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Time Step Selection for the Numerical Solution of Boundary Value Problems for Parabolic Equations

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Abstract—An algorithm is proposed for selecting a time step for the numerical solution of boundary value problems for parabolic equations. The solution is found by applying unconditionally stable implicit schemes, while the time step is selected using the solution produced by an explicit scheme. Explicit computational formulas are based on truncation error estimation at a new time level. Numerical results for a model parabolic boundary value problem are presented, which demonstrate the performance of the time step selection algorithm.

Keywords: implicit difference schemes, time step selection, parabolic equation, truncation error.

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INTRODUCTION

Primary attention in the approximate solution of boundary value problems for time-dependent equations is given to time approximations [1–3]. For second-order parabolic equations, unconditionally stable schemes are constructed using implicit approximations [4–6]. Two-level schemes are the most popular in numerical practice, while schemes with three and more time levels are used much less frequently. For unconditionally stable schemes, the choice of a time step is determined only by the accuracy of the approximate solution.

The problem of time step control has been relatively well studied in the case of the Cauchy problem for systems of differential equations [7–9]. The basic approach is that the error of the approximate solution is estimated at a new time level via additional computations, the time step is estimated using the theoretical asymptotic dependence of accuracy on the time step, and, if necessary, the time step is corrected and the computations are repeated.

Additional computations for estimating the error of the approximate solution can be based on different approaches. Specifically, an approximate solution can be obtained using two different schemes of the same theoretical order of accuracy. The best known example of this strategy is the solution of a problem at a certain time interval with the use of a given time step (first solution) and a halved time step (second solution). The approximate solution of Cauchy problems for systems of ordinary differential equations can also be based on nested methods. In this case, two approximate solutions with different orders of accuracy are compared.

In fact, the indicated techniques for time step selection belong to the class of a posteriori error estimation methods. In this case, whether the time step is suitable or has to be changed (increased or decreased and how much) and whether repeated computations are needed is decided after the solution has been computed. Similar strategies can also be used on the basis of a more advanced a posteriori analysis of approximate solutions of time-dependent boundary value problems (see [10–12]).

In fact, a priori time step selection for the approximate solution of boundary value problems for parabolic equations was considered in [13]. Some more general problems were addressed in [14]. The solution at a new time level was found by applying the standard implicit Euler scheme. The time step at the new level was explicitly calculated using the solutions from two previous time levels with variations in the coefficients of the equation and in the right-hand side taken into account.

In this paper, we explore new possibilities of time step estimation for the approximate solution of boundary value problems for parabolic equations based on an auxiliary solution produced by an explicit scheme. In [13, 14] the time step was estimated by comparing numerical solutions, namely, using an error estimate for the approximate solution. In this work, we use a more direct approach based on truncation error estimation.

1. FORMULATION OF THE PROBLEM

We consider the Cauchy problem for the linear equation

$$\frac{du}{dt} + A(t)u = f(t), \quad 0 < t \le T, \tag{1.1}$$

with the initial condition

$$u(0) = u_0. \tag{1.2}$$

The problem is studied in a finite-dimensional Hilbert space H. Assume that

$$A(t) \ge 0$$

in H. Since the operator A of problem (1.1), (1.2) is nonnegative, its solution satisfies the following estimate of stability with respect to the initial data and the right-hand side:

$$\|u(t)\| \le \|u_0\| + \int_0^t \|f(\theta)\| d\theta. \tag{1.3}$$

Problem (1.1), (1.2) arises when a finite difference, finite volume, or finite element approximation (lumped mass scheme [15]) is applied to boundary value problems for a second-order parabolic equation. In this problem, the unknown function $u(x,t)$ satisfies the equation

$$\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha}\left(k(\mathbf{x},t)\frac{\partial u}{\partial x_\alpha}\right) + p(\mathbf{x},t)u = f(\mathbf{x},t), \quad \mathbf{x} \in \Omega, \quad 0 < t \le T,$$

where $\underline{k} \le k(\mathbf{x},t) \le \overline{k}$, $\mathbf{x} \in \Omega$, $\underline{k} > 0$, and $p(\mathbf{x},t) \ge 0$. The equation is supplemented with the Dirichlet boundary conditions

$$u(\mathbf{x},t) = g(\mathbf{x},t), \quad \mathbf{x} \in \partial\Omega, \quad 0 < t \le T$$

and the initial condition

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

This time-dependent problem is solved numerically on a nonuniform grid in time:

$$t_0 = 0, \quad t_{n+1} = t_n + \tau_{n+1}, \quad n = 0,1,\ldots,N-1, \quad t_N = T.$$

Let $f_n = f(t_n)$. For problem (1.1), (1.2), we use a fully implicit scheme whereby the transition from one time level to another is given by the equality

$$\frac{y_{n+1} - y_n}{\tau_{n+1}} + A_{n+1}y_{n+1} = f_{n+1}, \quad n = 0,1,\ldots,N-1, \tag{1.4}$$

and the initial condition

$$y_0 = u_0. \tag{1.5}$$

Under the restriction $A_{n+1} \ge 0$, it follows directly from (1.4) that the approximate solution satisfies the levelwise estimate

$$\|y_{n+1}\| \le \|y_n\| + \tau_{n+1}\|f_{n+1}\|.$$

Thus, we obtain a discrete analogue of estimate (1.3) for problem (1.4), (1.5), namely,

$$\|y_{n+1}\| \le \|u_0\| + \sum_{k=0}^{n} \tau_{k+1}\|f_{k+1}\|. \tag{1.6}$$