FOURIER SERIES

IS3302

Complex Analysis and Mathematical Transforms

Department of Interdisciplinary Studies

Faculty of Engineering

University of Ruhuna

Content

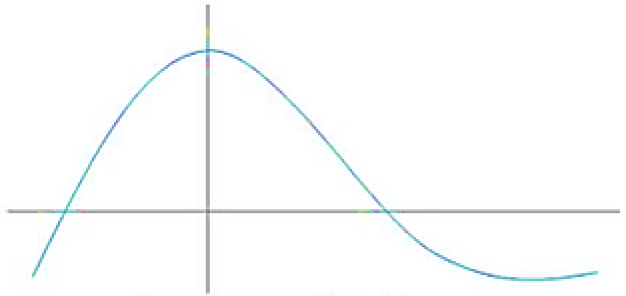
- Introduction to Signals and Mathematical Transforms
- Periodic Functions
- Harmonics, sinusoidal Functions and Non-Sinusoidal Functions
- Even and Odd Functions
- Representing Periodic Functions by Fourier Series
- Half – Range Series
- Fourier series for functions of general period
- Half-Range Cosine series for functions of general period
- Parseval’s Formula
- Complex form of Fourier Series

Essential Types of Signals in Signal Processing

- **Continuous-time and discrete-time signals.**
- **Even and Odd Signals.**
- **Periodic and non-periodic signals.**
- **Deterministic and random signals.**
- **Energy and Power types Signals.**

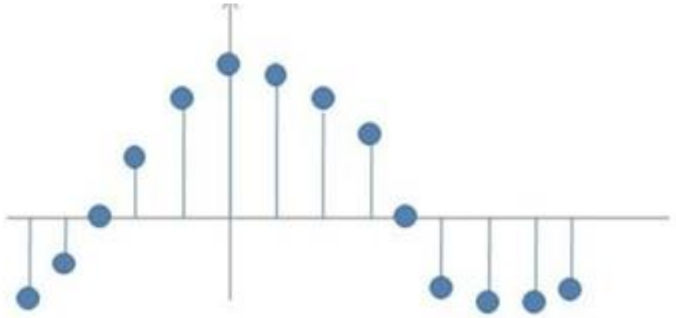
Essential Types of Signals in Signal Processing

- **Continuous-time**
  - A signal is considered to be a continuous time signal if it is defined over a continuum of the independent variable (If a signal can take any value on the x-axis (time axis) then it is called as continuous signal).



Continuous Signal

- **discrete-time signals**
  - A signal is considered to be discrete time if the independent variable only has discrete values (if it can only take finite values on x-axis (time axis) then it will be a discrete signal).



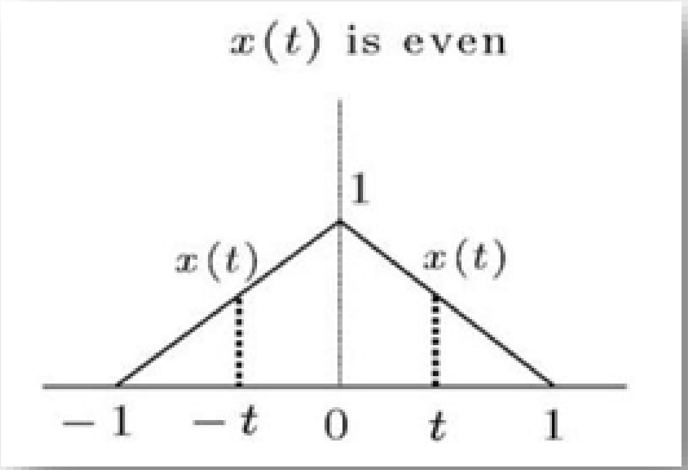
Discrete Signal

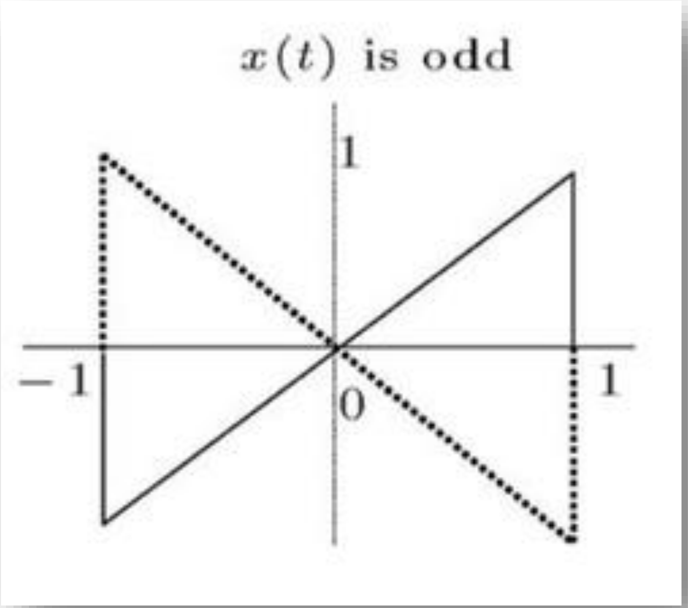
Essential Types of Signals in Signal Processing…

- **Even and Odd Signals.**
- Even signals are symmetric around vertical axis, and Odd signals are symmetric about origin.

    - Even Signal: A signal is referred to as an even if it is identical to its time-reversed counterparts; x(t) = x(-t).

    - Odd Signal: A signal is odd if x(t) = x(-t).





Essential Types of Signals in Signal Processing

- **Periodic signals**
- A periodic signal is one that repeats the sequence of values exactly after a fixed length of time, known as the period.
- $x(t) = x(t+T)$
- $f = \frac{1}{T}$; f= number of cycles per second, T=Period
- Periodic signals are characterized by their period, which is the time it takes for the signal to repeat itself, and their frequency, which is the number of cycles the signal goes through in a given time period.
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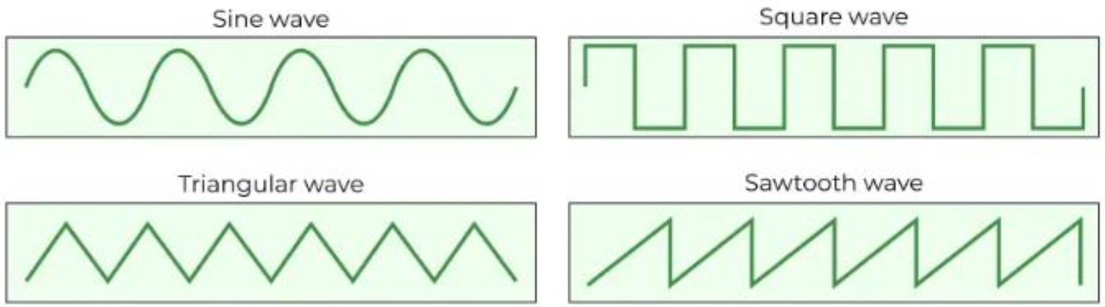
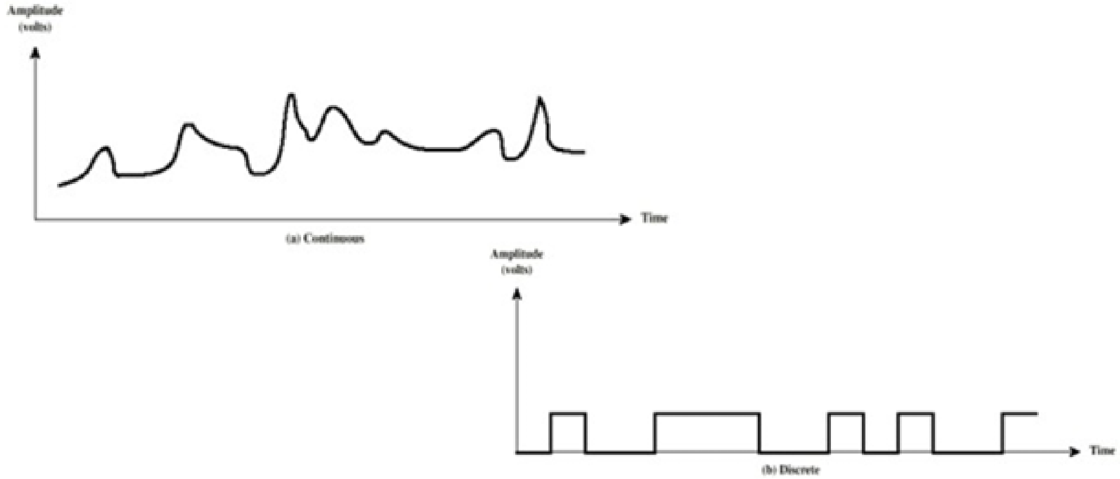
- The period and the frequency of a periodic signal are inversely related: a signal with a short period has a high frequency, and a signal with a long period has a low frequency.



Non-periodic signals

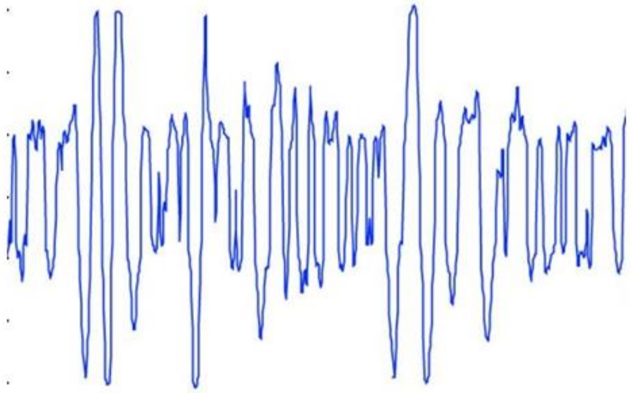
- **A non-periodic or aperiodic signal is one for which no value of T satisfies Equation**
- $\boldsymbol{x(t) = x(t+T)}$
- **a non-periodic signal is a signal that does not repeat itself at regular intervals.**
- **This means that it does not have a fundamental frequency, and its frequency spectrum is generally not composed of discrete frequencies.**
- **Non-periodic signals can be continuous or discrete, and they can be either deterministic or random.**
- **Examples of non-periodic signals include white noise, impulses, and arbitrary waveforms.**
- **Non-periodic signals are often encountered in real-world applications, such as speech, music, and environmental noise.**



Essential Types of Signals in Signal Processing

- **Deterministic signals**
  - A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time.
  - Example, $x(t) = \sin(3t)$

- **Random signals (**Non-deterministic signals **)**
  - Non-deterministic signals are random in nature hence they are called random signals.
  - Random signals cannot be described by a mathematical equation.



noise

Essential Types of Signals in Signal Processing

- **Energy types Signals.**
  - Signals that are both deterministic and non-periodic are energy signals.

- **Power types Signals**
  - Periodic and random signals are power signals.

MATHEMATICAL TRANSFORMS

- Mathematical transforms are essential tools in many scientific and engineering applications and are crucial in various fields, including **signal processing, image recognition, and data analysis**.
- They allow for representing complex signals and images in a more efficient and manageable form.

- By applying transforms,
    - it can **extract** valuable information,
    - **reduce noise**, and
    - **enhance the quality of the signal**.

- Transforms enable you to work with signals in a more abstract and manipulable way, making it easier to
    - **identify patterns,**
    - **analyze data, and**
    - **make predictions**.

- Mathematical transforms are **techniques used to represent functions or signals in a different domain**.

- Simply,
    - Fourier series/Fourier Transform: decomposes a function into its frequency components

    - Laplace Transform: used for continuous-time signals, transforms a function from the time domain to the s-domain

    - Z-Transform: used for discrete-time signals and has applications in signal processing and control systems

FOURIER SERIES

- The **Fourier series** is a fundamental concept in mathematics and signal processing!

- It's a way to represent a periodic function as a sum of simpler sine and cosine waves.

- It's used to **analyze and decompose signals, understand periodic phenomena, and solve partial differential equations**.

- The concept has numerous practical uses, such as **filtering noise, encrypting data, and even generating music and images.**

Fourier Series

- **Definition:** Represents a periodic function as a sum of sine and cosine functions.
- **Domain:** Time-domain to frequency-domain (for periodic signals).
- **Formula:** $f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$
- **Usage:**
  - Analyzing periodic signals in engineering (e.g., electrical circuits).
  - Signal processing and acoustics.

FOURIER TRANSFORMS

- **Fourier transforms** decomposes signals or functions into their constituent frequencies, providing valuable insights into the underlying structure and behavior.

- Fourier transforms are essential in many applications, such as
  - filtering,
  - modulation, and demodulation,
  - as well as in analyzing and understanding natural phenomena like sound, light, and vibrations.

Fourier Transform

- **Definition:** Extends the Fourier Series to non-periodic functions, transforming a time-domain signal into its frequency components.
- **Domain:** Time-domain to frequency-domain (for aperiodic signals).
- **Formula:** $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$
- **Usage:**
  - Signal processing, image analysis, and communication systems.
  - Analyzing frequency content of signals.

LAPLACE TRANSFORM

- The **Laplace transform** is a fundamental tool in the field of mathematics and electrical engineering.
- It transforms a **continuous-time signal into a frequency domain** representation, which makes it easier to analyze and manipulate the signal.
- The Laplace transform is used to **solve differential equations, determine the stability of systems, and analyze the frequency response of Linear Time-Invariant (LTI) systems**.
- It is also used in many real-world applications, such as signal processing, control systems, and communications.

Laplace Transform

- **Definition:** Transforms a time-domain function (typically of exponential growth) into a complex frequency domain.
- **Domain:** Time-domain to complex frequency-domain.
- **Formula:** $L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt$
- **Usage:**
  - Control systems and stability analysis.
  - Solving differential equations in engineering and physics.

Z-TRANSFORM

- The concept of **Z-transform** is a fundamental aspect of signal processing and control systems.
- The Z-transform is a **discrete-time variant of the Laplace transform**, used to analyze and design **digital control systems**.
- It transforms a **continuous-time signal into a discrete-time representation**, making it easier to analyze and process.
- The Z-transform is used to represent the frequency response of a system, allowing engineers to design and optimize their systems for specific frequency responses.
- It's a crucial tool in many fields, including control systems, signal processing, and digital filter design

Z Transform

- **Definition:** Discrete-time counterpart of the Laplace Transform, converting a discrete signal into a complex frequency domain.
- **Domain:** Discrete-time signals to complex frequency-domain.
- **Formula:** $Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$
- **Usage:**
  - Digital signal processing and control systems.
  - Analyzing and designing discrete-time systems.

Summary of Key Differences:

- **Nature of Signals:**
  - **Fourier Series:** Periodic signals.
  - **Fourier Transform:** Aperiodic signals.
  - **Laplace Transform:** Continuous-time signals (exponential growth).
  - **Z Transform:** Discrete-time signals.
- **Domain:**
  - Fourier Series and Transform work in the frequency domain, while Laplace and Z Transforms move into complex frequency domains.
- **Applications:**
  - Fourier methods are more common in signal analysis.
  - Laplace is heavily used in control systems and circuit analysis.
  - Z Transform is essential for digital signal processing.

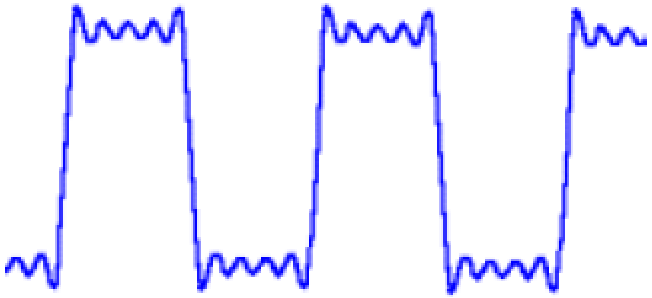
FOURIER SERIES

Introduction

- Fourier series are usually infinite series but involve sine and cosine functions (or their complex exponential equivalents) rather than polynomials.

- Fourier series are used for approximating periodic functions/in the analysis of **periodic** functions.

- Many of the phenomena studied in engineering and science are periodic in nature.



- The sum of these special trigonometric functions is called the **Fourier Series**.

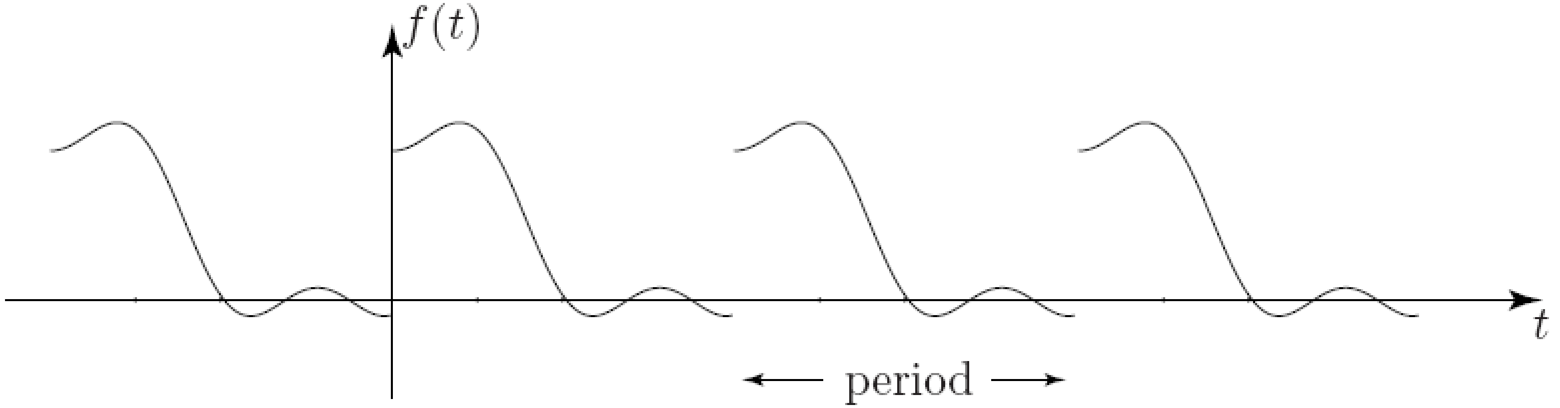
Periodic Functions

A $function\ f(t)$ is said to be **periodic** with **period** p if

$$f(t + p) = f(t)$$

for all values of t and if $p > 0$.

- The **period** of the function $f(t)$ is the interval between two successive repetitions.

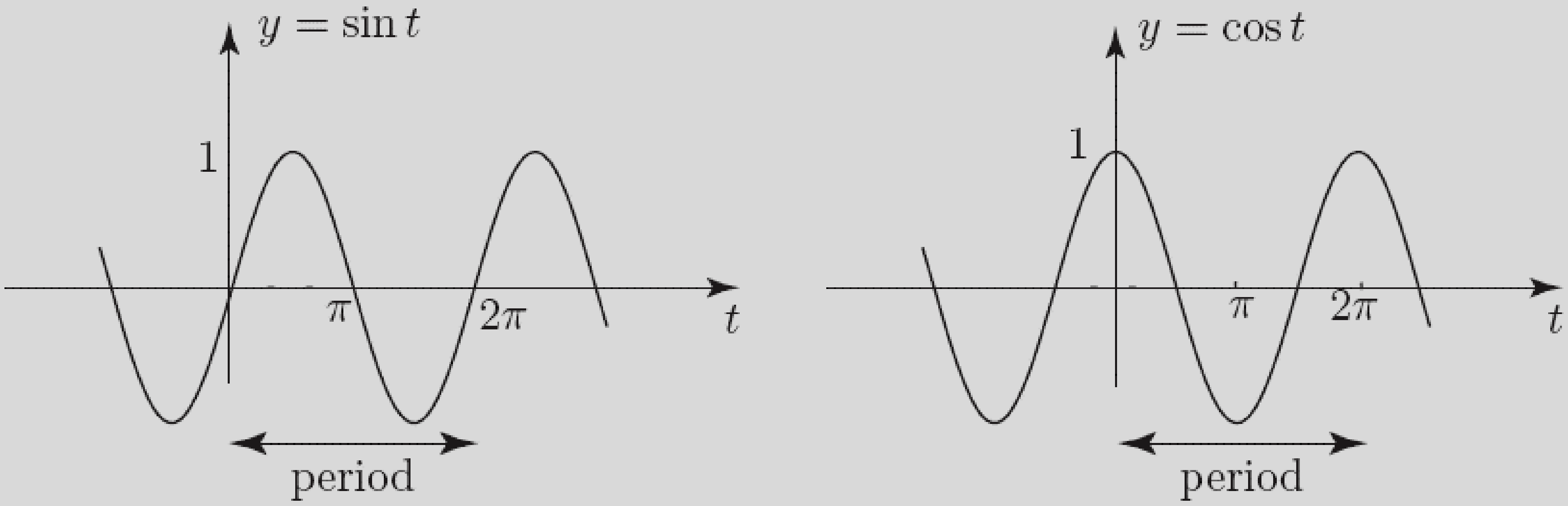


- Periodic functions have repetitive behaviour.

Periodic Functions…

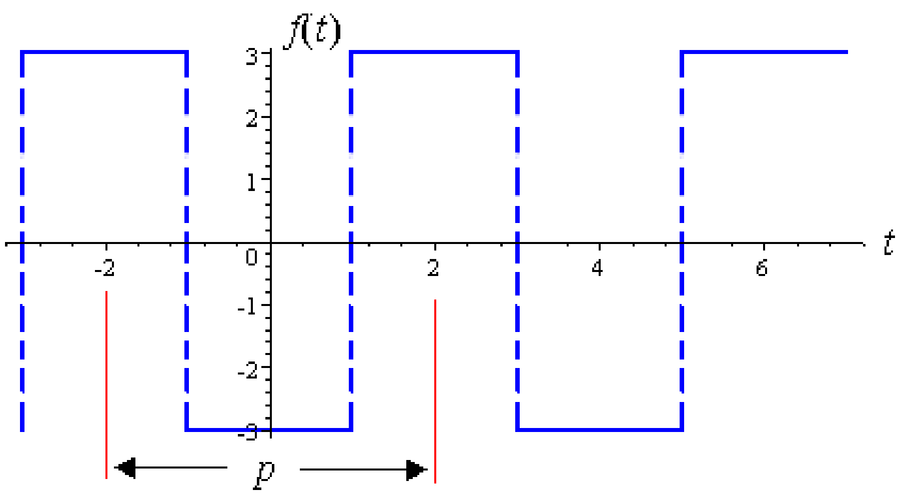
- The most obvious examples of periodic functions are the trigonometric functions $\sin t$ and $\cos t$, both of which have period 2π (using radian measure)

$$\sin(t + 2\pi) = \sin t \qquad \text{and} \qquad \cos(t + 2\pi) = \cos t$$



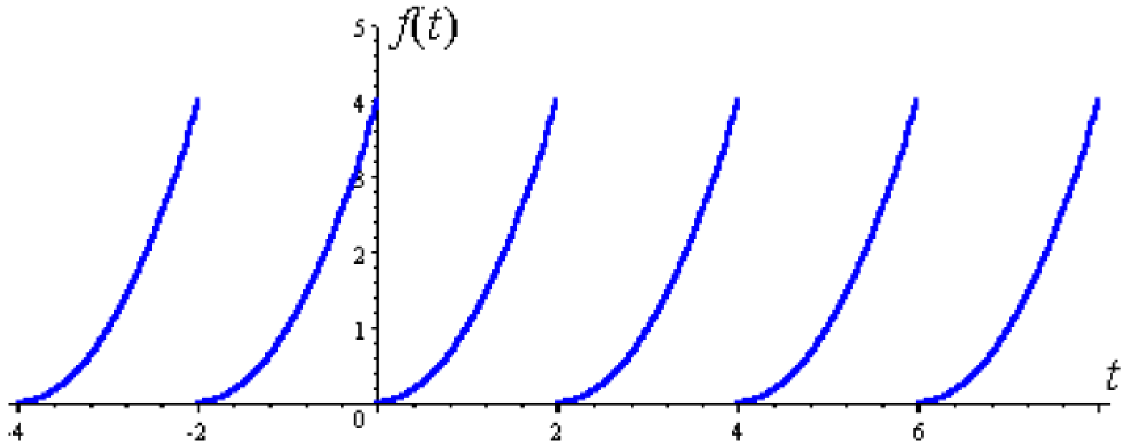
Periodic Functions…

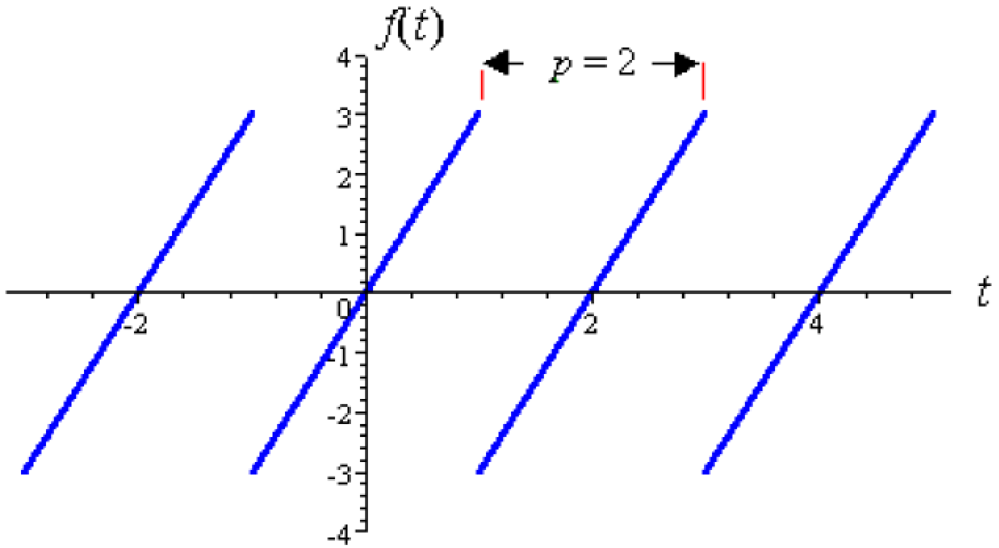
Square wave, period = 4.



Period = p = $2L$ = 4

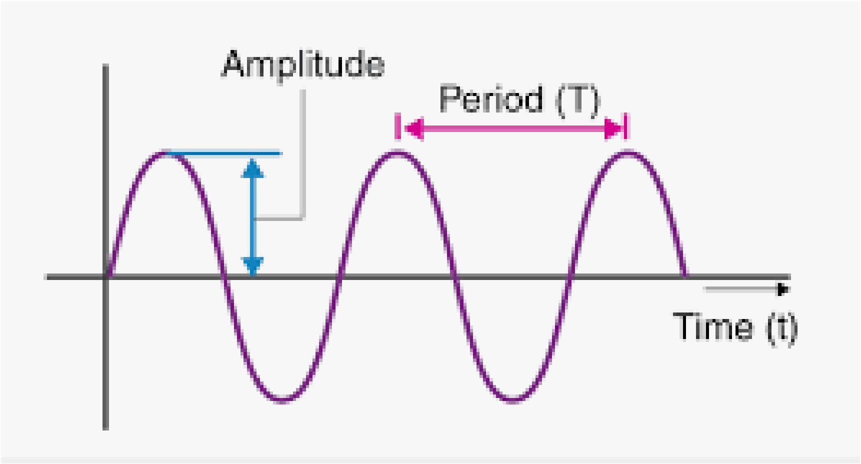
Parabolic, period = 2.





Periodic Functions…

- The amplitude of these sinusoidal functions is the maximum displacement from y = 0 and is clearly 1.
- More generally we can consider a sinusoid $y = A \sin nt$ which has maximum value, or amplitude, A and where n is usually a positive integer.



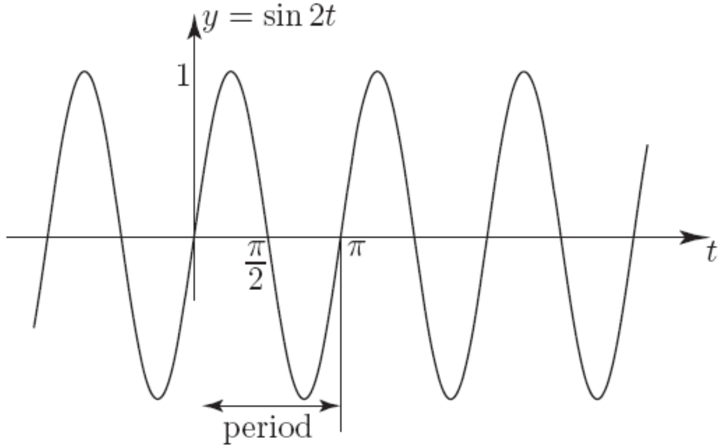
Example

$y = \sin 2t$ is a sinusoid of amplitude 1 and period $\frac{2\pi}{2} = \pi$

The fact that the period is π follows because

sin 2(t + π) = sin(2t + 2 π) = sin 2t

for any value of t.



Periodic Functions…

In general $y = A\sin nt$ has amplitude A, period $\frac{2\pi}{|n|}$ and completes n oscillations when t changes by 2.

Formally, we define the frequency of a sinusoid as the reciprocal of the period:

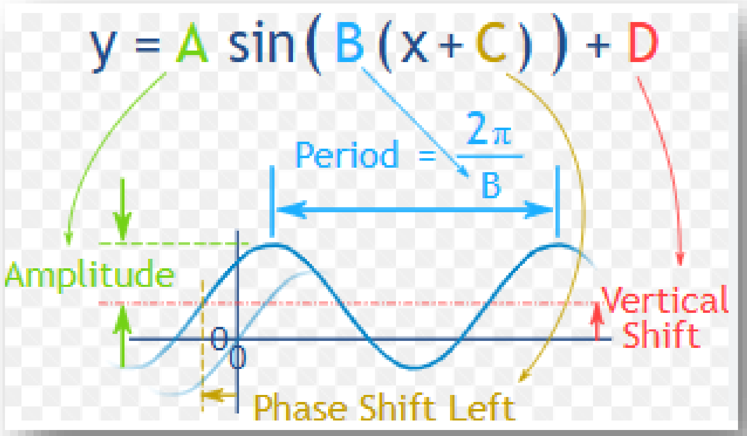
$$frequency(f) = \frac{1}{Period} \qquad f = 1/T$$

and the angular frequency, often denoted the Greek Letter ω (omega) as

$$angular\ frequency(\omega) = 2\pi \times frequency = \frac{2\pi}{period}$$

$$\omega = 2\pi f$$

Thus $y = A\sin nt$ has **frequency** $\boldsymbol{\frac{n}{2\pi}}$ **and angular frequency** $\boldsymbol{n}$**.**



State the amplitude, period, frequency and angular frequency of

1. $y = 5\cos 4t$ 2. $6\sin\frac{2t}{3}$

Sinusoids (A Sine Wave)

A curve similar to the sine function but possibly shifted in phase, period, amplitude, or any combination thereof.

i.e. if the function can be produced by stretching, shifting or compressing of sine function (y= $Asin(nt)$).

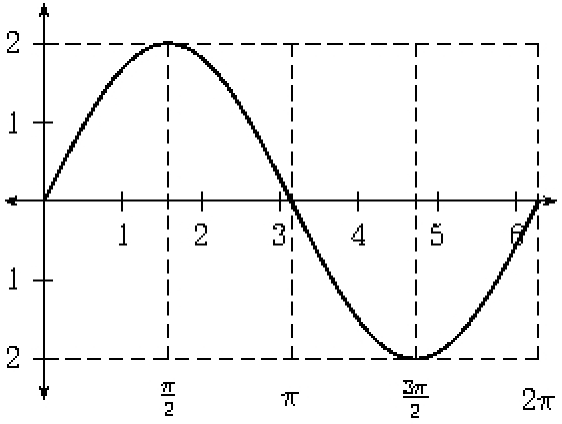
The general sinusoid of amplitude A, angular frequency ω, (and period $\frac{2\pi}{\omega}$), and phase c is given by

$$f(x) = Asin(\omega x + c)$$

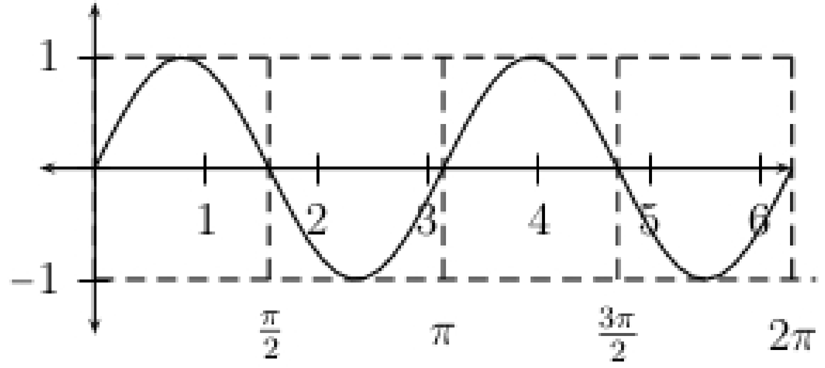
cosine is a sinusoidal function. You can think of it as the sine function with a phase shift of -pi/2 (or a phase shift of 3pi/2).

Examples

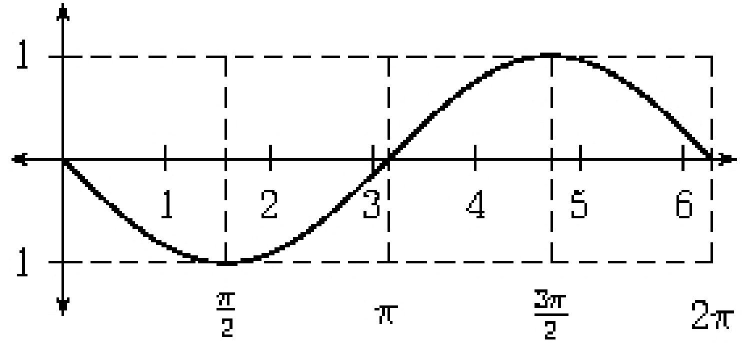
$y = 2sinx\ ; x = 0\ to\ x = 2\pi$



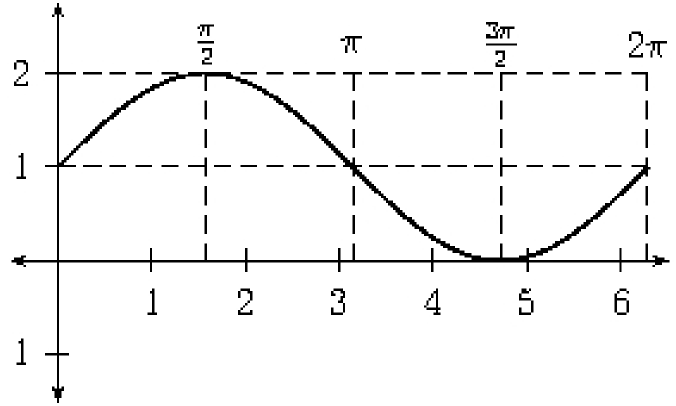
$y = sin2x\ ; x = 0\ to\ x = 2\pi$



$y = \sin(x - \pi)\ : x = 0\ to\ x = 2\pi$



$y = 1 + sinx\ ; x = 0\ to\ x = 2\pi$



Harmonics

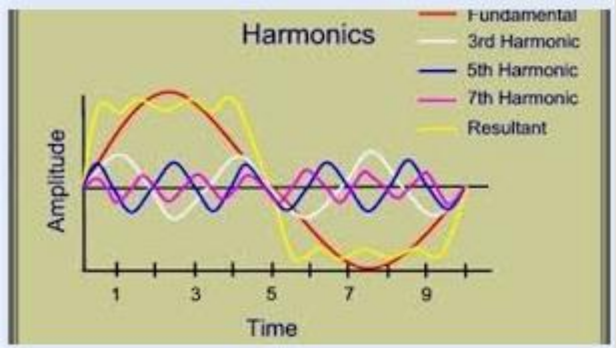
In representing a non-sinusoidal function of period 2π by a Fourier series we shall see shortly that only certain sinusoids will be required:

(a) $A_1 \cos t$ (and $B_1 \sin t$)

These also have period 2π and together are referred to as the **first harmonic** (or **fundamental harmonic**).

(b) $A_2 \cos 2t$ (and $B_2 \sin 2t$)

These have half the period, and double the frequency, of the first harmonic and are referred to as the **second harmonic**.



(c) $A_3 \cos 3t$ (and $B_3 \sin 3t$)

These have period $\dfrac{2\pi}{3}$ and constitute the **third harmonic**.

In general the Fourier series of a function of period 2π will require harmonics of the type

$A_n \cos nt$ (and $B_n \sin nt$) where $n = 1, 2, 3, \ldots$

Non-sinusoidal periodic functions

The following are examples of non-sinusoidal periodic functions (they are often called "waves").

Square wave

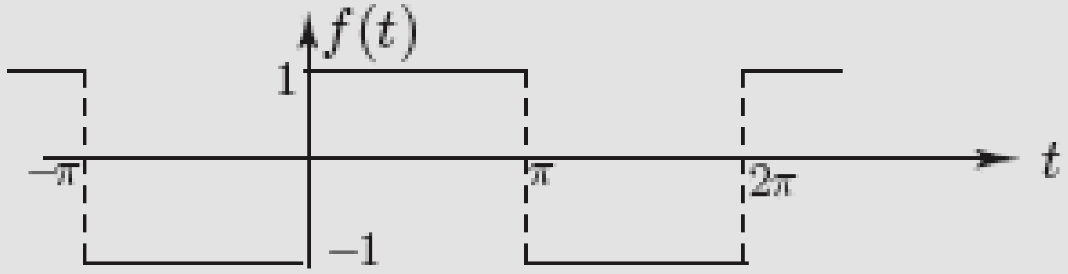


Figure 5

Analytically we can describe this function as follows:

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ +1 & 0 < t < \pi \end{cases} \qquad \text{(which gives the definition over one period)}$$

$$f(t + 2\pi) = f(t) \qquad \text{(which tells us that the function has period } 2\pi\text{)}$$

Non-sinusoidal periodic functions…

Saw-tooth wave

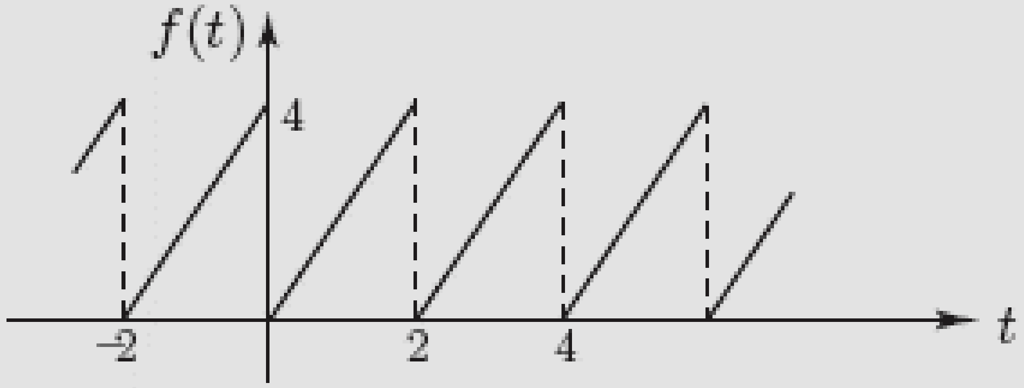


Figure 6

In this case we can describe the function as follows:

$$f(t) = 2t \qquad 0 < t < 2 \qquad f(t+2) = f(t)$$

Here the period is 2, the frequency is $\frac{1}{2}$ and the angular frequency is $\frac{2\pi}{2} = \pi$.

Non-sinusoidal periodic functions…

Triangular wave

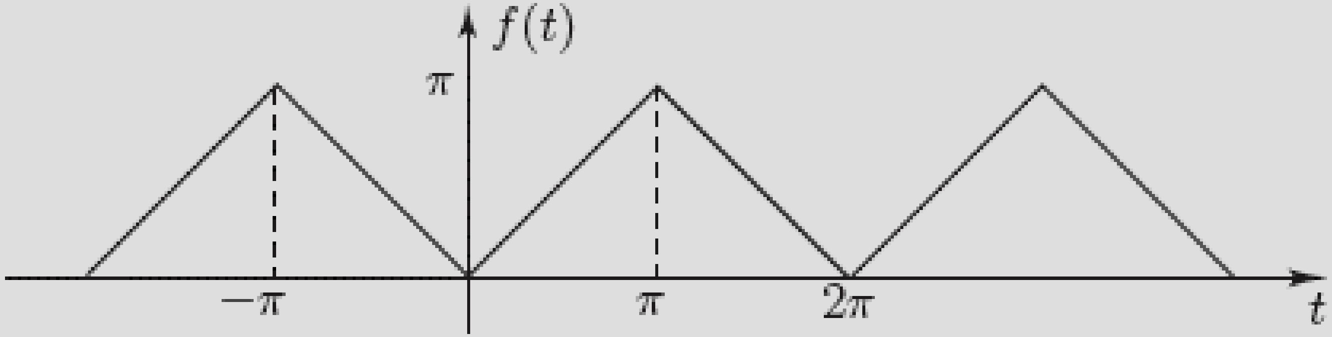


Figure 7

Here we can conveniently define the function using $-\pi < t < \pi$ as the "basic period":

$$f(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$$

or, more concisely,

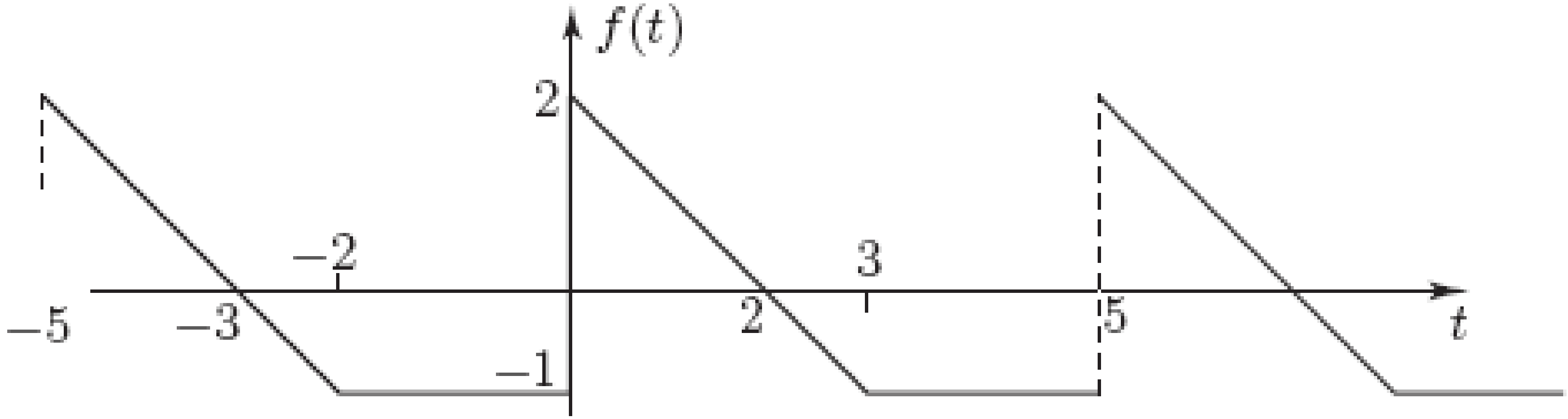
$$f(t) = |t| \qquad -\pi < t < \pi$$

together with the usual statement on periodicity

$$f(t + 2\pi) = f(t).$$

Non-sinusoidal periodic functions…

- Write down an analytic definition for the following periodic function:

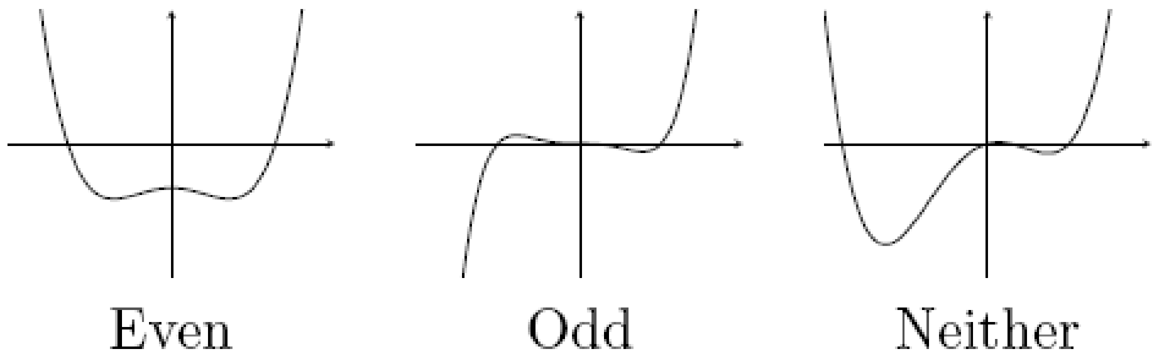


$$f(t) = \begin{cases} 2 - t & 0 < t < 3 \\ -1 & 3 < t < 5 \end{cases} \qquad f(t+5) = f(t)$$

Even and Odd Functions

- A function $f(x)$ is said to be even if $f(-x) = f(x)$.
- The function $f(x)$ is said to be odd if $f(-x) = -f(x)$.

- Graphically, even functions have symmetry about the y-axis, whereas odd functions have symmetry around the origin.



Even and Odd Functions… Examples

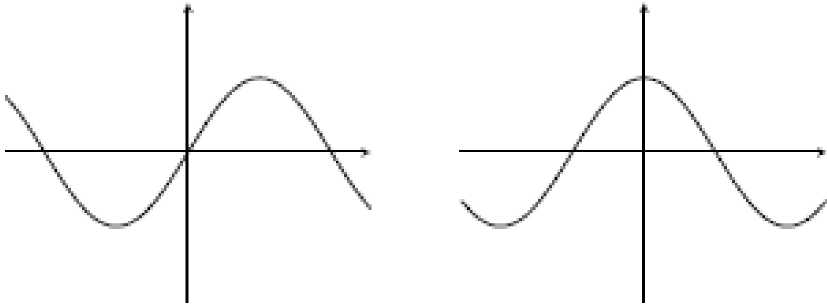
Products of functions

$$\begin{aligned}(\text{even}) \times (\text{even}) &= (\text{even})\\ (\text{even}) \times (\text{odd}) &= (\text{odd})\\ (\text{odd}) \times (\text{odd}) &= (\text{even})\end{aligned}$$

Sums of functions

$$\begin{aligned}(\text{even}) + (\text{even}) &= (\text{even})\\ (\text{even}) + (\text{odd}) &= (\text{neither})\\ (\text{odd}) + (\text{odd}) &= (\text{odd})\end{aligned}$$

- Sums of odd powers of x are odd: $5x^3 - 3x$
- Sums of even powers of x are even: $-x^6 + 4x^4 + x^2 - 3$
- $\sin x$ is odd, and $\cos x$ is even



$\sin x$ (odd) $\cos x$ (even)

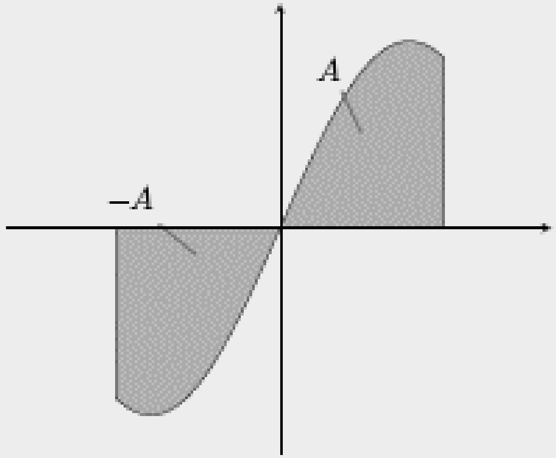
- The product of two odd functions is even: $x \sin x$ is even
- The product of two even functions is even: $x^2 \cos x$ is even
- The product of an even function and an odd function is odd: $\sin x \cos x$ is odd

Even and Odd Functions...

Let $p > 0$ be any fixed number. If $f(x)$ is an odd function, then

$$\int_{-p}^{p} f(x)\, dx = 0.$$

Intuition: The area beneath the curve on $[-p, 0]$ is the same as the area under the curve on $[0, p]$, but opposite in sign. So, they cancel each other out!

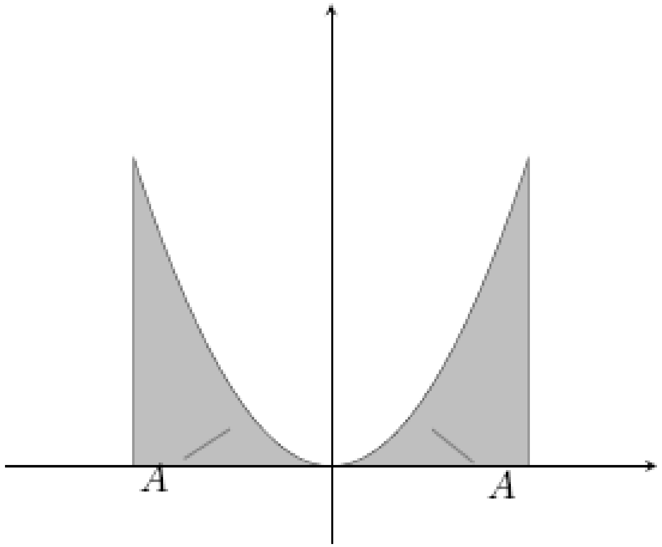


Even and Odd Functions…

Let $p > 0$ be any fixed number. If $f(x)$ is an even function, then

$$\int_{-p}^{p} f(x)\ dx = 2\int_{0}^{p} f(x)\ dx.$$

Intuition: The area beneath the curve on $[-p, 0]$ is the same as the area under the curve on $[0, p]$, but this time with the same sign. So, you can just find the area under the curve on $[0, p]$ and double it!



Representing Periodic Functions by Fourier Series

Here we discuss how a periodic function can be expressed as a series of sines and cosines.

If $f(t)$ is a periodic function, of period 2π, then the Fourier series expansion takes the form:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

Here for a_n ; $n = 0,1,2,3, \ldots$ and b_n ; $n = 1,2,3, \ldots$ are Fourier Coifficients.

Some useful Integrals

$\int_0^{2\pi} \sin nx \, dx = 0$

$\int_0^{2\pi} \cos nx \, dx = 0$

$\int_0^{2\pi} sin^2 nxdx = \pi$

$\int_0^{2\pi} cos^2 nxdx = \pi$

$\int_0^{2\pi} \sin nx \sin mx \, dx = 0$

$\int_0^{2\pi} \cos nx \cos mx \, dx = 0$

$\int_0^{2\pi} \sin nx \cos mx \, dx = 0$

$\int_0^{2\pi} \sin nx \cos nx \, dx = 0$

$\sin n\pi = 0, \qquad \cos n\pi = (-1)^n \; ; \; n \in I$

Calculation of Fourier coefficients

Consider the Fourier Series for a function $f(t)$ of period 2π ;

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

To obtain the coefficients a_n $(n = 1, 2, 3, \ldots)$, multiply both sides by $cosmt$ where m is some positive integer and Integrate both sides from $-\pi$ to π (or 0 to 2π).

$$\int_{-\pi}^{\pi} \mathrm{f(t)} \cos mt \, dt = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mt \, dt + \sum_{n=1}^{\infty} \left\{ a_n \int_{-\pi}^{\pi} \cos nt \cos mt \, dt + b_n \int_{-\pi}^{\pi} \sin nt \cos mt \, dt \right\}$$

Calculation of Fourier coefficients…

$$\int_{-\pi}^{\pi} \mathrm{f(t)cos}\, mt\, dt = a_m \pi \qquad \left(\int_{-\pi}^{\pi} cos^2 mtdt = \pi \right)$$

Rewriting m as n,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{f(t)cos}\, nt\, dt \quad ; \quad n = 1,2,3, \dots$$

Calculation of Fourier coefficients…

By multiplying the first equation by $\sin mt$, and integration,

$$\int_{-\pi}^{\pi} \mathrm{f(t)}\sin mt\,dt = \frac{a_0}{2}\int_{-\pi}^{\pi} \sin mt\,dt + \sum_{n=1}^{\infty}\left\{a_n \int_{-\pi}^{\pi} \cos nt \sin mt\,dt + b_n \int_{-\pi}^{\pi} \sin nt \sin mt\,dt\right\}$$

All the terms on the R.H.S equal to zero except the case $n = m$.

$$\int_{-\pi}^{\pi} \mathrm{f(t)}\sin mt\,dt = b_m\pi \qquad \left(\int_{-\pi}^{\pi} sin^2 mtdt = \pi\right)$$

Calculation of Fourier coefficients…

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{f(t)} sin\, mt\, dt$$

Rewriting m as n,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{f(t)} sin\, nt\, dt \quad ; \quad n = 1,2,3, \dots$$

Remarks

A function $f(t)$ with period 2π has a Fourier Series,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

The Fourier Coefficients are,

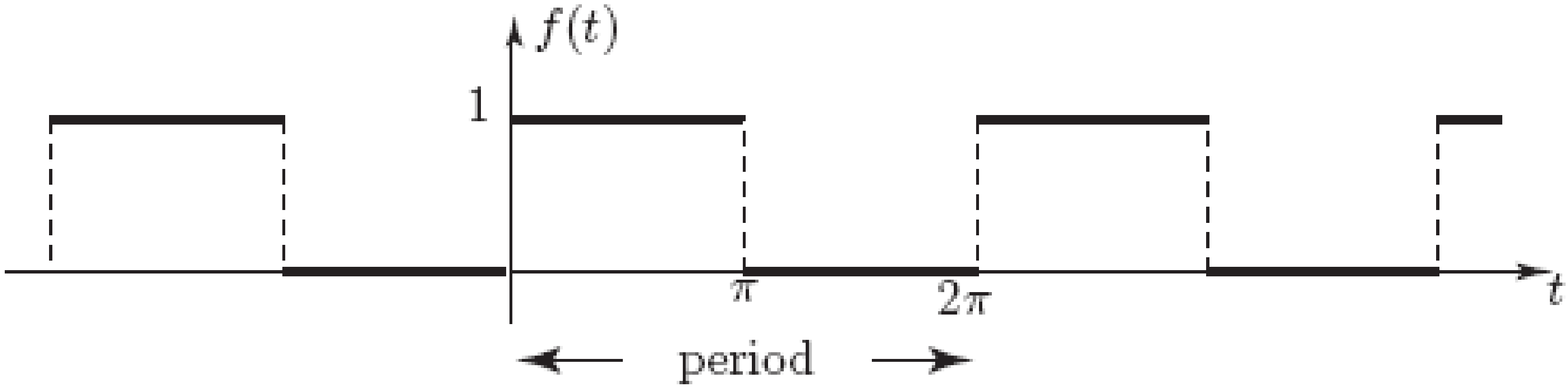
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{f(t)} \cos nt \, dt \quad ; \quad n = 0,1,2,3, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{f(t)} sin \, nt \, dt \quad ; \quad n = 1,2,3, \dots$$

Example

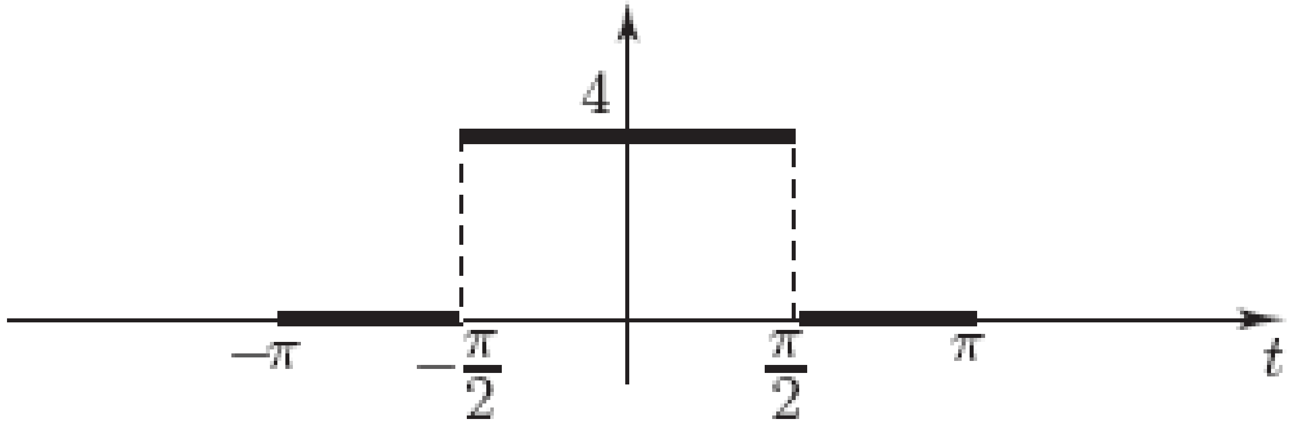
- Obtain the Fourier Series of the function $f(t)$ which is shown below.

$$f(t) = \begin{cases} 1\,; 0 < t < \pi \\ 0\,; \pi < t < 2\pi \end{cases}$$



Example

- Obtain the Fourier Series of the square wave one period of which is shown,



Expansion of Even Function

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt)$$

The Fourier Coefficients are,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(t) dt$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \mathrm{f(t)} \cos nt \, dt \quad ; \quad n = 0,1,2,3, \ldots$$

$$b_n = 0$$

Expansion of Odd Function

$$f(t) = \sum_{n=1}^{\infty} (b_n \sin nt)$$

The Fourier Coefficients are,

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \mathrm{f(t)} \sin nt \, dt \quad ; \quad n = 0,1,2,3,\ldots$$

Examples

- Find the Fourier series expansion of the periodic function of period 2π,

$$f(x) = x^2 \quad ; \quad -\pi \leq x \leq \pi$$

- Obtain the Fourier series expression for,

$$f(x) = x^3 \quad ; \quad -\pi \leq x \leq \pi$$

Half – Range Series, Period $0 \ \ to \ \ \pi$

- To get the series of Cosines only we assume that the function $f(x)$ is an even function in the interval $(-\pi, \pi)$.

  Then

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt)$$

The Fourier Coefficients are,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(t)dt$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \mathrm{f(t)} \cos nt \, dt \quad ; \quad n = 0,1,2,3, \dots$$

$$b_n = 0$$

Half – Range Series, Period $0 \;\; to \;\; \pi$

- To get the series of Sines only we assume that the function $f(x)$ is an odd function in the interval $(-\pi, \pi)$.

  Then

$$f(t) = \sum_{n=1}^{\infty} (b_n \sin nt)$$

The Fourier Coefficients are,

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \mathrm{f(t)} \sin nt \, dt \quad ; \quad n = 0,1,2,3, \ldots$$

Example

- Represent the following function by a Fourier sin series.

$$f(t) = \begin{cases} t\,; & 0 < t \leq \dfrac{\pi}{2} \\ \dfrac{\pi}{2}\,; & \dfrac{\pi}{2} < t \leq \pi \end{cases}$$

Fourier series for functions of general period

- a 2π periodic function $f(x)$ has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

- With

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad n = 0, 1, 2, \ldots \qquad\qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \qquad n = 1, 2, \ldots$$

π

- Thus $x = \pi$ corresponds to $z = c$ and $x = -\pi$ corresponds to $z = -c$
- Hence regarded as a function of z, we have a function with period $2c$.

Making the substitution $z = \frac{\pi x}{c}$ or $(x = \frac{zc}{\pi})$

and hence $\mathrm{dz} = \frac{\pi}{c} dx$

Then the function $f(x)$of period $2c$ is transformed to the function

$f\left(\frac{cz}{\pi}\right) = F(z)$ of period 2π.

Then

$$a_0 = \frac{1}{c}\int_0^{2c} f(x)dx \quad , a_n = \frac{1}{c}\int_0^{2c} \mathrm{f(x)cos}\frac{n\pi x}{c}dx, \quad b_n = \frac{1}{c}\int_0^{2c} \mathrm{f(x)sin}\frac{n\pi x}{c}dx$$

Half-Range Cosine series for functions of general period

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{c})$$

The Fourier Coefficients are,

$$a_0 = \frac{2}{c} \int_0^c f(x)dx$$

$$a_n = \frac{2}{c} \int_0^c \mathrm{f(x)} \cos \frac{n\pi x}{c} dx \quad ; \quad n = 0,1,2,3, \dots$$

$$b_n = 0$$

Half-Range Sine series for functions of general period

$$f(x) = \sum_{n=1}^{\infty} (b_n \sin \frac{n\pi x}{c})$$

The Fourier Coefficients are,

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{\pi} \int_0^c \mathrm{f(x)sin} \frac{n\pi x}{c} dx \quad ; \quad n = 0,1,2,3, \ldots$$

Example

- Find the Fourier series expansion of $f(x)$.

$$f(x) = |x| \quad , \qquad -2 < x < 2.$$

Parseval's Formula

$$\int_{-c}^{c} [f(x)]^2 dx = c\left\{\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$$

We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c})$$

Multiplying by $f(x)$ we have

$$[f(x)]^2 = \frac{a_0}{2}f(x) + \sum_{n=1}^{\infty}(a_n f(x)\cos\frac{n\pi x}{c} + b_n f(x)\sin\frac{n\pi x}{c})$$

Parseval's Formula…

- Integrating term by term from $-c\ to\ c$, we have

$$\int_{-c}^{c}[f(x)]^2dx = \frac{a_0}{2}\int_{-c}^{c}f(x)dx + \sum_{n=1}^{\infty}(a_n\int_{-c}^{c}f(x)\cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty}(b_n\int_{-c}^{c}f(x)\sin\frac{n\pi x}{c}dx$$

$$\int_{-c}^{c}[f(x)]^2dx = \frac{a_0}{2}a_0c + \sum_{n=1}^{\infty}(a_nca_n + b_ncb_n)$$

$$\int_{-c}^{c}[f(x)]^2dx = \frac{a_0^2}{2}c + \sum_{n=1}^{\infty}(a_n^2c + b_n^2c)$$

$$= c\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$$

Example

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \, . \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$

b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

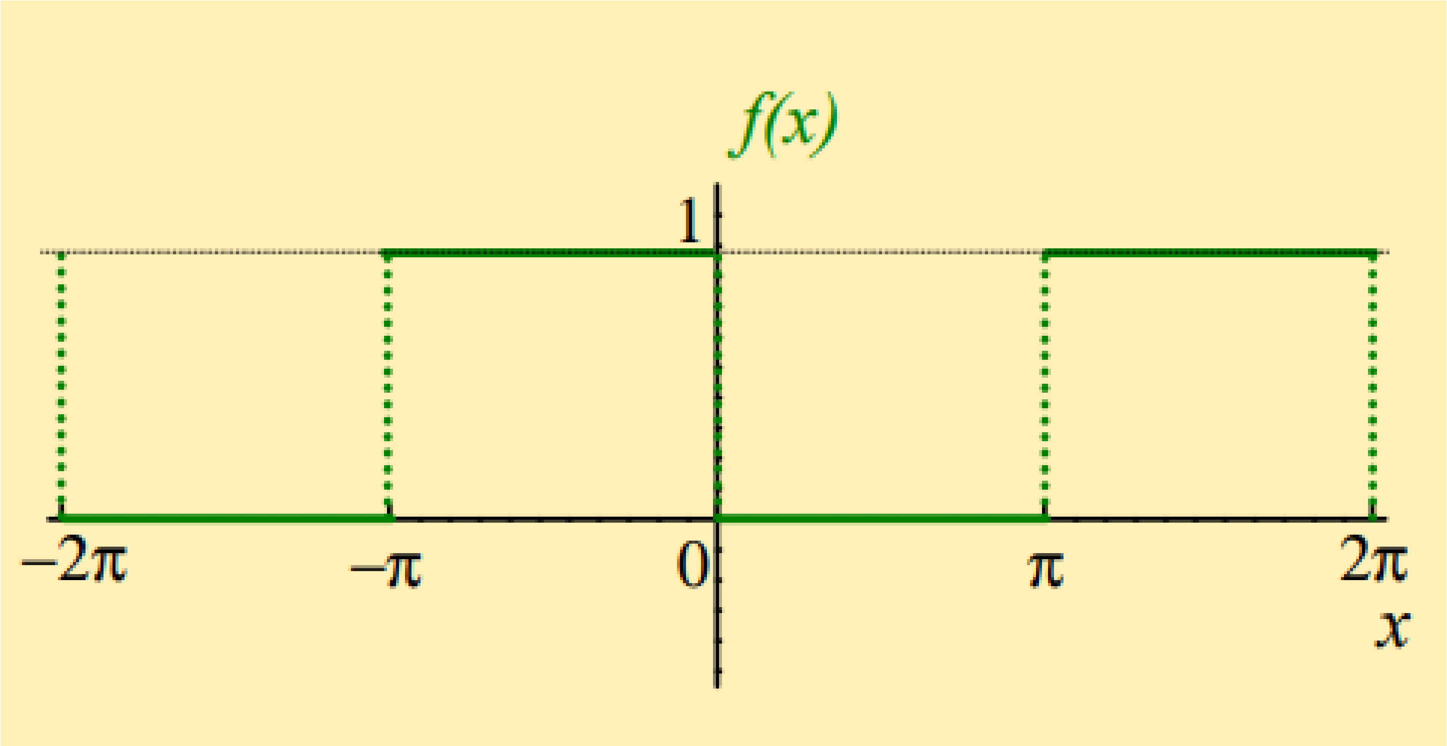
$$\frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right]$$

c) By giving an appropriate value to x, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Answer

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$
\begin{aligned}
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \mathrm{d}x \\
&= \frac{1}{\pi}\left[x\right]_{-\pi}^{0} \\
&= \frac{1}{\pi}\left(0-(-\pi)\right) \\
&= \frac{1}{\pi}\cdot(\pi) \\
\text{i.e. } a_0 &= 1\,.
\end{aligned}
$$

STEP TWO

$$
\begin{aligned}
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} = \frac{1}{n\pi}\left[\sin nx\right]_{-\pi}^{0} \\
&= \frac{1}{n\pi}\left(\sin 0 - \sin(-n\pi)\right) \\
&= \frac{1}{n\pi}(0+\sin n\pi) \\
\text{i.e. } a_n &= \frac{1}{n\pi}(0+0) = 0.
\end{aligned}
$$

STEP THREE

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\sin nx\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
\text{i.e. }\; b_n &= \frac{1}{\pi}\int_{-\pi}^{0}\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} \\
&= -\frac{1}{n\pi}\left[\cos nx\right]_{-\pi}^{0} = -\frac{1}{n\pi}\left(\cos 0 - \cos(-n\pi)\right) \\
&= -\frac{1}{n\pi}(1-\cos n\pi) = -\frac{1}{n\pi}\left(1-(-1)^n\right),\ \text{see } \textsc{Trig} \\
\text{i.e. }\; b_n &= \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n\pi} & , n \text{ odd} \end{cases} \quad , \text{since } (-1)^n = \begin{cases} 1 & , n \text{ even} \\ -1 & , n \text{ odd} \end{cases}
\end{aligned}
$$

We now have that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

with the three steps giving

$$a_0 = 1,\ \ a_n = 0\,,\ \text{and}\ \ b_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n\pi} & , n \text{ odd} \end{cases}$$

It may be helpful to construct a table of values of b_n

n	1	2	3	4	5
b_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi}\left(\frac{1}{3}\right)$	0	$-\frac{2}{\pi}\left(\frac{1}{5}\right)$

Substituting our results now gives the required series

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right]$$

c) Pick an appropriate value of x, to show that

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}$$

Comparing this series with

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right],$$

we need to introduce a minus sign in front of the constants $\frac{1}{3}, \frac{1}{7}, \dots$

So we need $\sin x = 1,\ \sin 3x = -1,\ \sin 5x = 1,\ \sin 7x = -1$, etc

The first condition of $\sin x = 1$ suggests trying $x = \frac{\pi}{2}$.

$$\begin{array}{lccccccc} \text{This choice gives} & \sin\frac{\pi}{2} & + & \frac{1}{3}\sin 3\frac{\pi}{2} & + & \frac{1}{5}\sin 5\frac{\pi}{2} & + & \frac{1}{7}\sin 7\frac{\pi}{2} \\ \text{i.e.} & 1 & - & \frac{1}{3} & + & \frac{1}{5} & - & \frac{1}{7} \end{array}$$

Looking at the graph of $f(x)$, we also have that $f(\frac{\pi}{2}) = 0$.

Picking $x = \frac{\pi}{2}$ thus gives

$$0 = \tfrac{1}{2} - \tfrac{2}{\pi}\Big[\sin\tfrac{\pi}{2} + \tfrac{1}{3}\sin\tfrac{3\pi}{2} + \tfrac{1}{5}\sin\tfrac{5\pi}{2} + \tfrac{1}{7}\sin\tfrac{7\pi}{2} + \ldots\Big]$$

$$\text{i.e. } 0 = \tfrac{1}{2} - \tfrac{2}{\pi}\Big[1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots\Big]$$

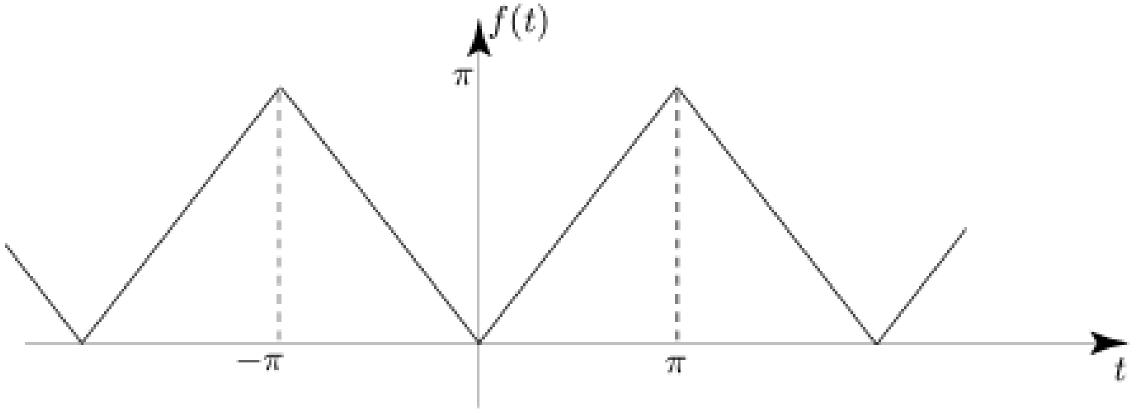
A little manipulation then gives a series representation of $\frac{\pi}{4}$

$$\frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right] = \frac{1}{2}$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.$$

Example

- Find the Fourier Series of the triangular signal shown below.



- Then use Parseval's Theorem to show that

$$\sum_{\substack{n=1 \\ (n \ \text{odd})}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}.$$

Answer

$$f(t) = |t| \qquad -\pi < t < \pi$$

$$f(t + 2\pi) = f(t)$$

We have $\dfrac{a_0}{2} = \dfrac{\pi}{2}$

$$a_n = \begin{cases} -\dfrac{4}{n^2\pi} & n = 1, 3, 5, \ldots \\ 0 & n = 2, 4, 6, \ldots \end{cases}$$

$$b_n = 0 \qquad n = 1, 2, 3, 4, \ldots$$

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ (\text{ odd } n)}}^{\infty} \frac{\cos nt}{n^2}.$$

We have $f^2(t) = t^2$ so

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{1}{2\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

The right-hand side of Parseval's theorem is

$$\frac{a_0^2}{4} + \sum_{n=1}^{\infty} a_n^2 = \frac{\pi^2}{4} + \frac{1}{2}\sum_{\substack{n=1\\(n\quad\text{odd})}}^{\infty}\frac{16}{n^4\pi^2}$$

Hence

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{\substack{n=1\\(n\quad\text{odd})}}^{\infty}\frac{1}{n^4} \qquad \therefore \qquad \frac{8}{\pi^2}\sum_{\substack{n=1\\(n\quad\text{odd})}}^{\infty}\frac{1}{n^4} = \frac{\pi^2}{12} \qquad \therefore \qquad \sum_{\substack{n=1\\(n\quad\text{odd})}}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}.$$

Complex form of Fourier Series

- Let the function $f\,(x)$ be defined on the interval $[-\pi, \pi]$. Using the well-known Euler's formulas
- $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$
- we can write the Fourier series of the function in complex form:

$$f\,(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{inx} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-inx} = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

- Here we have used the following notations:

$$c_0 = \frac{a_0}{2}, \;\; c_n = \frac{a_n - ib_n}{2}, \;\; c_{-n} = \frac{a_n + ib_n}{2}.$$

- The coefficients cn are called complex Fourier coefficients. They are defined by the formulas

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

- If necessary to expand a function f(x) of period 2L, we can use the following expressions:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}},$$

- where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

Example 1.

Using complex form, find the Fourier series of the function

$$f(x) = \operatorname{sign} x = \begin{cases} -1, & -\pi \le x \le 0 \\ 1, & 0 < x \le \pi \end{cases}.$$

Solution.

We calculate the coefficients c_0 and c_n for $n \neq 0$:

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)dx + \int_{0}^{\pi} dx\right] = \frac{1}{2\pi}\left[(-x)|_{-\pi}^{0} + x|_{0}^{\pi}\right] = \frac{1}{2\pi}(-\cancel{\pi} + \cancel{\pi}) = 0,$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)e^{-inx}dx + \int_{0}^{\pi} e^{-inx}dx\right] = \frac{1}{2\pi}\left[-\frac{\left(e^{-inx}\right)|_{-\pi}^{0}}{-in} + \frac{\left(e^{-inx}\right)|_{0}^{\pi}}{-in}\right] = \frac{i}{2\pi}$$

If $n = 2k$, then $c_{2k} = 0$. If $n = 2k-1$, then $c_{2k-1} = -\frac{2i}{(2k-1)\pi}$.

Hence, the Fourier series of the function in complex form is

$$f(x) = \operatorname{sign} x = -\frac{2i}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{2k-1} e^{i(2k-1)x}.$$

We can transform the series and write it in the real form. Rename: $n = 2k-1, n = \pm 1, \pm 2, \pm 3, \ldots$ Then

$$f(x) = \operatorname{sign} x = -\frac{2i}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{2k-1} e^{i(2k-1)x} = -\frac{2i}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n} = -\frac{2i}{\pi} \sum_{n=1}^{\infty} \left(\frac{e^{-inx}}{-n} + \frac{e^{inx}}{n} \right)$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{e^{inx} - e^{-inx}}{2in} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin (2k-1)x}{2k-1}.$$