MACT 4135 FINAL PROJECT
FALL 2018, AUC

BARNETTE'S CONJECTURE

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Abstract. In 1969, David Barnette conjectured that every 3-regular, 3-connected, bipartite, planar graph is Hamiltonian. The conjecture remains an open problem in graph theory to this day. In this project, we survey Barnette's conjecture, starting from its historical development and motivation, and covering several results throughout its almost 50-year history.

1. History and Motivation

Throughout this paper, we use the notation and terminology in [5]. Any terms not defined here are used as they are in [5]. We start by reviewing some important definitions. A graph $G = (V, E)$, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G, is connected if any pair of vertices $v, u \in V(G)$ can be connected by a path in G. A *cut vertex* in a connected graph G is a vertex $v \in V(G)$ such that $G - v$, the graph obtained by removing v and all its incident edges, is disconnected. A *cut set* in a connected graph G is a proper subset $S \subset V(G)$ of vertices whose removal disconnects G. We define the *connectivity* of G, denoted by $\kappa(G)$, as the size of the smallest possible cut set of G. We say that G is *k-connected* for a positive integer k if $k \leq \kappa(G)$, which means that no cut set of size less than k exists. An edge in G whose removal disconnects the graph is called a *bridge*.

We say that G is *k-regular* for a positive integer k if the degree of every vertex in G is k, which means that every vertex in G is incident to exactly k edges, or equivalently, has exactly k neighbors. A 3-regular graph is called a cubic graph. Note that in this paper, we only consider simple graphs that have no parallel edges or loops. A graph G is called *bipartite* if $V(G)$ can be partitioned into two sets of vertices X and Y, called its *partite sets*, such that every edge in $E(G)$ connects a vertex in X to a vertex in Y. G being bipartite is equivalent to G admitting a proper 2-coloring of its vertices. G is called *planar* if it can be drawn in the plane such that no edges cross each other. A *Hamiltonian cycle* is a cycle in G that visits every vertex exactly once. G is called Hamiltonian if it has a Hamiltonian cycle.

In 1884, Peter Tait [9] conjectured that every cubic 3-connected planar graph is Hamiltonian.

Conjecture 1.1. (Tait's conjecture, 1884)
Every cubic, 3-connected, planar graph is Hamiltonian.

Before we discuss subsequent developments, we take some time to comment on why this topic is interesting. We note two reasons. The first, more general reason is that this conjecture attempts to characterize a certain class of graphs as Hamiltonian, which furthers our understanding of the structure of Hamiltonian graphs. Hamiltonian graph theory is a topic of much interest due to the usefulness of Hamiltonicity in various applications, so discovering new classes of graphs that are Hamiltonian is an important goal. Other than practical applications, answers to such conjectures can yield new insights and discoveries in graph theory, moving the field forward. The second, more specific reason is that the validity of Tait's conjecture yields a simple and elegant proof of the infamous Four Color Problem! In fact, Tait came up with this conjecture when he was trying to find a proof of what was then the Four Color Conjecture. He proved that the Four Color Problem is equivalent to the problem of finding a 3-edge-coloring in bridgeless, cubic, planar graphs. Specifically, he proved the following theorem.

Theorem 1.2. ***(Tait's theorem, 1880)***
Let G be a bridgeless, cubic, planar graph. Then G has a 3-edge-coloring if and only if G has a 4-region-coloring.

Regions of a planar graph are commonly referred to as faces, but we use the term region in this paper following the terminology in [5]. A 4-region-coloring means a 4-coloring of the regions of the planar graph G. Thus, that every planar graph has a 4-region-coloring is the statement of the Four Color Theorem. But notice that the theorem states the equivalence only for bridgeless, cubic, planar graphs. Before proving Tait's theorem, we explain why this subset of planar graphs suffices. The first part of the answer lies in a reduction technique due to Cayley. He observed that if every cubic planar graph is 4-colorable, then every planar graph is 4-colorable, using the reduction technique in Figures 1.1 to 1.3 taken from [8]. We formalize Cayley's observation in the following lemma. The source [8] shows only the patching procedure in a series of images, but we provide a more formal and comprehensive proof.

Figure 1.1

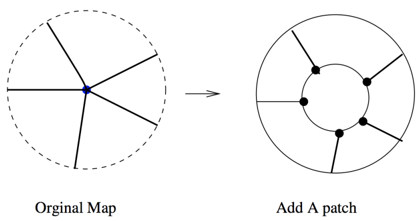


Figure 1.2

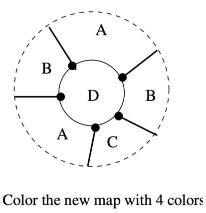
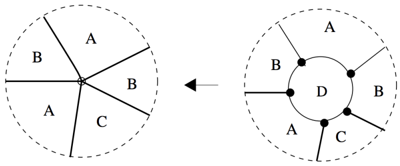


Figure 1.3



Lemma 1.3. ***(Cayley,*** [8]***)***
Every planar graph is 4*-region-colorable if and only if every cubic planar graph is* 4*-region-colorable.*

Proof. Assume that every cubic planar graph is 4-region-colorable. Let G be a planar graph that is not cubic. We start by adding vertices and edges to every vertex of degree less than 3 such that the resulting graph G' is cubic. (If no vertex of degree less than 3 exists, $G' = G$.) If G has no vertex of degree more than 3, then G' is cubic. By our assumption, G' is 4-region-colorable. Since G is a subgraph of G', it is also 4-region-colorable: we can keep the 4-region-coloring of G' and remove the vertices and edges we added earlier. Now assume that G has a vertex of degree more than 3, so that every vertex in G' has degree 3 or more. For every vertex v of degree more than 3 in G', replace v by a patch of $deg(v)$ vertices forming a cycle $C_{deg(v)}$ such that every vertex of the cycle is adjacent to exactly one of the neighbors of v in G', as shown in Figures 1.1 to 1.3 [8]. Call the resulting graph G''. It is easy to see that G'' is cubic. Now give G'' a 4-region-coloring, which exists by our assumption. Then simply replace each patch by the original vertex in G', while keeping the 4-region-coloring. Now G' is 4-region-colored.

Since cubic planar graphs are a subset of all planar graphs, the other direction is trivial. □

One more remark is in order before we proceed to the proof of Tait's theorem. The statement applies to every *bridgeless* cubic planar graph G. The following lemma, which we didn't find mentioned in any of the literature we reviewed likely due to its simplicity, shows that this subset of planar graphs is sufficient for our purposes.

Lemma 1.4. *Every planar graph is* 4*-region-colorable if and only if every bridgeless planar graph is* 4*-region-colorable.*

Proof. Assume that every bridgeless planar graph is 4-region-colorable. Let G be a planar graph that has a bridge e. Then e bounds only the external region of G, since if it bounds an internal region R, it is part of the cycle bounding R, contradicting that it is a bridge. This means that the existence of the bridge e doesn't require a new color for its bounded regions, since the only region it bounds is the external region, which must be colored in any region coloring of G. Specifically, assume we remove e from G, then successively remove any bridges from the resulting connected components until we have a set of connected components that are all bridgeless. Now we can give a 4-region-coloring to every connected component separately, by our assumption that every bridgeless planar graph is 4-region-colorable. We can also assign the colorings such that the external region is colored by the same color for every connected component. Now if we add back all the bridges we removed, the assigned colors would constitute a valid 4-region-coloring of G.

The other direction is trivial, since bridgeless planar graphs are a subset of all planar graphs. □

Now we prove Tait's theorem, following the illustrations in [8] and the explanation in [3]. As a reminder, Tait's theorem says that a bridgeless, cubic, planar graph has a 3-edge-coloring if and only if it has a 4-region-coloring.

Proof. Let G be a bridgeless, cubic, planar graph. Assume that G has a 4-region-coloring. Let A, B, C, and D be the colors for the regions. We will use colors 1, 2, and 3 to color the edges of G. First, note that since G is bridgeless and cubic, every edge bounds exactly two regions in G. Since we have a valid 4-region-coloring of G, we have that for every edge, the two regions it bounds have distinct colors. We define a term for convenience: if an edge bounds regions colored A and B, we say this edge is AB-bounding. We proceed to give a 3-edge-coloring of G. If an edge is AB-bounding or CD-bounding, color it with 1. If an edge is AC-bounding or BD-bounding, color it with 2. If an edge is AD-bounding or BC-bounding, color it with 3. Since this exhausts all possible color assignments of the two regions bounded by an edge, we have colored all the edges of G. We claim that this is a valid 3-edge-coloring of G. To see why, assume not, so that there are two adjacent edges e_1 and e_2 assigned the same color. Then according to the rules we used for coloring the edges, either both e_1 and e_2 bound the same two regions assigned the same two colors (eg. both e_1 and e_2 are AB-bounding), or the two regions bounded by e_1 and the two regions bounded by e_2 are disjoint (eg. e_1 is AB-bounding and e_2 is CD-bounding). The first case means that G has a vertex v of degree 2, the one incident to both e_1 and e_2 (see Figure 1.4). If it had degree 3, one of the regions bounded by e_1 must be colored differently from one of the regions bounded by e_2 since they would be two adjacent regions. That v has degree 2 contradicts the cubicness of G. The second case means that G has a vertex v of degree 4, the one incident to both e_1 and e_2. Since e_1 and e_2 are adjacent but the colors of their bounded regions are disjoint, there must be two more edges incident to v that separate the regions (see Figure 1.5). This also contradicts the cubicness of G. Therefore, we have provided a valid 3-edge-coloring of G.

The other direction makes use of a symmetric procedure, assigning pairs of region colors to each of the three edge colors and coloring the regions accordingly. It is left to the reader. □

Figure 1.4

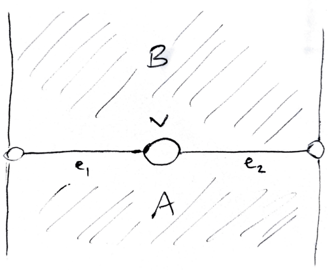


Figure 1.5

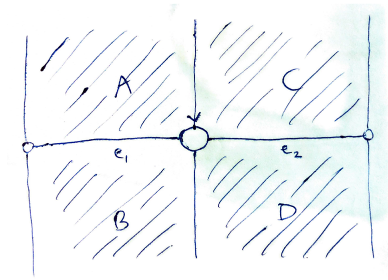
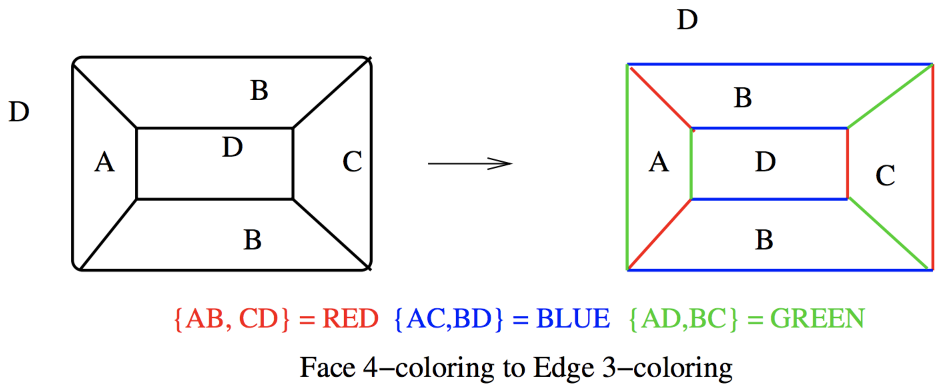


Figure 1.6, taken from [8], shows the procedures in the proof of Tait's theorem.

Figure 1.6



This concludes an excursion into the mechanics of Tait's theorem, and we now go back to our main topic, Hamiltonicity! Now that we've verified Tait's theorem in-depth, we connect all the dots in the following theorem. This is where Tait's conjecture comes into play, and his conjecture, more than the theorem and lemmas above, is what we really care about in this paper.

Theorem 1.5. *The validity of Tait's conjecture implies the Four Color Theorem.*

Proof. Tait's theorem proves that the Four Color Problem is equivalent to the problem of finding 3-edge-colorings in bridgeless, cubic, planar graphs. In an attempt to make use of this equivalence to prove the Four Color Theorem, he conjectured that every cubic, 3-connected, planar graph is Hamiltonian. First, note that a 3-connected graph is bridgeless; if it had a bridge e, then removing the two endpoints of e would disconnect the graph, contradicting that it is 3-connected. Now if Tait's conjecture were true, then every bridgeless, cubic, planar graph is Hamiltonian. In light of Tait's theorem, what remains to show is that the existence of a Hamiltonian cycle in a graph G implies that G is 3-edge-colorable. This is straightforward. Let C be a Hamiltonian cycle in G. Color the edges of C with alternating colors 1 and 2. Importantly, C has an even number of edges, so we need only use two colors. This is because a cubic graph must have an even number of vertices. Since every vertex has degree 3, we have $\sum_{v \in V(G)} deg(v) = 3n$ where n is the number of vertices. But by the handshaking lemma, $3n = 2|E(G)|$ must be even, so n must be even. Consequently, C must be an even cycle. Finally, color all the remaining edges in G with the third color 3. This produces a valid 3-edge-coloring because no two of the remaining edges share a vertex v on C. If they did, v would have degree 4, contradicting the cubicness of G. The Four Color Theorem now follows by Tait's theorem. □

Figure 1.7: Tutte's counterexample (taken from [1])

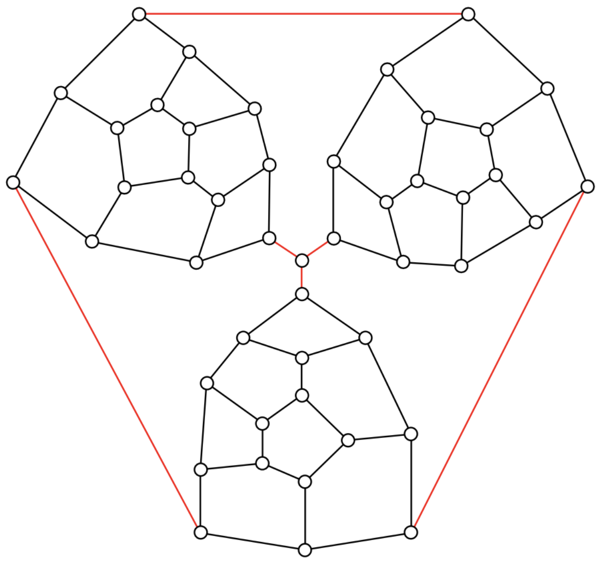
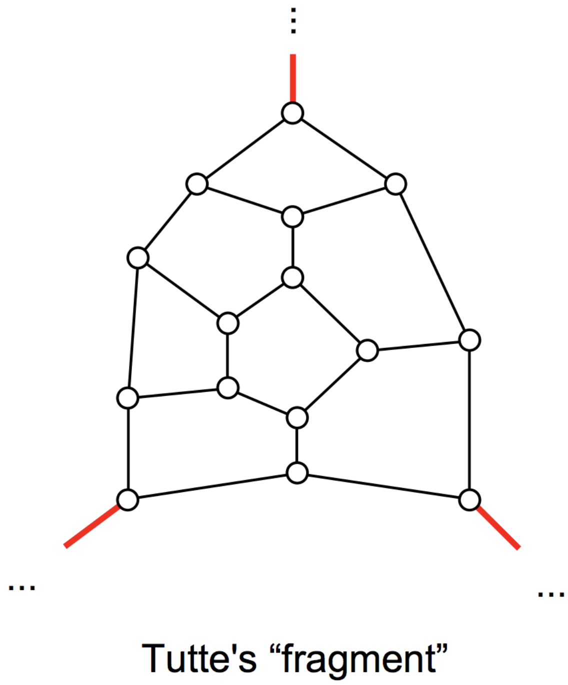


Figure 1.8 (taken from [1])



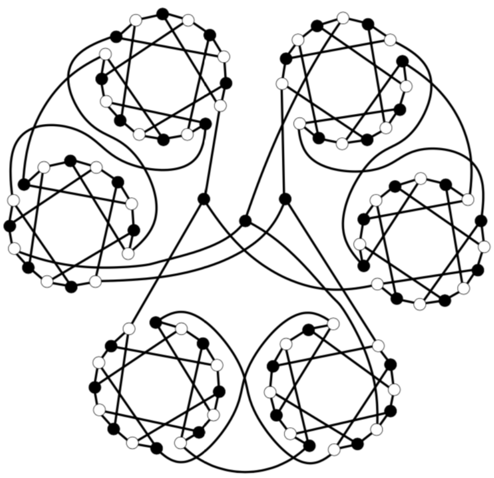
Unfortunately, Tait's conjecture is false. His work did lead to interesting work on edge colorings, but it couldn't crack the Four Color Theorem, which was still a conjecture at the time. In 1946, William Tutte disproved Tait's conjecture by showing the counterexample in Figure 1.7. The source of non-Hamiltonicity in Tutte's counterexample is Tutte's fragment, shown in Figure 1.8. A defining property of Tutte's fragment is that it's an "exclusive-OR" graph [4]: any Hamiltonian path starting from the bottom left or right vertices must exit the fragment through the top vertex. Combining three such fragments, we obtain a graph with no Hamiltonian cycles that satisfies all the properties in Tait's conjecture, thus disproving it. Since then, smaller counterexamples have been found [2].

Tutte formulated a conjecture of his own, in which he replaced planarity with bipartiteness.

Conjecture 1.6. (Tutte's conjecture, 1971)
Every cubic, 3-connected, bipartite graph is Hamiltonian.

This turned out to be false. It was disproved by Joseph Horton in 1982, who provided the non-Hamiltonian counterexample shown in Figure 1.9.

Figure 1.9: Horton's counterexample (taken from [6])



Now we introduce the main focus of this paper. In 1969, David Barnette conjectured a statement that is a combination of Tait and Tutte's conjectures (even though he made his conjecture before Tutte made his).

Conjecture 1.7. (Barnette's conjecture, 1969)
Every cubic, 3-connected, bipartite, planar graph is Hamiltonian.

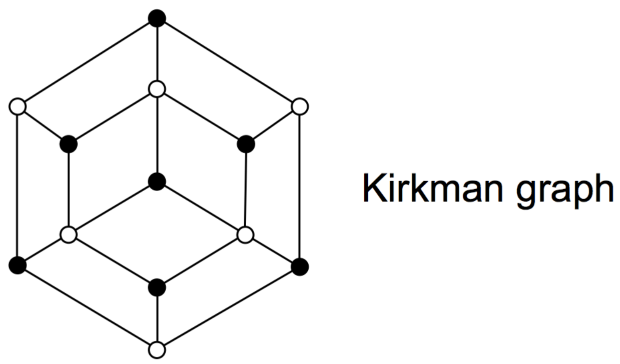
An equivalent statement is that every counterexample to Tait's conjecture is non-bipartite. This conjecture has eluded solutions for almost 50 years. In the next section, we describe a few partial and peripheral results obtained over the years. From now on, we will call cubic, 3-connected, bipartite, planar graphs by the name Barnette graphs.

2. Partial and Related Results

In [7], Holton et al. confirmed, through clever techniques and computer search, that all Barnette graphs with 64 or less vertices are Hamiltonian. In a later announcement, they extended this to 84 vertices, inclusive [6]. This means that the smallest counterexample to Barnette's conjecture, if it exists, must have at least 86 vertices. (Recall that cubic graphs have an even number of vertices.)

Perhaps one reason this conjecture is so elusive is that Barnette graphs seem to be right on the edge between Hamiltonicity and non-Hamiltonicity. If we strengthen just one of the conditions or add one more constraint, we have results that show that all graphs in this new, slightly restricted class of graphs are Hamiltonian. For example, in 1956 Tutte proved that every planar 4-*connected* graph is Hamiltonian [10]. Not only does this statement only slightly strengthen one of the four conditions of Barnette graphs (connectivity), but it also removes two others! Another example is Goodey's 1975 proof that every Barnette graph in which every region is bounded by either four or six edges is Hamiltonian. In this example, once we restrict the set of possible sizes of the regions in this way, we are able to prove Hamiltonicity. On the other hand, removing one property from the properties of Barnette graphs yields non-Hamiltonian graphs. We already know this is the case for planarity and bipartiteness. The graph in Figure 2.1, named the Kirkman graph, shows a non-Hamiltonian 3-connected bipartite planar graph, meaning that removing cubicness can result in non-Hamiltonicity [1]. Finally, if we remove 3-connectivity, we also get a set of graphs containing non-Hamiltonian graphs [6].

Figure 2.1 (taken from [1])



2.1. **Strengthening the conjecture.** One approach to making progress towards settling a conjecture is to *strengthen* the conjecture. This means that we prove that the statement of the conjecture is equivalent to another statement that is superficially stronger. This would make the conjecture easier to disprove, since we now have more properties that we can potentially break (if the conjecture is actually false). This was the approach of William Tutte, who described that he used a series of strengthening results of Tait's conjecture until he was able to find a counterexample. Researchers have been trying the same approach to attack Barnette's conjecture. The following strengthening results show that Barnette's conjecture is equivalent to statements about stronger versions of Hamiltonicity. These are useful advances for disproof attempts, since we're now trying to disprove a bolder statement.

In 1986, Kelmans proved an important strengthening result. We include it below, but we don't reproduce its proof. First, we use a definition from [1]. We say that a Barnette graph is $x - \overline{y}-$Hamiltonian if for every two edges x and y in a common region R there is a Hamiltonian cycle that contains x but not y.

Theorem 2.1. ***(Kelmans, 1986)***
Barnette's conjecture is true if and only if every Barnette graph is $x-\overline{y}-$Hamiltonian.

This theorem means that if Barnette's conjecture is true, we can pick any two edges in a common region in a Barnette graph G and remove one of them while keeping the other, without destroying the Hamiltonicity of G.

In [6], Hertel provided a new strengthening result. Before we describe it, we first need the following lemma.

Lemma 2.2. *A planar graph is bipartite if and only if all of its regions are bounded by an even number of edges.*

Proof. Let G be a planar bipartite graph. Then G has no odd cycles, by the well known result that a graph is bipartite if and only if it has no odd cycles. Since a region bounded by an odd number of edges would be surrounded by an odd cycle, this direction easily follows. Now suppose that G is a planar graph in which every region is bounded by an even number of edges. We need to prove that G has no odd cycles. Assume for the sake of contradiction that G has an odd cycle C. Since every region has even bound degree, C must go around more than one region. Suppose we form C by starting with a bounding even cycle of some region and extending around other regions. For every region R encompassed by C, a subset of the bounding edges of R are not counted in the number of edges of C due to being internal edges not included in the cycle C, and the rest are counted in the number of edges in C. But since every R has even bound degree, the number of internal edges that are not counted and the number of edges on C that are counted have the same parity, either both odd or both even, since their sum is even. Thus, for every new region encompassed by C, the number of edges in C maintains the same parity by losing k edges and gaining p edges where k and p have the same parity. Since C started as a bounding even cycle of one region, the parity of C must always be even. This contradiction proves that G has no odd cycles. Consequently, G is bipartite. □

Theorem 2.3. ***(Hertel, 2005)***
Barnette's conjecture is true if and only if every Barnette graph G is $\overline{x} - y - \overline{z}-$Hamiltonian where xyz is any path of length 3 that lies on a region in G (see Figure 2.2).

Proof. If every Barnette graph G is $\overline{x} - y - \overline{z}-$Hamiltonian where xyz is any path of length 3 that lies on a region in G, then every Barnette graph is Hamiltonian, so Barnette's conjecture is true. For the other direction, assume Barnette's conjecture is true, and let G be a Barnette graph. Let P be any path xyz of length 3 that lies on a region in G. Call the region to the left of y R_1 and the region to the right of y R_2. By Lemma 2.2, both R_1 and R_2 have even bound degree. Accordingly, in Figure

Figure 2.2 (taken from [1])

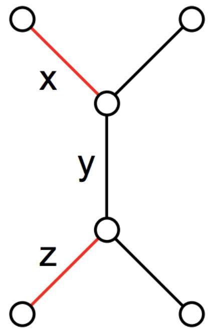
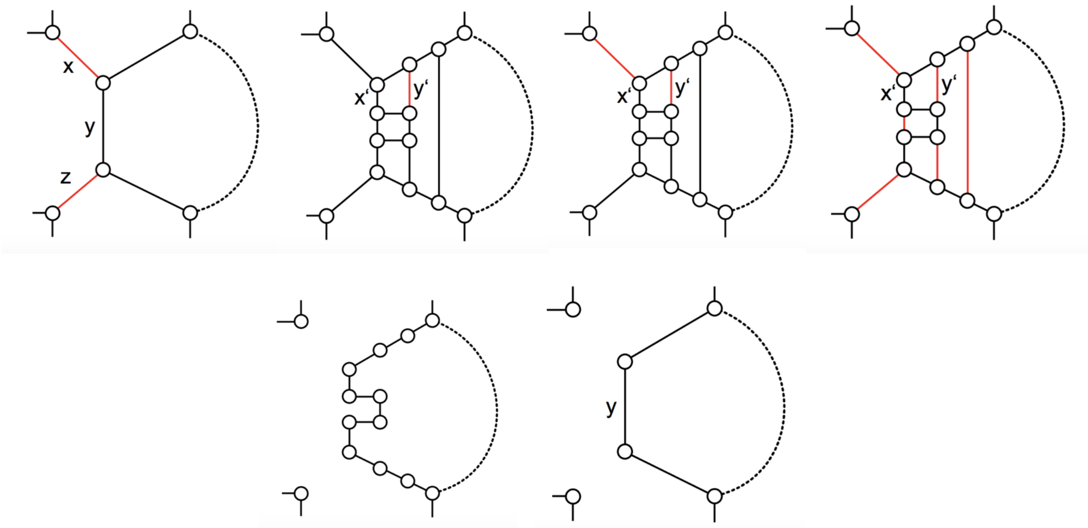


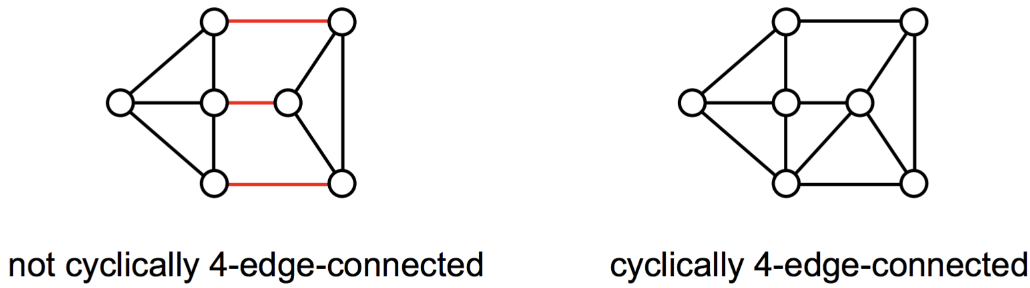
Figure 2.3 ([1])



2.3, taken from [1], the dotted line on one side of R_2 represents a path with an odd number of edges. Now subdivide the region R_2 to obtain the new graph G' depicted in Figure 2.3. It is easy to see that G' is still a Barnette graph. Since we assumed that Barnette's conjecture is true, we can use Theorem 2.1 to choose any two edges and delete one while keeping the other, knowing that this will not destroy Hamiltonicity. If we delete the edges indicated in Figure 2.3, we still have a Hamiltonian cycle. Now reverting back to the original graph G, we see that we have found a Hamiltonian cycle that goes through y and avoids x and z. Since we chose the path xyz arbitrarily, G is $\overline{x} - y - \overline{z}-$Hamiltonian, as required. $\square$

2.2. **Weakening the conjecture.** We now describe a *weakening* result. A result weakens a conjecture if it shows that it is equivalent to a superficially weaker statement. In this case, the result shows that Barnette's conjecture is equivalent to a statement on a subset of Barnette graphs. Such a result narrows down the space of possible counterexamples. This is not helpful for disproving the conjecture, since having more courses of action to take when looking for a counterexample is the better

Figure 2.4 (taken from [1])



situation. On the other hand, a weakening result could get us closer to *proving* Barnette's conjecture. With a weakening result like this one, we have more assumptions to use when trying to prove the conjecture.

This result is based on the cyclic edge connectivity of a graph. We say that a graph G is cyclically 4-edge-connected if every cycle-separating edge cut contains at least 4 edges [1]. In other words, at least 4 edges must be removed to disconnect G into two components that each contain a cycle (see Figure 2.4).

Theorem 2.4. *Barnette's conjecture is true if and only if every cyclically* 4*-edge-connected Barnette graph is* $x - \overline{y}-$*Hamiltonian.*

Proof. If Barnette's conjecture is true, then by Theorem 2.1, every Barnette graph is $x - \overline{y}-$Hamiltonian. Since cyclically 4-edge-connected Barnette graphs are a subset of Barnette graphs, this direction is trivial.

Now assume that every cyclically 4-edge-connected Barnette graph is $x-\overline{y}-$Hamiltonian. First note that all Barnette graphs are cyclically 3-edge-connected. So if a Barnette graph G is not cyclically 4-edge-connected, it must have a 3-edge-cut that disconnects G into two components that each have a cycle. We induct on the number of 3-edge-cuts in G. If there are none, G is $x - \overline{y}-$Hamiltonian by assumption. If there is one, then remove the 3 edges in that cut and replace them with two new vertices to form two components G_1 and G_2 as shown in Figure 2.5. It's easy to see that both components are also Barnette graphs, and since we've removed the only 3-edge-cut, each is cyclically 4-edge-connected. We can then take the two Hamiltonian cycles in each component and patch them back together to form a Hamiltonian cycle in the original graph G. In fact, every pair of Hamiltonian cycles in the newly formed components G_1 and G_2 has a corresponding Hamiltonian cycle in G, so G is $x - \overline{y}-$Hamiltonian. The induction can be completed using the same strategy, applying the inductive hypothesis on each of G_1 and G_2. This proves the theorem. □

References

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[2] https://ajc.maths.uq.edu.au/pdf/20/ocr-ajc-v20-p111.pdf.

Figure 2.5 (taken from [1])

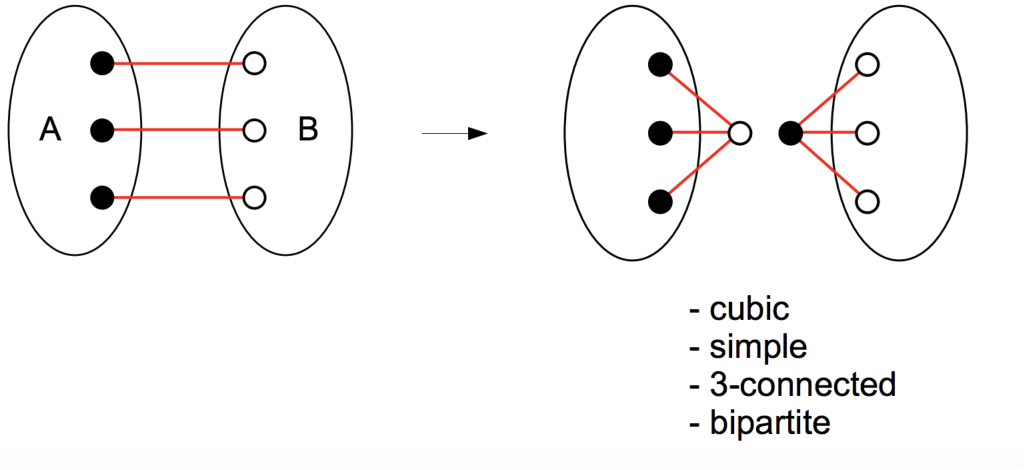
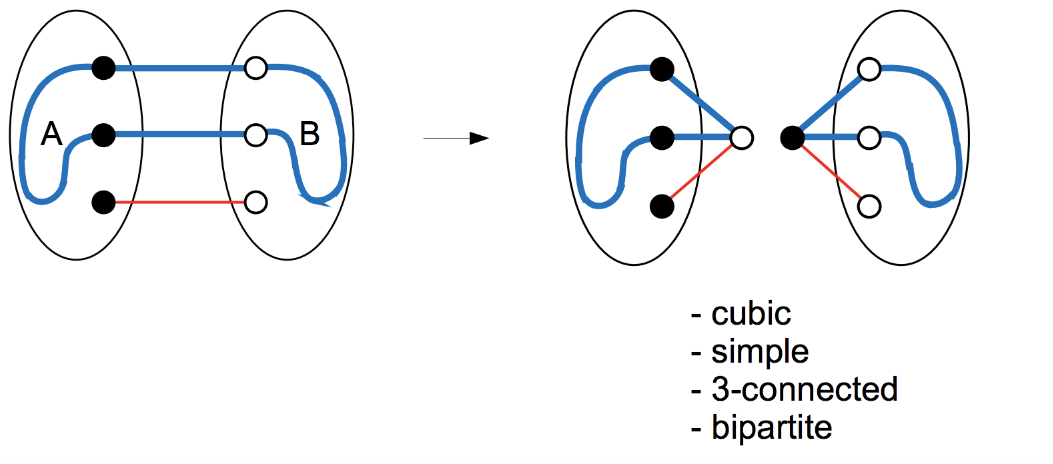


Figure 2.6: Forming a Hamiltonian cycle in G from the two Hamiltonian cycles in G_1, G_2 (taken from [1])



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