Engineering Mechanics: Statics in SI Units, 12e

2

Force Vectors

Chapter Objectives

- To show how to add forces and resolve them into components using the parallelogram Law

- To express force and position in cartesian vector form and explain how to determine the vector's magnitude and direction.

Chapter Outline

1. Scalars and Vectors
2. Vector Operations
3. Vector Addition of Forces
4. Addition of a System of Coplanar Forces
5. Cartesian Vectors
6. Addition and Subtraction of Cartesian Vectors
7. Position Vectors
8. Force Vector Directed along a Line

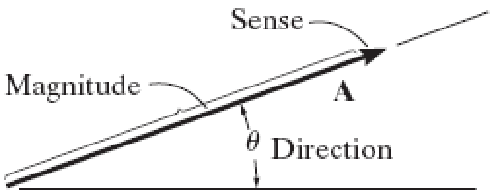
2.1 Scalars and Vectors

- Scalar
  - A quantity characterized by a positive or negative number
  - Indicated by letters in italic such as *A*

  e.g. Mass, volume and length

2.1 Scalars and Vectors

- Vector
  - A quantity that has magnitude and direction
    e.g. Position, force and moment
  - Represented by a letter with an arrow over it, $\vec{A}$
  - Magnitude is designated as $\left|\vec{A}\right|$
  - In this subject, vector is presented as **A** and its magnitude (positive quantity) as A



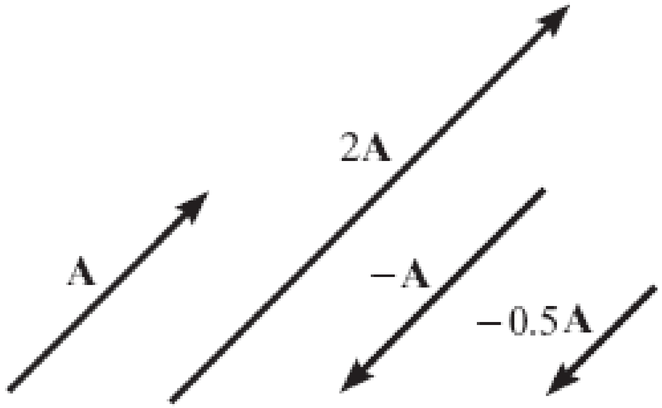
2.2 Vector Operations

- Multiplication and Division of a Vector by a Scalar

  - Product of vector **A** and scalar a = a**A**

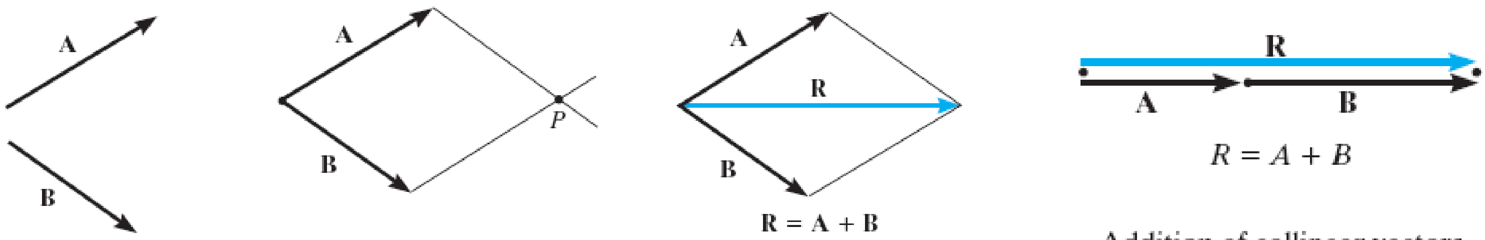
  - Magnitude = $\left|aA\right|$

  - Law of multiplication applies e.g. **A**/a = (1/a) **A**, a≠0



2.2 Vector Operations

- Vector Addition
  - Addition of two vectors **A** and **B** gives a resultant vector **R** by the *parallelogram law*
  - Result **R** can be found by *triangle construction*
  - Communicative e.g. **R** = **A** + **B** = **B** + **A**
  - Special case: Vectors **A** and **B** are *collinear* (both have the same line of action)



Parallelogram law

Addition of collinear vectors

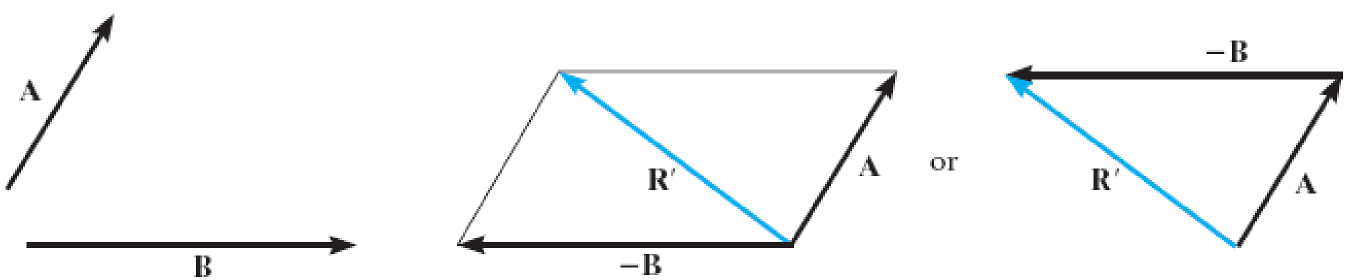
2.2 Vector Operations

- Vector Subtraction

  - Special case of addition

  e.g. $\mathbf{R}' = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$

  - Rules of Vector Addition Applies



Parallelogram law

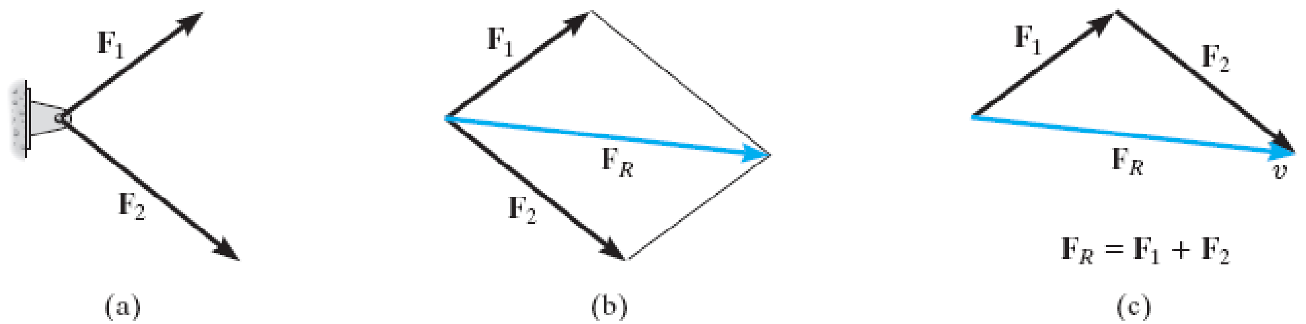
Triangle construction

Vector subtraction

2.3 Vector Addition of Forces

Finding a Resultant Force

- *Parallelogram law* is carried out to find the resultant force



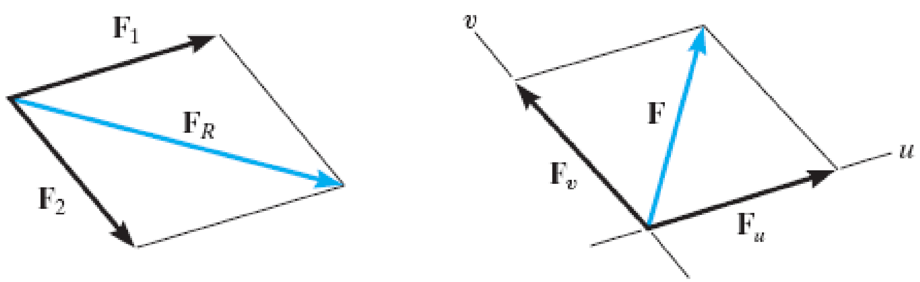
- Resultant,

  $\mathbf{F}_R = (\,\mathbf{F}_1 + \mathbf{F}_2\,)$

2.3 Vector Addition of Forces

Procedure for Analysis

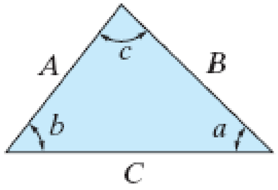
- Parallelogram Law
  - Make a sketch using the *parallelogram law*
  - 2 components forces add to form the resultant force
  - Resultant force is shown by the diagonal of the parallelogram
  - The components is shown by the sides of the parallelogram



2.3 Vector Addition of Forces

Procedure for Analysis

- Trigonometry
  - Redraw half portion of the parallelogram
  - Magnitude of the resultant force can be determined by the *law of cosines*
  - Direction of the resultant force can be determined by the *law of sines*
  - Magnitude of the two components can be determined by the *law of sines*



Cosine law:

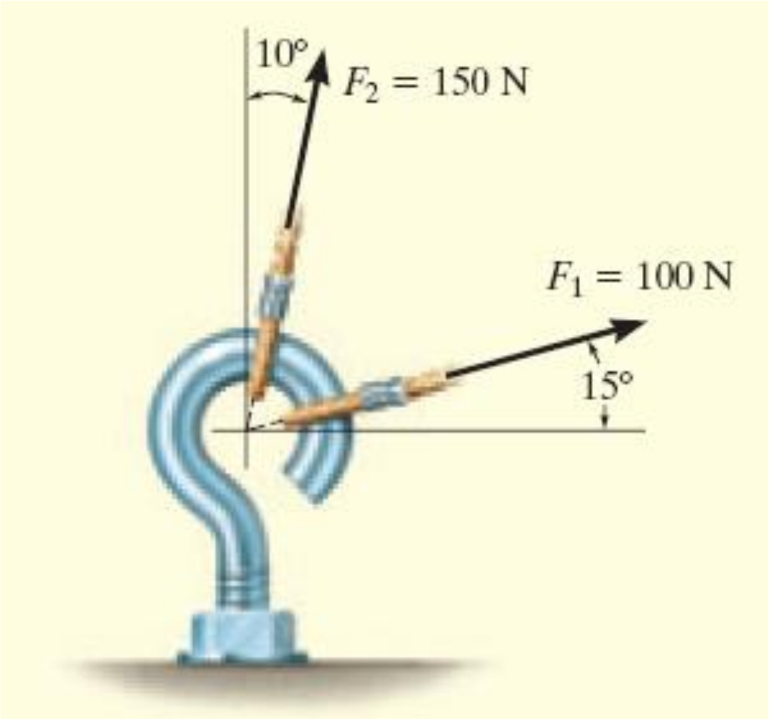
$$C = \sqrt{A^2 + B^2 - 2AB\cos c}$$

Sine law:

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$$

Example 2.1

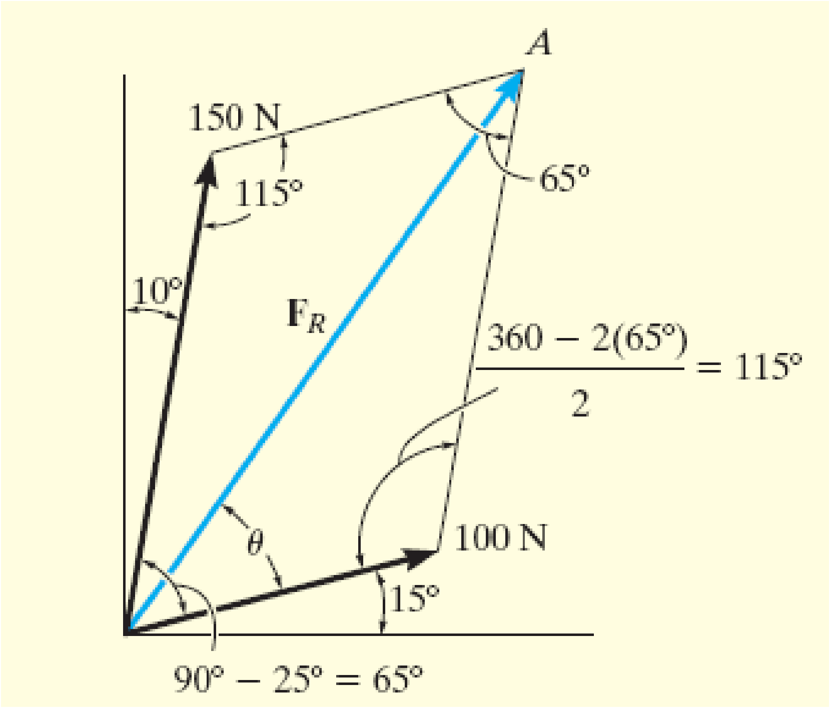
The screw eye is subjected to two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude and direction of the resultant force.



Solution

Parallelogram Law

Unknown: magnitude of **F**$_R$ and angle θ



Solution

Trigonometry

Law of Cosines

$$F_R = \sqrt{(100N)^2 + (150N)^2 - 2(100N)(150N)\cos 115^\circ}$$

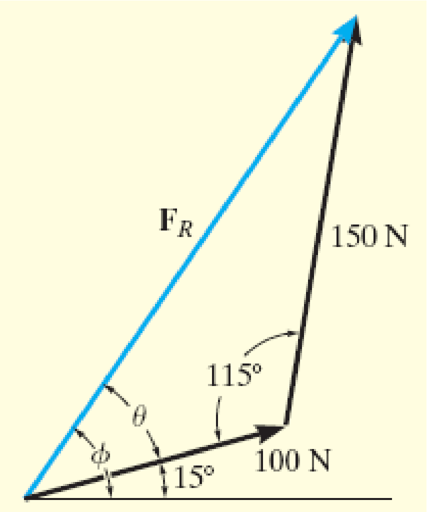
$$= \sqrt{10000 + 22500 - 30000(-0.4226)} = 212.6N = 213N$$

Law of Sines

$$\frac{150N}{\sin\theta} = \frac{212.6N}{\sin 115^\circ}$$

$$\sin\theta = \frac{150N}{212.6N}(0.9063)$$

$$\theta = 39.8^\circ$$



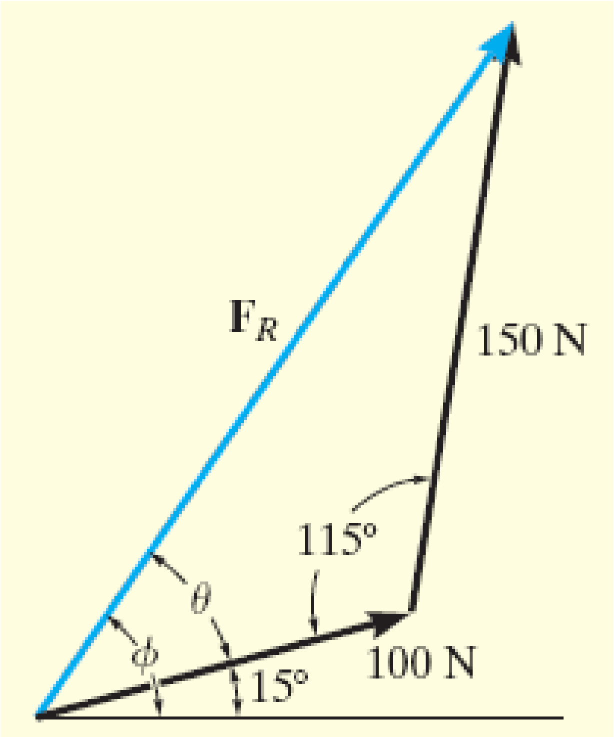
Solution

Trigonometry

Direction Φ of $\mathbf{F}_R$ measured from the horizontal

$$\phi = 39.8^\circ + 15^\circ$$

$$= 54.8^\circ \angle^{\phi}$$

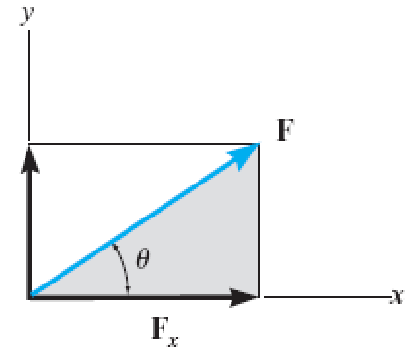


2.4 Addition of a System of Coplanar Forces

- Scalar Notation
  - x and y axis are designated positive and negative
  - Components of forces expressed as algebraic scalars

$$F = F_x + F_y$$

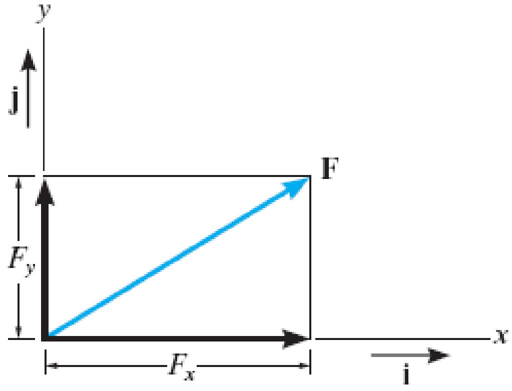
$$F_x = F\cos\theta \quad \text{and} \quad F_y = F\sin\theta$$



2.4 Addition of a System of Coplanar Forces

- Cartesian Vector Notation
  - Cartesian unit vectors **i** and **j** are used to designate the x and y directions
  - Unit vectors **i** and **j** have dimensionless magnitude of unity (= 1)
  - Magnitude is always a positive quantity, represented by scalars F_x and F_y

$$F = F_x i + F_y j$$



2.4 Addition of a System of Coplanar Forces

- Coplanar Force Resultants

  To determine resultant of several coplanar forces:

  – Resolve force into x and y components

  – Addition of the respective components using scalar algebra

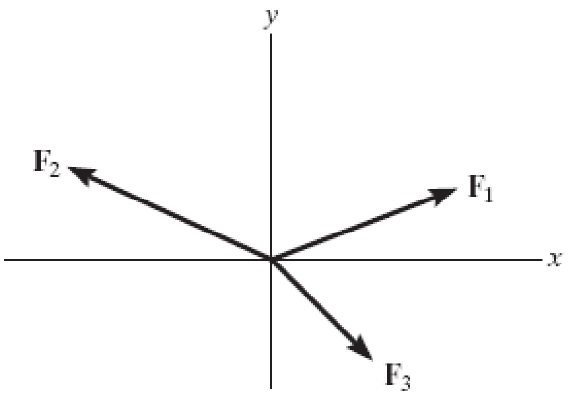
  – Resultant force is found using the parallelogram law

  – Cartesian vector notation:

$$F_1 = F_{1x} i + F_{1y} j$$

$$F_2 = -F_{2x} i + F_{2y} j$$

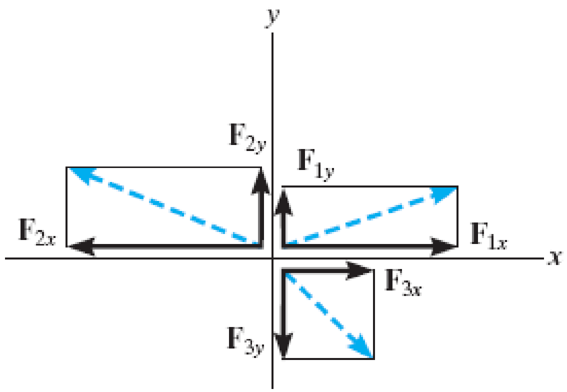
$$F_3 = F_{3x} i - F_{3y} j$$



2.4 Addition of a System of Coplanar Forces

- Coplanar Force Resultants
  - Vector resultant is therefore

$$F_R = F_1 + F_2 + F_3$$
$$= (F_{Rx})i + (F_{Ry})j$$



  - If scalar notation are used

$$F_{Rx} = F_{1x} - F_{2x} + F_{3x}$$
$$F_{Ry} = F_{1y} + F_{2y} - F_{3y}$$

2.4 Addition of a System of Coplanar Forces

- Coplanar Force Resultants
  - In all cases we have

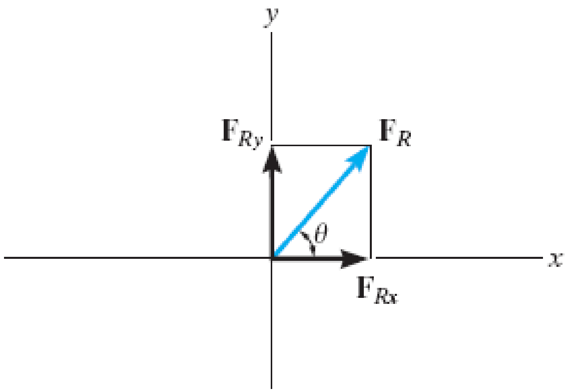
$$F_{Rx} = \sum F_x$$

$$F_{Ry} = \sum F_y$$

* Take note of sign conventions

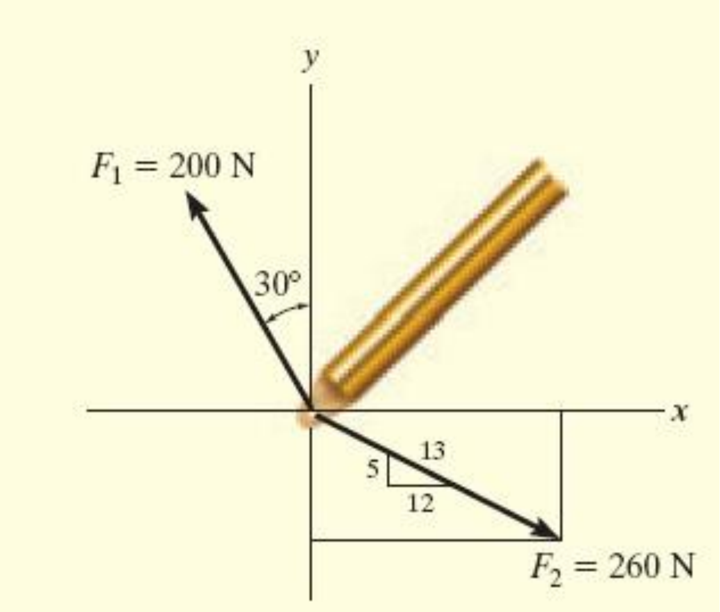
  - Magnitude of $\mathbf{F}_R$ can be found by Pythagorean Theorem

$$F_R = \sqrt{F_{Rx}^2 + F_{Ry}^2} \quad \text{and } \theta = \tan^{-1}\left|\frac{F_{Ry}}{F_{Rx}}\right|$$



Example 2.5

Determine x and y components of $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the boom. Express each force as a Cartesian vector.



Solution

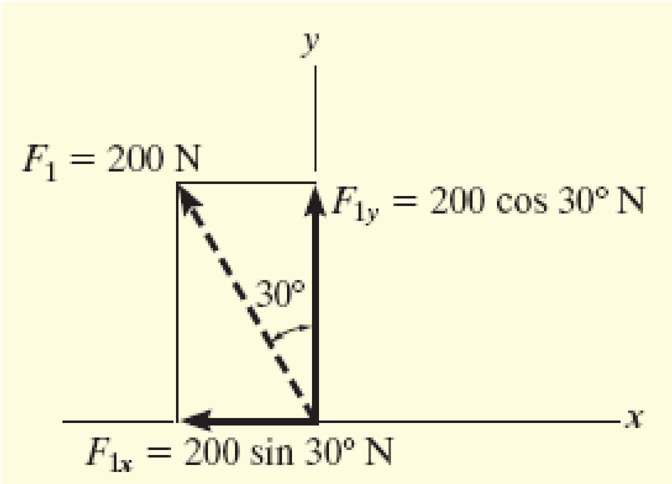
Scalar Notation

$$F_{1x} = -200\sin 30^\circ\ N = -100N = 100N \leftarrow$$

$$F_{1y} = 200\cos 30^\circ\ N = 173N = 173N \uparrow$$

Hence, from the slope triangle, we have

$$\theta = \tan^{-1}\left(\frac{5}{12}\right)$$

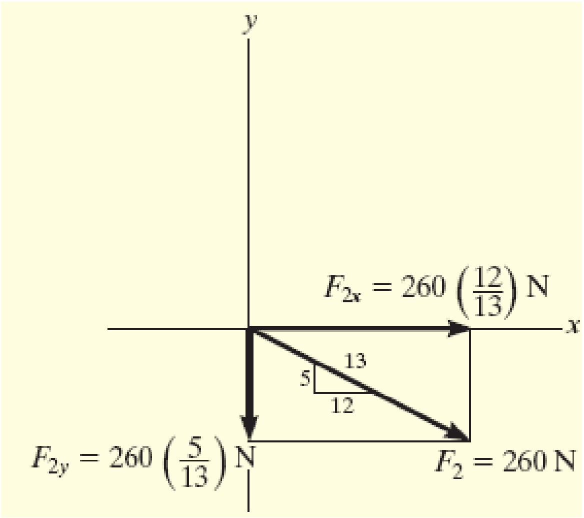


Solution

By similar triangles we have

$$F_{2x} = 260\left(\frac{12}{13}\right) = 240\text{N}$$

$$F_{2y} = 260\left(\frac{5}{13}\right) = 100\text{N}$$



Scalar Notation: $F_{2x} = 240N \rightarrow$

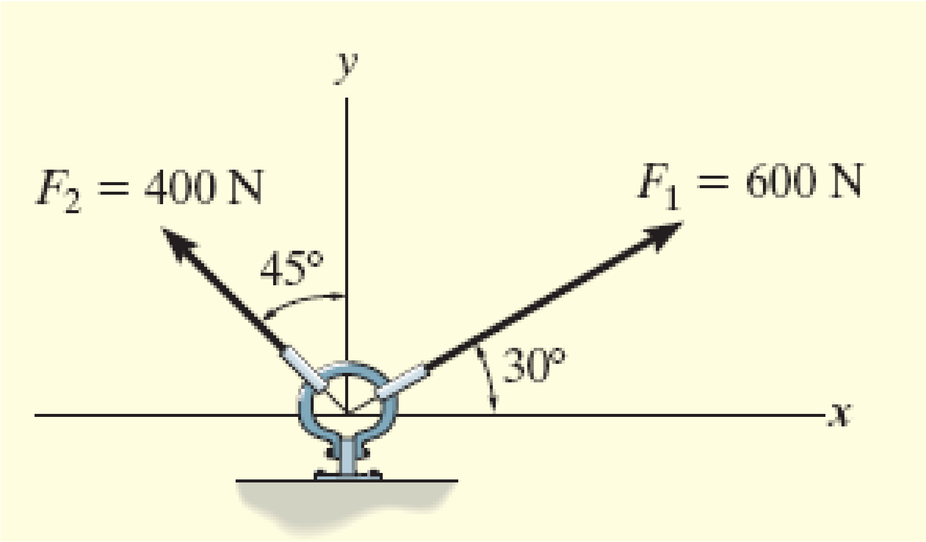
$$F_{2y} = -100N = 100N \downarrow$$

Cartesian Vector Notation: $F_1 = \{-100i + 173j\}N$

$$F_2 = \{240i - 100j\}N$$

Example 2.6

The link is subjected to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude and orientation of the resultant force.



Solution I

Scalar Notation:

$$F_{Rx} = \Sigma F_x :$$

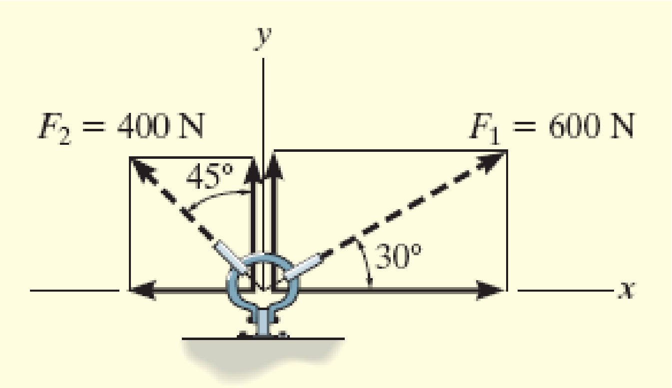
$$F_{Rx} = 600\cos 30^\circ N - 400\sin 45^\circ N$$

$$= 236.8N \rightarrow$$

$$F_{Ry} = \Sigma F_y :$$

$$F_{Ry} = 600\sin 30^\circ N + 400\cos 45^\circ N$$

$$= 582.8N \uparrow$$



Solution I

Resultant Force

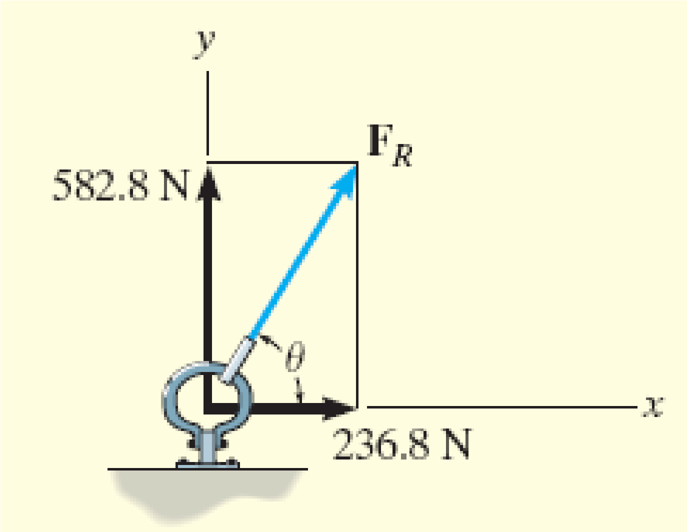
$$F_R = \sqrt{(236.8N)^2 + (582.8N)^2}$$

$$= 629N$$

From vector addition, direction angle θ is

$$\theta = \tan^{-1}\left(\frac{582.8N}{236.8N}\right)$$

$$= 67.9^\circ$$



Solution II

Cartesian Vector Notation

$\mathbf{F}_1 = \{ 600\cos30^\circ \; \mathbf{i} + 600\sin30^\circ \; \mathbf{j} \}$ N

$\mathbf{F}_2 = \{ -400\sin45^\circ \; \mathbf{i} + 400\cos45^\circ \; \mathbf{j} \}$ N

Thus,

$$\begin{aligned}\mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 \\ &= (600\cos30^\circ\text{N} - 400\sin45^\circ\text{N})\mathbf{i} \\ &\quad + (600\sin30^\circ\text{N} + 400\cos45^\circ\text{N})\mathbf{j} \\ &= \{236.8\mathbf{i} + 582.8\mathbf{j}\}\text{N}\end{aligned}$$

The magnitude and direction of $\mathbf{F}_R$ are determined in the same manner as before.

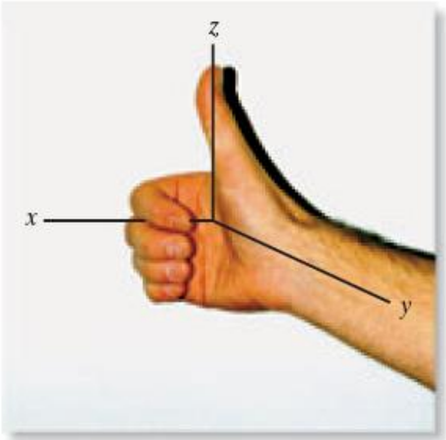
2.5 Cartesian Vectors

- Right-Handed Coordinate System

  A rectangular or Cartesian coordinate system is said to be right-handed provided:

  – Thumb of right hand points in the direction of the positive z axis

  – z-axis for the 2D problem would be perpendicular, directed out of the page.



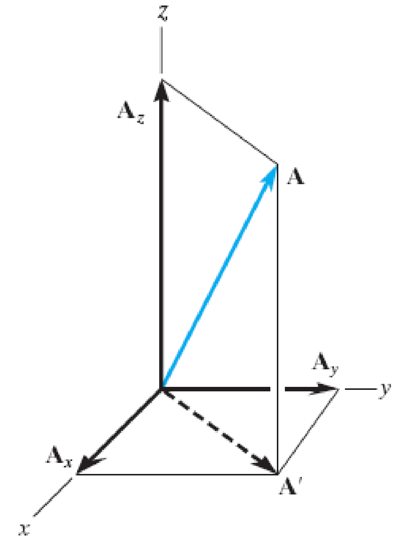
2.5 Cartesian Vectors

- Rectangular Components of a Vector
  - A vector **A** may have one, two or three rectangular components along the *x, y* and *z* axes, depending on orientation
  - By two successive application of the parallelogram law

    $\mathbf{A} = \mathbf{A}' + \mathbf{A}_z$

    $\mathbf{A}' = \mathbf{A}_x + \mathbf{A}_y$
  - Combing the equations,
    A can be expressed as

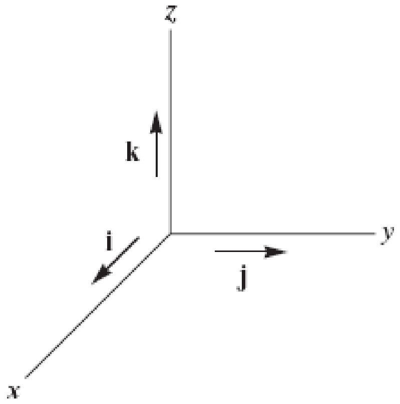
    $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z$



2.5 Cartesian Vectors

- Unit Vector
  - Direction of **A** can be specified using a unit vector
  - Unit vector has a magnitude of 1
  - If **A** is a vector having a magnitude of $A \neq 0$, unit vector having the same direction as **A** is expressed by $\mathbf{u}_A = \mathbf{A} / A$. So that

$$\mathbf{A} = A\,\mathbf{u}_A$$

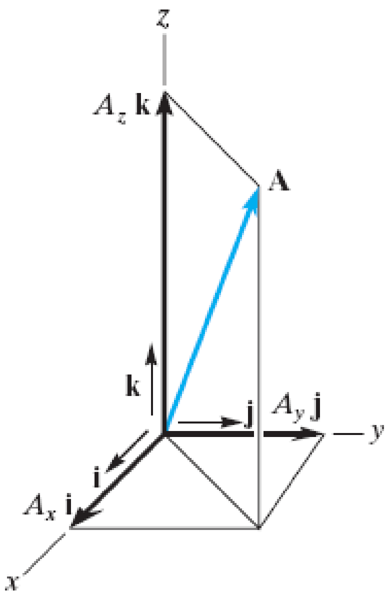


2.5 Cartesian Vectors

- Cartesian Vector Representations
  - 3 components of **A** act in the positive **i**, **j** and **k** directions

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

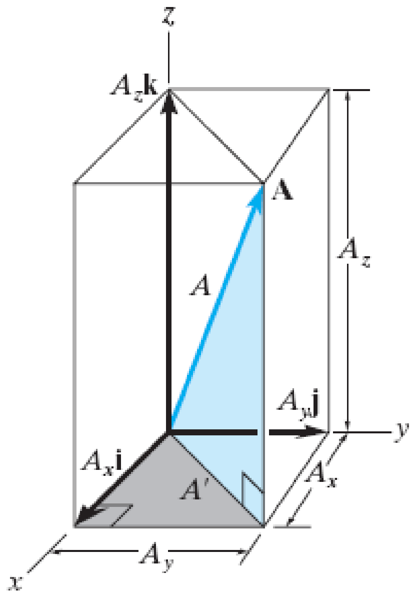
*Note the magnitude and direction of each components are separated, easing vector algebraic operations.



2.5 Cartesian Vectors

- Magnitude of a Cartesian Vector
  - From the colored triangle, $A = \sqrt{A'^2 + A_z^2}$
  - From the shaded triangle, $A' = \sqrt{A_x^2 + A_y^2}$
  - Combining the equations gives magnitude of **A**

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

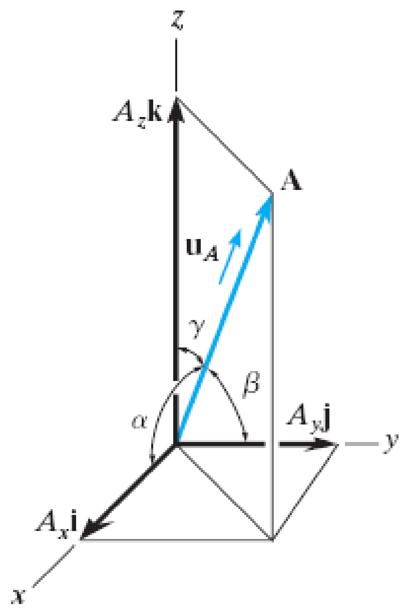


2.5 Cartesian Vectors

- Direction of a Cartesian Vector
  - Orientation of A is defined as the coordinate direction angles α, β and γ measured between the tail of A and the positive x, y and z axes
  - $0^\circ \leq \alpha, \beta$ and $\gamma \leq 180^\circ$
  - The *direction cosines* of **A** is

$$\cos\alpha = \frac{A_x}{A} \qquad \cos\gamma = \frac{A_z}{A}$$

$$\cos\beta = \frac{A_y}{A}$$



2.5 Cartesian Vectors

- Direction of a Cartesian Vector
  – Angles α, β and γ can be determined by the inverse cosines

Given

$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$

then,

$\mathbf{u}_A = \mathbf{A} / A = (A_x/A)\mathbf{i} + (A_y/A)\mathbf{j} + (A_z/A)\mathbf{k}$

where $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

2.5 Cartesian Vectors

- Direction of a Cartesian Vector
  - $\mathbf{u}_A$ can also be expressed as
    $\mathbf{u}_A = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$
  - Since $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ and $\mathbf{u}_A = 1$, we have

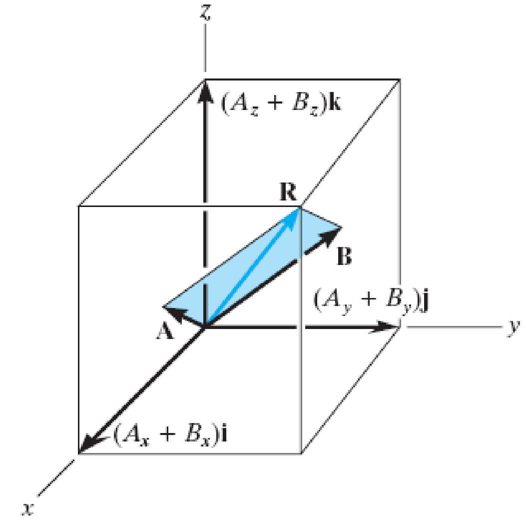
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

  - $\mathbf{A}$ as expressed in Cartesian vector form is
    $\mathbf{A} = A\mathbf{u}_A$
    $= A\cos\alpha\mathbf{i} + A\cos\beta\mathbf{j} + A\cos\gamma\mathbf{k}$
    $= A_x\mathbf{i} + A_y\mathbf{j} + A_Z\mathbf{k}$

2.6 Addition and Subtraction of Cartesian Vectors

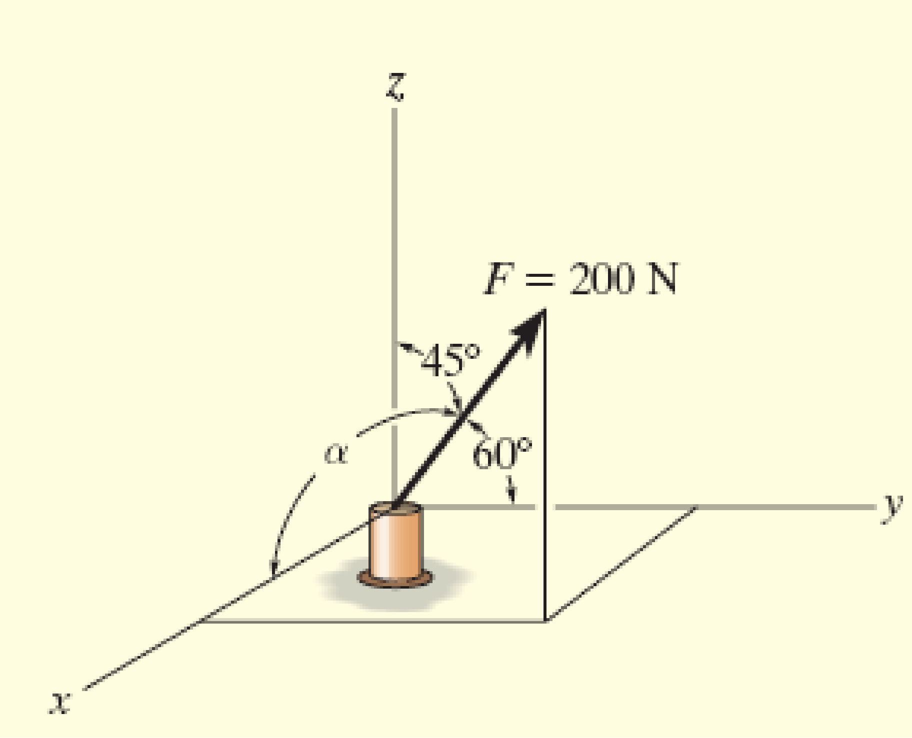
- Concurrent Force Systems
  - Force resultant is the vector sum of all the forces in the system

$$\mathbf{F}_R = \sum \mathbf{F} = \sum F_x\mathbf{i} + \sum F_y\mathbf{j} + \sum F_z\mathbf{k}$$



Example 2.8

Express the force **F** as Cartesian vector.



Solution

Since two angles are specified, the third angle is found by

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\cos^2\alpha + \cos^2 60^\circ + \cos^2 45^\circ = 1$$

$$\cos\alpha = \sqrt{1 - (0.5)^2 - (0.707)^2} = \pm 0.5$$

Two possibilities exit, namely

$$\alpha = \cos^{-1}(0.5) = 60^\circ$$

$$\alpha = \cos^{-1}(-0.5) = 120^\circ$$

Solution

By inspection, $\alpha = 60^o$ since $\mathbf{F}_x$ is in the +x direction

Given $F = 200N$

$$\mathbf{F} = F\cos\alpha\,\mathbf{i} + F\cos\beta\,\mathbf{j} + F\cos\gamma\,\mathbf{k}$$

$$= (200\cos60^oN)\mathbf{i} + (200\cos60^oN)\mathbf{j} + (200\cos45^oN)\mathbf{k}$$

$$= \{100.0\mathbf{i} + 100.0\mathbf{j} + 141.4\mathbf{k}\}N$$

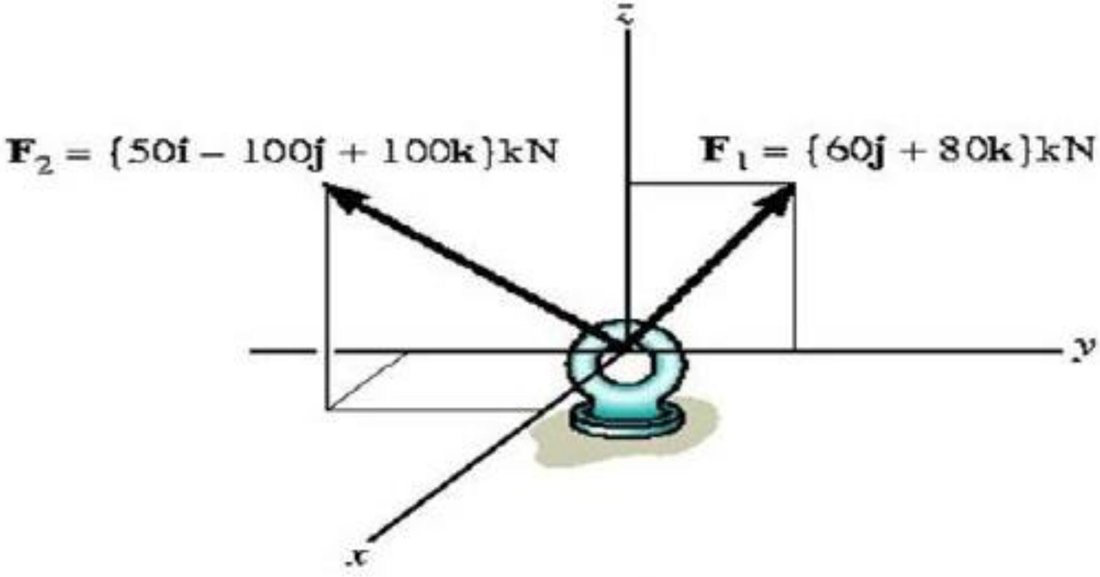
Checking:

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$= \sqrt{(100.0)^2 + (100.0)^2 + (141.4)^2} = 200N$$

Example 2.9

Determine the magnitude and coordinate direction angles of resultant force acting on the ring

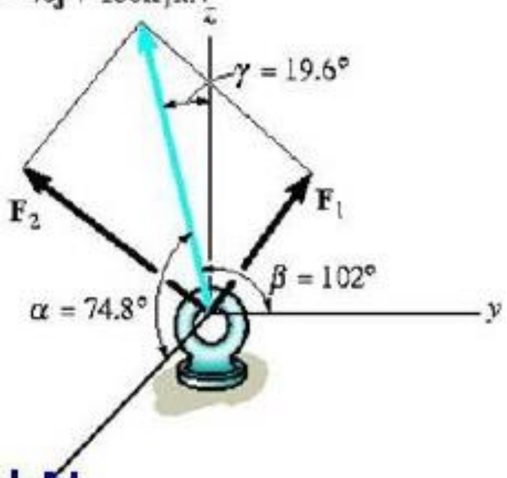


Solution

Solution

Resultant force

$$\mathbf{F}_R = \Sigma\mathbf{F}$$
$$= \mathbf{F}_1 + \mathbf{F}_2$$
$$= \{60\mathbf{j} + 80\mathbf{k}\}\text{kN}$$
$$+ \{50\mathbf{i} - 100\mathbf{j} + 100\mathbf{k}\}\text{kN}$$
$$= \{50.000\mathbf{i} - 40.000\mathbf{j} + 180.000\mathbf{k}\}\text{kN}$$



(b)

Magnitude of $\mathbf{F}_R$ is found by

$$F_R = \sqrt{(50.000)^2 + (-40.000)^2 + (180.000)^2}$$
$$= 191.050 = 191.0500kN$$

Solution

Solution

Unit vector acting in the direction of $\mathbf{F}_R$

$$\mathbf{u}_{FR} = \mathbf{F}_R / F_R$$

$$= (50.000/191.0500)\mathbf{i} + (-40.000/191.0500)\mathbf{j} + (180.000/191.0500)\mathbf{k}$$

$$= 0.261712\ \mathbf{i} - 0.20937\ \mathbf{j} + 0.942163\mathbf{k}$$

So that

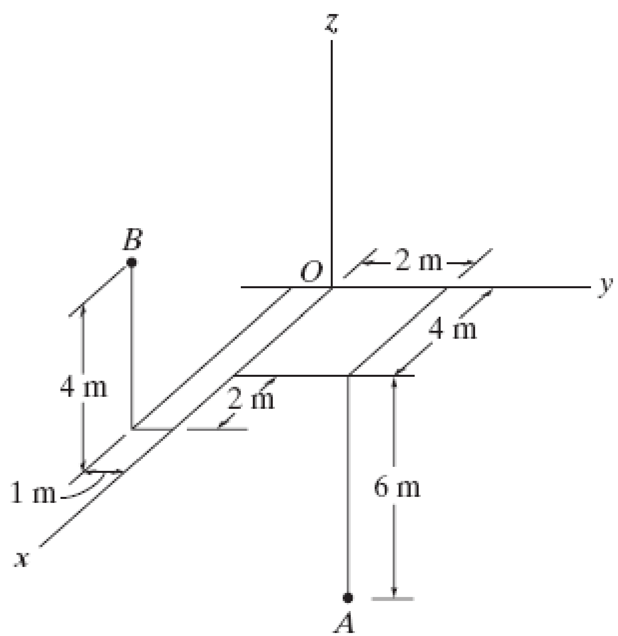
$\cos\alpha = 0.261712$ $\quad\alpha = 74.8283^\circ$

$\cos\beta = -0.20937$ $\quad\beta = 102.08500^\circ$

$\cos\gamma = 0.942163$ $\quad\gamma = 19.5820^\circ$

2.7 Position Vectors

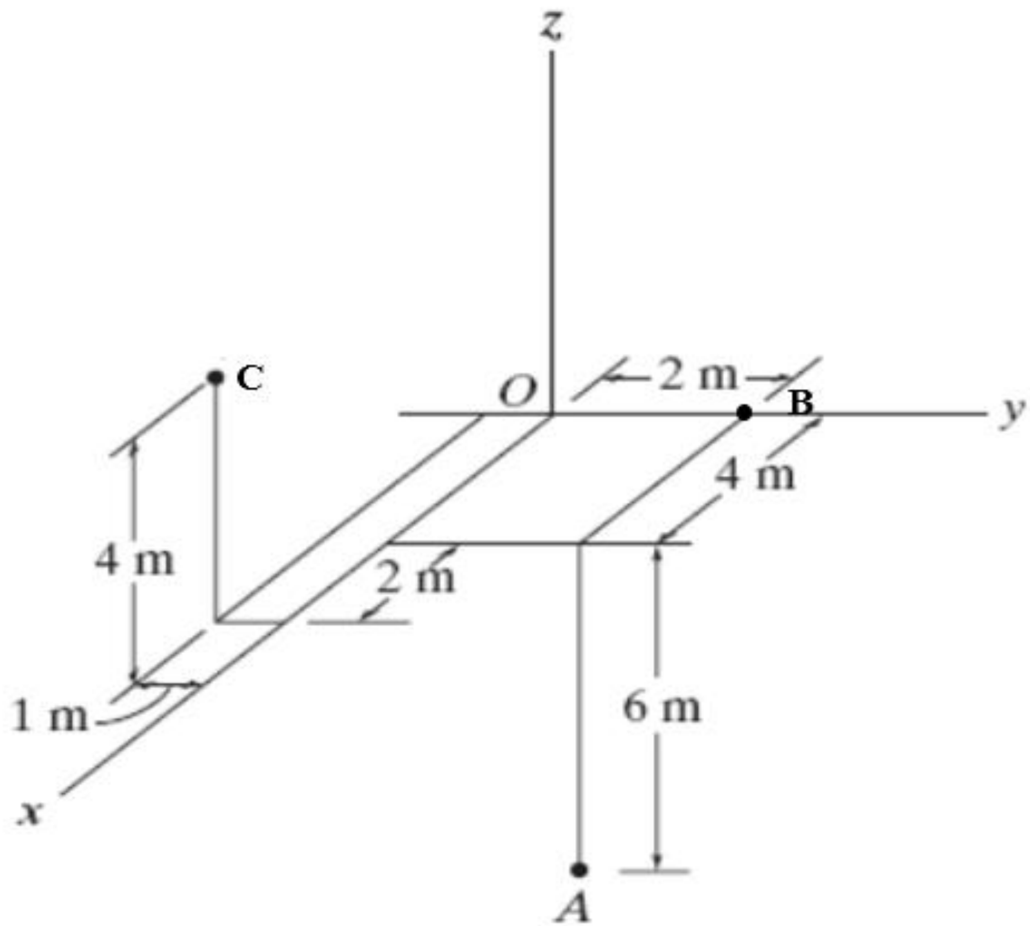
- *x,y,z* Coordinates
  - Right-handed coordinate system
  - Positive *z* axis points upwards, measuring the height of an object or the altitude of a point
  - Points are measured relative to the origin, O.



2.7 Position Vectors

Example 4:

Find the x, y, and z coordinates for points A, B and C



2.7 Position Vectors

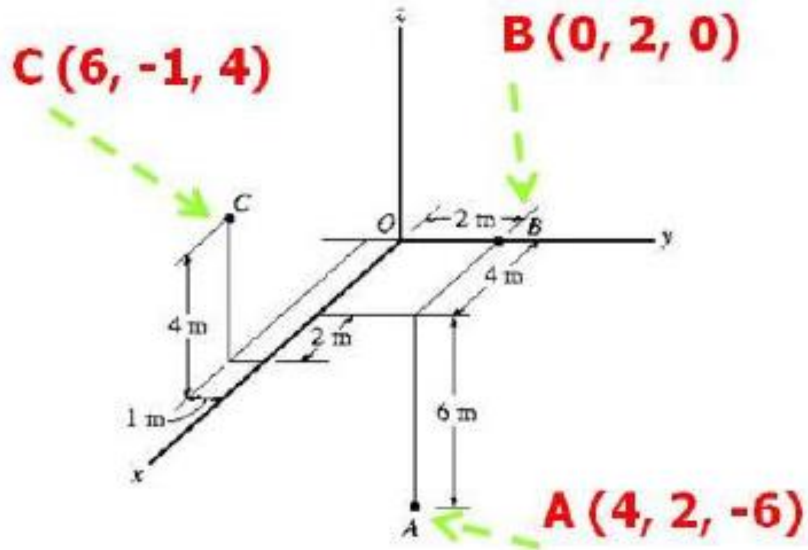
- *x,y,z* Coordinates

  Eg: For Point A, x_A = +4m along the x axis, y_A = 2m along the y axis and z_A = -6m along the z axis.

Thus, **A (4, 2, -6)**

Similarly, **B (0, 2, 0)**

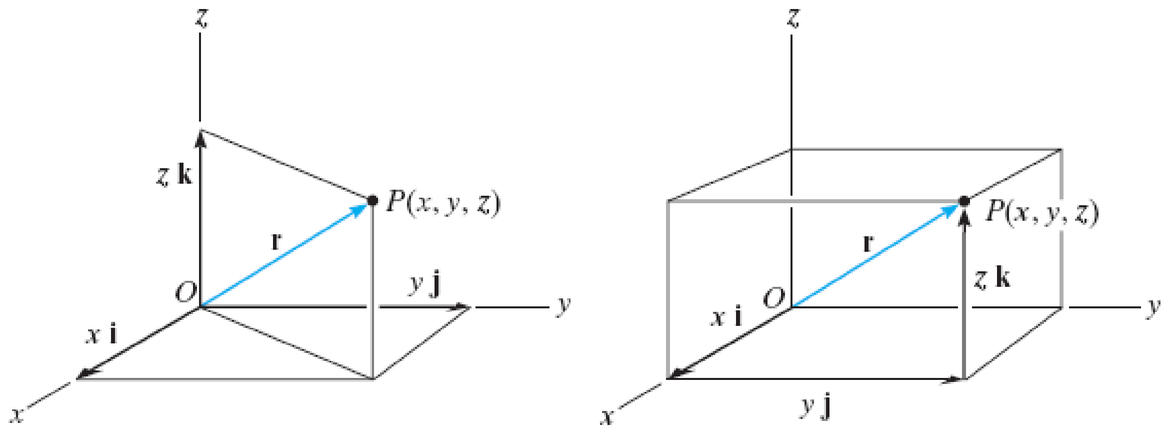
and **C (6, -1, 4)**



2.7 Position Vectors

Position Vector

- Position vector **r** is defined as a fixed vector which locates a point in space relative to another point.
- E.g. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



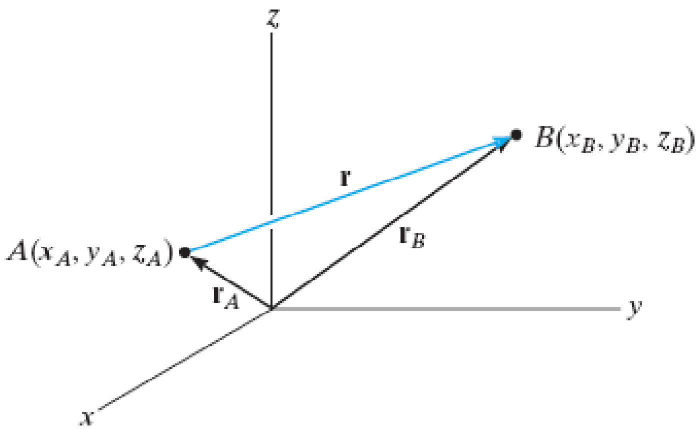
2.7 Position Vectors

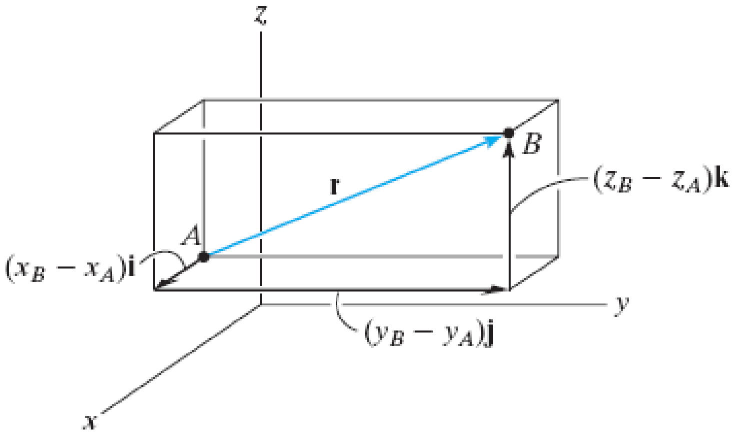
Position Vector

- Vector addition gives $\mathbf{r}_A + \mathbf{r} = \mathbf{r}_B$
- Solving

$$\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A = (x_B - x_A)i + (y_B - y_A)j + (z_B - z_A)k$$

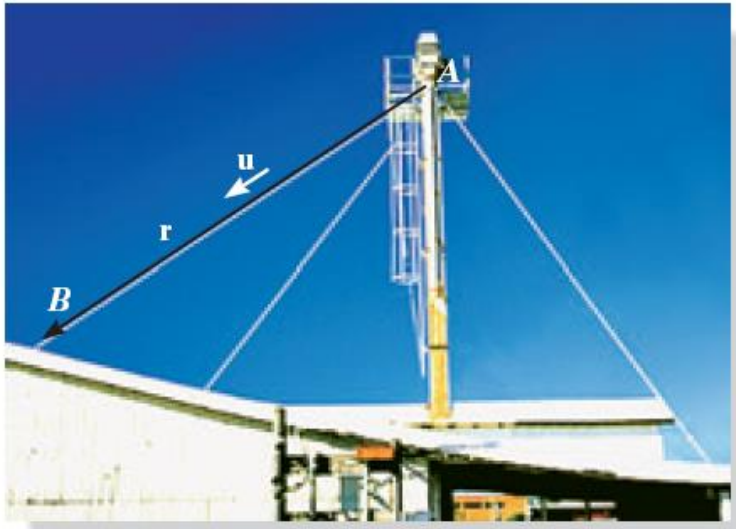
or $\quad \mathbf{r} = (x_B - x_A)i + (y_B - y_A)j + (z_B - z_A)k$





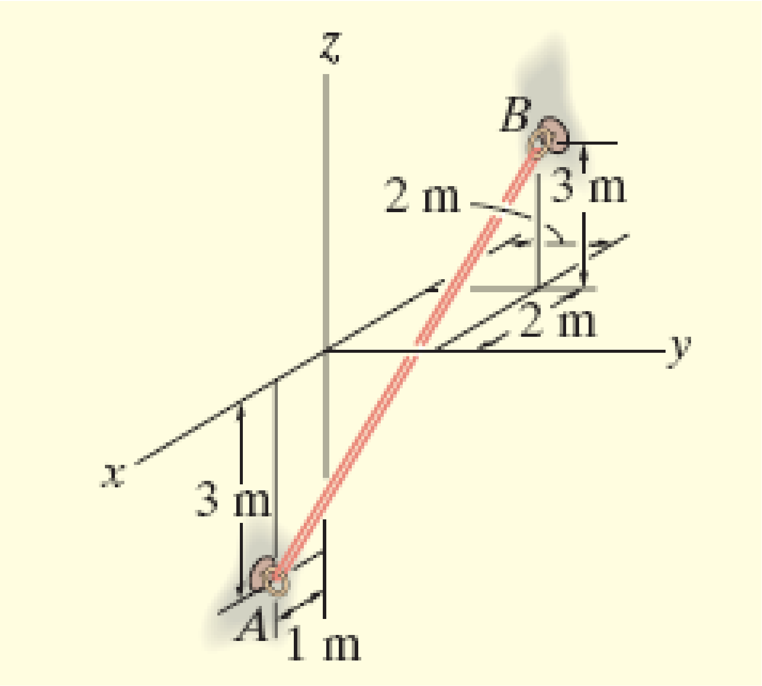
2.7 Position Vectors

- Length and direction of cable AB can be found by measuring A and B using the *x, y, z* axes
- Position vector **r** can be established
- Magnitude r represent the length of cable
- Angles, α, β and γ represent the direction of the cable
- Unit vector, **u** = **r**/r



Example 2.12

An elastic rubber band is attached to points A and B. Determine its length and its direction measured from A towards B.



Solution

Position vector

r = [-2m – 1m]i + [2m – 0]j + [3m – (-3m)]k

= {-3i + 2j + 6k}m

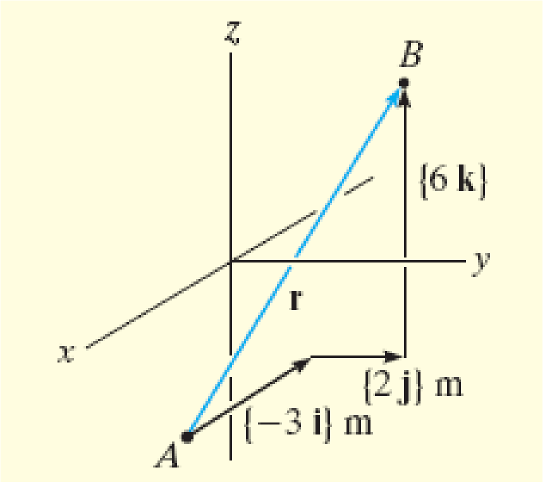
Magnitude = length of the rubber band

$$r = \sqrt{(-3)^2 + (2)^2 + (6)^2} = 7m$$

Unit vector in the direction of **r**

u = **r** /*r*

= -3/7**i** + 2/7**j** + 6/7**k**

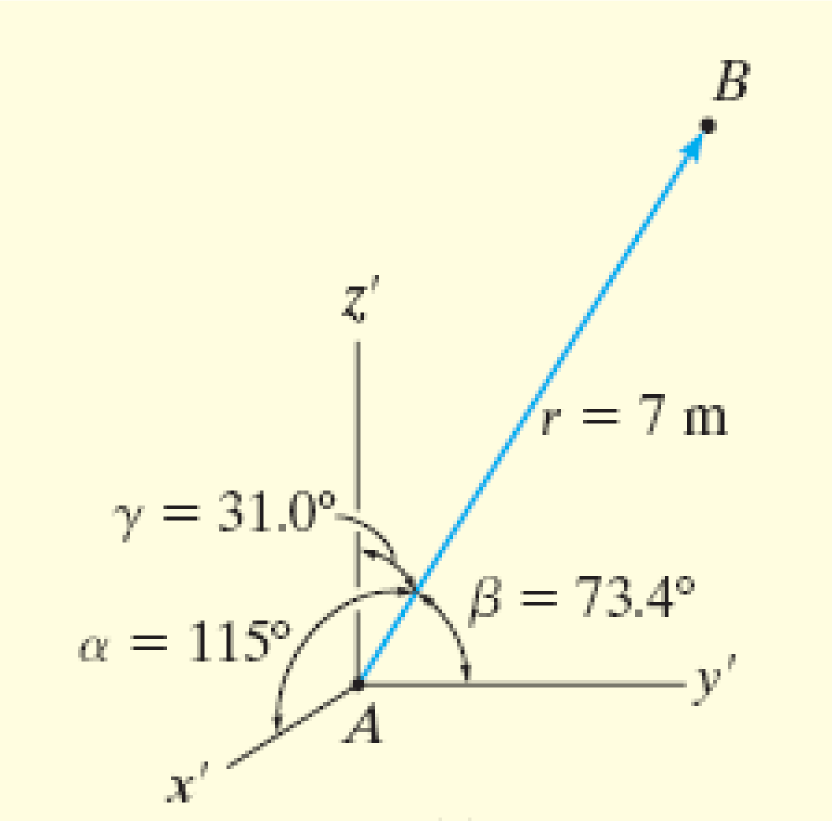


Solution

$\alpha = \cos^{-1}(-3/7) = 115^\circ$

$\beta = \cos^{-1}(2/7) = 73.4^\circ$

$\gamma = \cos^{-1}(6/7) = 31.0^\circ$

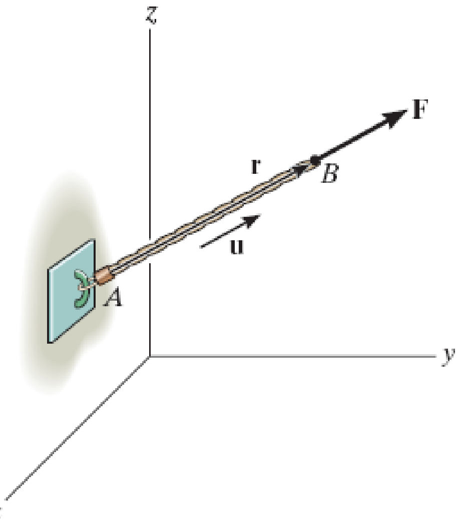


2.8 Force Vector Directed along a Line

- In 3D problems, direction of **F** is specified by 2 points, through which its line of action lies
- **F** can be formulated as a Cartesian vector

$$\mathbf{F} = F\,\mathbf{u} = F\,(\mathbf{r}/r)$$

- Note that **F** has units of forces (N) unlike **r**, with units of length (m)



2.8 Force Vector Directed along a Line

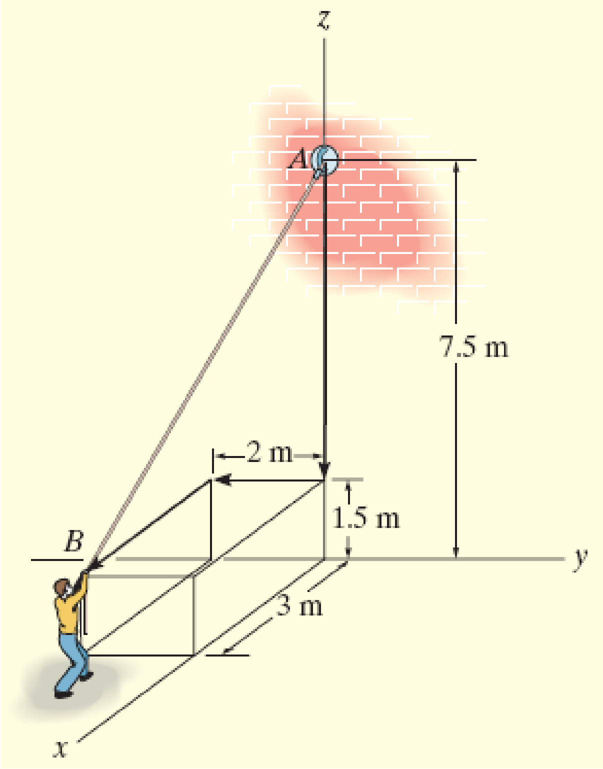
- Force **F** acting along the chain can be presented as a Cartesian vector by

  - Establish *x*, *y*, *z* axes

  - Form a position vector **r** along length of chain
- Unit vector, $\mathbf{u} = \mathbf{r}/r$ that defines the direction of both the chain and the force
- We get $\mathbf{F} = F\mathbf{u}$

Example 2.13

The man pulls on the cord with a force of 350N. Represent this force acting on the support A, as a Cartesian vector and determine its direction.



Solution

End points of the cord are A (0m, 0m, 7.5m) and B (3m, -2m, 1.5m)

r = (3m – 0m)**i** + (-2m – 0m)**j** + (1.5m – 7.5m)**k**
= {3**i** – 2**j** – 6**k**}m

Magnitude = length of cord AB

$$r = \sqrt{(3m)^2 + (-2m)^2 + (-6m)^2} = 7m$$

Unit vector,
u = **r** /*r*
= 3/7**i** - 2/7**j** - 6/7**k**

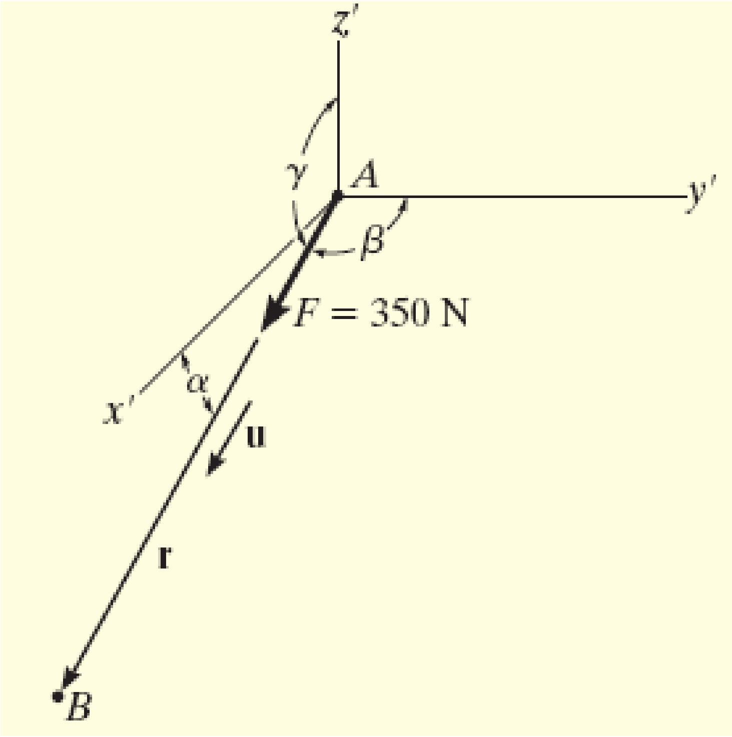
Solution

Force **F** has a magnitude of 350N, direction specified by **u.**

$\mathbf{F} = F\mathbf{u}$

$= 350\text{N}(3/7\mathbf{i} - 2/7\mathbf{j} - 6/7\mathbf{k})$

$= \{150\mathbf{i} - 100\mathbf{j} - 300\mathbf{k}\}\ \text{N}$



$\alpha = \cos^{-1}(3/7) = 64.6^\circ$

$\beta = \cos^{-1}(-2/7) = 107^\circ$

$\gamma = \cos^{-1}(-6/7) = 149^\circ$