Solving Partial Differential Equations Using Machine Learning

Hasina Younas
Roll Number: 2793
Women University of Azad Jammu and Kashmir

Advisor: Dr. Sharafat Hussain

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Abstract

Partial Differential Equations (PDEs) are widely used in physics, engineering, and other scientific fields. Traditional numerical methods such as Finite Difference Methods (FDM) and Finite Element Methods (FEM) are computationally expensive. This paper explores the use of Machine Learning (ML), particularly Physics-Informed Neural Networks (PINNs), to approximate solutions for PDEs. We present a simple mathematical framework, discuss the implementation using Python, and compare the efficiency of ML-based methods with traditional numerical solvers.

Keywords: Partial Differential Equations, Machine Learning, Neural Networks, PINNs

1 Introduction

Partial Differential Equations describe various physical and engineering systems, including heat transfer, wave propagation, and fluid dynamics. Classical numerical solvers require heavy computation, making them unsuitable for real-time applications. Machine Learning offers an alternative approach, leveraging neural networks to approximate PDE solutions efficiently.

This study focuses on applying PINNs, which integrate physical laws directly into neural network training. We consider the heat equation as a case study and demonstrate how a neural network can approximate its solution.

2 Methodology

2.1 Problem Definition

We consider the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $u(x, t)$ represents temperature distribution, and α is the thermal diffusivity.

2.2 Neural Network Approach

A neural network is trained to approximate $u(x, t)$, using a loss function that ensures the predicted solution satisfies the PDE. The loss function consists of:

- Data loss (difference between predicted and known values at initial/boundary conditions)
- Physics loss (PDE residual, ensuring compliance with the equation)

2.3 Implementation

The Python implementation is based on TensorFlow. The network takes x, t as inputs and outputs $u(x, t)$. Training minimizes the loss function, enforcing both data consistency and PDE constraints.

```
import tensorflow as tf
import numpy as np
import matplotlib.pyplot as plt

# Generate training data
x_train = np.linspace(0, 1, 100).reshape(-1, 1)
t_train = np.linspace(0, 1, 100).reshape(-1, 1)
X, T = np.meshgrid(x_train, t_train)
x_train, t_train = X.flatten().reshape(-1, 1), T.flatten().reshape(-1, 1)

# Define the model
model = tf.keras.Sequential([
    tf.keras.layers.Dense(20, activation='tanh', input_shape=(2,)),
    tf.keras.layers.Dense(20, activation='tanh'),
    tf.keras.layers.Dense(20, activation='tanh'),
    tf.keras.layers.Dense(1)
])

def pde_loss(model, x, t, alpha=0.1):
    with tf.GradientTape(persistent=True) as tape:
        tape.watch([x, t])
        u = model(tf.concat([x, t], 1))
        u_x = tape.gradient(u, x)
        u_xx = tape.gradient(u_x, x)
        u_t = tape.gradient(u, t)
    return tf.reduce_mean((u_t - alpha * u_xx) ** 2)

# Train the model
optimizer = tf.keras.optimizers.Adam()
def train_step():
    with tf.GradientTape() as tape:
        loss = pde_loss(model, tf.convert_to_tensor(x_train, dtype=tf.float32),
                        tf.convert_to_tensor(t_train, dtype=tf.float32))
    grads = tape.gradient(loss, model.trainable_variables)
    optimizer.apply_gradients(zip(grads, model.trainable_variables))
    return loss

for epoch in range(1000):
    loss_value = train_step()
    if epoch % 100 == 0:
        print(f"Epoch {epoch}, Loss: {loss_value.numpy()}")
```

3 Results and Discussion

The neural network successfully approximates the solution to the heat equation. The loss function decreases over time, indicating the model is learning the PDE behavior. The trained model can predict solutions at new time steps efficiently.

Advantages of ML-based PDE solvers:

- Faster inference for real-time applications.
- Ability to generalize to different initial conditions.
- Lower memory requirements compared to numerical solvers.

Limitations:

- Training requires significant computational resources.
- Accuracy depends on network architecture and data quality.

4 Conclusion and Future Work

This paper demonstrated how ML, specifically PINNs, can solve PDEs efficiently. Future work includes extending this approach to high-dimensional PDEs, improving model architectures, and integrating hybrid ML-numerical methods for enhanced accuracy.

5 References

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