Exploration 1.6: A Two-Parameter Family

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Consider the family of differential equations: $x' = f_{a,b} = ax - x^3 - b$ which depends on two parameters. The goal of this exploration is to combine all of the ideas in this chapter to put together a complete picture of the two-dimensional parameter for this differential equation.

1. First fix $a = 1$. Use the graph of $f_{1,b}$ to construct the bifurcation diagram for this family of differential equations depending on b.
2. Repeat the previous question for $a = 0$ and then for $a = -1$.
3. What does the bifurcation diagram look like for other values of a?
4. Now fix b and use the graph to construct the bifurcation diagram for this family, which this time depends on a.
5. In the ab–plane, sketch the regions where the corresponding differential equation has different numbers of equilibrium points, including a sketch of the boundary between these regions.
6. Describe, using phase lines and the graph of $f_{a,b}(x)$, the bifurcations that occur as the parameters pass through this boundary.
7. Describe in detail the bifurcations that occur at $a = b = 0$ as a and/or b vary.
8. Consider the differential equation $x' = x - x^3 - b\sin(2\pi t)$, where $|b|$ is small. What can you say about solutions of this equation? Are there any periodic solutions?
9. Experimentally, what happens as $|b|$ increases? Do you observe any bifurcations? Explain what you observe.

Figure 1: Exploration 1.6 questions

1. First fix $a = 1$. Use the graph of $f_{1,b}$ to construct the bifurcation diagram for this family of differential equations depending on b.
We fix a=1. Our differential equation takes the form $f_{1,b} = x - x^3 - b$. We construct the bifurcation diagram for the family of differential equations depending on b.
So, what we are seeing here through the arrows is essentially the phase lines with corresponding b values.

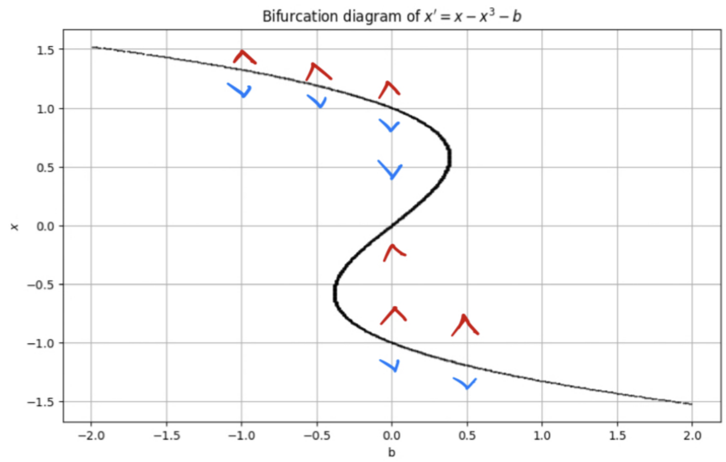

Figure 2: $a = 1$ bifurcation diagram

2. Repeat the previous question for $a = 0$ and then $a = -1$.
We fix $a = 0$ and draw the bifurcation diagram below.

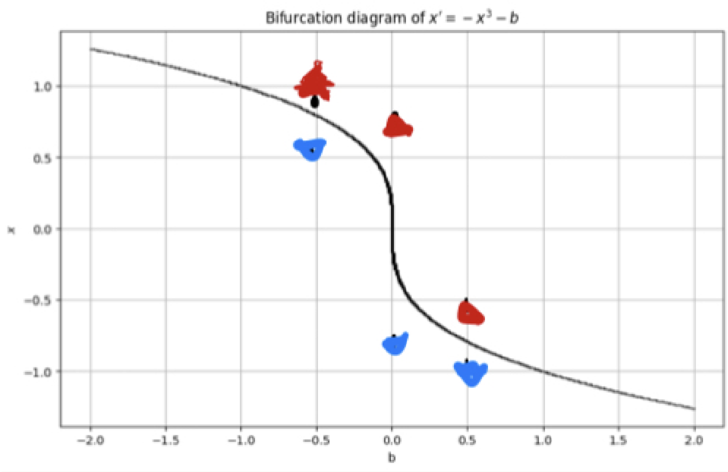

Figure 3: $a = 0$ bifurcation diagram

We then fix $a = -1$ and draw the bifurcation diagram below.

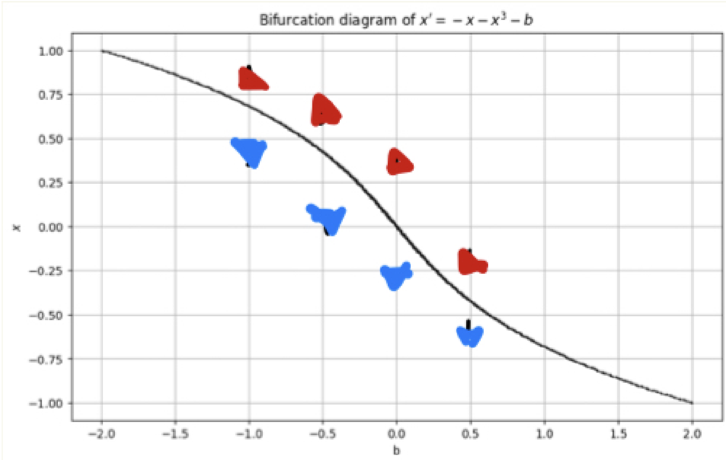

Figure 4: $a = -1$ bifurcation diagram

3. What does the bifurcation diagram look like for other values of a?
To create bifurcation diagrams for other values of a, I just write a for-loop that plots the solution to the differential equation for different values of a and plots them keeping b the x axis and x the y axis where $a \in [-2.0, 2.0]$ with 0.5 increments.

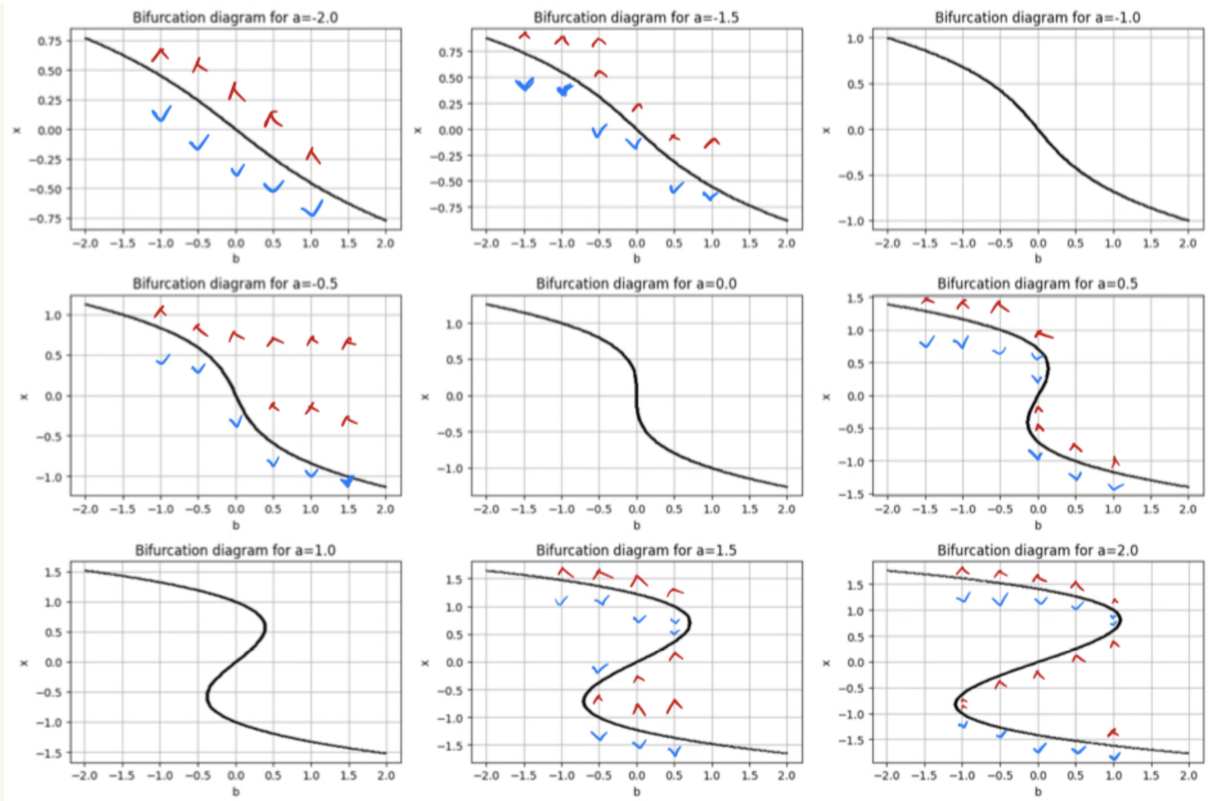

Figure 5: bifurcation diagram for $a \in [-2, 2]$

But this diagram only portrays the bifurcation diagrams for the abovementioned values of a. What above other cases?
So let's do observations. What is happening in the diagrams for cases when $a < 0$? For each b value there's only one solution - which are all **sources** and if you keep decreasing the values of a to $a < -2$ you see the same pattern. The curve keeps getting more

'straighter'- as in, eradicating any chance of multiple intersections as we draw the $b = m$ lines where m is any value, in this case, $m \in [-2, 2]$.
What about $a > 0$? Observe the diagrams for $a = 1.5$ and $a = 2$ closely- as a value increases our curve is more 'bumpy'- as in the 'wells' get deeper– meaning that the area in our curve where we will have 3 equilibriums spreads out. Such is the general case if you keep increasing $a > 2$, the 3 solution area for curve spreads out and the area where there's only 1 equilibriums shrinkens.

4. Now fix b and use the graph to construct the bifurcation diagram for this family, which this time depends on a.

So we fix b now and then create bifurcation diagrams which depend on a values.

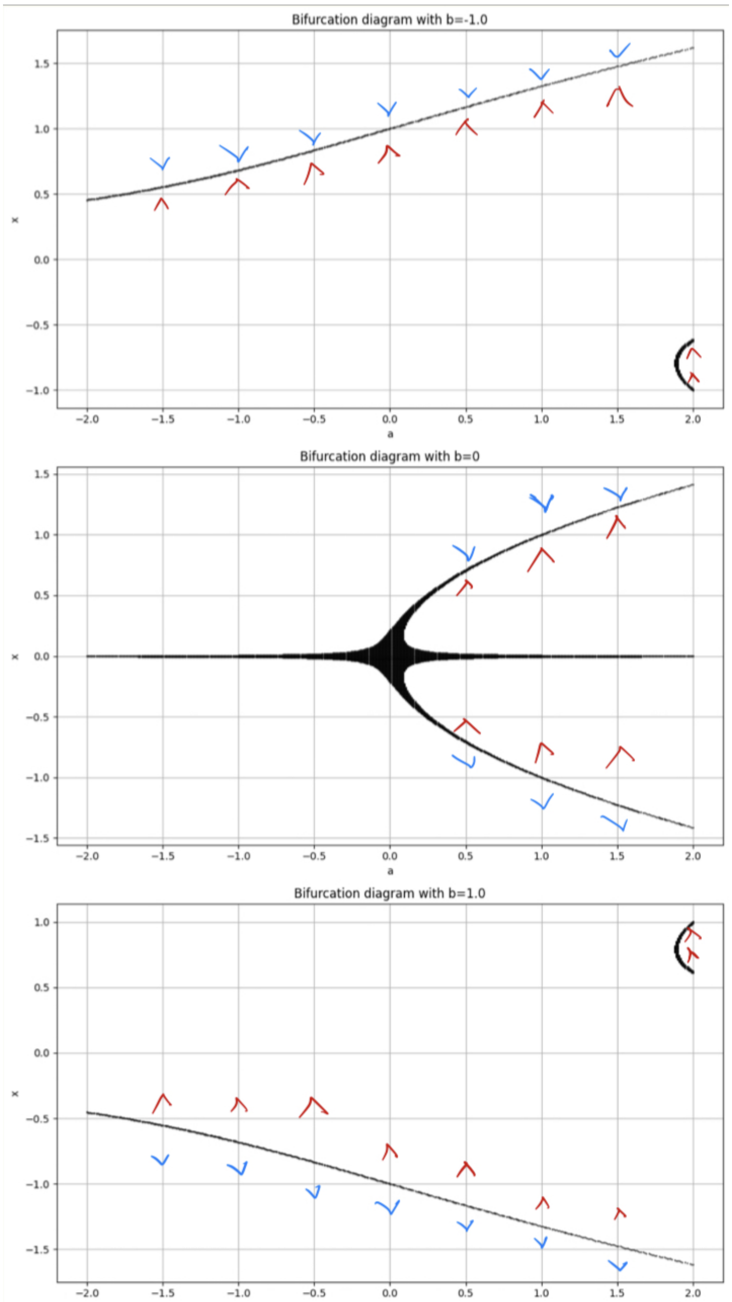

Figure 6: $a = -1$ bifurcation diagram

Disclaimer: More zoomed-in versions of these plots are available at step 6.

5. In the ab–plane, sketch the regions where the corresponding differential equation has different numbers of equilibrium points, including a sketch of the boundary between these regions.

We now construct an ab plane where we sketch the regions where the corresponding differential equation has different numbers of equilibrium points 3, 2,1. We denote the region of 1 equilibrium point with red lines, the boundary where there's 2 equilibrium points with a blue boundary and the region where there's 3 equilibrium points as black lines in the sketch below.

To explicitly find the boundary in the ab- plane, we need to find the discriminant of the equation $ax - x^3 - b = 0$. That's quite easy, because some smart people out there have already figured out the formula for discriminant for cubic equations. For a cubic equation $x^3 + ax^2 + bx + c = 0$, the determinant is $a^2b^2 + 18abc - 4b^3 - 4a^3c - 27c^2 = 0$. Using this formula, we find the determinant to be $4a^3 - 27b^2$. The boundary is the region where the discriminant is 0. So we plot the boundary by plotting $4a^3 - 27b^2 = 0$ below.

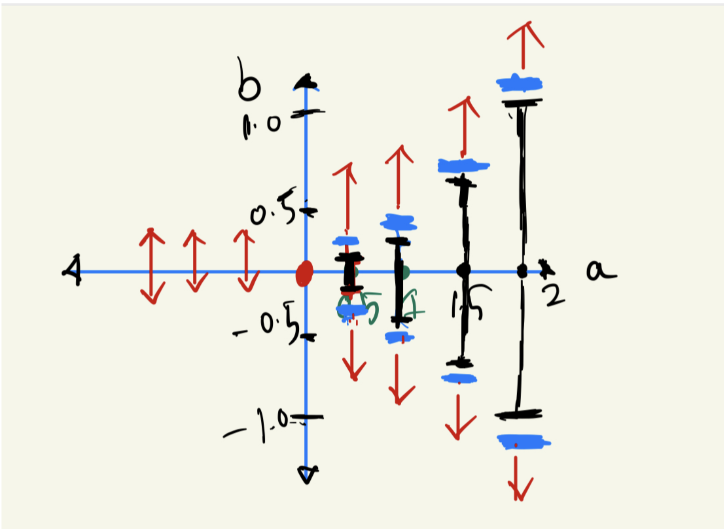

Figure 7: ab plane sketch

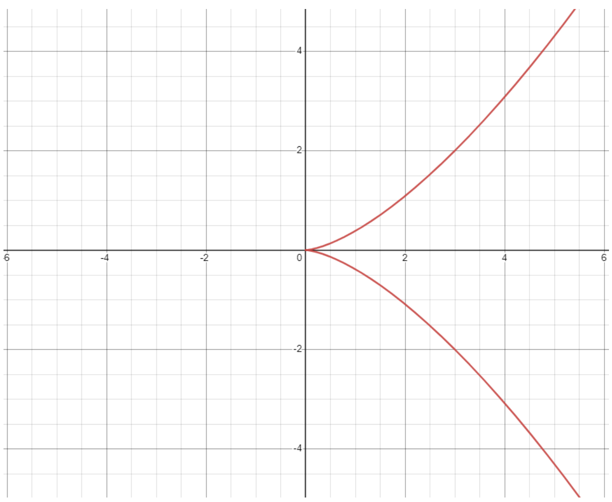

Figure 8: Explicit Plot of the boundary

6. Describe, using phase lines and the graph of $f_{a,b}(x)$, the bifurcations that occur as the parameters pass through this boundary.
So notice from the sketch at part 5 that for $a < 0$, we only have regions of 1 solutions- so there is no boundary of concern there where we see for different values of b- the number of solutions change. So from this sketch: let's take $a = 1$ for example. We have the differential equation $x' = x - x^3 - b$. The bifurcation diagram is symmetric with respect to $b = 0$ so doing bifurcation analysis on one particular (negative/positive) half is sufficient. We pick the positive half and vary our b values to notice how equilibrium points change behaviour.

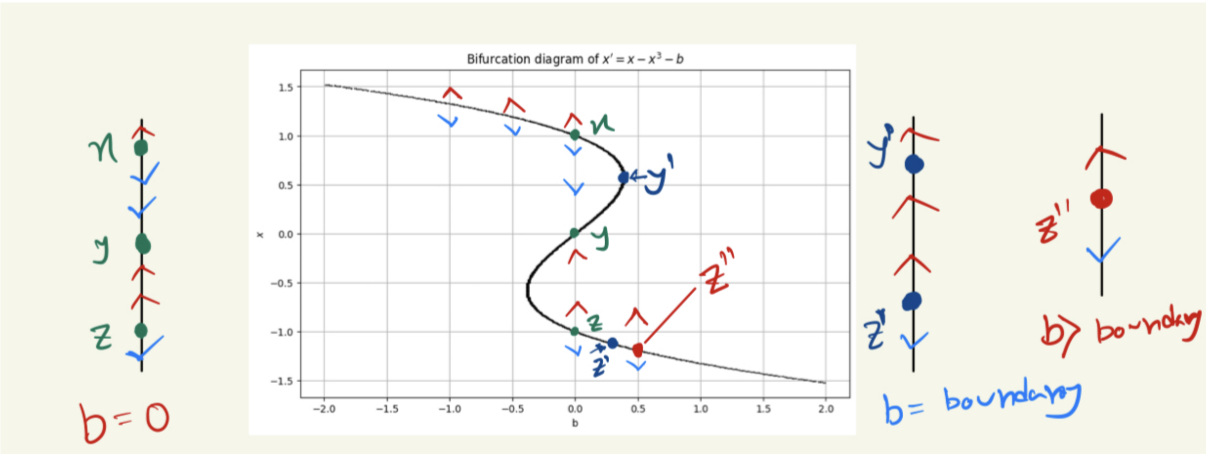

Figure 9: Bifurcation analysis for $f_{1,b}$

What I want you to take away from this figure: Notice at $b = 0$, we have 3 equilibrium points- x, y, z. As we keep increasing the value of b till what we call the boundary for b parameter- we see that x, y merge into a single equilibrium point y'. The reason turns out to be that x was an unstable equilibrium point and hence the merging happened as we perturbed b. Our equilibrium z shifts to z' and hence at boundary, we have 2 equilibrium points: y', z'. As we perturb b further beyond the boundary, we end up with a single equilibrium point-z''. The reason turns out to be that y' was a saddle equilibrium point so it vanished with further perturbation beyond the boundary.

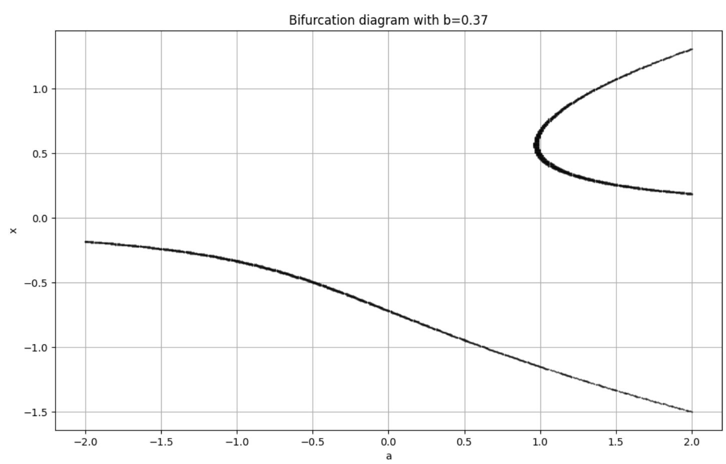

Figure 10: Bifurcation analysis for $f_{a,0.37}$

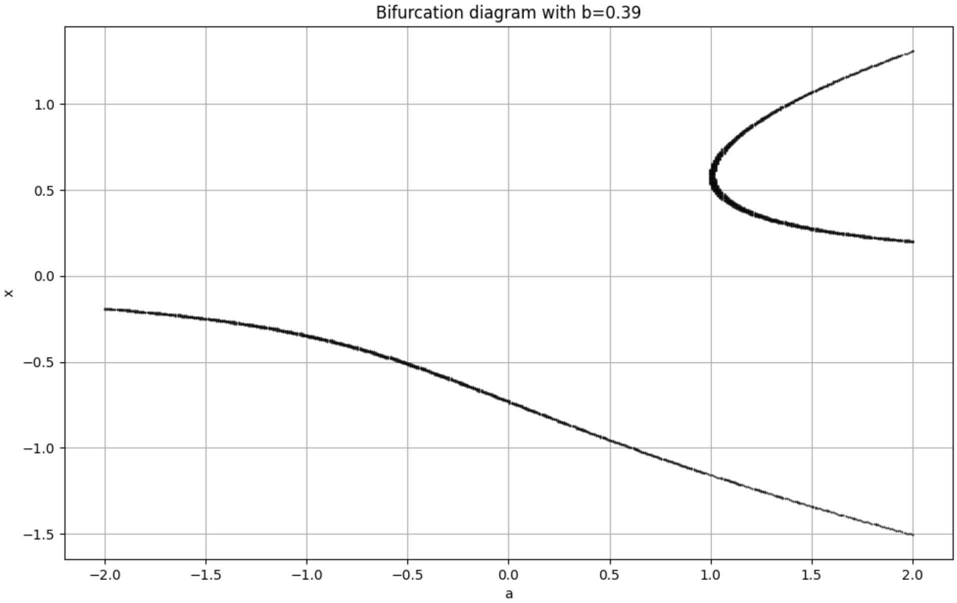

Figure 11: Bifurcation analysis for $f_{a,0.39}$

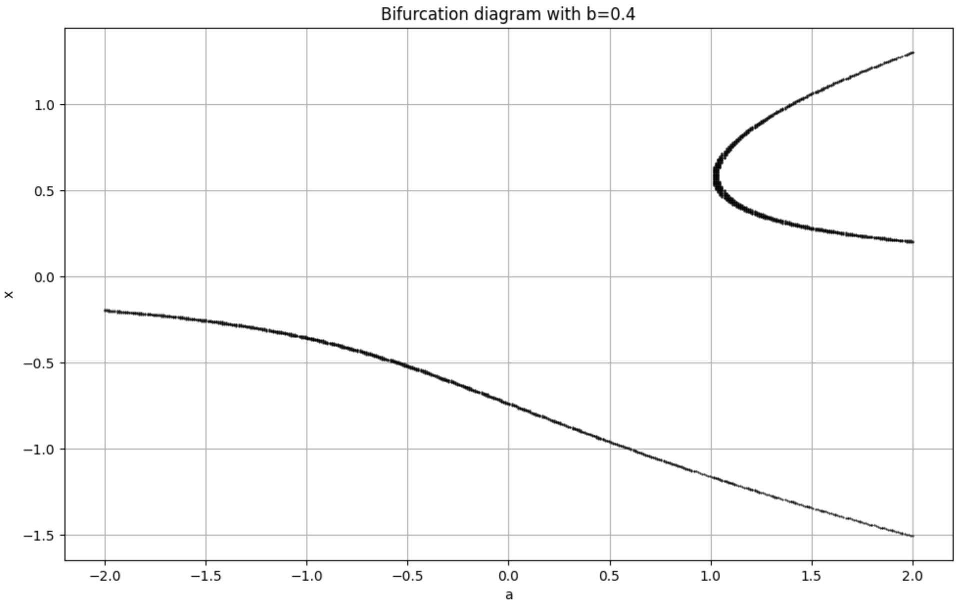

Figure 12: Bifurcation analysis for $f_{a,0.40}$

But we want to know what the boundary is, right? For this we plot bifurcation diagrams keeping b fixed and varying a- that way we will find the b value where the boundary behaviors happen.

If you notice that at $a = 1.0, b = 0.37$, we still see the existence of 3 equilibrium points at Fig. 10. What happens if we perturb b a little more? Find out in Fig. 11! Now we see that at $a = 1.0, b = 0.39$, we no longer find 3 equilibrium points– the merging to 2 equilibrium points claimed at Fig. 9 happens when $b \in (0.37, 0.39]$ [Fig. 11].

If we perturb b a little further- you guessed it- we only get one equilibrium around $a = 1.0, b = 0.4$ at Fig. 12.

7. Describe in detail the bifurcations that occur at a = b = 0 as a and/or b vary.

We want to check the cases of $a = b = 0$ and vary a/b.

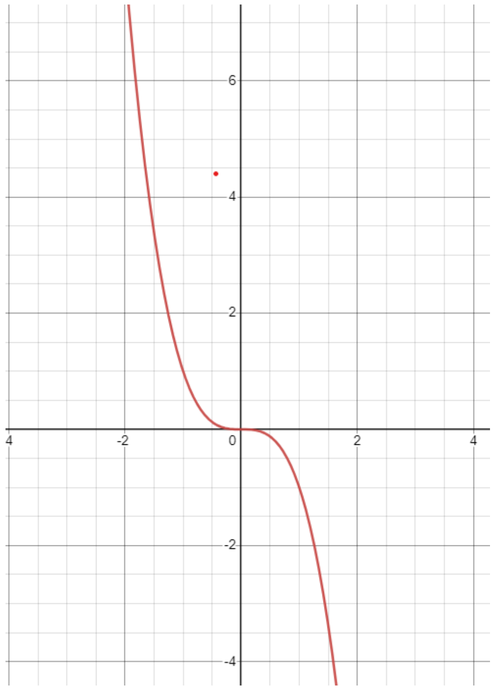

Figure 13: Plotting $ax - x^3 - b = 0; a = b = 0$

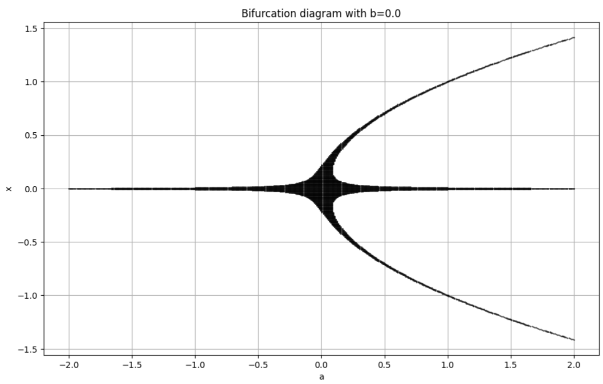

Figure 14: b=0 bifurcation

At Fig. 14, we encounter a significant phenomenon —a catastrophe. This catastrophe occurs at the point where equilibrium, represented by a cluster of points within a small range around $x = 0$ (denoted as $x \in [-\delta x, \delta x]$), undergoes a transformation as we perturb the parameter values a and b.

Initially, when a and b are both set to zero, this system exhibits a cluster of equilibrium points at or near the origin. These equilibrium points are within the range $x \in [-\delta x, \delta x]$. However, as we start to perturb the parameter a while keeping b constant, something remarkable happens. This cluster of equilibrium points begins to split into three distinct equilibrium points. Now we want to perturb b as well- but we notice that $b = 0.1, a = 0$ no longer gives us 3 equilibrium points [Fig. 15] . It's only when we get to $a \in [0.4, 0.5)$ we find the boundary where our equation has 2 equilibrium points. And then if we perturb

a just further beyond that boundary we find that the equation now has 3 equilibrium points.
If we now vary b to $b = 0.2$, we see the shift in boundary as well [Fig. 16]. As in the point (a, b) where we move from 1 solution to 2 solution and then later 3 solutions move to the right further.
Now what if we decrease from $a = b = 0$? Remember at part **5** when we sketched the $ab-$ plane? We found out that regardless of whatever values of b, $a < 0$ will always give only 1 equilibrium points. It's also reflected in Fig. 6.
Now what if we decrease from $b = 0$? Remember the symmetry we talked about before– that comes in to play. We see a mirror reflection of the bifurcation diagram along $x = 0$ line. See for yourself at Fig. 17 & 18. What this means is that our boundary still moves to the right as we decrease b.

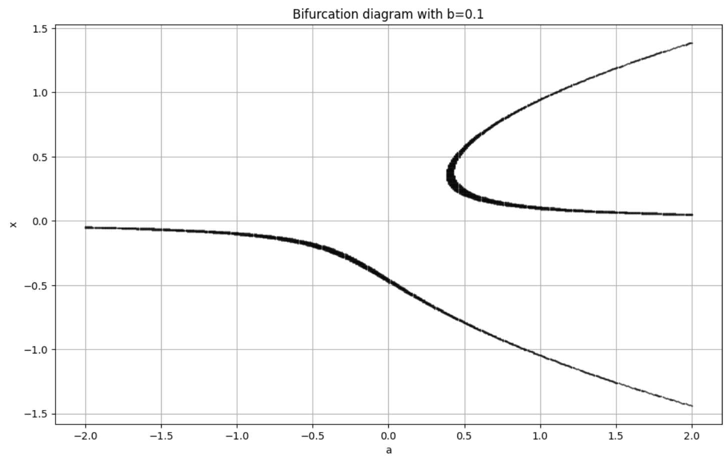

Figure 15: b=0.1 bifurcation

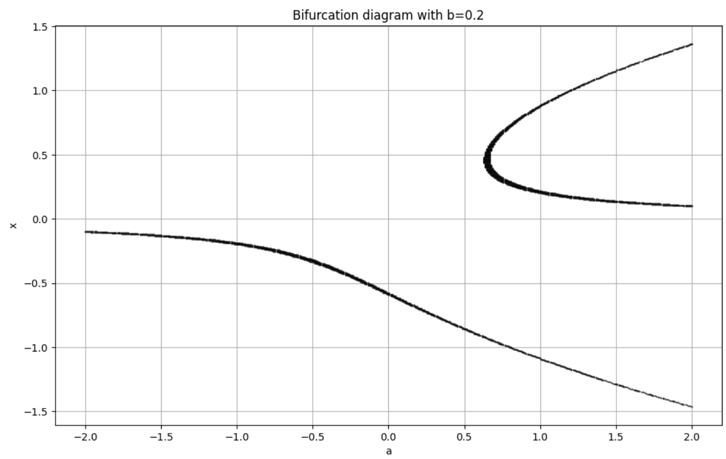

Figure 16: b=0.2 bifurcation

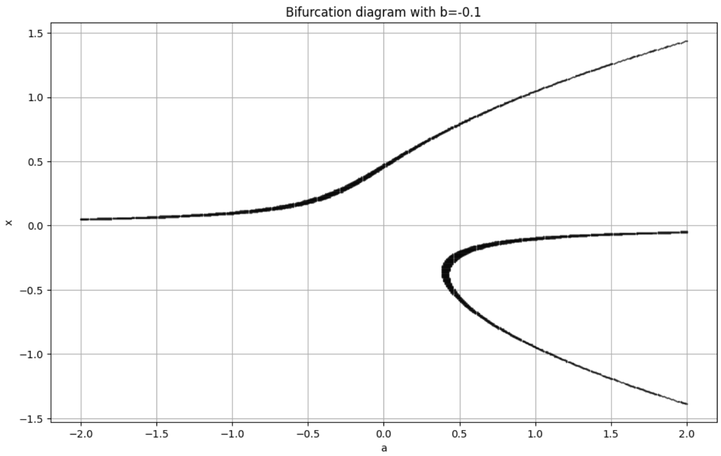

Figure 17: b=-0.1 bifurcation

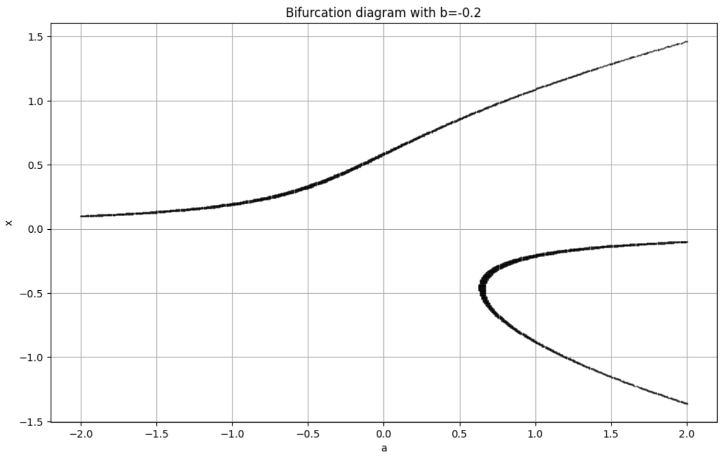

Figure 18: b=-0.2 bifurcation

8. Consider the differential equation $x' = x - x^3 - b\sin 2\pi t$, where $|b|$ is small. What can you say about the solutions of this equation? Are there any periodic solutions?

So if b is 0, we know the solutions already honestly- just by plotting the curve on desmos and seeing the solution points to be $-1, 0, 1$. But as soon you have nonzero b value- as in our differential equation becomes non-autonomous $f(x,t)$- we do not gain perspective from just plotting the equation rigorously, because now we have a temporal aspect that is changing the behavior of our differential equation. So what can we do?

This is where we invite our friend- The Poincare map. How do we make it? So we are given the non-autonomous differential equation $f(x,t) = x' = x - x^3 - b\sin 2\pi t$. The Poincaré map is generated by numerically solving the differential equation $\frac{dx}{dt} = x - x^3 - b\sin(2\pi t)$ using the 'solve-ivp' function from the SciPy library for different values of b. We solve the initial value problem at $t = 1$ with a range of initial values for $x(0) \in [-2, 2]$ and plot the solution points. Then we plot the $y = x$ line and we know from our text that the solution points which will intersect with this line- will give us the periodic solutions and then we generate a dynamic plot-gif which plots these solutions points to visualize the system's behavior as b varied.

So what can we say about the solutions for small b? More importantly, what is considered small in this case? - So if you notice the gif, you will see that up until around $b = 5$,

you will always see three solution points intersecting $y = x$ meaning that you will get 3 periodic solutions. We will define the range of b values from 0.1 to the neighborhood of $b = 5$ to be small values. This region will guarantee you 3 fixed points and therefore 3 periodic solutions.
Find the dynamic plot **here.** You will see in this plot how the fixed points sort of "drift" along the $y = x$ line.
What about solutions that aren't periodic?
In this case let's check our solutions for the autonomous case- $x(0) = -1, 0, 1$ where $b = 0.3$ and notice that how for initial value $x = 0$, the poincare map output is no longer periodic as after 5 seconds it doesn't come anywhere close to $p(x_0^0) = -0.078$, rather is attracted towards $p(x_0^{-1})$, where superscripts denote the corresponding initial values. This means the fixed point- $p(x_0^{-1}) = -0.961$ works as an attractor. The other non-periodic solutions in its neighborhood sort of drift towards it. Similarly, there are many other non-periodic solutions who get affected by nearby periodic solutions.

Initial Value of x	**Solution after 1 Second $p(x_0)$**	**Solution after 5 Seconds**
0	-0.078	-0.945
1	1.0369	1.042
-1	-0.961	-0.956

Table 1: Solutions for Different Initial Values in the case of $b = 0.3$

For a more visual experience, check out this plot below:

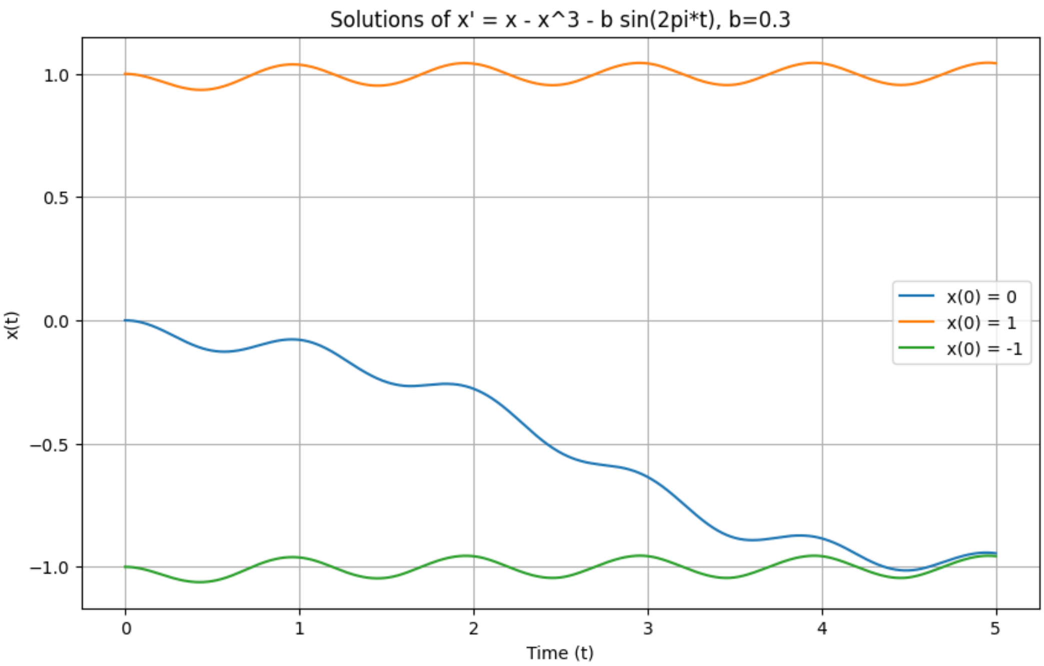

Figure 19: t= 5 seconds

9. Experimentally what happens as $|b|$ increases? Do you observe any bifurcations? Explain what you observe.
As we keep increasing the b value beyond the neighborhood of $b = 5$, we see how the solution points go through a weird trajectory. For $b \approx [4.55, 5.92]$, you no longer can find 3 fixed points rather a dense region of fixed points. That is sort of similar to the

catastrophe scene of bifurcation. As you perturb b a little further as in $b > 5.92$, you start seeing only one fixed point and this demonstrates the bifurcation we were looking for. Around $b = 10.31$ you no longer find any intersection with the $y = x$ line for the initial values of $x_0 \in [-1.5, 1.5]$. That means for any initial x in this region, your poincare map will no longer give you any periodic solutions for $b > 10.31$.