Automatic Time Step Selection for Numerical Solution of Neutron Diffusion Problems

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Introduction

In computational practice two-layer schemes are mostly used, compared with three-layered and multilayered schemes which are not so often used. The problem of controlling the time step is relatively well worked out for an approximate solution of the Cauchy problem for systems of differential equations.

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The algorithm takes into account the features of neutron diffusion problems, for instance, fast changes in the solution or instability with respect to the initial data.

Problem description

Consider second-order parabolic equation

$$\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha}\left(k(\boldsymbol{x}, t)\frac{\partial u}{\partial x_\alpha}\right) + s(\boldsymbol{x}, t)u = f(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega, \quad 0 < t \le T,$$

where $\underline{k} \le k(\boldsymbol{x}) \le \overline{k}$, $\boldsymbol{x} \in \Omega$, $\underline{k} > 0$.

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where $\underline{k} \le k(\boldsymbol{x}) \le \overline{k}$, $\boldsymbol{x} \in \Omega$, $\underline{k} > 0$.

Boundary condition

$$u(\boldsymbol{x}, t) = g(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \partial\Omega, \quad 0 < t \le T.$$

Initial condition

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

Operator notation

Cauchy problem

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Assume $A(t) \geq 0$ in H then

$$\|u(t)\| \leq \|u_0\| + \int_0^t \|f(\theta)\| d\theta.$$

Solution evaluation

Introduce irregular time grid

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$$\|y^{n+1}\| \leq \|y^n\| + \tau^{n+1} \|f^{n+1}\|.$$

Difference estimate

$$\|y^{n+1}\| \leq \|u^0\| + \sum_{k=0}^{n} \tau^{k+1} \|f^{k+1}\|.$$

Solution error

For $z^n = y^n - u^n$:

$$\frac{z^{n+1} - z^n}{\tau^{n+1}} + A^{n+1} z^{n+1} = \psi^{n+1}, \quad n = 0, 1, ..., N-1,$$

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Approximation error

$$\psi^{n+1} = f^{n+1} - \frac{u^{n+1} - u^n}{\tau^{n+1}} - A^{n+1} u^{n+1}.$$

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$$\|z^{n+1}\| \le \delta t^{n+1}.$$

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5. Solution on a new time layer y^{n+1}: an implicit scheme, $t^{n+1} = t^n + \tau^{n+1}$

Neutron diffusion equation

One group diffusion approximation with one group delayed neutron sources

$$\frac{1}{v}\frac{\partial \phi}{\partial t} - \nabla \cdot D\nabla\phi + \Sigma_a\phi = (1-\beta)\nu\Sigma_f\phi + \lambda c,$$

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Boundary condition

$$D\frac{\partial \phi}{\partial n} + \gamma \phi = 0.$$

Initial conditions

$$\phi(0) = \phi^0, \; c(0) = c^0.$$

Calculated formulas

Denote vectors and matrix $\boldsymbol{\varphi} = \{\varphi, s\}$, $\boldsymbol{\psi} = \{\psi_1, \psi_2\}$,

$$A = \begin{pmatrix} -\nabla \cdot D\nabla + \Sigma_a - (1-\beta)\nu\Sigma_f - \lambda & 0 \\ 0 & \lambda - \beta\nu\Sigma_f \end{pmatrix}.$$

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The approximation error

$$\begin{aligned} \widetilde{\boldsymbol{\psi}}^{\boldsymbol{n+1}} &= (A^{n+1} - A^n)\boldsymbol{\varphi}^{\boldsymbol{n}} + A^{n+1}(\widetilde{\boldsymbol{\varphi}}^{\boldsymbol{n+1}} - \boldsymbol{\varphi}^{\boldsymbol{n}}) \\ &= \widetilde{\tau}^{n+1} \left(\frac{A^{n+1} - A^n}{\widetilde{\tau}^{n+1}} \boldsymbol{\varphi}^{\boldsymbol{n}} + A^{n+1} \frac{\widetilde{\boldsymbol{\varphi}}^{\boldsymbol{n+1}} - \boldsymbol{\varphi}^{\boldsymbol{n}}}{\widetilde{\tau}^{n+1}} \right). \end{aligned}$$

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Error $\widetilde{\boldsymbol{\psi}}^{n+1}$ compare with $\widetilde{\tau}^{n+1}$, and $\boldsymbol{\psi}^{n+1}$ with step τ^{n+1}:

$$\bar{\tau}^{n+1} = \gamma_{n+1}\tau^n, \quad \gamma_{n+1} = \frac{\delta}{\|\widetilde{\boldsymbol{\psi}}^{n+1}\|}\gamma.$$

The needed time step

$$\tau^{n+1} \leq \bar{\tau}^{n+1}, \quad \tau^{n+1} \leq \widetilde{\tau}^{n+1}, \quad \tau^{n+1} = \max\left\{\tau^0, \min\{\gamma_{n+1}, \gamma\}\tau^n\right\}.$$

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In view of this, we set

$$\|\widetilde{\boldsymbol{\psi}}^{\boldsymbol{n+1}}\| \leq \|(A^{n+1} - A^n)\boldsymbol{\varphi}^{\boldsymbol{n}} + A^{n+1}(\widetilde{\boldsymbol{\varphi}}^{\boldsymbol{n+1}} - \boldsymbol{\varphi}^{\boldsymbol{n}})\|.$$

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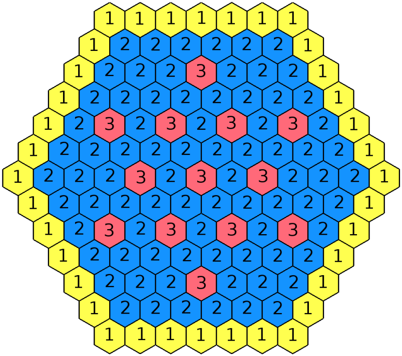
Calculated formula for time step

$$\gamma_{n+1} = \frac{\delta}{\|(A^{n+1} - A^n)\boldsymbol{\varphi}^{\boldsymbol{n}} + A^{n+1}(\widetilde{\boldsymbol{\varphi}}^{\boldsymbol{n+1}} - \boldsymbol{\varphi}^{\boldsymbol{n}})\|}\gamma.$$

IAEA-2D benchmark



- Without reflector

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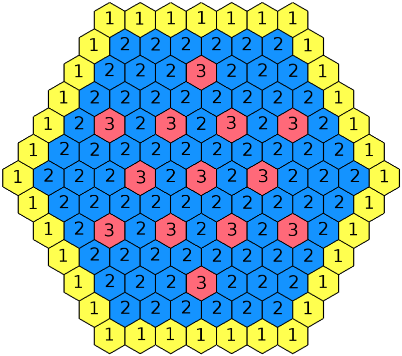

- Without reflector
- One group of *instantaneous* and *delayed* neutrons

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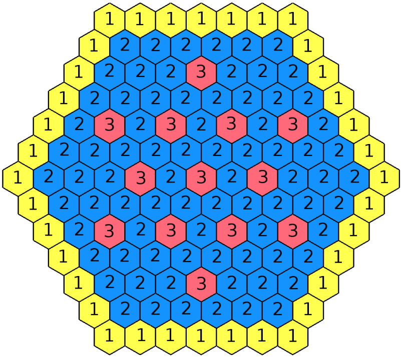

- Without reflector
- One group of *instantaneous* and *delayed* neutrons
- Modeling effect of *immersion* or *extraction* of control rods

Scenario

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1. The spectral problem is solved (initial condition);

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3. At a moment of 0.1 sec the value Σ_a for the zone 3 changes to $\pm$ 0.000625.

Software

gmsh

Software

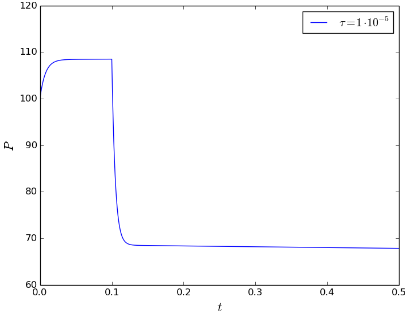
gmsh

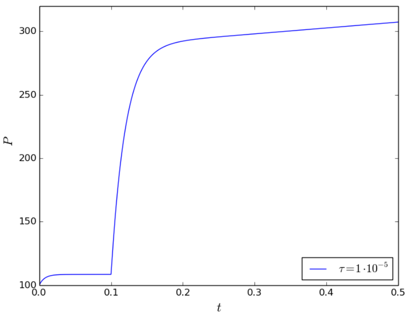
SLEPc

Software

Software

Nuclear power



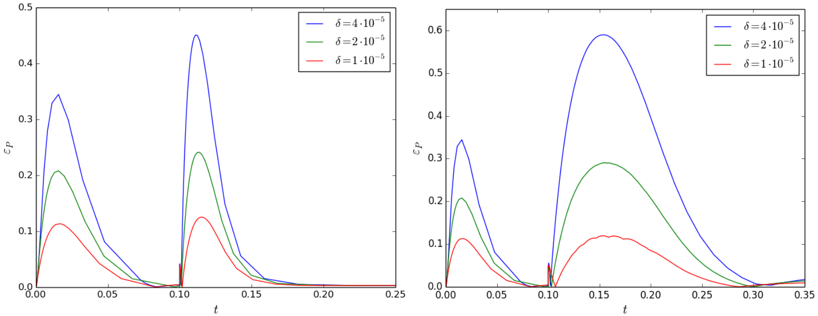


Nuclear power for immersion (left) and extraction (right).

$$P(t) = a \sum_{g=1}^{G} \int_{\Omega} \Sigma_{fg} \phi_g d\boldsymbol{x},$$

where a – normalization factor.

Error

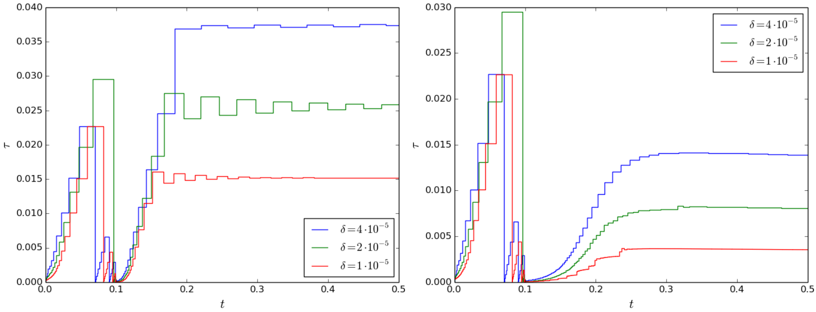


Error for immersion (left) and extraction (right).

$$\epsilon_P(t) = |P_{ref} - P|,$$

where P_{ref} – reference solution.

Time step



Time steps for immersion (left) and extraction (right).

Counting time and number of steps

	immersion			extraction		
δ	$\max(\epsilon_P)$	n	t, sec	$\max(\epsilon_P)$	n	t, sec
$4 \cdot 10^{-5}$	0.450	136	16	0.590	241	35
$2 \cdot 10^{-5}$	0.241	159	20	0.290	373	62
$1 \cdot 10^{-5}$	0.125	270	37	0.120	773	145

Reference solution: fixed time step 10^{-5}, number of steps – 50000, counting time – 2130 sec.

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Thank you for your attention!