M350: Ordinary Differential Equations

Lawrence University
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Alexander Micheal Heaton

Exploration 7.5 : Numerical Methods

Hasif Ahmed

Your goal in this exploration is to compare the effectiveness of these three methods by evaluating the errors made in carrying out this procedure in several examples. We suggest that you use a spreadsheet to make these lengthy calculations.

1. First, just to make sure you comprehend the rather terse descriptions of the preceding three methods, draw a picture in the tx-plane that illustrates the process of moving from (t_k, x_k) to (t_{k+1}, x_{k+1}) in each of the three cases.
2. Now let's investigate how the various methods work when applied to an especially simple differential equation, $x' = x$.

   (a) Find the explicit solution $x(t)$ of this equation satisfying the initial condition $x(0) = 1$ (now there's a free gift from the math department...).

   (b) Use Euler's method to approximate the value of $x(1) = e$ using the step size $\Delta t = 0.1$. That is, recursively determine t_k and x_k for $k = 1, \ldots, 10$ using $\Delta t = 0.1$ and starting with $t_0 = 0$ and $x_0 = 1$.

   (c) Repeat the previous step with Δt half the size, namely 0.05.

   (d) Again use Euler's method, this time reducing the step size by a factor of 5, so that $\Delta t = 0.01$ to approximate $x(1)$.

Figure 1: Exploration 7.5

(e) Repeat the previous three steps using the Improved Euler's method with the same step sizes.

(f) Repeat using Runge–Kutta.

(g) You now have nine different approximations for the value of $x(1) = e$, three for each method. Calculate the error in each case. For the record, use the value $e = 2.71828182845235360287\ldots$ in calculating the error.

(h) Calculate how the error changes as you change the step size from 0.1 to 0.05 and then from 0.05 to 0.01. That is, if ρ_Δ denotes the error made using step size Δ, compute both $\rho_{0.1}/\rho_{0.05}$ and $\rho_{0.05}/\rho_{0.01}$.

3. Repeat the previous exploration, this time for the nonautonomous equation $x' = 2t(1 + x^2)$. Use the value $\tan 1 = 1.557407724654\ldots$
4. Discuss how the errors change as you shorten the step size by a factor of two or a factor of five. Why, in particular, is the Runge–Kutta method called a "fourth-order" method?

Figure 2: Exploration 7.5

1

We draw pictures in the tx plane that illustrates the process of moving from (t_k, x_k) to (t_{k+1}, x_{k+1}) for the methods: Euler Method, Improved Euler Method and Runge-Kutta Method.

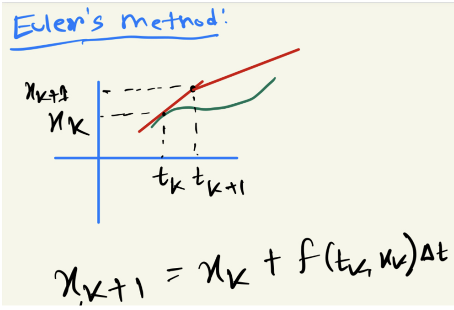

Figure 3: Euler's Method

Firs we demonstrate the picture of Euler's method:
Then, we demonstrate the Improved Euler Method:

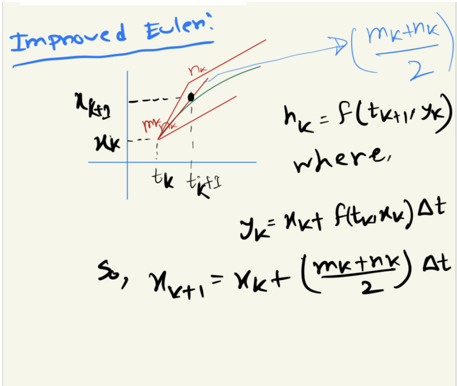

Figure 4: Improved Euler's Method

As for the picture of Runge-Kutta method- it's a little too tedious to draw by hand. So we use internet resources to portray it.

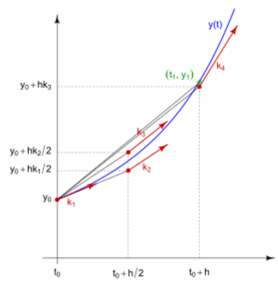

Figure 5: Runge-Kutta Method

Here assume the y axis in the photo represents our x axis and $k_1 = m_k, k_2 = n_k, k_3 = p_k, q_k = k_4$ following textbook notation.

2a

For the given differential equation $\frac{dx}{dt} = x(t)$ with initial condition x(0)=1 we have the analytical solution $x(t) = e^t$

We want to approximate the value at $t = 1$ as in the value of e.

For part 2b-2g, we demonstrate our results and errors in the notebook linked **here.**

For 2h, notice the ratios computed in Table 1.

3

Now we look at the second ODE $x' = 2t(1 + x^2)$. Taking the initial condition $x(0) = 0$, we get the analytical solution to be $x(t) = tan(t^2)$. The repitition of varying stepsizes and all are also included in the **notebook.**

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Table 1: Error Ratio Comparison for Different Numerical Methods for 1st ODE

Method	**Ratio 1 (0.1 to 0.05)**	**Ratio 2 (0.05 to 0.01)**
Euler	1/1.916	1/4.82
Improved Euler	1/3.85	1/24.25
Runge-Kutta	1/15.34	1/604.52

Table 2: Error Ratio Comparison for Different Numerical Methods for 2nd ODE

Method	**Ratio 1 (0.1 to 0.05)**	**Ratio 2 (0.05 to 0.01)**
Euler	1/1.6532	1/4.0701
Improved Euler	1/3.8284	1/25.4665
Runge-Kutta	1/4.1495	1/312.2825

The fourth-order Runge-Kutta (RK4) method is a "fourth-order" method because its global truncation error behaves roughly as $O(h^4)$, where h is the step size. This means that, if the method is consistent in ideal conditions, when you halve the step size (i.e., reduce the step size by a factor of two), the error should decrease by a factor of $2^4 = 16$. And we do notice that in our case for the first ODE (around 1/15.34 th and 1/604.52th which are around $\frac{1}{2^4}$ and $\frac{1}{5^4}$). In the case of second ODE that ratio goes down a bit to 1/4.15 and 1/312.28 but that is due to the nonlinear nature of the problem. It exhibits nonlinearity due to the y^2 term. Nonlinear problems can have more complex error behaviors when solved with numerical methods. Depending on the specific properties of the problem, the observed error reduction might not perfectly match the theoretical expectation.