LAPLACE TRANSFORM

CONTENT

What is Laplace Transform?

The **Laplace transform** is a mathematical operation that transforms a function of time (usually denoted as f(t)) into a function of a complex variable (denoted as s).

This transformation is widely used in engineering, physics, and control theory to analyze linear time-invariant systems, such as electrical circuits, mechanical systems, and signal processing problems.

Definition:

The Laplace transform L{f(t)} of a function f(t) is given by the following integral:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

Where:

- f(t) is the original function of time.
- e^{-st} is a decaying exponential.
- s is a complex number refers to the complex frequency or the Laplace variable, where s=σ+jω, where σ and ω are real numbers and j is the imaginary unit.
- F(s) is the Laplace-transformed function in the complex s-domain.

Linear Time-Invariant Systems

Linear Time-Invariant (LTI) systems are a fundamental class of systems in signal processing, control theory, and electrical engineering. These systems are characterized by two key properties: **linearity** and **time invariance**. Let's break down these properties:

1. Linearity

A system is linear if it satisfies the principle of superposition. This means that the response to a weighted sum of inputs is equal to the weighted sum of the responses to each individual input. Mathematically, if we have two inputs $x1(t)$x 1 (t) and $x2(t)$x 2 (t) and their corresponding outputs $y1(t)$y 1 (t) and $y2(t)$y 2 (t), a linear system satisfies the following:

$$\text{If } x(t) = a \cdot x_1(t) + b \cdot x_2(t),$$

then the output $y(t)$ will be:

$$y(t) = a \cdot y_1(t) + b \cdot y_2(t),$$

where a and b are constants.

2. Time Invariance

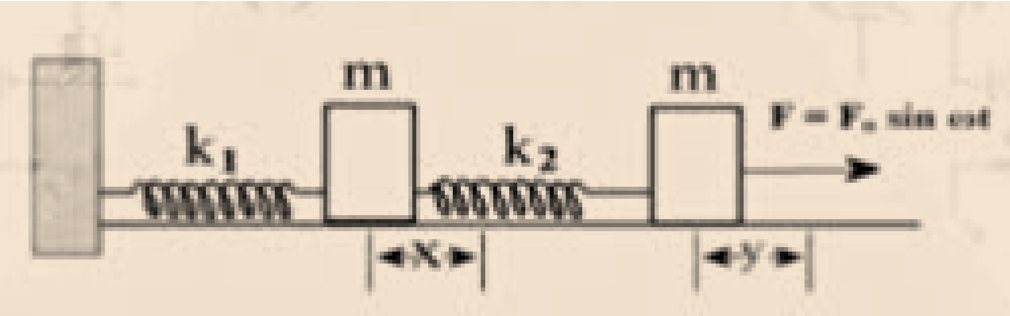
A system is time-invariant if a time shift in the input results in an identical time shift in the output. In other words, if the input $x(t)$ produces an output $y(t)$, then shifting the input by $t0$ (i.e., using $x(t-t0)$) results in the same output shifted by $t0$ (i.e., $y(t-t0)$). Mathematically:

$$\text{If } x(t) \text{ produces output } y(t),$$

then

$$x(t - t_0) \text{ produces output } y(t - t_0).$$

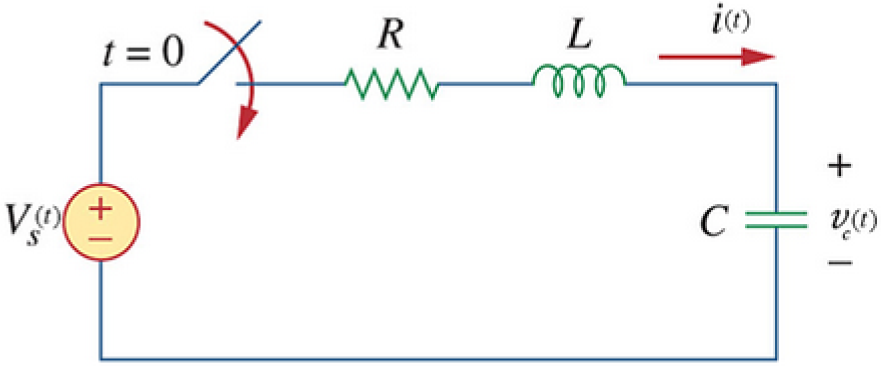
INTRODUCTION



The Laplace Transform is a widely used integral transform in mathematics.

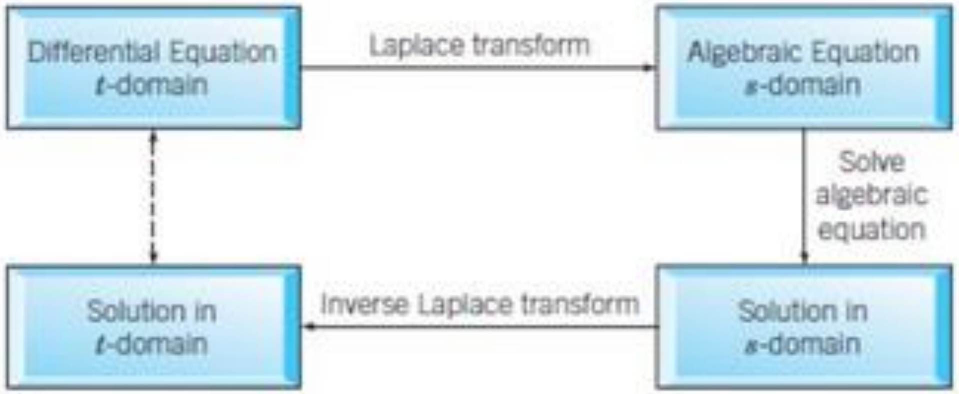
Applications in science and engineering such as

- **Electric circuit analysis,**
- **Communication engineering,**
- **Control engineering and,**
- **Nuclear physics.**



Laplace Transform is used to

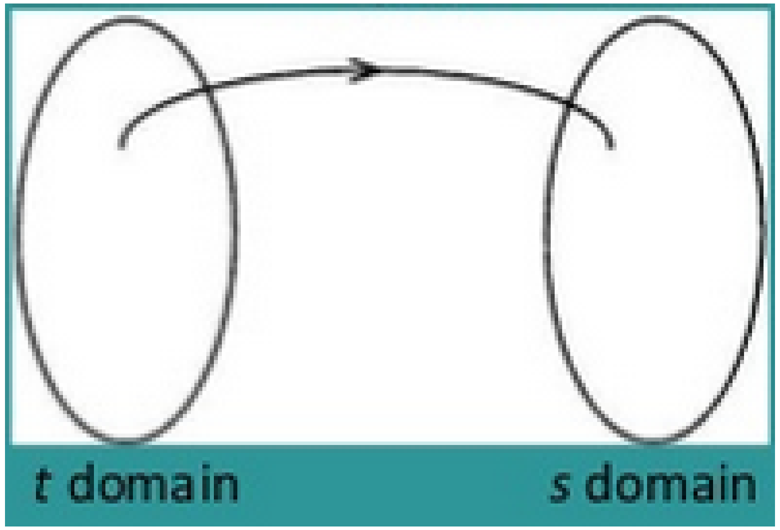
- solve linear ordinary differential equations.
- analysis and design of engineering systems.

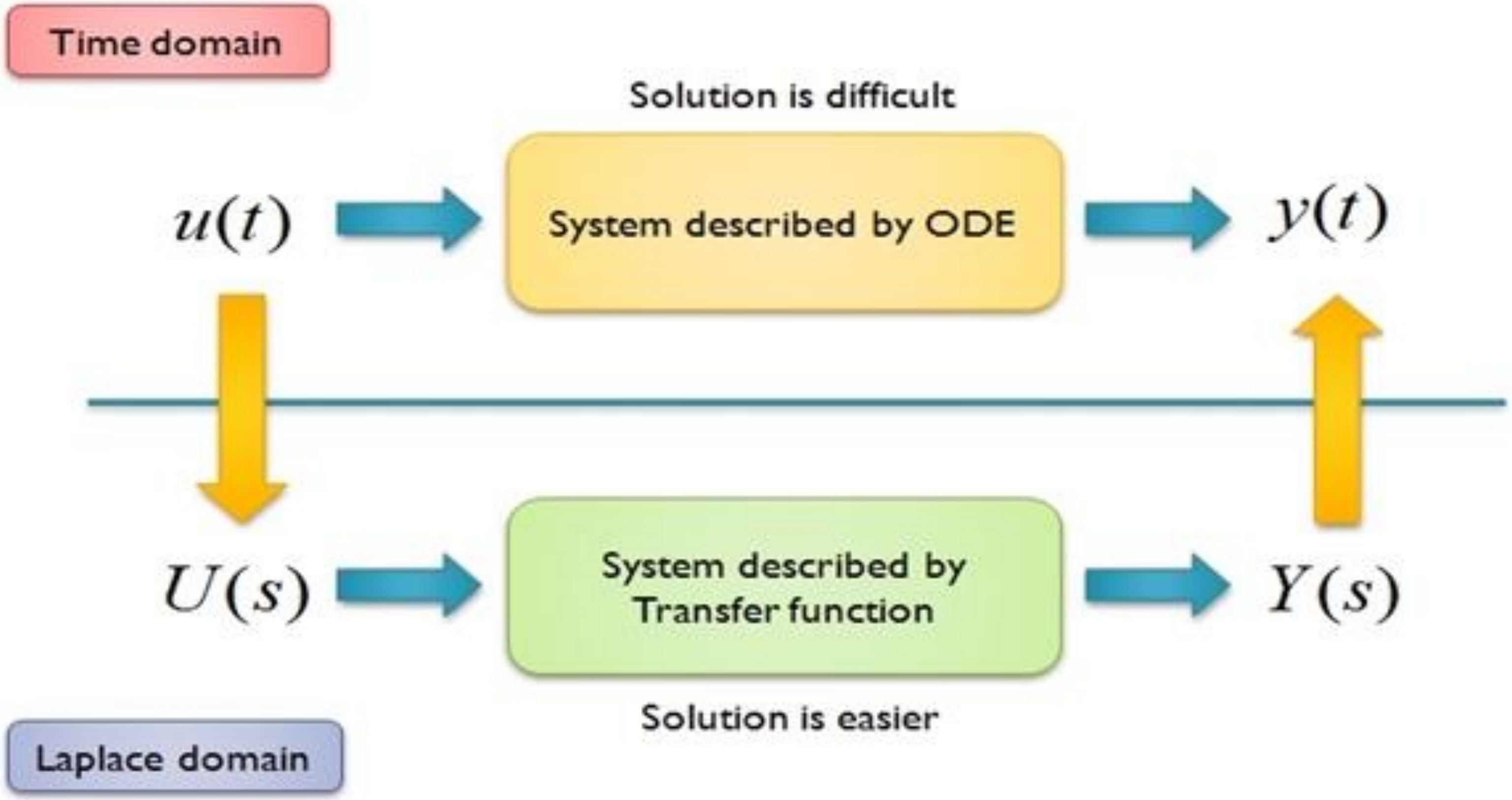


INTRODUCTION…

- It can be interpreted as a transformation from time domain (where inputs and outputs are functions of time) to the frequency domain (where inputs and outputs are functions of complex angular frequency).

- Laplace transformation is normally applied causal or one-sided functions.





LAPLACE TRANSFORM

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

Let $f(t)$ be a function of t $(t > 0)$, then the integral $\int_0^\infty e^{-st} f(t)dt$ is called Laplace Transform of $f(t)$.

We denote it as $L[\mathrm{f(t)}]$ or $F(s)$.

i.e.

$$L[\mathrm{f(t)}] = \int_0^\infty e^{-st} f(t)dt = F(s)$$

LAPLACE TRANSFORM

Find the Laplace Transform of the following function $f(t)$:

- $f(t) = 1$ (ans: ${}^{1}/_{s}$)
- $f(t) = e^{at}$ (ans: ${}^{1}/_{s-a}$)
- $f(t) = \mathrm{t}$ (ans: ${}^{1}/_{s^2}$)
- $f(t) = \cos at$ (ans: ${}^{s}/_{(s^2+a^2)}$)
- $f(t) = e^{at}u(t)$
- $f(t) = e^{-at}u(t)$
- $f(t) = e^{at}u(-t)$
- $f(t) = e^{-at}u(-t)$

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{\left(s^2+a^2\right)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{\left(s^2+a^2\right)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{\left(s^2+a^2\right)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{\left(s^2+a^2\right)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s\left(s^2-a^2\right)}{\left(s^2+a^2\right)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s\left(s^2+3a^2\right)}{\left(s^2+a^2\right)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$

	$f(t)$	$F(s)$		$f(t)$	$F(s)$
17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^nf(t),\quad n=1,2,3,\ldots$	$(-1)^nF^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

The following table highlights some of the important properties of Laplace transform

Property	Function $x(t)$	Laplace Transform $X(s)$
Notation	$x_1(t)$	$X_1(s)$
	$x_2(t)$	$X_2(s)$
Scalar Multiplication	$kx(t)$	$kX(s)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$
Time Shifting	$x(t - t_0)$	$e^{-t_0 s} X(s)$
Frequency Shifting	$e^{-at} x(t)$	$X(s+a)$
Time Scaling	$x(at)$	$\frac{1}{\|a\|} X(\frac{s}{a})$
	$\frac{d}{dt} x(t)$	$sX(s) - x(0^-)$

properties of Laplace transform

Differentiation in Time Domain	$\frac{d^2}{dt^2}\, x(t)$	$s^2 X(s) - sx(0^-) - \frac{d}{dt}\, x(0^-)$
	$\frac{d^n}{dt^n}\, x(t)$	$s^n X(s) - s^{(n-1)} x(0^-) - \dots - \frac{d^{(n-1)}}{dt^{(n-1)}} x(0^-)$
Integration in Time Domain	$\int_{0^-}^{t} x(\tau)\, d\tau$	$\frac{X(s)}{s}$
	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(s)}{s} + \frac{1}{s} \int_{-\infty}^{0^-} x(\tau)\, d\tau$
Differentiation in Frequency Domain	$tx(t)$	$-\frac{d}{ds} X(s)$
	$t^n x(t)$	$(-1)^n \frac{d^n}{ds^n} X(s)$
Integration in Frequency Domain	$\frac{x(t)}{t}$	$\int_s^{\infty} X(s)\, ds$
Time Convolution	$x_1(t) * x_2(t)$	$X_1(s) X_2(s)$
Convolution in Frequency Domain	$x_1(t) x_2(t)$	$\frac{1}{2\pi j} X_1(s) * X_2(s) = \frac{1}{2\pi j} \int_{(c-j\infty)}^{(c+j\infty)} X_1(p) X_2(s-p)\, dp$

Linearity Property:

If $f(t)$ and $g(t)$ are any two functions of t and α , β are any two constant then, $L[\alpha f(t) + \beta g(t)] = \alpha L[f(t)] + \beta L[g(t)]$

- **Examples:**
- Find $L\{10t - 5\} = \frac{10}{s^2} - \frac{5}{s}$
- Find L$\{cos^2 t\}$
- Find $L\{2e^{-5t}u(t)$) - $15e^{4t}u(-t)$) }

$$L[x(t)] = L[x_1(t)] + L[x_2(t)] = \frac{2}{(s+5)} + \frac{15}{(s-4)}$$

$$\Rightarrow L[x(t)] = L[2e^{-5t}u(t) - \ 15e^{4t}u(-t)] = \frac{17s-83}{s^2+s-20}$$

PROPERTIES OF LAPLACE TRANSFORM

Shift in time: If f(t) is shifted by t_0, the Laplace transform becomes:

L{f(t–t_0)}=e^{-st_0}L{f(t)}

Show it!

Find $\boldsymbol{L}\{\mathrm{u}(\mathrm{t}{-}5)\}$

Shifting Property:

If $L[\mathrm{f(t)}]{=}F(s)$, then $L[e^{at}\mathrm{f(t)}]{=}F(s-a)$

Example: Find $L[e^{3t}\sin t\,]$

PROPERTIES OF LAPLACE TRANSFORM

Multiplication by t^n Property:

If $L[\mathrm{f(t)}]=F(s)$, then $L[t^n\mathrm{f(t)}]=(-1)^n\frac{d^n}{ds^n}[F(s)]$

Examples

Find $L\{t\cos 2t\}$

Find $L\{t^2\cos 3t\}$

Examples

Find the Laplace Transform of the following functions:

- $f(t) = e^{at}t^n$
- $f(t) = e^{at}\cos bt$
- $f(t) = e^{at}\sin bt$
- $f(t) = e^{at}\sinh bt$

- $f(t) = tsinhat$
- $f(t) = \mathrm{t}\cos at$
- $f(t) = t^2 e^t \sin 4t$

Laplace Transform of a Derivative of $f(t)$

- **If** $L[\mathrm{f(t)}]=F(s)$, then

$$L[f^n(\mathrm{t})]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\cdots-f^{n-1}(0)$$

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i.e.

$$L[f'(\mathrm{t})]=sF(s)-f(0)$$
$$L[f''(\mathrm{t})]=s^2F(s)-sf(0)-f'(0)$$
$$L[f'''(\mathrm{t})]=s^3F(s)-s^2f(0)-sf'(0)-f''(0)$$

Show that $L[f'(\mathrm{t})]=sF(s)-f(0)$

$$\begin{aligned}\mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^\infty \frac{df}{dt}e^{-st}\,dt \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt \\ &= -f(0)+sF(s).\end{aligned}$$

Further Laplace Transforms

- Laplace Transform of Integral of $f(t)$:

If $L[\mathrm{f(t)}]=F(s)$, then $L\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}F(s) = \frac{1}{s}L[\mathrm{f(t)}]$

- Laplace Transform of $\frac{1}{t}f(t)$:

If $L[\mathrm{f(t)}]=F(s)$, then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)ds$

CAUSAL FUNCTIONS

These are functions f(t) of a single variable t such that f(t) = 0 if t < 0.

In particular we consider the simplest causal function: the unit step function (often called the Heaviside function) u(t):

$$u(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

Use this function to show how signals (functions of time t) may be 'switched on' and 'switched off'.

Examples for Causal Functions

$$u(t-3) = \begin{cases} 1 \;\; if\; t-3 \geq 0 \\ 0 \;\; if\; t-3 < 0 \end{cases} \quad \text{or} \quad u(t-3) = \begin{cases} 1 \;\; if\; t \geq 3 \\ 0 \;\; if\; t < 3 \end{cases}$$

$$u(t-a) = \begin{cases} 1 \;\; if\; t-a \geq 0 \\ 0 \;\; if\; t-a < 0 \end{cases} \quad \text{or} \quad u(t-a) = \begin{cases} 1 \;\; if\; t \geq a \\ 0 \;\; if\; t < a \end{cases}$$

The step-function has a useful property: Multiplying an ordinary function $f(t)$ by the step function $u(t)$ changes into a causal function.

i.e. If $f(t) = sint$ then, $sint\, u(t)$ is causal.

Laplace Transform of Unit Function

$$u(t-a) = \begin{cases} 1 \ \ if \ t \geq a \\ 0 \ \ if \ t < a \end{cases}$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a) dt$$

$$= \int_0^a e^{-st} 0 \, dt + \int_a^\infty e^{-st} 1 \, dt$$

$$= \frac{e^{-as}}{s}$$

Second Shifting Property

If $L[\mathrm{f(t)}]=F(s)$ then,

$$L\{f(t-a)u(t-a)\}=e^{-as}F(s)$$

Examples

1. $L\{\sin(t-2)u(t-2)\}$

2. $L\{\sin\left(\frac{\pi}{2}t\right)u(t-3)\}$
3. $L\{e^{t}(1-u(t-2))\}$
4. $\mathrm{L}\{\sin 2t\, u(t-\pi)\}$

$L\{t^2u(t-3)\}$

$L\{t\text{^}2u(t-3)\} = L\{\left((t-3)+3\right)^2 u(t-3)\}$ then expand and solve.

Or

$L\{t^2u(t-3)\} = e^{-3s}L\{(t+3)^2\}$

$$u_\pi(t) = \begin{cases} 0 & \text{if } t < \pi, \\ 1 & \text{if } t \geq \pi. \end{cases}$$

$L\{\sin(t)\, u(t-\pi)\}$

$\sin(t-\pi) = \sin t \cos \pi - \sin \pi \cos t = -\sin t$

Hence,

$L\{\sin(t)\, u(t-\pi)\} = L\{-\sin(t-\pi)u(t-\pi)$

$= e^{-\pi s}\frac{1}{s^2+1}$

Additional formulae

$\sin(A+B) = \sin A \cos B + \sin B \cos A$

$\sin(A-B) = \sin A \cos B - \sin B \cos A$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$L\{e^{-2t}u_{\pi}(t)\}$

$L\{e^{-t}u_{\pi}(t)\} = L\left\{e^{-t}u(t-\pi)\right\} = e^{-\pi s}L\{e^{-(t+\pi)}\}$

Or

$L\{e^{-t}u_{\pi}(t)\} = L\left\{e^{-t}u(t-\pi)\right\} = L\{e^{-(t-\pi+\pi)}u(t-\pi)\}$

Find

$L\{e^{-2t}u_{\pi}(t)\}$

The step function enables us to represent piecewise continuous functions conveniently. For example, consider the function

$$f(t) = \begin{cases} f_0(t), & 0 \le t < t_1, \\ f_1(t), & t \ge t_1, \end{cases}$$

where we assume that f_0 and f_1 are defined on $[0,\infty)$, even though they equal f only on the indicated intervals. This assumption enables us to rewrite Equation

$$f(t) = f_0(t) + \big(f_1(t) - f_0(t)\big)u(t - t_1)$$

Examples

Express the following functions in terms of unit step functions and find its Laplace Transforms.

1. $f(t) = \begin{cases} 1 \ \ if \ 0 < t \leq 1 \\ t \ \ if \ \ 1 < t \leq 2 \\ t^2 \ \ if \ \ \ t > 2 \end{cases}$

$$f(t) = 1 + (t-1)u(t-1) + (t^2 - t)u(t-2)$$

Ans: $\frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$

Examples

Express the following functions in terms of unit functions and find its Laplace Transforms.

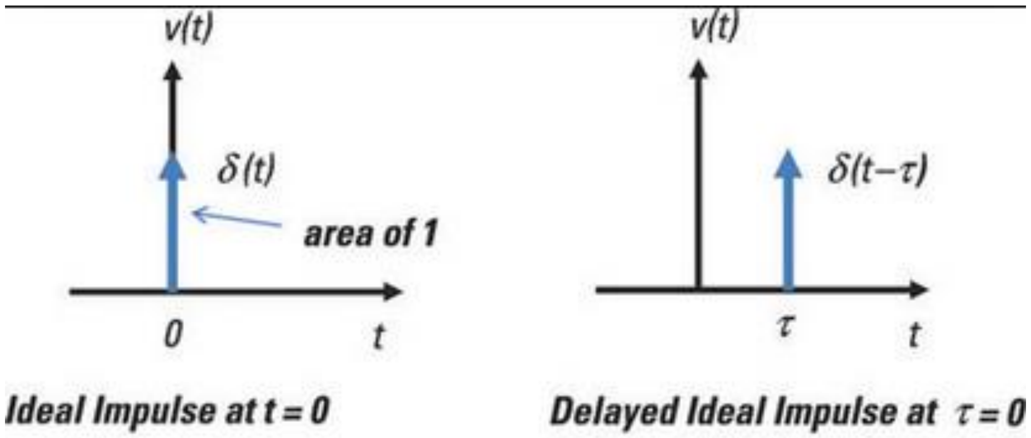
1. $f(t) = \begin{cases} t - 1 & if\ 1 < t < 2 \\ 3 - t & if\ 2 < t < 3 \end{cases}$

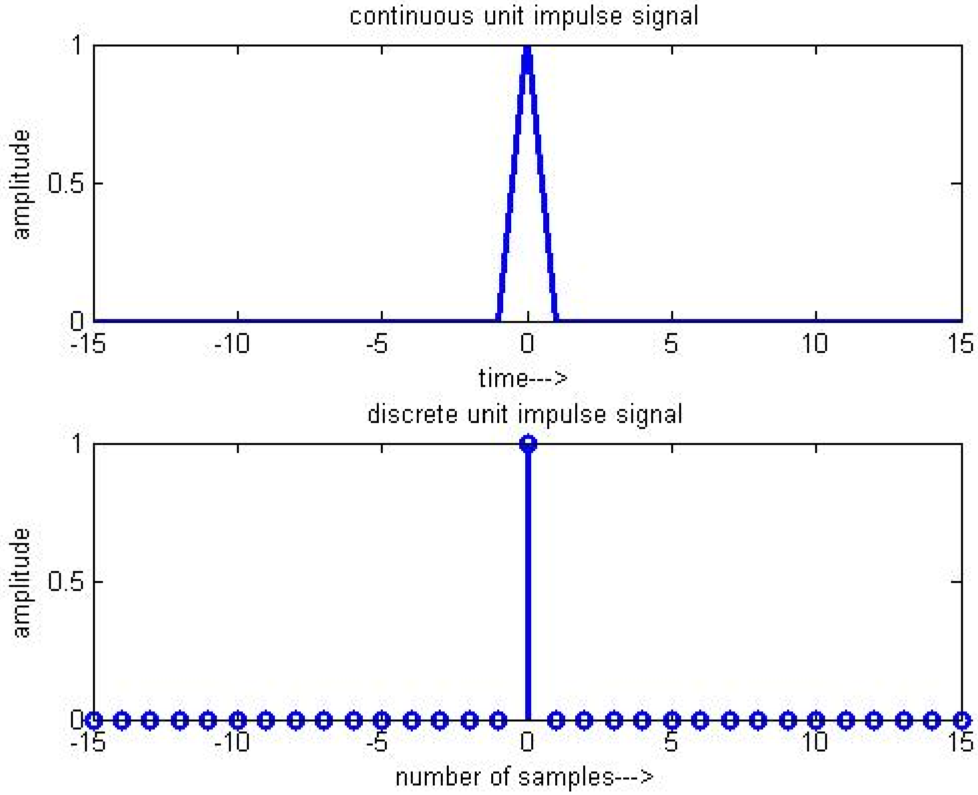
2. $f(t) = \begin{cases} 8 & if\ t < 2 \\ 6 & if\ t \geq 2 \end{cases}$

3. $f(t) = \begin{cases} E & if\ a < t < b \\ 0 & if\ \quad t > b \end{cases}$

Impulse (Delta) Functions

- Forcing functions that model impulsive actions – external forces of very short duration (and usually of very large amplitude).
- The idealized impulsive forcing function is the *Dirac delta function* (or the *unit impulse function*), denotes $\delta(t)$.





Dirac delta function and the unit impulse function

The **Dirac delta function** and the **unit impulse function** are terms that are often used interchangeably, but they have different meanings depending on the context.

1. **Dirac Delta Function** (δ(t)):

The **Dirac delta function** is a mathematical construct that is not a traditional function in the classical sense but is rather a **distribution** or a **generalized function**. It is usually denoted as δ(t), and its key properties are:

- **Sifting Property**: The Dirac delta function has the property that for any continuous function f(t),

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$$

- **Zero Everywhere Except at Zero**: The Dirac delta function is zero everywhere except at t=0, but it is not a regular function in the usual sense, as its value at t=0 is undefined.
- **Integral Equals One**: The Dirac delta function has the property that the integral over the entire real line is equal to one:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Dirac delta function and the unit impulse function

2. Unit Impulse Function:

The term **unit impulse function** is often used in the context of **signal processing** and **systems theory** to refer to the same mathematical object as the Dirac delta function. However, in certain contexts, the unit impulse is treated as a signal that "acts" like a pulse at a specific moment in time.

- **Discrete Impulse (Kronecker Delta)**: In discrete systems, the unit impulse is represented by the **Kronecker delta** δ[n], which is defined as:
- The Kronecker delta function is used in discrete-time systems and represents an impulse that occurs at n=0.

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

- **Continuous Impulse (Dirac Delta)**: In continuous-time systems, the unit impulse function is typically represented by the **Dirac delta function** δ(t), which is the continuous counterpart of the discrete unit impulse.

Impulse (Delta) Functions…

- $\delta(t) = \begin{cases} \infty \ \ for\, t = 0 \\ 0 \ \ for\, t \neq 0 \end{cases}$

- It is defined by the two properties

$$\delta(t) = 0 \quad if \ \ t \neq 0 \quad and \quad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

That is, it is a force of zero duration that is only non-zero at the exact moment t = 0, and has strength (total impulse) of 1 unit.

Translation of $\delta(t)$

The impulse can be located at arbitrary time, rather than just at $t = 0$.

For an impulse at $t = c$, we just have:

$$\delta(t-c) = 0 \quad if \;\; t \neq c \quad and$$

$$\int_{-\infty}^{\infty} \delta(t-c)dt = 1$$

$$\delta(t-c) = \begin{cases} \infty \;\; for\, t = c \\ 0 \;\; for\, t \neq c \end{cases}$$

Laplace transforms of Dirac delta functions

$L\{\delta(t)\} = 1 \quad , \quad L\{\delta(t-c)\} = e^{-cs} \quad , \quad c > 0$

An important and interesting property of the Dirac delta function:

If $f(t)$ is any continuous function, then

$$\int_{-\infty}^{\infty} \delta(t-c) f(t) dt = f(c)$$

$$\mathcal{L}\{\delta(t-c)\, f(t)\} = \int_0^{\infty} \delta(t-c)\, f(t)\, e^{-st} dt = f(c)\, e^{-cs}$$

Examples

1. $\int_{-\infty}^{\infty} e^{-5t}\delta(t-2)dt$

2. $L\{t^3\delta(t-4)\}$

INVERSE LAPLACE TRNASFORMS

The Laplace transform takes a causal function $f(t)$ and transforms it into a function of s, $F(s)$:

$$L\{f(t)\} = F(s)$$

The inverse Laplace transform operator is denoted by L^{-1} and involves recovering the original causal function $f(t)$. That is,

$$L^{-1}\{F(s)\} = f(t) \text{ where } L\{f(t)\} = F(s)$$

Examples

1. $L^{-1}\left\{\frac{s}{s^2+4}\right\}$

2. $L^{-1}\left\{\frac{1}{s-2}\right\}$

3. $L^{-1}\left\{\frac{1}{s^2-9}\right\}$

4. $L^{-1}\left\{\frac{1}{s^2+25}\right\}$

5. $L^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}$

6. $L^{-1}\left\{\frac{s+2}{(s+2)^2-25}\right\}$

7. $L^{-1}\left\{\frac{1}{2s-7}\right\}$

CONVOLUTION

This section describes the convolution of two functions f(t), g(t) which we denote by

$$(f * g)(t).$$

The convolution is an important construct because of the convolution theorem which allows us to find the inverse Laplace transform of a product of two transformed functions:

$$L^{-1}\{F(s)G(s)\} = (f * g)(t)$$

In mathematics convolution is a mathematical operation on two functions to produce a third function that expresses how the shape of one is modified by the other.

CONVOLUTION ...

Let f(t) and g(t) be two functions of t. The convolution of f(t) and g(t) is also a function of t, denoted by $(f\ *\ g)(t)$ and is defined by the relation

$$(f\ *\ g)(t)\ =\ \int_{-\infty}^{\infty} f(t\ -\ x)g(x)\,dx$$

However if f and g are both causal functions then (strictly) f(t), g(t) are written f(t)u(t) and g(t)u(t) respectively, so that

$$(f\ *\ g)(t) =\ \int_{-\infty}^{\infty} f(t\ -\ x)u(t-x)g(x)u(x)\,dx = \int_{0}^{t} f(t\ -\ x)g(x)\,dx$$

because of the properties of the step functions.

If f(t) and g(t) are causal functions then their convolution is defined by:

$$(f\ *\ g)(t)\ =\ \int_{0}^{t} f(t\ -\ x)g(x)\,dx$$

The convolution theorem

Let f(t) and g(t) be causal functions with Laplace transforms F(s) and G(s) respectively.

i.e.

L{f(t)} = F(s) and L{g(t)} = G(s). Then it can be shown that

$$L^{-1}\{F(s)G(s)\} = (f * g)(t)$$

or equivalently

$$L\{(f * g)(t)\} = F(s)G(s)$$

Commutativity Property of Convolution

$$(f * g)(t) = (g * f)(t)$$

The convolution of f(t) with g(t) is the same as the convolution of g(t) with f(t).

Examples

1. Find the convolution of f and g if $f(t) = tu(t)$ and $g(t) = t^2u(t)$.

2. Find the convolution of $f(t) = t.u(t)$ and $g(t) = \sin t.u(t)$.

3. Obtain the Laplace transforms of $f(t) = t.u(t)$ and $g(t) = \sin t.u(t)$ and $(f * g)(t)$.

Examples

1. Find the inverse transform of $\dfrac{6}{s(s^2+9)}$.

(a) Using partial fractions (b) Using the convolution theorem.

1. Use the convolution theorem to find the inverse transform of
$$H(s) = \frac{s}{(s-1)(s^2+1)}.$$

Solving Ordinary Differential Equations using Laplace Transform

This section explain how to use Laplace transform to solve constant coefficient ordinary differential equations. Particularly, initial value problems.

Procedure:

- Take the Laplace transform of each term in the differential equation.
- If the unknown function is y(t) then, on taking the transform, obtain an algebraic equation involving Y (s) = L{y(t)}.
- Solve the equation for Y (s).
- Obtain required solution y(t) by taking Inverse Laplace $y(t) = L^{-1}\{Y\ (s)\}$.

Examples

use the Laplace transform method to solve the followings..

1. $\frac{dy}{dt} + 2y = 12e^{3t} \quad ; \; y(0) = 3$

2. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = e^{-t} \quad ; \; y(0) = 0, \; y'(0) = 0$

3. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 3\delta; \quad y(0) = 0, \; y'(0) = 0$

Solving systems of differential equations

The Laplace transform method is also well suited to solving systems of differential equations.

Procedure

- Let x(t), y(t) be two independent functions which satisfy the coupled differential equations.
- Taking the Laplace transform converts a system of differential equations into a system of algebraic simultaneous equations.
- solve these algebraic equations and then apply Inverse Laplace Transforms to obtain x(t), y(t).

Examples

1. $\frac{dx}{dt} + y = e^{-t} \quad ;$

   $\frac{dy}{dt} - x = 3e^{-t} \quad ; \quad x(0) = 0, \quad y(0) = 1$

2. $\frac{dy}{dt} - x = 0 \quad ;$

   $\frac{dx}{dt} + y = 1 \quad ; \qquad x(0) = -1, \; y(0) = 1$

Applications of Systems of Differential Equations

Electrical circuits

Consider the RL (resistance/inductance) circuit with a voltage $v(t)$ applied as shown in Figure 17.

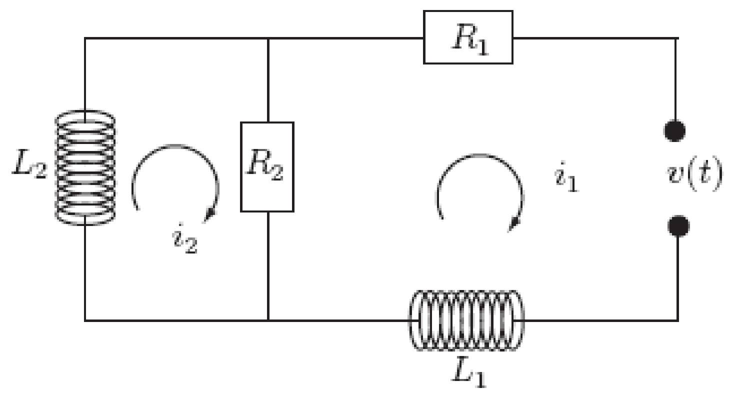


Figure 17

If i_1 and i_2 denote the currents in each loop we obtain, using Kirchhoff's voltage law:

(i) in the right loop: $$L_1\frac{di_1}{dt} + R_2(i_1 - i_2) + R_1 i_1 = v(t)$$

(ii) in the left loop: $$L_2\frac{di_2}{dt} + R_2(i_2 - i_1) = 0$$

Suppose, in the above circuit, that

$L_1 = 0.8$ henry, $L_2 = 1$ henry, $R_1 = 1.4\ \Omega$ $R_2 = 1\ \Omega$.

Assume zero initial conditions: $i_1(0) = i_2(0) = 0$.

Suppose that the applied voltage is constant: $v(t) = 100$ volts $t \geq 0$.

Solve the problem by Laplace transforms.

Applications of Systems of Differential Equations…

Two masses on springs

Consider the vibrating system shown:

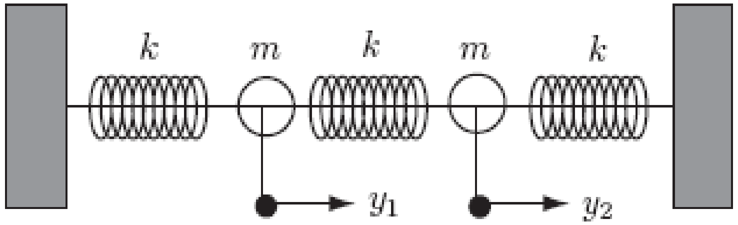


Figure 18

As you can see, the system consists of two equal masses, both m, and 3 springs of the same stiffness k. The governing differential equations can be obtained by applying Newton's second law ('force equals mass times acceleration'): (recall that a single spring of stiffness k will experience a force $-ky$ if it is displaced a distance y from its equilibrium.)

In our system therefore

$$m\frac{d^2y_1}{dt^2} = -ky_1 + k(y_2 - y_1)$$

$$m\frac{d^2y_2}{dt^2} = -k(y_2 - y_1) - ky_2$$

which is a **pair** of second order differential equations.

For the above system, if $m = 1$, $k = 2$ and the initial conditions are

$$y_1(0) = 1 \quad y_1'(0) = \sqrt{6} \quad y_2(0) = 1 \quad y_2'(0) = -\sqrt{6}$$

use Laplace transforms to solve the system of differential equations to find $y_1(t)$ and $y_2(t)$.