Exercise for the Lecture on Materials Science

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Exercise Sheet 3

Solution: Appendix

1. Model Assumptions

Each Kelvin-Voigt element satisfies:

$$\sigma = E_i\varepsilon_i + \eta_i\dot{\varepsilon}_i \quad \text{(I)} \tag{1}$$

For the series configuration, the following holds:

$$\sigma = \sigma_1 = \sigma_2, \tag{2}$$
$$\varepsilon = \varepsilon_1 + \varepsilon_2, \tag{3}$$
$$\dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 \tag{4}$$

2. Rearrangement and Substitution

From equation (I), for each element:

$$\varepsilon_1 = \frac{\sigma - \eta_1\dot{\varepsilon}_1}{E_1}, \tag{5}$$
$$\varepsilon_2 = \frac{\sigma - \eta_2\dot{\varepsilon}_2}{E_2} \tag{6}$$

Substitute into the total strain:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \tag{7}$$
$$= \frac{\sigma - \eta_1\dot{\varepsilon}_1}{E_1} + \frac{\sigma - \eta_2\dot{\varepsilon}_2}{E_2} \tag{8}$$

Multiplying by E_1E_2:

$$(E_1 + E_2)\sigma = E_1E_2\varepsilon + E_2\eta_1\dot{\varepsilon}_1 + E_1\eta_2\dot{\varepsilon}_2 \tag{9}$$

Use:

$$\dot{\varepsilon}_2 = \dot{\varepsilon} - \dot{\varepsilon}_1 \tag{10}$$

Substitute and simplify:

$$(E_1 + E_2)\sigma = E_1E_2\varepsilon + E_2\eta_1\dot{\varepsilon}_1 + E_1\eta_2(\dot{\varepsilon} - \dot{\varepsilon}_1) \tag{11}$$
$$= E_1E_2\varepsilon + (E_2\eta_1 - E_1\eta_2)\dot{\varepsilon}_1 + E_1\eta_2\dot{\varepsilon} \tag{12}$$

Solving for $\dot{\varepsilon}_1$:

$$\dot{\varepsilon}_1 = \frac{(E_1 + E_2)\sigma - E_1E_2\varepsilon - E_1\eta_2\dot{\varepsilon}}{E_2\eta_1 - E_1\eta_2} \tag{13}$$

3. Time Derivative of the Equation

Differentiate equation (10) with respect to time:

$$(E_1 + E_2)\dot{\sigma} = E_1 E_2 \dot{\varepsilon} + (E_2\eta_1 - E_1\eta_2)\ddot{\varepsilon}_1 + E_1\eta_2\ddot{\varepsilon} \tag{14}$$

Differentiating $\dot{\varepsilon}_1$:

$$\ddot{\varepsilon}_1 = \frac{(E_1 + E_2)\dot{\sigma} - E_1 E_2 \dot{\varepsilon} - E_1\eta_2\ddot{\varepsilon}}{E_2\eta_1 - E_1\eta_2} \tag{15}$$

Substitute into equation (13):

$$(E_1 + E_2)\dot{\sigma} = E_1 E_2 \dot{\varepsilon} + (E_2\eta_1 - E_1\eta_2) \cdot \left(\frac{(E_1 + E_2)\dot{\sigma} - E_1 E_2 \dot{\varepsilon} - E_1\eta_2\ddot{\varepsilon}}{E_2\eta_1 - E_1\eta_2} \right) + E_1\eta_2\ddot{\varepsilon} \tag{16}$$

$$= (E_1 + E_2)\dot{\sigma} \quad \text{(simplifies)} \tag{17}$$

Conclusion: The difference cancels out — the equation is consistent.

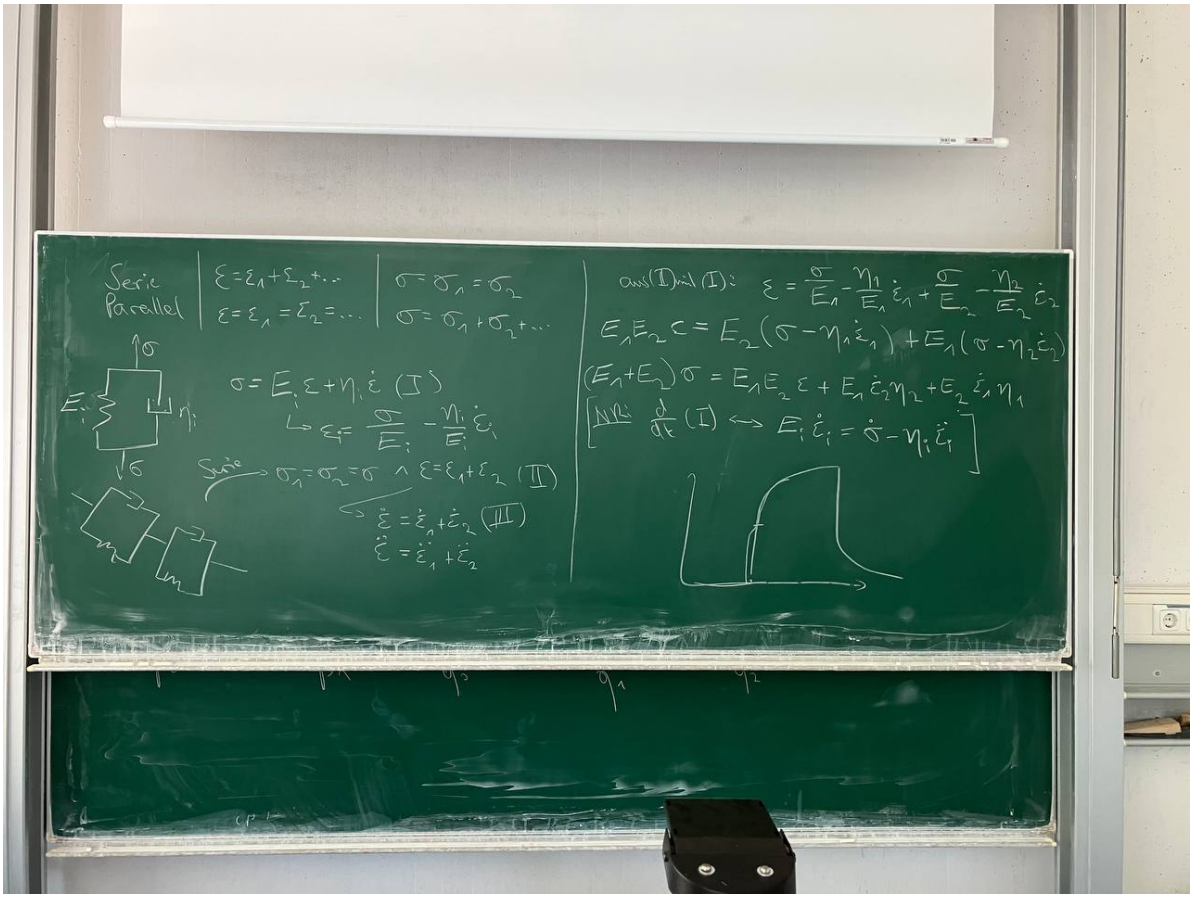
4. Final Form of the Differential Equation

Alternative expression by substitution:

$$(E_1 + E_2)\sigma + (\eta_1 + \eta_2)\dot{\sigma} = E_1 E_2 \varepsilon + (E_1\eta_2 + E_2\eta_1)\dot{\varepsilon} + \eta_1\eta_2\ddot{\varepsilon} \tag{18}$$

5. Definition of the Coefficients

$$\begin{aligned} p_0 &= E_1 + E_2 \\ p_1 &= \eta_1 + \eta_2 \\ q_0 &= E_1 E_2 \\ q_1 &= E_1\eta_2 + E_2\eta_1 \\ q_2 &= \eta_1\eta_2 \end{aligned}$$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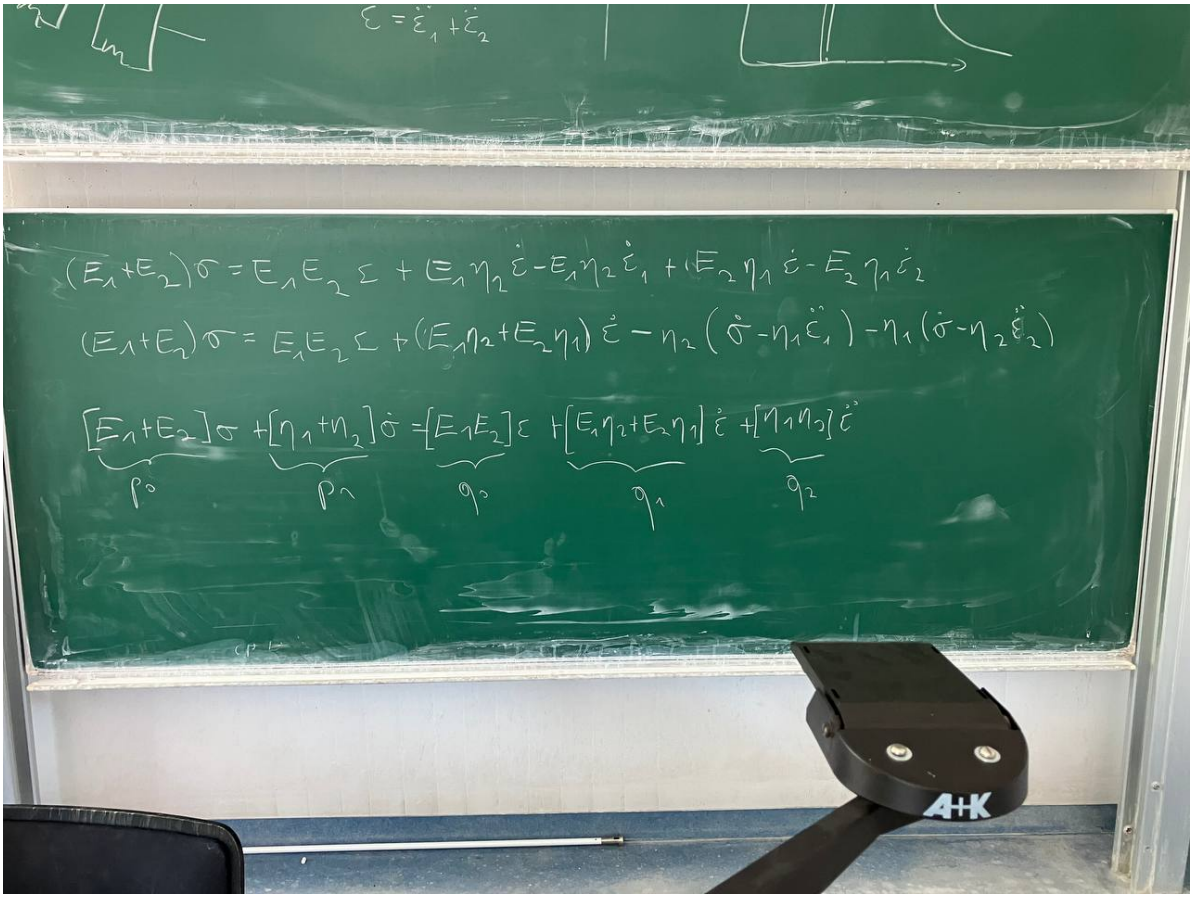
Parallel
$\varepsilon = \varepsilon_1 + \varepsilon_2 + \dots$
$\varepsilon = \varepsilon_1 = \varepsilon_2 = \dots$
$\sigma = \sigma_1 = \sigma_2$
$\sigma = \sigma_1 + \sigma_2 + \dots$
$\sigma = E_i \varepsilon + \eta_i \dot{\varepsilon}$ (I)
$\varepsilon = \frac{\sigma}{E_i} - \frac{\eta_i}{E_i}\dot{\varepsilon}_i$
Série $\sigma_1 = \sigma_2 = \sigma \wedge \varepsilon = \varepsilon_1 + \varepsilon_2$ (II)
$\dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2$ (III)
$\ddot{\varepsilon} = \ddot{\varepsilon}_1 + \ddot{\varepsilon}_2$
$\varepsilon = \frac{\sigma}{E_1} - \frac{\eta_1}{E_1}\dot{\varepsilon}_1 + \frac{\sigma}{E_2} - \frac{\eta_2}{E_2}\dot{\varepsilon}_2$
$E_1 E_2 \varepsilon = E_2(\sigma - \eta_1 \dot{\varepsilon}_1) + E_1(\sigma - \eta_2 \dot{\varepsilon}_2)$
$(E_1 + E_2)\sigma = E_1 E_2 \varepsilon + E_1 \dot{\varepsilon}_2 \eta_2 + E_2 \dot{\varepsilon}_1 \eta_1$
$\frac{d}{dt}$ (I) $\leftrightarrow$ $E_i \dot{\varepsilon}_i = \dot{\sigma} - \eta_i \ddot{\varepsilon}_i$


$\ddot{\varepsilon} = \ddot{\varepsilon}_1 + \ddot{\varepsilon}_2$
$(E_1+E_2)\sigma = E_1 E_2 \varepsilon + E_1\eta_2 \dot{\varepsilon} - E_1\eta_2 \dot{\varepsilon}_1 + E_2\eta_1 \dot{\varepsilon} - E_2\eta_1 \dot{\varepsilon}_2$
$(E_1+E_2)\sigma = E_1 E_2 \varepsilon + (E_1\eta_2 + E_2\eta_1)\dot{\varepsilon} - \eta_2(\dot{\sigma} - \eta_1 \ddot{\varepsilon}_1) - \eta_1(\dot{\sigma} - \eta_2 \ddot{\varepsilon}_2)$
$[E_1+E_2]\sigma + [\eta_1+\eta_2]\dot{\sigma} = [E_1 E_2]\varepsilon + [E_1\eta_2 + E_2\eta_1]\dot{\varepsilon} + [\eta_1\eta_2]\ddot{\varepsilon}$
p_0 p_1 q_0 q_1 q_2
A+K