Exercise for the Lecture on Materials Science

Prof. Peter Müller-Buschbaum, Lea Westphal, Ziyan Zhang, Doan Duy Ky Le

Exercise Sheet 3
Solution

Exercise 1: Tensile Test

1.1 Calculation of Engineering Stress and Strain

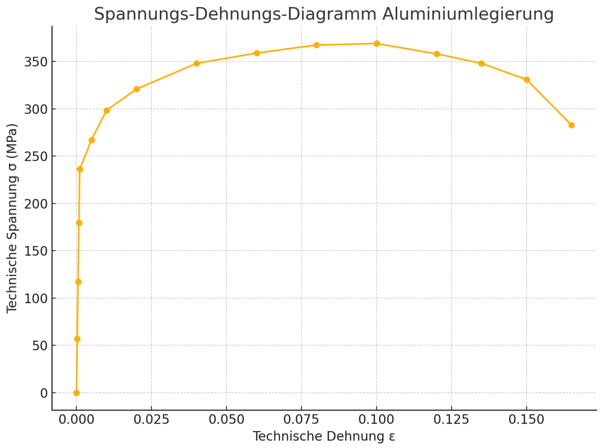


Figure 1: Stress-strain diagram of the Al alloy

First, the cross-sectional area of the cylindrical specimen is calculated. The diameter is $d = 12{,}8\,\text{mm}$, resulting in an initial area:

$$A_0 = \pi \cdot \left(\frac{d}{2}\right)^2 = \pi \cdot (6{,}4\,\text{mm})^2 \approx 128{,}68\,\text{mm}^2$$

For each measurement, the **engineering strain** is calculated as:

$$\varepsilon = \frac{l - l_0}{l_0}$$

Here, $l_0 = 50{,}800\,\text{mm}$ is the initial gauge length. The **engineering stress** is calculated as:

$$\sigma = \frac{F}{A_0}$$

As an example, for the second measurement point with $F = 7330\,\text{N}$ and $l = 50{,}813\,\text{mm}$:

$$\varepsilon = \frac{50{,}813 - 50{,}800}{50{,}800} \approx 0{,}000256, \quad \sigma = \frac{7330}{128{,}68} \approx 56{,}96\,\text{MPa}$$

1.2 Significance of the 0.2% Proof Stress

Since many metals do not exhibit a clearly defined yield point, the so-called **0.2% proof stress** is used instead. The linear initial portion of the stress-strain curve is offset by $\varepsilon = 0{,}002$ to the right. The intersection of this **offset line** with the curve defines the yield strength. This method is standardized and provides a reliable value for the onset of plastic deformation.

1.3 Material Properties from the Diagram

- **Elastic Modulus** E:
$$E \approx 219{,}03\,\text{GPa}$$
- **Yield Strength** (0.2% offset method):
$$\sigma_{0,2} \approx 267{,}33\,\text{MPa} \quad \text{at} \quad \varepsilon \approx 0{,}005$$
- **Ultimate Tensile Strength**:
$$R_m \approx 369{,}13\,\text{MPa} \quad \text{at} \quad \varepsilon \approx 0{,}10$$
- **Fracture Strain** (last measurement point):
$$\varepsilon_{\text{fracture}} \approx 0{,}165 \quad \text{(equivalent to } 16{,}5\%\text{)}$$

1.4 Volume Change and Shear Modulus

Assuming isotropic behavior and a Poisson's ratio of $\nu = 0{,}33$, the relative volume change in the elastic range is:

$$\frac{\Delta V}{V} \approx (1 - 2\nu) \cdot \varepsilon \approx (1 - 2 \cdot 0{,}33) \cdot 0{,}165 \approx 0{,}0561\ (5{,}61\,\%)$$

The **shear modulus** is given by:

$$G = \frac{E}{2(1+\nu)} = \frac{219{,}03\,\text{GPa}}{2(1+0{,}33)} \approx 82{,}34\,\text{GPa}$$

1.5 Springback Modulus

The so-called **springback modulus** E_r describes the elastic energy potential after plastic deformation:

$$E_r = \frac{\sigma_y^2}{2E} = \frac{(267{,}33\,\text{MPa})^2}{2 \cdot 219{,}03 \cdot 10^3\,\text{MPa}} \approx 0{,}163\,\text{MPa}$$

Exercise 2: Elasticity of a Rubber Strip

An elastic rubber strip with an initial length of $L_0 = 12\,\text{cm}$ and a cross-sectional area of $A = 1\,\text{cm}^2$ is stretched at room temperature by a tensile stress of $\sigma = 2\,\text{MPa}$ to a length of $L = 30\,\text{cm}$.

2.1 Calculation of the Elastic Modulus E and Crosslink Density n

First, the stretch ratio is calculated:

$$\lambda = \frac{L}{L_0} = \frac{30}{12} = 2{,}5$$

For elastomers, the stress can be described using the entropic elastic behavior of ideally crosslinked chains:

$$\sigma = nRT(\lambda - \lambda^{-2})$$

Where:

- $\sigma = 2 \cdot 10^6\,\mathrm{Pa}$
- $R = 8{,}314\,\mathrm{J/(mol \cdot K)}$
- $T = 293\,\mathrm{K}$

Substituting gives:

$$2 \cdot 10^6 = n \cdot 8{,}314 \cdot 293 \cdot (2{,}5 - 0{,}16) \Rightarrow n \approx \frac{2 \cdot 10^6}{2437 \cdot 2{,}34} \approx 351{,}2\,\mathrm{mol/m^3}$$

The elastic modulus is then:

$$E = 3nRT \approx 3 \cdot 351{,}2 \cdot 8{,}314 \cdot 293 \approx 2{,}57 \cdot 10^6\,\mathrm{Pa} = 2{,}57\,\mathrm{MPa}$$

2.2 Calculation of Stress at Other Temperatures and Lengths

a) For $L = 20\,\mathrm{cm}$ at $T = 293\,\mathrm{K}$:

$$\lambda = \frac{20}{12} \approx 1{,}667, \quad \lambda^{-2} \approx 0{,}36$$

$$\sigma = \frac{E}{3}(\lambda - \lambda^{-2}) = \frac{2{,}57}{3}(1{,}667 - 0{,}36) \approx 1{,}12\,\mathrm{MPa}$$

b) For $L = 30\,\mathrm{cm}$ at $T = 373\,\mathrm{K}$:

$$E = 3nRT = 3 \cdot 0{,}3512\,\mathrm{mol/cm^3} \cdot 8{,}314\,\mathrm{cm^3\,MPa/(mol\,K)} \cdot 373{,}15\,\mathrm{K} \approx 3{,}26\,\mathrm{MPa}$$

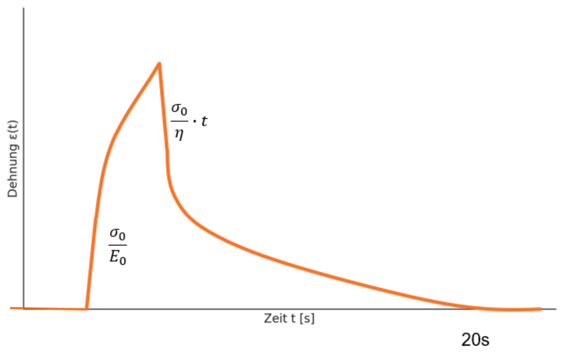
$$\lambda = 2{,}5, \quad \lambda^{-2} \approx 0{,}16$$

$$\sigma = \frac{E}{3}(\lambda - \lambda^{-2}) = \frac{3{,}26}{3}(2{,}5 - 0{,}16) \approx 2{,}55\,\mathrm{MPa}$$

Summary of Results

Case	λ	T [K]	σ [MPa]
30 cm at 20 °C	2,5	293	2,00
20 cm at 20 °C	1,667	293	1,12
30 cm at 100 °C	2,5	373	2,55

Exercise 3: Theory of Viscoelasticity



A viscoelastic material is subjected to a sudden stress increase of

$$\sigma_0 = 1000\,\mathrm{N/m^2} = 1\,\mathrm{kPa}$$

The relaxation time is:

$$\tau = 20\,\mathrm{s}$$

3.1 Qualitative Behavior of Strain $\varepsilon(t)$

The behavior of a viscoelastic material after a sudden stress jump can be described by the **Maxwell model**, consisting of a spring (elastic element) and a dashpot (viscous element) in series.

Immediately after applying the stress σ_0, the material responds with an instantaneous elastic strain:

$$\varepsilon(0) = \frac{\sigma_0}{E}$$

Subsequently, due to the viscous component, the material continues to deform linearly with time:

$$\varepsilon(t) = \frac{\sigma_0}{E} + \frac{\sigma_0}{\eta} \cdot t$$

Note on Relaxation Time τ:
In the Maxwell model, the relaxation time is defined as:

$$\tau = \frac{\eta}{E}$$

In this case, a stress-controlled (creep) experiment is considered, so τ does not explicitly appear in the strain formula. However, it characterizes the relaxation behavior under a strain step and is useful to compute η or E if the other is known.

3.2 Comparison with Ideal Elastic and Ideal Viscous Materials

Ideal Elastic:
Strain ε appears immediately and remains constant:

$$\varepsilon(t) = \frac{\sigma_0}{E}$$

Ideal Viscous:
Strain grows linearly with time:

$$\varepsilon(t) = \frac{\sigma_0}{\eta} \cdot t$$

Viscoelastic (Maxwell):
Strain is the sum of an immediate elastic component and a time-dependent viscous component:

$$\varepsilon(t) = \frac{\sigma_0}{E} + \frac{\sigma_0}{\eta} \cdot t$$

Exercise 4: Modeling Time-Dependent Deformation of a Viscoelastic Material

a) Derivation of the Differential Equation

Given: Two Kelvin-Voigt elements in series with the following parameters:

$$E_1 = 5\,\mathrm{MPa}, \quad \eta_1 = 1.2 \times 10^5\,\mathrm{Pa\,s}$$
$$E_2 = 20\,\mathrm{MPa}, \quad \eta_2 = 5 \times 10^5\,\mathrm{Pa\,s}$$

Since the elements are in series, the total strain is:

$$\varepsilon = \varepsilon_{\mathrm{KV1}} + \varepsilon_{\mathrm{KV2}}$$

The stress is equal in both elements:

$$\sigma = E_1\varepsilon_{\mathrm{KV1}} + \eta_1\dot{\varepsilon}_{\mathrm{KV1}} = E_2\varepsilon_{\mathrm{KV2}} + \eta_2\dot{\varepsilon}_{\mathrm{KV2}}$$

Substituting:

$$\varepsilon_{\mathrm{KV2}} = \varepsilon - \varepsilon_{\mathrm{KV1}}, \quad \dot{\varepsilon}_{\mathrm{KV2}} = \dot{\varepsilon} - \dot{\varepsilon}_{\mathrm{KV1}}$$
$$\dot{\varepsilon}_{\mathrm{KV1}} = \frac{\sigma - E_1\varepsilon_{\mathrm{KV1}}}{\eta_1}$$

into the constitutive law of KV2 gives:

$$\sigma = E_2(\varepsilon - \varepsilon_{\mathrm{KV1}}) + \eta_2\left(\dot{\varepsilon} - \frac{\sigma - E_1\varepsilon_{\mathrm{KV1}}}{\eta_1}\right)$$

Solving for $\varepsilon_{\mathrm{KV1}}$ and its derivative:

$$\varepsilon_{\mathrm{KV1}} = \frac{1}{E_2\eta_1 - E_1\eta_2}\left(E_2\eta_1\varepsilon + \eta_1\eta_2\dot{\varepsilon} - (\eta_1+\eta_2)\sigma\right)$$
$$\dot{\varepsilon}_{\mathrm{KV1}} = \frac{1}{E_2\eta_1 - E_1\eta_2}\left(E_2\eta_1\dot{\varepsilon} + \eta_1\eta_2\ddot{\varepsilon} - (\eta_1+\eta_2)\dot{\sigma}\right)$$

Substituting into the equation of KV1:

$$\sigma = E_1\varepsilon_{\mathrm{KV1}} + \eta_1\dot{\varepsilon}_{\mathrm{KV1}}$$

leads to the following differential equation:

$$\left(\frac{E_1+E_2}{E_1E_2}\right)\sigma + \left(\frac{\eta_1+\eta_2}{E_1E_2}\right)\dot{\sigma} = \varepsilon + \left(\frac{\eta_1}{E_1} + \frac{\eta_2}{E_2}\right)\dot{\varepsilon} + \left(\frac{\eta_1\eta_2}{E_1E_2}\right)\ddot{\varepsilon}$$

The coefficients are:

$$p_0 = \frac{E_1+E_2}{E_1E_2}$$
$$p_1 = \frac{\eta_1+\eta_2}{E_1E_2}$$
$$q_0 = 1$$
$$q_1 = \frac{\eta_1}{E_1} + \frac{\eta_2}{E_2}$$
$$q_2 = \frac{\eta_1\eta_2}{E_1E_2}$$

Substituting the given values:

$$\begin{aligned}
p_0 &= 2{,}5 \times 10^{-7}\,\mathrm{Pa}^{-1} \\
p_1 &= 6{,}2 \times 10^{-9}\,\mathrm{s} \cdot \mathrm{Pa}^{-1} \\
q_0 &= 1 \\
q_1 &= 4{,}9 \times 10^{-2}\,\mathrm{s} \\
q_2 &= 6{,}0 \times 10^{-4}\,\mathrm{s}^2
\end{aligned}$$

b) Qualitative Strain Behavior $\varepsilon(t)$

1. $t = 0$: Instantaneous elastic strain due to the springs.
2. $0 < t < 0.5s$: Time-dependent strain increase (creep) due to the dashpots.
3. $t = 0.5s$: Unloading $\rightarrow$ immediate elastic recovery.
4. $t > 0.5s$: Slow viscous recovery due to the dashpots.

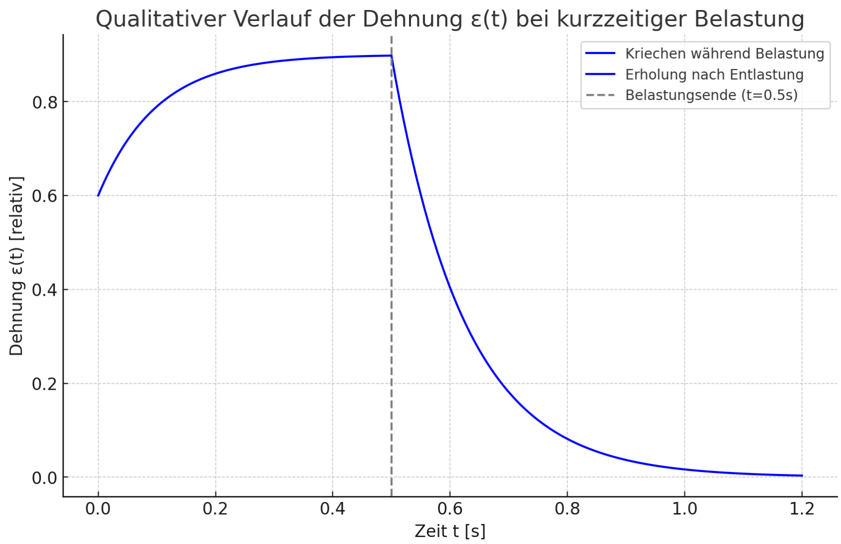


Figure 2: Qualitative strain behavior $\varepsilon(t)$ during and after loading.

c) Influence of Increasing η_1

1. **Shock Absorption:** Increasing η_1 $\rightarrow$ stronger viscous damping $\rightarrow$ more energy absorption $\rightarrow$ reduced shock impact.
2. **Energy Return:** Increasing η_1 $\rightarrow$ delayed recovery $\rightarrow$ less elastic energy return $\rightarrow$ shoe feels less "responsive."

Conclusion: A higher η_1 increases comfort but reduces responsiveness – suitable for comfort-oriented shoes, less so for performance-oriented ones.