# M350: Ordinary Differential Equations

Lawrence University
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Alexander Micheal Heaton

# Exploration 7.6: Numerical Methods and Chaos

**Hasif Ahmed**

**7.6 Exploration: Numerical Methods and Chaos**

In this exploration we will see how numerical methods can sometimes fail dramatically. This is also our first encounter with chaos, a topic that will reappear numerous times later in the book.

1. Consider the "simple" nonautonomous differential equation

$$\frac{dy}{dt} = e^t \sin y.$$

Sketch the graph of the solution to this equation that satisfies the initial condition $y(0) = 0.3$.

2. Use Euler's method with a step size $\Delta t = 0.3$ to approximate the value of the previous solution at $y(10)$. It is probably easiest to use a spreadsheet to carry out this method. How does your numerical solution compare to the actual solution?
3. Repeat the previous calculation with step sizes $\Delta t = 0.001, 0.002$, and $0.003$. What happens now? This behavior is called *sensitive dependence on initial conditions*, the hallmark of the phenomenon known as chaos.
4. Repeat step 2 but now for initial conditions $y(0) = 0.301$ and $y(0) = 0.302$. Why is this behavior called senstitive dependence on initial conditions?

Figure 1: Exploration 7.6

5. What causes Euler's method to behave in this manner?
6. Repeat steps 2 and 3, now using the Runge–Kutta method. Is there any change?
7. Can you come up with other differential equations for which these numerical methods break down?

Figure 2: Exploration 7.6

Consider the differential equation: $\frac{dy}{dt} = e^t \sin(y)$.

**1. Consider the "simple" nonautonomous differential equation $\frac{dy}{dt} = e^t \sin y$. Sketch the graph of the solution to this equation that satisfies the initial condition $y(0) = 0.3$.**

We present the sketch of the solution that satisfies the initial condition $y(0) = 0.3$

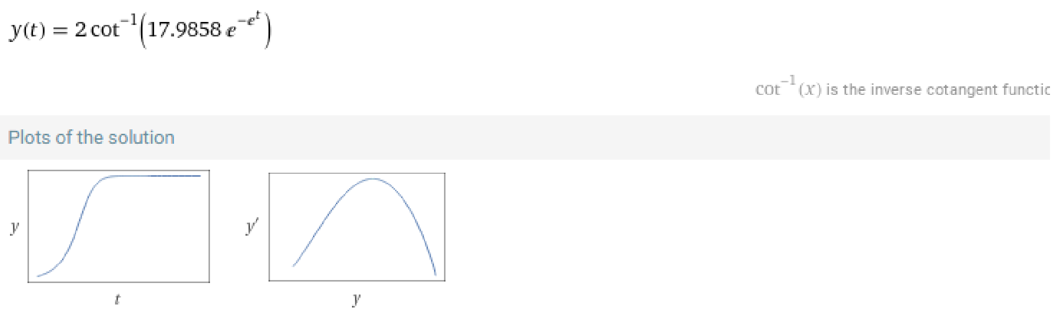


Figure 3: Sketch of the solution computed by Wolfram Alpha

For steps 2 to 4, please refer to the computations made in this **Jupyter notebook.**
**Step 2: Use Euler's method with a step size $\Delta t = 0.3$ to approximate the value of the previous solution at $y(10)$. It is probably easiest to use a spreadsheet to carry out this method. How does your numerical solution compare to the actual solution?**
So from the computation we made in the notebook, Approximate value of y(10) using Euler's method: 10909.871541508808. Analytical prediction of y(10) is 3.141592653589793. Ratio of numerical with respect to analytical prediction is 3472.719968657446. This shows how the numerical solution brutally fails in approximating the solution.
**Repeat the previous calculation with step sizes $\Delta t = 0.001, 0.002, 0.003$. What happens now? This behavior is called sensitive dependence on initial conditions, the hallmark of the phenomenon known as chaos.**
Referring to the notebook computations, We see that for little perturbations to initial conditions the y(10) varies drastically– For 0.001, the prediction is -1179.6728318223827 and For stepsize 0.002, the prediction is 522.1741061875068 and for stepsize 0.003 the prediction is -462.84192139051936. This shows the sensitivity of initial condition variance.

**4. Repeat step 2 but now for initial conditions $y(0) = 0.301$ and $y(0) = 0.302$. Why is this behavior called senstitive dependence on initial conditions?** We see that, Approximate value of $y(10)$ using Euler's method: -683.260730706581 for $y_0 = 0.301$. Furthermore, Approximate value of $y(10)$ using Euler's method: 12467.542923116842. Notice how varying initial conditions from $y(0) = 0.301$ to $y(0) = 0.302$ changes our numerical approximation drastically. Given subtle perturbations to initial conditions bring these drastic changes we can claim that it exhibits sensitive dependence on initial conditions.

**5. What causes Euler's method to behave in this manner?** We notice the drastic changes of numerical approximations due to perturbations in the initial conditions. When you perturb the initial conditions slightly and observe a drastic change in the numerical solutions, a couple of things might be happening:
The differential equation itself here is sensitive to initial conditions, suggesting chaotic behavior. The method (Euler's in this case) is compounding or amplifying errors due to its inherent limitations, leading to a divergence in trajectories that should have been closer.
**6. Repeat steps 2 and 3, now using the Runge–Kutta method. Is there any change?** As we use Runge-Kutta method to numerically approximate our solution, we still fail as the results are drastically far from analytical predictions and similar nature

of initial condition sensitivity is shown. For reference, Approximate value of y(10) using 4th order Runge-Kutta method: 13386.789636711563 with step size 0.3, which is really far from the analytical solution of $\pi$.
Similarly, you also notice sensitive dependence on initial conditions and all chaotic behaviors. Please refer to the jupyter notebook for rigorous computations.
**7.Can you come up with other differential equations for which these numerical methods break down?** In terms of other similar natured differential equation, we check that $y'(t) = e^t cos(y)$ with initial condition $y(0) = 1$ also exhibit chaotic properties as displayed in **our Jupyter notebook.**
For reference, with stepsize 0.300, the approximation value of $y(10)$ is -1348.0925793566998 and for stepsize 0.301 approximate value of y(10) using 4th order Runge-Kutta method: -5256.425904070151– exhibiting the chaotic behavior as we noticed before throughout the exploration.