Analysis

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Contents

Chapter 5

Differentiation

5.1 Basic Definitions and Examples

Definition 1 (A function being differentiable at x_0). Let I be a non-degenerate interval (i.e. containing more than one point) in $\mathbb{R}$, $x_0 \in I$ and let $f : I \to \mathbb{R}$ be a function. We say that f is *differentiable* at x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f^{`}(x_0)$$

exists and is finite. This limit $f^{`}(x_0)$ is then called the derivative of f at x_0. (In the case where x_0 is an endpoint of I this is a one-sided limit)

Definition 2. Let I be an interval in $\mathbb{R}$, $X_0 \in I$ and let $f : I \to \mathbb{R}$ be a function. We say that f is differentiable at x_0 from the left if the limit

$$(left\ derivative) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

exists and is finite. Similarly, f is differentiable at x_0 from the right if the limit

$$(right\ derivative) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

exists and is finite.

Theorem 1 (Chain Rule). Let $I, J \subseteq \mathbb{R}$ be intervals and $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ functions such that $f(I) \subset J$. Assume that f is differentiable at $x_0 \in I$ and g is differentiable at $f(x_0)$. Then $g \circ f$ is differentiable at x_0 and $(g \circ f)^{'}(x_0) = g^{'}(f(x_0))\, f^{'(x_0)}$.

Proof. Since f is differentiable at x_0, there is a (chord function) $\hat{f} : I \to \mathbb{R}$

such that $\hat{f}$ is continious at x_0,

$$f(x) - f(x_0) = (x - x_0)\hat{f}(x) \tag{5.1}$$

for all x in I, and

$$\hat{f}(x_0) = f^{'}(x_0)$$

Since g is differentiable at $f(x_0)$, there is a (chord function) $\hat{g} : J \to \mathbb{R}$ such that $\hat{g}$ is continious at $f(x_0)$,

$$g(y) - g(f(x_0)) = (y - f(x_0))\,\hat{g}(y) \tag{5.2}$$

for all y in J and

$$\hat{g}(f(x_0)) = g^{'}(f(x_0))\,.$$

Can we find a suitable chord function $g \circ f$?

Set $h(x) = g(f(x_0))$ $(x \in I)$ and so $h = g \circ f$. We are looking for a chord function $\hat{h} : I \to \mathbb{R}$ such that

- $\hat{h}$ is continious at x_0
- $h(x) - h(x_0) = (x - x_0)\,\hat{h}(x)$ $(x \in I)$

If we can do this, what we know about chord functions tells us h is differentiable at x_0, and

$$\begin{aligned}
\text{We have } h(x) - (x_0) &= g(f(x)) - g(f(c_0)) \\
\text{(Apply (2) with } y = f(x)) \\
&= (f(x) - f(x_0))\,\hat{g}(f(x)) \\
&= (x - x_0)\,\hat{f}(x)\,\hat{g}(f(x)) \\
&= (x - x_0)\,\hat{f}(x)\,\hat{g}(f(x))
\end{aligned}$$

Claim $\hat{h}(x) = \hat{f}(x)\,\hat{g}(f(x))$ works.

$\hat{f}$ is continious at x_0 because f is differentiable at x_0 (see above). $\hat{f}(x_0) = f^{'}(x_0)$. Since f is differentiable at x_0, we also have f is continious at x_0. Since $\hat{g}$ is continious at $f(x_0)$ with $\hat{g}(f(x_0)) = g^{'}(f(x_0))$ is continious at x_0 **Conclusion:** $h = g \circ f$ is differentiable at x_0, and

$$\begin{aligned}
h^{'}(x_0) &= \hat{h}(x_0) \\
&= \hat{f}(x_0)\,\hat{g}(f(x_0)) \\
&= fdt'(x_0)\,g^{'}(f(x_0))\,.
\end{aligned}$$

□