Math 530: Differential Geometry

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I hereby reaffirm the Lawrence University Honor Code. In particular, I understand that people who know the solution are only allowed to (1) suggest a simpler problem, but not help me solve it, and (2) suggest a textbook, but not certain pages or chapters.

1. I worked with Connor Phelps for problems: 1.5
2. I used material from the following textbooks: *Curvatures in Mathematics and Physics* by Sternberg, *Linear Algebra* by Serge Lang, *Differential Topology* by Alan Pollack and Victor Guillemin
3. I used material from the following websites: for 1.4.

Problem (6.2). Tensor Calculus 2. The video discusses how basis vectors are like partial derivatives. They also discuss how the basis vectors in polar coordinates change direction and lengths from point to point in space. Look up *local frame* in one of Lee's books on manifolds. Explain how the basis vectors in this video are really a chart-induced local frame.

Solution.

A local frame for M is an ordered n-tuple of vector fields $(E_1, \ldots, E_n)$ defined on an open subset $U \subseteq M$ that is linearly independent and spans the tangent bundle; thus the vectors $(E_{1,p}, \ldots, E_{n,p})$ form a basis for T_pM at each $p \in U$.

The union of all tangent spaces at all points of a smooth manifold can be "glued together" to form a new manifold- called the tangent bundle.

A local chart is a domain $U \subset \mathbb{R}^n$ together with a one-to-one mapping $\phi : W \mapsto U$ of a subset W of the manifold M onto U. Essentially, a chart is a map that associates a subset of a manifold with an open set in Euclidean space, which enables us to define local coordinates on a manifold.

The basis vectors discussed are the partial derivatives of a function with respect to its coordinates. These partial derivatives form a local frame that is induced by the coordinate system itself. This means that at each point in space, the partial derivatives form a basis for the tangent space, and the choice of basis depends on the choice of coordinates used to describe the space. Essentially, the local frame is a map from the manifold to the tangent bundle, so it's values are vectors in each tangent space.

The local frame induced by the Cartesian coordinate chart is a collection of basis vectors that are defined as the partial derivatives with respect to $x - y$ coordinates of that chart, but which vary from point to point in space. For the $x - y$ coordinate system we saw in the video you can imagine the basis vectors $\vec{e_x}$ and $\vec{e_y}$ to be linearly independent and span $\mathbb{R}^2$. This local frame is tangent to the manifold $\subseteq \mathbb{R}^2$ at each point and provides a basis for the tangent space at that point. Essentially the outputs of the local frame here are vectors in each tangent space. Hence, basis vectors in the case of cartesian coordinates are really a chart-induced local frame.

For the argument of polar coordinates: Take a chart $f : (r, \theta) \mapsto (rcos(\theta), rsin(\theta)$. This chart takes inputs in the polar coordinates and spits outputs in the cartesian coordinates. What we want to show is that the partial derivatives of the chart defined form a local frame induced by the polar coordinate system itself.

The partial derivative of f will be the jacobian: $J_f = \begin{bmatrix} \cos(\theta) & -rsin(\theta) \\ \sin(\theta) & rcos(\theta) \end{bmatrix}$

Now notice that the column vectors $\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$, $r\begin{bmatrix}-sin(\theta)\\ cos(\theta)\end{bmatrix}$ are linearly independent and span $\mathbb{R}^2$. In essence, the partial derivative of our defined chart forms a local frame induced by the polar coordinate system itself.

Problem (1.4). (On page 24, the book considers a map $X : M \to \mathbb{R}^3$ defined on a subset M of the plane $\mathbb{R}^2$. They assume the Jacobian matrix of this map has rank two at all points of M. The *critical points* of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ are defined as those points $x \in \mathbb{R}^n$ where the Jacobian fails to be full rank. The *critical values* are the images of critical points under the map. Problem (from V.I. Arnold's *Trivium*): Find the critical points and the critical values of the map $z \mapsto z^2 + 2\bar{z}$. Have a computer draw them.)

Solution. Let $z = x + iy$ be a complex number, where $x, y \in \mathbb{R}$. Then, we can write $z^2 + 2\overline{z}$ as:

$$z^2 + 2\overline{z} = (x + iy)^2 + 2(x - iy) \qquad = (x^2 - y^2 + 2x) + i(2xy - 2y)$$

So, we have a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$f(x, y) = \begin{pmatrix}u(x, y)\\ v(x, y)\end{pmatrix} = \begin{pmatrix}x^2 - y^2 + 2x\\ 2xy - 2y\end{pmatrix}$

The Jacobian matrix of f is given by:

$$J_f(x, y) = \begin{pmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{pmatrix}.$$

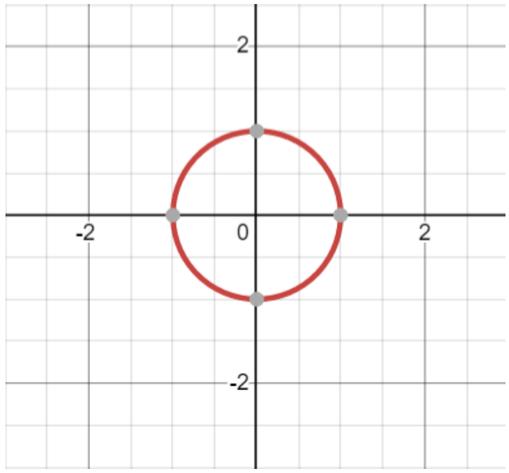
In the case of the specific function $z \mapsto z^2 + 2\bar{z}$, we have $u(x, y) = x^2 - y^2 + 2x$ and $v(x, y) = 2xy - y$. So, the Jacobian matrix is given by

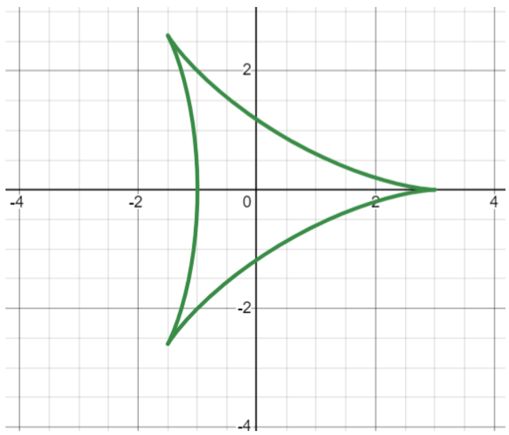
$$J_f(x, y) = \begin{pmatrix}2x + 2 & -2y\\ 2y & 2x - 2\end{pmatrix}.$$

Now it's time to use some smart theorems. From this link., you can see that if A is an n x n matrix then rank (A) = n iff A has an inverse. Applying this theorem to the $J_f(x, y)$ case, if our jacobian matrix has a determinant of zero, then it fails to be invertible and hence fails to be of full rank. And this is how we will solve for the critical points.

$detJ_f(x, y) = (2x + 2)(2x - 2) + 4y^2 = 0$

This equation is same as $x^2 + y^2 = 1$, which is the equation of a circle with center $(0, 0)$ and radius 1. Therefore, the critical points of f lie on this circle. So, the critical values of f are the images of the points on the circle under f.





The critical points of f lie on the red circle and the critical values of f lie on the green shape shown in the second shape plot.

Problem (1.7). (If γ is a curve in the manifold with $\gamma(0) = x$ and $\gamma'(0) = v$ then the book defines $dG_x(v)$ as $\frac{d(G\circ\gamma)}{dt}(0)$. Verify this definition does not depend on the choice of representing curve γ.)

Solution. To verify that the definition of $dG_x(v)$ as $\frac{d(G\circ\gamma)}{dt}(0)$ does not depend on the choice of the representing curve γ, we can show that it produces the same result for any two different curves γ_1 and γ_2 that satisfy the conditions $\gamma_1(0) = x$, $\gamma_1'(0) = v$, $\gamma_2(0) = x$, and $\gamma_2'(0) = v$.

Then, by the chain rule, we have:

$\frac{d(G\circ\gamma_1)}{dt} = \nabla G(\gamma_1(t)) \cdot \gamma_1'(t)$,
where ∇G denotes the gradient of G with respect to the coordinates on the manifold. Similarly, we have:

$\frac{d(G\circ\gamma_2)}{dt} = \nabla G(\gamma_2(t)) \cdot \gamma_2'(t)$.
Since $\gamma_1(0) = \gamma_2(0) = x$ and $\gamma_1'(0) = \gamma_2'(0) = v$, we know that $\gamma_1(t)$ and $\gamma_2(t)$ pass through the same point x with the same tangent vector v at $t = 0$. Therefore, $\gamma_1'(0)$ and $\gamma_2'(0)$ are equal, and $\nabla G(\gamma(t)) \cdot \gamma'(t)$ is the same for both curves at $t = 0$.

Thus, we have:
$\frac{d(G\circ\gamma_1)}{dt}(0) = \nabla G(\gamma_1(0)) \cdot \gamma_1'(0) = \nabla G(x) \cdot v$

and
$\frac{d(G\circ\gamma_2)}{dt}(0) = \nabla G(\gamma_2(0)) \cdot \gamma_2'(0) = \nabla G(x) \cdot v.$

Therefore, the definition of $dG_x(v)$ as $\frac{d(G\circ\gamma)}{dt}(0)$ does not depend on the choice of the representing curve γ, as it produces the same result for any two curves that satisfy the conditions $\gamma(0) = x$ and $\gamma'(0) = v$.

Problem (1.5). (On pages 25-26 the book gives three descriptions of the tangent space at a point. Show that all three describe the same vector space.)

Solution.

Let Y be a hypersurface parametrized by a map X and given implicitly by a function $F = 0$. Let V be the span of the $(n-1)$ partial derivatives of the map X which parametrizes the hypersurface Y. So, $V = \text{span}\left(\frac{\partial X}{\partial y_1}(y), \ldots, \frac{\partial X}{\partial y_{n-1}}(y)\right)$. Let W be the span of all tangent vectors $\gamma'(0)$ to curves $\gamma : \mathbb{R} \to Y$. Let V_x be all vectors $v \in \mathbb{R}^n$ such that the directional derivative of F in the direction of v is zero. First we will show that V and W represent the same vector space which is the tangent space T_{Y_x}.

Show that any tangent vector $v \in W$ to Y at x can be expressed as a linear combination of $\frac{\partial X}{\partial y_1}(y), \ldots, \frac{\partial X}{\partial y_{n-1}}(y)$. That means the vector space W has the same basis as of V and hence represent the same vector space.

Let $v \in W$ be a tangent vector to Y at x. By definition of the space W, there exists a smooth curve $\gamma(t)$ lying in Y such that $\gamma(0) = x$ and $\gamma'(0) = v$. Let $X^{-1} : R^n \mapsto M$ be a differentiable inverse map to X. This is locally true by X because of implicit function theorem and X being regular. Let $y(t) = X^{-1}(\gamma(t))$ be a smooth curve in $M \subseteq \mathbb{R}^{n-1}$ passing through x at $t = 0$, i.e., $X(y(0)) = x$. By the chain rule of differentiation, it follows that $\gamma'(0) = \frac{d}{dt}X(y(t))\Big|_{t=0}$.

Now, consider the coordinates $y_1, \ldots, y_{n-1}$ on M near x induced by the map X, i.e., $y = (y_1, \ldots, y_{n-1})$ are the local coordinates on M near x such that $X(y) = x$. As $y(t)$ lies entirely within $M \subseteq \mathbb{R}^{n-1}$ where each coordinate function $y_i(t)$ maps $\mathbb{R}$ to $\mathbb{R}$, for $i = 1, \ldots, n-1$, we have $\frac{d}{dt}X(y(t))\Big|_{t=0} = \sum_{i=1}^{n-1} \frac{\partial X}{\partial y_i}(y)\frac{dy_i}{dt}(0)$.

Since $v = \gamma'(0)$ and $\gamma'(0) = \frac{d}{dt}X(y(t))\Big|_{t=0}$, v can be expressed as a linear combination of $\frac{\partial X}{\partial y_1}(y), \ldots, \frac{\partial X}{\partial yn-1}(y)$, with coefficients given by $\frac{dy_i}{dt}(0)$ for $i = 1, \ldots, n-1$. This shows that any tangent vector v to Y at x can be expressed as a linear combination of $\frac{\partial X}{\partial y_1}(y), \ldots, \frac{\partial X}{\partial y_{n-1}}(y)$.

That means V and W share the same basis and hence they refer to the same vector space.

Now We want to prove that W, and V_x represent the same vector space.

To show this, we start by picking an arbitrary $v \in W$, which is a tangent vector to Y at x. By definition of the space W, there exists a smooth curve $\gamma(t)$ lying in Y such that $\gamma(0) = x$ and $\gamma'(0) = v$. Using the implicit equation of the surface, we have $F(\gamma(t)) = 0$. Taking a time derivative of this equation and evaluating it at $t = 0$, we get $\nabla F(\gamma(0)) \cdot \gamma'(0) = 0$, which gives us $\nabla F(x) \cdot v = 0$. This means that $v \in V_x$. So, $W \subset V_x$.

Next, we show that V_x has dimension $(n-1)$. Since V and W both have dimension $n-1$, we can pick $v_1 = \nabla F(x)$ and choose e_1 as a unit basis vector in the direction of v_1. As the ambient space is $\mathbb{R}^n$, we can use the Gram-Schmidt process to find $e_2, e_3, \ldots, e_n$ as basis vectors that are all orthogonal to e_1 and span a $\mathbb{R}^{n-1}$ space where v lies. This is precisely the space V_x which includes v, and we have proved that $\dim(V_x) = n - 1 = \dim(V) = \dim(W)$.

Using the fact that $W \subset V_x$ and $\dim(V_x) = \dim(W) = n - 1$ and using theorem 3.29 from Meckes' text, we can conclude that $V_x = W$. So, $V = W = V_x$

Problem (1.11). (Prove that if $(W_x u, v) = (u, W_x v)$ for all $u, v \in T_x M$ then W_x is diagonalizable and has all real eigenvalues. Give also a real matrix with complex eigenvalues. Do most real matrices have real or complex eigenvalues?)

Solution.

By set-up and definition, W_x is self-adjoint. We want to show W_x is diagonalizable and has all real eigenvalues. We will show the case for any self-adjoint operator and automatically the case applies to W_x.

Let T be an arbitrary self-adjoint linear map from a finite-dimensional vector space V to itself. We want to show that T has an eigenbasis and all real eigenvalues.

Let λ be an eigenvalue of T. So, $T(v) = \lambda v$ for some $v \neq 0$
We have: $T(v) = \lambda v$ and $T^*(v) = \bar{\lambda} v$. Since, $T = T^*$, it follows that $\lambda v = \bar{\lambda} v$ and hence $\lambda = \bar{\lambda}$

But it's only possible if λ is real. So, T has real eigenvalues.

We want to assume something at this point: T has at least one eigenvalue. [It's provable easily but I want to save time as it is a well-established theorem in functional analysis]

We want to show that there is an orthonormal basis B for V consisting of eigenvectors of T. In essence, T is diagonalizable.

Proof Use induction on $n = dim(V)$ Base case: $n = 1$: $T : V \mapsto V$ where $dim(V) = 1$ is given by $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$. Let $v_0 \in V$ with $v_0 \neq 0$ and let $v_1 = \frac{v_0}{|v_0|}$. So Gram-Schmidt process gives us $B = \{v_1\}$. Now assume that the statement holds for $n \in \mathbb{N}$.

Suppose now, $dim(V) = n + 1$.

By your assumption, T has an eigenvalue λ_1. Let v_1 be a unit eigenvector corresponding to λ_1. Let $J = \{j \in V \mid \langle j, v_1 \rangle = 0\}$.

I claim that J is T-invariant.

Let $j \in J$ be arbitrary. Then,

$$\langle T(j), v_1 \rangle = \langle j, T^*(v_1) \rangle \qquad = \langle j, T(v_1) \rangle \ = \langle j, \lambda_1 v_1 \rangle \qquad = \bar{\lambda}_1 \langle j, v_1 \rangle \ = 0$$

Hence we know that $T(j) \subseteq J$.

Note that J is also an inner-product space, subspace of V by set-up, and we have that $T_j : J \mapsto J$ is self-adjoint.

Moreover, since $v_1 \notin J$, $dim(J) < dim(V) = n + 1$.

By induction hypothesis, there is an orthonormal basis $B_J = \{v_2, \ldots, v_{n+1}\}$ for J consisting of eigenvectors of T_J.

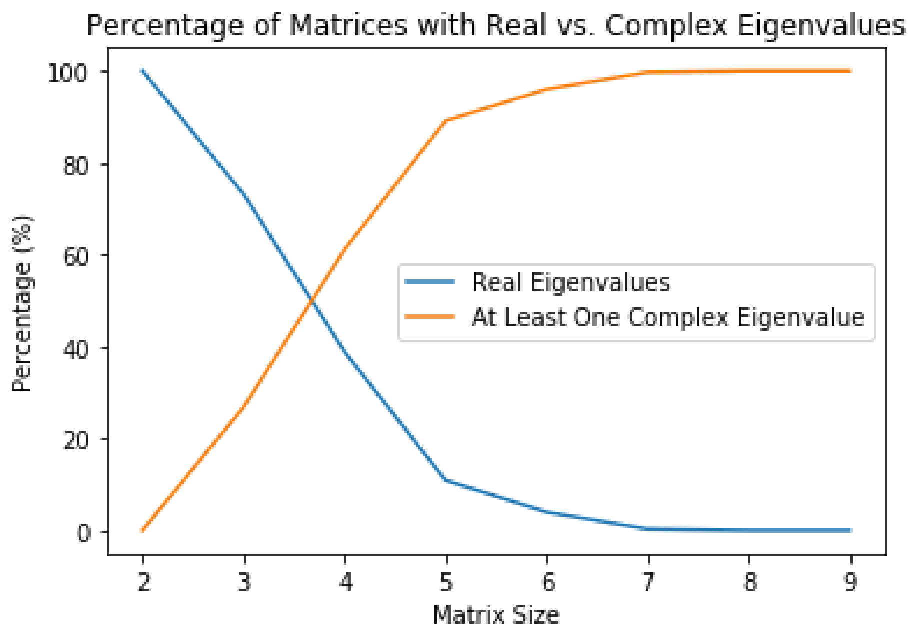
Hence, $B = \{v_1, \ldots, v_{n+1}\}$ is an orthonormal basis for V consisting of eigenvectors of T. Hence T is diagonalizable.

And since W_x is also a self-adjoint operator for our case. Following the general proof we laid out, we can say that W_x is diagonalizable and has all real eigenvalues.

A real matrix with complex eigenvalues is $M = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

the eigenvalues of the matrix M are $-1 + 2i$ and $-1 - 2i$.

I claim that most real matrices have complex eigenvalues. So, I wrote a python code that generates a 1000 matrices randomly of different sizes and computes their eigenvalues and spit out their eigenvalues and track how many of them are real vs how many matrices have at least one complex eigenvalue. Then from the pattern of the data it gives out, it seems that bigger sized matrices tend to have a higher percentage of complex eigenvalues- and that makes sense as for bigger sized matrices you would have a characteristic polynomial that is of higher order powers- which would lead to a polynomial whose roots are less likely to be found on the real numbers set and they tend to have complex roots and thus complex eigenvalues.



Problem (1.12). ((a) For $n = 2, 3, 4, 5, 6$ write out the $H_0, H_1, \dots, H_{n-1}$ explicitly in each case. (b) Relate the coefficients of a polynomial to its roots using symmetric polynomials.)

Solution.

(a)

For $n = 2$:

$$H_0 = 1$$

$$H_1 = \sum_{1 \le i_1 < i_2 \le 1} k_{i_1} \cdot k_{i_2} = k_1$$

For $n = 3$:

$$H_0 = 1$$

$$H_1 = \frac{1}{2}(k_1 + k_2)$$

$$H_2 = \frac{1}{\binom{3-1}{2}} \sum_{1 \le i_1 < i_2 < i_3 \le 2} k_{i_1} \cdot k_{i_2} = k_1 \cdot k_2$$

For $n = 4$:

$$H_0 = 1$$

$$H_1 = \frac{1}{3}(k_1 + k_2 + k_3)$$

$$H_2 = \frac{1}{\binom{4-1}{2}} \sum_{1 \le i_1 < i_2 \le 3} k_{i_1} \cdot k_{i_2} = \frac{1}{3}(k_1 k_2 + k_2 k_3 + k_3 k_1)$$

$$H_3 = k_1.k_2.k_3$$

For $n = 5$:

$$H_0 = 1$$

$$H_1 = \frac{1}{4}(k_1 + k_2 + k_3 + k_4)$$

$$H_2 = \frac{1}{\binom{5-1}{2}} \sum_{1 \le i_1 < i_2 \le 5-1} k_{i_1} \cdot k_{i_2} = \frac{1}{6}(k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4)$$

$$H_3 = \frac{1}{\binom{5-1}{3}} \sum_{1 \le i_1 < i_2 < i_3 \le 5-1} k_{i_1} \cdot k_{i_2} \cdot k_{i_3} = \frac{1}{4}(k_1k_2k_3 + k_1k_3k_4 + k_1k_2k_4 + k_2k_3k_4)$$

$$H_4 = k_1 \cdot k_2 \cdot k_3 \cdot k_4$$

For n=6:

$$H_0 = 1$$

$$H_1 = \frac{1}{5}(k_1 + k_2 + k_3 + k_4 + k_5)$$

$$H_2 = \frac{1}{\binom{6-1}{2}} \sum_{1 \le i_1 < i_2 \le 5} k_{i_1} \cdot k_{i_2} = \frac{1}{10}(k_1k_2 + k_1k_3 + k_1k_4 + k_1k_5 + k_2k_3 + k_2k_4 + k_2k_5 + k_3k_4 + k_3k_5 + k_4k_5)$$

$$H_3 = \frac{1}{\binom{6-1}{3}} \sum_{1 \le i_1 < i_2 < i_3 \le 5} k_{i_1} \cdot k_{i_2} \cdot k_{i_3} = \frac{1}{10}(k_1k_2k_3 + k_1k_2k_4 + k_1k_2k_5k_1k_3k_4 + k_1k_3k_5 + k_1k_4k_5 + ...)$$

$$H_4 = \frac{1}{5}(k_1k_2k_3k_4 + k_1k_2k_3k_5 + k_1k_2k_4k_5 + k_1k_3k_4k_5 + k_2k_3k_4k_5)$$

$$H_5 = k_1 \cdot k_2 \cdot k_3 \cdot k_4 \cdot k_5$$

(b)

What this question is basically hinting towards Vieta's formula for symmetric polynomials. Using vieta's formula, you can solve for polynomials' roots by relating the coefficients of the polynomial.

For say, a general degree two polynomial looks like: $P(x) = ax^2 + bx + c$. To solve for its roots, we can use vieta's formula to see the roots represented using the coefficients of the polynomial. For example: $x^2 - 5x + 6 = 0$ where its roots are related with its coefficients in the manner : $r_1 + r_2 = \frac{-b}{a}$ and $r_1r_2 = \frac{c}{a}$ in our case $(r_1, r_2) = (3, 2)$

Problem (1.8). (Use a computer to visualize the image of the Gauss map $\nu : Y \to S^{n-1}$ as a subset of S^{n-1} when $n = 3$. Find three examples where the image of ν is qualitatively different for each.)

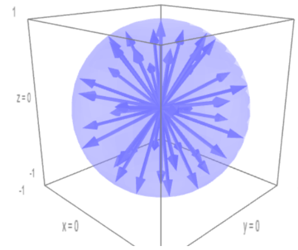
Example 1: Sphere

The Gauss map for a sphere is given by:

$$\nu(\mathbf{p}) = \frac{\mathbf{p}}{r}$$

where $\mathbf{p} = (x, y, z)$ is a point on the sphere and r is the radius of the sphere.

The image is essentially the whole of unit sphere S^2

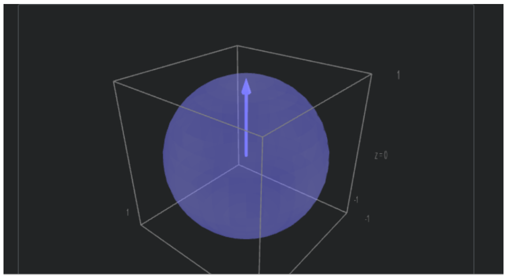


Example 2: Plane

The Gauss map for a plane with equation $z = 0$ is given by:

$$\nu(\mathbf{p}) = (0, 0, 1)$$

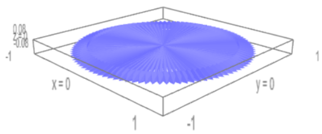
where $\mathbf{p} = (x, y, z)$ is a point on the plane.



What we have is a constant normal vector which makes sense for a plane $z = 0$.

Example 3: Cylinder

The Gauss map of a cylinder, on the other hand, consists of a collection of straight lines in three-dimensional space, each passing through the origin and a point on the unit circle in the plane perpendicular to the cylinder's axis.



Problem (1.13). (Write out formula (1.4) (reproduced below) explicitly for the cases $n = 2, 3, 4, 5, 6$ without summation notation, without binomial coefficient notation, and replacing the H_j with their explicit formulas from the previous problem.

$$V_n(Y_h) = \frac{1}{n} \sum_{i=1}^{n} \binom{n}{i} h^i \int_Y H_{i-1}\, d^{n-1}A$$

)

Solution.

For n=2, we have : $V_2(Y_h) = \frac{1}{2} * 2h * \int_Y dA + h^2 \int_Y k_1 dA$
For $n = 3$:

$$V_3(Y_h) = \frac{1}{3}\left(3h\int_Y d^2A + 3h^2\int_Y H_1 d^2A + h^3\int_Y H_2 d^2A\right)$$

$$= \frac{1}{3}\left(3h\int_Y d^2A + 3h^2\int_Y \frac{1}{2}(k_1+k_2)d^2A + h^3\int_Y k_1\cdot k_2 d^2A\right)$$

For $n = 4$:

$$V_4(Y_h) = \frac{1}{4}\left(4h\int_Y d^3A + 6h^2\int_Y H_1 d^3A + 4h^3\int_Y H_2 d^3A + h^4\int_Y H_3, d^3A\right)$$

$$= \frac{1}{4}\left(4h\int_Y d^3A + 6h^2\int_Y \frac{1}{3}(k_1+k_2+k_3)d^3A + 4h^3\int_Y \frac{1}{3}(k_1k_2+k_2k_3+k_3k_1)d^3A + h^4\int_Y k_1.k_2.k_3 d^3A\right)$$

For $n = 5$:

$$V_5(Y_h) = \frac{1}{5}\left(5h\int_Y d^4A + 10h^2\int_Y H_1 d^4A + 10h^3\int_Y H_2 d^4A + 5h^4\int_Y H_3 d^4A + h^5\int_Y H_4 d^4A\right)$$

$$\begin{aligned} V_5(Y_h) = \frac{1}{5}\Bigg(&5h\int_Y d^4A + 10h^2\int_Y \frac{1}{4}(k_1+k_2+k_3+k_4)d^4A \\ &+ 10h^3\int_Y \frac{1}{6}(k_1k_2+k_1k_3+k_1k_4+k_2k_3+k_2k_4+k_3k_4)d^4A \\ &+ 5h^4\int_Y \frac{1}{4}(k_1k_2k_3+k_1k_3k_4+k_1k_2k_4+k_2k_3k_4)d^4A \qquad + h^5\int_Y k_1\cdot k_2\cdot k_3\cdot k_4 d^4A\Bigg) \end{aligned}$$

For $n = 6$:

$$V_6(Y_h) = \frac{1}{6}\left(6h\int_Y d^5A + 15h^2\int_Y H_1 d^5A + 20h^3\int_Y H_2 d^5A + 15h^4\int_Y H_3 d^5A + 6h^5\int_Y H_4 d^5A + h^6\int_Y H_5 d^5A\right)$$

$$\begin{aligned} V_6(Y_h) = \frac{1}{6}\Bigg(&6h\int_Y d^5A + 15h^2\int_Y \frac{1}{5}(k_1+k_2+k_3+k_4+k_5)d^5A \\ &+ 20h^3\int_Y \frac{1}{10}(k_1k_2+k_1k_3+k_1k_4+k_1k_5+k_2k_3+k_2k_4+k_2k_5+k_3k_4+k_3k_5+k_4k_5)d^5A \\ &+ 15h^4\int_Y \frac{1}{10}(k_1k_2k_3+k_1k_2k_4+k_1k_2k_5k_1k_3k_4+k_1k_3k_5+k_1k_4k_5+\ldots)d^5A \\ &+ 6h^5\int_Y \frac{1}{5}(k_1k_2k_3k_4+k_1k_2k_3k_5+k_1k_2k_4k_5+k_1k_3k_4k_5+k_2k_3k_4k_5)d^5A \\ &+ h^6\int_Y k_1\cdot k_2\cdot k_3\cdot k_4\cdot k_5 d^5A\Bigg) \end{aligned}$$

Problem (1.15). (Explain in detail the last equation $W_x X_i = N_i$ from page 30. Why is it true? Basically, I'm asking you to unpack the notation, definitions, and the chain rule.)

Solution.

So we have a map defined $N := \nu \circ X$. We get the equation $W_x X_i = N_i$ from page 30 by taking derivatives using chain rule and introducing some notations.

$$dN(e_i) = N_i = (\nu \circ X(y))' = \nu'(X(y)) \cdot X'(y) = d\nu_{X(y)} \cdot dX_y = W_x X_i$$

By definition: The weingarten map, W_x, is the differential of the gauss map, $W_x = d\nu_x$.

The book then does some notation introduction: $dX_y(e_i) = X_i(y), dN(e_i) = N_i$ where the subscript i denotes the partial derivative with respect to the ith coordinate. This is because of the map : $dX_y : R^{n-1} \mapsto TY_x$ takes the standard basis vectors of R^{n-1} and spits out vectors in the tangent space TY_x which are the column vectors $X_i(y)$. Essentially, these column vectors then build up the matrix representations of the jacobian of X. Similar is the case for $dN(e_i) = N_i$ and the column vectors build up J_N.

Problem (1.9). (Compute the Weingarten map $d\nu_x$ for the three examples you used above (or at least three examples different than the book's examples). By "compute" I mean find its matrix in some chosen basis.)

Solution.

Let's compute the Weingarten map of a cylinder first.

Say there exists a parametrization: $X(u,v) = \begin{bmatrix} rcosu \\ rsinu \\ v \end{bmatrix}$, where X is a map from R^2 to R^3.

The jacobian of X looks like : $J_X = \begin{bmatrix} -rsinu & 0 \\ rcosu & 0 \\ 0 & 1 \end{bmatrix}$. Notice that the two column vectors in J_X's matrix representation lies in the tangent space of our cylinder in R^3. But we are interested in the vector perpendicular to the tangent space because we want to follow the recipe of the textbook's page 31 to solve for the weingarten map. So, how do we get the normal vector here to the tangent space?

Simple- cross product. We take the cross product of the column vectors(v_1, v_2- corresponding to the first and second columns of J_X). That gives us $v_3 = \begin{bmatrix} rcosu \\ rsinu \\ 0 \end{bmatrix}$

Now from text book we know that N is described by : $N : M \mapsto S^{n-1}$. The textbook also gives us $dX(e_i) = X_i$, $dN(e_i) = N_i$ and $W_x X_i = N_i$. The notation is quite weird here, so I want to make it more convenient and say that, all these mean $J_N = J_X W$. We already have J_X, but what's J_N? We know that N maps from the parameter space to the unit sphere. And if you have followed my argument, you should see that the image would be the normal vector we computed with a normalization factor: $N(u,v) = \frac{1}{r}\begin{bmatrix} rcosu \\ rsinu \\ 0 \end{bmatrix}$

Computing the jacobian of N gives us : $J_N = \frac{1}{r}\begin{bmatrix} -rcosu & 0 \\ rsinu & 0 \\ 0 & 0 \end{bmatrix}$

Now we have the matrix equation

$$\frac{1}{r}\begin{bmatrix} -rcosu & 0 \\ rsinu & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -rsinu & 0 \\ rcosu & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{1}$$

Solving for this system of equation gives us the weingarten map $W = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & 0 \end{bmatrix}$

Now we solve for the weingarten map of the $z = 0$ plane.

We follow the same recipe as for all the cases. For the plane, we have a parametrization : $X(x, y) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. It follows that $J_X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Now we know that : $N(x, y) = \frac{\vec{v_3}}{v_3} =$ where $\vec{v_3} = \vec{v_1} \times \vec{v_2}$ and $\vec{v_1}, \vec{v_2}$ are the corresponding column vectors of J_X

So, $N(x, y) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

It follows that $J_N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

So we have the system of equations:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

Solving for this, gives us the weingarten map $W_{plane} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Although it looks trivial, it shows that the plane $z = 0$ has no curvatures but in an elegant mathematical form.

Now we solve for the weingarten map of the sphere

Say there exists a parametrization for a sphere $X(x.y) = \begin{bmatrix} x \\ y \\ \sqrt{r^2 - x^2 - y^2} \end{bmatrix}$

So, $J_X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{r^2-x^2-y^2}} & \frac{-y}{\sqrt{r^2-x^2-y^2}} \end{bmatrix}$

Now we know that : $N(x, y) = \frac{\vec{v_3}}{v_3} =$ where $\vec{v_3} = \vec{v_1} \times \vec{v_2}$ and $\vec{v_1}, \vec{v_2}$ are the corresponding column vectors of J_X

Now the computation for the norm starts off a bit scary but simplifies later. I do not want to write that computation bits rather i ask you to trust me as I claim that $N(x, y) = \begin{bmatrix} \frac{x}{r} \\ \frac{y}{r} \\ 1 \end{bmatrix}$

So it follows that $J_N = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \\ 0 & 0 \end{bmatrix}$

Now our system of equations look like:

$$J_N = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{r^2-x^2-y^2}} & \frac{-y}{\sqrt{r^2-x^2-y^2}} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

Solving for this system of equations gives us the weingarten map $W_{Sphere} = \frac{1}{r}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

As for other three examples: For the plane $x + y + z = 1$: we also get the zero weingarten map as we got for $z = 0$ which makes sense too as it has no curvatures.

For the weingarten map of an ellipsoid: we can parametrize the shape by two parameters: $X(u, v) = \begin{pmatrix} asinucosv \\ bsinusinv \\ ccosu \end{pmatrix}$ and solving for the weingarten map we get:

$W = \frac{1}{\sqrt{(asinucosv)^2+(bsinusinv)^2+(ccosu)^2}} \begin{bmatrix} \frac{-bccosucosv}{a} & \frac{-absinv}{c} \\ \frac{-acsinv}{b} & \frac{abcosucosv}{c} \end{bmatrix}$

Note that the Weingarten map is not diagonal, reflecting the fact that the ellipsoid has varying Gaussian curvature along its length

For another example, we have a parametrized shape: $X(x, y) = \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix}$. Following the recipe written above we solve for the weingarten map to be $W = \frac{-2}{\sqrt{1+4(x^2+y^2)}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Problem (1.10). (For the three examples you used in the previous problem, compute the second fundamental form and give an explicit basis in which it diagonalizes. Therefore you should also find the principal curvatures, the mean curvature, and the Gaussian curvature.)

Solution.

Firstly, for our plane $x + y + z = 1$, we get a weingarten map of 4 zeros. So plugging that into $L = QW$, we can see that the second fundamental form is also all zeros. So obviously it diagonalizes. And therefore the principal curvatures, mean curvature and the gaussian curvature are all zeros.

y

We have another parametrization $X(x, y) = (x, y, e^x)$. The weingarten map computed for this parametrization $W_{exp} = \begin{bmatrix} \frac{-e^x}{(1+e^{2x})^{3/2}} & 0 \\ 0 & 0 \end{bmatrix}$. So the principal curvatures are the values on our diagonalized weingarten map in the parameter basis which are $\frac{-e^x}{(1+e^{2x})^{3/2}}, 0$. The mean curvature is $\frac{1}{2}\frac{-e^x}{(1+e^{2x})^{3/2}}$ and the Gaussian curvature is 0 following the outline set up in page 29.

We have another parametrization $X(x, y) = (x, y, x^2)$. The weingarten map computed for this surface is $W_{x^2} = \begin{bmatrix} \frac{-2}{(1+4x)^{3/2}} & 0 \\ 0 & 0 \end{bmatrix}$ and it's diagonal on the parameter basis. So the principal curvatures are $\frac{-2}{(1+4x)^{3/2}}, 0$. The mean curvature is $\frac{1}{2}\frac{-2}{(1+4x)^{3/2}}$ and the gaussian curvature is 0.

Problem (1.25). (Prove that $X_{uu} \cdot (X_u \times X_v) = \det(X_{uu}, X_u, X_v)$ from page 36.)

Solution.

We know that X_{uu}, X_u, X_v are all column vectors and the general triple product rule applies here. So basically for proving this case, we can just prove the case for general vectors living in R^3 as the column vectors mentioned in the problem are vectors in R^3 from the textbook examples.

We can prove this case for $a, b, c \in R^3$ and $a \cdot (b \times c) = \det(a, b, c)$

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\mathbf{i} + (b_3c_1 - b_1c_3)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}$ We then take the dot product of

a with this expression:

$$\begin{aligned}
a \cdot (b \times c) &= a \cdot ((b_2c_3 - b_3c_2)\mathbf{i} + (b_3c_1 - b_1c_3)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}) \\
&= (a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)) \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\
&= \det(a, b, c)
\end{aligned}$$

where the last step follows from the definition of the determinant of a matrix. Therefore, we have shown that $a \cdot (b \times c) = \det(a, b, c)$, which is the triple product rule.

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \det(a, b, c)$$

So it follows that $X_{uu} \cdot (X_u \times X_v) = \det(X_{uu}, X_u, X_v)$.

Problem (1.16). (Could the matrices L_{ij} and W_{ij} ever be the same? Describe exactly when this happens, or prove that it never does.)

Solution.

From the relation $L = QW$, we can see that L_{ij} and W_{ij} can ever be the same if Q is the identity matrix. This happens exactly when $X_u \cdot X_u = X_v \cdot X_v = 1$ and $X_u . X_v = 0$. This is the case when the basis vectors that span the tangent space of our surface each have unit norm and the basis is orthogonal.

Now we want to investigate the other direction. Assume $L = W$, does it imply that Q is the identity always? The answer is no and you will discover it through different examples.

If $L = W$, then $W = QW$. So Solve for the following system of equations: $\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$

This gives you, $\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} Ew_{11} + Fw_{21} & Ew_{12} + Fw_{22} \\ Fw_{11} + Gw_{21} & Fw_{12} + Gw_{22} \end{bmatrix}$

In the case where all the W_{ij} are nonzero, you get that Q is the identity matrix with $E = G = 1$ and $F = 0$. Notice that since L will always be a symmetric matrix, then also W must be symmetric. Therefore you may assume $W_{12} = W_{21}$ because we are assuming L=W.

For the example when $w_{22} = w_{12} = w_{21} = 0$ and a nonzero w_{11}, we get the solution of the system to be $E = 1, F = 0, G = x$ where x is a free variable as in it could be anything. In this case, Q is no longer an identity matrix and the basis vectors of tangent space are no longer orthonormal in all cases.

Another interesting case is the trivial weingarten map with all zeroes. Precisely, one of the plane. In this case, the solution to the system is all of E, F, G are free variables and so the basis vectors of our tangent space have no restrictions on them and thus the basis vectors of tangent space are no longer orthonormal in all cases.

Problem (7.19). (Let $e_1, \ldots, e_n$ and $\widetilde{e}_1, \ldots, \widetilde{e}_n$ be two bases for the vector space V. Let $\varepsilon_1, \ldots, \varepsilon_n$ and $\widetilde{\varepsilon}_1, \ldots, \widetilde{\varepsilon}_n$ be their respective dual bases. Say that $\widetilde{e}_j = \sum_i c_{ij} e_i$ and that $e_\ell = \sum_k d_{k\ell} \widetilde{e}_k$. (a) How are the $d_{k\ell}$ related to the c_{ij}? (a) The components of a vector change when you change basis. How is this related to the c_{ij}? (b) The dual space V^* is a vector space with the above two bases. How is changing from one dual basis to the other dual basis related to the c_{ij}? (c) How do the components of $\alpha \in V^*$ change between the two dual bases, in terms of the c_{ij}?)

Solution. a1

$\bar{e_j} = \Sigma_i c_{ij} e_i$, $e_l = \Sigma_k d_{kl} \bar{e_k}$
Now it follows that $e_i = \Sigma_k d_{ki} \bar{e_k}$.

$$\begin{aligned}\bar{e_j} &= \Sigma_i c_{ij} e_i \\ &= \Sigma_i c_{ij} \Sigma_k d_{ki} \bar{e_k} \\ &= \Sigma_k \Sigma_i d_{ki} c_{ij} \bar{e_k} \\ &= \Sigma_k (\Sigma_i d_{ki} c_{ij}) \bar{e_k} \\ &= \Sigma_k \delta_{kj} \bar{e_k}\end{aligned}$$

That means the sum over k reduces to a single nonzero term, namely $\bar{e_j}$. So, the relation therefore is : $(\Sigma_i d_{ki} c_{ij}) = \delta_{kj}$

a2

Take any $v \in V$. So, $v = \Sigma_k v_k e_k = \Sigma_j \bar{v_j} \bar{e_j}$

$$\begin{aligned}v &= \Sigma_j \bar{v_j} \bar{e_j} \\ &= \Sigma_j \bar{v_j} \Sigma_k c_{kj} e_k \\ &= \Sigma_k (\Sigma_j c_{kj} \bar{v_j}) e_k \\ \Sigma_k v_k e_k &= \Sigma_k (\Sigma_j c_{kj} \bar{v_j}) e_k\end{aligned}$$

So following from this equality, it has to follow that: $v_k = \Sigma_j c_{kj} \bar{v_j}$

b

We want to compute $\bar{\epsilon}(\bar{e_d})$

$$\begin{aligned}\bar{\epsilon}(\bar{e_d}) &= \Sigma_z g_{zc}\epsilon_z(\bar{e_d})\textit{[How I am choosing to define it]}\\ &= \Sigma_z g_{zc}\epsilon_z(\Sigma_x c_{xd}e_x)\\ &= \Sigma_z g_{zc}\Sigma_x c_{xd}\epsilon_z(e_x)\end{aligned}$$

Now the term $\epsilon_z(e_x) = 0 when x \neq z$. So for the sum we are considering, the non-zero items show up when we consider $z = x$ and we get:

$$\begin{aligned}\bar{\epsilon}(\bar{e_d}) &= g_{xc}c_{xd}\\ &= g_{cx}c_{xd}\end{aligned}$$

Now the left hand side equals 1 when $c = d$ and else it's 0. So , take $c = d$ and we get: $g_{cx}c_{xd} = 1$. So the transformation from one dual basis to another (g) is the inverse of the transform indexed wth c_{ij}

c

I am going to work with a instead of alpha because it's easier for me to work with and index.

Take any $a \in V^*$. By definition $a = \Sigma_z a_z\epsilon_z = \Sigma_j \bar{a_j}\bar{\epsilon}_j$

We know from our set-up $\bar{\epsilon}_j = \Sigma_z g_{zj}\epsilon_z$

$$\begin{aligned}\Sigma_z a_z\epsilon_z &= \Sigma_j \bar{a_j}\bar{\epsilon}_j\\ &= \Sigma_j \bar{a_j}\Sigma_z g_{zj}\epsilon_z\\ \Sigma_z a_z\epsilon_z &= \Sigma_z(\Sigma_j \bar{\epsilon}_j g_{zj})\epsilon_z\end{aligned}$$

By observation we get : $a_z = (\Sigma_j \bar{\epsilon}_j g_{zj})$

But from part (b) we know that $g_{zj}c_{zj} = 1$

so we get $a_z = (\Sigma_j \bar{\epsilon}_j {c_{zj}}^{-1})$

So we are done.

Problem (1.30). (Explain in words what is happening in each of the five equals signs below.

$$\begin{aligned}X_{uu}\cdot X_{vv} - X_{uv}\cdot X_{uv} &= (X_u\cdot X_{vv})_u - X_u\cdot X_{vvu} - (X_u\cdot X_{uv})_v + X_u\cdot X_{uvv}\\ &= (X_u\cdot X_{vv})_u - (X_u\cdot X_{uv})_v\\ &= ((X_u\cdot X_v)_v - X_{uv}\cdot X_v)_u - \frac{1}{2}(X_u\cdot X_u)_{vv}\\ &= (X_u\cdot X_v)_{vu} - \frac{1}{2}(X_v\cdot X_v)_{uu} - \frac{1}{2}(X_u\cdot X_u)_{vv}\\ &= -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu}.\end{aligned}$$

)

Solution.

So, from what it looks like and from my checks, the equal signs are just using chain and product rule really smartly. It requires a lot of thinking ahead - and what I did was go backwards to make those thinking ahead steps make sense.

$X_{uu}.X_{vv} - X_{uv}.X_{uv} = X_{uu}.X_{vv} + X_u.X_{uvv} - X_u.X_{uvv} - X_{uv}.X_{uv} - X_{uvv}.X_u + X_u.X_{uvv}$ [We just added and substracted some elements, both sides are equal]
$= (X_u.X_{vv})_u - X_u.X_{vvu} - (X_u.X_{uv})_v + X_u.X_{uvv}$ [Notice that the product rule expansion of this is equal to the previous line]
$= (X_u.X_{vv})_u - (X_u.X_{uv})_v$ [Again use the differentiation product rule and see that it is equal to the previous line]

We have to think backwards for the next equal sign- yet again. So we want to use differential chain rule on line 3 to find out that equals line 2. And it does:
$((X_u.X_v)_v - X_{uv}.X_v)_u - \frac{1}{2}(X_u.X_u)_{vv}$
$= (X_{uv}.X_v + X_u.X_{vv} - X_{uv}.X_v)_u - \frac{1}{2}[X_{uv}.X_u + X_u.X_{uv}]_v$

$= (X_u.X_{vv})_u - (X_u.X_{uv})_v$

For the fourth equal sign we use the chain rule again!:

$((X_u.X_v)_v - X_{uv}.X_v)_u - \frac{1}{2}(X_u.X_u)_{vv} = (X_u.X_v)_{uv} - (X_{uuv}.X_v + X_{uv}.X_{uv}) - \frac{1}{2}(X_u.X_u)_{vv}$
$= (X_u.X_v)_{uv} - \frac{1}{2}(X_v.X_v)_{uu} - \frac{1}{2}(X_u.X_u)_{vv}$
Here the essential trick was how $\frac{1}{2}(X_v.X_v)_{uu}$ equals $(X_{uuv}.X_v + X_{uv}.X_{uv})$- the trick is also product rule and it just takes 1 line of expansion to see that!

Then, we go back to page 35 and use the definitions of E, F, G to replace and get the last equality. We are done.

Problem (5.1). (Write out all $\binom{n}{k}$ dx^I with $I = (i_1, i_2, \dots)$ but $i_1 < i_2 < \cdots$ explicitly for $n = 1, 2, 3, 4$ and all possible $1 \le k \le n$.)

Solution.

For $n = 1$, we only get 1 form on a 1 space dx^1
For $n = 2$, we have:

we have dx^1, dx^2 for 1 forms on 2 space and $dx^1 \wedge dx^2$ are 2 forms on 2 space. For $n = 3$:

dx^1, dx^2, dx^3- 1 forms on 3space,
$dx^1 \wedge dx^2, dx^2 \wedge dx^3, dx^1 \wedge dx^3$ - 2 forms and
$dx^1 \wedge dx^2 \wedge dx^3$- 3 forms
For $n = 4$, we have :

So 1 form on 4 space: dx^1, dx^2, dx^3, dx^4
2 form on 4 space: $dx^1 \wedge dx^2, dx^2 \wedge dx^3, dx^3 \wedge dx^4, dx^1 \wedge dx^4, dx^2 \wedge dx^4, dx^1 \wedge dx^3$
3 form on 4 space: $dx^1 \wedge dx^2 \wedge dx^3, dx^2 \wedge dx^3 \wedge dx^4, dx^1 \wedge dx^3 \wedge dx^4, dx^1 \wedge dx^2 \wedge dx^4$
4-form on a 4 space: $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$

Problem (4.2). ((num. 2 page 125) With the above choice of spherical coordinates along the geodesic, show that the g_{00} and g_{33} equations become

$$h\frac{dt}{ds} = E \qquad\qquad r^2\frac{d\phi}{ds} = L$$

where E and L are constants called the "energy" and "angular momentum". Notice that for $L > 0$, as we shall assume, $d\phi/ds > 0$, so we can use ϕ as a parameter on the orbit, if we like. These functions, energy and angular momentum, are constant along solutions of the differential equations, and are sometimes called "first integrals" in classical literature. Knowing them can help enormously in solving the differential equations. Noether proved that for every symmetry of the odes there is some function which is constant along the solution curves.)

Solution. This should be a breeze. Just use 4.4 to get the results.

$$g_{00} = -h = -\left(1 - \frac{2GM}{r}\right), \; g_{11} \quad = h^{-1} = \left(1 - \frac{2GM}{r}\right)^{-1}, \; g_{22} = r^2, \; g_{33} \quad = r^2 \sin^2 \theta.$$

The equation 4.4 goes like this:

$$\frac{d}{ds}\left(g_{kk}\frac{dx^k}{ds}\right) = \frac{1}{2}\sum_j \frac{\partial g_{jj}}{\partial x^k}\left(\frac{dx^j}{ds}\right)^2$$

Let's solve 4.4 for g_{00}:

$$\begin{aligned} \frac{d}{ds}(-h\frac{dt}{ds}) &= \frac{1}{2}(0+0+0+0) \\ \frac{d}{ds}(h\frac{dt}{ds}) &= 0 \\ h\frac{dt}{ds} &= E \end{aligned}$$

where E is a constant.

Let's solve 4.4 for g_{33}:

$$\begin{aligned} \frac{d}{ds}(r^2 \sin^2 \theta \frac{d\phi}{ds}) &= \frac{1}{2}(0+0+0+0) \\ \frac{d}{ds}(r^2 \sin^2 \theta \frac{d\phi}{ds}) &= 0 \\ r^2 \sin^2 \theta \frac{d\phi}{ds} &= \text{Constant} \end{aligned}$$

Now we just saw from the last problem $\theta(s) = \frac{\pi}{2}$ along the whole geodesic as in here we can think of it to be s-invariant. So we can absorb it into another constant, namely- L:

$$r^2 \frac{d\phi}{ds} = L$$

Problem (4.1). ((num.1 page 125) The Schwarzschild metric is $ds^2 = -hdt^2 + h^{-1}dr^2 + r^2 d\sigma^2$ where $h(r) = 1 - \frac{2GM}{r}$ and $d\sigma^2$ is the invariant metric on the ordinary sphere. Show that the equation involving g_{22} on the left of

$$\frac{d}{ds}\left(g_{kk}\frac{dx^k}{ds}\right) = \frac{1}{2}\sum_j \frac{\partial g_{jj}}{\partial x^k}\left(\frac{dx^j}{ds}\right)^2$$

is equal to

$$\frac{d}{ds}\left(r^2 \frac{d\theta}{ds}\right) = r^2 \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2.$$

Conclude from the uniqueness theorem for differential equations that if $\theta(0) = \pi/2$ and $\dot\theta(0) = 0$ then $\theta(s) = \pi/2$ along the whole geodesic. Conclude from rotational invariance that all geodesics must lie in a plane, i.e. by suitable choice of poles of the sphere we can arrange that $\theta(s) = \pi/2$. Also, I'm adding one last part to this problem. Find a rotation matrix which sends the point $(1, 2, 3)/\sqrt{14}$ to the point $(1, 0, 0)$.)

Solution.

To show that the equation involving g_{22} is equal to $\frac{d}{ds}\left(r^2 \frac{d\theta}{ds}\right) = r^2 \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2$, let's calculate the relevant terms step by step.

First, let's compute the components of the metric tensor g_{jk}. For the Schwarzschild metric, we have:

$$g_{00} = -h = -\left(1 - \frac{2GM}{r}\right), \; g_{11} \qquad = h^{-1} = \left(1 - \frac{2GM}{r}\right)^{-1}, \; g_{22} = r^2, \; g_{33} \qquad = r^2 \sin^2\theta.$$

Now, let's calculate the left-hand side of the given equation:

$$\frac{d}{ds}\left(g_{22}\frac{dx^2}{ds}\right) = \frac{d}{ds}\left(r^2 \frac{d\theta}{ds}\right)$$

.

Next, we calculate the right-hand side of the equation:

$$\begin{aligned}\frac{1}{2}\sum_j \frac{\partial g_{jj}}{\partial x^2}\left(\frac{dx^j}{ds}\right)^2 &= \frac{1}{2}\left(\frac{\partial g_{22}}{\partial\theta}\left(\frac{d\theta}{ds}\right)^2 + \frac{\partial g_{22}}{\partial\phi}\left(\frac{d\phi}{ds}\right)^2\right) \\ &= \frac{1}{2}\left(0 + 2r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2\right) \qquad = r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2.\end{aligned}$$

Comparing the left-hand side and the right-hand side, we see that they are equal:

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) = r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2.$$

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) = 2r\frac{dr}{ds}\frac{d\theta}{ds} + r^2\frac{d^2\theta}{ds^2}$$

Using chain rule we get:

$$\begin{aligned}2r\frac{dr}{ds} + r^2\frac{d^2\theta}{ds^2} &= r^2\sin\theta\cos\theta(\frac{d\phi}{ds})^2 \\ \frac{d^2\theta}{ds^2} &= \sin\theta\cos\theta(\frac{d\phi}{ds})^2 - \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds}\end{aligned}$$

So now we have : $\frac{d^2\theta}{ds^2} = f(s, \theta, \dot{\theta})$

So we know by observation that f is continuous because ϕ, θ are all smooth parametrizations of s and sine and cosine are smooth functions. So, the existence and uniqueness theorem for differential equations guarantees a unique solution which is: if $\theta(0) = \pi/2$ and $\dot{\theta}(0) = 0$ then $\theta(s) = \pi/2$ along the whole geodesic.

Given that the geodesic equations are second-order differential equations, we need to specify both the initial position and the initial first derivatives / velocity to uniquely determine a particular geodesic. However, the rotational invariance of the metric implies that the choice of the initial direction, as characterized by $\theta(0)$ and $\dot{\theta}(0)$, can be altered through a rotation without changing the form of the metric or the geodesic equations. This means that because the rotation is θ-invariant, even if you have different initial conditions for the parameter θ, you can always rotate the sphere accordingly so that fixes $\theta(0)$. In the case where you have a different initial condition for $\theta'(0)$, you can intoduce a new coordinate system conveniently: For example, Take ϕ to be the rotation around a plane on a sphere and θ to be the rotation around the azimuthal axis so if you have a geodesic in the sphere, I claim that you can parametrize its velocity vector v using a size-2 tuple- $(\dot{\phi}(0), \dot{(\theta(0))})$ because, for rotations, time and radius parameters aren't relevant here. And because of rotational invariance, it's possible to introduce a new coordinate system where $v = (\dot{(\phi(0))}, \dot{(\theta(0))}) = (v', 0)$. The way you would do that is by aligning the ϕ' coordinate parallel to the velocity vector and so your initial $\theta'(0)$ condition would satisfy as it would go to zero. And as we have been able to find the sweet initial conditions after a particular coordinate changing process, using the power of uniqueness theorem, we have found the solution for all $\theta(s)$.

As for the rotation matrix, I just used mathematica to solve for the matrix. The way mathematica would be solving this is: For all the entries of the particular rotation matrix, mathematica takes a product of the rotation matrices (pictured below), call it Q. So, given the length of the vectors are equal, the initial vector you give me only needs to move along θ, ϕ . First, we calculate the rotation angle needed to align the z-axis of the coordinate system with the line connecting P and P'. The angle can be calculated as follows:

$$\cos\phi = \frac{P \cdot P'}{\|P\|\|P'\|} \quad = \frac{\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) \cdot (1,0,0)}{\left\|\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right\| \cdot \|1,0,0\|} = \frac{\frac{1}{\sqrt{14}}}{\sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2} \cdot 1} \quad = \frac{1}{\sqrt{14}}.$$

And similarly you will compute for θ and then plug them into the product of rotation matrices that we have.

1. The first rotation is counterclockwise through an angle ϕ about the x_3'-axis (Figure 11-9a) to transform the x_i' into the x_i''. Because the rotation takes place in the x_1'-x_2' plane, the transformation matrix is

$$\boldsymbol{\lambda}_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.91}$$

and

$$\mathbf{x}'' = \boldsymbol{\lambda}_\phi \mathbf{x}' \tag{11.92}$$

2. The second rotation is counterclockwise through an angle θ about the x_1''-axis (Figure 11-9b) to transform the x_i'' into the x_1'''. Because the rotation is now in the x_2''-x_3'' plane, the transformation matrix is

$$\boldsymbol{\lambda}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{11.93}$$

and

$$\mathbf{x}''' = \boldsymbol{\lambda}_\theta \mathbf{x}'' \tag{11.94}$$

```
In[1]:= point1 = {1 / Sqrt[14], 2 / Sqrt[14], 3 / Sqrt[14]};
        point2 = {1, 0, 0};

        rotationMatrix = RotationMatrix[{point1, point2}]
```

Out[3]= $\left\{\left\{\frac{1}{\sqrt{14}}, \sqrt{\frac{2}{7}}, \frac{3}{\sqrt{14}}\right\}, \left\{-\sqrt{\frac{2}{7}}, \frac{819+26\sqrt{14}}{1183}, \frac{3\left(-182+13\sqrt{14}\right)}{1183}\right\}, \left\{-\frac{3}{\sqrt{14}}, \frac{3\left(-182+13\sqrt{14}\right)}{1183}, \frac{728+117\sqrt{14}}{2366}\right\}\right\}$

```
In[5]:= MatrixForm[rotationMatrix]
```

Out[5]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{14}} & \sqrt{\frac{2}{7}} & \frac{3}{\sqrt{14}} \\ -\sqrt{\frac{2}{7}} & \frac{819+26\sqrt{14}}{1183} & \frac{3\left(-182+13\sqrt{14}\right)}{1183} \\ -\frac{3}{\sqrt{14}} & \frac{3\left(-182+13\sqrt{14}\right)}{1183} & \frac{728+117\sqrt{14}}{2366} \end{pmatrix}$$

Problem (4.4). ((num 4 page 129) It is now convenient to introduce the variable $u = 1/r$ instead of r. The book explains how the following equations arise:

$$\left(\frac{du}{d\phi}\right)^2 = 2GMQ, \qquad\qquad Q := u^3 - \frac{u^2}{2GM} + \beta_1 u + \beta_0$$

where β_0, β_1 are constants, which are combinations of the constants E, L, GM as $\beta_1 = 1/L^2$ and $\beta_0 = (E^2-1)/2GML^2$ and so we introduce the β_j to hide that complexity. The polynomial Q is cubic in u. Call the maximum value u_1 of u along the orbit, and the minimum u_2, which must be roots of Q since those are turning points where the left-side of (4.14) in the book vanishes. The third root is $1/2GM - u_1 - u_2$ since the roots add to $1/2GM$. The left equation above becomes

$$\left(\frac{du}{d\phi}\right)^2 = 2GM(u-u_1)(u-u_2)(u - \frac{1}{2GM} + u_1 + u_2)$$

By ignoring terms linear in $2GM(u + u_1 + u_2)$, since we are interested in the region where this is very small, we obtain the approximation:

$$\left|\frac{d\phi}{du}\right| = \frac{1}{\sqrt{(u_1 - u)(u - u_2)}}$$

where e and ℓ are determined from the equations $u_1 = \frac{1}{\ell}(1+e)$ and $u_2 = \frac{1}{\ell}(1-e)$ so that the mean distance $a = \frac{1}{2}\left(\frac{1}{u_1} + \frac{1}{u_2}\right) = \frac{\ell}{1-e^2}$. Show that the ellipse $u = \frac{1}{\ell}(1 + e\cos\phi)$ is a solution of the equation above. This is the approximating ellipse with the same maximum and minimum distance to the sun as the true orbit, if we choose our angular coordinate ϕ so that the x-axis is aligned with the axis of the ellipse.)

Solution.

We want to show that $u = \frac{1}{\ell}(1 + e\cos\phi)$ is a solution to the given differential equation.

$$u = \frac{1}{\ell}(1 + e\cos\phi)$$
$$\phi = \cos^{-1}(\frac{lu - 1}{e})$$

Now, take derivative w.r.t u:

$$\frac{d\phi}{du} = \frac{-l}{\sqrt{e^2 - (lu-1)^2}}$$
$$|\frac{d\phi}{du}| = \frac{l}{\sqrt{e^2 - (lu-1)^2}}$$

Now We wish to compute $\frac{1}{\sqrt{(u_1-u)(u-u_2)}}$. Substitute $u_1 = \frac{1}{\ell}(1+e)$ and $u_2 = \frac{1}{\ell}(1-e)$ and find that:

$$\frac{1}{\sqrt{(u_1 - u)(u - u_2)}} = \frac{l}{\sqrt{((e + lu - 1)(e - lu + 1))}}$$
$$= \frac{l}{\sqrt{(e^2 - (lu-1)^2)}} = |\frac{d\phi}{du}|$$

So We have showed that $u = \frac{1}{\ell}(1 + e\cos\phi)$ is a solution to the given differential equation.