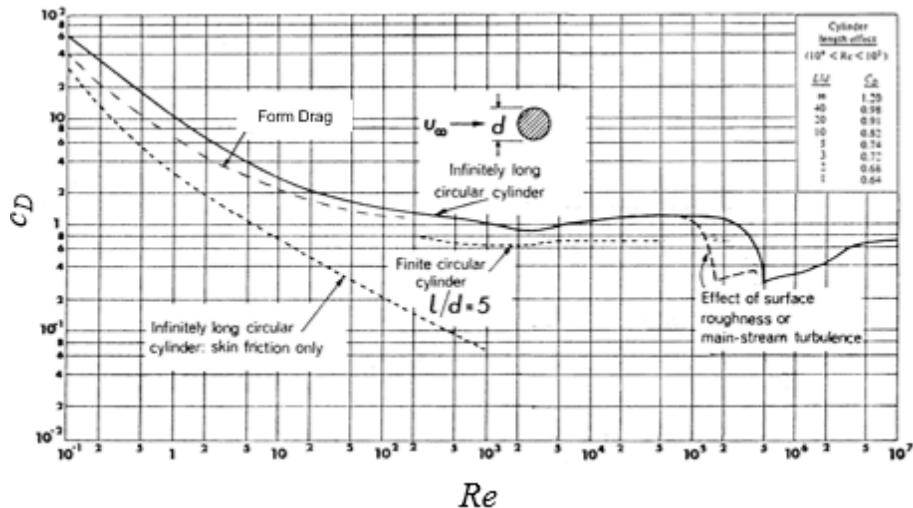


Figure 6.3: Left: laminar, right: turbulent

The boundary layer has a high momentum deficit and in the case of laminar flow, it is unable to adjust to the increasing pressure and separates. A turbulent boundary layer is more resistant to separation than a laminar one. Due to the chaotic nature of turbulence, a turbulent boundary layer has a great capacity for mixing and absorbing energy from the free stream (this is obvious from its profile and its rate of growth compared with a laminar layer). The effect of increased mixing is increased transport of momentum (due to turbulent stresses) between the free-stream and the flow near the wall. This increased transport of momentum from the free-stream to the wall increases the stream-wise momentum in the boundary layer allowing the flow to overcome  $\frac{dp}{dx}$  for longer and to separate further downstream (larger velocities near the wall are not so easily halted).



## 6.5 Flow past a cylinder - Drag coefficient

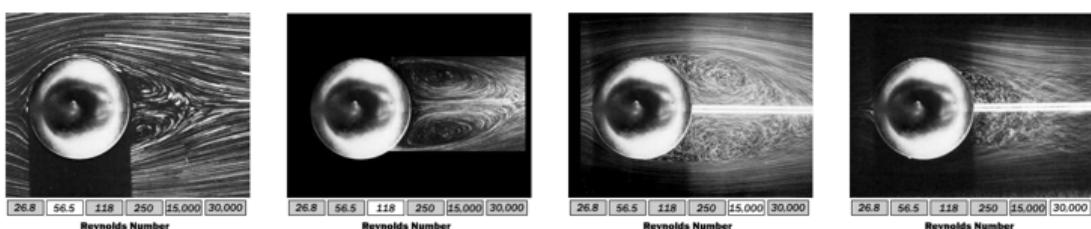


For a cylinder,

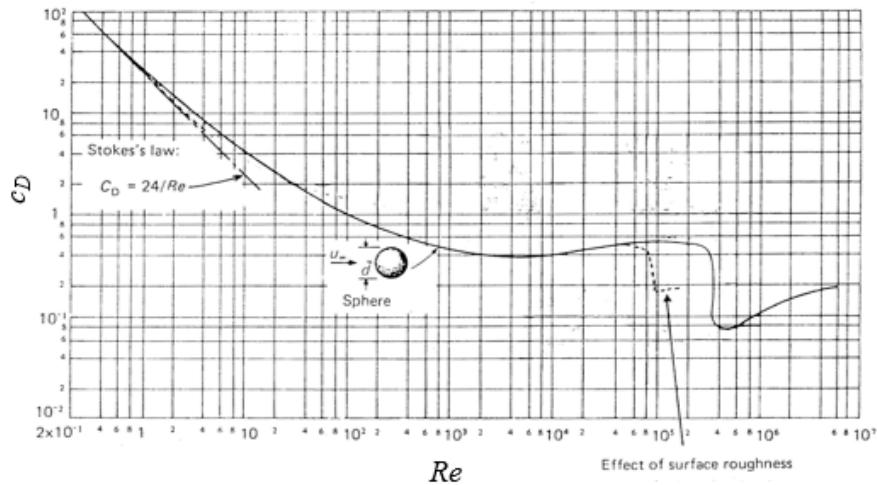
- $c_D$  is nearly constant when  $Re = 10^3 - 10^5$ , but varies considerably as the boundary layer becomes turbulent between  $Re = 10^5 - 10^6$
- At  $Re \approx 2000$ ,  $c_D$  reaches minimum of  $\approx 0.9$
- At  $Re \approx 30000$ ,  $c_D$  rises to  $\approx 1.2$  (partly due to increasing turbulence in the wake)
- At  $Re \approx 200000$ ,  $c_D$  drops suddenly to  $\approx 0.3$
- At  $Re > 200000$ ,  $c_D$  increases slowly.
- The form drag is  $\approx 50\%$  of the profile drag as  $Re \rightarrow 0$
- By  $Re \approx 200$  the form drag is  $\approx 90\%$  of the profile drag
- At  $Re \approx 30000$ , skin friction is insignificant.

## 6.6 Flow past a sphere

The flow over a sphere starts to separate from the surface at  $Re \approx 20$ . As  $Re$  increases, a pair of recirculating vortices becomes visible in the wake region. At flows up to  $Re \approx 200$  the flow is still steady but beyond this value oscillations start to appear and the separation point moves upstream of the top of the sphere, increasing the drag.



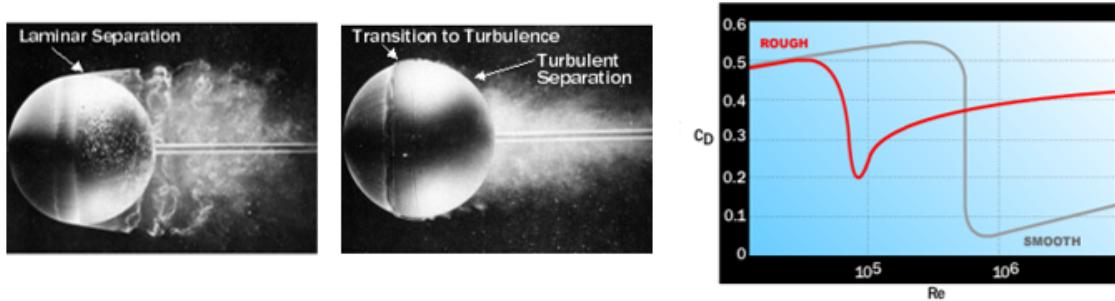
### 6.6.1 Drag coefficient



For a sphere,  $c_D$  is nearly constant for  $Re = 10^3 - 10^5$  but varies considerably as the boundary layer becomes turbulent between  $Re = 10^5 - 10^6$  and, in fact, this transition causes  $c_D$  to dramatically decrease. As  $Re \rightarrow 0$ ,  $c_D$  is given by Stokes's law ( $c_D = \frac{24}{Re}$ )

### 6.6.2 Drag reduction

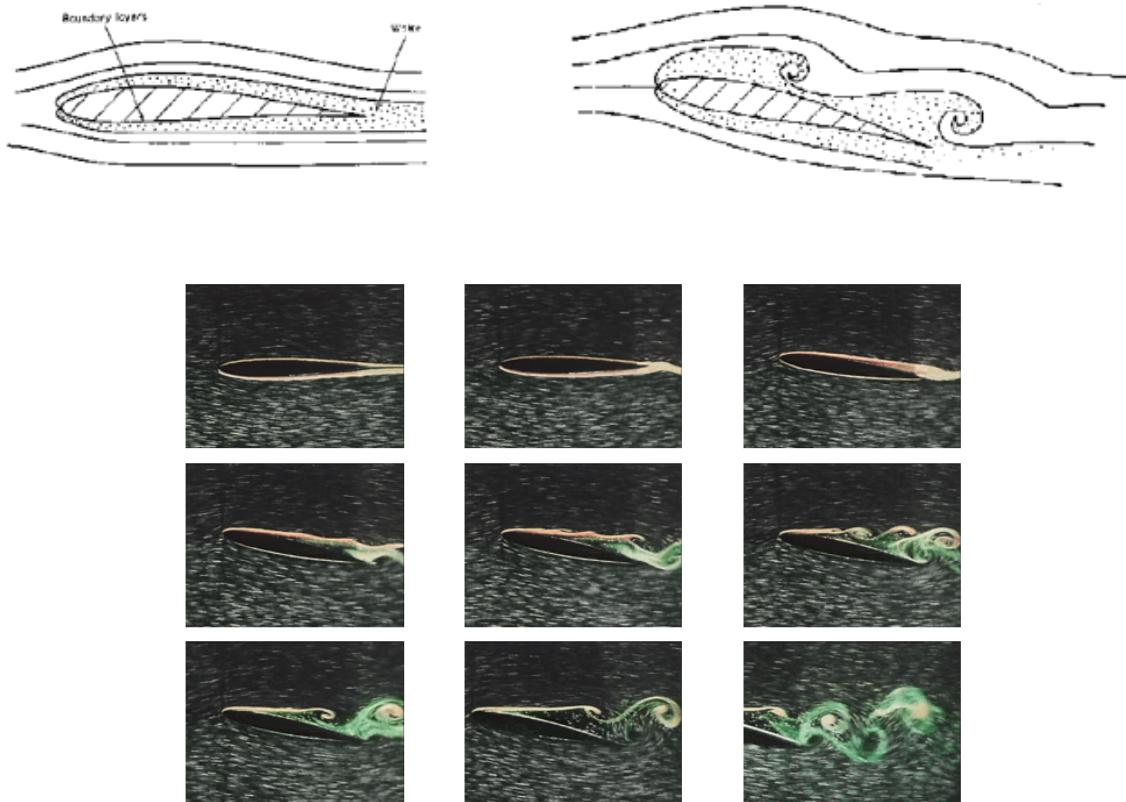
Deliberately tripping the boundary layer from laminar to turbulent (i.e. by a trip wire) results in an increase in skin drag (due to increased shear stresses), but a decrease in form drag (due to delayed separation point). The reduction in form drag more than compensates for the increased drag due to turbulent shearing, resulting in a greatly reduced overall drag coefficient  $c_D$ . Drag reduction can also be accomplished by roughening the surface and, thereby, lowering  $Re$  at which transition occurs. However, the drag coefficient for a roughened surface will typically rise faster in comparison to a smooth surface, once the transition point is passed.



## 6.7 Flow past an aerofoil - stalling

If the angle of attack becomes too great and boundary layer separation occurs on top of the aerofoil, the pressure pattern will change dramatically. In such a case, the lift component is insufficient to overcome the weight of one aircraft and disaster

is imminent. This phenomenon is known as **stalling**. When stalling occurs, all, or most, of the 'suction' pressure is lost, and the plane will suddenly drop from the sky. The solution to this is to put the plane into a dive to regain the boundary layer. A transverse lift force is then exerted on the wing which gives the pilot some control and allows the plane to be pulled out of the dive.



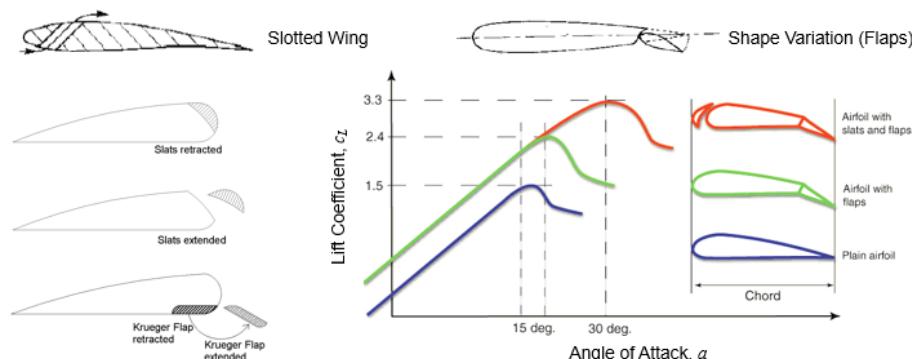
## 6.8 Boundary layer control

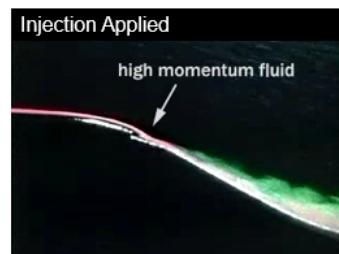
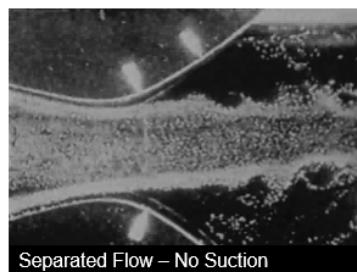
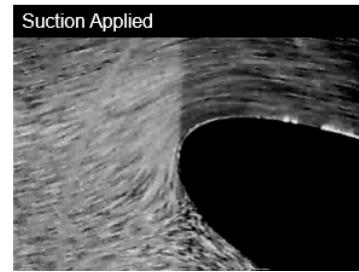
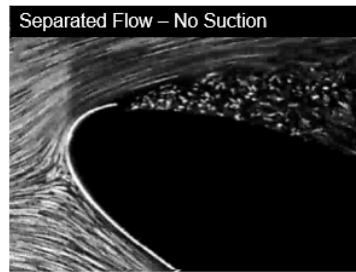
There are mechanisms for preventing the boundary layer from separating in the first place.

- Surface roughening
  - To initiate turbulent boundary layer, thus delay separation and achieve a narrower wake of higher pressure
- Shape variation
  - Putting a flap at the end of the wing and tilting it before separation occurs increases the velocity over the top of the wing, reduces the pressure and chance of separation
- Fluid injection
  - Accelerate boundary layer in direction of flow (but this will produce very turbulent flow and more skin friction)
- Fluid removal

- Suction through small holes or a porous surface
- Downstream of suction position, boundary layer is thinner and faster thus better able to withstand adverse pressure gradient. Suction delays laminar to turbulent transition and leads to less skin friction
- Slotted wing
  - Slower moving air on the upper surface can be accelerated by bringing air from the high-pressure area on the bottom of the wing through slots. Pressure decreases on the top and adverse pressure gradient reduces
- Engine arrangement
  - Engine intakes draw slow air from the boundary layer at the rear of the wing through small holes. This keeps the boundary layer close to the wing and greater pressure gradients can be maintained before separation

Control systems are often costly and difficult to install. Advantages for some conditions, hindrance of others (advantages maybe offset by the added drag). Planes take advantage of the cambered aerofoil's high lift characteristics by enabling shape variation of the basic wing. Leading edge 'Krüger' flaps or slats and trailing edge 'Fowler' flaps are extended from the wing and change the aerofoil's shape into a cambered form, generating greater lift during low-speed flight.





# Chapter 7

## Boundary Layer of a Flat Plate

01/12/2020

### 7.1 Boundary Layer Theory

x-direction:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad (7.1.1)$$

y-direction:

$$\frac{\partial p}{\partial y} = 0 \quad (7.1.2)$$

Pressure does no change along the thickness of the boundary layer. Therefore, I can estimate the pressure outside of boundary layer using Bernoulli (inviscid flow theory), and then apply that pressure on the body wall. By definition, outside of the boundary layer:

$$\frac{\partial}{\partial y} = 0 \quad \& \quad v = 0 \quad (7.1.3)$$

$$U(x) = \frac{\partial U(x)}{\partial x} = \frac{\partial^{\frac{1}{2}} U(x)^2}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (7.1.4)$$

$$p + \frac{1}{2} \rho U(x)^2 = \text{const} \quad (\text{from } \frac{\partial^{\frac{1}{2}} U(x)^2}{\partial x}) \quad (7.1.5)$$

### 7.2 Flat plate boundary layer

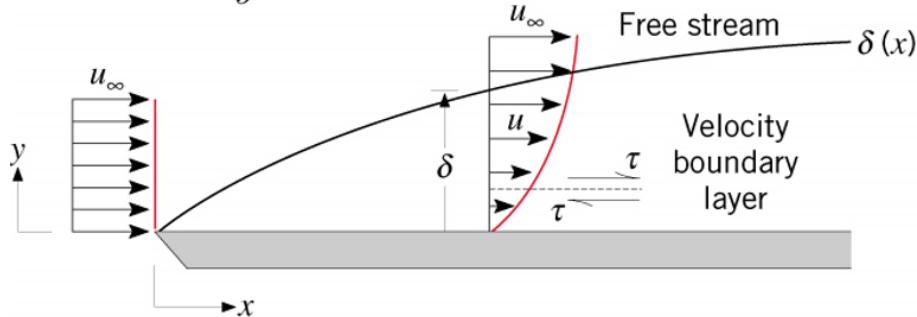
Inside the boundary layer:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad (7.2.1)$$

$$\frac{\partial p}{\partial y} = 0 \quad (7.2.2)$$

Pressure does not change along boundary layer thickness:

$$U_\infty = \frac{\partial U_\infty}{\partial x} = 0 = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (7.2.3)$$



### 7.2.1 Analysis of order of magnitude

$$v \ll u \quad (7.2.4)$$

$$\left| \frac{\partial}{\partial x} \right| \approx \frac{1}{L} \quad (7.2.5)$$

$$\left| \frac{\partial}{\partial y} \right| \approx \frac{1}{\delta} \quad (7.2.6)$$

Inside the boundary layer, viscous diffusion is large and of the same order of magnitude of inertial terms, therefore:

$$\frac{u \frac{\partial u}{\partial x}}{\nu \frac{\partial^2 u}{\partial y^2}} \approx 1 \begin{cases} u \approx U_\infty \\ y \approx \delta \\ x \approx L \end{cases} \rightarrow \frac{\frac{U_\infty^2}{L}}{\frac{\nu U_\infty}{\delta^2}} = \frac{\delta^2}{L^2} Re \approx 1 \quad (7.2.7)$$

## 7.3 Blasius solution

In 1908, Blasius found the analytical solution to the flat plate boundary layer problem. The velocity profiles are self-similar at different axial distances from the leading edge.

$$\eta = \frac{y}{\delta(x)} = \frac{y \sqrt{Re_x}}{x} = \frac{y}{x} \sqrt{\frac{U_\infty x}{\nu}} = y \sqrt{\frac{U_\infty}{x \nu}} \quad (7.3.1)$$

$$\frac{u}{U_\infty} = f'(\eta) \quad (7.3.2)$$

$$\frac{v}{U_\infty} = \frac{1}{2} \left( \frac{\nu}{x U_\infty} \right)^{0.5} (\eta f'(\eta) - f(\eta)) \quad (7.3.3)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \rightarrow \frac{1}{2} f(\eta) f''(\eta) + f'''(\eta) = 0 \quad (7.3.4)$$

Boundary conditions:

$$\begin{cases} u = v = 0 & @ y = 0 \\ u = U_\infty & @ y = \infty \end{cases} \rightarrow \begin{cases} f = f' = 0 & @ \eta = 0 \\ f' = 1 & @ \eta = \infty \end{cases} \quad (7.3.5)$$

$$\frac{\delta}{x} = \frac{A}{\sqrt{Re_x}} \rightarrow A = ? \quad (7.3.6)$$

Blasius solved this problem and derived the analytical solution, which is shown below:

$y[U/(\nu x)]^{1/2}$	$u/U$	$y[U/(\nu x)]^{1/2}$	$u/U$
0.0	0.0	2.8	0.81152
0.2	0.06641	3.0	0.84605
0.4	0.13277	3.2	0.87609
0.6	0.19894	3.4	0.90177
0.8	0.26471	3.6	0.92333
1.0	0.32979	3.8	0.94112
1.2	0.39378	4.0	0.95552
1.4	0.45627	4.2	0.96696
1.6	0.51676	4.4	0.97587
1.8	0.57477	4.6	0.98269
2.0	0.62977	4.8	0.98779
2.2	0.68132	5.0	0.99155
2.4	0.72899	$\infty$	1.00000
2.6	0.77246		

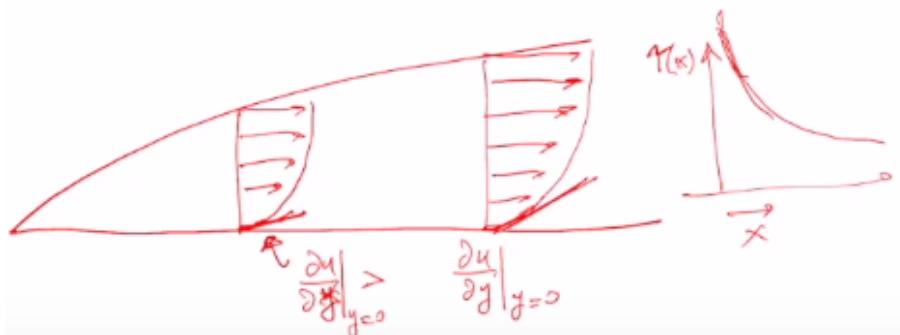
The boundary layer corresponds to when we have  $0.99U_\infty$ , we have  $A = 5$ , hence:

$$\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \quad (7.3.7)$$

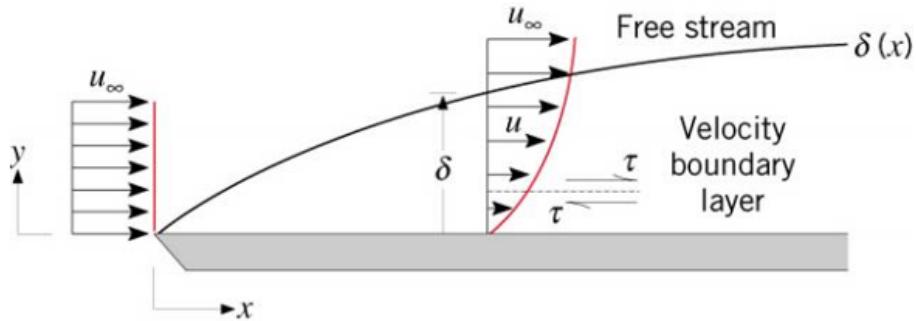
Let us look at the shear stress and the friction coefficient:

$$\tau(x) = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \rightarrow c_f = \frac{\tau(x)}{\frac{1}{2} \rho U_\infty^2} = \frac{0.664}{\sqrt{Re_x}} \quad (7.3.8)$$

How does the shear stress,  $\tau(x)$ , vary with distance from the leading edge? If we look at our  $Re_x$  term, we see that as  $x$  increases  $c_f$  decreases. Looking at the gradients of  $\frac{\partial u}{\partial y}$  along our boundary layer, we see that the gradient is steeper at  $y = 0$ . We can also plot  $\tau(x)$  against  $x$ .



Let us look at the friction drag on the flat plate:

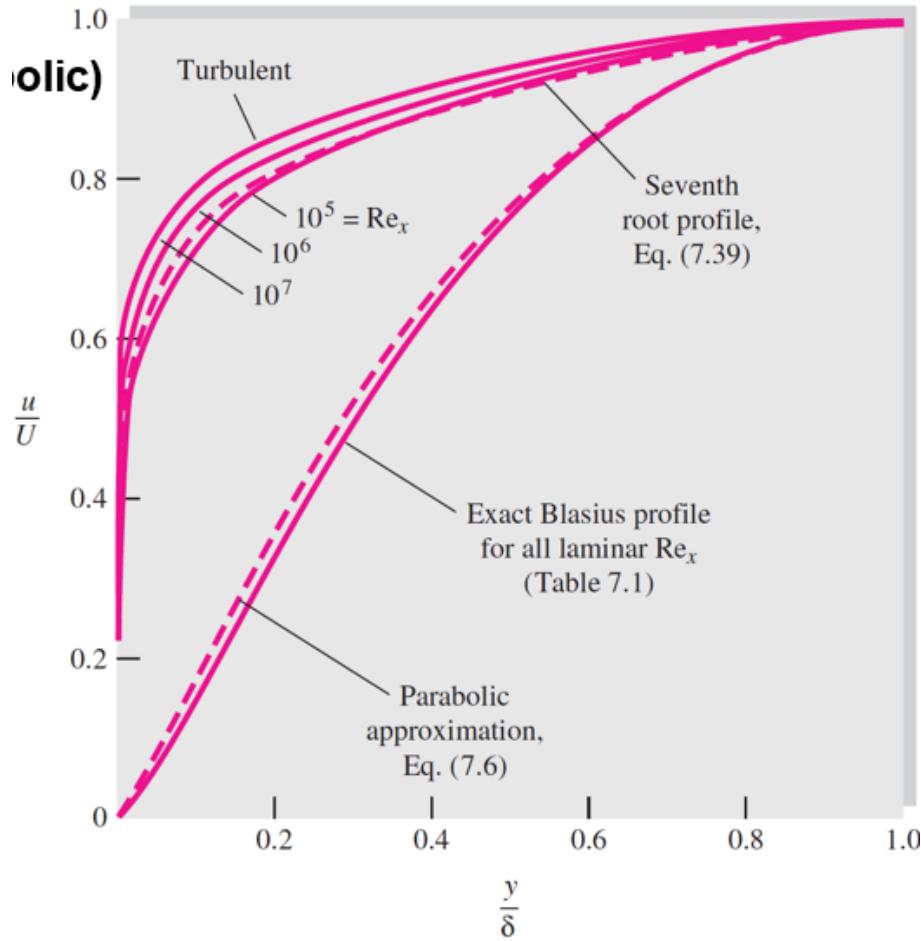


What is the friction drag on the flat plate per unit of span?

$$D = \int_0^L \tau(x) dx = 0.664 \rho U_\infty^2 \frac{L}{\sqrt{Re_L}} \quad (7.3.9)$$

$$c_D = \frac{D}{\frac{1}{2} \rho U_\infty^2 L} = \frac{1.328}{\sqrt{Re_L}} = 2 c_f(L) \quad (7.3.10)$$

## 7.4 Laminar and turbulent profiles



Our parabolic approximation is given by the Karman approximation:

$$\frac{u}{U_\infty} = \left( 2 \frac{y}{\delta} - \frac{y^2}{\delta^2} \right) \quad (7.4.1)$$

Turbulent flow can be described as:

$$\frac{u}{U_\infty} = \left( \frac{y}{\delta} \right)^{\frac{1}{5}} \quad (7.4.2)$$

The Prandtl approximation is given as:

$$\frac{u}{U_\infty} = \left( \frac{y}{\delta} \right)^{\frac{1}{7}} \quad (7.4.3)$$

In Figure (??), we see that with increasing  $Re$ , the bulge in our curve also increases. This also means that  $\frac{\partial u}{\partial y}$  is also increasing, meaning our shear stress also increases with  $Re$ .

### 7.4.1 Turbulent friction and drag coefficients

For  $n = \frac{1}{5}$  (fully turbulent flow):

$$\frac{\delta}{x} = \frac{0.385}{(Re_x)^{\frac{1}{5}}} \quad (7.4.4)$$

$$c_f = \frac{\tau(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{0.0594}{(Re_x)^{\frac{1}{5}}} \quad (7.4.5)$$

$$c_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 L} = \frac{0.074}{(Re_L)^{\frac{1}{5}}} = \frac{5}{4} c_f(L) \quad (7.4.6)$$

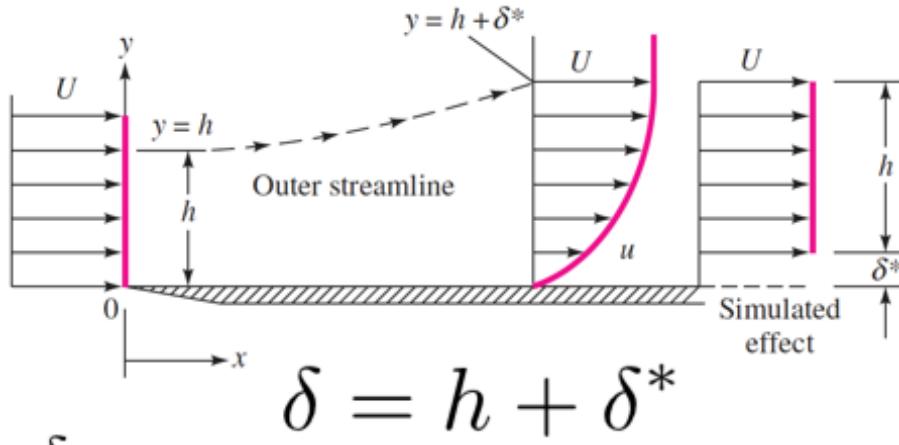
For  $n = \frac{1}{7}$  (fully turbulent flow):

$$\frac{\delta}{x} = \frac{0.16}{(Re_x)^{\frac{1}{7}}} \quad (7.4.7)$$

$$c_f = \frac{\tau(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{0.027}{(Re_x)^{\frac{1}{7}}} \quad (7.4.8)$$

$$c_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 L} = \frac{0.031}{(Re_L)^{\frac{1}{7}}} = \frac{7}{6} c_f(L) \quad (7.4.9)$$

## 7.5 Displacement thickness



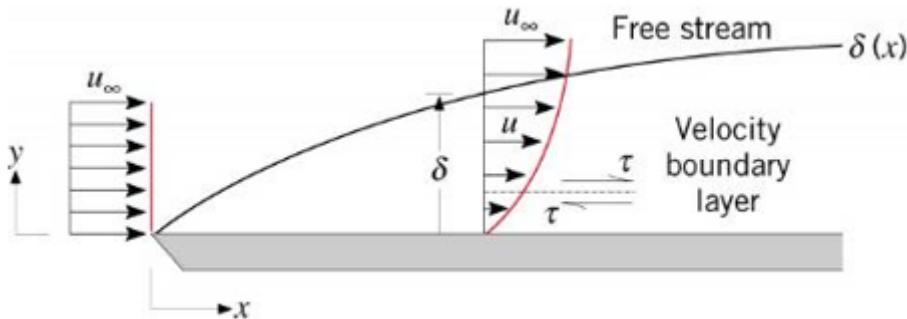
Based on the conservation of mass; the outer streamline must displace outwards to conserve the mass flow rate. The displacement thickness measure the streamline displacement for different  $x$ :

$$\rho U h = \int_0^\delta \rho u \, dy = \int_0^\delta \rho (u + U - U) \, dy \quad (7.5.1)$$

$$Uh = U(h + \delta^*) + \int_0^\delta (u - U) \, dy \quad (7.5.2)$$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) \, dy \quad (7.5.3)$$

## 7.6 Momentum thickness



Based on momentum loss; the total loss os momentum is equivalent to the loss of momentum occurring over a distance  $\theta$ , denoted as the "momentum thickness."

$$\rho U^2 \theta = \int_0^\delta \rho u (U - u) \, dy \quad (7.6.1)$$

$$\theta = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) \, dy \quad (7.6.2)$$

- The "momentum thickness" measures the deficit in the rate of transport momentum in the layer compared with the rate of transport of momentum in the absence of the layer.
- Alternatively it can be seen as the thickness of a completely stagnant layer of fluid giving the same deficit in momentum as the actual profile.

For a flat plate (zero pressure gradient), the momentum loss is also equal to the friction drag along the flat plate.

$$D = \int_0^\delta \rho u (U_\infty - u) dy \quad (7.6.3)$$

$$D = \int_0^x \tau_w dx \quad (7.6.4)$$

$$\tau_w = \frac{dD}{dx} = \rho U_\infty^2 \frac{d\theta}{dx} \quad (7.6.5)$$

$$\frac{\tau_w}{\rho U_\infty^2} = \frac{d\theta}{dx} \quad (7.6.6)$$

The following equations are valid:

$$c_f = \frac{\tau_w}{\frac{1}{2} \rho U_\infty^2} = 2 \frac{d\theta}{dx} \quad (7.6.7)$$

If we know the momentum thickness, I can find the shear stress/friction coefficient on the plate and vice versa.

## 7.7 General case with pressure gradient & shape factor

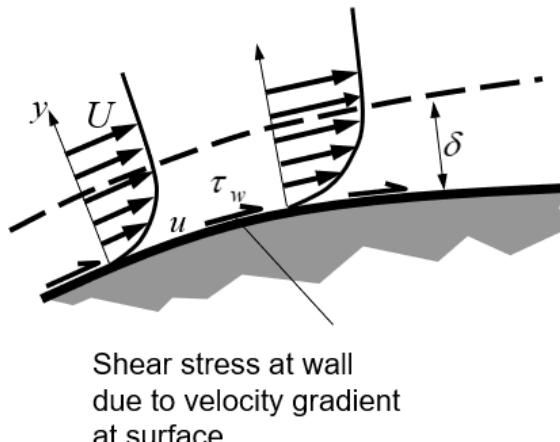
The shape factor is the ratio between the displacement thickness and the momentum thickness.

$$H = \frac{\delta^*}{\theta} = \frac{\int_0^\delta \left(1 - \frac{u}{U}\right) dy}{\int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy} > 1 \quad (7.7.1)$$

This leads to:

$$\frac{\tau_w}{\rho U_\infty^2} = \frac{d\theta}{dx} + U \frac{dU}{dx} (H+2)\theta \quad (7.7.2)$$

where  $U \frac{dU}{dx} (H+2)\theta$  is an extra term added due to the presence of a pressure gradient.



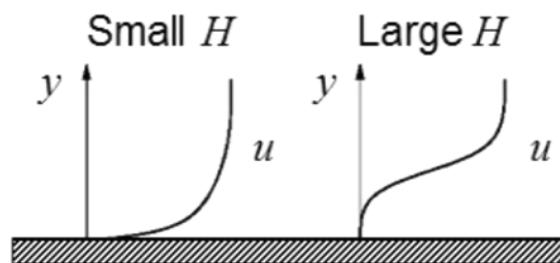
For laminar flow (flat plate)

$$H = 2.59 \quad (7.7.3)$$

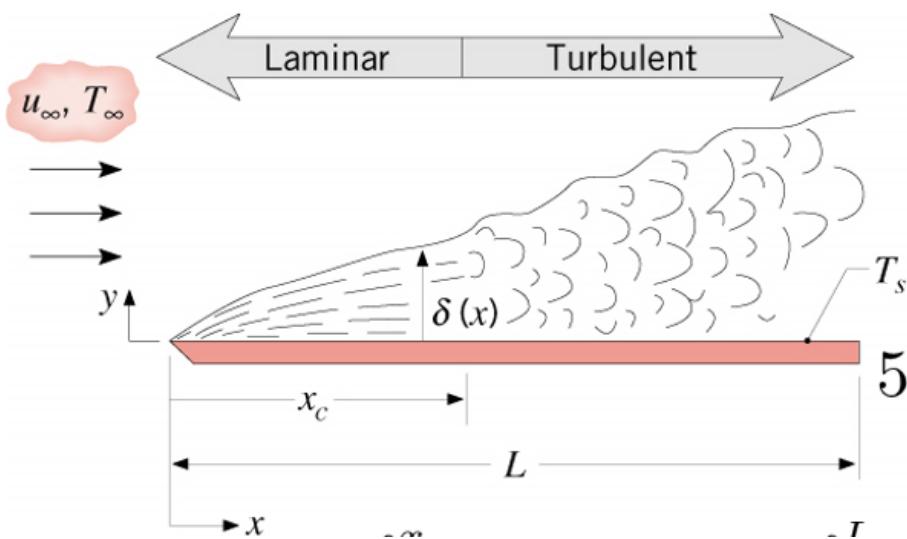
For turbulent flow (flat plate)

$$H = 1.29 \quad (7.7.4)$$

A large shape factor is indicative of boundary layer separation.



## 7.8 Laminar - turbulent flow transition



$x_c$  is defined as the point along our plate where our flow turns turbulent.

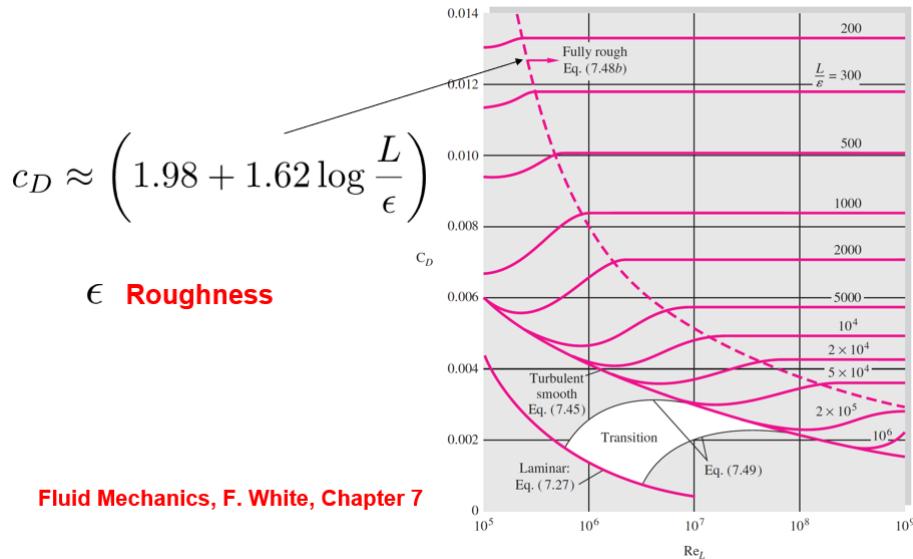
$$Re = \frac{U_\infty x_c}{\nu} \quad (7.8.1)$$

This is normally in the region where  $Re = 5 \times 10^5 - 8 \times 10^7$ .

$$D = \int_0^{x_c} \tau_{\text{lam}}(x) dx + \int_{x_c}^L \tau_{\text{turb}}(x) dx \quad (7.8.2)$$

$$c_D = \frac{1}{L} \left( \int_0^{x_c} c_{fl}(x) dx + \int_{x_c}^L c_{ft}(x) dx \right) \quad (7.8.3)$$

## 7.9 Flat plate drag coefficient



# Chapter 8

## Boundary Layer and Flow in Pipes

08/12/2020

### 8.1 Reynold's number

Reynolds conducted experiments, in which he measured pressure drop and critical velocity in a variety of pipe diameters and with different fluids, and verified the importance of the parameter  $\frac{\rho d \vec{U}}{\mu}$  later to be given his name. He found that, instead of a different critical velocity for each fluid and pipe, the onset of turbulence (i.e. transition) was determined by the achievement of the same critical value of Reynolds number, usually quoted as:

$$Re_{d,crit} = 2300 \quad (8.1.1)$$

We note the confirmation of the empirical results regarding critical velocity because if:

$$Re_{d,crit} = \frac{\rho d \vec{U}_{crit}}{\mu} = 2300 \quad (8.1.2)$$

then  $\vec{U} \propto \frac{1}{d}$  for given  $\rho$  and  $\mu$  and  $\vec{U}_{crit} \propto \mu$  for given  $d$  and  $\mu$ .

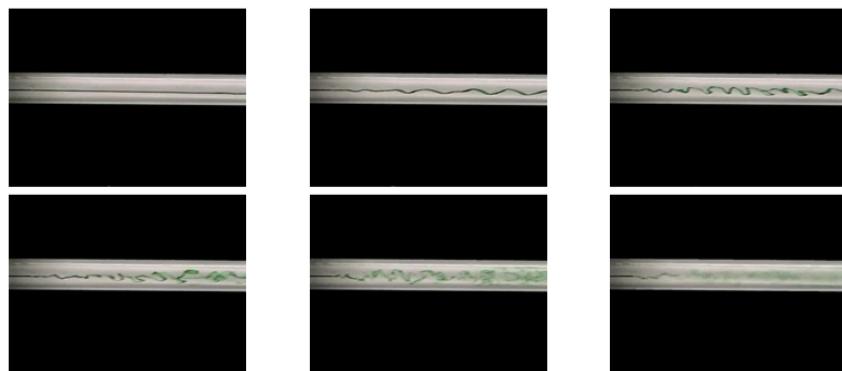


Figure 8.1:

- In this experiment water flows through a clear pipe with increasing speed
- Dye is injected through a small diameter tube at the left portion of the screen
- At low speed ( $Re < 2300$ ) the flow is laminar and the dye is a straight line
- As the speed increases, the dye stream becomes wavy (oscillatory laminar flow)

- At still higher speeds ( $Re > 4000$ ) the flow becomes turbulent and the dye stream is dispersed randomly throughout the flow

## 8.2 Flow in pipes

### 8.2.1 Entrance region

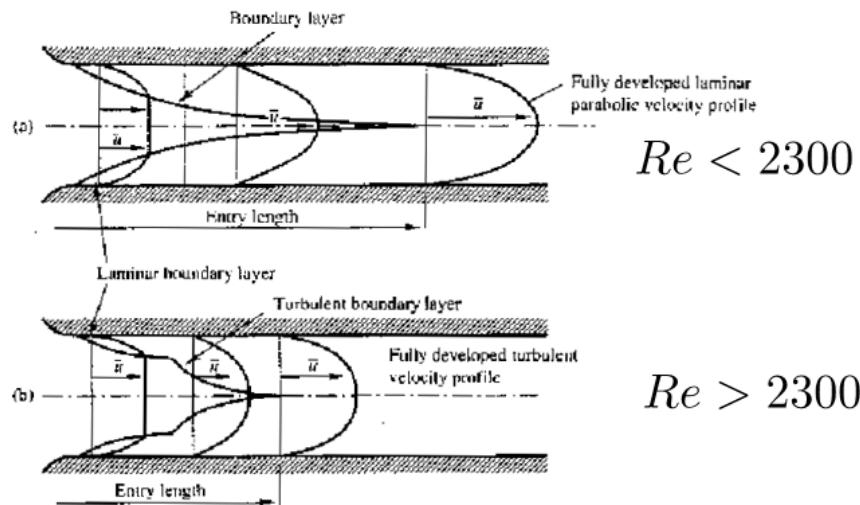


Figure 8.2:

Reynolds number:  $Re = \frac{\rho \vec{U} D}{\mu}$  where  $D$  is the diameter.

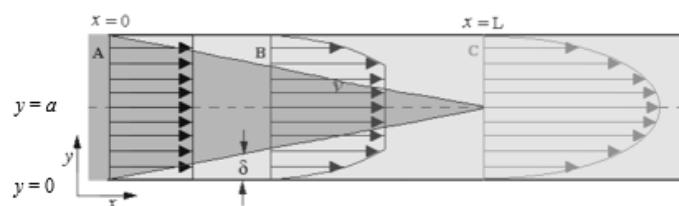


Figure 8.3:

- The nature of the flow in the entrance region and hence its length, depend on whether the fully-developed flow is laminar or turbulent
  - $Re_d < 2300$ : Fully-developed flow should be laminar
  - $Re_d > 2300$ : Fully-developed flow should be turbulent
  - $2000 < Re_d < 3000$ : Transitional flow...
- Figures given for laminar entry length variety
  - Laghaar quotes: Entry length =  $0.057dRe_d$
  - Rule of thumb: Entry length =  $100d$  minimum
- For turbulent flow the fully-developed state is achieved much sooner (entry length less dependent on  $Re_d$ )
  - Rule of thumb: Entry length =  $50d$  minimum

### 8.2.2 Shear stress

What is the shear stress distribution in fully-developed pipe flows? Consider an axial cylindrical element of length  $L$  within a pipe.

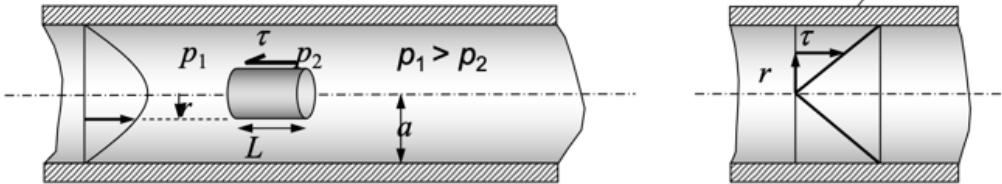


Figure 8.4:

The forces due to pressure and shear (from velocity gradient) must balance out:

$$p_1 \pi r^2 - p_2 \pi r^2 - \tau 2\pi r L = 0 \rightarrow \tau = \frac{p_1 - p_2}{L} \frac{r}{2} = \frac{\Delta p}{L} \frac{r}{2} \quad (8.2.1)$$

This relationship is valid for both laminar and turbulent motion and shows that shear stress must vary linearly with radius  $r$  in the pipe. The value of  $\tau$  at the wall ( $r = a$ ) is:  $\tau_w = \frac{\Delta p}{L} \frac{a}{2}$  so that:  $\frac{\tau}{\tau_w} = \frac{r}{a}$ . Wall shear stress can be measured by the pressure gradient in the pipe.

## 8.3 Laminar flow in pipes

### 8.3.1 Velocity profile

Laminar motion ( $Re < 2300$ ). What is the velocity profile? If the motion is laminar, the shear stress is easily related to the velocity gradient through the dynamic viscosity,  $\mu$ .

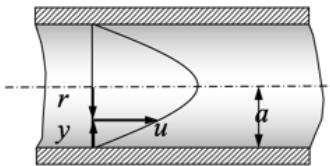


Figure 8.5:

$$\tau = \mu \frac{du}{dy} \text{ where } y = a - r \text{ such that: } dy = -dr \quad (8.3.1)$$

- From previous result:  $\tau = -\mu \frac{du}{dr} = \frac{\Delta p}{L} \frac{r}{2}$
- Thus:  $du = -\frac{\Delta p}{2L\mu} r dr$  which we can integrate to get:  $u = -\frac{\Delta p}{4L\mu} r^2 + C$
- No slip boundary condition says:  $u = 0$  at  $r = a \rightarrow C = \frac{\Delta p}{4L\mu} a^2$
- Thus, finally:  $u = \frac{1}{4\mu} \frac{\Delta p}{L} (a^2 - r^2)$  Parabolic velocity profile (*Hagen-Poiseuille Law*)

Segment V6.6 Laminar flow (Related to textbook section 6.9.3 - Steady, Laminar flow in circular tubes) The velocity distribution is parabolic for steady, laminar flow in circular tubes. A filament of dye is placed across a circular tube containing a very viscous liquid which is initially at rest. With the opening of a valve at the bottom of the tube the liquid starts to flow, and the parabolic velocity distribution is revealed. Although the flow is actually unsteady, it is quasi-steady since it is only slowly changing. Thus, at any instant in time the velocity distribution corresponds to the characteristic steady-flow parabolic distribution.

### 8.3.2 Flow rate

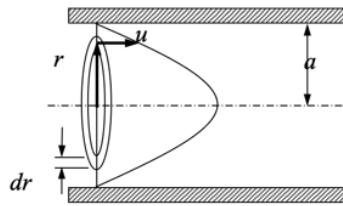


Figure 8.6:

Parabolic velocity profile:

$$u = \frac{1}{4\mu} \frac{\Delta p}{L} (a^2 - r^2) \quad (8.3.2)$$

The maximum velocity  $U_{max}$  occurs on the centre-line of the pipe (i.e. at  $r = 0$ ):

$$U_{max} = \frac{1}{4\mu} \frac{\Delta p}{L} a^2 \quad (8.3.3)$$

$$\frac{u}{U_{max}} = \frac{a^2 - r^2}{a^2} \rightarrow \frac{u}{U_{max}} = 1 - \frac{r^2}{a^2} \quad (8.3.4)$$

The mean velocity/flow rate is given by:

$$\vec{U} = \frac{Q}{A} = \frac{\int_0^a (2\pi r u) dr}{\pi a^2} \quad (8.3.5)$$

$$\vec{U} = \frac{2\pi}{\pi a^2} \int_0^a (ru) dr = \frac{2}{a^2} \int_0^a \left( U_{max} \left( 1 - \frac{r^2}{a^2} \right) r \right) dr \quad (8.3.6)$$

$$\vec{U} = \frac{2U_{max}}{a^2} \int_0^a \left( r - \frac{r^3}{a^2} \right) dr \quad (8.3.7)$$

$$\vec{U} = \frac{2U_{max}}{a^2} \left[ \frac{r^2}{2} - \frac{r^4}{4a^2} \right]_0^a = \frac{2U_{max}}{a^2} \left[ \frac{a^2}{2} - \frac{a^4}{4a^2} \right] = \frac{U_{max}}{2} \quad (8.3.8)$$

For a parabolic profile the mean velocity is half the maximum velocity:

$$u = \frac{1}{4\mu} \frac{\Delta p}{L} (a^2 - r^2) \quad U_{max} = \frac{1}{4\mu} \frac{\Delta p}{L} a^2 \quad \vec{U} = \frac{1}{8\mu} \frac{\Delta p}{L} a^2 \quad (8.3.9)$$

The volumetric flow rate is:

$$Q = \pi a^2 \vec{U} = \frac{\pi}{8\mu} \frac{\Delta p}{L} a^4 \quad (8.3.10)$$

This can be used as a basis for measuring  $\mu$  if all the other parameters can be measured sufficiently accurately.

## 8.4 Friction factor (flow in pipes)

We have seen that the pressure loss in a pipe (whether laminar or turbulent) is related to the wall shear stress by:

$$\tau_w = \frac{\Delta p}{L} \frac{a}{2} \quad (8.4.1)$$

A dimensionless representation of the wall shear stress (and therefore the pressure gradient) is given by the defining a friction factor as the ratio of wall shear stress to dynamic pressure in the flow (based on mean velocity).

$$f = \frac{\tau_w}{\frac{1}{2} \rho \vec{U}^2} \quad (8.4.2)$$

### 8.4.1 Pressure drop due to friction

Generalised expression for pressure drop due to friction. We have seen from equilibrium considerations (for laminar or turbulent flow) that:

$$\tau_w = \frac{\Delta p}{L} \frac{a}{2} = \frac{\delta p}{L} \frac{d}{4} \quad (8.4.3)$$

Thus,

$$\Delta p = \frac{4L\tau_w}{d} \rightarrow \Delta p = \frac{4L}{d} f \frac{1}{2} \rho \vec{U}^2 \quad (8.4.4)$$

Hence:

$$h_f = \frac{\Delta p}{\rho g} \rightarrow h_f = \frac{4L}{d} f \frac{\vec{U}^2}{2g} \quad (8.4.5)$$

Eq.?? is known as *Darcy's* Formula. To work out the pressure drop due to friction in pipe flows, we need to know accurately the friction factor  $f$ .

## 8.5 Friction factor (laminar flow in pipes)

Friction factor as a function of Reynolds number. Definition of friction factor (laminar or turbulent flow):

$$f = \frac{\tau_w}{\frac{1}{2} \rho \vec{U}^2} \quad (8.5.1)$$

For laminar flow we have:

$$u = U_{max} \left( 1 - \frac{r^2}{a^2} \right) \quad \vec{U} = \frac{U_{max}}{2} \quad (8.5.2)$$

Also:

$$\tau_w = -\mu \left[ \frac{du}{dr} \right]_{r=a} = -\mu \left[ \frac{-2\mu U_{max} a}{a^2} \right]_{r=a} = \frac{2\mu U_{max} a}{a^2} = \frac{2\mu U_{max}}{a} = \frac{4\mu U_{max}}{d} \quad (8.5.3)$$

$$\tau_w = \frac{8\mu \vec{U}}{d} \quad (8.5.4)$$

Thus, finally:

$$f = \frac{\frac{8\mu \vec{U}}{d}}{\frac{1}{2}\rho \vec{U}^2} \rightarrow f = \frac{16\mu}{\rho d \vec{U}} \rightarrow f = \frac{16}{Re_d} \quad (8.5.5)$$

## 8.6 Laminar vs Turbulent profiles

Segment V8.3 Laminar/Turbulent Velocity profiles (Related to textbook section 8.3.3 - Turbulent velocity profiles) The velocity profile for laminar flow in a pipe is quite different than that for turbulent flow. An approximation to the velocity profile in a pipe is obtained by observing the motion of a dye streak placed across the pipe. With a viscous oil at Reynolds number of about 1, viscous effects dominate and it is easy to inject a relatively straight dye streak. The resulting laminar flow profile is parabolic. With water at Reynolds number of about 10000, inertial effects dominate and it is difficult to inject a straight dye streak. It is clear, however, that the turbulent velocity profile is not parabolic, but is more nearly uniform than for laminar flow.

## 8.7 Turbulent flow in pipes

### 8.7.1 Velocity profile

Due to chaotic nature of turbulence, turbulent flows are difficult to analyse and empirical relations mostly necessary. The time average velocity profile is much flatter than a laminar paraboloid:

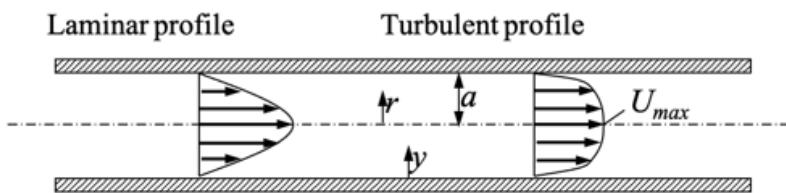


Figure 8.7:

Approximate power laws used, depending on  $Re_d$ :

$$\frac{u}{U_{max}} = \left(\frac{y}{a}\right)^{\frac{1}{n}} \text{ or } \frac{u}{U_{max}} = \left(\frac{a-r}{a}\right)^{\frac{1}{n}} \rightarrow \frac{u}{U_{max}} = \left(1 - \frac{r}{a}\right)^{\frac{1}{n}} \quad (8.7.1)$$

$n = 6 - 10$  but often taken as  $n = 7$ , known as the "One-Seventh Power Law". It can be shown that the ratio of the mean velocity to the maximum velocity is:

$$\frac{\vec{U}}{U_{max}} = \frac{2n^2}{(n+1)(2n+1)} \approx 0.817 \text{ for } n = 7 \quad (8.7.2)$$

Also note that:

$$\frac{du}{dy} = \frac{d}{dy} \left( U_{max} \frac{y^{\frac{1}{n}}}{a^{\frac{1}{n}}} \right) = \frac{U_{max}}{a^{\frac{1}{n}}} \frac{d(y^{\frac{1}{n}})}{dy} = \frac{U_{max}}{a^{\frac{1}{n}}} \frac{1}{n} y^{\frac{1}{n}-1} = \frac{U_{max}}{na^{\frac{1}{n}}} y^{\frac{1}{n}-1} \quad (8.7.3)$$

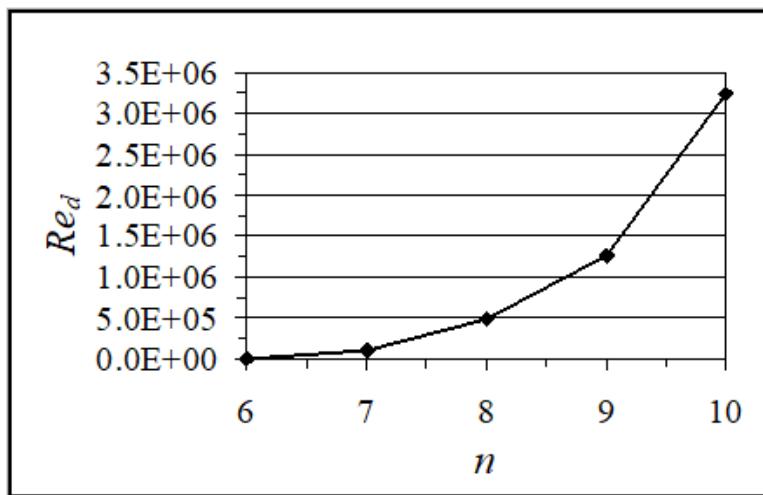
Is the power law valid everywhere?

- The power law is not valid at the wall
  - $y = 0 \rightarrow \frac{du}{dy} = \infty \rightarrow \tau_w = \infty$ , whereas in practice  $\frac{du}{dy}$  = finite at the wall
- The power law is not valid on the centre-line
  - $y = a \rightarrow \frac{du}{dy} = \text{finite} \rightarrow \tau \neq 0$ , whereas in practice  $\frac{du}{dy} = 0$  on the centre-line

The power law was derived empirically by Nikuradse for  $4 \times 10^3 < Re_d < 3.24 \times 10^6$ .

<b>n</b>	$Re_d$
6	$4.00 \times 10^3$
7	$1.10 \times 10^5$
8	$4.80 \times 10^5$
9	$1.26 \times 10^6$
10	$3.24 \times 10^6$

Table 8.1: Dependence of  $n$  on Reynolds Number  $Re_d$

Figure 8.8: Dependence of  $n$  on Reynolds Number  $Re_d$ 

### 8.7.2 Friction factor

The friction factor can be found empirically

- Very approximately:  $2\sqrt{f} \approx \frac{1}{n}$
- *Blasius* ( $Re_d < 8 \times 10^4$ ):  $f = 0.0791 Re_d^{-0.25}$
- *Lees* ( $Re_d < 4 \times 10^5$ ):  $f = 0.0019 + 0.153 Re_d^{-0.35}$

Shear stress at wall:

$$\tau_w = f \frac{1}{2} \rho \vec{U}^2 \quad (8.7.4)$$

Using *Blasius's* friction factor and  $n = 7$  (i.e.  $\vec{U} \approx 0.817 U_{max}$ )

$$\tau_w = \left[ 0.0791 \left( \frac{\rho \vec{U} d}{\mu} \right)^{-0.25} \right] \frac{1}{2} \rho \vec{U}^2 \rightarrow \tau_w = 0.0225 \rho U_{max}^2 \left( \frac{\rho U_{max} d}{\mu} \right)^{-0.25} \quad (8.7.5)$$

## 8.8 Laminar flow in rough pipes

To be truly smooth, the perturbations on the pipe wall surface would have to be small compared to the fluid molecules, i.e. no pipe is truly smooth in practice! *Nikuradse* conducted experiments using pipes artificially roughened with sand grains. Roughness =  $k$ , Relative roughness =  $\frac{k}{d}$ . Alternative notation:  $k_s$  = uniform bumps (sand grains),  $k$  = more random (typical) distribution of size. Pipes of all  $\frac{k}{d}$  behave in same manner for laminar flow,  $f = f(Re_d)$ :

$$f = \frac{16}{Re_d} \quad (8.8.1)$$

The critical  $Re_d$  number for laminar to turbulent transition is independent of  $\frac{k}{d}$ . Once the flow is turbulent, the factor that determines flow behaviour is that sand grain height relative to the viscous sublayer thickness  $\delta_L$ .

## 8.9 Turbulent flow

### 8.9.1 Universal laws

Improved profiles by subdivision of flow: *Universal Laws*. The universal laws are semi-empirical: The expressions have been derived theoretically but the constants have been found experimentally. Three layer concept:

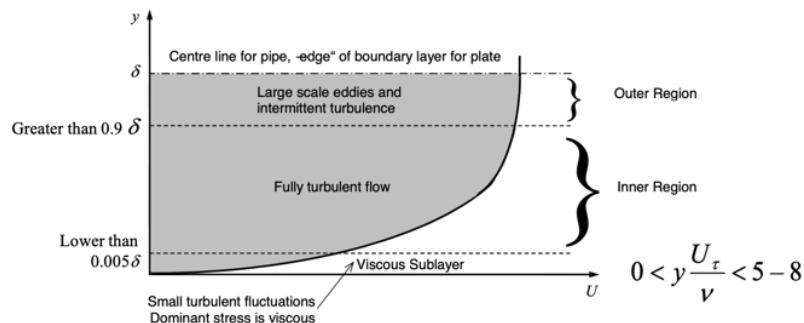


Figure 8.9:

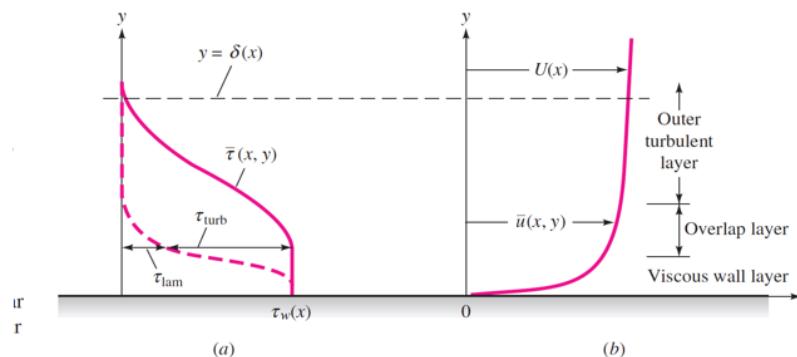


Figure 8.10:

Friction velocity:

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}} \quad (8.9.1)$$

Sublayer thickness:

$$\delta_L = \frac{5v}{U_\tau} \quad (8.9.2)$$

## 8.10 Turbulent flow in rough pipes

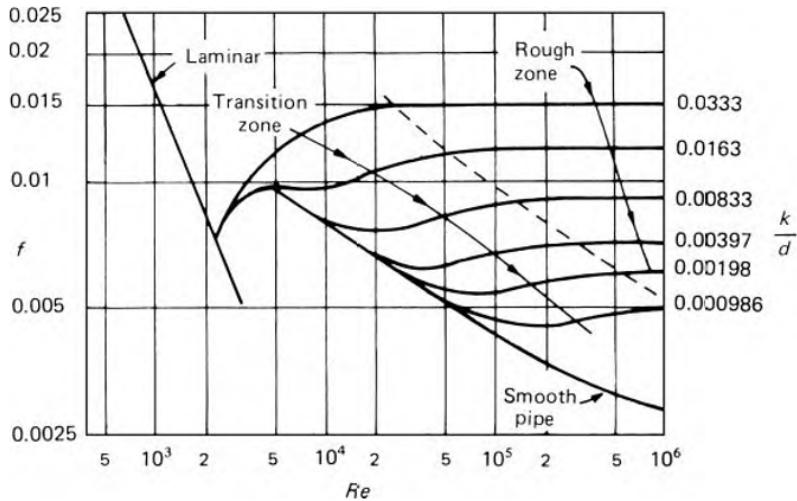


Figure 8.11:

In the turbulent region there is a range of  $Re_d$  over which a pipe of a given  $\frac{k}{d}$  behaves as if it were a smooth pipe: the *Hydraulically Smooth* pipe,  $f = f(Re_d)$  only. The hydraulically smooth regime exists when  $k < \delta_L$ :

$$k < \delta_L \rightarrow \frac{kU_\tau}{v} < 5 \text{ or } k^+ < 5 : \frac{1}{\sqrt{5}} = 4.0 \log \left( Re_d \sqrt{f} \right) - 0.4 \quad (8.10.1)$$

Roughness elements immersed in viscous sublayer (no effect of roughness). At a particular  $Re_d$  (which decreases as  $\frac{k}{d}$  increases) the resistance law deviates from the smooth pipe law: the *Smooth-Rough* transition  $f = f(Re_d, \frac{k}{d})$

$$5 < \frac{kU_\tau}{v} < 70 \text{ or } 5 < k^+ < 70 \quad (8.10.2)$$

Roughness elements partly outside sublayer and provide additional resistance. Ultimately a region is reached where the friction factor is a function of relative roughness only: the *Fully Rough* flow.  $f = f(\frac{k}{d})$  only.

$$\frac{kU_\tau}{v} > 70 \text{ or } k^+ > 70 : \frac{u}{U_\tau} = 5.75 \log \left( \frac{y}{k} \right) + 8.48 \frac{1}{\sqrt{f}} = 4/0 \log \left( \frac{d}{2k} \right) + 3.48 \quad (8.10.3)$$

All roughness elements extend well beyond sublayer (no effect of  $Re_d$ )

## 8.11 Flow in rough pipes

The friction factor for rough-pipe flows is typically given by charts as a function of the Reynolds number and pipe-wall roughness. These charts are based on experimental data and empirical correlations. Same "smooth pipe" line for all fluids, all

pipe diameters and all flow velocities. Different behaviour for laminar and turbulent flows as expected, with ill-definition in transition region ( $Re_{d,crit} = 2300$ ).

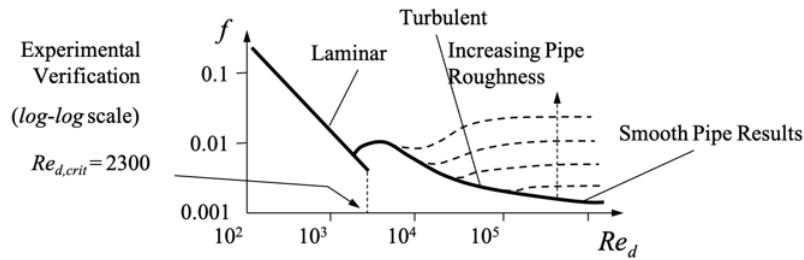


Figure 8.12:

## 8.12 the Moody chart for pipe flows

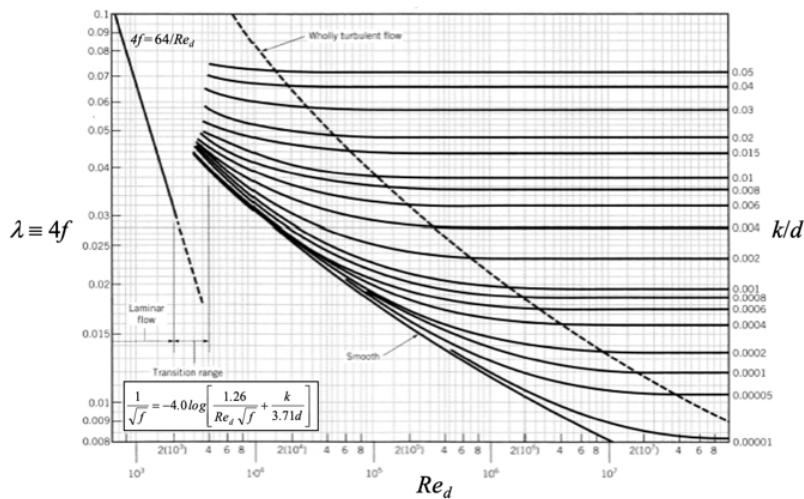


Figure 8.13: Composite log-Law for Smooth and Rough pipes: Moody Chart ( $\lambda = 4f$  on vertical axis)

## 8.13 The Haaland approximation for pipe flows

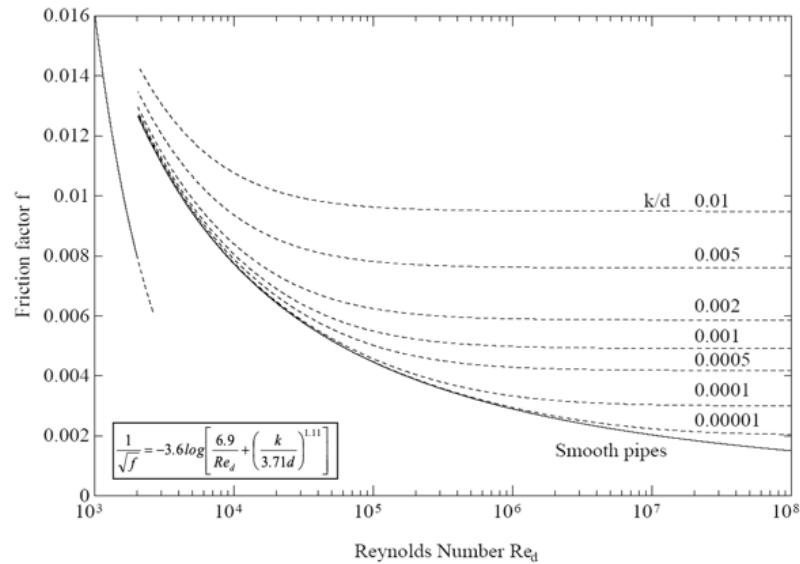


Figure 8.14: Composite log-law for smooth and rough pipes: Haaland Formula chart ( $f$  on vertical axis)

# **Part II**

# **Thermodynamics**

# Chapter 9

## Ideal Gas Mixtures

19/01/2021

### 9.1 Review on Thermodynamics

#### 9.1.1 System, Surroundings and Boundary

**Thermodynamic system** - An amount of matter “or” a region of space chosen for study

**Surroundings** - The region of space outside the arbitrary System boundary

**Boundary** - surface that separates the system from surrounding:

- Real or imaginary
- Massless
- Fixed (rigid tank) or movable (piston), flexible (balloon)

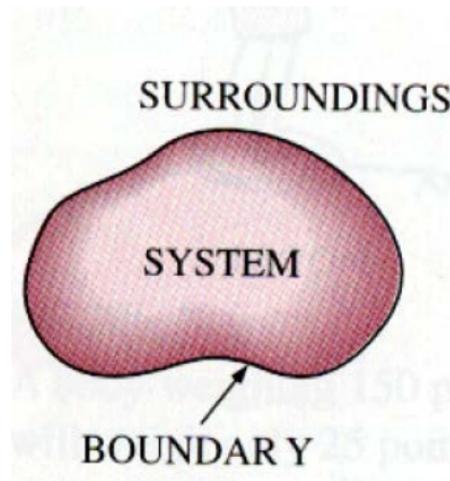


Figure 9.1:

Identification of system and its surroundings is the most critical first step to begin thermodynamic analysis.

### 9.1.2 System Classifications

Different types of systems can be found, depending on whether work, heat or mass cross the system boundary.

Definition	Work	Heat	Mass
Isolated System	No	No	No
Closed System (Control Mass)	Yes	Yes	No
Open System (Control Volume)	Yes	Yes	Yes

### 9.1.3 Thermodynamic Properties - Extensive vs. Intensive

Extensive properties are additive (i.e. depending on size/mass of system)

- U: internal energy [kJ]
- H: enthalpy [kJ]
- S: entropy [kJ/K]
- V: volume [ $\text{m}^3$ ]

Intensive properties are non-additive (i.e. independent of size/mass of system)

- T: temperature [K]
- h: specific enthalpy [kJ/kg]
- P: pressure [Pa]
- s: specific entropy [kJ/(kg K)]
- x: quality [-]
- v: specific volume [ $\text{m}^3/\text{kg}$ ]
- u: specific internal energy [kJ/kg]
- $\rho$ : density [ $\text{kg}/\text{m}^3$ ]

Extensive properties per unit mass or mole are called specific properties, which become intensive properties.

### 9.1.4 State, Equilibrium State, Steady State

**State** of a system is the condition characterized by a given set of properties, as well as their distributions in space. For a simple system of pure homogeneous substance, the state is uniquely specified by just two properties such as T and P or V and S.

**Thermodynamic equilibrium (state)** implies that a system state cannot change in the absence of unbalanced driving **forces** between the system and surroundings. It requires the following equilibrium within the system:

- Thermal equilibrium (constant temperature)
- Mechanical equilibrium (no pressure difference without gravity)
- Chemical equilibrium (no chemical reactions)
- Phase equilibrium (no phase change)

**Steady state** requires none of its properties changes with time, but within the system, the properties at different locations can be different.

## 9.2 Ideal Gas

Ideal gas is an imaginary substance whose properties satisfy the following state equation:

$$PV = nRT \quad (9.2.1)$$

Some Assumptions:

- No molecular volume: molecules do not occupy space
- No intermolecular forces: intermolecular force can be neglected

These assumptions are not always valid. Substances may be treated as ideal gases for engineering applications at sufficiently low pressure and high temperatures (i.e. low density)

### 9.2.1 Ideal Gas Equation of State

On a molar basis:

$$PV = NR_uT \quad (9.2.2)$$

$$P\bar{v} = R_uT \quad (9.2.3)$$

On a mass basis:

$$PV = mRT \quad (9.2.4)$$

$$Pv = RT \quad (9.2.5)$$

Where:

- $N$  = number of moles of gas [kmol]
- $T$  = absolute temperature [K]
- $R_u$  = universal gas constant = 8.314 [kJ kmol<sup>-1</sup> K<sup>-1</sup>]
- $V$  = volume [m<sup>3</sup>]
- $v$  = specific volume [m<sup>3</sup> kg<sup>-1</sup>];  $\bar{v}$  [m<sup>3</sup> kmol<sup>-1</sup>]
- $R = R_u/M$  = specific gas constant [kJ/kgK]
- $M$  = molecular weight of gas [kg/kmol]

### 9.2.2 Limitations on Ideal Gas EoS

- Usually good for air, N<sub>2</sub>, O<sub>2</sub>, H<sub>2</sub>, He, Ar, Ne at low pressures and high temperatures (less than 1% error)
- Works well for water vapor below 10kPa irrespective of temperature (less than 0.1% error)
- Fails close to critical point and saturated vapor line
- Definitely not applicable for water vapor in steam power plants owing to very high pressures

## 9.3 Ideal Gas Mixtures

Aim: To develop rules for evaluating the properties of gas mixtures using the properties of individual gases (called components or constituents).

Assumptions:

- Each gas in the mixture behaves like an ideal gas.
- The overall mixture behaves like an ideal gas.

Examples:

- Air
- Syngas
- Exhaust gases of a combustion process

## 9.4 Composition of Mixtures

State of a mixture is fully determined by:

- Composition: mass  $m_i$  or number of moles  $n_i$  of each component;
- Two independent intensive properties (such as  $p$  and  $T$  or  $v$  and  $T$ ).

Mass  $m_i$ , number of moles  $n_i$  and molecular weight  $M_i$  of each component  $i$  are related by:

$$n_i = \frac{m_i}{M_i} \quad (9.4.1)$$

Mass fraction,  $x_i$ :

$$x_i = \frac{m_i}{m} \longrightarrow \sum_{i=1}^k x_i = 1 \quad (9.4.2)$$

Mass fraction based analysis → gravimetric analysis

Mole fraction,  $y_i$ :

$$y_i = \frac{n_i}{n} \longrightarrow \sum_{i=1}^k y_i = 1 \quad (9.4.3)$$

Molar analysis → volumetric analysis

Apparent or Average molecular weight,  $M$ :

$$M = \frac{m}{n} = \frac{\sum_{i=1}^k m_i}{n} = \frac{\sum_{i=1}^k n_i M_i}{n} = \sum_{i=1}^k y_i M_i \quad (9.4.4)$$

Total mass of the mixture:

$$m = m_1 + m_2 + \cdots + m_k = \sum_{i=1}^k m_i \quad (9.4.5)$$

Mass fraction of  $i^{\text{th}}$  component:

$$x_i = \frac{m_i}{m} \longrightarrow \sum_{i=1}^k x_i = 1 \quad (9.4.6)$$

Total number of moles:

$$n = n_1 + n_2 + \cdots + n_k = \sum_{i=1}^k n_i \quad (9.4.7)$$

Mole fraction of  $i^{\text{th}}$  component:

$$y_i = \frac{n_i}{n} \longrightarrow \sum_{i=1}^k y_i = 1 \quad (9.4.8)$$

Molar mass:

$$M_i = \frac{m_i}{n_i} \text{ (single component)} \quad M = \frac{m}{n} \text{ (mixture)} \quad (9.4.9)$$

$$m = n_1 M_1 + n_2 M_2 + \cdots + n_k M_k = nM \quad (9.4.10)$$

### 9.4.1 Composition of Air

In practical calculations, air can be assumed to consist of 21% oxygen and 79% nitrogen by volume.

Constituent	Mass Fraction $x_i$	Molar Mass $M_i$	Moles $n_i$ per kg of mixture	Mole Fraction $y_i$
O <sub>2</sub>	0.232	32	0.00725	0.210
N <sub>2</sub>	0.756	28.01	0.02699	0.781
Ar	0.012	39.94	0.00300	0.009
Air	1	28.96 kg kmol <sup>-1</sup>	0.03454 kmol/kg <sub>air</sub>	1.0

$$M_{\text{air}} = \frac{m_{\text{air}}}{n_{\text{air}}} = \frac{1}{0.03454} = 28.96 \text{ kg kmol}^{-1} \quad (9.4.11)$$

## 9.5 P-v-T Relationships of Ideal Gas Mixtures

### 9.5.1 Dalton's Law

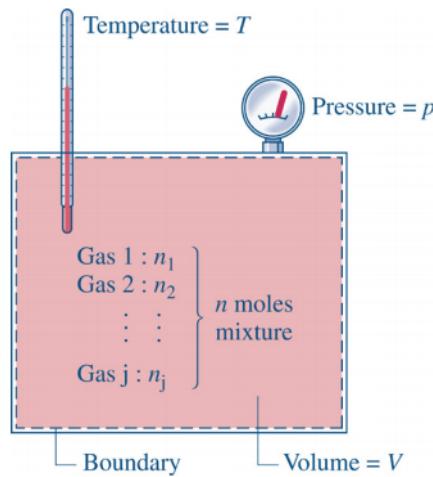


Figure 9.2:

The pressure of a gas mixture is equal to the sum of the pressures each gas would exert if it existed alone at the mixture temperature and volume.

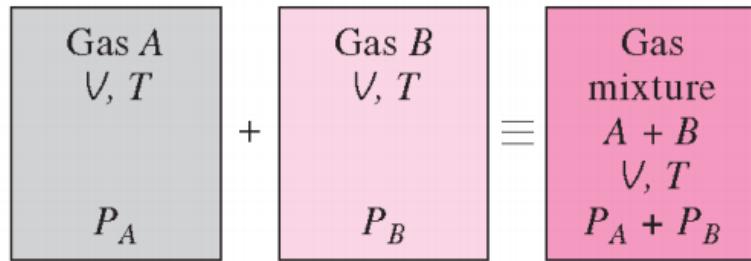


Figure 9.3:

Ideal gas law and the universal gas constant are, respectively:

$$p_i V = n_i \bar{R} T \quad (9.5.1)$$

$$\bar{R} = R_u = R_0 = 8.314 \text{ kJ kmol}^{-1} \text{ K}^{-1} \quad (9.5.2)$$

The pressure of each component occupying the whole volume of the mixture at the same mixture temperature is a **partial pressure**:

$$p_i = \frac{n_i \bar{R} T}{V} \quad (9.5.3)$$

$$p = \frac{n \bar{R} T}{V} \quad (9.5.4)$$

$$\frac{p_i}{p} = \frac{n_i}{n} = y_i \quad (9.5.5)$$

$$p_i = y_i p \quad (9.5.6)$$

$$\sum_{i=1}^j p_i = \sum_{i=1}^j y_i p = p \sum_{i=1}^j y_i = p \quad (9.5.7)$$

### 9.5.2 Amagat's Law

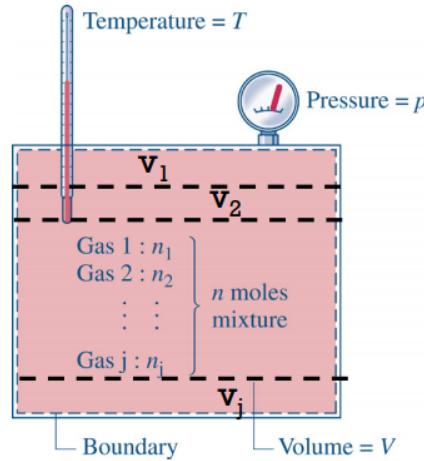


Figure 9.4:

The volume of a gas mixture is equal to the sum of the volumes each gas would occupy if it existed alone at the mixture temperature and pressure.

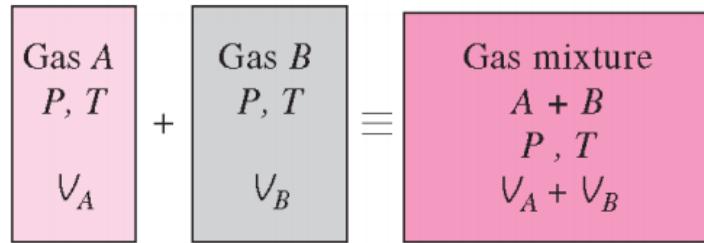


Figure 9.5:

Ideal gas law and the universal gas constant are, respectively:

$$pV_i = n_i \bar{R}T \quad (9.5.8)$$

$$\bar{R} = R_u = R_0 = 8.314 \text{ kJ kmol}^{-1} \text{ K}^{-1} \quad (9.5.9)$$

The volume occupied by each component at the same pressure  $p$  and temperature  $T$  of the mixture is a **partial volume**:

$$V_i = \frac{n_i \bar{R}T}{p} \quad (9.5.10)$$

$$V = \frac{n \bar{R}T}{p} \quad (9.5.11)$$

$$\frac{V_i}{V} = \frac{n_i}{n} = y_i \quad (9.5.12)$$

$$V_i = y_i V \quad (9.5.13)$$

$$\sum_{i=1}^j V_i = \sum_{i=1}^j y_i V = V \sum_{i=1}^j y_i = V \quad (9.5.14)$$

## 9.6 Properties of Ideal Gas Mixtures

Extensive properties  $U, H, S$  and intensive properties  $u, h, s$ :

### Internal Energy

$$U = U_1 + U_2 + \cdots + U_k = \sum_{i=1}^k U_i \quad (9.6.1)$$

$$U = m_1 u_1 + m_2 u_2 + \cdots + m_k u_k = \sum_{i=1}^k m_i u_i = m u \quad (9.6.2)$$

$$U = n_1 \bar{u}_1 + n_2 \bar{u}_2 + \cdots + n_k \bar{u}_k = \sum_{i=1}^k n_i \bar{u}_1 = n \bar{u} \quad (9.6.3)$$

$$u = \sum_{i=1}^k x_i u_i \text{ kJ kg}^{-1} \quad (9.6.4)$$

$$\bar{u} = \sum_{i=1}^k y_i \bar{u}_i \text{ kJ kmol}^{-1} \quad (9.6.5)$$

### Enthalpy

$$H = H_1 + H_2 + \cdots + H_k = \sum_{i=1}^k H_i \quad (9.6.6)$$

$$H = m_1 h_1 + m_2 h_2 + \cdots + m_k h_k = \sum_{i=1}^k m_i h_i = m h \quad (9.6.7)$$

$$H = n_1 \bar{h}_1 + n_2 \bar{h}_2 + \cdots + n_k \bar{h}_k = \sum_{i=1}^k n_i \bar{h}_1 = n \bar{h} \quad (9.6.8)$$

$$h = \sum_{i=1}^k x_i h_i \text{ kJ kg}^{-1} \quad (9.6.9)$$

$$\bar{h} = \sum_{i=1}^k y_i \bar{h}_i \text{ kJ kmol}^{-1} \quad (9.6.10)$$

## Entropy

$$S = S_1 + S_2 + \cdots + S_k = \sum_{i=1}^k S_i \quad (9.6.11)$$

$$S = m_1 s_1 + m_2 s_2 + \cdots + m_k s_k = \sum_{i=1}^k m_i s_i = ms \quad (9.6.12)$$

$$S = n_1 \bar{s}_1 + n_2 \bar{s}_2 + \cdots + n_k \bar{s}_k = \sum_{i=1}^k n_i \bar{s}_i = n\bar{s} \quad (9.6.13)$$

$$s = \sum_{i=1}^k x_i s_i \text{ kJ kg}^{-1} \quad (9.6.14)$$

$$\bar{s} = \sum_{i=1}^k y_i \bar{s}_i \text{ kJ kmol}^{-1} \quad (9.6.15)$$

For all extensive (depending on system size) properties:

$$A = ma = n\bar{a} = \sum_{i=1}^k A_i = \sum_{i=1}^k m_i a_i = \sum_{i=1}^k n_i \bar{a}_i \quad (9.6.16)$$

For all specific ( $u, h, s, V, \phi, \psi$ ) properties:

$$a = \sum_{i=1}^k x_i a_i \quad (9.6.17)$$

$$\bar{a} = \sum_{i=1}^k y_i \bar{a}_i \quad (9.6.18)$$

### 9.6.1 Change in Properties

Given:

$$U = \sum_{i=1}^k m_i u_i \quad (9.6.19)$$

Differentiation:

$$dU = d\left(\sum_{i=1}^k m_i u_i\right) = \sum_{i=1}^k d(m_i u_i) \quad (9.6.20)$$

$$dU = \sum_{i=1}^k m_i du_i + \sum_{i=1}^k u_i dm_i \quad (9.6.21)$$

Change in mixture properties are due to 2 parts:

- changing component property  $\rightarrow \mathrm{d}u_i$
- changing composition  $\rightarrow \mathrm{d}m_i$

For nonreacting mixture, fixed composition  $\rightarrow \mathrm{d}m_i = 0$ . Hence:

$$\mathrm{d}U = \sum_{i=1}^k m_i \mathrm{d}u_i \quad (9.6.22)$$

$$\mathrm{d}u = \sum_{i=1}^k x_i \mathrm{d}u_i \quad (9.6.23)$$

In general:

$$\mathrm{d}A = \sum_{i=1}^k m_i \mathrm{d}a_i \quad (9.6.24)$$

$$\mathrm{d}a = \sum_{i=1}^k x_i \mathrm{d}a_i \quad (9.6.25)$$

## 9.6.2 Specific Heats of Ideal Gas Mixture

### Evaluating $C_v$ and $C_p$ of the mixture (volumetric analysis)

Given:

$$\bar{u} = \sum_{i=1}^k y_i \bar{u}_i \quad \text{and} \quad \bar{h} = \sum_{i=1}^k y_i \bar{h}_i \quad (9.6.26)$$

Differentiation with respect to temperature leads to:

$$\bar{c}_v = \sum_{i=1}^k y_i \bar{c}_{v,i} \quad \text{and} \quad \bar{c}_p = \sum_{i=1}^k y_i \bar{c}_{p,i} \quad (9.6.27)$$

The specific ratio of heats of the mixture:

$$\gamma = \frac{\bar{c}_p}{\bar{c}_v} \quad (9.6.28)$$

### Evaluating $C_v$ and $C_p$ of the mixture (gravimetric analysis)

Define specific heats for single component  $i$  as:

$$\mathrm{d}u_i = c_{v,i} \mathrm{d}T_i \quad \text{and} \quad \mathrm{d}h_i = c_{p,i} \mathrm{d}T_i \quad (9.6.29)$$

For equilibrium conditions:

$$\mathrm{d}T_i = \mathrm{d}T \quad (9.6.30)$$

$$\mathrm{d}u = \sum_{i=1}^k x_i c_{v,i} \mathrm{d}T \quad \text{and} \quad \mathrm{d}h = \sum_{i=1}^k x_i c_{p,i} \mathrm{d}T \quad (9.6.31)$$

Also:

$$du = c_v dT \quad \text{and} \quad dh = c_p dT \quad (9.6.32)$$

Therefore:

$$c_v = \sum_{i=1}^k x_i c_{v,i} \quad \text{and} \quad c_p = \sum_{i=1}^k x_i c_{p,i} \quad (9.6.33)$$

### 9.6.3 Change in Properties of Mixture (non-reacting)

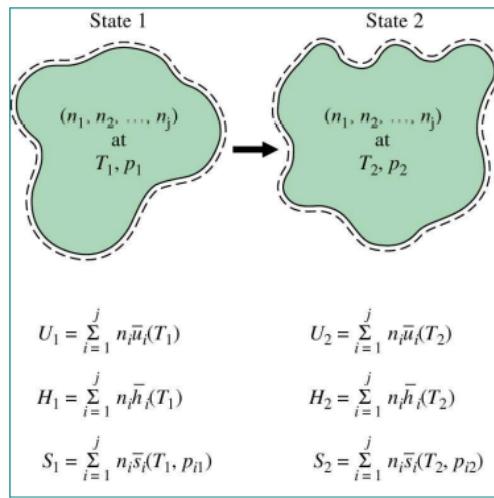


Figure 9.6:

$$U_2 - U_1 = \sum_{i=1}^j n_i [\bar{u}_i(T_2) - \bar{u}_i(T_1)] \quad (9.6.34)$$

$$\Delta \bar{u} = \sum_{i=1}^j y_i [\bar{u}_i(T_2) - \bar{u}_i(T_1)] \quad (9.6.35)$$

$$H_2 - H_1 = \sum_{i=1}^j n_i [\bar{h}_i(T_2) - \bar{h}_i(T_1)] \quad (9.6.36)$$

$$\Delta \bar{h} = \sum_{i=1}^j y_i [\bar{h}_i(T_2) - \bar{h}_i(T_1)] \quad (9.6.37)$$

$$S_2 - S_1 = \sum_{i=1}^j n_i [\bar{s}_i(T_2, p_{i2}) - \bar{s}_i(T_1, p_{i1})] \quad (9.6.38)$$

$$\Delta \bar{s} = \sum_{i=1}^j y_i [\bar{s}_i(T_2, p_{i2}) - \bar{s}_i(T_1, p_{i1})] \quad (9.6.39)$$

Changes in mixture properties are written in terms of the respective property changes of the components, weighted by the mole fraction.

Changes in properties for component  $i$  can be evaluated by:

- Either integration of the relevant equations or
- Tabulation of ideal gas properties

#### 9.6.4 Change in Entropy of Ideal Gas Mixtures

At fixed composition:

$$ds = \sum_{i=1}^k x_i ds_i \quad (9.6.40)$$

For an ideal gas mixture:

$$ds = c_p \frac{dT}{T} - R \frac{dp}{p} \quad (9.6.41)$$

$$\Delta s = \int_{T_1}^{T_2} c_p \frac{dT}{T} - R \ln \left( \frac{p_2}{p_1} \right) \quad (9.6.42)$$

$$\Delta s = s_2^\circ - s_1^\circ - R \ln \left( \frac{p_2}{p_1} \right) \quad (9.6.43)$$

Where:

$$c_p = \sum_{i=1}^k x_i c_{p,i} \quad \text{and} \quad R = \frac{R_u}{M} \quad (9.6.44)$$

For constant specific heat:

$$\Delta s = c_p \ln \left( \frac{T_2}{T_1} \right) - R \ln \left( \frac{p_2}{p_1} \right) \quad (9.6.45)$$

Then:

$$\Delta s = \sum_{i=1}^k x_i \Delta s_i^\circ - \sum_{i=1}^k x_i R_i \ln \left( \frac{p_{i,2}}{p_{i,1}} \right) \quad (9.6.46)$$

For constant specific heat:

$$\Delta s = \sum_{i=1}^k x_i c_{p,i} \ln \left( \frac{T_2}{T_1} \right) - \sum_{i=1}^k x_i R_i \ln \left( \frac{p_{i,2}}{p_{i,1}} \right) \quad (9.6.47)$$

On a molar basis:

$$\Delta \bar{s} = \sum_{i=1}^k y_i \Delta \bar{s}_i^\circ - \sum_{i=1}^k y_i R_u \ln \left( \frac{p_{i,2}}{p_{i,1}} \right) \quad (9.6.48)$$

# Chapter 10

## Psychrometrics

26/01/2021

The study of systems involving **moist air** (**dry air + water vapour**) is called psychrometrics.

### 10.1 Air-Water Vapour Mixtures

Moist air is found in:

- Nature;
- HVAC (heating, ventilation and air-conditioning) processes, where moisture (water vapour) is added or removed from the air;
- Steam turbines; and so on.

Moist air is a mixture of dry air and water vapour, where the **amount of dry air** is usually kept **constant** and the **amount of water vapour** may **change**. Two assumption are employed for air-water vapour properties :

1. **Dalton's law** applies for the mixture, i.e. partial pressure  $\rightarrow p = p_a + p_v$
2. Air and water vapour are each treated as **ideal gases**, so is their **mixture**.

### 10.2 Moist Air

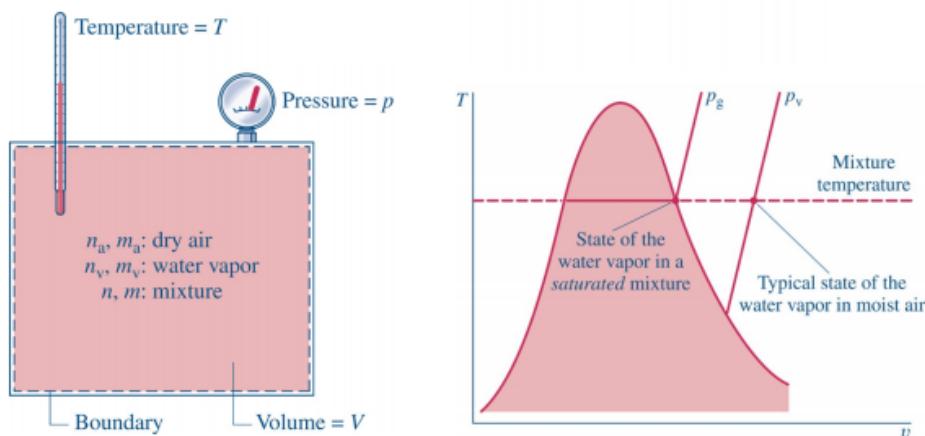


Figure 10.1:

Equation of state of ideal gases:

$$p = \frac{nR_u T}{V} = \frac{m(\frac{R_u}{M})T}{V} \quad (10.2.1)$$

$$p_a = \frac{n_a R_u T}{V} = \frac{m_a(\frac{R_u}{M_a})T}{V} \quad p_a = y_a p \quad (10.2.2)$$

$$p_v = \frac{n_v R_u T}{V} = \frac{m_v(\frac{R_u}{M_v})T}{V} \quad p_v = y_v p \quad (10.2.3)$$

$p_v$  and  $T$ :

- Typical state: vapor is superheated;
- At  $p_v = p_g$  (saturation pressure at  $T$ )  $\rightarrow$  Saturated Air (mixture of dry air and saturated water vapour); the mixture is said to be **saturated**.

### 10.2.1 Moist Air Properties

For a single component:

- **Two** independent, intrinsic properties are needed to define the state

For a two-component mixture (e.g. air-water vapor):

- **Three** independent, intrinsic properties are needed to define the state
  - Usually, two out of temperature, pressure and specific volume.
  - Plus humidity ratio  $\omega$  or relative humidity  $\phi$

## 10.3 Specific Humidity

The **specific humidity** or **humidity ratio**  $\omega$  is defined as:

$$\omega = \frac{m_v}{m_a} = \frac{\text{mass of water vapour}}{\text{mass of dry air}} \quad (10.3.1)$$

Using ideal gas law for both water vapour and dry air:

$$\omega = \frac{R_u T}{M_a p_a V} \frac{M_v p_v V}{R_u T} = \frac{M_v p_v}{M_a p_a} \quad (10.3.2)$$

Where,  $T$  is temperature of the mixture measured by a conventional thermometer, called the dry bulb temperature denoted by  $T_{db}$ .

The molecular weights:

$$M_v = 18 \quad M_a = 28.96 \quad (10.3.3)$$

$$\frac{M_v}{M_a} \approx 0.622 \quad (10.3.4)$$

Dalton's Law:

$$p = p_a + p_v \quad (10.3.5)$$

$$\therefore p_a = p - p_v \quad (10.3.6)$$

Specific Humidity:

$$\omega = 0.622 \frac{p_v}{p - p_v} \quad (10.3.7)$$

### 10.3.1 Moist Air Properties

The humidity ratio is a parameter for determining other quantities:

Mass:

$$m = m_m = m_a + m_v = m_a + \omega m_a \quad (10.3.8)$$

$$\therefore m = (1 + \omega)m_a \quad (10.3.9)$$

Energy:

$$H = H_m = m_a h_a + m_v h_v \quad (10.3.10)$$

$$\therefore H = m_a(h_a + \omega h_v) \quad (10.3.11)$$

$$h = h_m = \frac{H}{m_a} = h_a + \omega h_v \quad (10.3.12)$$

Humidity ratio is measured by a **hygrometer** in which a moist air sample is exposed to suitable chemicals until the moisture present is absorbed. The amount of water vapour is determined by weighing the chemicals.

## 10.4 Relative Humidity

$$\phi = \frac{\text{partial pressure of water vapour at a temperature } T \text{ (in superheated state)}}{\text{saturation pressure of water vapour at the same temperature } T \text{ (or } T_{db})} \quad (10.4.1)$$

$$\phi = \frac{p_v}{p_g} \quad (10.4.2)$$

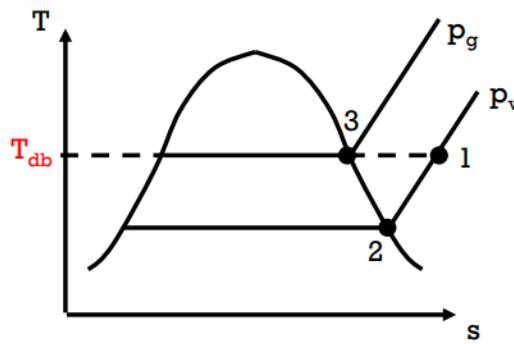


Figure 10.2:

Consider ideal gas mixture:

$$\phi = \frac{p_v}{p_g} = \frac{R_v T_v}{v_v} \frac{v_g}{R_v T_v} = \frac{v_g}{v_v} \quad (10.4.3)$$

$$\phi = \frac{v_g}{v_v} = \frac{p_v}{p_g} \leq 1.0 \quad \text{always} \quad (10.4.4)$$

### 10.4.1 An Alternative Definition

$$\phi = \frac{\text{mole fraction of water vapour at a given } T \text{ and } P \text{ (in superheated state)}}{\text{mole fraction of water vapour at saturation at the same } T \text{ and } P} \quad (10.4.5)$$

$$\phi = \frac{y_v}{y_{v,sat}} = \frac{y_v p}{y_{v,sat} p} = \frac{p_v}{p_g} \quad (10.4.6)$$

Relative humidity can be measured by **transducers** whose electrical characteristics change with relative humidity.

### 10.4.2 Condensation of Moist Air

Partial condensation of moist air can occur when temperature is reduced below a certain level:



Figure 10.3: Left: Condensate on windowpanes — Right: Dew on grass

The condensation process can be divided into the following stages:

- Superheated water vapour (state 1) is **cooled under constant system pressure** (thus the composition of moist air and partial pressure of vapour remain constant), until it reaches **the dew point** (state d);
  - The saturation temperature corresponding to  $p_v$  is called the dew point temperature ( $T_{dp}$ ).
- Vapour starts to **condense when the moist air is cooled below  $T_{dp}$** .
- Further cooling leads to more condensate until the dry air, saturated vapour and liquid water **reach equilibrium**.
- At the final state, the system consists of **saturated air** (state 2) and **saturated liquid** (state 3) at the final temperature. The saturated vapour (state 2) has a partial pressure of the saturation pressure  $p_{g2}$

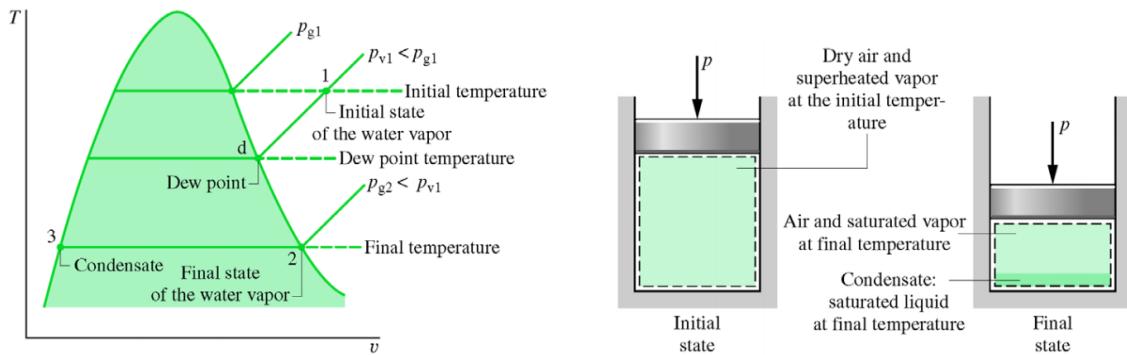


Figure 10.4:

## 10.5 Adiabatic Saturation Temperature $T_{as}$

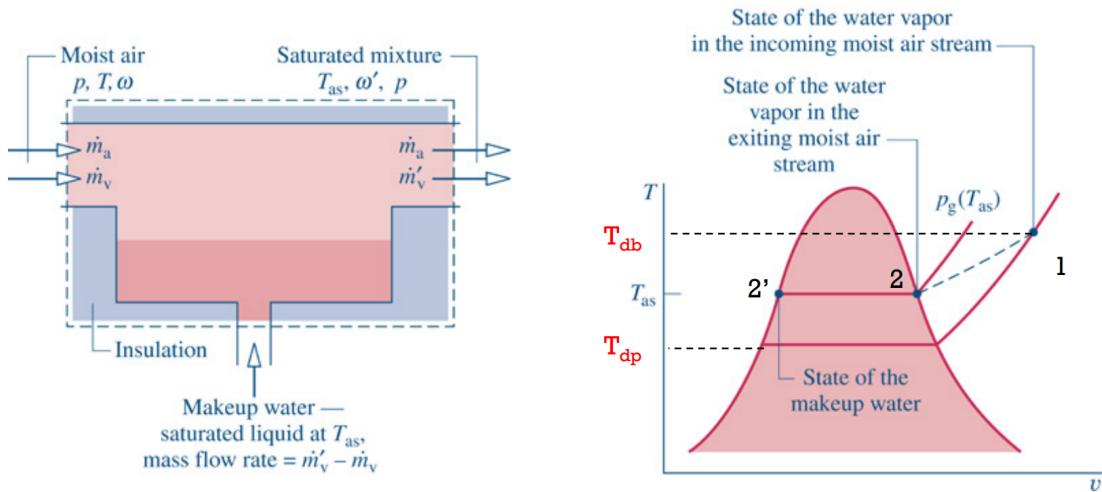


Figure 10.5:

The idea of the above **Adiabatic Saturator** is to determine the unknown humidity ratio at the inlet based on values of  $p$ ,  $T$  and **Adiabatic Saturation Temperature**  $T_{as}$ .

### 10.5.1 The Adiabatic Saturation Analysis

Assumptions:

- Steady state
- $\Delta KE = \Delta PE = 0$
- Adiabatic process
- Constant pressure process
- Air and water vapor are ideal gases

Mass Balances:

$$\text{Dry air balance: } \dot{m}_{a,1} = \dot{m}_{a,2} = \dot{m}_a \quad (10.5.1)$$

$$\text{Water balance: } \dot{m}_{v,1} + \dot{m}_L = \dot{m}_{v,2} \quad (10.5.2)$$

$$\rightarrow \omega \dot{m}_a + \dot{m}_L = \omega' \dot{m}_a \quad (10.5.3)$$

$$\rightarrow \dot{m}_L = \dot{m}_a(\omega' - \omega) = \dot{m}_a(\omega_2 - \omega_1) \quad (10.5.4)$$

1st Law for Open System, energy balance:

$$\frac{dE_{CV}}{dt} = \dot{Q} + \dot{W} + \sum_{in} \dot{m}_{in} \left( h + \frac{v^2}{2} + gz \right)_{in} - \sum_{out} \dot{m}_{out} \left( h + \frac{v^2}{2} + gz \right)_{out} \quad (10.5.5)$$

$$\cancel{\frac{dE_{CV}}{dt}} = \cancel{\dot{Q}} + \cancel{\dot{W}} + \sum_{in} \dot{m}_{in} \left( h + \frac{v^2}{2} + gz \right)_{in} - \sum_{out} \dot{m}_{out} \left( h + \frac{v^2}{2} + gz \right)_{out} \quad (10.5.6)$$

$$0 = \dot{m}_a h_{a,1} + \dot{m}_{v,1} h_{v,1} + \dot{m}_L h_L - \dot{m}_a h_{a,2} - \dot{m}_{v,2} h_{v,2} \quad (10.5.7)$$

$$0 = \dot{m}_a h_{a,1} + \omega_1 \dot{m}_a h_{v,1} + \dot{m}_a (\omega_2 - \omega_1) h_L - \dot{m}_a h_{a,2} - \omega_2 \dot{m}_a h_{v,2} \quad (10.5.8)$$

$$0 = h_{a,1} + \omega h_{v,1} + (\omega' - \omega) h_{f,2} - h_{a,2} - \omega' h_{v,2} \quad (10.5.9)$$

$$\rightarrow \omega h_{v,1} - \omega h_{f,2} = (h_{a,2} - h_{a,1}) + \omega' (h_{v,2} - h_{f,2}) \quad (10.5.10)$$

$$\rightarrow \omega = \frac{(h_{a,2} - h_{a,1}) + \omega' (h_{v,2} - h_{f,2})}{h_{v,1} - h_{f,2}} \quad (10.5.11)$$

$$h_v \approx h_g(T) \quad (10.5.12)$$

$$\omega = \frac{[h_a(T_{as}) - h_a(T)] + \omega' [h_g(T_{as}) - h_f(T_{as})]}{h_g(T) - h_f(T_{as})} \quad (10.5.13)$$

$$\omega' = 0.622 \frac{p_g(T_{as})}{p - p_g(T_{as})} \quad (10.5.14)$$

## 10.6 Wet-bulb Temperature $T_{wb}$

**Dry-bulb Temperature  $T_{db}$**

- Measured by a normal thermometer placed in the mixture.

**Wet-bulb Temperature  $T_{wb}$**

- Measured by a wet-bulb thermometer, which is an ordinary liquid-in-glass thermometer whose bulb is enclosed by a wick moistened with water.
- $T_{wb}$  is a close approximation of the adiabatic saturation temperature.
- $T_{wb}$  is used in place of  $T_{as}$  to determine  $\omega$  in the adiabatic saturator.

$$\omega = \frac{[h_a(T_{wb}) - h_a(T_{db})] + \omega' [h_g(T_{wb}) - h_f(T_{wb})]}{h_g(T_{db}) - h_f(T_{wb})} \quad (10.6.1)$$

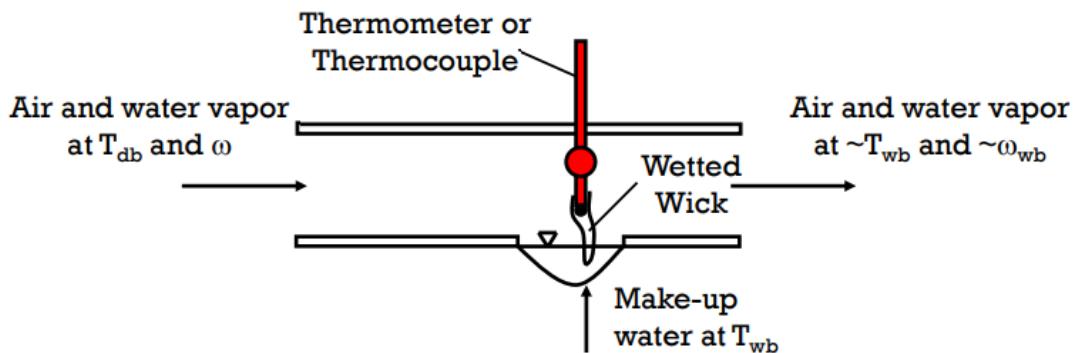


Figure 10.6:

## 10.7 Psychrometers

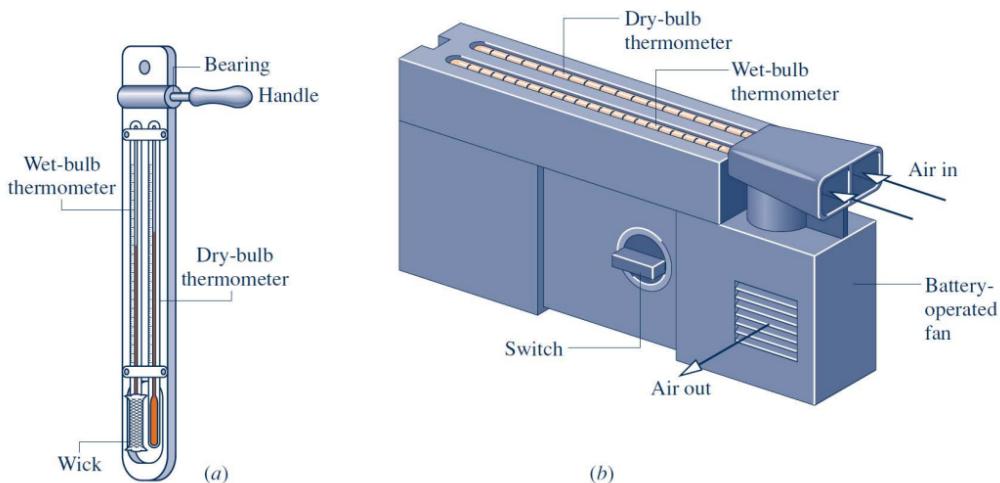


Figure 10.7: (a) Sling Psychrometers — (b) Aspirating Psychrometers

These are the devices to measure temperature and provide the humidity ratio.

02/02/2021

## 10.8 Psychrometric Chart

### 10.8.1 Air-Water Vapour Properties

How to evaluate air-water vapor properties?

- Psychrometric relationships
- Psychrometric chart
- Computer programme/database

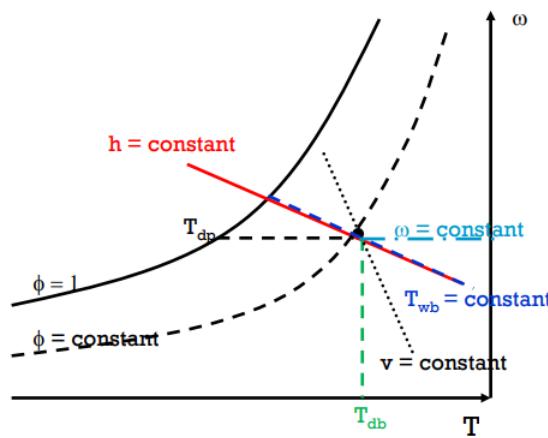


Figure 10.8:

**Psychrometric Chart:**

- Graphical relationship between:  $T_{db}$ ,  $T_{wb}$ ,  $T_{dp}$ ,  $\omega$ ,  $\phi$ ,  $h$ ,  $v$
- Constructed for pressure at 1 bar
- Valid for mixture pressure around 1 bar, for engineering analysis
- Useful for visualizing air-water vapour processes at constant pressure
- The specific volume ( $v = V/m_a$ ) has the unit of  $\text{m}^3 \text{kg}^{-1}$  dry air
- $h$  gives the mixture enthalpy per unit mass of dry air in the mixture:

$$h = \frac{H}{m_a} = h_a + \omega h_v \quad (\text{kJ kg}^{-1} \text{ dry air}) \quad (10.8.1)$$

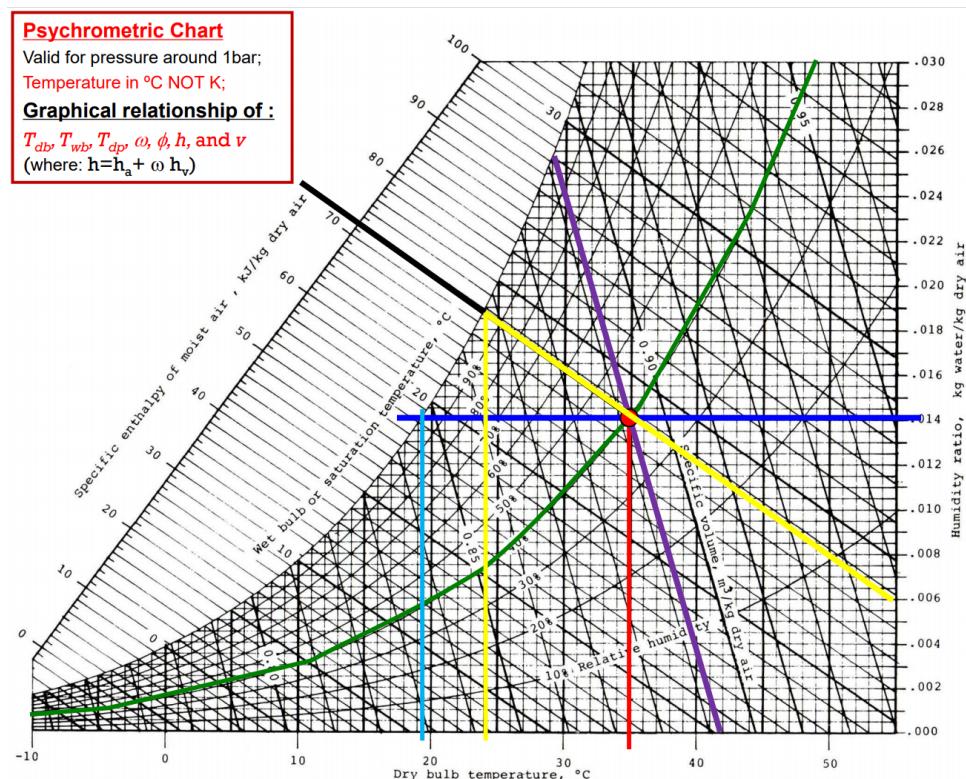


Figure 10.9:

## 10.9 Psychrometric Applications

### 10.9.1 Summer and Winter Comfort Zones

**Comfort zones:** Acceptable ranges of operative temperature and humidity for people in typical summer and winter clothing during primarily sedentary activity.

**Air-conditioning:** To change the temperature and water vapour fraction in the moist air

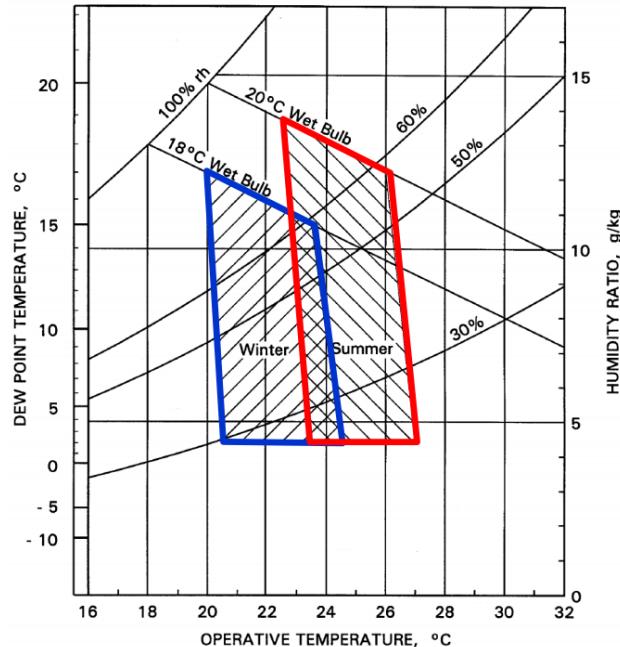


Figure 10.10:

### 10.9.2 List of Psychrometric Applications

- Cooling with and without Dehumidification
- Cooling with Dehumidification and Reheating
- Evaporative Cooling
- Humidification
- Humidification with Heating
- Adiabatic Mixing
- Cooling Tower

The first 5 of these applications are known of **Refrigerant, Air-Conditioning Processes**.

### 10.9.3 Air-Conditioning Processes

- Simple Heating and Cooling ( $\omega=\text{constant}$ ), no change in the amount of water vapor.
- Cooling with Dehumidification
- Heating with Humidification

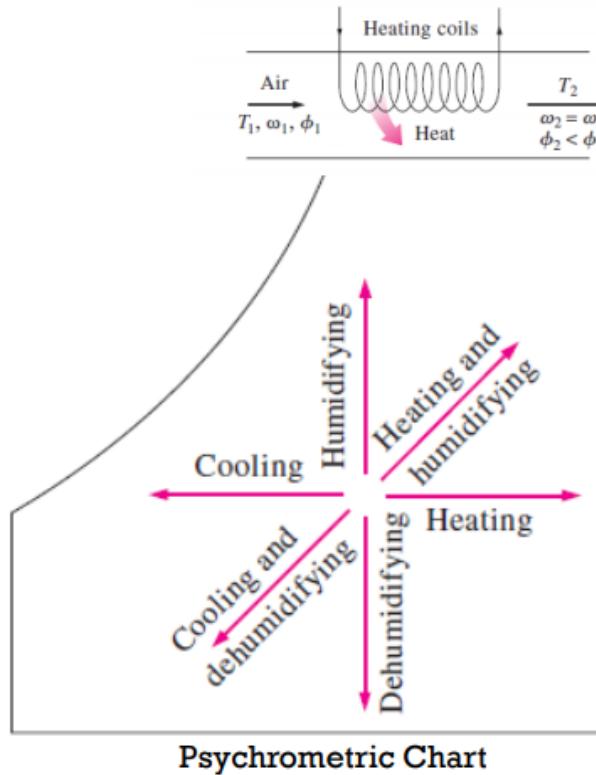


Figure 10.11:

### Mass and Energy Balance for Open Systems:

Mass balance for multiple inlets and exits:

$$\frac{dm_{cv}}{dt} = \sum_i \dot{m}_i - \sum_e \dot{m}_e \quad (10.9.1)$$

Energy balance for multiple inlets and exits:

$$\frac{dE_{cv}}{dt} = \dot{Q}_{cv} + \dot{W}_{cv} + \sum_i \dot{m}_i \left( h_i + \frac{V_i^2}{2} + gz_i \right) - \sum_e \dot{m}_e \left( h_e + \frac{V_e^2}{2} + gz_e \right) \quad (10.9.2)$$

Steady state:

$$\frac{dm_{cv}}{dt} = 0 \quad (10.9.3)$$

$$\frac{dE_{cv}}{dt} = 0 \quad (10.9.4)$$

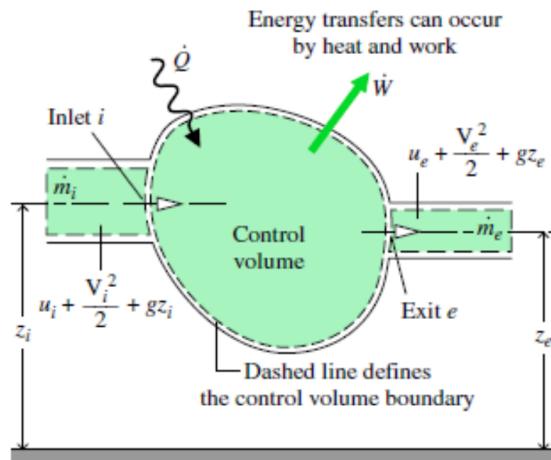


Figure 10.12:

### Air-Conditioning Processes:

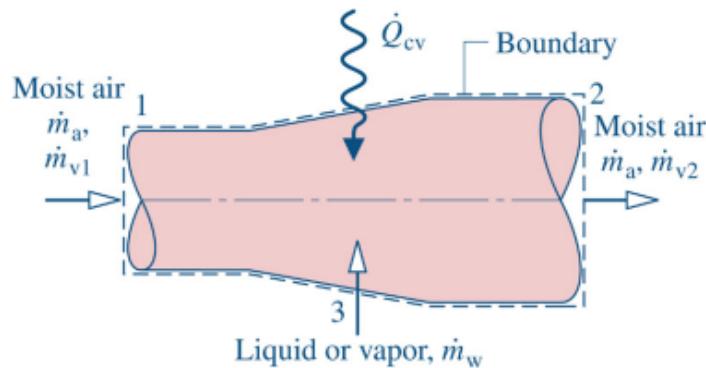


Figure 10.13:

- On control volume basis;
- Steady State;
- Assume
  - $W_{cv} = 0$
  - $\Delta KE = 0$
  - $\Delta PE = 0$
- Heat transfer between the control volume and its surroundings;
- Liquid water (mist) or vapour added

Mass Balance:

$$\dot{m}_{a1} = \dot{m}_{a2} = \dot{m}_a \quad (\text{dry air}) \quad (10.9.5)$$

$$\dot{m}_{v1} + \dot{m}_w = \dot{m}_{v2} \quad (\text{water}) \quad (10.9.6)$$

$$\dot{m}_{v1} = \omega_1 \dot{m}_a \quad \text{and} \quad \dot{m}_{v2} = \omega_2 \dot{m}_a \quad (10.9.7)$$

$$\therefore \dot{m}_w = \dot{m}_a(\omega_2 - \omega_1) \quad (\text{water}) \quad (10.9.8)$$

Energy Balance:

$$0 = \dot{Q}_{cv} + (\dot{m}_a h_{a1} + \dot{m}_{v1} h_{v1}) + \dot{m}_w h_w - (\dot{m}_a h_{a2} + \dot{m}_{v2} h_{v2}) \quad (10.9.9)$$

$$0 = \dot{Q}_{cv} + \dot{m}_a [(h_{a1} + \omega_1 h_{v1}) + (\omega_2 - \omega_1)h_w - (h_{a2} + \omega_2 h_{v2})] \quad (10.9.10)$$

$$0 = \dot{Q}_{cv} + \dot{m}_a [h_1 + (\omega_2 - \omega_1)h_w - h_2] \quad (10.9.11)$$

The  $(h_{a1} + \omega_1 h_{v1})$  term represents the **total specific enthalpy of the mixture** at Point 1, and the  $(h_{a2} + \omega_2 h_{v2})$  term represents the **total specific enthalpy of the mixture** at Point 2. They can be simplified by writing in  $h_1$  and  $h_2$  respectively. The quantities  $h_1$  and  $h_2$  can be obtained from the Psychrometric Chart, while the enthalpy of water,  $h_w$ , can be obtained from the Property Tables.

Alternatively, enthalpies of the water vapour can be treated as the saturated vapour enthalpies at the corresponding temperatures:

$$h_{v1} \approx h_{g1} \quad (10.9.12)$$

$$0 = \dot{Q}_{cv} + \dot{m}_a [(h_{a1} - h_{a2}) + \omega_1 h_{g1} + (\omega_2 - \omega_1)h_w - \omega_2 h_{g2}] \quad (10.9.13)$$

All the properties above can be found from the relevant tables:

- $(h_{a1} - h_{a2}) \rightarrow$  Dry Air: Table A-22 — OR:  $h_2 - h_1 = c_p(T_2 - T_1)$
- $\omega_1 h_{g1} + (\omega_2 - \omega_1)h_w - \omega_2 h_{g2} \rightarrow$  Steam Table

#### 10.9.4 Cooling with and without Dehumidification

Refrigeration:

- During the evaporation of the refrigerant, heat is absorbed from an air stream
- Air side process = cooling process with or without dehumidification
- Dehumidification occurs only if air stream temperature drops below dew point temperature
- Otherwise, air side process = cooling process at constant humidity ratio

Analysis:

- **Dry coil analysis:** no dehumidification, air stream temperature stays above dew point temperature of incoming air stream
- **Wet coil analysis:** dehumidification, air stream temperature drops below dew point temperature of incoming air stream

## Cooling without Dehumidification

### Cooling Process, Dry Coil Analysis:

Assumptions:

- $\Delta KE = \Delta PE = 0$
- Steady state

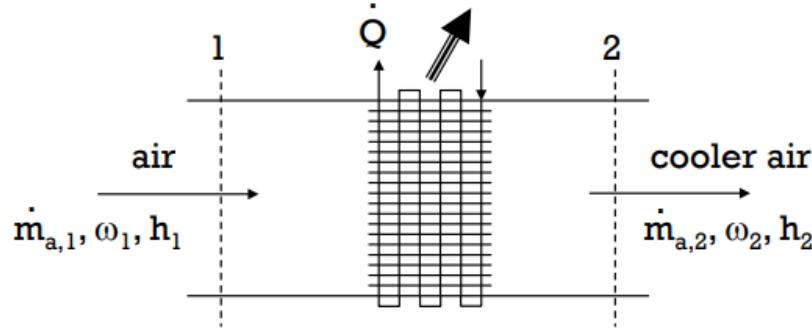


Figure 10.14:

$$\text{Dry air mass balance: } \dot{m}_{a,1} = \dot{m}_{a,2} = \dot{m}_a \quad (10.9.14)$$

$$\text{Water mass balance: } \dot{m}_a \omega_1 = \dot{m}_a \omega_2 \quad (10.9.15)$$

$$\text{Energy balance: } \dot{Q} = \dot{m}_a (h_2 - h_1) \quad (10.9.16)$$

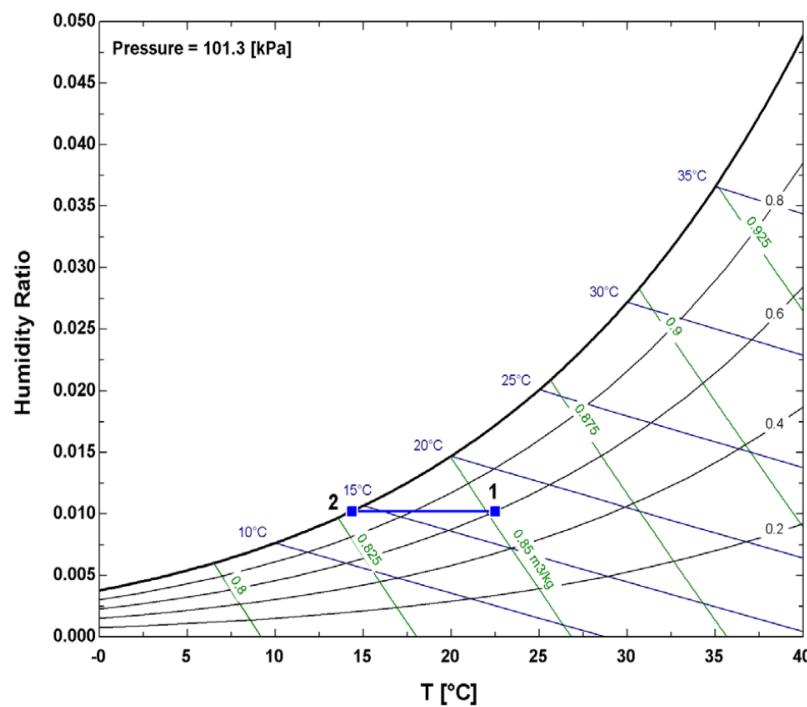


Figure 10.15:

## Cooling with Dehumidification

### Cooling Process, Wet Coil Analysis:

Assumptions:

- $\Delta KE = \Delta PE = 0$
- Steady state

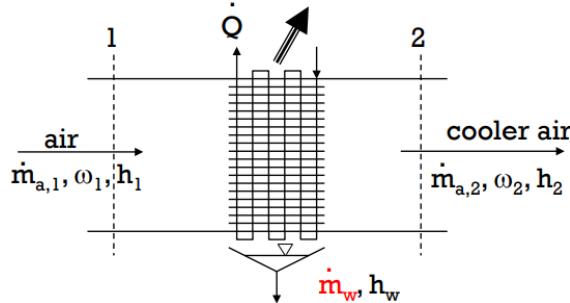


Figure 10.16:

$$\text{Dry air mass balance: } \dot{m}_{a,1} = \dot{m}_{a,2} = \dot{m}_a \quad (10.9.17)$$

$$\text{Water mass balance: } \dot{m}_a \omega_1 = \dot{m}_a \omega_2 + \dot{m}_w \quad (10.9.18)$$

$$\rightarrow \dot{m}_w = \dot{m}_a (\omega_1 - \omega_2) \quad (10.9.19)$$

$$\text{Energy balance: } \dot{Q} = \dot{m}_a (h_2 - h_1) + \dot{m}_w h_w \quad (10.9.20)$$

$$= \dot{m}_a [(h_2 - h_1) + (\omega_1 - \omega_2) h_w] \quad (10.9.21)$$

Note - Outlet state is saturated:  $\phi_2 = 1.0$

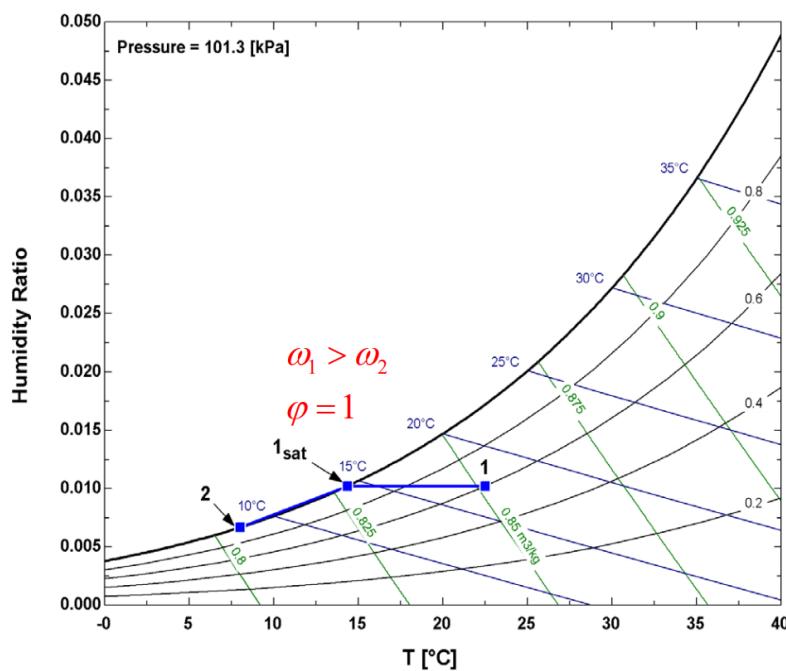


Figure 10.17:

**Real** air coil with dehumidification:

- Bulk air stream temperature between two fins may not reach dew point temperature  $\rightarrow$  no dehumidification
- Air stream temperature close to the fin surface drops below dew point temperature  $\rightarrow$  dehumidification
- Combined air stream leaving the coil may not be saturated!

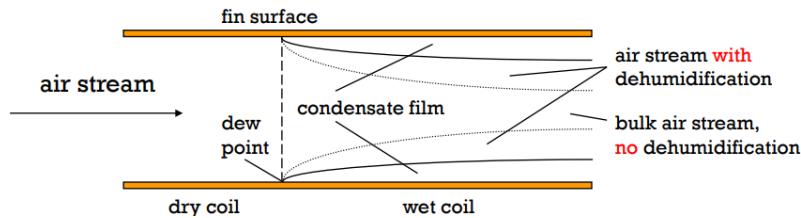


Figure 10.18:

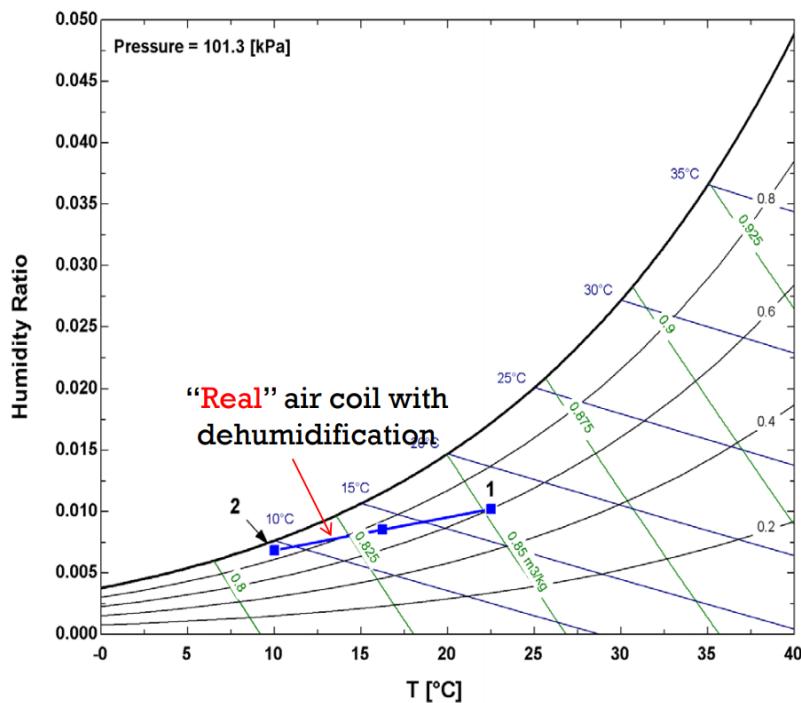


Figure 10.19:

### 10.9.5 Humidification

Difference with respect to evaporative cooling (Use water vapour (steam) instead of liquid water to humidify the air)

Assumptions:

- $\Delta KE = \Delta PE = 0$
- Steady state
- $p = \text{constant}$

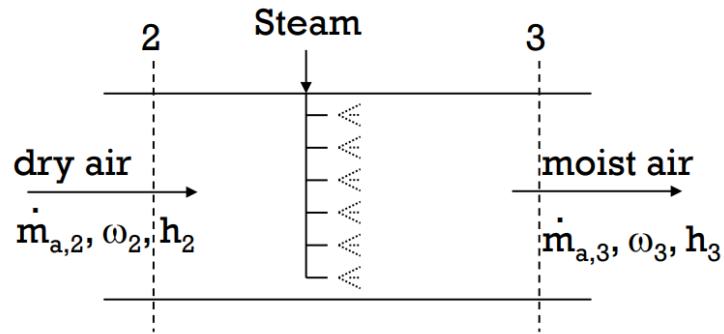


Figure 10.20:

Mass Balances:

$$\dot{m}_{a,2} = \dot{m}_{a,3} = \dot{m}_a \quad (\text{dry air}) \quad (10.9.22)$$

$$\dot{m}_w = \dot{m}_a(\omega_3 - \omega_2) \quad (\text{water}) \quad (10.9.23)$$

Energy Balance:

$$h_3 = h_2 + (\omega_3 - \omega_2)h_{st} \quad (10.9.24)$$

For Steam Humidification:

$$h_{st} = h_{vapor}(T_{st}; p_{st}) \longrightarrow T_3 > T_2 \quad (10.9.25)$$

For Evaporative Cooler:

$$h_{st} = h_w = h_f(T_3) \longrightarrow T_3 < T_2 \quad (10.9.26)$$

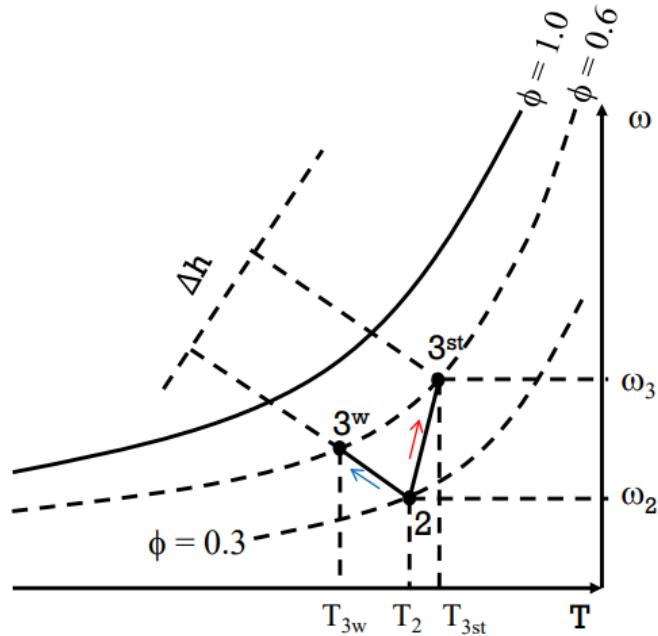


Figure 10.21:

### 10.9.6 Humidification with Heating

During winter time, when the outdoor humidity is low, typically **both** heating and humidification are needed in door.

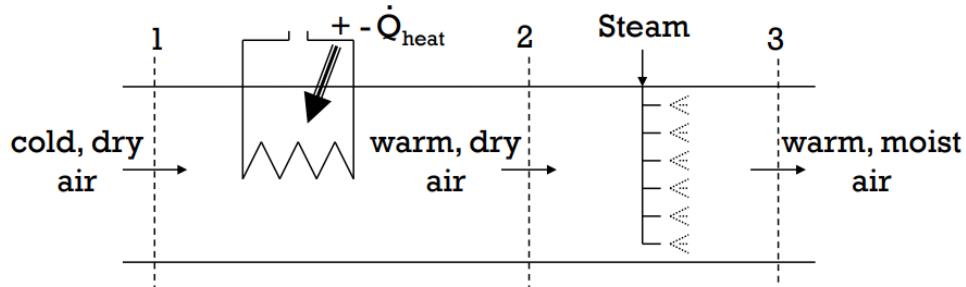


Figure 10.22:

Heating Coil:

- Hot water
- Steam
- Combustion gases
- Electric heater

Dry air mass balance:

$$\dot{m}_{a,1} = \dot{m}_{a,2} = \dot{m}_{a,3} = \dot{m} \quad (10.9.27)$$

Water mass balance:

$$\omega_1 \dot{m}_a = \omega_2 \dot{m}_a \longrightarrow \omega_1 = \omega_2 \quad (10.9.28)$$

$$\omega_2 \dot{m}_a + \dot{m}_{st} = \omega_3 \dot{m}_a \longrightarrow \dot{m}_{st} = \dot{m}_a (\omega_3 - \omega_2) \quad (10.9.29)$$

Energy balance on heater:

$$\dot{Q}_{heat} = \dot{m}_a (h_2 - h_1) \quad (10.9.30)$$

Energy balance on humidification:

$$h_3 = h_2 + (\omega_3 - \omega_2) h_{st} \quad (10.9.31)$$

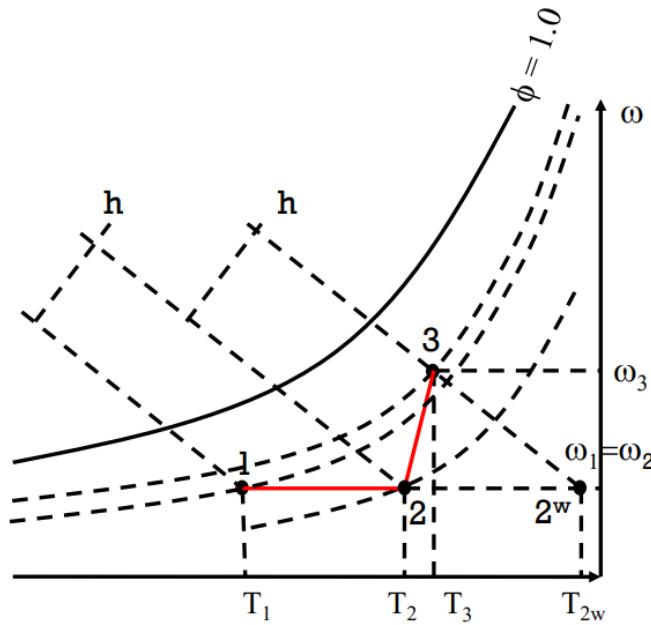


Figure 10.23:

### 10.9.7 Evaporative Cooling

Used in desert climates, where it is hot and dry (low relative humidity).

Analysis is just like **adiabatic saturator** analysis, except that the supply water temperature is independent of saturation temperature.

Assumptions:

- $\Delta KE = \Delta PE = 0$
- Steady state
- $p = \text{constant}$

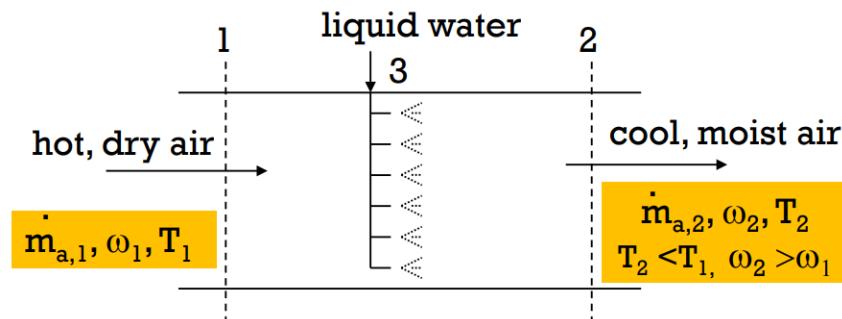


Figure 10.24:

Mass Balances:

$$\dot{m}_{a,1} = \dot{m}_{a,2} = \dot{m}_a \quad (\text{dry air}) \quad (10.9.32)$$

$$\dot{m}_w = \dot{m}_a(\omega_2 - \omega_1) \quad (\text{water}) \quad (10.9.33)$$

Energy Balance:

$$(h_{a2} + \omega_2 h_{v2}) = (h_{a1} + \omega_1 h_{v1}) + (\omega_2 - \omega_1)h_f \quad (10.9.34)$$

$$h_2 = h_1 + (\omega_2 - \omega_1)h_f \quad (10.9.35)$$

The  $(\omega_2 - \omega_1)h_f$  term is much smaller than the other two terms:

$$\therefore h_2 \approx h_1 \text{ is a good approximation} \quad (10.9.36)$$

$$\therefore h \approx \text{constant} \quad \& \quad T_{wb} \approx \text{constant} \quad (10.9.37)$$

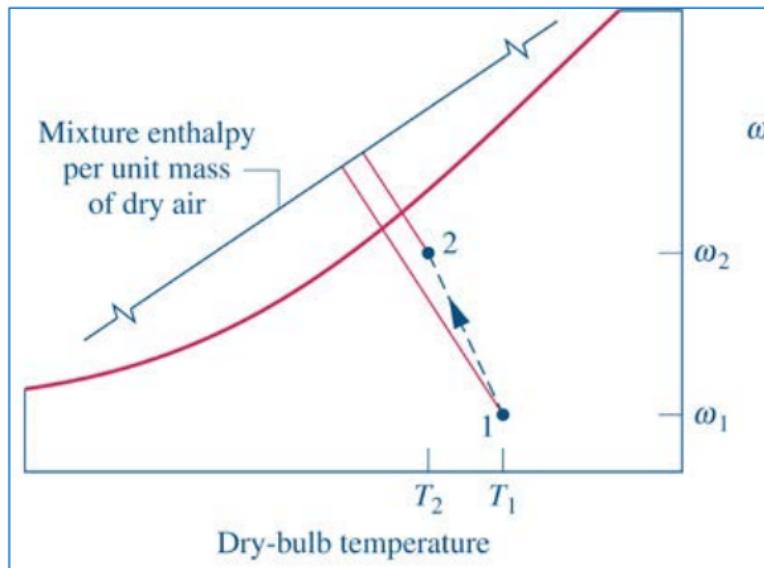


Figure 10.25:

### 10.9.8 Adiabatic Mixing of Two Moist Air Streams

Adiabatic Mixing of state 1 and state 2:

- Assume:
  - $\dot{Q} = 0$
  - $\dot{W} = 0$
- Mixing state is **on a straight line** between the two inlet state points

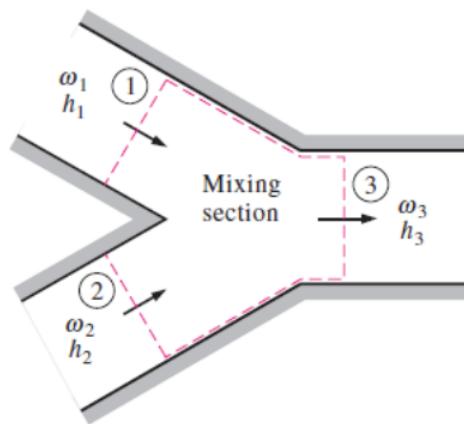


Figure 10.26:

Dry air mass balance:

$$\dot{m}_{a,3} = \dot{m}_{a,1} + \dot{m}_{a,2} \quad (10.9.38)$$

Vapour mass balance:

$$\omega_3 \dot{m}_{a,3} = \omega_1 \dot{m}_{a,1} + \omega_2 \dot{m}_{a,2} \quad (10.9.39)$$

Energy balance:

$$h_3 \dot{m}_{a,3} = h_1 \dot{m}_{a,1} + h_2 \dot{m}_{a,2} \quad (10.9.40)$$

$$\frac{\dot{m}_{a1}}{\dot{m}_{a2}} = \frac{\omega_2 - \omega_3}{\omega_3 - \omega_1} = \frac{h_2 - h_3}{h_3 - h_1} \quad (10.9.41)$$

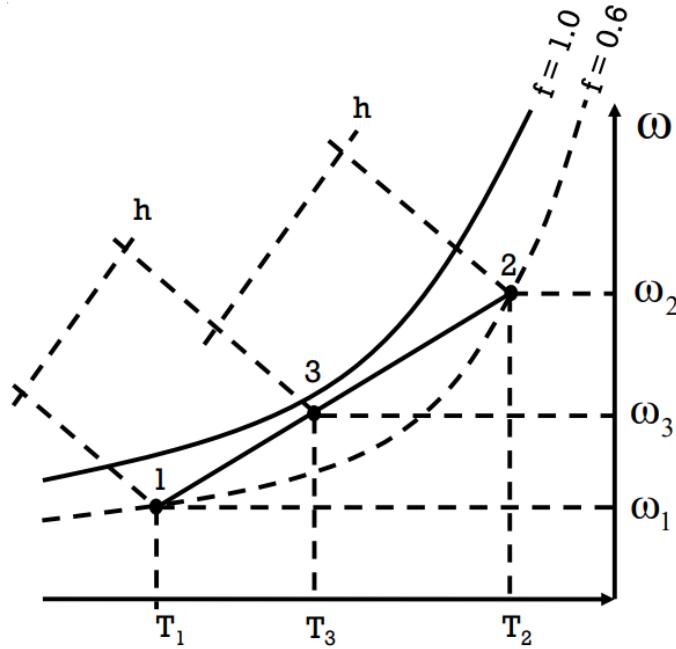


Figure 10.27:

### 10.9.9 Cooling Towers

- Cooling towers are used to:
  1. Release power plant waste heat at acceptable  $T$  to surroundings
  2. Provide chilled water
- Operate by natural or forced convection of air
- Counterflow, cross flow or hybrid configuration

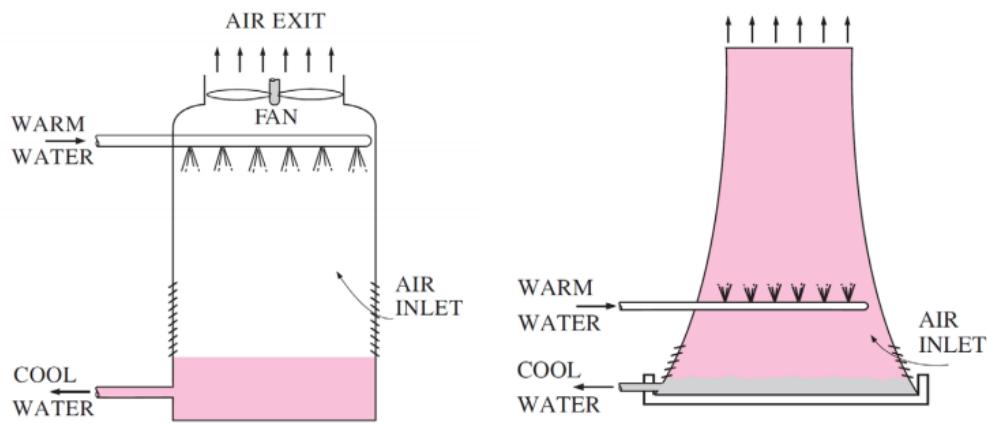


Figure 10.28: Left: An induced-draft counterflow cooling tower — Right: A natural-draft cooling tower

- Steady state analysis of mass and energy balance, usually;
- Heat transfer to surroundings is neglected, usually.
- Fan power may be considered

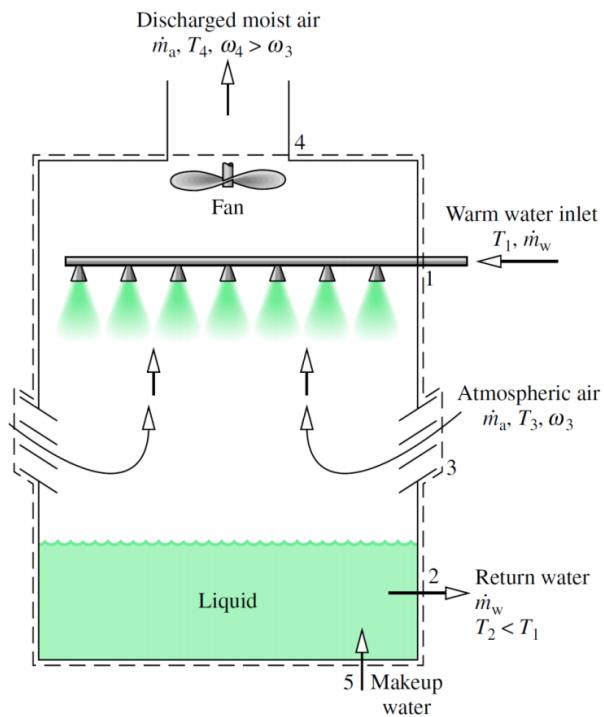


Figure 10.29:

# Chapter 11

## Elementary Combustion

09/02/2021

### Importance of Combustion

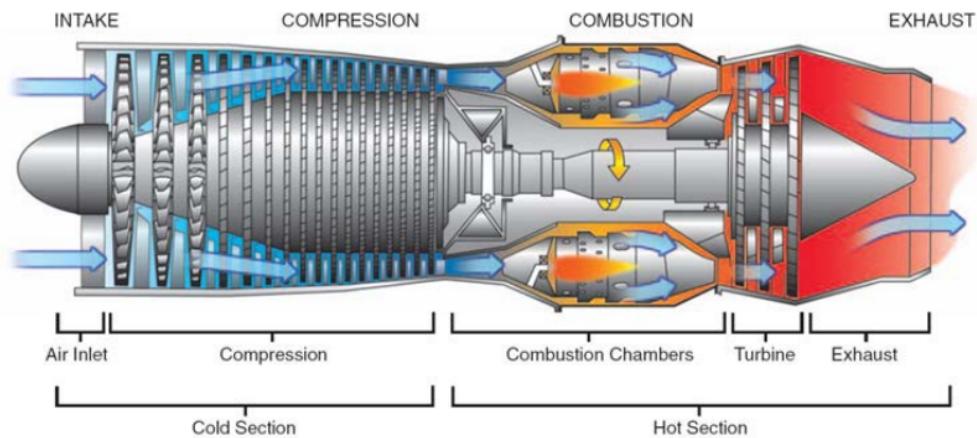


Figure 11.1:

Combustion is predominant source of useful energy, and predominant source of pollutant and CO<sub>2</sub> emissions.

### What Is Combustion?

Chemical Reaction:

- Bonds within the molecules of the reactants are broken.
- Atoms and electrons are rearranged to **form new chemical species** called products.

Combustion:

- Rapid oxidation of fuel for heat release:



## 11.1 Global and Elementary Reactions

### 11.1.1 Study of Chemical Reactions

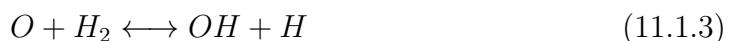
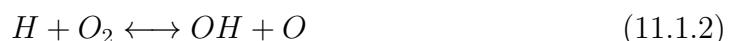
#### Overall/Global Reaction

Statement of mass and number of atoms conservation.



#### Elementary Reaction

Direct results of collisions between reactant molecules. Combustion usually consists of a large set of elementary reactions, e.g. hydrogen combustion:



Chemical Kinetics: Study of mechanisms and rates of chemical change.

### 11.1.2 Chemical Complexity

**Reality:** Detailed chemical mechanisms involve a huge number of elementary reactions and intermediate species.

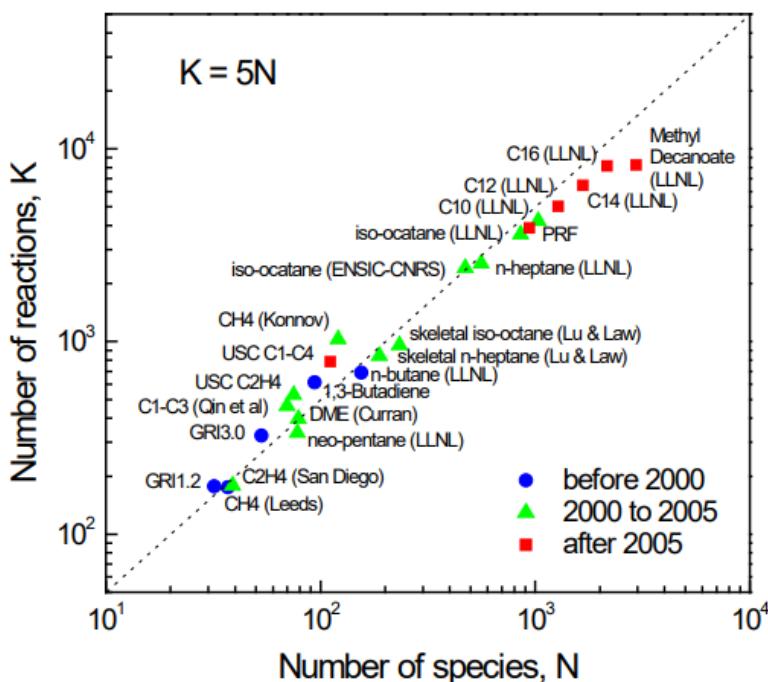


Figure 11.2:

### 11.1.3 Fuel and Oxidizer

Fuel:

- Liquids: e.g. gasoline, diesel, kerosene;
- Gases: e.g. natural gas;
- Solids: e.g. coal.
- Most common fuels : **Hydrocarbons ( $C_xH_y$ )**, e.g., Methane ( $CH_4$ ), propane ( $C_3H_8$ ), octane ( $C_8H_{18}$ ), gasoline (average  $C_{7.2}H_{12.6}$ ), diesel (average  $C_{12}H_{23}$ )
- Bio-fuels : **Usually ( $C_xH_yO_z$ )**, e.g., Methanol ( $CH_3OH$ ), ethanol ( $C_2H_5OH$ )

Oxidizer:

- In most combustion applications, air provides the needed oxygen;
- However, pure oxygen ( $O_2$ ) can be used, called oxy-fuel combustion, for carbon capture and storage

### 11.1.4 Fuel Composition

Fuel	Hydrogen/Carbon Ratio	Formula
Hydrocarbons	$y/x$	$C_xH_y$
Gasoline/Petrol	1.87	$C_xH_{1.87x}$
Diesel	1.7 ~ 1.8	$C_xH_{1.7-1.8x}$
Natural Gas (Methane)	4	$CH_4$
Coal	0.8	$C_xH_{0.8x}$
Hydrogen	$\infty$	$H_2$
Alcohol (Methanol)	4	$CH_4O$

Figure 11.3:

## 11.2 Combustion in Air

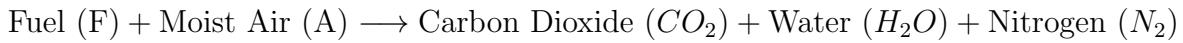
**Combustion in Dry Air:**



Dry Air (A) = 21% by volume of Oxygen ( $O_2$ ) + 79% by volume of Nitrogen ( $N_2$ )

$$= 1 \text{ unit volume of Oxygen (O}_2\text{)} + 3.76 \text{ unit volume of Nitrogen (N}_2\text{)}$$

### Combustion in Moist Air:



$$\begin{aligned} \text{Moist Air (A)} &= 1 \text{ unit mass of Oxygen (O}_2\text{)} + 3.29 \text{ unit mass of Nitrogen (N}_2\text{)} \\ &\quad + 4.29\omega \text{ unit mass of Water Vapour (H}_2\text{O)} \end{aligned}$$

#### 11.2.1 Air/Fuel Ratio and Fuel/Air Ratio

Air/Fuel Ratio by Mass:

$$AF = \frac{\text{mass of air (}m_{air}\text{)}}{\text{mass of fuel (}m_{fuel}\text{)}} = \frac{x_{air}}{x_{fuel}} = \frac{n_{air}}{n_{fuel}} \frac{M_{air}}{M_{fuel}} \quad (11.2.1)$$

Fuel/Air Ratio by Mass:

$$FA = \frac{1}{AF} \quad (11.2.2)$$

Air/Fuel Ratio by Volume:

$$\overline{AF} = \frac{\text{volume of air (}V_{air}\text{)}}{\text{volume of fuel (}V_{fuel}\text{)}} = \frac{n_{air}}{n_{fuel}} = \frac{y_{air}}{y_{fuel}} \quad (11.2.3)$$

Relationship:

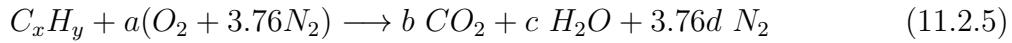
$$AF = \overline{AF} \frac{M_{air}}{M_{fuel}} \quad (11.2.4)$$

#### 11.2.2 Stoichiometric Combustion in Air

**Stoichiometry:**

Chemically exact proportion of fuel and air for complete burning of fuel.

#### Stoichiometric Combustion of Hydrocarbons in Dry Air:



Balance of atom number for each element:

$$C : \quad x = b \quad (11.2.6)$$

$$H : \quad y = 2c \quad (11.2.7)$$

$$O : \quad 2a = 2b + c \quad (11.2.8)$$

$$N : \quad 2 \cdot 3.76a = 2 \cdot 3.76d \quad (11.2.9)$$

Therefore, the stoichiometric coefficients:

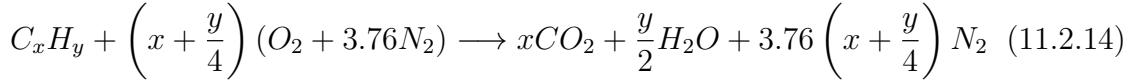
$$b = x \quad (11.2.10)$$

$$c = \frac{y}{2} \quad (11.2.11)$$

$$a = x + \frac{y}{4} \quad (11.2.12)$$

$$d = a \quad (11.2.13)$$

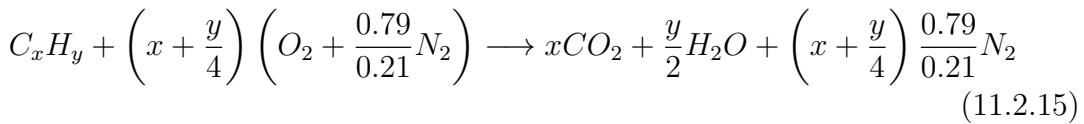
The mass balance equation for stoichiometric combustion:



Note:

- The mass, atom/mole number of each element are conserved.
- The total mass of reactants equals that of the products.
- The total mole numbers of reactants and products are not usually the same.

### Stoichiometric Combustion of Hydrocarbons in Air:



Stoichiometric Air/Fuel Ratio (by volume):

$$\overline{AF}_{stoi} = \left(x + \frac{y}{4}\right) \left(1 + \frac{0.79}{0.21}\right) \quad (11.2.16)$$

Stoichiometric Air/Fuel Ratio (by mass):

$$AF_{stoi} = \frac{\left(x + \frac{y}{4}\right) \left(32 + \frac{0.79}{0.21} \cdot 28\right)}{12x + y} \quad (11.2.17)$$

For typical petroleum-based fuels,  $AF_{stoi} = 14 \sim 15$ .

### 11.2.3 Equivalence Ratio

Equivalence ratio is a measure of the proportion of fuel and air in a reactive mixture relative to its stoichiometric value.

Equivalence Ratio (by Mass):

$$\phi = \frac{(FA)_{actual}}{(FA)_{stoi}} = \frac{(AF)_{stoi}}{(AF)_{actual}} \quad (11.2.18)$$

Equivalence Ratio (by Volume):

$$\bar{\phi} = \frac{(\overline{FA})_{actual}}{(\overline{FA})_{stoi}} = \frac{(\overline{AF})_{stoi}}{(\overline{AF})_{actual}} \quad (11.2.19)$$

$\phi > 1.0 \rightarrow$  fuel-rich mixture/combustion

$\phi = 1.0 \rightarrow$  stoichiometric mixture/combustion

$\phi < 1.0 \rightarrow$  fuel-lean mixture/combustion

### 11.2.4 Actual Combustion Process

In an actual process:

- Combustion takes place in **tens or hundreds of elementary reactions**.
- Usually a **huge number of intermediate species** are generated.
- A combustion process is almost always **incomplete**.
- In addition to the usual products  $CO_2$  and  $H_2O$ , there are **harmful emissions** from combustion such as unburned  $HC$ ,  $CO$ ,  $NO_x$  and soot.
- The combustion process is strongly affected by fluid dynamics (in particular, **turbulence**) and **mixing**.

## 11.3 Open System Energy Analysis of Combustion Processes

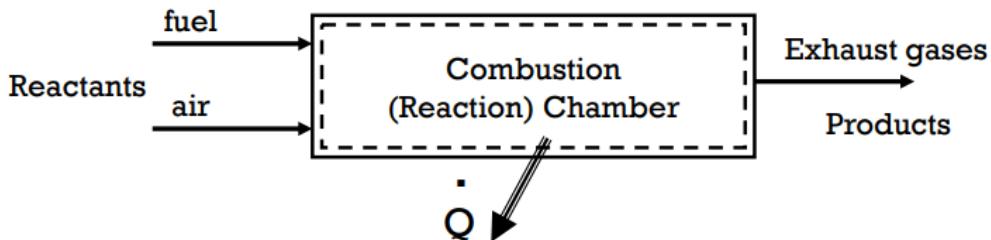


Figure 11.4:

Assumptions:

- SSSF (steady state simple fluid)
- $\Delta KE = \Delta PE = 0$
- No mechanical work

Steady State Energy Balance:

$$\frac{dE_{\text{tot}}}{dt} = \dot{Q} + \dot{W} + \sum_{in} \dot{m}_{in} \left( h + \frac{v^2}{2} + gz \right)_{in} - \sum_{out} \dot{m}_{out} \left( h + \frac{v^2}{2} + gz \right)_{out} \quad (11.3.1)$$

$$\dot{Q} = \sum_{out} \dot{m}_{out} h_{out} - \sum_{in} \dot{m}_{in} h_{in} \quad (11.3.2)$$

Combustion heat release (on a mass basis):

$$Q = \sum_P (m_i h_i) - \sum_R (m_i h_i) = H_P - H_R \quad (11.3.3)$$

Combustion heat release (on a molar basis):

$$Q = \sum_P (n_i \bar{h}_i) - \sum_R (n_i \bar{h}_i) = H_P - H_R \quad (11.3.4)$$

### 11.3.1 Energy Change in a Chemical Reaction

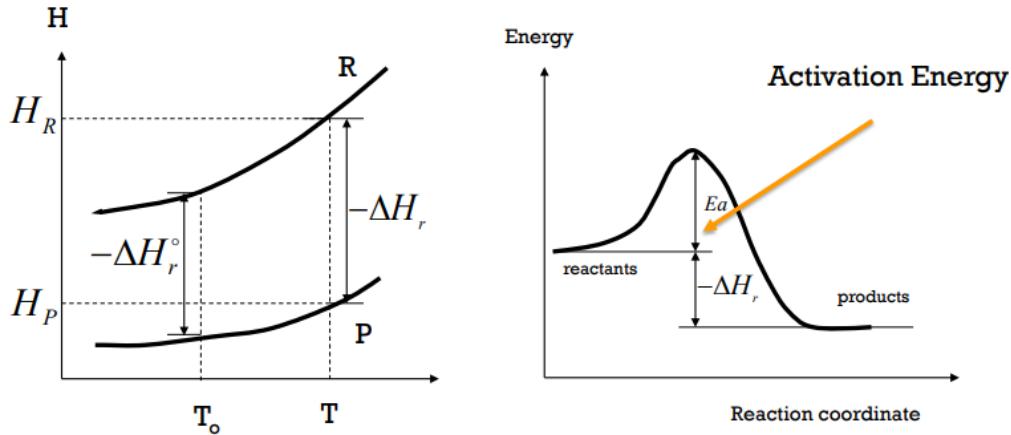


Figure 11.5:

$\Delta H_r^{\circ}$  is enthalpy change in combustion at standard reference temperature  $T_0$ .  
 $\Delta H_r$  is enthalpy change in combustion at an arbitrary temperature  $T$ .

## 11.4 Enthalpy of Formation

The **Enthalpy of Formation of a chemical compound** is the enthalpy change associated with the reaction of forming one mole of the compound from its elements in their standard states (gas or solid) at the standard reference conditions:

$$\bar{h}_{f,i}^{\circ} = \Delta \bar{h}_{f,i}^{\circ} = \Delta \bar{h}_i \text{ (298.15K; 1 atm)} \quad (11.4.1)$$

- In exothermic (or endothermic) reaction, the energy is released (or absorbed) when the compound is formed from its elements, so the enthalpy of formation is negative (or positive).
- By definition, the enthalpy of formation for any element (such as  $H_2$ ,  $O_2$ ) at its standard state is zero.

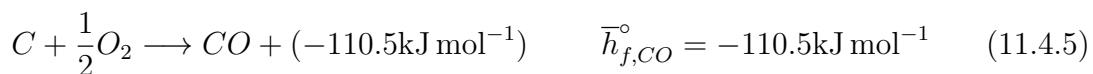
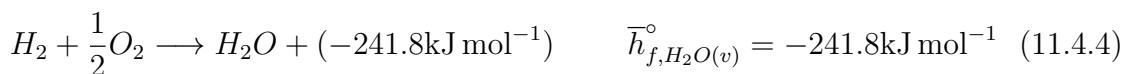
Thus, the enthalpy of any compound  $i$  at an arbitrary state becomes:

$$\bar{h}_i(T, p) = \bar{h}_{f,i}^{\circ} + [\bar{h}_i(T, p) - \bar{h}_i(T_{ref}, p_{ref})] \quad (11.4.2)$$

### 11.4.1 Enthalpy of Formation - Example

In the following reactions:





Substance	Formula	Molar Mass, <i>M</i> (kg/kmol)	Enthalpy of Formation, $\bar{h}_f^\circ$ (kJ/kmol)	Gibbs Function of Formation, $\bar{g}_f^\circ$ (kJ/kmol)	Absolute Entropy, $\bar{s}^\circ$ (kJ/kmol · K)	Heating Values	
						Higher, HHV (kJ/kg)	Lower, LHV (kJ/kg)
Carbon	C(s)	12.01	0	0	5.74	32,770	32,770
Hydrogen	H <sub>2</sub> (g)	2.016	0	0	130.57	141,780	119,950
Nitrogen	N <sub>2</sub> (g)	28.01	0	0	191.50	—	—
Oxygen	O <sub>2</sub> (g)	32.00	0	0	205.03	—	—
Carbon monoxide	CO(g)	28.01	-110,530	-137,150	197.54	—	—
Carbon dioxide	CO <sub>2</sub> (g)	44.01	-393,520	-394,380	213.69	—	—
Water	H <sub>2</sub> O(g)	18.02	-241,820	-228,590	188.72	—	—
Water	H <sub>2</sub> O(l)	18.02	-285,830	-237,180	69.95	—	—
Hydrogen peroxide	H <sub>2</sub> O <sub>2</sub> (g)	34.02	-136,310	-105,600	232.63	—	—
Ammonia	NH <sub>3</sub> (g)	17.03	-46,190	-16,590	192.33	—	—
Oxygen	O(g)	16.00	249,170	231,770	160.95	—	—
Hydrogen	H(g)	1.008	218,000	203,290	114.61	—	—
Nitrogen	N(g)	14.01	472,680	455,510	153.19	—	—
Hydroxyl	OH(g)	17.01	39,460	34,280	183.75	—	—
Methane	CH <sub>4</sub> (g)	16.04	-74,850	-50,790	186.16	55,510	50,020
Acetylene	C <sub>2</sub> H <sub>2</sub> (g)	26.04	226,730	209,170	200.85	49,910	48,220
Ethylene	C <sub>2</sub> H <sub>4</sub> (g)	28.05	52,280	68,120	219.83	50,300	47,160
Ethane	C <sub>2</sub> H <sub>6</sub> (g)	30.07	-84,680	-32,890	229.49	51,870	47,480
Propylene	C <sub>3</sub> H <sub>6</sub> (g)	42.08	20,410	62,720	266.94	48,920	45,780
Propane	C <sub>3</sub> H <sub>8</sub> (g)	44.09	-103,850	-23,490	269.91	50,350	46,360
Butane	C <sub>4</sub> H <sub>10</sub> (g)	58.12	-126,150	-15,710	310.03	49,500	45,720
Pentane	C <sub>5</sub> H <sub>12</sub> (g)	72.15	-146,440	-8,200	348.40	49,010	45,350
Octane	C <sub>8</sub> H <sub>18</sub> (g)	114.22	-208,450	17,320	463.67	48,260	44,790
Octane	C <sub>8</sub> H <sub>18</sub> (l)	114.22	-249,910	6,610	360.79	47,900	44,430
Benzene	C <sub>6</sub> H <sub>6</sub> (g)	78.11	82,930	129,660	269.20	42,270	40,580
Methyl alcohol	CH <sub>3</sub> OH(g)	32.04	-200,890	-162,140	239.70	23,850	21,110
Methyl alcohol	CH <sub>3</sub> OH(l)	32.04	-238,810	-166,290	126.80	22,670	19,920
Ethyl alcohol	C <sub>2</sub> H <sub>5</sub> OH(g)	46.07	-235,310	-168,570	282.59	30,590	27,720
Ethyl alcohol	C <sub>2</sub> H <sub>5</sub> OH(l)	46.07	-277,690	174,890	160.70	29,670	26,800

Figure 11.6: Table A-25 — Thermochemical Properties of Selected Substances at 298K and 1 atm

### 11.4.2 Evaluating Enthalpy of Ideal Gases

Enthalpy of any ideal gas *i* at temperature *T* (independent of *P*):

$$\bar{h}_i(T) = \bar{h}_{f,i}^\circ + [\bar{h}_i(T) - \bar{h}_i(T_{ref})] \quad (11.4.6)$$

$$\text{Absolute enthalpy} = \text{Enthalpy of formation} + \text{Sensible enthalpy} \quad (11.4.7)$$

- Enthalpy of formation from its elements (see Table A-25 in text book)
- Enthalpy change from reference temperature of 298.15 K

1. Evaluated from ideal gas Table A-22, A-23 in text book OR

2. By calculation:

$$\Delta\bar{h}_i(T) = \bar{h}_i(T) - \bar{h}_i(T_{ref}) = \int_{298.15K}^T \bar{C}_p \, dT \quad (11.4.8)$$

## 11.5 Closed System Energy Analysis of Combustion

### Closed Systems:

In the absence of kinetic and potential energy changes as well as mechanical work, the energy balance equation can be written as:

$$Q + \mathcal{W} = \Delta U_{CM} + \Delta KE_{CM} + \Delta PE_{CM} \quad (11.5.1)$$

$$Q = U_P - U_R \quad (11.5.2)$$

$$Q = \sum_P n_i \bar{u}_i - \sum_R n_i \bar{u}_i \quad (11.5.3)$$

$$Q = \sum_P n_i (\bar{h}_i - R_u T) - \sum_R n_i (\bar{h}_i - R_u T) \quad (11.5.4)$$

$$Q = \sum_P n_i (\bar{h}_{f,i}^\circ + \Delta\bar{h}_i - R_u T) - \sum_R n_i (\bar{h}_{f,i}^\circ + \Delta\bar{h}_i - R_u T) \quad (11.5.5)$$

## 11.6 Heat of Reaction

The **heat of reaction** (or **heat of combustion** or **chemical heat release**) is the energy released by the oxidation of a fuel under a specified (constant) temperature.

The heat of reaction is therefore measured by **keeping the temperature of the reactants and products the same**. It is also dependent on the process (or path) by which it is measured.

In a constant-volume process, the **heat of reaction** is the internal energy change, called **the internal energy of reaction**:

$$(Q_r)_{T,V} = (\Delta U_r)_{T,V} = U_P - U_R = \sum_P n_i \bar{u}_i(T) - \sum_R n_i \bar{u}_i(T) \quad (11.6.1)$$

$$= \sum_P n_i (\bar{h}_{f,i}^\circ + \Delta\bar{h}_i(T) - R_u T) - \sum_R n_i (\bar{h}_{f,i}^\circ + \Delta\bar{h}_i(T) - R_u T) \quad (11.6.2)$$

In a constant-pressure process, the **heat of reaction** is the enthalpy change, called **the enthalpy of reaction**.

$$(Q_r)_{T,P} = (\Delta H_r)_{T,P} = H_P - H_R = \sum_P n_i \bar{h}_i(T) - \sum_R n_i \bar{h}_i(T) \quad (11.6.3)$$

$$= \sum_P n_i (\bar{h}_{f,i}^\circ + \Delta \bar{h}_i(T)) - \sum_R n_i (\bar{h}_{f,i}^\circ + \Delta \bar{h}_i(T)) \quad (11.6.4)$$

$\Delta U_r, \Delta H_r < 0$  for exothermic reaction

$\Delta U_r, \Delta H_r > 0$  for endothermic reaction

### 11.6.1 Heat Release of Reaction between Arbitrary Temperatures

In reality, reactants and products are rarely at the same temperature, not to mention the reference temperature.

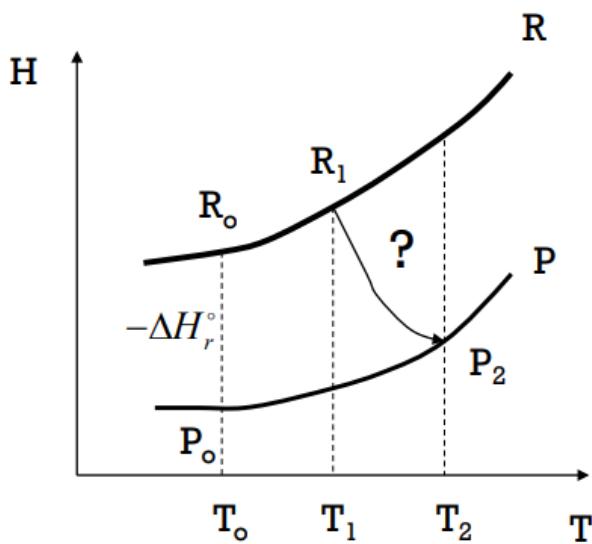


Figure 11.7:

### 11.6.2 Hess's Law

The enthalpy of reaction for a given reaction is the same whether it takes one or several stages to form the same product.

That is, the enthalpy of reaction is independent of the path of the reaction, but only dependent on the initial and final states.

For example:



is equilave to:



### 11.6.3 Heat Release of Reaction between Arbitrary Temperatures

For a reaction of reactants at  $T_1$  and products at  $T_2$ , the enthalpy of reaction can be determined as follows:

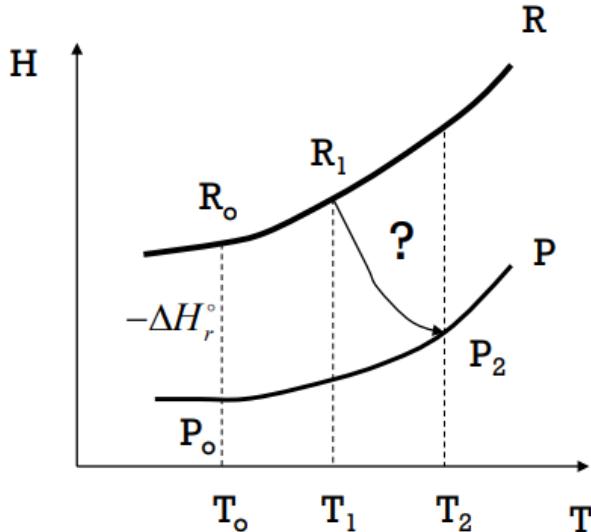


Figure 11.8:

$$(Q_r)_{T_1, T_2} = \Delta H_{T_1}^{T_2} = H_P(T_2) - H_R(T_1) \quad (11.6.8)$$

$$= H_P(T_2) - H_P(T_o) + H_P(T_o) - H_R(T_o) + H_R(T_o) - H_R(T_1) \quad (11.6.9)$$

On a mass basis:

$$= \sum_P \int_{T_o}^{T_2} m_i C_{p,i} dT + \Delta H_r^° + \sum_R \int_{T_1}^{T_o} m_i C_{p,i} dT \quad (11.6.10)$$

On a molar basis:

$$= \sum_P \int_{T_o}^T T n_i \bar{C}_{p,i} dT + \Delta \bar{H}_r^° + \sum_R \int_{T_1}^{T_o} n_i \bar{C}_{p,i} dT \quad (11.6.11)$$

The enthalpy of reaction at standard reference conditions:

$$\Delta H_r^° = \sum_P m_i h_{f,i}^° - \sum_R m_i h_{f,i}^° \quad (11.6.12)$$

$$\text{OR} \quad (11.6.13)$$

$$\Delta \bar{H}_r^° = \sum_P n_i \bar{h}_{f,i}^° - \sum_R n_i \bar{h}_{f,i}^° \quad (11.6.14)$$

### 11.6.4 State of Water in Combustion Products

Water may exist as liquid or vapour in combustion products, depending on pressure and temperature.

Relationship between vapour phase and liquid phase enthalpies is given by:

$$\left(\Delta h_f^\circ\right)_{H_2O,vapour} = \left(\Delta h_f^\circ\right)_{H_2O,liquid} + (\Delta h_{fg})_{H_2O} \quad (11.6.15)$$

$$\left(\Delta \bar{h}_f^\circ\right)_{H_2O,vapour} = \left(\Delta \bar{h}_f^\circ\right)_{H_2O,liquid} + (\Delta \bar{h}_{fg})_{H_2O} \quad (11.6.16)$$

Where  $(\Delta h_{fg})_{H_2O}$  and  $(\Delta \bar{h}_{fg})_{H_2O}$  are the enthalpy of evaporation in  $\text{J kg}^{-1}$  water and  $\text{J kmol}^{-1}$ , respectively.

Note that in text books,  $\Delta h_f^\circ$  and  $h_f^\circ$  are used interchangeably.

### 11.7 Heating Values of Fuels

The heating value of a fuel is the magnitude of heat of reaction of a unit composite fuel usually measured in a calorimeter under standard reference conditions (298.15K, 1 atm).

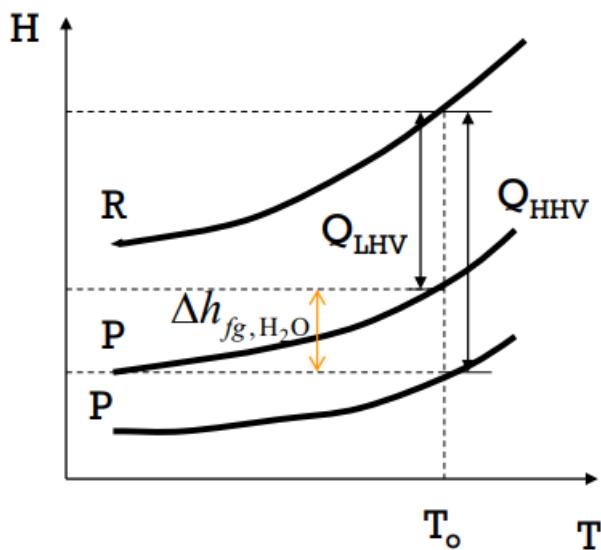


Figure 11.9:

1. The fuel composition is usually unknown;
2. The heating value is also called the calorific value;
3. HHV - higher heating value;
4. LHV – lower heating value;
5.  $h_{fg,H_2O}$  - enthalpy of evaporation;
6. For constant volume,  $Q_{HV} = |\Delta U_r|$
7. For constant pressure,  $Q_{HV} = |\Delta H_r|$

$$Q_{HHV} = Q_{LHV} + n_{H_2O} \Delta \bar{h}_{fg, H_2O} \longrightarrow (\text{J kmol}^{-1}) \quad (11.7.1)$$

$$Q_{HHV} = Q_{LHV} + m_{H_2O} \Delta h_{fg, H_2O} \longrightarrow (\text{J kg}^{-1}) \quad (11.7.2)$$

The values for HHV and LHV for specific substances can be found from Table A-25 — Thermochemical Properties of Selected Substances at 298K and 1 atm.

## 11.8 Adiabatic Flame Temperature

If all heat evolved during the reaction is used **solely** for raising the products temperature without heat loss, the final temperature  $T_2$  is called the adiabatic flame temperature  $T_F$ .

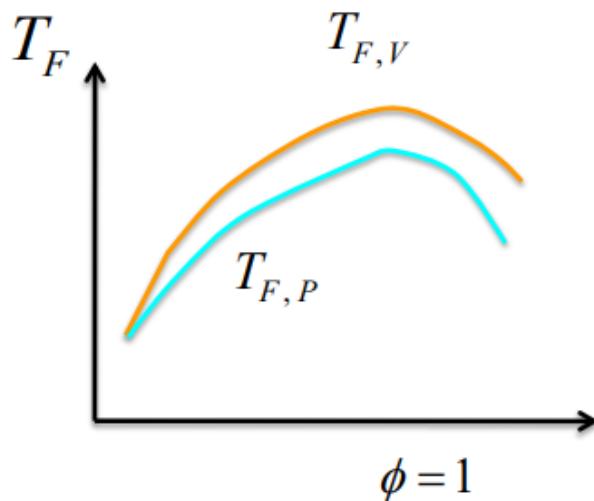


Figure 11.10:

The adiabatic flame temperature is obtained by setting:

$$\Delta \bar{H}_{T_1,r}^{T_2} = 0 \quad (11.8.1)$$

And solving the following equation iteratively,

$$\sum_P \int_{T_o}^{T_2} n_i \bar{C}_{p,i} dT + \sum_R \int_{T_1}^{T_o} n_i \bar{C}_{p,i} dT = -\Delta \bar{H}_r^\circ \quad (11.8.2)$$

$T_{F,P}$  is constant-pressure adiabatic flame temperature.

$T_{F,V}$  is constant-volume adiabatic flame temperature.

# Chapter 12

## Steam Turbines

02/03/2021

### 12.1 Overview of a Steam Power Plant

#### 12.1.1 Schematic of A Steam Power System

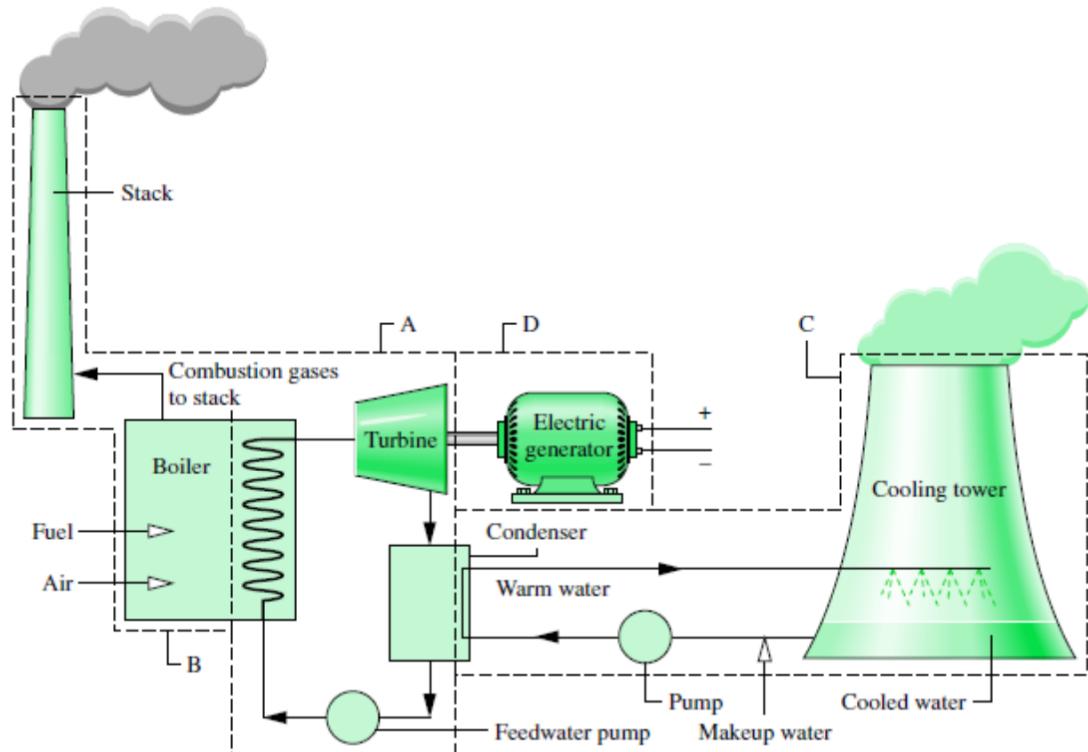


Figure 12.1:

## 12.1.2 Pressurised-water Reactor Nuclear Steam Power Plant

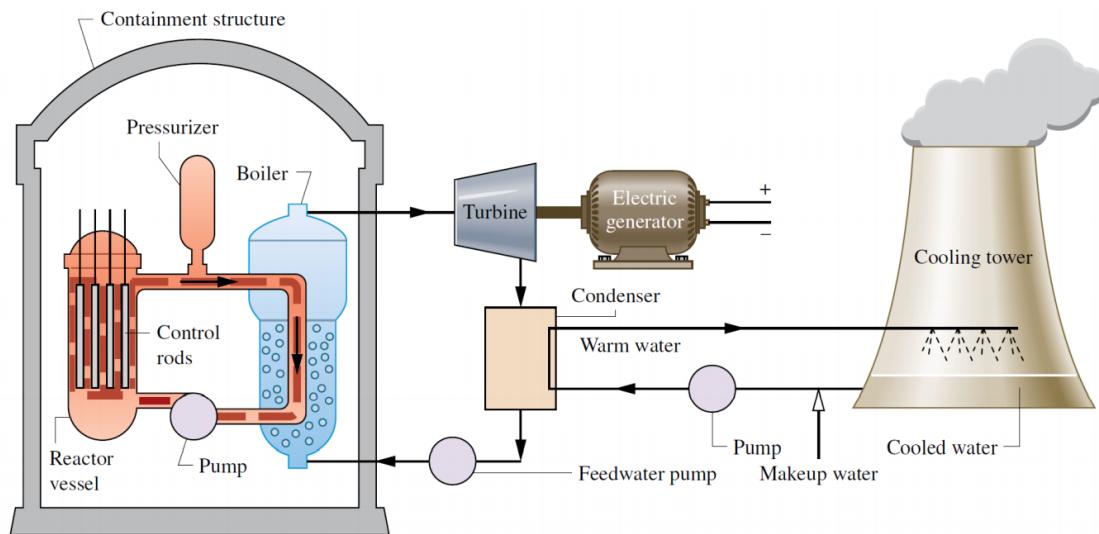


Figure 12.2:

## 12.1.3 Concentrating Solar Thermal Steam Power Plant

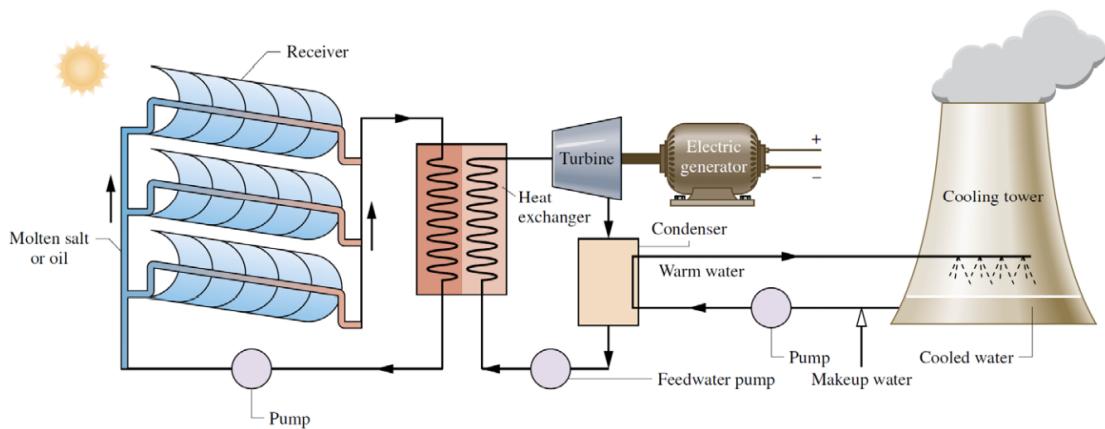


Figure 12.3:

### 12.1.4 Geothermal Vapour Power Plant

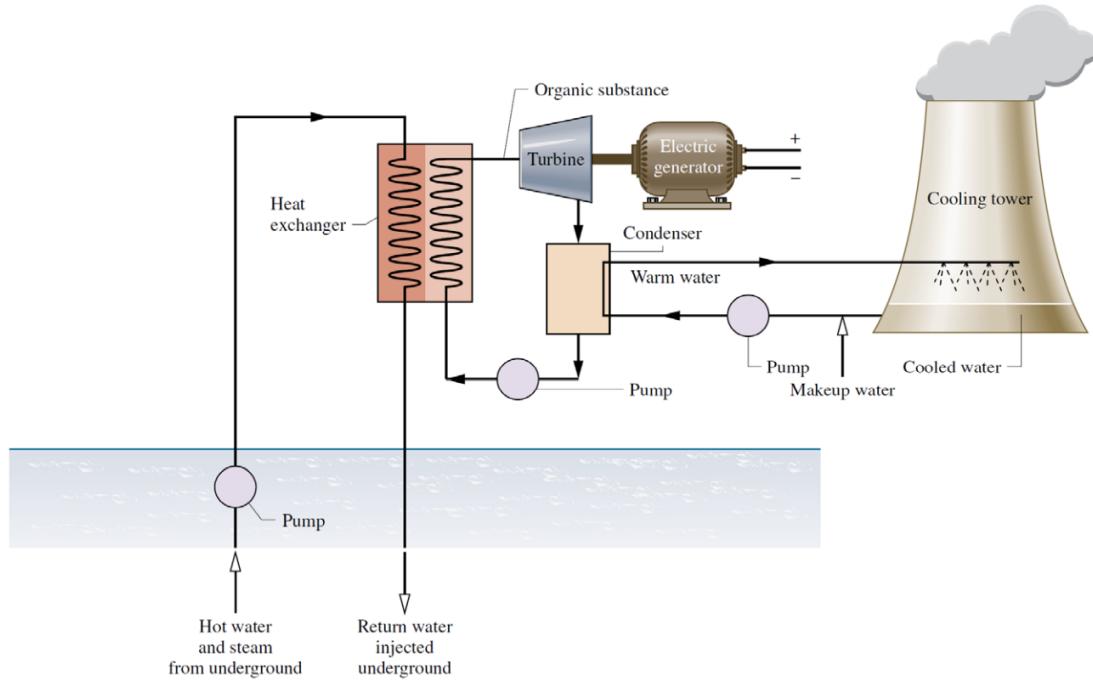


Figure 12.4:

### 12.1.5 Main Components of a Steam Power System

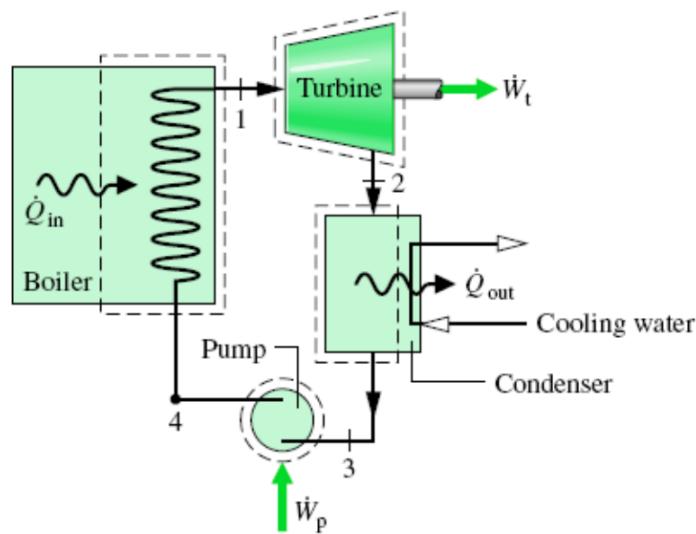


Figure 12.5:

- Turbine
- Condenser
- Pump
- Boiler (Steam Generator)

### 12.1.6 Work and Heat Transfer

Turbine:

$$0 = \dot{Q}_{CV} - \dot{W}_t + \dot{m} \left[ h_1 - h_2 + \frac{V_1^2 - V_2^2}{2} + g(z_1 - z_2) \right] \quad (12.1.1)$$

$$\frac{\dot{W}_t}{\dot{m}} = h_1 - h_2 \quad (12.1.2)$$

Condenser:

$$\frac{\dot{Q}_{out}}{\dot{m}} = h_2 - h_3 \quad (12.1.3)$$

Pump:

$$\frac{\dot{W}_p}{\dot{m}} = h_4 - h_3 \quad (12.1.4)$$

Boiler:

$$\frac{\dot{Q}_{in}}{\dot{m}} = h_1 - h_4 \quad (12.1.5)$$

## 12.2 The Carnot Vapour Power Cycle

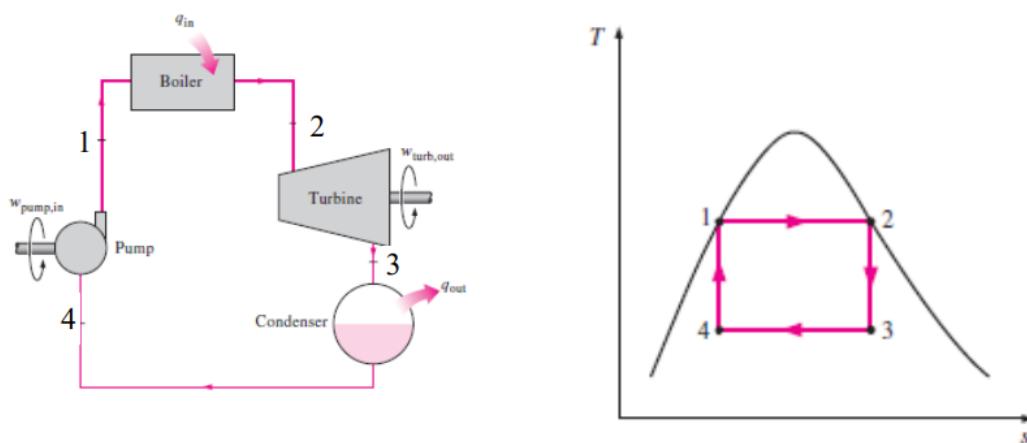


Figure 12.6:

- **Process 1 → 2:** The working fluid is heated reversibly and isothermally in a boiler
- **Process 2 → 3:** then expanded isentropically in a turbine
- **Process 3 → 4:** subsequently condensed reversibly and isothermally in a condenser

- Process 4 → 1: and finally compressed isentropically by a compressor to the initial state

### 12.2.1 Problems with the Carnot Vapour Power Cycle

The quality of the steam decreases during **isentropic expansion process (process 2-3)**, leading to steam with a high moisture content. The impingement of **liquid droplets** on the turbine blades causes **erosion** and is a major source of wear. Usually, steam with quality less than about **90 percent** cannot be tolerated in the operation of power plants.

The **isentropic compression process (process 4-1)** involves the compression of a liquid-vapor mixture to a saturated liquid. Problems:

- It is **not easy to control the condensation process** so precisely as to end up with the desired quality at state 4.
- It is **not practical to design a compressor that handles two phases**.

Thus, the idealised Carnot Vapour Cycle is not practical!

## 12.3 Rankine Vapour Power Cycle

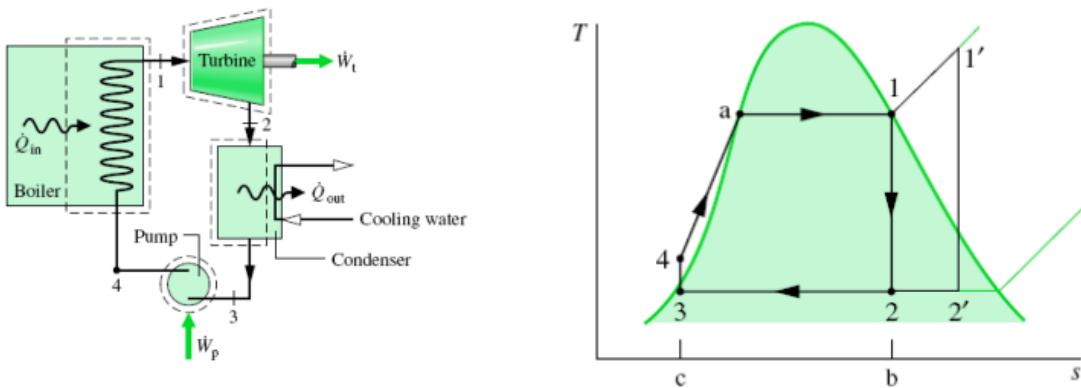


Figure 12.7:

- **Process 1 → 2 or (1' → 2')**: Isentropic expansion of the working fluid (**Saturated or Superheated Vapor**) through the turbine to the condenser pressure.
- **Process 2 → 3 (2' → 3')**: Heat transfer from the working fluid as it flows at constant pressure through the condenser with **saturated liquid at state 3**.
- **Process 3 → 4**: Isentropic compression in the pump to state 4 in the compressed liquid region.
- **Process 4 → 1 (4' → 1')**: Heat transfer to the working fluid as it flows at constant pressure through the boiler to complete the cycle.

Area 1-b-c-4-a-1 represents the heat transfer to the working fluid passing through the boiler:

$$q_{in} = (h_1 - h_4) \quad (12.3.1)$$

Area 2–b–c–3–2, is the heat transfer from the working fluid passing through the condenser, each per unit of mass flowing:

$$q_{out} = (h_2 - h_3) \quad (12.3.2)$$

The enclosed area 1–2–3–4–a–1 can be interpreted as the net heat input or, equivalently, the net work output, per unit mass of the working fluid:

$$w_{net} = (h_1 - h_4) - (h_2 - h_3) \quad (12.3.3)$$

### 12.3.1 Key Parameters of Ideal Rankine Cycle:

#### Thermal Efficiency:

$$\eta = \frac{\frac{\dot{W}_t}{\dot{m}} - \frac{\dot{W}_p}{\dot{m}}}{\frac{\dot{Q}_{in}}{\dot{m}}} = \frac{(h_1 - h_2) - (h_4 - h_3)}{h_1 - h_4} \quad (12.3.4)$$

$$\eta = \frac{\frac{\dot{Q}_{in}}{\dot{m}} - \frac{\dot{Q}_{out}}{\dot{m}}}{\frac{\dot{Q}_{in}}{\dot{m}}} = 1 - \frac{\frac{\dot{Q}_{out}}{\dot{m}}}{\frac{\dot{Q}_{in}}{\dot{m}}} \quad (12.3.5)$$

$$= 1 - \frac{(h_2 - h_3)}{(h_1 - h_4)} \quad (12.3.6)$$

$$\text{Neglect Pump Power:} \quad (12.3.7)$$

$$\eta = \frac{h_1 - h_2}{h_1 - h_3} \quad (12.3.8)$$

#### The Back Work Ratio:

$$bwr = \frac{\frac{\dot{W}_p}{\dot{m}}}{\frac{\dot{W}_t}{\dot{m}}} = \frac{(h_4 - h_3)}{(h_1 - h_2)} \quad (12.3.9)$$

#### Pump Power:

$$\left( \frac{\dot{W}_p}{\dot{m}} \right)_{rev} = \int_3^4 v \, dp \quad (12.3.10)$$

$$\longrightarrow \text{Incompressible Assumption:} \quad (12.3.11)$$

$$\left( \frac{\dot{W}_p}{\dot{m}} \right)_{rev} \approx v_3(p_4 - p_3) \quad (12.3.12)$$

### 12.3.2 The Effect of Temperature on Thermal Efficiency

Heat input (internally reversible process):

$$\left( \frac{\dot{Q}_{in}}{\dot{m}} \right)_{rev}^{int} = \int_4^1 T \, ds = \text{area 1-b-c-4-a-1} \quad (12.3.13)$$

Average high temperature:

$$\left( \frac{\dot{Q}_{in}}{\dot{m}} \right)_{rev}^{int} = \bar{T}_{in}(s_1 - s_4) \quad (12.3.14)$$

Heat output (internally reversible process):

$$\left( \frac{\dot{Q}_{out}}{\dot{m}} \right)_{rev}^{int} = T_{out}(s_2 - s_3) = \text{area 2-b-c-3-2} \quad (12.3.15)$$

Low temperature:

$$\left( \frac{\dot{Q}_{out}}{\dot{m}} \right)_{rev}^{int} = T_{out}(s_1 - s_4) \quad (12.3.16)$$

For internally reversible cycle:

$$\eta_{ideal} = 1 - \frac{\left( \frac{\dot{Q}_{out}}{\dot{m}} \right)_{intrev}}{\left( \frac{\dot{Q}_{in}}{\dot{m}} \right)_{intrev}} = 1 - \frac{T_{out}}{\bar{T}_{in}} \quad (12.3.17)$$

### 12.3.3 Increasing the Efficiency of the Rankine Cycle

**Lowering the Condenser Pressure (Lowering the Low temperature):**

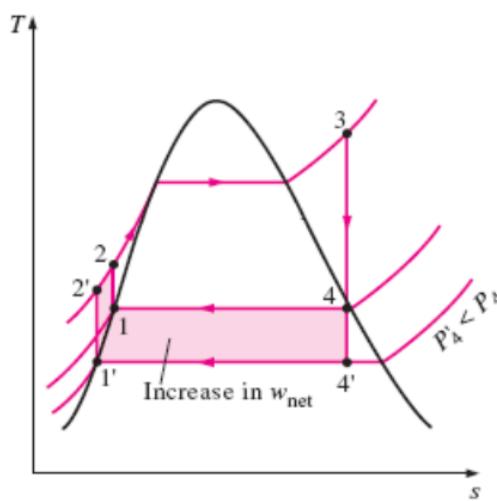


Figure 12.8:

The colored area on this diagram represents the increase in net work output as a result of lowering the condenser pressure.

The heat input requirements also increase but this increase is very small.

**The overall effect of lowering the condenser pressure is an increase in the thermal efficiency of the cycle.**

**The condenser pressure cannot be lower than the saturation pressure corresponding to the ambient temperature.**

### Superheating the Steam to Higher Temperatures:

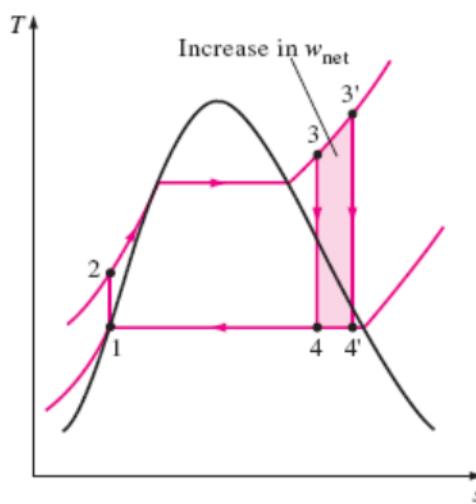


Figure 12.9:

Both the net work and heat input increase, however, the overall effect is an increase in thermal efficiency since the **average high temperature** increases.

Another desirable effect: It decreases the moisture content of the steam at the turbine exit.

The temperature to which steam can be superheated is limited by metallurgical considerations:

- Presently the highest steam temperature allowed at the turbine inlet is about **620C**
- Any increase in this value depends on improving the materials used or finding new ones that can withstand higher temperatures
- **Ceramics** are very promising in this regard

## Increasing the Boiler Pressure:

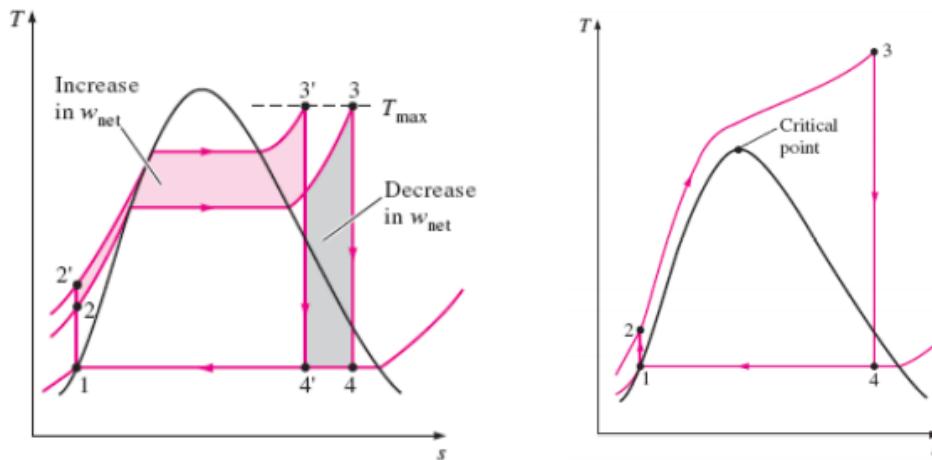


Figure 12.10: A supercritical Rankine cycle

Raises the average temperature at which heat is transferred to the steam and thus raises the thermal efficiency of the cycle.

Notice that for a fixed turbine inlet temperature, the cycle shifts to the left and **the moisture content of steam at the turbine exit increases**. This undesirable side effect can be corrected, however, by reheating the steam, as discussed in the next section.

Today advanced modern steam power plants operate at **supercritical pressures (up to 30 MPa)** and have thermal efficiencies of up to **47 percent** for fossil fuel plants and **35 percent** for nuclear plants.

### 12.3.4 The Ideal Reheat Rankine Cycle

Superheat the steam to very high temperatures before it enters the turbine. Expand the steam in the turbine in two stages, and reheat it in between.

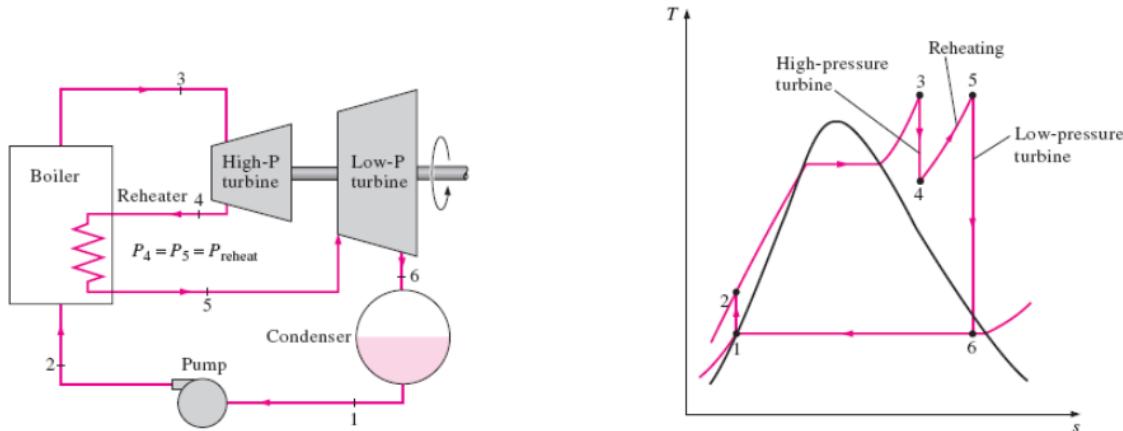


Figure 12.11:

Total heat input:

$$q_{in} = q_{primary} + q_{reheat} = (h_3 - h_2) + (h_5 - h_4) \quad (12.3.18)$$

Total turbine work output:

$$w_{turb,out} = w_{turb,I} + w_{turb,II} = (h_3 - h_4) + (h_5 - h_6) \quad (12.3.19)$$

The incorporation of a single reheat stage in a modern power plant improves the cycle efficiency by 4 to 5 percent. The average temperature during the reheat process can be increased by increasing the number of expansion and reheat stages:

- The effect of performance improvement decreases with the increasing number of the reheat
- Normally, 1-2 stages of reheat are used in practice.

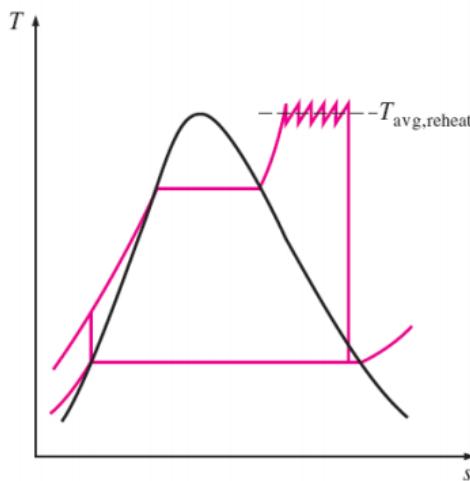


Figure 12.12:

### 12.3.5 Principal Irreversibilities and Losses

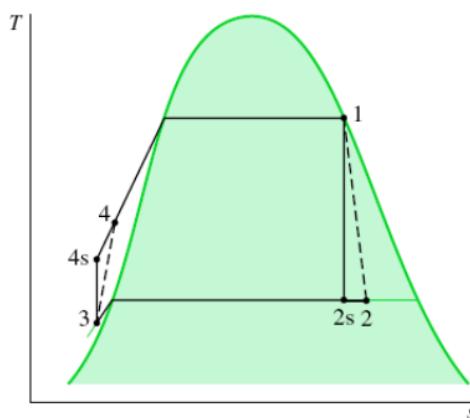


Figure 12.13:

**Turbine:**

Heat transfer from the turbine to the surroundings represents a loss, but since it is usually of secondary importance, it often can be neglected.

The principal irreversibility experienced by the working fluid is associated with the expansion through the turbine. The isentropic turbine efficiency allows the effect of irreversibilities within the turbine to be accounted for in terms of the actual and isentropic work amounts:

$$\eta_t = \frac{\left(\frac{\dot{W}_t}{\dot{m}}\right)}{\left(\frac{\dot{W}_t}{\dot{m}}\right)_s} = \frac{h_1 - h_2}{h_1 - h_{2s}} \quad (12.3.20)$$

**Pump:**

The work input to the pump required to overcome frictional effects also reduces the net power output of the plant. The isentropic pump efficiency is:

$$\eta_p = \frac{\left(\frac{\dot{W}_p}{\dot{m}}\right)_s}{\left(\frac{\dot{W}_p}{\dot{m}}\right)} = \frac{h_{4s} - h_3}{h_4 - h_3} \quad (12.3.21)$$

**Other Losses:**

The combustion of the fuel and the subsequent heat transfer from the hot combustion products to the cycle working fluid (**external irreversibilities**).

The energy discharge to the cooling water as the working fluid condenses. Although considerable energy is carried away by the cooling water, its **utility is extremely limited because its temperature is low**.

**Heat transfer from the outside surfaces** of the plant components has detrimental effects on performance.

Frictional effects resulting in **pressure drops** are sources of **internal irreversibilities** as the working fluid flows through the boiler, condenser, and piping connecting the various components

## 12.4 Regenerative Vapour Power Cycle

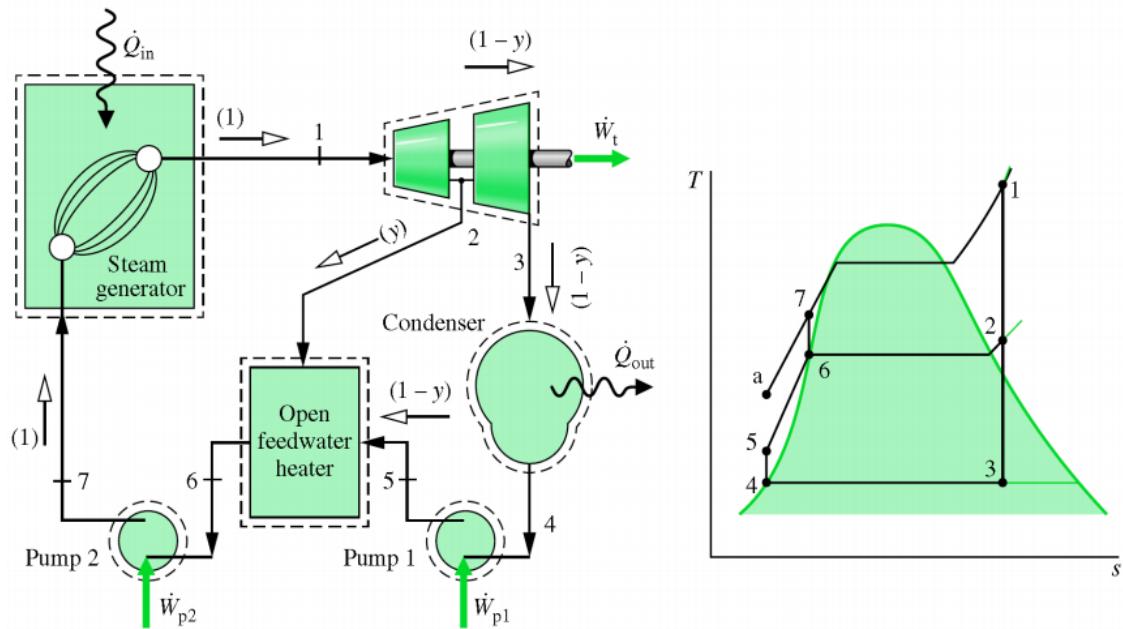


Figure 12.14: Regenerative vapour power cycle with one open feedwater heater

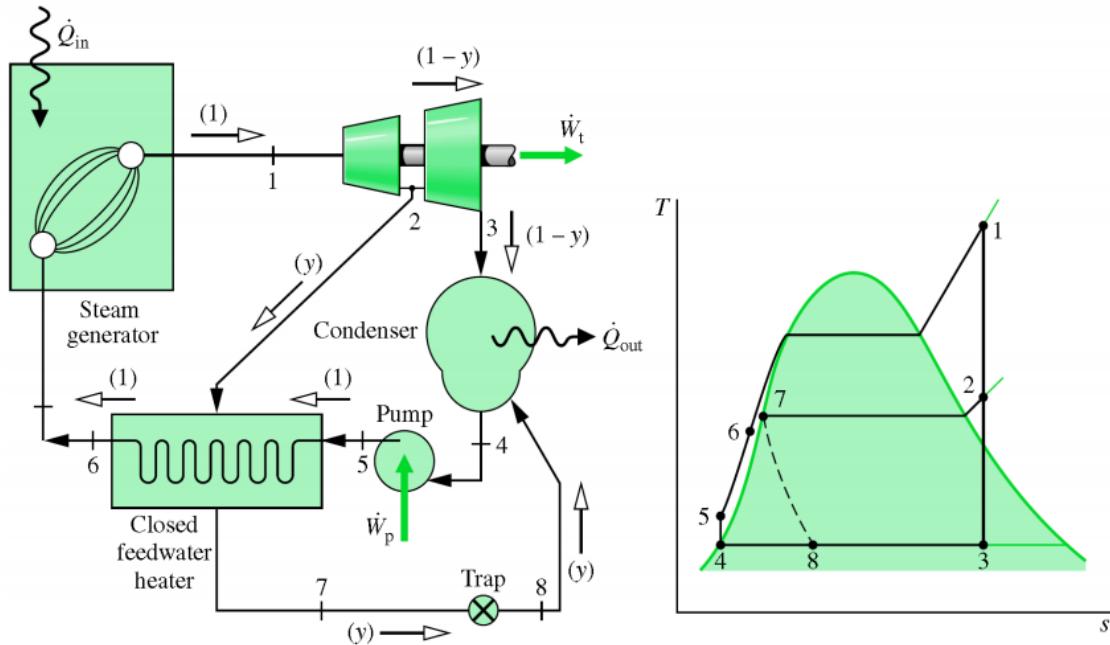


Figure 12.15: Regenerative vapour power cycle with one closed feedwater heater

# Chapter 13

## Gas Turbines

09/03/2021

### 13.1 Overview of Gas Turbine Systems

Applications of Gas Turbines:

- Aircraft propulsion
- Marine propulsion
- Stationary power plants for electricity generation

#### 13.1.1 Schematic of a Gas Turbine Aeroengine

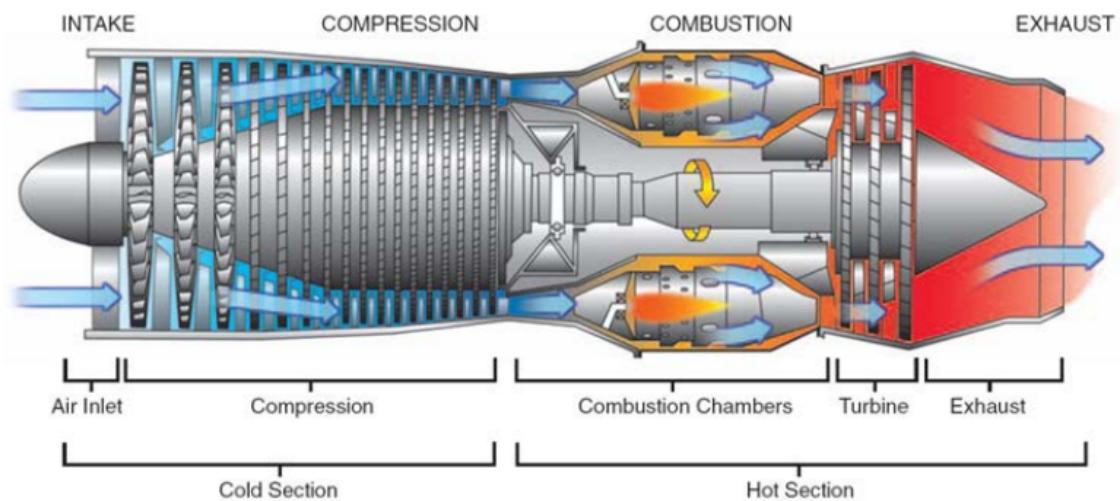


Figure 13.1:

### 13.2 Ideal Air-Standard Joule-Brayton Cycle

Assumptions:

- The working fluid is air, which is assumed to be an ideal gas.

- The temperature rise that would be brought about by combustion is accomplished by a heat transfer from an external source.
- Regard the turbine exhaust air as restored to the compressor inlet state by passing through a heat exchanger

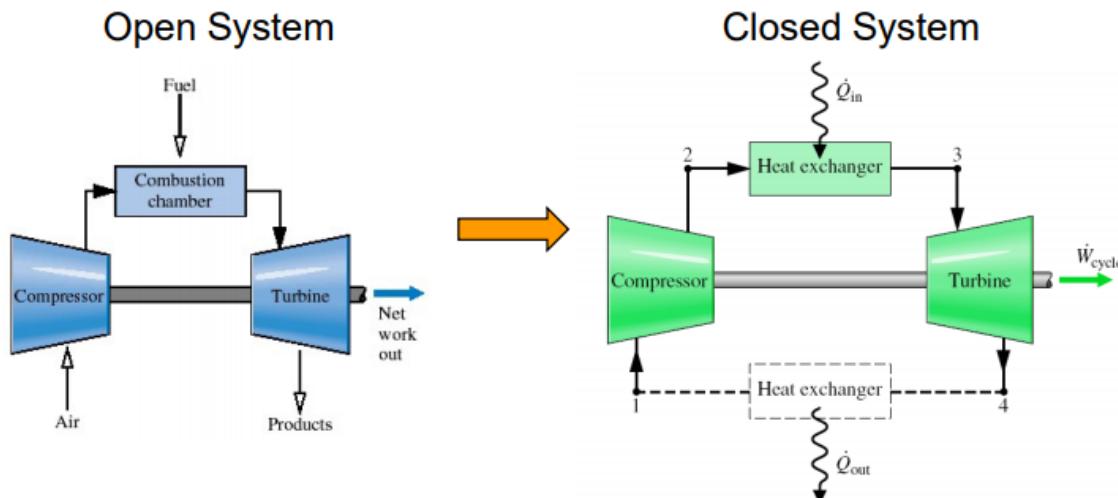


Figure 13.2:

Ignoring irreversibilities in the various component:

- No frictional pressure drops through heat exchangers.
- No heat transfers to the surroundings in heat exchangers.
- Isentropic expansion and compression in turbine and compressor.

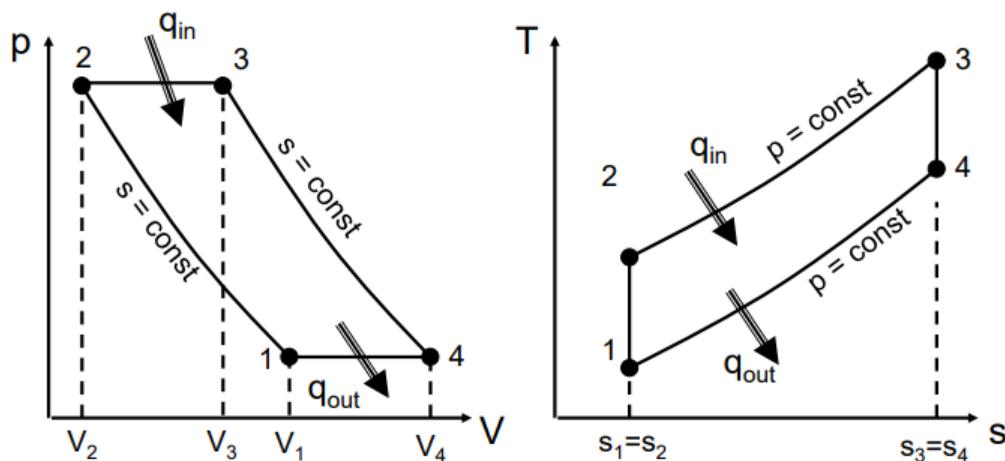


Figure 13.3:

### 13.2.1 Brayton Cycle

Comparison with vapour power cycle:

- **High power output-to-weight ratio** → lighter and more compact

- **Lower pressure ratios**, higher volume based on gas vs. liquid compression (Liquid is difficult to be compressed)
- **Lower efficiencies** based on non-isothermal heat addition and heat rejection

Comparison with Otto and Diesel cycles:

- 4 → 1 : constant pressure vs. constant volume process (both Otto and Diesel cycles)
- 2 → 3 : constant pressure vs. constant volume process (Otto cycle only)

In general, pressure rise much smaller for turbo-machinery than for reciprocating devices

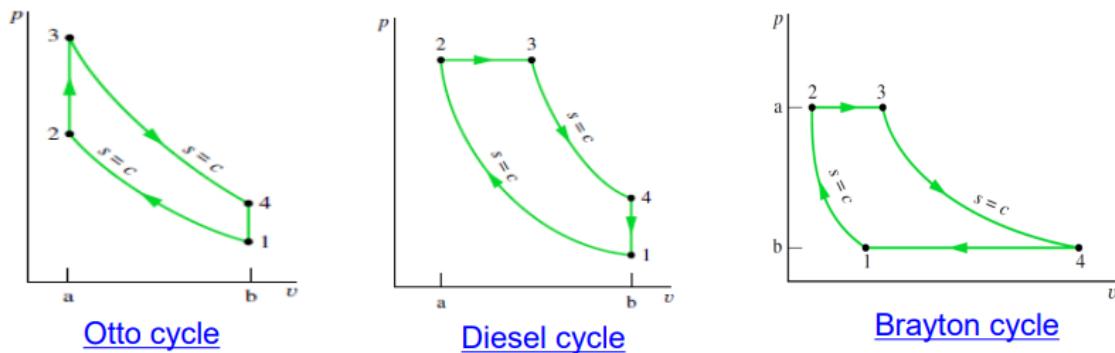


Figure 13.4:

### 13.2.2 First Law Analysis of Brayton Cycle

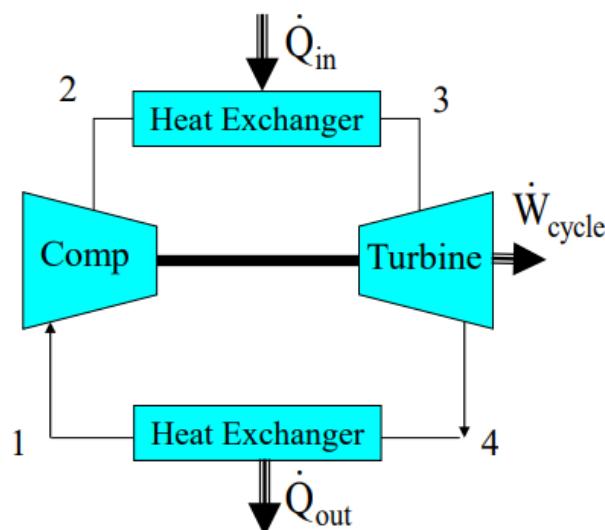


Figure 13.5:

#### 1st Law, Energy Balance for Each Component:

Work (Turbine and Compressor):

$$\frac{\dot{W}_T}{\dot{m}} = h_3 - h_4 \quad (13.2.1)$$

$$\frac{\dot{W}_C}{\dot{m}} = h_2 - h_1 \quad (13.2.2)$$

Heat Transfer:

$$\frac{\dot{Q}_{in}}{\dot{m}} = h_3 - h_2 \quad (13.2.3)$$

$$\frac{\dot{Q}_{out}}{\dot{m}} = h_4 - h_1 \quad (13.2.4)$$

Back Work Ratio:

$$r_{BW} = \frac{\dot{W}_C}{\dot{W}_T} = \frac{h_2 - h_1}{h_3 - h_4} \quad (13.2.5)$$

- For gas turbine, typically 40–80%
- For vapour cycle, 1–2%

**Thermal Efficiency:**

$$\eta_{th} = \frac{w_{net}}{q_{in}} = \frac{w_T - w_C}{q_{in}} = \frac{q_{in} - |q_{out}|}{q_{in}} = 1 - \frac{|q_{out}|}{q_{in}} \quad (13.2.6)$$

For closed cycle, SSSF (Steady-State, Steady-Flow), and  $\Delta KE = \Delta PE = 0$

$$\eta_{th} = 1 - \frac{|q_{out}|}{q_{in}} = 1 - \frac{h_4 - h_1}{h_3 - h_2} \quad (13.2.7)$$

For constant specific heats (**cold air-standard analysis**):

$$\eta_{th} = 1 - \frac{c_p(T_4 - T_1)}{c_p(T_3 - T_2)} = 1 - \frac{T_1}{T_2} \cdot \frac{\left(\frac{T_4}{T_1} - 1\right)}{\left(\frac{T_3}{T_2} - 1\right)} \quad (13.2.8)$$

For an **isentropic process**:

$$pv^k = \text{constant} \quad (13.2.9)$$

$$k = \frac{c_p}{c_v} \quad (13.2.10)$$

$$\text{Process } 1 \longrightarrow 2 : p_1 v_1^k = p_2 v_2^k \quad (13.2.11)$$

$$\longrightarrow \frac{v_1^k}{v_2^k} = \frac{p_2}{p_1} = \frac{\left(\frac{RT_1}{p_1}\right)^k}{\left(\frac{RT_2}{p_2}\right)^k} = \left(\frac{T_1}{T_2} \frac{p_2}{p_1}\right)^k \quad (13.2.12)$$

$$\longrightarrow \left(\frac{p_2}{p_1}\right)^{1/k} = \frac{T_1}{T_2} \frac{p_2}{p_1} \quad (13.2.13)$$

$$\rightarrow \frac{T_2}{T_1} = \left( \frac{p_2}{p_1} \right)^{k/k} \cdot \left( \frac{p_2}{p_1} \right)^{-1/k} = \left( \frac{p_2}{p_1} \right)^{\frac{k-1}{k}} \quad (13.2.14)$$

$$\frac{T_4}{T_3} = \left( \frac{p_4}{p_3} \right)^{\frac{k-1}{k}} = \left( \frac{p_1}{p_2} \right)^{\frac{k-1}{k}} = \frac{T_1}{T_2} \quad (13.2.15)$$

$$\therefore \frac{T_4}{T_1} = \frac{T_3}{T_2} \quad (13.2.16)$$

Therefore, going back to Equation (??):

$$\eta_{th} = 1 - \frac{T_1}{T_2} \cdot \frac{\left( \frac{T_4}{T_1} - 1 \right)}{\left( \frac{T_3}{T_2} - 1 \right)} = 1 - \frac{T_1}{T_2} \quad (13.2.17)$$

$$\frac{T_2}{T_1} = \left( \frac{p_2}{p_1} \right)^{\frac{k-1}{k}} \quad (13.2.18)$$

Define temperature ratio:

$$\tau = \frac{T_3}{T_1} \quad (13.2.19)$$

Define pressure ratio:

$$r_p = \frac{p_2}{p_1} \quad (13.2.20)$$

$$\therefore \eta_{th} = 1 - \frac{1}{r_p^{\frac{k-1}{k}}} \quad (13.2.21)$$

### 13.2.3 Effects of Pressure Ratio on Ideal Cold Air-standard Brayton Cycle

- If  $r_p \uparrow$ , then  $\eta_{th} \uparrow$
- If  $r_p$  is fixed,  $q_{in} \uparrow$ , then  $w_{net} \uparrow$  but  $\eta_{th}$  is not changed

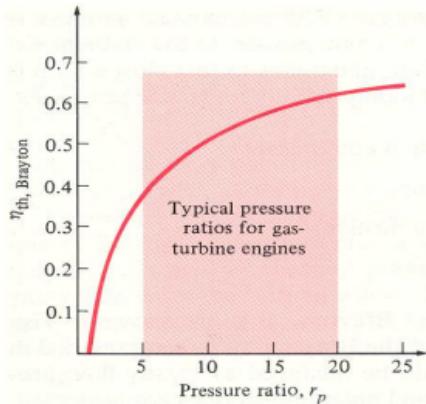


Figure 13.6:

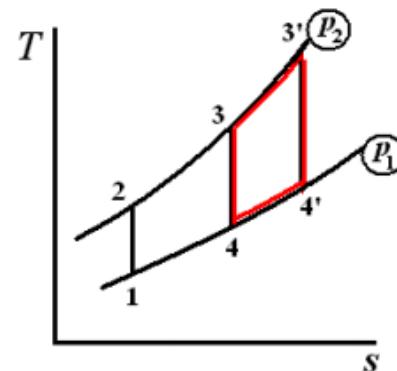


Figure 13.7:

In practice, work output and efficiency are both important

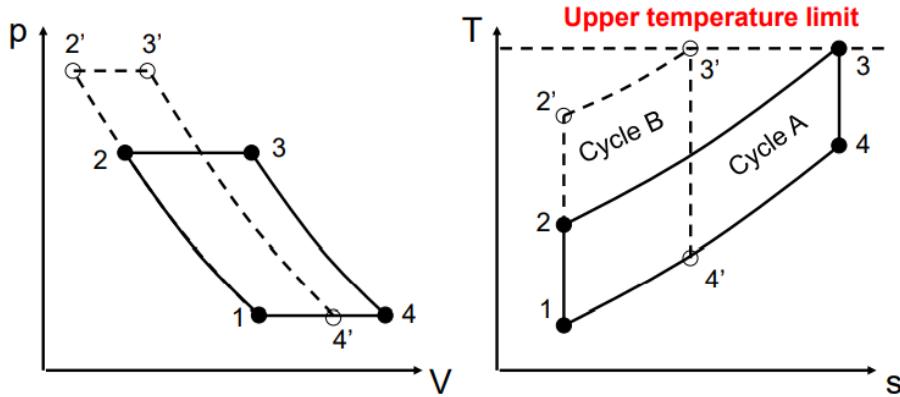


Figure 13.8:

### 13.2.4 Maximize Work Output with Fixed Temperature Ratio

For given  $T_1$  and  $T_3$  (or temperature ratio), find:

- The optimal  $T_2$ , or
- The optimal pressure ratio  $r_p$  that maximizes work output

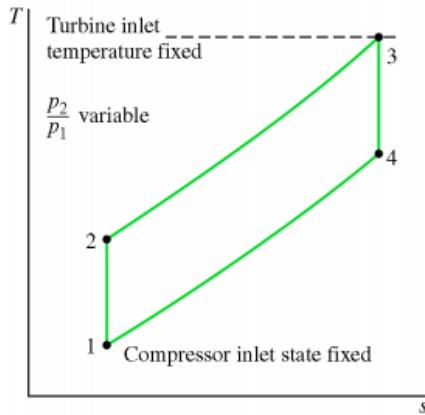


Figure 13.9:

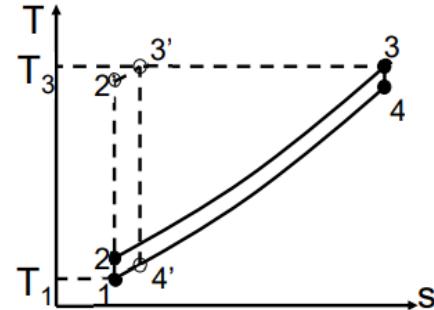


Figure 13.10:

Limiting Cases:

1.  $T_2 = T_3 \rightarrow w_{net} = 0, \eta_{th} = \eta_{max}$
2.  $T_2 = T_1 \rightarrow w_{net} = 0, \eta_{th} = 0$

**Find Optimal  $T_2$**

$$w_{net} = w_T - |w_C| = q_{in} - |q_{out}| \quad (13.2.22)$$

For constant specific heats:

$$w_{net} = c_p(T_3 - T_2) - c_p(T_4 - T_1) \quad (13.2.23)$$

For maximum  $w_{net}$ :

$$\frac{dw_{net}}{dT_2} = 0 = c_p \left( \frac{dT_3}{dT_2} - 1 - \frac{dT_4}{dT_2} + \frac{dT_1}{dT_2} \right) \quad (13.2.24)$$

Recalling:

$$\frac{T_4}{T_1} = \frac{T_3}{T_2} \longrightarrow T_4 = \frac{T_1 T_3}{T_2} \longrightarrow \frac{dT_4}{dT_2} = -\frac{T_1 T_3}{T_2^2} \quad (13.2.25)$$

Hence:

$$\frac{w_{net}}{dT_2} = 0 = -1 - \frac{dT_4}{dT_2} = -1 + \frac{T_1 T_3}{T_2^2} \quad (13.2.26)$$

$$\therefore T_2 = \sqrt{T_1 T_3} \quad (13.2.27)$$

**Find Optimal  $r_p$**

$$w_{net} = q_{in} - q_{out} = cp \left[ (T_3 - T_2) - (T_4 - T_1) \right] \quad (13.2.28)$$

$$= c_p T_1 \left[ \left( \frac{T_3}{T_1} - \frac{T_2}{T_1} \right) - \left( \frac{T_4}{T_1} - 1 \right) \right] \quad (13.2.29)$$

$$= c_p T_1 \left[ \left( \frac{T_3}{T_1} - \frac{T_2}{T_1} \right) - \left( \frac{T_3}{T_2} - 1 \right) \right] \quad (13.2.30)$$

$$= c_p T_1 \left[ \left( \frac{T_3}{T_1} - \frac{T_2}{T_1} \right) - \left( \frac{T_3 T_1}{T_1 T_2} - 1 \right) \right] \quad (13.2.31)$$

$$\tau = \frac{T_3}{T_1} \quad (13.2.32)$$

$$= c_p T_1 \left[ \tau - r_p^{\frac{k-1}{k}} - \left( \tau r_p^{-\frac{k-1}{k}} - 1 \right) \right] \quad (13.2.33)$$

Let:

$$\frac{\partial w_{net}}{\partial r_p} = 0 \quad (13.2.34)$$

Hence:

$$r_p = \tau^{\frac{k}{2(k-1)}} \implies w_{net} \rightarrow w_{net,max} \quad (13.2.35)$$

$$w_{net} = c_p T_1 \left( r_p^{\frac{k-1}{k}} - 1 \right) \left( \frac{\tau}{r_p^{\frac{k-1}{k}}} - 1 \right) \quad (13.2.36)$$

For each temperature ratio, there is an optimum pressure ratio to achieve the maximum work output.

The larger the temperature ratio, the larger the optimum pressure ratio, i.e., the larger the thermal efficiency when the maximum output is acquired.

$T_3$  is a key parameter in practice

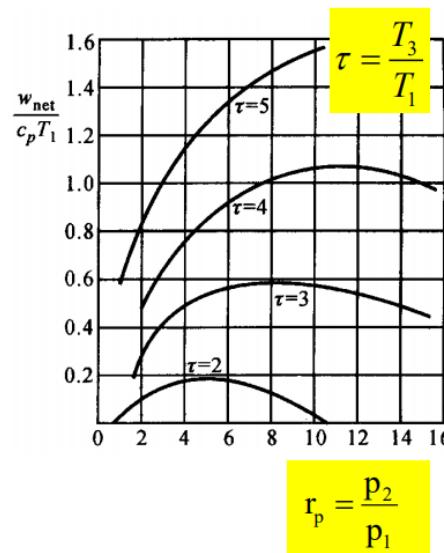


Figure 13.11:

### 13.3 Brayton Cycles with Irreversibilities

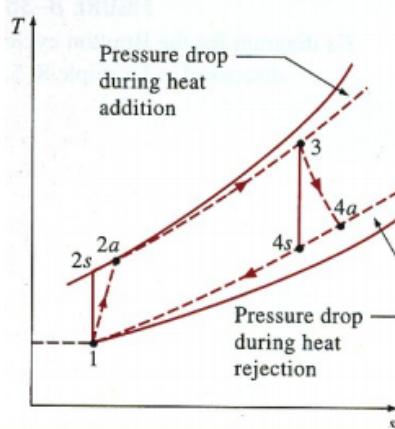


Figure 13.12:

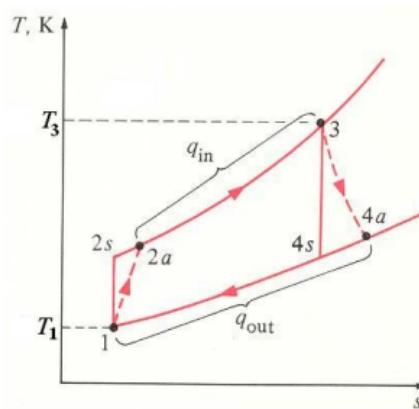


Figure 13.13:

Deviation of actual gas-turbine cycles from idealized ones:

$$\eta_C = \frac{w_s}{w_a} = \frac{h_{2s} - h_1}{h_{2a} - h_1} \quad (13.3.1)$$

$$\eta_T = \frac{w_a}{w_s} = \frac{h_3 - h_{4a}}{h_3 - h_{4s}} \quad (13.3.2)$$

Note:

- $T_2 > T_{2s}$  if  $p_2 = p_{2s}$
- $T_4 > T_{4s}$  if  $p_4 = p_{4s}$

Irreversibilities convert mechanical energy to thermal loss through friction.

### 1st Law, Energy Balance for Each Component:

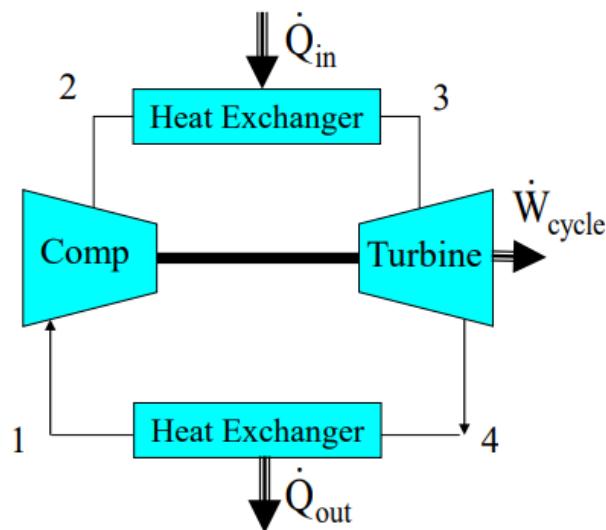


Figure 13.14:

Work (Turbine and Compressor):

$$\frac{\dot{W}_T}{\dot{m}} = h_3 - h_{4a} \quad (13.3.3)$$

$$\frac{\dot{W}_C}{\dot{m}} = h_{2a} - h_1 \quad (13.3.4)$$

Heat Transfer:

$$\frac{\dot{Q}_{in}}{\dot{m}} = h_3 - h_{2a} \quad (13.3.5)$$

$$\frac{\dot{Q}_{out}}{\dot{m}} = h_{4a} - h_1 \quad (13.3.6)$$

Back Work Ratio:

$$r_{BW} = \frac{\dot{W}_C}{\dot{W}_T} = \frac{h_{2a} - h_1}{h_3 - h_{4a}} \quad (13.3.7)$$

Compressor Efficiency and Work:

$$\eta_C = \frac{w_{CS}}{w'_C} = \frac{h_{2s} - h_1}{h_{2a} - h_1} \quad (13.3.8)$$

$$w'_C = \frac{h_{2s} - h_1}{\eta_C} \quad (13.3.9)$$

$$h_{2a} = h_1 + \frac{h_{2s} - h_1}{\eta_C} \quad (13.3.10)$$

Turbine Efficiency and Work:

$$\eta_T = \frac{w'_{t,T}}{w_{t,T}} = \frac{h_3 - h_{4a}}{h_3 - h_{4s}} \quad (13.3.11)$$

$$w'_{t,T} = \eta_T(h_3 - h_{4s}) \quad (13.3.12)$$

$$h_{4a} = h_3 - \eta_T(h_3 - h_{4s}) \quad (13.3.13)$$

Actual Efficiency, Work and Heat Transfer with Irreversibilities:

$$\eta_i = \frac{w'_{net}}{q'_{in}} \quad (13.3.14)$$

$$w'_{net} = w'_{t,T} - w'_C = \eta_T(h_3 - h_{4s}) - \frac{h_{2s} - h_1}{\eta_C} \quad (13.3.15)$$

$$q'_{in} = h_3 - h_2 = h_3 - h_1 - \frac{h_{2s} - h_1}{\eta_C} \quad (13.3.16)$$

$$\eta_i = \frac{\eta_T(h_3 - h_{4s}) - \frac{h_{2s} - h_1}{\eta_C}}{h_3 - h_1 - \frac{h_{2s} - h_1}{\eta_C}} \quad (13.3.17)$$

**Example:**

$$T_1 = 22^\circ\text{C}, p_1 = 0.95 \text{ bars}, r_p = p_2/p_1 = 6, T_3 = 1100 \text{ K}$$

	$\eta$	$w_C/w_T$
Ideal	0.38	0.44
$\eta_C = 0.82$ and $\eta_T = 0.85$	0.23	0.62

Figure 13.15:

Notes:

- The overall efficiency is reduced by 40%

- Gas turbine performance is very sensitive to turbine and compressor efficiencies, because the overall efficiency is low.

## 13.4 Regenerative Brayton Cycle

Introduce “regeneration” to boost overall efficiency:

- Idea: reclaim “waste” heat in the exhaust normally lost to the ambient.

Regenerative open Brayton cycle:

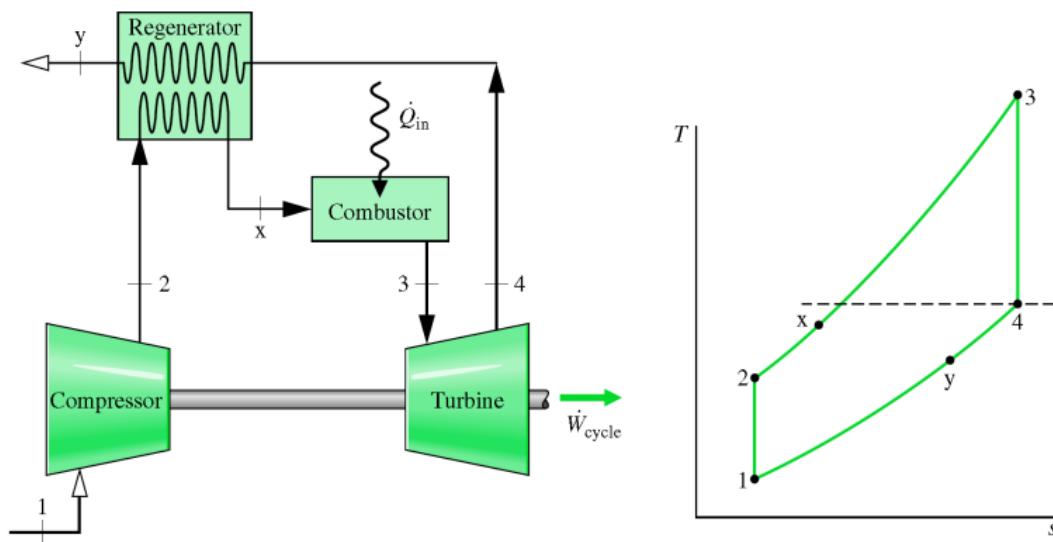


Figure 13.16:

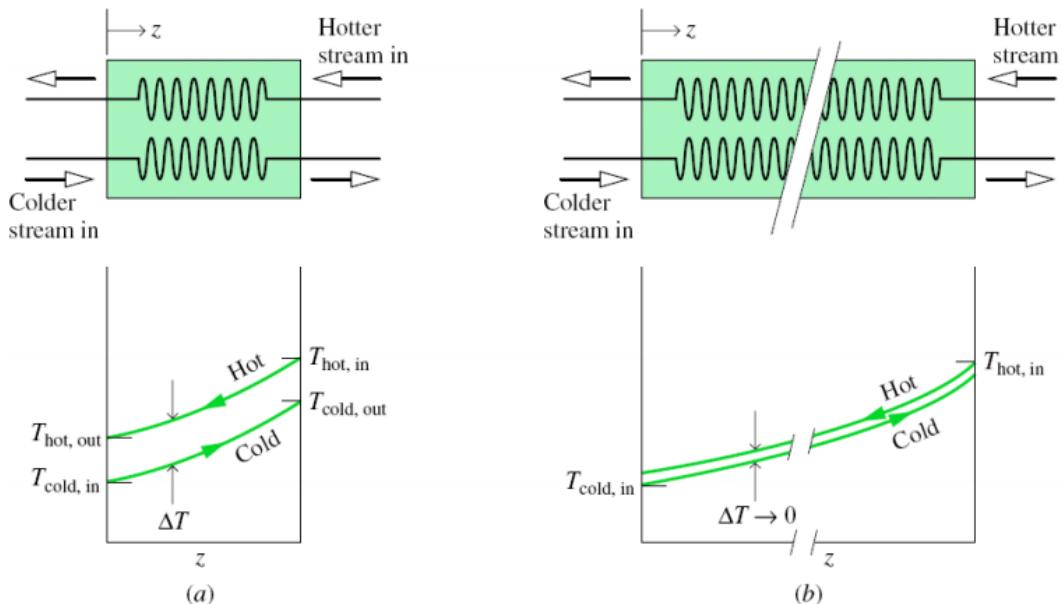


Figure 13.17: Temperature distributions in counterflow heat exchangers (a) Actual — (b) Reversible

### 13.4.1 Regenerator Effectiveness

Regenerator Effectiveness is defined as:

$$\eta_{reg} = \frac{h_x - h_2}{h_4 - h_2} \quad (13.4.1)$$

$$\eta_{reg} = 60 \approx 80\% \text{ typically} \quad (13.4.2)$$

### 13.4.2 Thermal Efficiency of Ideal Regenerative Brayton Cycle

Work and heat analysis:

$$\frac{\dot{W}_T}{\dot{m}} = h_3 - h_4 \quad (13.4.3)$$

$$\frac{\dot{W}_C}{\dot{m}} = h_2 - h_1 \quad (13.4.4)$$

$$\frac{\dot{Q}_{in}}{\dot{m}} = h_3 - h_x \quad (13.4.5)$$

Thermal efficiency:

$$\eta = \frac{\frac{\dot{W}_T}{\dot{m}} - \frac{\dot{W}_C}{\dot{m}}}{\frac{\dot{Q}_{in}}{\dot{m}}} = \frac{(h_3 - h_4) - (h_2 - h_1)}{h_3 - h_x} \quad (13.4.6)$$

Regenerator Effectiveness:

$$\eta_{reg} = \frac{h_x - h_2}{h_4 - h_2} \quad (13.4.7)$$

$$h_x = \eta_{reg}(h_4 - h_2) + h_2 \quad (13.4.8)$$

Finally, ideal thermal efficiency:

$$\eta = \frac{(h_3 - h_4) - (h_2 - h_1)}{h_3 - h_x} = \frac{(h_3 - h_4) - (h_2 - h_1)}{h_3 - \eta_{reg}(h_4 - h_2) - h_2} \quad (13.4.9)$$

### 13.4.3 Thermal Efficiency of Non-ideal Regenerative Brayton Cycle

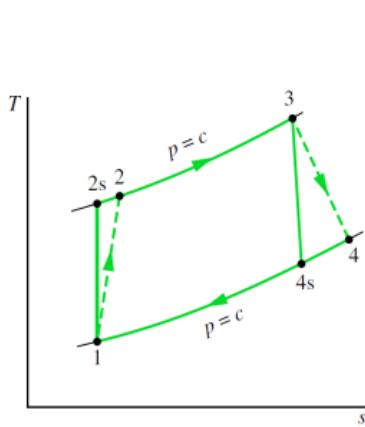


Figure 13.18:

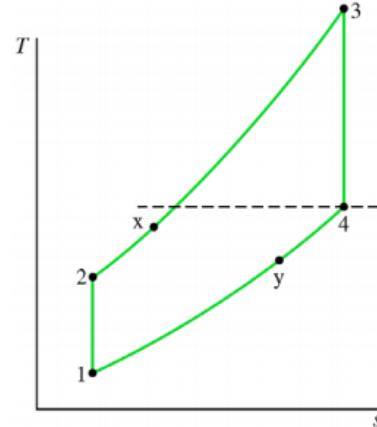


Figure 13.19:

Efficiency of turbine:

$$\eta_T = \frac{h_3 - h_4}{h_3 - h_{4s}} \quad (13.4.10)$$

$$h_4 = h_4 - \eta_T(h_3 - h_{4s}) \quad (13.4.11)$$

Efficiency of compressor:

$$\eta_C = \frac{h_{2s} - h_1}{h_2 - h_1} \quad (13.4.12)$$

$$h_2 = h_1 + \frac{h_{2s} - h_1}{\eta_C} \quad (13.4.13)$$

Finally, non-ideal thermal efficiency:

$$\eta = \frac{(h_3 - h_4) - (h_2 - h_1)}{h_3 - h_x} = \frac{(h_3 - h_4) - (h_2 - h_1)}{h_3 - \eta_{reg}(h_4 - h_2) - h_2} \quad (13.4.14)$$

$$\eta = \frac{\eta_T(h_3 - h_{4s}) - \frac{h_{2s} - h_1}{\eta_C}}{h_3 - \eta_{reg} \left[ h_3 - \eta_T(h_3 - h_{4s}) - h_1 - \frac{h_{2s} - h_1}{\eta_C} \right] \frac{h_{2s} - h_1}{\eta_C}} \quad (13.4.15)$$

## 13.5 Gas Turbines with Reheat and Intercooling

### 13.5.1 Gas Turbines with Reheat

- Two-stage gas turbine with reheat in between
- The net work per cycle is increased with reheat than without
- The thermal efficiency may or may not increase due to reheat
- Gas turbine with reheat has greater potential for regeneration

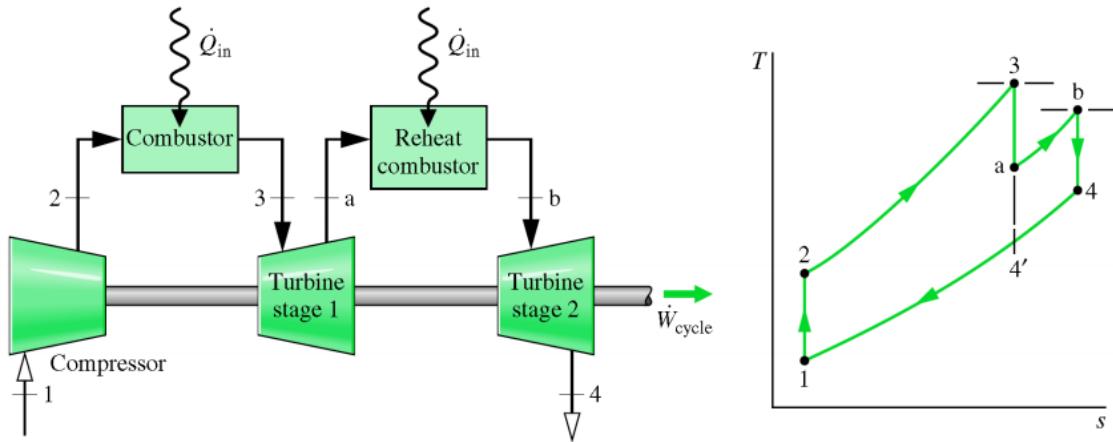


Figure 13.20:

### 13.5.2 Compression with Intercooling

- Compression work is significant in gas turbines
- Adiabatic compression requires more work than compression with cooling
- Cooling during compression is hard to achieve due to heat transfer rate limit
- Intercooling between compression stages is more efficient

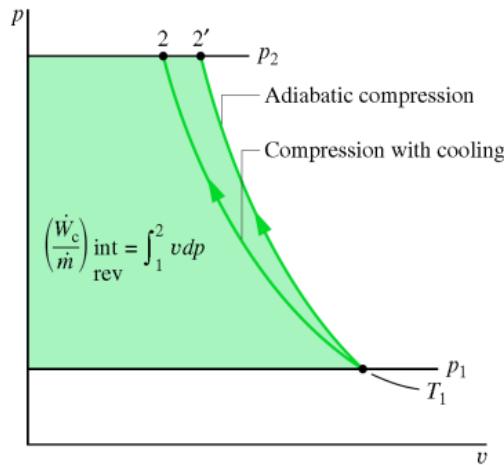


Figure 13.21: Internally reversible compression between two fixed pressures

### 13.5.3 Two-stage Compressor with Intercooler

- Process 1-c and d-2: isentropic compression
- Process c-d: constant-pressure cooling
- The net work of gas turbine is increased due to decreased compression work
- Cycle efficiency is not necessarily increased due to lower temperature at combustor inlet
- Potential for regeneration is increased

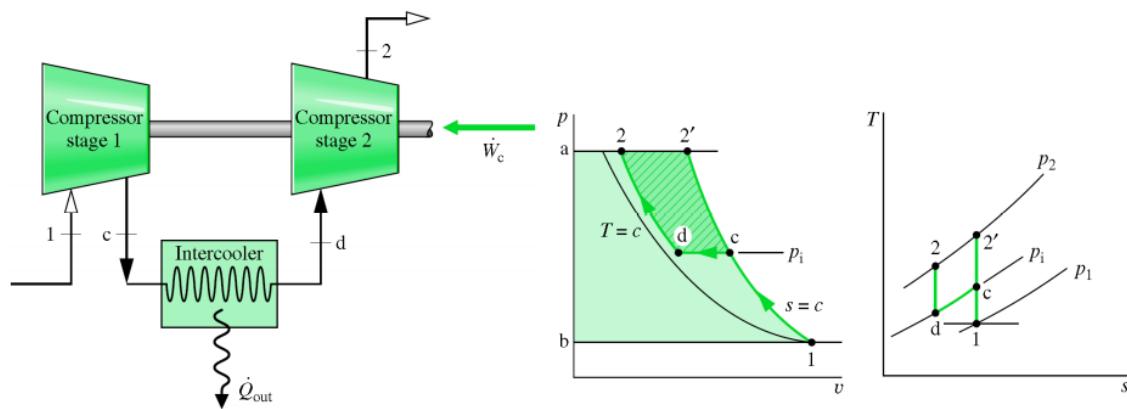


Figure 13.22:

### 13.5.4 Regenerative GT with Reheat & Intercooling

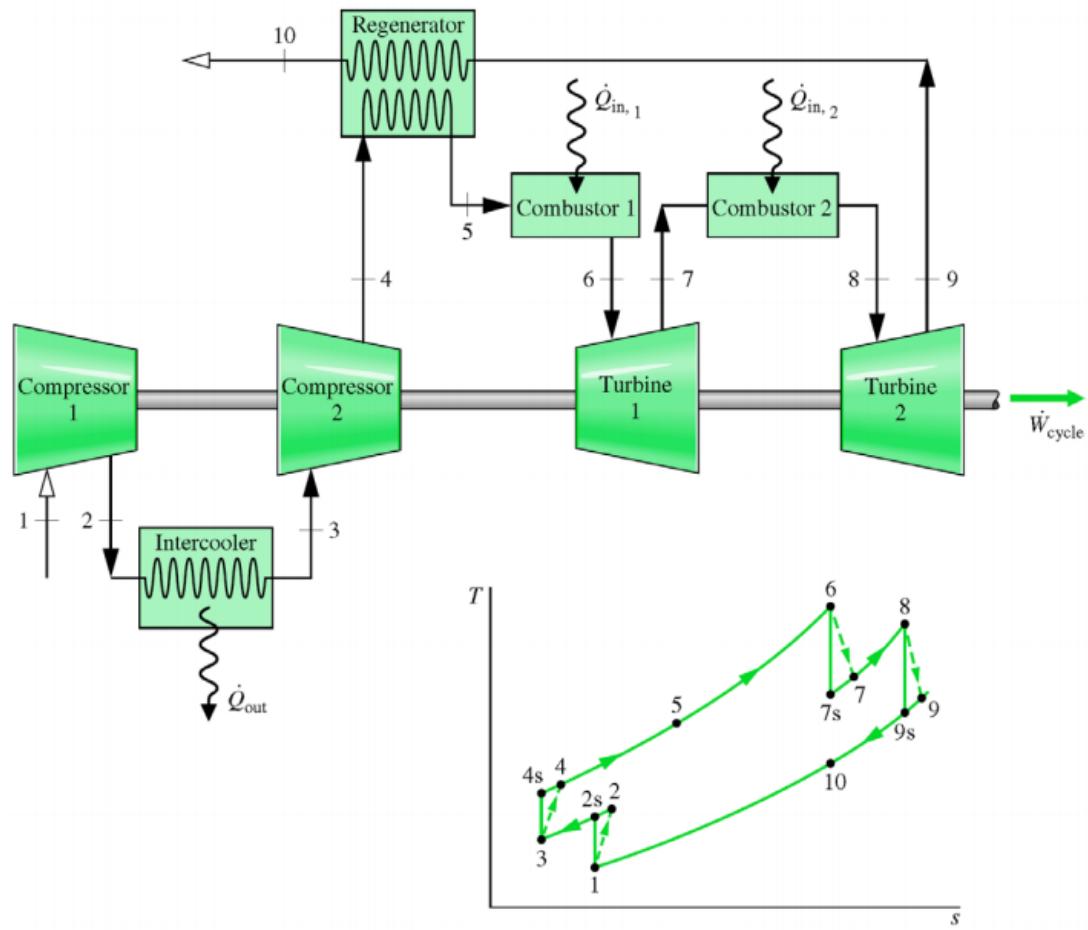


Figure 13.23:

## 13.6 Gas Turbine – Steam Turbine Combined Cycle

The combustion of fossil fuel can produce much higher temperature than that of evaporation temperature of the steam (due to the limitation of critical temperature of the water), potentially producing large exergy destruction during heat transfer process in boiler( 30%).

The gas turbine can operate at higher temperature, reducing exergy destruction due to heat transfer.

The exhaust gas temperature after expansion in gas turbine is high enough to transfer heat to the water to generate steam for the vapour power cycle.

By combining gas turbine plant cycle with steam power plant cycle, the use of thermal energy is more efficient.

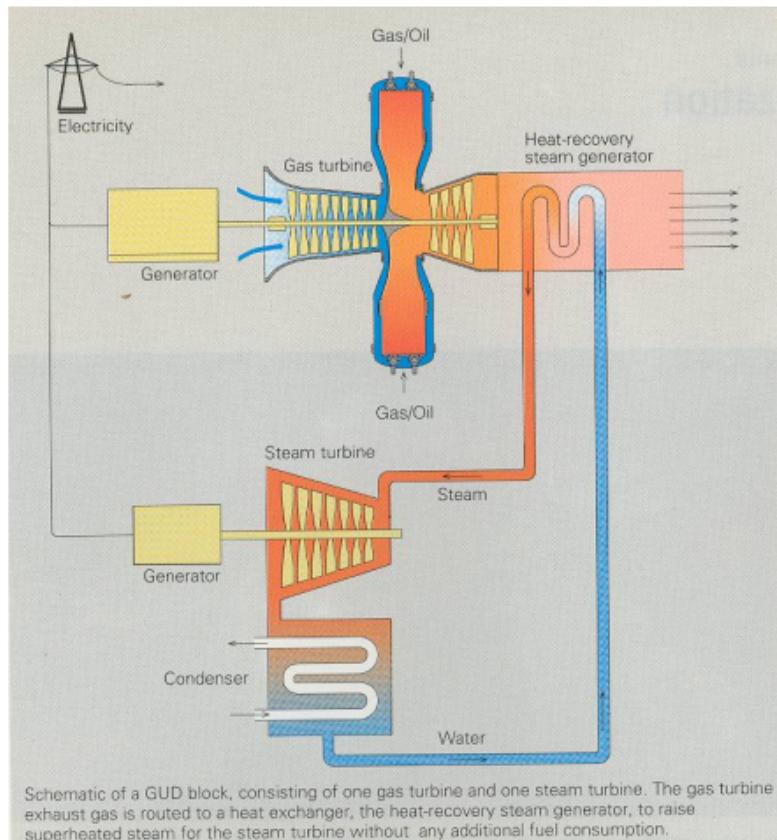


Figure 13.24:

The energy balance in heat recovery steam generator:

$$\dot{m}_v(h_7 - h_6) = \dot{m}_g(h_4 - h_5) \quad (13.6.1)$$

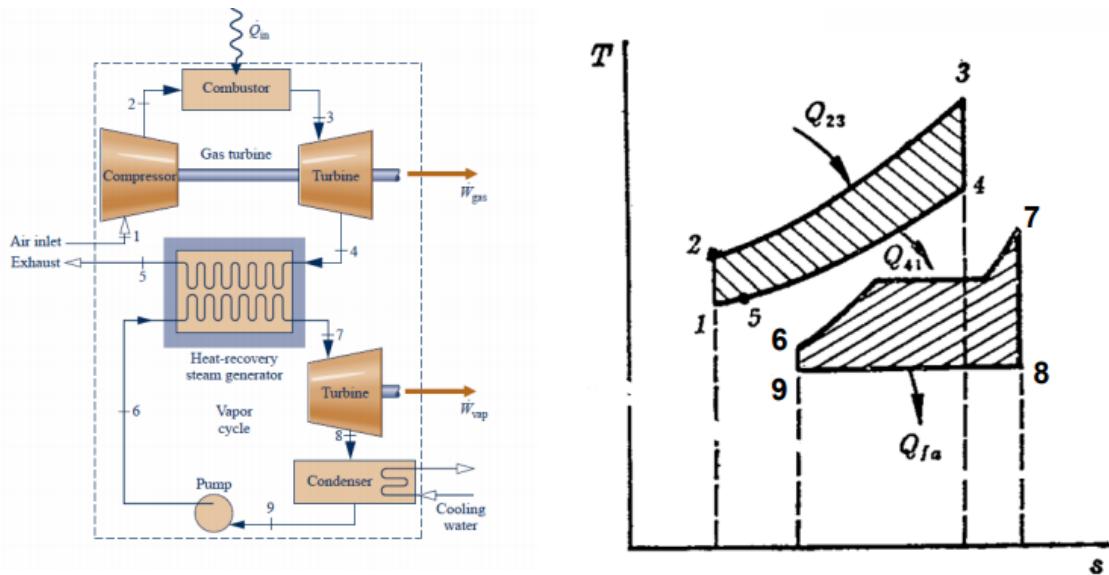


Figure 13.25:

The overall thermal efficiency:

$$\eta_t = 1 - \frac{\dot{Q}_{out}}{\dot{Q}_{in}} = 1 - \frac{\dot{m}_g(h_5 - h_1) + \dot{m}_v(h_8 - h_9)}{\dot{m}_g(h_3 - h_2)} \quad (13.6.2)$$

### 13.6.1 Gas Turbine – Steam Turbine Combined Power Cycle with Cogeneration

Generate both electricity and heat to cater for the demand of electricity, steam for industry, and steam for domestic heating purpose at the same time.

Higher overall “energy” utilization efficiency, because less heat is rejected to the environment.

If steam for industry and for domestic heating be generated through independent boiler, a large amount of “exergy” will be wasted.

## 13.7 Summary

- Thermal efficiency of ideal air-standard Joule-Brayton Cycle is dependent on pressure ratio of the compressor
- Regeneration in gas turbines increases thermal efficiency
- Reheat and compression intercooling increase work output
- Gas turbine and steam turbine combined cycle has high overall energy efficiency