M350: Ordinary Differential Equations
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Exploration 4.3 : A 3D Parameter Space

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4.3 Exploration: A 3D Parameter Space

Consider the three-parameter family of linear systems given by

$$X' = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} X,$$

where a, b, and c are parameters.

1. First fix $a > 0$. Describe the analogue of the trace–determinant plane in the bc-plane. That is, identify the bc-values in this plane where the corresponding system has saddles, centers, spiral sinks, and so on. Sketch these regions in the bc-plane.
2. Repeat the previous task when $a < 0$ and when $a = 0$.
3. Describe the bifurcations that occur as a changes from positive to negative.
4. Now put all of the previous pieces of information together and give a description of the full three-dimensional parameter space for this system. You could build a 3D model of this space, create a flip-book animation of the changes as, say, a varies, or use a computer model to visualize this image. In any event, your model should accurately capture all of the distinct regions in this space.

Figure 1: Exploration 4.3

1. First, fix $a > 0$. Describe the analogue of the trace–determinant plane in the bc-plane. That is, identify the bc-values in this plane where the corresponding system has saddles, centers, spiral sinks, and so on. Sketch these regions in the bc-plane.

We fix $a > 0$. Compute Trace $(T) = a$ and Determinant $D = -bc$. So just like the trace-determinant plane, we compute $T^2 - 4d = a^2 + 4bc$. The curve $a^2 + 4bc = 0$ is portrayed in the figure below.

The red marked region in Fig 29 is the place where $a^2 + 4bc > 0$ and this is the region of real and distinct eigenvalues. If the determinant $D = -bc < 0$, then we will have saddles. It follows directly that regions where $b, c > 0$ or $b, c < 0$, we will have saddles. This is because our determinant is negative in that region- meaning one of eigenvalues being positive and another positive- therefore, saddles.

What about the region where $b > 0, c < 0$ and $b < 0, c > 0$? This is the region where our determinant $-bc > 0$ and as our trace $a > 0$ and determinant $-bc > 0$, we have real sources in this region. Notice it's real because it's part of the red marked $a^2 + 4bc > 0$ region. What about sinks? The existence of sinks don't apply here as our $T = a > 0$.

The green marked region in Fig 29 is the place where $a^2 + 4bc < 0$ and this is the region of complex eigenvalues with nonzero imaginary part. Given $T = a > 0$, we find spiral sources in this region.

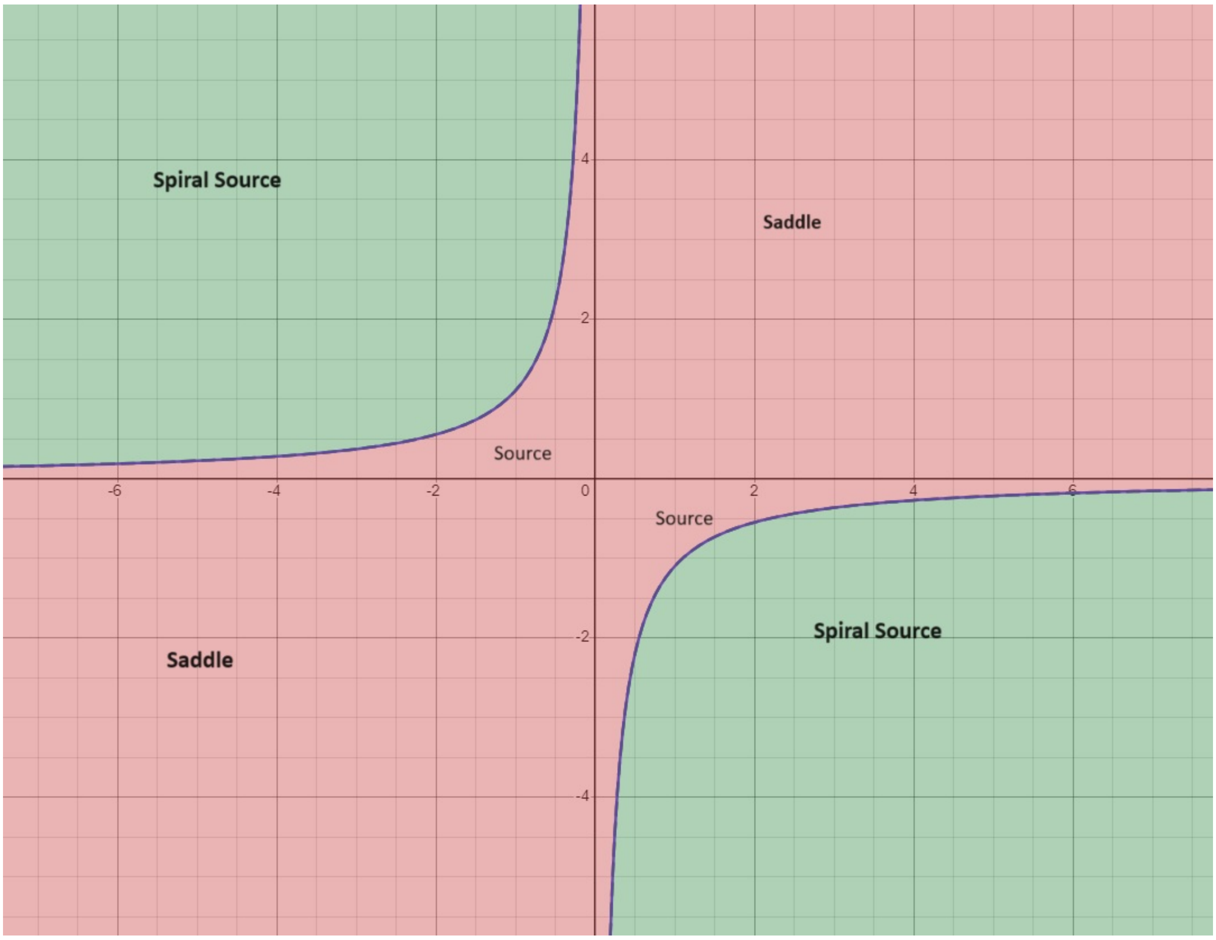

Figure 2: a>0 case, b lies in x axis and c in y axis

2. Repeat the previous task when $a < 0$ and when $a = 0$
For $a < 0$:

Similar to part **1**, The red marked region in Fig 30 is the place where $a^2 + 4bc > 0$ and this is the region of real and distinct eigenvalues. If the determinant $D = -bc < 0$, then we will have saddles. It follows directly that regions where $b, c > 0$ or $b, c < 0$, we will have saddles. This is because our determinant is negative in that region- meaning one of eigenvalues being positive and another positive- therefore, saddles.
What about the region where $b > 0, c < 0$ and $b < 0, c > 0$? This is the region where our determinant $-bc > 0$. As our trace $a < 0$ and determinant $-bc > 0$ as well, we have real sinks in this region. Notice it's real because it's part of the red marked $a^2 + 4bc > 0$ region.
The green marked region in Fig 30 is the place where $a^2 + 4bc < 0$ and this is the region of complex eigenvalues with nonzero imaginary part. Given $T = a < 0$, we find spiral sinks in this region.

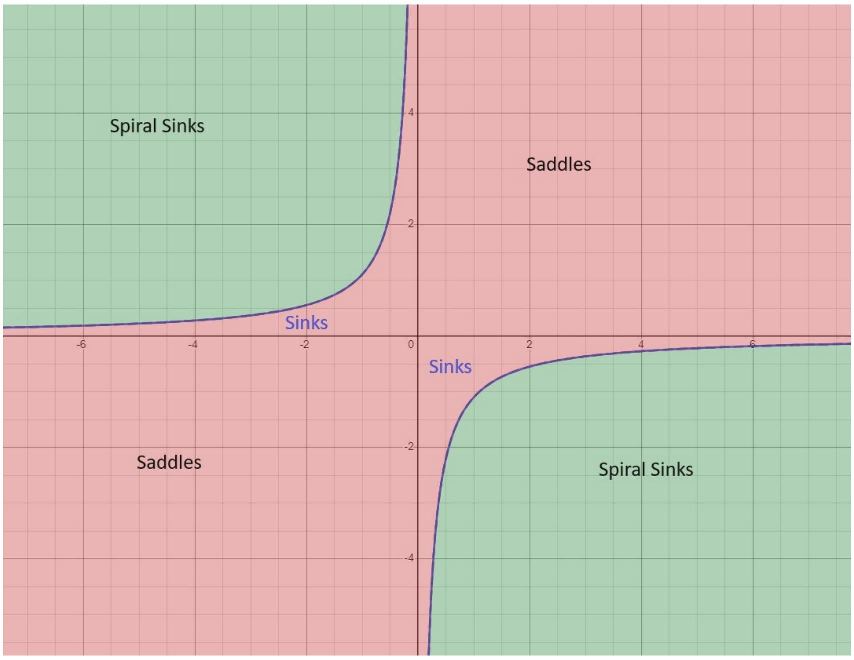

Figure 3: a<0 case

For a=0:
Notice Figure below.
Similar to part **1**, we have saddles in the $b, c > 0$ and $b, c < 0$ region.
The dark region in Fig 31 is the place where $4bc < 0$ and this is the region where we find centers as $T = a = 0$.

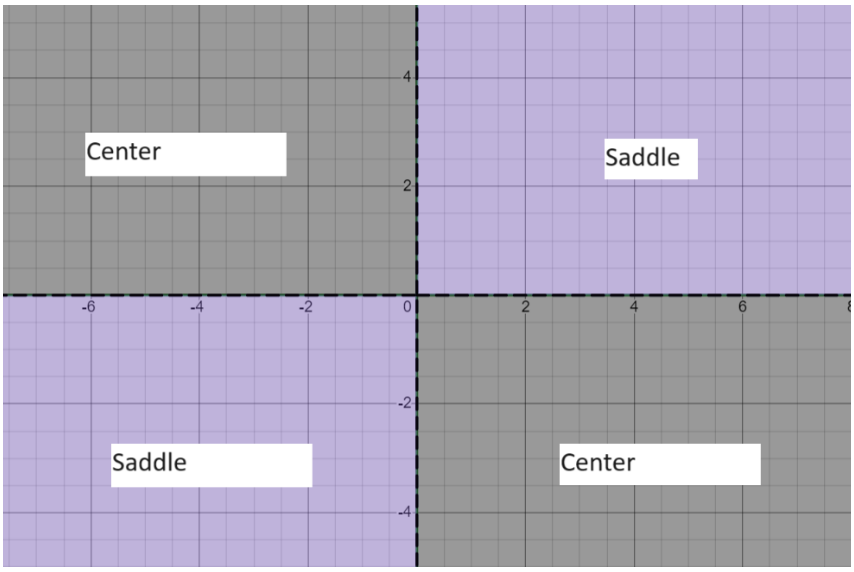

Figure 4: a=0 case

3. Describe the bifurcations that occur as a changes from positive to negative
From the description above in 1 and 2, it's clear that if you start in the 1st and 3rd quadrants, regardless of a values, you stay in saddles region.
So Fix a point B(1,-1) in the 4th quadrant. When a is near positive infinity, you get real sources at B. As a decreases to a=2, you have repeated eigenvalues as B is on the

$a^2 + 4bc = 0$ curve. As a falls below 2, then at B you get spiral sources. As a gets to a=0, you get centers at B. Then a goes negative, you start seeing spiral sinks at B. As a goes to a=-2, you get repeated eigenvalues at B. As a falls lower towards negative infinity, you start getting real sinks.
What's the picture when you start varying from negative infinity for a value? The 1st and 3rd quadrant still gave you saddles.
Now fix a point in the second quadrant C(-1,1). Now we start varying from negative infinity. For these large negative a values, you get real sinks at C. As you come to a=-2, you get repeated eigenvalues due to being on the boundary curve. Then as you increase a to a=0, you get centers at C. Then as you increase a from a=0 till a=2, you have spiral sources at C. As you go past a=2, you start getting spiral sources at C.
4. Now put all of the previous pieces of information together and give a description of the full three-dimensional parameter space for this system. You could build a 3D model of this space, create a flip-book animation of the changes as, say, a varies, or use a computer model to visualize this image. In any event, your model should accurately capture all of the distinct regions in this space.
So if you want to have a 3d model to view this parameter space (b,c,a). We need to make sure we explain what happens in all the 8 'octants' now that we are in 3d space.
Let's first take a look at the boundary region in our 3D space. The cone represents the $a^2 + 4bc = 0$ boundary region and the blue line represents $4bc = 0$ boundary region when a=0. These are the places where you have real and repeated eigenvalues.

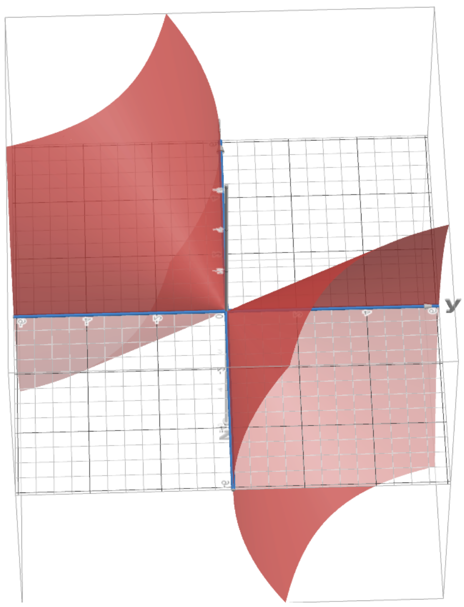

Figure 5: The Boundary regions

1. (b,c,a) all positive: If the determinant $D = -bc < 0$, then we will have saddles. It follows directly that regions where $b, c > 0$ or $b, c < 0$, we will have saddles. This is because our determinant is negative in that region- meaning one of eigenvalues being positive and another positive- therefore, saddles.

2. (b<0, c<0, a>0): This is also saddles region– see no.1.

3. (b>0, c<0, a>0): This is the region where our determinant $-bc > 0$ and as our trace $a > 0$ and determinant $-bc > 0$, we have real sources in this region.

   Now, there's a caveat here- as you cross the $a^2+4bc = 0$ curve to $a^2+4bc < 0$ region- this is where you get spiral sources because you now have complex eigenvalues with non-zero imaginary part.

4. (b<0, c>0,a>0):This is the region where our determinant $-bc > 0$ and as our trace $a > 0$ and determinant $-bc > 0$, we have real sources in this region. Same caveat as no.3 applies here- as you cross the $a^2 + 4bc = 0$ curve to $a^2 + 4bc < 0$ region- this is where you get spiral sources because you now have complex eigenvalues with non-zero imaginary part.

   Now what about $a = 0$ case? This will be a 2-dimensional subspace in our 3D picture. Instead of the $a^2+4bc = 0$ as a boundary cone, you have $4bc = 0$ boundary region- what you might call a degenerate cone if you want. The boundary regions are where you have real and repeated eigenvalues.

   we have saddles in the $b, c > 0$ and $b, c < 0$ region.

   The dark region in Fig 31 is the place where $4bc < 0$ and this is the region where we find centers as $T = a = 0$.

5. (b>0, c>0): Saddles

6. (b<0,c<0) : Saddles

7. (b>0, c<0): Centers

8. (b<0, c>0): centers

   Now we can take a bottom view of our 3 dimensional space as in $a < 0$

9. (b>0,c>0, a<0):If the determinant $D = -bc < 0$, then we will have saddles. It follows directly that regions where $b, c > 0$ or $b, c < 0$, we will have saddles. This is because our determinant is negative in that region- meaning one of eigenvalues being positive and another positive- therefore, saddles.

10. (b<0, c<0, a<0): This is also saddles region

11. (b>0, c<0, a>0): This is the region where our determinant $-bc > 0$ and as our trace $a < 0$ and determinant $-bc > 0$, we have real sinks in this region.

    Now, there's a caveat here- as you cross the $a^2 + 4bc = 0$ curve to $a^2 + 4bc < 0$ region- this is where you get spiral sinks because you now have complex eigenvalues with non-zero imaginary part.

12. (b<0, c>0,a>0): This is also the region where our determinant $-bc > 0$ and as our trace $a < 0$ and determinant $-bc > 0$, we have real sources in this region. Same caveat as no.7 applies here- as you cross the $a^2 + 4bc = 0$ curve to $a^2 + 4bc < 0$ region- this is where you get spiral sinks because you now have complex eigenvalues with non-zero imaginary part.

For a 3d live-time experience, view this desmos live action **plot.**