Linear Algebra

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Contents

Chapter 1

Spring 1

Lecture 1: Introduction

Lecture 7

1.1 Basis and dimension theory

Keep thinking back to $\mathbb{R}^n$

Definition 1.1 (Basis of V). Let V be a vector space. A subset $B \subseteq V$ is a basis of v if B is:

- linearly independent
- and spans V

Example. Let $V = M_{m,n}$. This is a vector space. The standard basis is the set

$$\{E_{ij} \mid 1 \leq i \leq m, 1 \leq j, +n\}.$$

where E_{ij} is the matrix with 0's everywhere except in $(i,j)^{th}$ entry where it equals 1.

For example, in the case M_{22} we have

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Linear independece is clear. We can show it spans M_{22}

$$\text{Span: } a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{22}$$

◇

Example. Let $V = \mathbb{R}^n$. The standard basis $\{e_1, e_2, \ldots, e_n\}$. ◇

Example. Let $V \leq \mathbb{R}^5$ be the space of vectors $(x_1, \ldots, x_5)$ such that

$$\begin{aligned} x_1 + x_2 - x_3 + x_5 &= 0 \\ x_1 + 2x_2 + x_4 + 3x_5 &= 0 \\ x_2 + x_3 + x_4 + 2x_5 &= 0. \end{aligned}$$

We write this system as

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 1 \\ 1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Placing this in reduced row echelon form gives:

$$\begin{pmatrix} 1 & 0 & -2 & -1 & -1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Assign parameters α, β, γ to x_3, x_4, x_5 :

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &= (2\alpha + \beta + \gamma, -\alpha - \beta - 2\gamma, \alpha, \beta, \gamma) \\ &= \alpha\,(2, -1, 1, 0, 0) + \beta\,(1, -1, 0, 1, 0) + \gamma\,(1, -2, 0, 0, 1)\end{aligned}$$

Hence a basis for V is $\{(2, -1, 1, 0, 0)\,, (1, -1, 0, 1, 0)\,, (1, -2, 0, 0, 1)$ $\diamond$

Proposition 1.2. Let V be a finite-dimensional vector space, $V \neq \{0_v\}$. Suppose $S = \{v_1, \ldots, v_s$ is a spanning set of V. Moreover, if $L \subseteq S$ in any linearly independent set then we can choose B to contain L.

Proof. We only prove the final statement, since the rest follows by setting $L = \emptyset$. Let $B \subseteq S$ be a maximal subset containing L, such that B is linearly independent. We claim B is a basis. We need to show $\langle B \rangle = V$. Suppose not. Hence $\exists v_i \notin \langle B \rangle$. But then $B \cup \{v_i\}$ is a strictly larger linearly independent set.

To show $B \cup \{v_i\}$ is linearly independent:

$$\lambda v_i + \sum \mu_j b_j = 0_v$$

with $b_j \in B$ and $\lambda, \mu_j \in \mathbb{R}$ is a dependence relation.

If $\lambda = 0$ this contradicts B being linearly independent. Hence $\lambda \neq 0$. So

$$v_i = \frac{1}{\lambda} \sum \mu_j b_j$$

But this would mean that $v_i \in \langle b \rangle$, again resulting in a contradiction $\square$

Lecture 8

Remark. Says that every finite-dimensional vector space has a basis.

Lemma 1.3 (Steinitz Exchange Lemma)**.** Let V be a vector space. Let $X = \{v_1, .., v_N) \subseteq V$. Suppose $u \in \langle X \rangle$, but $u \notin \langle X \ \ \{v_i\} \rangle$ fro some $1 \leq i \leq n$. Then let $Y = (X \ \ \{v_i\}) \cup \{u\}$. We have $\langle Y \rangle = \langle X \rangle$.

Proof.

- Since $u \in \langle X \rangle \exists \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$u = \lambda_1 v_! + \ldots +_n v_n \tag{1.1}$$

By assumtion we have $u \notin \langle X \ \{v_i\} \rangle$. Withought loss of generality assume $i = n$. Since $u \notin \langle X \ \{v_i\} \rangle$, so $_n \neq 0$. Rearranging (1) we get

$$v_n = \frac{1}{\lambda_n} (u - \lambda_1 v_1 - \ldots - \lambda_{n-1} v_{n-1}) \tag{1.2}$$

- Let $w \in \langle Y \rangle$. We can express w as a linear combination of $v_1, \ldots, v_{n-1}, u$. We can replace u with (1), hence write w as a linear combination of $v_1, \ldots, v_{n-1}, v_n$. Hence $w \in \langle X \rangle$. So $\langle Y \rangle \subseteq \langle X \rangle$.
- Let $w \in \langle X \rangle$. Then we have an expression for w as a linear combination of $v_1, \ldots, v_n$. By (2) we can replace v_n with a linear combination of $v_1, \ldots, v_{n-1}, k$. Hence $w \in \langle Y \rangle$. So $\langle X \rangle \subseteq \langle Y \rangle$.

□

Example. Let $V = \mathbb{R}^3$. $B = \{e_1, e_2, e_3\}$ the standard basis. Let $u = 2e_1 - 3e_2$. The Steinitz Exchange Lemma tells us

$$B_1 = \{u, e_2, e_3\} \qquad\qquad B_2 + \{e_1, u, e_3\}$$

both span V. (You should check this later!)

Although the Steinitz Exchange Lemma says nothing about linear independence, infact both B_1, B_2 are. Hence B_1, B_2 are bases for V. (Again check this!)

What abut if we exchange e_3 for u?

$$\text{i.e. } \{e_1, e_2, u\} = \{(1,0,0), (0,1,0), (2,-3,0)\}$$

Since the third coordinate is zero, this cannot span V. ◇

Theorem 1.4. Let V be a vector space and $S, T \subseteq V$ are finite subsets. If S is linearly independent and if $\langle T \rangle = V$, then $\mid S \mid \leq \mid T \mid$.

Proof.

- Let $S = \{u_1, \ldots, u_m\}$, $T = \{v_1, \ldots, v_n\}$
- We will use Steinitz Exchange Lemma to swap elements of T for those in S. This will finish exhausting S, hence $\mid S \mid \leq \mid T \mid$.
- Let $T_0 = \{v_1, \ldots, vn\}$. We have $\langle T_0 \rangle = V_1$, so $u_1 \in \langle v_1, \ldots, v_i \rangle$ but $u_1 \notin \langle v_1, \ldots, v_{i-1} \rangle$. By the Steinitz Exchange Lemma,

$$\langle v_1, \ldots, v_i \rangle = \langle u_1, v_1, \ldots, v_{i-1} \rangle.$$

Hence $V = \langle u_1, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n \rangle$ withought loss of generality we can relable the v_j to get u_1 has been exchanged for v_1e set

$$T_1 = \{u_1, v_2, \ldots, v_n\}, \qquad\qquad \langle T_1 \rangle = V.$$

- Proceeding by induction we have

$$T_k = \{u_1, \ldots, u_k, v_{k+1}, \ldots, v_n\}, \qquad \langle T_k \rangle = V.$$

- At each stage $u_{k+1} \in \langle T_k \rangle$, but $u_{k+1} \notin \langle u_1, \ldots, u_k \rangle$ because S is linearly independent. Hence we continue to make subsitution. This can only terminate when S is exhausted, $m \le n$.

- Hence $\mid S \mid \le \mid T \mid$.

$\square$

Lecture 9

Let B_2 be another basis of V. Recall that this means both B_1 and B_2 are spanning and are linearly independent.

$$\left.\begin{array}{c} B_2 \text{ is linearly independent} \\ \langle B_1 \rangle = V \end{array}\right\} \Rightarrow \mid B_2 \mid \le \mid B_1 \mid$$

$$\left.\begin{array}{c} B_1 \text{ is linearly independent} \\ \langle B_2 \rangle = V \end{array}\right\} \Rightarrow \mid B_1 \mid \le \mid B_2 \mid$$

$$\Rightarrow \mid B_1 \mid = \mid B_2 \mid$$

In particular, B_2 is also finite since B_1 is finite.

Definition 1.5 (Dimension of V)**.** Let V be a finite-dimensional vector space. The dimension of V is defined to be $\dim(V) := |B|$ for some basis B of V

Example.

$$\begin{aligned} S :&= \{(0,1,2,3), (1,2,3,4), (2,3,4,5)\} \\ &= \{r_1, r_2, r_3\} \text{ where } r_1, r_2, r_3 \subseteq \mathbb{R}^4 \end{aligned}$$

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix} \in M_{34}$$

$$\begin{aligned} \text{Then the Rowspan}\,(A) &= \langle S \rangle \subseteq \mathbb{R}^4 \\ &= \langle r_1, r_2, r_3 \rangle. \end{aligned}$$

$$\text{RRE}\,(A) = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} r_1^{'} \\ r_2^{'} \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} r_1^{'} = (1,0,-1,-2) = -r_1 - r_2 + r_3 \\ r_2^{'} = (0,1,2,3) = r_1 \end{array}\right\} S^{'} := \left\{r_1^{'}, r_2^{'}\right\}$$

$$\begin{aligned} \Rightarrow & \langle S^{'} \rangle \subseteq \langle S \rangle \\ & \langle r_1^{'}, r_2^{'} \rangle \subseteq \langle r_1, r_2, r_3 \rangle \end{aligned}$$

$$
\begin{aligned}
\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = A &= \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix}^{-1} \operatorname{RRE}(A) \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} r_1^{'} \\ r_2^{'} \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\left.\begin{aligned} r_1 &= r_2^{'} \\ r_2 &= r_1^{'} + 2r_2^{'} \\ r_3 &= 2r_1^{'} + 3r_2^{'} \end{aligned}\right\} \Rightarrow \langle S \rangle \leq \langle S^{'} \rangle
$$

$$
\Rightarrow \langle S \rangle = \langle S^{'} \rangle
$$

Notice that $\left\{ r_1^{'}, r_2^{'} \right\}$ is spanning $\langle S \rangle$. Notice that if

$$
\begin{aligned}
\lambda_1 r_1^{'} + \lambda_2 r_2^{'} = \underline{0} &= (\lambda_1, \lambda_2, -\lambda_1 + 2\lambda_2, -2\lambda_1 + 3\lambda_2) = \underline{0} \\
&\Rightarrow \lambda_1 = \lambda_2 = 0
\end{aligned}
$$

So $\left\{ r_1^{'}, r_2^{'} \right\}$ is linearly independent. That is, $\left\{ r_1^{'}, r_2^{'} \right) = S^{'}$ is a basis of $\text{Rowspan}(A)^{'}$. Hence, $\dim \text{Rowspan}(A) = 2$. ◇

Example (Generalisation). Let $A \in M_{nn}$

$$
\operatorname{RRE}(A) = BA = \begin{bmatrix} r_1^{'} \\ \vdots \\ r_k^{'} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ with } r_1^{'} \neq \underline{0}, \ldots, r_k^{'} \neq \underline{0}
$$

Like above we have $\text{Rowspan}(A) = \langle r_1^{'}, \ldots, r_k^{'} \rangle$

$$
\begin{aligned}
\text{row rank of } A &= \dim \text{Rowspan}(A) \\
&= \text{non-zero rows in } \operatorname{RRE}(A) \\
&= k
\end{aligned}
$$

◇

1.2 Subspaces and Dimension Theory

Proposition 1.6. Let V be a finite-dimensional vector space and $U \leq V$ is a subspace. Then

- U is finite-dimensional
- $\dim U \leq \dim V$

Furthermore, if $\dim U = \dim V$, then $U = V$.

Proof. Let $S \subseteq U$ be the largest linearly independent subset. Let B be a basis of V, say B with $|B| = n = \dim(V)$

$$\left.\begin{array}{c} S \subseteq V \text{ is linearly independent.} \\ \langle B \rangle = V \end{array}\right\} \Rightarrow |S| \leq |B|$$

Assume towards a contradiction that $\langle S \rangle \not\subseteq U$. Then there exists $u \in U\ \langle S \rangle$

<u>**Claim:**</u> $S \cup \{u\}$ is linearly independent.

> **Proof.** Say $S = \{s_1, \ldots, s_k\}$.
> Let $\lambda_1, \ldots, \lambda_k, \lambda \in \mathbb{R}$ such that $\lambda_1 s_1 + \ldots + \lambda_k s_k + \lambda u = \underline{0}$. There are two cases:
>
> - $\lambda = 0$
>
> $$\Rightarrow u = \frac{-1}{\lambda}(\lambda_1 s_1 + \ldots + \lambda_k s_k) \in \langle S \rangle$$
>
> which is a contradiction.
>
> - $\lambda \neq 0$
>
> $$\begin{aligned} \Rightarrow \underline{0} &= \lambda_1 s_1 + \ldots + \lambda_k s_k + \lambda_u \text{ where } \lambda_u = 0 \\ &= \lambda_1 s_1 + \ldots + \lambda_k s_k \end{aligned}$$
>
> Since $s_{1,k}$ are linearly independent so $\lambda_1 = \ldots = \lambda_k = 0$ That is all coefficients are zero.
>
> □

Hence, $S \not\subseteq S \cup \{u\}$ is linearly independent. This is a contradiction to S being maximal, which implies $U = \langle S \rangle$ and S is a basis of U, which implies $\dim(U) = |S| \leq |B| = n = \dim(V)$

□

Lecture 10

Theorem 1.7 (Dimension Formula). Let V be a finite-dimensional vector space. Let $U, W \leq V$ be subspaces. Then

$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$$

Example. Suppose $V = \mathbb{R}_3$, $U = \langle (1,0,0) \rangle$, $W = \langle (1,0,1) \rangle$. Have $\dim(U) = 1$, $\dim W = 1$.

$$U \cap W = \langle (1,0,0) \rangle \cap \langle (1,0,1) \rangle = \{\underline{0}\}$$

Hence $\dim(U \cap W = 0)$. By the dimension formula

$$\begin{aligned} \dim(U + W) &= \dim(U) + \dim(W) - \dim(U \cap W) \\ &= 1 + 1 - 0 \\ &= 2 \end{aligned}$$

◇

Proof. Let $\{v_1, \ldots, v_m\}$ be a basis for $U \cap W$. Have $U \cap W$. Have $UW \leq U$, $U \cap W \leq W$. We can extend our basis to give a basis

$$\{v_1, \ldots, v_m, u_1, \ldots, u_p) \text{ of } U$$

$$\{v_1, \ldots, v_m, w_1, \ldots, w_q\} \text{ of } W$$

Hence $\dim(U) = m + p$, $\dim(W) = m + q$

$$\dim(U \cap W) = m$$

We claim $S = \{v_1, \ldots, v_m, u_1, \ldots, u_p, w_1, \ldots, w_q\}$ is a basis for $U + W$. Hence $\dim(U + W) = m + p + q$ as required.

Spanning:
Take $x \in U + W$. Then x = u+w, for some $u \in U$, $w \in W$.

$$u = a_1 v_1 + \ldots + a_m v_m + a'_1 u_1 + \ldots + a'_p u_p$$

$$w = b_1 v_1 + \ldots + b_m v_m + b'_1 w_1 + \ldots + b'_q w_q$$

for some $a_i, a'_i, b_i, b'_i \in \mathbb{R}$. Then

$$\begin{aligned} x &= u + w \\ &= (a_1 + b_1) v_1 + \ldots + (a_m + b_m) v_m + a'_1 u_1 + \ldots + a'_p u_p + b'_1 w_1 + \ldots + b'_q w_1 + \ldots + b'_q w_q \end{aligned}$$

Hence $x \in \langle S \rangle$.

Linear Independence:
Suppose

$$a_1 v_1 + \ldots + a_m v_m + b_1 u_1 + \ldots + b_p u_p + c_1 w_1 + \ldots + c_q w_q = 0$$

for some $a_i, b_i, c_i \in \mathbb{R}$. Have

$$(a_1 v_1 + \ldots + a_m v_m + b_1 u_1 + \ldots + b_p u_p) \in U = -(c_1 w_1 + \ldots + c_q w_q) \in W$$

so this lies in $U \cap W$. Since $\{v_1, \ldots, v_m, w_1, \ldots, w_q\}$ is a basis of W, so

$$c_1 = \ldots = c_q = d_1 = \ldots = d_m = 0.$$

Hence we have:

$$a_1 v_1 + \ldots + a_m v_m + b_1 u_1 + \ldots + b_p u_p = 0$$

Since $\{v_1, \ldots, v_m, u_1, \ldots, u_p\}$ is a basis of U, so

$$a_1 = \ldots = a_m = b_1 = \ldots = b_p = 0.$$

Hence it is linearly independent. $\square$

Definition 1.8 (Direct Sum). Let V be a vector space $U, W \leq V$. If $U \cap W = \{\underline{0}\}$ and $U + W = V$. We write $V = U \bigoplus W$ and we say V is the direct sum of U and W.