Summer 2016
Math 132 Complex Analysis for Applications
Exam II General Information

What to do if you have an exam conflict

As announced in the course syllabus, Exam II will be given in lecture on **Thursday, July 21, 2016** at the usual lecture time **11:00am-12:00pm)** in **Boelter 5249**. If you cannot make the final exam due to serious illness requiring a doctor's note or a family emergency, you must notify me by email as soon as possible. There are no makeup exams. We will be having lecture in Boelter 5249 right after the exam.

What to bring to the exam

Be sure to bring enough pens, pencils, and erasers. No calculators, notes, or electronic devices will be allowed. All cell phones and portable electronic devices must be turned off and put away during the entire exam.

Tips on exam taking

Don't rush. The exam will be designed to take 50 minutes, but you will have an extra 10 minutes to work and check your solutions. If you get stuck on a problem, it is best to move on to the next problem. Do all the problems you understand well first, and then come back to finish the ones you find more challenging to maximize the number of points you will receive on the exam. You must write a complete organized solution in order to receive full credit.

Exam content

Many of the exam questions will be similar to problems on the homework assignments, examples covered in class, and examples in the textbook. There will be five problems (some with multiple parts). Some problems may be theoretical, while others may be computational in nature. The exam covers the material from the sections of our textbook listed below.

- **II.7 Fractional Linear Transformations**
  Definition of a fractional linear transformation f, its inverse f^{-1}, and its derivative f'.

$$f(z) = \frac{az+b}{cz+d} \text{ where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0$$

Associations between FLTs and invertible 2×2 matrices with complex entries:

$$f(z) = \frac{az+b}{cz+d} \text{ and } ad - bc \neq 0 \text{ with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } ad - bc \neq 0$$

$$f^{-1}(z) = \frac{dz-b}{-cz+a} \quad \text{since} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}$$

$$f'(z) = \frac{ad-bc}{(cz+d)^2}$$

Properties of FLT's

- maps circles and lines in $\mathbb{C} \cup \{\infty\}$ to circles and lines in $\mathbb{C} \cup \{\infty\}$ or equivalently, circles on the Riemann sphere to circles on the Riemann sphere
- preserves orientation
- is conformal (one-to-one, onto, preserves angles)
- is uniquely determined by the image of three points in $\mathbb{C} \cup \{\infty\}$
- is a composition of translations, dilations, and inversions

The Cayley Transform:

$$f(z) = \frac{z-i}{z+i}$$

(as well as where the Cayley Transform sends the upper half-plane, real axis, imaginary axis, and unit circle), Constructing an FLT mapping based on its images of three points in $\mathbb{C} \cup \{\infty\}$, Given an FLT, determine the images of circles, lines, and regions in $\mathbb{C} \cup \{\infty\}$ under that FLT.

- **III.1 Line Integrals and Green's Theorem**
  Definition of a differential $Pdx + Qdy$ where $P(x, y)$ and $Q(x, y)$ are continuous complex-valued functions. Definition of the line integral of $Pdx + Qdy$ along a piecewise smooth planar curve γ. Evaluating line integrals by parametrization of γ by $\gamma(t) = (x(t), y(t))$ for $a \leq t \leq b$:

$$\int_\gamma Pdx + Qdy = \int_a^b P(x(t), y(t)) \frac{dx}{dt} dt + \int_a^b Q(x(t), y(t)) \frac{dy}{dt} dt$$

Evaluating line integrals using Green's Theorem:

$$\int_{\partial D} Pdx + Qdy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

for continuously differentiable complex-valued P and Q on $D \cup \partial D$, where D is a bounded domain in the plane and ∂D is a piecewise smooth positively-oriented smooth curve boundary (D is on the left side ∂D). Definition of the differential of a continuously differentiable complex-valued function $h(x, y)$:

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$$

Definition of a closed differential on D, definition of an exact differential on D.

- **III.2 Independence of Path**
  Definition of a line integral that is path independent from A to B. Definition of a line integral that is path independent in domain D. Fundamental Theorem of Calculus for Line Integrals. Equivalence of path independence in D of $\int P dx + Q dy$ and exactness of $P dx + Q dy$. Exact implies closed. For continuously differentiable P and Q on a star-shaped domain D, $P dx + Q dy$ closed on D implies exact on D. Continuous deformation of path γ_0 in D into path γ_1 in D yields

$$\int_{\gamma_0} P dx + Q dy = \int_{\gamma_1} P dx + Q dy$$

- **III.3 Harmonic Conjugates** If u is harmonic on domain D, then the following differential is closed on D:

$$-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

  If, in addition, D is a star-shaped domain, then it is exact so

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

  for some continuously differentiable complex-valued v. v is a harmonic conjugate conjugate for u on D so $u + iv$ is analytic on D. Thus any harmonic u on a star-shaped domain D has a harmonic conjugate v on D:

$$v(B) = \int_A^B dv + v(A) = \int_{x_0}^x -\frac{\partial u}{\partial y}(s, y_0) ds + \int_{y_0}^y \frac{\partial u}{\partial x}(x, t) dt + v(x_0, y_0)$$

  where $A(x_0, y_0)$ and $B(x, y)$ are points in D.

- **III.4 The Mean Value Property**
  If u is harmonic on domain D containing $|z - z_0| < \rho$, then for any r with $0 < r < \rho$,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta = \text{ mean value of } u \text{ on the circle } |z - z_0| = r$$

- **IV.1 Complex Line Integrals**
  Definition of the complex line integral of a complex valued function $f = u + iv$ along a piecewise smooth curve γ:

$$\int_\gamma f(z) dz = \int_\gamma f(z) dx + f(z) i dy = \int_\gamma (u + iv) dx + (iu - v) dy$$

Note: $dz = dx + idy$. When γ is a closed curve, we denote the complex line integral as

$$\oint_\gamma f(z)dz$$

Complex line integral with respect to the arc-length measure $|dz| = \sqrt{dx^2 + dy^2} = ds$

$$\int_\gamma f(z)|dz| = \int_a^b f(z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

where $\gamma(t) = z(t) = x(t) + iy(t)$ for $a \le t \le b$. When $f(z) = 1$ on γ, we get the length L of γ:

$$L = \int_\gamma |dz| = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

Triangle Inequality for Complex Line Integrals: If f is continuous on piecewise smooth curve γ,

$$\left|\int_\gamma f(z)dz\right| \le \int_\gamma |f(z)||dz|$$

If, in addition $|f(z)| \le M$ on γ, then we get the following ML-estimate

$$\left|\int_\gamma f(z)dz\right| \le ML$$

- **IV.2 Fundamental Theorem of Calculus for Analytic Functions**
  Part I: If f is continuous on domain D and F is a complex antiderivative for f on D, then

  $$\int_A^B f(z)\,dz = F(B) - F(A)$$

  where the integral is path independent in D.

  Part II: Let f be analytic on a star-shaped domain D, then f has a complex antiderivative on D (unique up to an additive constant)

  $$F(z) = \int_{z_0}^z f(\zeta)\,d\zeta \qquad \text{for any } z \in D$$

  where z_0 is any fixed point in D and the integral is path independent in D.

- **IV.3 Cauchy's Theorem**
  Let D be a bounded domain with piecewise smooth positively-oriented boundary ∂D. If f is analytic on D and extends smoothly to ∂D, then

  $$\int_{\partial D} f(z)\,dz = 0$$

  Reason: If f is continuously differentiable on a domain D, then f is analytic on D if and only if the differential $f(z)\,dz$ is closed.

- **IV.4 The Cauchy Integral Formula**
  Let D be a bounded domain with piecewise smooth positively oriented boundary ∂D. If f is analytic on D and extends smoothly to ∂D, then
  $$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw \quad \text{for any } z \in D$$
  Furthermore, f has complex derivatives of all orders on D. For any $m \in \mathbb{N} \cup \{0\}$,
  $$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\,dw \quad \text{for any } z \in D$$
  which all are analytic on D.
  Note that since f is analytic on D, it is harmonic on D and thus for $z \in D$,
  $$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{i\theta})d\theta = \text{ mean value of } u \text{ on the circle } |w-z| = \epsilon$$
  for $\epsilon > 0$ small enough so that the open disk $\{w : |w-z| < \epsilon\}$ is in D.

- **IV.5 Liouville's Theorem**
  Cauchy Estimates: Say f is analytic on domain D containing the closed disk $|z-z_0| \leq R$. Suppose $|f(z)| \leq M$ for all z in $|z-z_0| = R$. Then for all $m \in \mathbb{N}$,
  $$|f^{(m)}(z_0)| \leq \frac{m!M}{R^m}$$
  Definition and examples of entire functions. Sums, differences, products, and compositions of entire functions are entire. Liouville's Theorem: Any bounded entire function is constant.

- **IV.6 Morera's Theorem**
  If f is continuous on a domain D and for every closed rectangle R in D with sides parallel to the axes,
  $$\int_{\partial R} f(z)\,dz = 0$$
  then f is analytic on D.

- **V.1 Infinite Series**
  Definition of an infinite series as the sequence S_N of Nth-partial sums of an infinite series. Definition of convergence of a series, sum if a series, divergence of a series. Equivalent notions of convergence of a series:
  $$\begin{aligned}\sum_{n=0}^{\infty} a_n \text{ converges} \quad &\Longleftrightarrow \quad \sum_{n=0}^{\infty} \overline{a_n} \text{ converges} \quad \Longleftrightarrow \quad \sum_{n=0}^{\infty} \operatorname{Re} a_n \text{ and } \sum_{n=0}^{\infty} \operatorname{Im} a_n \text{ converge} \\ &\Longleftrightarrow \quad c\sum_{n=0}^{\infty} a_n \text{ converges for some } c \neq 0 \\ &\Longleftrightarrow \quad \sum_{n=k}^{\infty} a_n \text{ converges for some } k \in \mathbb{N}\end{aligned}$$

n-term Divergence Test for Infinite Series. Definition of a positive series ($a_n \geq 0$ for all n). Positive series either converge or diverge to ∞. Comparison Test for positive series. Absolute convergence. Triangle inequality for Series. Absolute convergence implies convergence. Geometric Series. Formula for N-th partial sum of a geometric series:

$$S_N = \sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z} \text{ for } z \neq 1$$

Formula for the sum of a convergent geometric series:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z} \text{ for } |z| < 1$$

- **V.2 Sequences and Series of Functions**
  Sequences $\{f_n\}$ of complex functions. Definition of pointwise convergence on a set E: $f_n \to f$ pointwise on E if for each $z \in E$,

$$\lim_{n \to \infty} f_n(z) = f(z)$$

Definition of uniform convergence on a set E: $f_n \to f$ uniformly on E if

$$\lim_{n \to \infty} \left[\sup_{z \in E} |f_n(z) - f(z)| \right] = 0$$

Uniform convergence implies pointwise convergence. If $f_n \to f$ uniformly on E and f_n continuous on E for all n, then f is continuous on E. If $f_n \to f$ uniformly on a piecewise smooth curve γ and f_n continuous on γ for all n, then

$$\lim_{n \to \infty} \int_\gamma f_n(z)\, dz = \int_\gamma \lim_{n \to \infty} f_n(z)\, dz = \int_\gamma f(z)\, dz$$

If $f_n \to f$ uniformly on E and f_n analytic on E for all n, then f is analytic on E. Definition of a series S of complex functions formed from the sequence f_n of complex functions:

$$S(z) = \sum_{n=0}^{\infty} f_n(z)$$

The Nth-partial sum of a series of complex functions:

$$S_N(z) = \sum_{n=0}^{N} f_n(z)$$

Pointwise convergence of a series of complex functions on E:

$$\lim_{N \to \infty} S_N(z) = S(z)$$

Uniform convergence of a series of complex functions on E: $S_N \to S$ uniformly on E.

Weierstrass M-Test for uniform convergence of a series of complex functions.

If f_n analytic on $|z - z_0| \leq R$ for all n and $f_n \to f$ uniformly on $|z - z_0| \leq R$, then for every $m \in \mathbb{N}$ and every r with $0 < r < R$, we have $f_n^{(m)} \to f^{(m)}$ uniformly on $|z - z_0| \leq r$.

- **V.3 Power Series**

  Definition of a power series centered at $z_0 \in \mathbb{C}$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where the radius of convergence (called R) is defined to be

$$R = \sup\{r \geq 0 : |a_n| r^n \text{ is bounded for all } n \in \mathbb{N} \cup \{0\}\}$$

For every r with $0 < r < R$, the power series converges uniformly on $|z - z_0| \leq r$, absolutely on $|z - z_0| < R$, and diverges on $|z - z_0| > R$. It may or may not converge if $|z - z_0| = R$. Formulae for the radius of convergence:

$$R = \frac{1}{\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right|} = \frac{1}{\lim\limits_{n \to \infty} |a_n|^{1/n}}$$

provided these limits are positive real numbers. If $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 0$ or $\lim\limits_{n \to \infty} |a_n|^{1/n} = 0$, we define $R = \infty$. If $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \infty$ or $\lim\limits_{n \to \infty} |a_n|^{1/n} = \infty$, we define $R = 0$. If neither of these limits exist, we can still find R using the Cauchy-Hadamard Formula:

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

Power series are analytic on $|z - z_0| < R$ and their derivatives may be found by differentiating termwise:

$$f^{(m)}(z) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-(m-1)) a_n (z - z_0)^{n-m}$$

From this, we get a formula for the mth coefficient of the power series centered at z_0:

$$a_m = \frac{f^{(m)}(z_0)}{m!}$$

An antiderivative F of the power series f may be found by antidifferentiating termwise:

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

If $f(z) = \sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$ has radius of convergence R, then so does its derivative f' and antiderivative F.

- **V.4 Power Series Expansion of an Analytic Function**
  If f is analytic on $|z-z_0|<\rho$, then f has a power series expansion centered at z_0:

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ on } |z-z_0|<\rho$$

whose radius of convergence R satisfies $\rho \le R$ and whose coefficients a_n are

$$a_n=\frac{f^{(n)}(z_0)}{n!}=\frac{1}{2\pi i}\oint_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{n+1}}\,dw$$

for any fixed r with $0<r<\rho$. Furthermore, if $|f(w)|\le M$ on $|w-z_0|=r$, then

$$|a_n|\le \frac{M}{r^n}$$

If f and g are analytic on $|z-z_0|<r$ and $f^{(n)}(z_0)=g^{(n)}(z_0)$ for all $n\in\mathbb{N}\cup\{0\}$, then $f=g$ on $|z-z_0|<r$. The radius of convergence R is the largest $r\in\mathbb{R}\cup\{\infty\}$ such that f extends to be analytic on $|z-z_0|<r$. R could be viewed as the distance from the center z_0 to the nearest singularity of f.

Some common power series expansions centered at 0:

$$e^z=\sum_{n=0}^{\infty}\frac{1}{n!}z^n=1+z+\frac{z^2}{2}+\frac{z^3}{3!}+\cdots \qquad |z|<\infty$$

$$\sin z=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots \qquad |z|<\infty$$

$$\cos z=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots \qquad |z|<\infty$$

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n=1+z+z^2+z^3+\cdots \qquad |z|<1$$

$$-\mathrm{Log}(1-z)=\sum_{n=0}^{\infty}\frac{1}{n+1}z^{n+1}=z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots \qquad |z|<1$$

$$\mathrm{Tan}^{-1}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}z^{2n+1}=z-\frac{z^3}{3}+\frac{z^5}{5}-\cdots \qquad |z|<1$$

- **V.5 Power Series Expansion at Infinity**
  Definition of analytic at ∞: $f(z)$ is analytic at ∞ if $g(w)=f(1/w)$ is analytic at $w=0$. If f is analytic at ∞, then g has a power series expansion centered at 0:

$$g(w)=\sum_{n=0}^{\infty}b_n w^n \text{ on } |w|<\rho$$

Letting $w = 1/z$, we get a convergent power series expansion of f in the variable $1/z$:

$$f(z) = g(1/z) = \sum_{n=0}^{\infty} b_n \left(\frac{1}{z}\right)^n \quad \text{on } |z| > \frac{1}{\rho}$$

which converges absolutely on $|z| > \frac{1}{\rho}$ and uniformly on $|z| \geq r$ for any $r > \frac{1}{\rho}$. It may or may not converge on $|z| = \frac{1}{\rho}$. The coefficients b_n are given by

$$b_n = \frac{1}{2\pi i} \oint_{|z|=r} f(z) z^{n-1}\, dz$$

- **V.6 Manipulation of Power Series**

If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n \quad |z| < R$$

then

$$\begin{aligned}
f(z) + g(z) &= \sum_{n=0}^{\infty} (a_n + b_n) z^n \quad |z| < R \\
c f(z) &= \sum_{n=0}^{\infty} c a_n z^n \quad |z| < R \\
f(z) g(z) &= \sum_{n=0}^{\infty} c_n z^n \quad |z| < R \text{ where } c_n = \sum_{k=0}^{n} a_{n-k} b_k \\
\frac{1}{g(z)} &= \frac{1}{1 + \left(\sum_{n=1}^{\infty} b_n z^n\right)} = 1 - \left(\sum_{n=1}^{\infty} b_n z^n\right) + \left(\sum_{n=1}^{\infty} b_n z^n\right)^2 - \cdots \text{ if } b_0 = 1 \text{ for } |z| \text{ small} \\
&= \sum_{n=0}^{\infty} d_n z^n \\
\frac{f(z)}{g(z)} &= \frac{f(z)}{b_0} \frac{1}{\frac{g(z)}{b_0}} \text{ if } b_0 \neq 0
\end{aligned}$$

- **V.7 The Zeros of an Analytic Function**

Definition of a zero of order N at z_0 of a nonzero function f analytic at z_0

$$\begin{aligned}
f \text{ has a zero of order } N \text{ at } z_0 &\iff f(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^n \text{ with } a_N \neq 0 \\
&\iff f(z) = (z - z_0)^N \sum_{n=0}^{\infty} a_{n+N} (z - z_0)^n \text{ with } a_N \neq 0 \\
&\iff f(z) = (z - z_0)^N g(z)
\end{aligned}$$

where $g(z)$ is analytic at z_0 and $g(z_0) = a_N \neq 0$

Definition of f analytic at ∞ having a zero of order N at ∞

$$
\begin{aligned}
f \text{ has a zero of order } N \text{ at } \infty &\iff g(w) = f(1/w) \text{ has a zero at } w = 0 \text{ of order } N \\
&\iff g(w) = \sum_{n=N}^{\infty} b_n w^n \text{ with } b_N \neq 0 \quad |w| < \frac{1}{R} \\
&\iff f(z) = g(1/z) = \sum_{n=N}^{\infty} b_n \left(\frac{1}{z}\right)^n \text{ with } b_N \neq 0 \quad |z| > R
\end{aligned}
$$