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Numerical simulation of time fractional Cable equations and convergence analysis

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In this article, the numerical solution of time fractional Cable equation is considered. We convert the time fractional Cable equations into equivalent integral equations with singular kernel, then propose a spectral collection method in both time and space discretizations with a spectral expansion of Lagrange interpolation polynomial for this equation. The convergence of the method is rigorously established. Numerical tests are carried out to confirm the theoretical results.

KEYWORDS
convergence analysis, Jacobi collocation method, time fractional Cable equation

1 | INTRODUCTION

The Cable equation is one of the most fundamental equations for modelling neuronal dynamics. When the ions are undergoing anomalous subdiffusion, the standard or normal models may lead to incorrect or misleading diffusion coefficient values and models, so it is better to use the model incorporate with anomalous diffusion. A recent study on spiny Purkinje cell dendrites that spines trap and release diffusing molecules resulting in anomalously show molecular diffusion along the dendrite [1]. The diffusive spatial variance $< r^2(t) >$ of an inert tracer was found to evolve as a sublinear power law in time, that is, $< r^2(t) >\sim t^{\gamma}$ with $0 < \gamma < 1$. Henry et al. [2] derived a fractional Cable equation from the fractional Nernst-Planck equations to model anomalous electrodiffusion of ions in spiny dendrites. They subsequently found a fractional Cable equation by treating the neuron and its membrane as two separate materials governed separate fractional Nernst-Planck equations and used a small ionic concentration gradient assumption [3, 4]. The resulting equation involves two fractional temporal Riemann-Liouville derivatives.

In this article, we consider the following time fractional Cable equation

$$u_t(x,t) = {}^R D_t^{1-\mu_1} K u_{xx}(x,t) - \lambda {}^R D_t^{1-\mu_2} u(x,t) + f(x,t), \quad 0 < \mu_2 \le \mu_1 < 1, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = \phi(x), \quad a < x < b, \tag{1.2}$$

and boundary conditions

$$u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), \quad 0 < t < T, \tag{1.3}$$

where $K > 0$ and λ are constants, ${}^R D_t^{\mu}$ is defined as Riemann-Liouville fractional derivative of order μ,

$${}^R D_t^{\mu} P(t) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_0^t (t-\tau)^{-\mu} P(\tau) d\tau, \quad \mu \in (0,1). \tag{1.4}$$

In the order, the well-known Caputo and Riemann-Liouville also have a big problem that their kernel although nonlocal but is singular. This weakness has effect when modeling real world problems. Recently, some newly concept of fractional calculus are proposed to answer some outstanding questions [5–9].

The numerical method to fractional differential equations has been recently discussed by numerous researchers. Some different numerical methods for solving time or space fractional partial differential equations have been proposed [10–14]. Liu and Yang [15], proposed two new implicit numerical methods with convergence analysis for the fractional Cable equation. Ibrahim and Nurdane [16] constructed a new difference scheme based on Crank-Nicholson difference scheme for solution of fractional Cable equation involving Caputo fractional derivative. Bu [17, 18] considered finite element multigrid method for time fractional advection diffusion equations. Recently, we provided Jacobi spectral-collocation method [19, 20] for time-fractional equations. Bhrawy and Zaky [21] reported a spectral collocation method based on shifted Jacobi collocation procedure in conjunction with the shifted Jacobi operational matrix for solving one and two-dimensional variable-order fractional nonlinear Cable equations. Sweilam and Khader [22] introduced numerical study for the fractional Cable equation using weighted average of finite difference methods. Mao and Xiao [23, 24] investigated the numerical solution for a class of fractional diffusion-wave equations based on the sinc-collocation technique. Yang and Xiao [25, 26] discussed the cubic spline collocation method with two parameters for solving the initial value problems of fractional differential equations.

The main purpose of this article is to solve and analyse the fractional Cable problem by Jacobi spectral collocation method. One contribution of this article is that we transform the fractional Cable equation into an integro-differential equation with weakly singular kernel and propose a Jacobi spectral collocation method in both time and space discretizations with a spectral expansion of Lagrange interpolation polynomial for this equation. Another contribution of this article is that we provide a rigorous error analysis for the proposed method, which shows that the numerical errors decay exponentially in both the infinity norm and the weighted L_2 norms.

2 | JACOBI COLLOCATION METHOD FOR THE TIME-FRACTIONAL CABLE EQUATION

First, we give the properties of the fractional derivative and integral, we can transform the time-fractional Cable problem into a Volterra integro-differential equation with a weakly singular kernel equivalently. Therefore, a number of the numerical schemes developed for Volterra-type integro-differential equation with weakly singular kernel can be applied to find numerical solution of fractional differential equations.

First, we define $\mathbf{J}_t^{\mu}$ as the Riemann-Liouville fractional integral of order μ

$$\mathbf{J}_t^{\mu} P(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} P(\tau) d\tau, \quad \mu \in (0,1). \tag{2.1}$$

Letting $\mathbf{J}_t^{1-\mu_1}$ act on both sides of (1.1), we find

$$\mathbf{J}_t^{1-\mu_1} u_t(x,t) = \mathbf{J}_t^{1-\mu_1} \left({}^R D_t^{1-\mu_1} K u_{xx}(x,t) - \lambda {}^R D_t^{1-\mu_2} u(x,t) + f(x,t) \right), \tag{2.2}$$

using the following properties

$$\begin{aligned} \mathbf{J}_t^{\mu} ({}^R D_t^{\mu} P(t)) &= P(t) - [\mathbf{J}_t^{1-\mu} P(t)]_{t=0} \frac{t^{\mu-1}}{\Gamma(\mu)}, \\ \mathbf{J}_t^{1-\mu} P_t(t) &= {}^R D_t^{\mu} P(t) - \frac{P(0) t^{-\mu}}{\Gamma(1-\mu)}, \end{aligned} \tag{2.3}$$

(2.2) becomes

$$\begin{aligned} {}^R D_t^{\mu} u(x,t) - \frac{u(x,0) t^{-\mu}}{\Gamma(1-\mu)} &= K \left(u_{xx}(x,t) - \frac{t^{-\mu_1}}{\Gamma(1-\mu_1)} \mathbf{J}_t^{\mu_1} u_{xx}(x,t) \Big|_{t=0} \right) \\ &\quad - \lambda \left(\mathbf{J}_t^{\mu_1-\mu_2} u(x,t) - \frac{t^{-\mu_1}}{\Gamma(1-\mu_1)} \mathbf{J}_t^{\mu_1-\mu_2} u(x,t) \Big|_{t=0} \right) \\ &\quad + \mathbf{J}_t^{\mu_1} f(x,t). \end{aligned} \tag{2.4}$$

As $u(x,t)$ is at least a continuous function w.r.t. t, then

$$\lim_{t\to 0} |\mathbf{J}_t^{\mu} u(x,t)| = \lim_{t\to 0} \left| \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} u(x,\tau) d\tau \right| \le \lim_{t\to 0} M \frac{t^{\mu}}{\Gamma(1+\mu)} = 0,$$

where $M = \max_{t\in[0,T]} u(x,t)$ for fixed x, and T is any small fixed positive number. Then, we can rewrite (2.4) in the following equivalent form

$${}^R D_t^{\mu} u(x,t) - \frac{u(x,0) t^{-\mu}}{\Gamma(1-\mu)} = K u_{xx}(x,t) - \lambda \mathbf{J}_t^{\mu_1-\mu_2} u(x,t) + \mathbf{J}_t^{\mu_1} f(x,t). \tag{2.5}$$

We define D_t^μ as Caputo fractional derivative of order μ,

$$D_t^\mu P(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t (t-\tau)^{-\mu} P_\tau(\tau) d\tau, \quad \mu \in (0,1), \tag{2.6}$$

and note that the following properties of Caputo fractional derivative hold

$$\begin{aligned} D_t^\mu P(t) &= {}^R D_t^\mu P(t) - \frac{P(0)t^{-\mu}}{\Gamma(1-\mu)}, \\ \mathbf{J}_t^\mu \left(D_t^\mu P(t)\right) &= P(t) - P(0). \end{aligned} \tag{2.7}$$

So, letting $\mathbf{J}^{\mu_1}$ act on both sides of (2.5), we find the fractional Cable equation can be written in the integrated form with weakly singular kernel $(t-\tau)^{\mu-1}$ finally

$$\begin{aligned} u(x,t) &= \mathbf{J}_t^{\mu_1} K u_{xx}(x,\tau) - \lambda \mathbf{J}_t^{2\mu_1-\mu_2} u(x,t) + \mathbf{J}_t^{2\mu_1} f(x,t) + u(x,0) \\ &= K \frac{1}{\Gamma(\mu_1)} \int_0^t (t-\tau)^{-\gamma_1} u_{xx}(x,\tau) d\tau \\ &\quad - \lambda \frac{1}{\Gamma(2\mu_1-\mu_2)} \int_0^t (t-\tau)^{-\gamma_2} u(x,\tau) d\tau + \mathbf{J}_t^{2\mu_1} f(x,t) + u(x,0), \\ &\quad -1 < -\gamma_1 = \mu_1 - 1 < -\gamma_2 = 2\mu_1 - \mu_2 - 1 < 0, \end{aligned} \tag{2.8}$$

subject to the initial condition

$$u(x,0) = \phi(x), \quad a < x < b, \tag{2.9}$$

and boundary conditions

$$u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), \quad 0 < t < T, \tag{2.10}$$

where $g(x,t) = \mathbf{J}_t^{2\mu_1} f(x,t)$.

In this article, we consider the numerical solution of integral equations with singular kernel (2.8), which is equivalent to the time fractional Cable equation (1.1). The numerical treatment of the Equation 2.8 is not simple, mainly due to the fact that the solutions of (2.8) usually have a weak singularity near $t = 0^+$. Recently, we provided Jacobi spectral-collocation method [27, 28] or spectral Petrov-Galerkin method [29, 30] and convergence analysis for integro-differential equations. We propose a spectral collocation method in both temporal and spatial discretizations with a spectral expansion of Jacobi interpolation polynomial for this equation. The main advantage of the present scheme is that it gives very accurate convergence by choosing less number of grid points and the problem can be solved up to big time, and the storage requirement due to the time memory effect can be considerably reduced. At the same time, the technique is simple and easy to apply to multidimensional problems.

Let $\omega^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$ be a weight function in the usual sense, for $\alpha, \beta > -1$. The set of Jacobi polynomials $\left\{J_n^{\alpha,\beta}(x)\right\}_{n=0}^\infty$ forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)$—orthogonal system, where $L^2_{\omega^{\alpha,\beta}}(-1,1)$ is a weighted space defined by

$$L^2_{\omega^{\alpha,\beta}}(-1,1) = \left\{v : v \text{ is measurable and } \|v\|_{\omega^{\alpha,\beta}} < \infty\right\},$$

equipped with the inner product

$$(u,v)_{\omega^{\alpha,\beta}} = \int_{-1}^{1} u(x)v(x)\omega^{\alpha,\beta}(x)dx \quad \forall u,v \in L^2_{\omega^{\alpha,\beta}}(-1,1),$$

and the norm

$$\|v\|_{\omega^{\alpha,\beta}} = (v,v)^{\frac{1}{2}}_{\omega^{\alpha,\beta}}.$$

For a given $N \geq 0$, we denote by $\{\theta_k\}_{k=0}^N$ the Legendre points, and by $\{\omega_k\}_{k=0}^N$ the corresponding Legendre weights $\left(\text{i.e., Jacobi weights } \left\{\omega_k^{0,0}\right\}_{k=0}^N\right)$. Then, the Legendre-Gauss integration formula is

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=0}^{N} f(\theta_k)\omega_k. \tag{2.11}$$

Similarly, we denote by $\left\{\tilde{\theta}_k\right\}_{k=0}^N$ the Jacobi-Gauss points, and by $\left\{\omega_k^{\alpha,\beta}\right\}_{k=0}^N$ the corresponding Jacobi weights. Then, the Jacobi-Gauss integration formula is

$$\int_{-1}^{1} f(x)\omega^{\alpha,\beta}(x)dx \approx \sum_{k=0}^{N} f(\tilde{\theta}_k)\omega_k^{\alpha,\beta}. \tag{2.12}$$

For a given positive integer N, we denote the collocation points by $\left\{x_i^{\alpha,\beta}\right\}_{i=0}^N$, which is the set of $(N+1)$ Jacobi-Gauss points, corresponding to the weight $\omega^{\alpha,\beta}(x)$. Let $\mathcal{P}_N$ denote the space of all polynomials of degree not exceeding N. For any $v \in C[-1,1]$, we can define the Lagrange interpolating polynomial $I_N^{\alpha,\beta}v \in \mathcal{P}_N$, satisfying

$$I_N^{\alpha,\beta}v(x_i^{\alpha,\beta}) = v(x_i^{\alpha,\beta}), \quad 0 \leq i \leq N.$$

The Lagrange interpolating polynomial can be written in the form

$$I_N^{\alpha,\beta}v(x) = \sum_{i=0}^{N} v(x_i^{\alpha,\beta})F_i(x), \quad 0 \leq i \leq N,$$

where $F_i(x)$ is the Lagrange interpolation basis function associated with $\left\{x_i^{\alpha,\beta}\right\}_{i=0}^N$.

To apply the theory of orthogonal polynomials, we make following variable change

$$\begin{aligned}
x &= \frac{b-a}{2}(1+\bar{x}) + a, \quad \bar{x} = \frac{2(x-a)}{b-a} - 1, \quad \bar{x} \in [-1,1], \\
t &= \frac{T}{2}(1+\bar{t}), \quad \bar{t} = \frac{2t}{T} - 1 \quad \bar{t} \in [-1,1], \\
\tau &= \frac{T}{2}(1+s), \quad s = \frac{2\tau}{T} - 1, \quad s \in [-1,y],
\end{aligned}$$

then the singular problems (2.8) can be rewritten as

$$\bar{u}(\bar{x},\bar{t}) = \frac{K}{\Gamma(\mu_1)}\left(\frac{T}{2}\right)^{\mu_1} \int_{-1}^{\bar{t}} (\bar{t}-s)^{-\gamma_1}\bar{u}_{\bar{x}\bar{x}}(\bar{x},s)ds$$

$$- \frac{\lambda}{\Gamma(2\mu_1 - \mu_2)} \left(\frac{T}{2}\right)^{2\mu_1-\mu_2} \int_{-1}^{\bar{t}} (\bar{t} - s)^{-\gamma_2} \bar{u}(\bar{x}, s) ds$$
$$+ \bar{g}(\bar{x}, \bar{t}) + \overline{\phi}(\bar{x}), \tag{2.13}$$

with boundary conditions

$$\bar{u}(-1, \bar{t}) = \overline{\psi}_1(\bar{t}), \quad \bar{u}(1, \bar{t}) = \overline{\psi}_2(\bar{t}), \quad -1 < \bar{t} < 1, \tag{2.14}$$

where

$$\bar{u}(\bar{x}, \bar{t}) = u\left(\frac{b-a}{2}(1+\bar{x}) + a, \frac{T}{2}(1+\bar{t})\right),$$
$$\bar{u}_{\bar{x}\bar{x}}(\bar{x}, s) = \frac{4}{(b-a)^2} u\left(\frac{b-a}{2}(1+\bar{x}) + a, \frac{T}{2}(1+s)\right),$$
$$\bar{g}(\bar{x}, \bar{t}) = g\left(\frac{b-a}{2}(1+\bar{x}) + a, \frac{T}{2}(1+\bar{t})\right),$$
$$\overline{\phi}(\bar{x}) = \phi\left(\frac{b-a}{2}(1+\bar{x}) + a\right), \quad \overline{\psi}_i(\bar{t}) = \psi_i\left(\frac{T}{2}(1+\bar{t})\right), \quad i = 1, 2.$$

2.1 | Spectral collocation method in time

For the collocation methods in time, (2.13) holds at Jacobi collocation points $\{\bar{t}_j\}_{j=0}^M$ with Jacobi weight functions $\omega^{\overline{\alpha},\overline{\beta}}(\bar{t}) = (1-\bar{t})^{\overline{\alpha}}(1+\bar{t})^{\overline{\beta}}$ on $[-1, 1]$, namely,

$$\bar{u}(\bar{x}, \bar{t}_j) = \zeta \int_{-1}^{\bar{t}_j} (\bar{t}_j - s)^{-\gamma_1} \bar{u}_{\bar{x}\bar{x}}(\bar{x}, s) ds + \eta \int_{-1}^{\bar{t}_j} (\bar{t}_j - s)^{-\gamma_2} \bar{u}(\bar{x}, s) ds$$
$$+ \bar{g}(\bar{x}, \bar{t}_j) + \overline{\phi}(\bar{x}) \tag{2.15}$$

where $\zeta = \frac{K}{\Gamma(\mu_1)}\left(\frac{T}{2}\right)^{\mu_1}$, $\eta = -\frac{\lambda}{\Gamma(2\mu_1-\mu_2)}\left(\frac{T}{2}\right)^{2\mu_1-\mu_2}$.

To use some appropriate quadrature rule, we will transfer the integral interval $[-1, \bar{t}_j]$ to a fixed interval $[-1, 1]$ and j, we make a simple linear transformation

$$s = s_j(\theta) = \frac{1+\bar{t}_j}{2}\theta + \frac{\bar{t}_j - 1}{2}, \quad \theta \in [-1, 1], \tag{2.16}$$

we can rewrite (2.15) as follows:

$$\bar{u}(\bar{x}, \bar{t}_j) = \zeta_j \int_{-1}^{1} (1-\theta)^{-\gamma_1} \bar{u}_{\bar{x}\bar{x}}(\bar{x}, s_j(\theta)) d\theta + \eta_j \int_{-1}^{1} (1-\theta)^{-\gamma_2} \bar{u}(\bar{x}, s_j(\theta)) d\theta$$
$$+ \bar{g}(\bar{x}, \bar{t}_j) + \overline{\phi}(\bar{x}), \tag{2.17}$$

with $\zeta_j = \zeta\left(\frac{1+\bar{t}_j}{2}\right)^{1-\gamma_1}$, $\eta_j = \eta\left(\frac{1+\bar{t}_j}{2}\right)^{1-\gamma_2}$.

Next, using a $(L+1)$-point Gauss quadrature formula, the first integration term in (2.17) can be approximated by

$$\int_{-1}^{1} (1-\theta)^{-\gamma_1} \bar{u}_{\bar{x}\bar{x}}(\bar{x}, s_j(\theta)) d\theta \approx \sum_{k=0}^{L} \bar{u}_{\bar{x}\bar{x}}(\bar{x}, s_j(\theta_k)) \omega_k^{-\gamma_1,0}, \tag{2.18}$$

where $L \geq M$, and $\{\theta_k\}_{k=0}^{L}$ is the set of Jacobi-Gauss points corresponding to the weights $\omega^{-\gamma_1,0}(\theta) = (1-\theta)^{-\gamma_1}(1+\theta)^0$ and $\omega_k^{-\gamma_1,0} = \omega^{-\gamma_1,0}(\theta_k)$ on $[-1,1]$. Similarly, the second integration term in (2.17) can be approximated by

$$\int_{-1}^{1} (1-\theta)^{-\gamma_2}\bar{u}(\bar{x}, s_j(\theta))d\theta \approx \sum_{k=0}^{P} \bar{u}(\bar{x}, s_j(\vartheta_k))\omega_k^{-\gamma_2,0}, \tag{2.19}$$

where $P \geq M$, and $\{\vartheta_k\}_{k=0}^{P}$ is the set of Jacobi-Gauss points corresponding to the weights $\omega^{-\gamma_2,0}(\theta) = (1-\vartheta)^{-\gamma_2}(1+\vartheta)^0$ and $\omega_k^{-\gamma_2,0} = \omega^{-\gamma_2,0}(\vartheta_k)$ on $[-1,1]$.

We first use $\bar{u}^j(\bar{x})$ to indicate the approximate value for $\bar{u}(\bar{x}, \bar{t}_j)$, and

$$\bar{u}(\bar{x}, \bar{t}) \approx \bar{u}^M(\bar{x}, \bar{t}) = \sum_{j=0}^{M} \bar{u}^j(\bar{x})F_j(\bar{t}), \tag{2.20}$$

where $F_j(\bar{t})$ is the j th Lagrange interpolation polynomial associated with the collocation points $\{\bar{t}_j\}_{j=0}^{M}$. Combining the above equations, the semidiscretized problem (2.17) can be approximated by

$$\begin{aligned}\bar{u}^j(\bar{x}) = \zeta_j \sum_{k=0}^{L} \bar{u}_{\bar{x}\bar{x}}(\bar{x}, s_j(\theta_k))\omega_k^{-\gamma_1,0} + \eta_j \sum_{k=0}^{P} \bar{u}(\bar{x}, s_j(\vartheta_k))\omega_k^{-\gamma_2,0} \\ + \bar{g}(\bar{x}, \bar{t}_j) + \overline{\phi}(\bar{x}).\end{aligned} \tag{2.21}$$

2.2 | Spectral collocation method in space

For Jacobi collocation methods in space, (2.21) holds at the Jacobi-Gauss collocation points $\{\bar{x}_i\}_{i=0}^{N}$ with Jacobi weight functions $\omega^{\tilde{\alpha},\tilde{\beta}}(\bar{x}) = (1-\bar{x})^{\tilde{\alpha}}(1+\bar{x})^{\tilde{\beta}}$ on $[-1,1]$,

$$\begin{aligned}\bar{u}^j(\bar{x}_i) = \zeta_j \sum_{k=0}^{L} \bar{u}_{\bar{x}\bar{x}}(\bar{x}_i, s_j(\theta_k))\omega_k^{-\gamma_1,0} + \eta_j \sum_{k=0}^{P} \bar{u}(\bar{x}_i, s_j(\vartheta_k))\omega_k^{-\gamma_2,0} \\ + \bar{g}(\bar{x}_i, \bar{t}_j) + \overline{\phi}(\bar{x}_i).\end{aligned} \tag{2.22}$$

Use $\bar{u}_i^j$ to approximate the function $\bar{u}^j(\bar{x}_i)$, and

$$\bar{u}^j(\bar{x}) \approx \sum_{i=0}^{N} \bar{u}_i^j H_i(\bar{x}), \quad \bar{u}(\bar{x}, \bar{t}) \approx \bar{u}_N^M(\bar{x}, \bar{t}) = \sum_{i=0}^{N}\sum_{j=0}^{M} \bar{u}_i^j H_i(\bar{x})F_j(\bar{t}), \tag{2.23}$$

where $H_i(\bar{x})$ is the i th Lagrange interpolation polynomial associated with the collocation points $\{\bar{x}_i\}_{i=0}^{N}$. The discretized problem in space (2.22) can be approximated by

$$\begin{aligned}\bar{u}_i^j = \zeta_j \sum_{k=0}^{L} \bar{u}_{N\,\bar{x}\bar{x}}^M(\bar{x}_i, s_j(\theta_k))\omega_k^{-\gamma_1,0} + \eta_j \sum_{k=0}^{P} \bar{u}_N^M(\bar{x}_i, s_j(\vartheta_k))\omega_k^{-\gamma_2,0} \\ + \bar{g}(\bar{x}_i, \bar{t}_j) + \overline{\phi}(\bar{x}_i).\end{aligned} \tag{2.24}$$

Then the solution of the full-discrete problem (2.13) using collocation spectral method is to seek $\bar{u}_N^M(\bar{x},\bar{t})$ of the form (2.23) such that $\bar{u}_i^j$ satisfies the above collocation Equations 2.24 for $1 \leq i \leq N-1$, $0 \leq j \leq M$.

For completeness sake, the implementation is briefly described here. To simplify the computation, we rewrite the above collocation Equations 2.24 into the following:

$$\begin{aligned}
\bar{u}_i^j &= \zeta_j \sum_{n=0}^{N}\sum_{m=0}^{M} \bar{u}_n^m \bar{H}_n''(\bar{x}_i) \sum_{k=0}^{L} F_m(s_j(\theta_k))\omega_k^{-\gamma_1,0} \\
&\quad + \eta_j \sum_{n=0}^{N}\sum_{m=0}^{M} \bar{u}_n^m \bar{H}_n(\bar{x}_i) \sum_{k=0}^{P} F_m(s_j(\vartheta_k))\omega_k^{-\gamma_2,0} + \bar{g}(\bar{x}_i,\bar{t}_j) + \overline{\phi}(\bar{x}_i) \\
&= \sum_{n=0}^{N}\sum_{m=0}^{M} b_{i,n}c_{j,m}\bar{u}_n^m + \bar{g}_i^j + \overline{\phi}_i \\
&= \sum_{n=1}^{N-1}\sum_{m=0}^{M} b_{i,n}c_{j,m}\bar{u}_n^m + \sum_{m=0}^{M} \left(b_{i,0}\bar{u}_0^m + b_{i,N}\bar{u}_N^m\right) c_{j,m} + \bar{g}_i^j + \overline{\phi}_i, \\
&\quad i = 1,\cdots,N-1, \quad j = 0,\cdots,M,
\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}
&b_{i,n} = H_n''(\bar{x}_i) + H_n(\bar{x}_i), \quad \bar{g}_i^j = \bar{g}(\bar{x}_i,\bar{t}_j), \quad \overline{\phi}_i = \overline{\phi}(\bar{x}_i) \\
&c_{j,m} = \zeta_j \sum_{k=0}^{L} F_m(s_j(\theta_k))\omega_k^{-\gamma_1,0} + \eta_j \sum_{k=0}^{P} F_m(s_j(\vartheta_k))\omega_k^{-\gamma_2,0}.
\end{aligned}$$

The Dirichlet boundary conditions (2.10) are directly applied in (2.25) and give numerical solutions on boundary in the following way

$$\bar{u}_0^m = \psi_1\left(\frac{T(1+\bar{t}_m)}{2}\right), \quad \bar{u}_N^m = \psi_2\left(\frac{T(1+\bar{t}_m)}{2}\right), \quad m = 0,\cdots,M. \tag{2.26}$$

Let us set

$$\begin{aligned}
&\bar{d}_i^j = \sum_{m=0}^{M} \left(b_{i,0}\bar{u}_0^m + b_{i,N}\bar{u}_N^m\right) c_{j,m} + \bar{g}_i^j, \\
&\overline{\mathbf{U}}_{\mathbf{N}}^{\mathbf{M}} = \left[\bar{u}_1^0,\cdots,\bar{u}_{N-1}^0,\bar{u}_1^1,\cdots,\bar{u}_{N-1}^1,\cdots,\bar{u}_1^M,\cdots,\bar{u}_{N-1}^M\right]^T, \\
&\overline{\mathbf{F}}_{\mathbf{N}}^{\mathbf{M}} = \left[\overline{\phi}_1,\cdots,\overline{\phi}_{N-1},\overline{\phi}_1,\cdots,\overline{\phi}_{N-1},\cdots,\overline{\phi}_1,\cdots,\overline{\phi}_{N-1}\right]^T, \\
&\overline{\mathbf{D}}_{\mathbf{N}}^{\mathbf{M}} = \left[\bar{d}_1^0,\cdots,\bar{d}_{N-1}^0,\bar{d}_1^1,\cdots,\bar{d}_{N-1}^1,\cdots,\bar{d}_1^M,\cdots,\bar{d}_{N-1}^M\right]^T, \\
&\mathbf{B} = (b_{i,n}), \quad i = 1,\cdots,N-1, \quad n = 1,\cdots,N-1, \\
&\mathbf{A} = \left[c_{j,m}\mathbf{B}\right], \quad j = 0,\cdots,M, \quad m = 0,\cdots,M, \\
&\mathbf{A} \text{ is a matrix of } (N-1)\times(M+1) \text{ by } (N-1)\times(M+1).
\end{aligned} \tag{2.27}$$

Thus, the numerical scheme (2.10) leads to a system of equation of the form

$$\overline{\mathbf{U}}_{\mathbf{N}}^{\mathbf{M}} = \mathbf{A}\overline{\mathbf{U}}_{\mathbf{N}}^{\mathbf{M}} + \overline{\mathbf{D}}_{\mathbf{N}}^{\mathbf{M}} + \overline{\mathbf{F}}_{\mathbf{N}}^{\mathbf{M}}. \tag{2.28}$$

3 | CONVERGENCE ANALYSIS

In this section, we will carry out error estimations for the solution of the semidiscretized full-discretized problems. First, we will provide some elementary lemmas, which are important for the derivation of the main results. Let $I := (-1, 1)$.

Lemma 3.1 (*see* [31]) *Assume that an* $(N+1)$ *-point Gauss quadrature formula relative to the Jacobi weight is used to integrate the product* $u\varphi$*, where* $u \in H^{m,N}_{\omega^{\alpha,\beta}}$ *with* I *for some* $m \geq 1$ *and* $\varphi \in \mathcal{P}_N$*. Then there exists a constant* C *independent of* N *such that*

$$\left| \int_{-1}^{1} u(x)\varphi(x)dx - (u,\varphi)_N \right| \leq CN^{-m}|u|_{H^{m,N}_{\omega^{\alpha,\beta}}(I)}\|\varphi\|_{L^2_{\omega^{\alpha,\beta}}(I)}, \tag{3.1}$$

where

$$|u|_{H^{m,N}_{\omega^{\alpha,\beta}}(I)} = \left(\sum_{j=\min(m,N+1)}^{m} \|u^{(j)}\|^2_{L^2_{\omega^{\alpha,\beta}}(I)} \right)^{1/2},$$

$$(u,\varphi)_N = \sum_{j=0}^{N} u(x_j)\varphi(x_j)\omega_j. \tag{3.2}$$

Lemma 3.2 (*see* [31]) *Assume that* $u \in H^{m,N}_{\omega^{\alpha,\beta}}(I)$ *and denote by* $I_N^{\alpha,\beta}u$ *its interpolation polynomial associated with the* $(N+1)$ *Jacobi-Gauss points* $\{x_j\}_{j=0}^{N}$*, namely,*

$$I_N^{\alpha,\beta}u = \sum_{i=0}^{N} u(x_i)F_i(x).$$

Then the following estimates hold:

$$\|u - I_N^{\alpha,\beta}u\|_{L^2_{\omega^{\alpha,\beta}}(I)} \leq CN^{-m}|u|_{H^{m,N}_{\omega^{\alpha,\beta}}(I)}, \tag{3.3a}$$

$$\|u - I_N^{\alpha,\beta}u\|_{L^\infty(I)} \leq \begin{cases} CN^{\frac{1}{2}-m}|u|_{H^{m,N}_{\omega^c}(I)}, & -1 \leq \alpha,\beta < -\frac{1}{2}, \\ CN^{1+\varepsilon-m}logN|u|_{H^{m,N}_{\omega^c}(I)}, & otherwise, \end{cases} \tag{3.3b}$$

where $\varepsilon = \max(\alpha,\beta)$ *and* $\omega^c = \omega^{-\frac{1}{2},-\frac{1}{2}}$ *denotes the Chebyshev weight function.*

Lemma 3.3 (*see* [32]) *For every bounded function* v*, there exists a constant* C*, independent of* v*, such that*

$$\sup_N \left\| \sum_{j=0}^{N} v(x_j)F_j(x) \right\|_{L^2_{\omega^{\alpha,\beta}}(I)} \leq C\max_{x\in[-1,1]}|v(x)|,$$

where $F_j(x)$, $j = 0, 1, \cdots, N$*, are the Lagrange interpolation basis functions associated with the Jacobi collocation points* $\{x_j\}_{j=0}^{N}$

Lemma 3.4 (*see* [33]) *Assume that* $\{F_j(x)\}_{j=0}^{N}$ *are the* N*–th degree Lagrange basis polynomials associated with the Gauss points of the Jacobi polynomials. Then,*

$$\|I_N^{\alpha,\beta}\|_{L^\infty(I)} \leq \max_{x\in[-1,1]} \sum_{j=0}^{N} |F_j(x)| = \begin{cases} \mathcal{O}(\log N), & -1 < \alpha, \beta \leq -\frac{1}{2}, \\ \mathcal{O}(N^{\varepsilon+\frac{1}{2}}), & otherwise. \end{cases} \tag{3.4}$$

Lemma 3.5 (*Gronwall inequality, see* [34] Lemma 7.1.1) *Suppose* $L \geq 0$, $0 < \mu < 1$, *and* u *and* v *are a non-negative, locally integrable functions defined on* $[-1, 1]$ *satisfying*

$$u(x) \leq v(x) + L\int_{-1}^{x} (x-\tau)^{-\mu} u(\tau)d\tau.$$

Then there exists a constant $C = C(\mu)$ *such that*

$$u(x) \leq v(x) + CL\int_{-1}^{x} (x-\tau)^{-\mu} v(\tau)d\tau, \quad for \quad -1 \leq x < 1.$$

If a nonnegative integrable function $E(x)$ *satisfies*

$$E(x) \leq L\int_{-1}^{x} E(s)ds + J(x), \quad -1 < x \leq 1,$$

where $J(x)$ *is an integrable function, then*

$$\begin{aligned} \|E\|_{L^\infty(-1,1)} &\leq C\|J\|_{L^\infty(-1,1)}, \\ \|E\|_{L^p_{\omega^{\alpha,\beta}}(-1,1)} &\leq C\|J\|_{L^p_{\omega^{\alpha,\beta}}(-1,1)}, \quad p \geq 1. \end{aligned} \tag{3.5}$$

Lemma 3.6 (*see* [35, 36]) *For a nonnegative integer* r *and* $\kappa \in (0, 1)$*, there exists a constant* $C_{r,\kappa} > 0$ *such that for any function* $v \in C^{r,\kappa}([-1, 1])$*, there exists a polynomial function* $\mathcal{T}_N v \in \mathcal{P}_N$ *such that*

$$\|v - \mathcal{T}_N v\|_{L^\infty(I)} \leq C_{r,\kappa} N^{-(r+\kappa)} \|v\|_{r,\kappa}, \tag{3.6}$$

where $\|\cdot\|_{r,\kappa}$ *is the standard norm in* $C^{r,\kappa}([-1, 1])$*,* $\mathcal{T}_N$ *is a linear operator from* $C^{r,\kappa}([-1, 1])$ *into* $\mathcal{P}_N$*, as stated in [35, 36].*

Lemma 3.7 (*see* [37]) *Let* $\kappa \in (0, 1)$ *and let* $\mathcal{M}$ *be defined by*

$$(\mathcal{M}v)(x) = \int_{-1}^{x} (x-\tau)^{-\mu} K(x, \tau)v(\tau)d\tau.$$

Then, for any function $v \in C([-1, 1])$*, there exists a positive constant* C *such that*

$$\frac{|\mathcal{M}v(x') - \mathcal{M}v(x'')|}{|x' - x''|} \leq C \max_{x\in[-1,1]} |v(x)|,$$

under the assumption that $0 < \kappa < 1 - \mu$*, for any* $x', x'' \in [-1, 1]$ *and* $x' \neq x''$*. This implies that*

$$\|\mathcal{M}v\|_{0,\kappa} \leq C \max_{x\in[-1,1]} |v(x)|, \quad 0 < \kappa < 1 - \mu.$$

Now, we devote to providing a convergence analysis for the numerical scheme. The goal is to show that the rate of convergence is exponential, that is, that spectral accuracy can be obtained for the proposed approximations.

Theorem 3.1 *Let $\bar{u}(\bar{x},\bar{t})$ be the solution of the continuous problem (2.13), $\bar{u}^M(\bar{x},\bar{t}) = \sum_{m=0}^{M} \bar{u}^m(\bar{x})F_m(\bar{t})$ be the time-discrete solution of (2.21), $\overline{\varepsilon} = \max(\overline{\alpha},\overline{\beta})$. If $\bar{u}(\bar{x},\bar{t})$, $\bar{u}_{\bar{x}}(\bar{x},\bar{t})$ and $\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t}) \in H^{m,M}_{\omega^{\overline{\alpha},\overline{\beta}}}(I)$ for t, $m > 1$, then for M sufficiently large,*

$$\|\bar{u}(\bar{x},\bar{t}) - \bar{u}^M(\bar{x},\bar{t})\|_{L^\infty} \leq \begin{cases} CM^{\frac{1}{2}-m}Q_1(\bar{x}), & -1 \leq \overline{\alpha},\overline{\beta} < -\frac{1}{2}, \\ CM^{1+\overline{\varepsilon}-m}\log M Q_1(\bar{x}), & -\frac{1}{2} \leq \overline{\varepsilon} < \frac{1}{2}-\mu, \end{cases} \tag{3.7a}$$

$$|\bar{u}(\bar{x},\bar{t}) - \bar{u}^M(\bar{x},\bar{t})\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} \leq \begin{cases} CM^{-m}\left(Q_2(\bar{x}) + M^{\frac{1}{2}-\kappa_1}Q_1(\bar{x})\right), \\ \qquad -1 \leq \overline{\alpha},\overline{\beta} < -\frac{1}{2}, \\ CM^{-m}\log M\left(Q_2(x) + M^{1+\overline{\varepsilon}-\kappa_1}Q_1(\bar{x})\right), \\ \qquad -\frac{1}{2} \leq \overline{\varepsilon} < \frac{1}{2}-\gamma_1, \end{cases} \tag{3.7b}$$

for $\kappa_1 \in (0, 1-\gamma_1)$, where C is a constant independent of M,

$$Q_1(x) = |\bar{u}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}} + |\bar{u}_{\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}} + |\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}},$$
$$Q_2(x) = |\bar{u}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^{\overline{\alpha},\overline{\beta}}}} + |\bar{u}_{\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^{\overline{\alpha},\overline{\beta}}}} + |\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^{\overline{\alpha},\overline{\beta}}}}.$$

Proof Using the boundary conditions, we have

$$\begin{aligned} \bar{u}(\bar{x},\bar{t}_j) &= \int_{-1}^{\bar{x}} \bar{u}_\tau(\tau,\bar{t}_j)d\tau + \overline{\psi}_1(\bar{t}_j), \\ \bar{u}_{\bar{x}}(\bar{x},\bar{t}_j) &= \int_{\xi}^{\bar{x}} \bar{u}_{\tau\tau}(\tau,\bar{t}_j)d\tau + \frac{\overline{\psi}_2(\bar{t}_j) - \overline{\psi}_1(\bar{t}_j)}{2}, \end{aligned} \tag{3.8}$$

where $\partial_{\bar{x}}\bar{u}(\bar{x},\bar{t})|_{\bar{x}=\xi} = \frac{\overline{\psi}_2(\bar{t})-\overline{\psi}_1(\bar{t})}{2}$. Consequently, $\bar{u}^M(\bar{x},\bar{t})$ satisfies the following collocation equations for $0 \leq j \leq M$,

$$\begin{aligned} \bar{u}^j(x) &= \int_{-1}^{\bar{x}} \bar{u}^M_\tau(\tau,\bar{t}_j)d\tau + \overline{\psi}_1(\bar{t}_j), \\ \bar{u}^M_{\bar{x}}(x,\bar{t}_j) &= \int_{\xi}^{\bar{x}} \bar{u}^M_{\tau\tau}(\tau,\bar{t}_j)d\tau + \frac{\overline{\psi}_2(\bar{t}_j) - \overline{\psi}_1(\bar{t}_j)}{2}. \end{aligned} \tag{3.9}$$

As $L \geq M, P \geq M$ $\bar{u}^M(\bar{x},\bar{t}) = \sum_{m=0}^{M} \bar{u}^m(\bar{x})F_m(\bar{t}) \in \mathcal{P}_M(\bar{t})$, we have

$$\begin{aligned} \int_{-1}^{1} (1-\theta)^{-\gamma_1}\bar{u}^M_{\bar{x}\bar{x}}(\bar{x},s_j(\theta))d\theta &= \sum_{k=0}^{L} \bar{u}^M_{\bar{x}\bar{x}}(\bar{x},s_j(\theta_k))\omega_k^{-\gamma_1,0}, \\ \int_{-1}^{1} (1-\theta)^{-\gamma_2}\bar{u}^M(\bar{x},s_j(\theta))d\theta &= \sum_{k=0}^{P} \bar{u}^M(\bar{x},s_j(\vartheta_k))\omega_k^{-\gamma_2,0}. \end{aligned}$$

Subtracting (2.17) from (2.21), subtracting (3.8) from (3.9), yields

$$
\begin{aligned}
\bar{u}(\bar{x},\bar{t}_j) - \bar{u}^j(\bar{x}) &= \zeta_j \int_{-1}^{\bar{t}_j} (\bar{t}_j - s)^{-\gamma_1} \left(\bar{u}_{\bar{x}\bar{x}}(\bar{x},s) - \bar{u}^M_{\bar{x}\bar{x}}(\bar{x},s)\right) ds \\
&\quad + \eta_j \int_{-1}^{\bar{t}_j} (\bar{t}_j - s)^{-\gamma_2} \left(\bar{u}(\bar{x},s) - \bar{u}^M(\bar{x},s)\right) ds \\
\bar{u}(\bar{x},\bar{t}_j) - \bar{u}^j(\bar{x}) &= \int_{-1}^{\bar{x}} \bar{u}_\tau(\tau,\bar{t}_j) d\tau - \int_{-1}^{\bar{x}} \bar{u}^M_\tau(\tau,\bar{t}_j) d\tau, \\
\bar{u}_{\bar{x}}(\bar{x},\bar{t}_j) - \bar{u}^j_{\bar{x}}(\bar{x}) &= \int_{\xi}^{\bar{x}} \bar{u}_{\tau\tau}(\tau,\bar{t}_j) d\tau - \int_{\xi}^{\bar{x}} \bar{u}^M_{\tau\tau}(\tau,\bar{t}_j) d\tau.
\end{aligned} \tag{3.10}
$$

Multiplying by $F_j(\bar{t})$ both sides of (3.10) and summing from 0 to M for j yield

$$
\begin{aligned}
I_M^{\bar{\alpha},\bar{\beta}} \bar{u}(\bar{x},\bar{t}) - \bar{u}^M(\bar{x},\bar{t}) &= I_M^{\bar{\alpha},\bar{\beta}} \zeta(\bar{t}) \int_{-1}^{\bar{t}} (\bar{t} - s)^{-\gamma_1} \left(\bar{u}_{\bar{x}\bar{x}}(\bar{x},s) - \bar{u}^M_{\bar{x}\bar{x}}(\bar{x},s)\right) ds \\
&\quad + I_M^{\bar{\alpha},\bar{\beta}} \eta(\bar{t}) \int_{-1}^{\bar{t}} (\bar{t}_j - s)^{-\gamma_2} \left(\bar{u}(\bar{x},s) - \bar{u}^M(\bar{x},s)\right) ds, \\
I_M^{\bar{\alpha},\bar{\beta}} \bar{u}(\bar{x},\bar{t}) - \bar{u}^M(\bar{x},\bar{t}) &= I_M^{\bar{\alpha},\bar{\beta}} \int_{-1}^{\bar{x}} \bar{u}_\tau(\tau,\bar{t}) d\tau - \int_{-1}^{\bar{x}} \bar{u}^M_\tau(\tau,\bar{t}) d\tau, \\
I_M^{\bar{\alpha},\bar{\beta}} \bar{u}_{\bar{x}}(\bar{x},\bar{t}) - \bar{u}^M_{\bar{x}}(\bar{x},\bar{t}) &= I_M^{\bar{\alpha},\bar{\beta}} \int_{\xi}^{\bar{x}} \bar{u}_{\tau\tau}(\tau,\bar{t}) d\tau - \int_{\xi}^{\bar{x}} \bar{u}^M_{\tau\tau}(\tau,\bar{t}) d\tau.
\end{aligned} \tag{3.11}
$$

Let

$$
\begin{gathered}
E(\bar{x},\bar{t}) = \bar{u}(\bar{x},\bar{t}) - \bar{u}^M(\bar{x},\bar{t}), \quad E_{\bar{x}}(\bar{x},\bar{t}) = \bar{u}_{\bar{x}}(\bar{x},\bar{t}) - \bar{u}^M_{\bar{x}}(\bar{x},\bar{t}), \\
E_{\bar{x}\bar{x}}(\bar{x},\bar{t}) = \bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t}) - \bar{u}^M_{\bar{x}\bar{x}}(\bar{x},\bar{t}).
\end{gathered}
$$

It follows from (3.11) that

$$
\begin{aligned}
E(\bar{x},\bar{t}) &= \zeta(\bar{t}) \int_{-1}^{\bar{t}} (\bar{t} - s)^{-\gamma_1} E_{\bar{x}\bar{x}}(\bar{x},s) ds + \eta(\bar{t}) \int_{-1}^{\bar{t}} (\bar{t} - s)^{-\gamma_2} E(\bar{x},s) ds \\
&\quad + K_1(\bar{x},\bar{t}) + K_2(\bar{x},\bar{t}) + K_3(\bar{x},\bar{t}), \\
E(\bar{x},\bar{t}) &= \int_{-1}^{\bar{x}} E_\tau(\tau,\bar{t}) d\tau + K_1(\bar{x},\bar{t}) + K_4(\bar{x},\bar{t}), \\
E_{\bar{x}}(\bar{x},\bar{t}) &= \int_{\xi}^{\bar{x}} E_{\tau\tau}(\tau,\bar{t}) d\tau + K_5(\bar{x},\bar{t}) + K_6(\bar{x},\bar{t}),
\end{aligned} \tag{3.12}
$$

where

$$
\begin{aligned}
K_1(\bar{x},\bar{t}) &= \bar{u}(\bar{x},\bar{t}) - I_M^{\bar{\alpha},\bar{\beta}} \bar{u}(\bar{x},\bar{t}), \\
K_2(\bar{x},\bar{t}) &= I_M^{\bar{\alpha},\bar{\beta}} \zeta(\bar{t}) \int_{-1}^{\bar{t}} (\bar{t} - s)^{-\gamma_1} E_{\bar{x}\bar{x}}(\bar{x},s) ds - \zeta(\bar{t}) \int_{-1}^{\bar{t}} (\bar{t} - s)^{-\gamma_1} E_{\bar{x}\bar{x}}(\bar{x},s) ds, \\
K_3(\bar{x},\bar{t}) &= I_M^{\bar{\alpha},\bar{\beta}} \eta(\bar{t}) \int_{-1}^{\bar{t}} (\bar{t} - s)^{-\gamma_2} E(\bar{x},s) ds - \eta(\bar{t}) \int_{-1}^{\bar{t}} (\bar{t} - s)^{-\gamma_2} E(\bar{x},s) ds, \\
K_4(\bar{x},\bar{t}) &= I_M^{\bar{\alpha},\bar{\beta}} \int_{-1}^{\bar{x}} E_\tau(\tau,\bar{t}) d\tau - \int_{-1}^{\bar{x}} E_\tau(\tau,\bar{t}) d\tau,
\end{aligned}
$$

$$K_5(\bar{x},\bar{t}) = \bar{u}_{\bar{x}}(\bar{x},\bar{t}) - I_M^{\overline{\alpha},\overline{\beta}}\bar{u}_{\bar{x}}(\bar{x},\bar{t}),$$
$$K_6(\bar{x},\bar{t}) = I_M^{\overline{\alpha},\overline{\beta}}\int_{\xi}^{\bar{x}}\bar{u}_{\tau\tau}^M(\tau,\bar{t})d\tau - \int_{\xi}^{\bar{x}}\bar{u}_{\tau\tau}^M(\tau,\bar{t})d\tau.$$

Furtherly, we have

$$|E(\bar{x},\bar{t})| \le C_1\int_{-1}^{\bar{t}}(\bar{t}-s)^{-\gamma_1}|E(\bar{x},s)|ds + C_2\int_{-1}^{\bar{t}}(\bar{t}-s)^{-\gamma_2}|E(\bar{x},s)|ds + \sum_{i=1,5,7,8,9}|K_i(\bar{x},\bar{t})|, \tag{3.13}$$

where

$$K_7(\bar{x},\bar{t}) = \bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t}) - I_M^{\overline{\alpha},\overline{\beta}}\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t}),$$
$$K_8(\bar{x},\bar{t}) = I_M^{\overline{\alpha},\overline{\beta}}C_1\int_{-1}^{\bar{t}}(\bar{t}-s)^{-\gamma_1}|E(\bar{x},s)|ds - C_1\int_{-1}^{\bar{t}}(\bar{t}-s)^{-\gamma_1}|E(\bar{x},s)|ds,$$
$$K_9(\bar{x},\bar{t}) = I_M^{\overline{\alpha},\overline{\beta}}C_2\int_{-1}^{\bar{t}}(\bar{t}-s)^{-\gamma_2}|E(\bar{x},s)|ds - C_2\int_{-1}^{\bar{t}}(\bar{t}-s)^{-\gamma_2}|E(\bar{x},s)|ds.$$

Using the Gronwall inequality in Lemma 3.5, we deduce that

$$\|E(\bar{x},\bar{t})\|_{L^\infty(I)} \le \sum_{i=1,5,7,8,9}|K_i(\bar{x},\bar{t})|_{L^\infty(I)}. \tag{3.14}$$

and

$$\|E(\bar{x},\bar{t})\|_{L^2_{\overline{\alpha},\overline{\beta}}(I)} \le \sum_{i=1,5,7,8,9}|K_i(\bar{x},\bar{t})|_{L^2_{\overline{\alpha},\overline{\beta}}}. \tag{3.15}$$

Using the L^∞ and weighted $L^2_{\omega^{\alpha,\beta}}$ error bounds for the interpolation polynomials (see Lemma 3.2) gives

$$\|K_1(\bar{x},\bar{t})\|_{L^\infty} \le \begin{cases} CM^{\frac{1}{2}-m}|\bar{u}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}}, & -1 \le \overline{\alpha},\overline{\beta} < -\frac{1}{2}, \\ CM^{1+\bar{\varepsilon}-m}\log M|\bar{u}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}}, & \text{otherwise}, \end{cases} \tag{3.16}$$

$$\|K_5(\bar{x},\bar{t})\|_{L^\infty} \le \begin{cases} CM^{\frac{1}{2}-m}|\bar{u}_{\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}}, & -1 \le \overline{\alpha},\overline{\beta} < -\frac{1}{2}, \\ CM^{1+\bar{\varepsilon}-m}\log M|\bar{u}_{\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}}, & \text{otherwise}, \end{cases} \tag{3.17}$$

$$\|K_7(\bar{x},\bar{t})\|_{L^\infty} \le \begin{cases} CM^{\frac{1}{2}-m}|\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}}, & -1 \le \overline{\alpha},\overline{\beta} < -\frac{1}{2}, \\ CM^{1+\bar{\varepsilon}-m}\log M|\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}}, & \text{otherwise}, \end{cases} \tag{3.18}$$

$$\|K_1(\bar{x},\bar{t})\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} \le CM^{-m}|\bar{u}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^{\overline{\alpha},\overline{\beta}}}}. \tag{3.19}$$

$$\|K_5(\bar{x},\bar{t})\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} \le CM^{-m}|\bar{u}_{\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^{\overline{\alpha},\overline{\beta}}}}. \tag{3.20}$$

$$\|K_7(\bar{x},\bar{t})\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} \le CM^{-m}|\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^{\overline{\alpha},\overline{\beta}}}}. \tag{3.21}$$

Finally, it follows from Lemma 3.6, Lemma 3.7, and Lemma 3.4 that

$$
\begin{aligned}
\|K_8(\bar{x},\bar{t})\|_{L^\infty} &= \|(I_M^{\overline{\alpha},\overline{\beta}} - I)\mathcal{M}|E(\bar{x},s)|\|_{L^\infty} \\
&= \|(I_M^{\overline{\alpha},\overline{\beta}} - I)(\mathcal{M}|E(\bar{x},s)| - \mathcal{T}_M\mathcal{M}|E(\bar{x},s)|)\|_{L^\infty} \\
&\le \left(1 + \|I_M^{\overline{\alpha},\overline{\beta}}\|_{L^\infty(I)}\right) CM^{-\kappa_1}\|\mathcal{M}|E(\bar{x},s)|\|_{0,\kappa_1}, \quad \kappa_1 \in (0, 1-\gamma_1) \\
&\le \begin{cases} CM^{-\kappa_1}\log M\||E(\bar{x},s)|\|_{L^\infty}, & -1 \le \overline{\alpha},\overline{\beta} < -\frac{1}{2}, \\ CM^{\frac{1}{2}+\overline{\varepsilon}-\kappa_1}\||E(\bar{x},s)|\|_{L^\infty}, & -\frac{1}{2} \le \overline{\varepsilon} < \frac{1}{2} - \gamma_1, \end{cases} \qquad (3.22)\\
\|K_8(\bar{x},\bar{t})\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} &= \|(I_M^{-\mu,-\mu} - I)\mathcal{M}|E(\bar{x},s)|\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} \\
&= \|(I_M^{-\mu,-\mu} - I)(\mathcal{M}|E(\bar{x},s)| - \mathcal{T}_M|E(\bar{x},s)|\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} \\
&\le \|I_M^{\overline{\alpha},\overline{\beta}}(\mathcal{M}|E(\bar{x},s)| - \mathcal{T}_M|E(\bar{x},s)|)\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} + \|\mathcal{M}|E(\bar{x},s)| - \mathcal{T}_M|E(\bar{x},s)|\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} \\
&\le C\|\mathcal{M}|E(\bar{x},s)| - \mathcal{T}_M|E(\bar{x},s)|\|_{L^\infty} \\
&\le CM^{-\kappa_1}\|\mathcal{M}|E(\bar{x},s)|\|_{0,\kappa_1} \\
&\le CM^{-\kappa_1}\||E(\bar{x},s)|\|_{L^\infty}. \qquad (3.23)
\end{aligned}
$$

Similarly, we have

$$
\|K_9(\bar{x},\bar{t})\|_{L^\infty} \le \begin{cases} CM^{-\kappa_2}\log M\||E(\bar{x},s)|\|_{L^\infty}, & -1 \le \overline{\alpha},\overline{\beta} < -\frac{1}{2}, \\ CM^{\frac{1}{2}+\overline{\varepsilon}-\kappa_2}\||E(\bar{x},s)|\|_{L^\infty}, & -\frac{1}{2} \le \overline{\varepsilon} < \frac{1}{2} - \gamma_2, \end{cases} \qquad (3.24)
$$

$$
\|K_9(\bar{x},\bar{t})\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} \le CM^{-\kappa_2}\||E(\bar{x},s)|\|_{L^\infty}, \quad \kappa_2 \in (0, 1-\gamma_2). \qquad (3.25)
$$

Noting $\gamma_1 > \gamma_2$, estimates (3.16), (3.17), (3.18), (3.22), and (3.24) lead to (3.7a), estimates (3.19), (3.20), (3.21), (3.23), and (3.25), lead to (3.7b). ∎

Theorem 3.2 *Let $\bar{u}^M(\bar{x},\bar{t})$ be the time-discrete solution of (2.21) and $\bar{u}_N^M(\bar{x},\bar{t})$ the solution of the full-discrete problem (2.24) with boundary conditions (2.14). If $\bar{u}^M(\bar{x},\bar{t})$, $\bar{u}_{\bar{x}}^M(\bar{x},\bar{t})$ and $\bar{u}_{\bar{x}\bar{x}}^M(\bar{x},\bar{t}) \in H^{n,N}_{\omega^{\tilde{\alpha},\tilde{\beta}}}$ for $n > 1$, $\tilde{\varepsilon} = \max(\tilde{\alpha},\tilde{\beta})$, $\overline{\varepsilon} = \max(\overline{\alpha},\overline{\beta})$, then the following error estimate holds*

$$
\|\bar{u}^M(\bar{x},\bar{t}) - \bar{u}_N^M(\bar{x},\bar{t})\|_{L^\infty} \le \begin{cases} CN^{\frac{1}{2}-n}\log MV_1, & -1 \le \tilde{\alpha},\tilde{\beta},\overline{\alpha},\overline{\beta} < -\frac{1}{2}, \\ CN^{1+\tilde{\varepsilon}-n}\log NM^{1+\overline{\varepsilon}}V_1, & \textit{otherwise}, \end{cases}
$$

$$
\|\bar{u}^M(\bar{x},\bar{t}) - \bar{u}_N^M(\bar{x},\bar{t})\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} \le CN^{-n}\log MV_2, \qquad (3.26)
$$

where

$$
V_1 = \max_{\bar{t}\in[-1,1]}\left(|\bar{u}^M(\bar{x},\bar{t})|_{H^{n,N}_{\omega^c}} + |\bar{u}_{\bar{x}\bar{x}}^M(\bar{x},\bar{t})|_{H^{n,N}_{\omega^c}}\right),
$$

$$
V_2 = \max_{\bar{t}\in[-1,1]}\left(|\bar{u}^M(\bar{x},\bar{t})|_{H^{n,N}_{\omega^{\tilde{\alpha},\tilde{\beta}}}} + |\bar{u}_{\bar{x}\bar{x}}^M(\bar{x},\bar{t})|_{H^{n,N}_{\omega^{\tilde{\alpha},\tilde{\beta}}}}\right),
$$

C is independent of N and M

Proof By subtracting (2.22) from (2.24), we obtain

$$
\begin{aligned}
\bar{u}^j(\bar{x}_i) - \bar{u}_i^j &= \zeta \int_{-1}^{\bar{t}_j} (\bar{t}_j - s)^{-\gamma_1} \left(\bar{u}^M_{\bar{x}\bar{x}}(\bar{x}_i, s) - \bar{u}^M_{N\,\bar{x}\bar{x}}(\bar{x}_i, s)\right) ds \\
&\quad + \eta \int_{-1}^{\bar{t}_j} (\bar{t}_j - s)^{-\gamma_2} \left(\bar{u}^M(\bar{x}_i, s) - \bar{u}^M_N(\bar{x}_i, s)\right) ds \\
&= \zeta \int_{-1}^{\bar{t}_j} (\bar{t}_j - s)^{-\gamma_1} \sum_{m=0}^{M} \left(\bar{u}^m_{\bar{x}\bar{x}}(\bar{x}_i) - \sum_{n=0}^{N} \bar{u}^m_n H_n''(\bar{x}_i)\right) F_m(s) ds \\
&\quad + \eta \int_{-1}^{\bar{t}_j} (\bar{t}_j - s)^{-\gamma_2} \sum_{m=0}^{M} \left(\bar{u}^m(\bar{x}_i) - \sum_{n=0}^{N} \bar{u}^m_n H_n(\bar{x}_i)\right) F_m(s) ds \\
&= \sum_{m=0}^{M} \left[C_1^{jm} \left(\bar{u}^m_{\bar{x}\bar{x}}(\bar{x}_i) - \sum_{n=0}^{N} \bar{u}^m_n H_n''(\bar{x}_i)\right) + C_2^{jm} \left(\bar{u}^m(\bar{x}_i) - \bar{u}^m_i)\right)\right],
\end{aligned} \tag{3.27}
$$

where

$$
C_1^{jm} = \zeta \int_{-1}^{\bar{t}_j} (\bar{t}_j - s)^{-\gamma_1} F_m(s) ds, \quad C_2^{jm} = \eta \int_{-1}^{\bar{t}_j} (\bar{t}_j - s)^{-\gamma_2} F_m(s) ds.
$$

Multiplying by $H_i(x)$ both sides of (3.27) and summing from 0 to N for i, and noting that $\sum_{n=0}^{N} \bar{u}^m_n H_n''(\bar{x}) \in \mathcal{P}_N(\bar{x})$, then $I_N^{\tilde{\alpha},\tilde{\beta}} \sum_{n=0}^{N} \bar{u}^m_n H_n''(\bar{x}_i) = \sum_{n=0}^{N} \bar{u}^m_n H_n''(\bar{x})$, yield

$$
\begin{aligned}
&I_N^{\tilde{\alpha},\tilde{\beta}} \bar{u}^j(\bar{x}) - \sum_{i=0}^{N} \bar{u}^j_i H_i(\bar{x}) \\
&= \sum_{m=0}^{M} \left[C_1^{jm} \left(I_N^{\tilde{\alpha},\tilde{\beta}} \bar{u}^m_{\bar{x}\bar{x}}(\bar{x}) - \sum_{n=0}^{N} \bar{u}^m_n H_n''(\bar{x}) \right) + C_2^{jm} \left(I_N^{\tilde{\alpha},\tilde{\beta}} \bar{u}^m(\bar{x}) - \sum_{n=0}^{N} \bar{u}^m_n H_n(\bar{x}) \right) \right].
\end{aligned} \tag{3.28}
$$

Let

$$
\begin{aligned}
e(\bar{x}, \bar{t}_j) = e^j(\bar{x}) &= \bar{u}^j(\bar{x}) - \sum_{i=0}^{N} \bar{u}^j_i H_i(\bar{x}), \\
e_{\bar{x}}(\bar{x}, \bar{t}_j) = e^j_{\bar{x}}(\bar{x}) &= \bar{u}^j_{\bar{x}}(\bar{x}) - \sum_{i=0}^{N} \bar{u}^j_i H_i'(\bar{x}) \\
e_{\bar{x}\bar{x}}(\bar{x}, \bar{t}_j) = e^j_{\bar{x}\bar{x}}(\bar{x}) &= \bar{u}^j_{\bar{x}\bar{x}}(\bar{x}) - \sum_{i=0}^{N} \bar{u}^j_i H_i''(\bar{x}).
\end{aligned}
$$

Thus, we have

$$
\begin{gathered}
|e^j| \leq C(|J_1(\bar{x})| + |J_2(\bar{x})|) \\
J_1(x) = \bar{u}^j(\bar{x}) - I_N^{\tilde{\alpha},\tilde{\beta}} \bar{u}^j(\bar{x}), \quad J_2(x) = \bar{u}^j_{\bar{x}\bar{x}}(\bar{x}) - I_N^{\tilde{\alpha},\tilde{\beta}} \bar{u}^j_{\bar{x}\bar{x}}(\bar{x}).
\end{gathered} \tag{3.29}
$$

Similar to the proof of Theorem 3.1, using the L^∞ and $L^2_{\omega^{\alpha,\beta}}$ error bounds for the interpolation polynomials in Lemma 3.2, gives

$$\begin{aligned}
\|J_1(\bar{x})\|_{L^\infty} &\le \begin{cases} CN^{\frac{1}{2}-n}|\bar{u}^j(\bar{x})|_{H^{n,N}_{\omega^c}}, & -1 \le \tilde{\alpha},\tilde{\beta} < -\frac{1}{2}, \\ CN^{1+\varepsilon_x-n}\log N|\bar{u}^j(\bar{x})|_{H^{n,N}_{\omega^c}}, & \text{otherwise}, \end{cases} \\
\|J_2(\bar{x})\|_{L^\infty} &\le \begin{cases} CN^{\frac{1}{2}-n}|\bar{u}^j_{\bar{x}\bar{x}}(\bar{x})|_{H^{n,N}_{\omega^c}}, & -1 \le \tilde{\alpha},\tilde{\beta} < -\frac{1}{2}, \\ CN^{1+\varepsilon_x-n}\log N|\bar{u}^j_{\bar{x}\bar{x}}(\bar{x})|_{H^{n,N}_{\omega^c}}, & \text{otherwise}, \end{cases}
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
\|J_1(x)\|_{L^2_{\tilde{\alpha},\tilde{\beta}}} &\le CN^{-n}|\bar{u}^j(\bar{x})|_{H^{n,N}_{\omega\tilde{\alpha},\tilde{\beta}}}, \\
\|J_2(x)\|_{L^2_{\omega^{\tilde{\alpha},\tilde{\beta}}}} &\le CN^{-n}|\bar{u}^j_{\bar{x}\bar{x}}(\bar{x})|_{H^{n,N}_{\omega\tilde{\alpha},\tilde{\beta}}}.
\end{aligned} \tag{3.31}$$

We note that

$$\begin{aligned}
\bar{u}^M(\bar{x},\bar{t}) - \bar{u}^M_N(\bar{x},\bar{t}) &= \sum_{j=0}^{M}\left(\bar{u}^j(\bar{x}) - \sum_{i=0}^{N}\bar{u}^j_i H_i(\bar{x})\right)F_j(\bar{t}) \\
&= \sum_{j=0}^{M} e(\bar{x},\bar{t}_j)F_j(\bar{t}) = I_M^{\overline{\alpha},\overline{\beta}} e(\bar{x},\bar{t}).
\end{aligned}$$

Using Lemma 3.4, we have

$$\|\bar{u}^M(\bar{x},\bar{t}) - \bar{u}^M_N(\bar{x},\bar{t})\|_{L^\infty} \le \begin{cases} C\ \log M\ \max_{\bar{t}\in[-1,1]}|e(\bar{x},\bar{t})|, & -1 \le \overline{\alpha},\overline{\beta} < -\frac{1}{2}, \\ CM^{1+\overline{\epsilon}}\max_{\bar{t}\in[-1,1]}|e(\bar{x},\bar{t})|, & \text{otherwise}. \end{cases} \tag{3.32}$$

By Lemma 3.3, we obtain

$$\|\bar{u}^M(\bar{x},\bar{t}) - \bar{u}^M_N(\bar{x},\bar{t})\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} \le C\ \log M \max_{\bar{t}\in[-1,1]}|e(\bar{x},\bar{t})|. \tag{3.33}$$

Then using (3.29)–(3.33) and above inequality, we will obtain estimate (3.26), the proof is completed. ∎

For the full-discrete error, we state the following result.

Theorem 3.3 *Let $\bar{u}(\bar{x},\bar{t})$ be the solution of the continuous problem (2.13) and let $\bar{u}^M_N(\bar{x},\bar{t})$ be the solution of the full-discrete problem (2.24) with the initial condition (1.2) and boundary conditions (2.10). If $\bar{u}(\bar{x},\bar{t})$, $\bar{u}_{\bar{x}}(\bar{x},\bar{t})$ and $\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t}) \in H^{n,N}_{\omega\tilde{\alpha},\tilde{\beta}} \otimes H^{m,M}_{\omega^{\overline{\alpha},\overline{\beta}}}$ for $\bar{x}$ and $\bar{t}$, with $n,m > 1$, $\tilde{\epsilon} = \max(\tilde{\alpha},\tilde{\beta})$, $\overline{\epsilon} = \max(\overline{\alpha},\overline{\beta})$, then for M and N sufficiently large, the following error estimate holds,*

$$\|\bar{u}(\bar{x},\bar{t}) - \bar{u}^M_N(\bar{x},\bar{t})\|_{L^\infty} \le \begin{cases} CM^{\frac{1}{2}-m}Q_1 + CN^{\frac{1}{2}-n}\log MV_1, \\ \qquad -1 \le \tilde{\alpha},\tilde{\beta},\overline{\alpha},\overline{\beta} < -\frac{1}{2}, \\ CM^{1+\overline{\epsilon}-m}\log MQ_1 + CN^{1+\tilde{\epsilon}-n}\log NM^{1+\epsilon_y}V_1, \\ \qquad -\frac{1}{2} \le \overline{\epsilon} < \mu - \frac{1}{2}, \end{cases}$$

$$\|\bar{u}(\bar{x},\bar{t})-\bar{u}_N^M(\bar{x},\bar{t})\|_{L^2_{\omega^{\overline{\alpha},\overline{\beta}}}} \le \begin{cases} CM^{-m}\left(Q_2+M^{\frac{1}{2}-\kappa}Q_1\right)+CN^{-n}\log MV_2, \\ \qquad -1\le\tilde{\alpha},\tilde{\beta},\overline{\alpha},\overline{\beta}<-\frac{1}{2}, \\ CM^{-m}\log M\left(Q_2+M^{1+\overline{\varepsilon}-\kappa}Q_1\right)+CN^{-n}\log MV_2, \\ \qquad -\frac{1}{2}\le\overline{\varepsilon}<\mu-\frac{1}{2}, \end{cases} \tag{3.34}$$

for $\kappa\in(0,1-\gamma_1)$

Proof Using the triangle inequality

$$|\bar{u}(x,y)-\bar{u}_N^M(x,y)|\le|\bar{u}(\bar{x},\bar{t})-\bar{u}^M(\bar{x},\bar{t})|+|\bar{u}^M(\bar{x},\bar{t})-\bar{u}_N^M(\bar{x},\bar{t})|,$$

Theorem 3.1 and Theorem 3.2, leads to estimation (3.34). ∎

4 | NUMERICAL EXPERIMENTS

In this section, we present numerical results obtained by the proposed space-time spectral method to support our theoretical statements. The main purpose is to check the convergence behavior of the numerical solution with respect to the time polynomial degree M and space polynomial degree N used in the calculation.

Example 4.1 Consider the following initial and boundary problems of the fractional cable equation

$$u_t(x,t)={}^RD_t^{1-\mu_1}u_{xx}(x,t)-{}^RD_t^{1-\mu_2}u(x,t),$$

with the initial and boundary condition

$$u(x,0)=10\delta(x-5),\quad 0\le x\le 10,$$
$$u(0,t)=u(10,t)=0,\quad 0\le t\le 0.5,$$

where $\delta(x)$ is the Dirac delta function. This problem does not exist an analytical solution.

In order to test the effectiveness of our method, we set $\mu_1=\mu_2=0.5$. Frist, we fix the number of time collocation points $M=36$, and investigate the behavior of the numerical solutions for $N=4,6,8,10$ with $t=0.5$. Form Figure 1, we can see that the numerical solution of $N=8$ is mostly overlapped with that of $N=10$. We also study the mesh refinement in time with fixing space points $N=38$, it shows similar results. Thus, we obtain that the numerical solution is convergent with the collocation points increased.

Example 4.2 Consider the following initial-boundary value problem of the fractional Cable equation

$$u_t(x,t)={}^RD_t^{1-\mu_1}u_{xx}(x,t)-{}^RD_t^{1-\mu_2}u(x,t)+f(x,t),$$

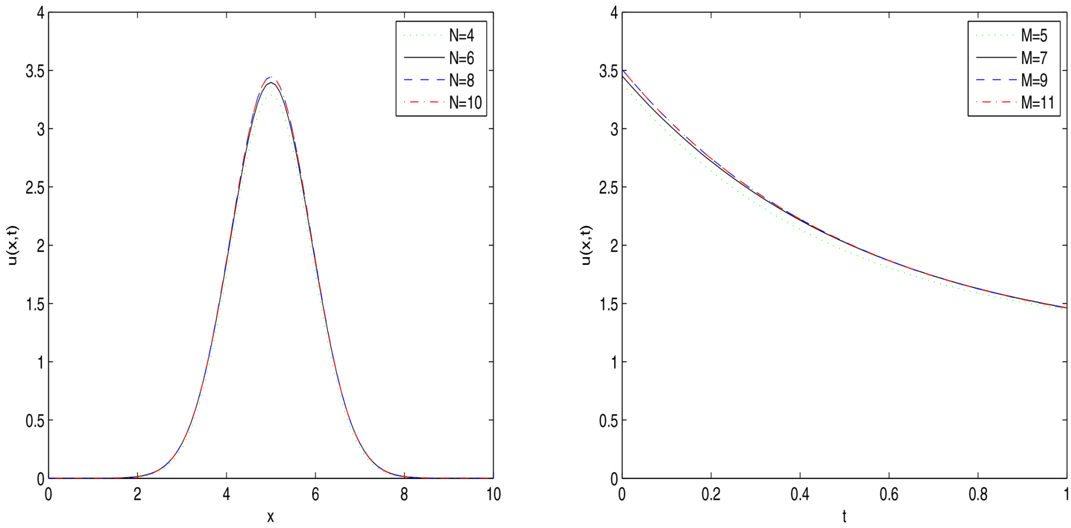

FIGURE 1 The behavior of numerical solutions for different N at $t = 0.5$ (left) and different M (right) at $x = 5$ with $\mu_1 = \mu_2 = 0.5$ [Color figure can be viewed at wileyonlinelibrary.com]

with the initial and boundary condition

$$u(x,0) = 0, \quad 0 \le x \le 1,$$
$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le 1,$$

where

$$f(x,t) = \left(2t + \frac{2\pi^2 t^{\mu_1+1}}{\Gamma(2+\mu_1)} + \frac{2t^{\mu_2+1}}{\Gamma(2+\mu_2)}\right)\sin(\pi x).$$

The exact solution is

$$u(x,t) = t^2 \sin(\pi x).$$

We first investigate the temporal errors. In this test, we fix the number of space collocation points $M = 24$, a value large enough such that the space discretization error is negligible as compared with the time error. In Figures 2, we plot the errors in the L^∞ norm and weighted L^2-norm in semilog scale varying with various of the number of time collocation points N and various values of the fractional derivatives μ_1 and μ_2. Similarly, we investigate the spatial error by letting the number of space collocation points M vary and fixing the number of time collocation points $N = 24$ to avoid contamination of the temporal errors. In Figure 3, we plot the errors as functions of M for $\alpha = \beta = 0.5$. It is clear that the spectral convergence is achieved both of spatial and temporal errors. This indicates that the convergence in space and time of the time-space collocation spectral method is exponential, as in these semilog representation one observes that the error variations are essentially linear versus the degrees of polynomial. This is the so-called spectral accuracy, which can be expected for smooth solutions.

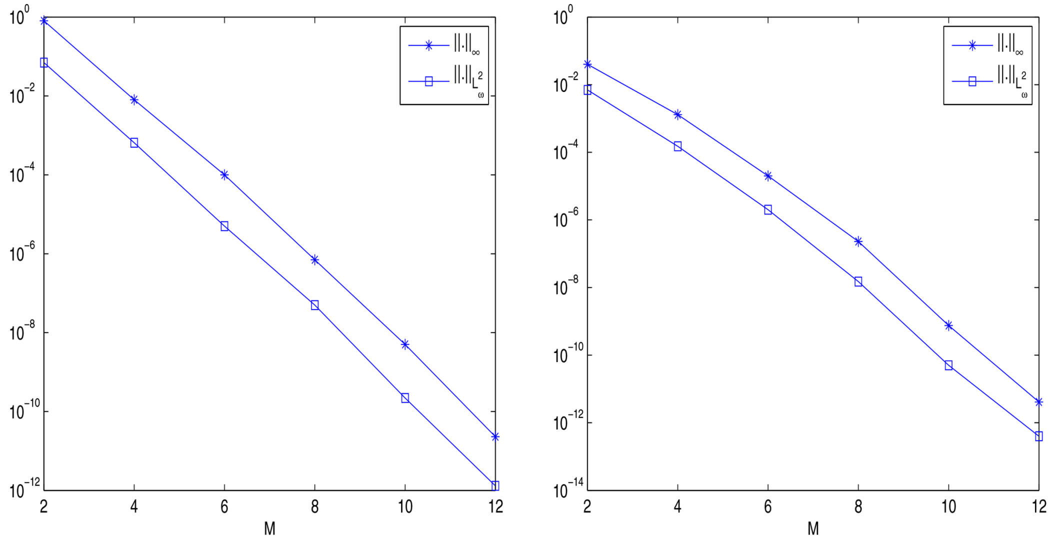

FIGURE 2 The errors as a function of time collocation points M for $\mu_1 = 0.2, \mu_2 = 0.8$ (left) and $\mu_1 = 0.9, \mu_2 = 0.1$ (right) with $N = 24$ [Color figure can be viewed at wileyonlinelibrary.com]

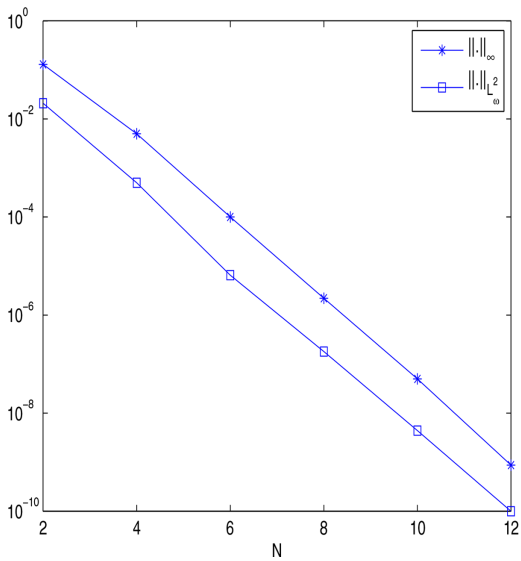

FIGURE 3 The errors as a function of space collocation points N for $\mu_1 = 0.5, \mu_2 = 0.5$ with $M = 24$ [Color figure can be viewed at wileyonlinelibrary.com]

5 | CONCLUSION

In this article, we consider the numerical solution of the time fractional Cable equation. First, we convert the time fractional Cable equation into an equivalent integral equation with singular kernel, then we propose a spectral collocation method in both time and space discretizations with a spectral expansion of Jacobi interpolation polynomial for this equation. The convergence of the method is rigorously established. Numerical tests are performed to confirm the theoretical results. The main advantage of the present scheme is that it gives very accurate convergency by choosing less number of grid points and the problem can be solved up to big time, and the storage requirement due to the time memory effect can be considerably reduced. At the same time, the technique is simple and easy to apply to multidimensional problems.

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REFERENCES

[1] F. Santamaria, S. Wils, E. Schutte, G.J. Augustine, *Anomalous diffusion in Purkinje cell dendrites caused by spines*, Neuron vol. 52 (2006) pp. 635-648.

[2] B. Henry, T. Langlands, S. Wearne, *Fractional diffusion-weighted MRI to anomalous diffusion imagine*, Magn. Reson. Med. vol. 59 (2008) pp. 447–455.

[3] T. Langlands, B. Henry, S. Wearne, *Fractional cable equation models for Anomalous electrodiffusion in nerve cells: Infinite domain solutions*, J. Math. Biol. vol. 59 (2009) pp. 761–808.

[4] T. Langlands, B. Henry, S. Wearne, *Fractional cable equation models for Anomalous electrodiffusion in nerve cells: Finite domain solutions*, SIAM J. Appl. Math. vol. 71 (2011) pp. 1168–1203.

[5] A. Atangana, D. Baleanu, *New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model*, Thermal Sci. vol. 20 (2016) pp. 763–769.

[6] A. Atangana, I. Koca, *Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order*, Chaos Solitons Fractals, vol. 89 (2016) pp. 447–454.

[7] A. Atangana, I. Koca, *On the new fractional derivative and application to nonlinear Baggs and Freedman model*, J. Nonlinear Sci. Appl. vol. 9 (2016) pp. 2467–2480.

[8] B. S. T. Alkahtani, A. Atangana, *Controlling the wave movement on the surface of shallow water with the Caputo-Fabrizio derivative with fractional order*, Chaos Solitons Fractals vol. 89 (2016) pp. 539–546.

[9] A. Atangana, D. Baleanu, *Caputo-Fabrizio derivative applied to groundwater flow within confined aquifer*, J. Eng. Mech. vol. 143 (2017) pp. D4016005.

[10] Z. Ding, A. Xiao, M. Li, *Weighted finite difference methods for a class of space fractional partial differential equations with variable coefficients*, J. Comput. Appl. Math. vol. 233 (2010) pp. 1905–1914.

[11] Y. Yang, Y. Chen, Y. Huang, *Spectral-collocation method for fractional Fredholm integro-differential equations*, J. Korean Math. Soc. vol. 51 (2014) pp. 203–224.

[12] S. Yang, A. Xiao, *Convergence of the variational iteration method for solving multi-order fractional differential equations*, Comput. Math. Appl. vol. 60 (2010) pp. 2871–2879.

[13] Z. Mao, A. Xiao, D. Wang, Z. Yu, L. Shi, *Exponentially accurate Rayleigh-Ritz method for fractional variational problems*, J. Comput. Nonlinear Dyn. vol. 10 (2015) pp. 051009.

[14] Y. Yang, Y. Chen, Y. Huang, *Convergence analysis of the Jacobi spectral-collocation method for fractional integro-differential equations*, Acta Math. Sci. vol. 34B (2014) pp. 673–690.

[15] F. Liu, Q. Yang, I. Turner, *Two new implicit numerical methods for the fractional cable equation*, J. Comput. Nonlinear Dyn. vol. 6 (2010) pp. 011009.

[16] I. Karatay, N. Kale, *A new difference scheme for fractional Cable equation and stability analysis*, Int. J. Appl. Math. Res. vol. 4 (2015) pp. 52–57.

[17] W. Bu, X. Liu, Y. Tang, J. Yang, *Finite element multigrid method for multi-term time fractional advection diffusion equations*, Int. J. Model. Simul. Sci. Comput. vol. 6 (2015) pp. 1540001.

[18] W. Bu, A. Xiao A, W. Zeng, *Finite difference/finite element methods for distributed-order time fractional diffusion equations*, J. Sci. Comput. vol. 72 (2017) pp. 422–441.

[19] Y. Yang, Y. Huang, Y. Zhou, *Numerical solutions for solving time fractional Fokker-Plank equations based on spectral collocation methods*, J. Comput. Appl. Math., 2017, in press, http://dx.doi.org/10.1016/j.cam.2017.04.003

[20] Y. Yang, Y. Chen, Y. Huang, H. Wei, *Spectral collocation method for the time-fractional diffusion-wave equation and convergence analysis*, Comput. Math. Appl. vol. 73 (2017) pp. 1218–1232.

[21] A. Bhrawy, M. Zaky, *Numerical simulation for two-dimensional variable-order fractional nonlinear Cable equation*, Nonlinear Dyn. vol. 80 (2015) pp. 101–116.

[22] N. Sweilam, M. Khader, M. Adel, *Numerical simulation of fractional Cable equation of spiny neuronal dendrites*, J. Adv. Res. vol. 5 (2014) pp. 253–259.

[23] Z. Mao, A. Xiao, Z. Yu, L. Shi, *Sinc-Chebyshev collocation method for a class of fractional diffusion-wave equations*, Sci. World J. vol. 2014 (2014) pp. 7.

[24] Z. Mao, A. Xiao, Z. Yu, L. Shi, *Finite difference and sinc-collocation approximations to a class of fractional diffusion-wave equations*, J. Appl. Math. vol. 2014 (2014) pp. 11.

[25] S. Yang, A. Xiao, *Cubic spline collocation method for fractional differential equations*, J. Appl. Math. vol. 2013 (2013) pp. 20.

[26] S. Yang, A. Xiao, *An efficient numerical method for fractional differential equations with two Caputao derivatives*, J. Comput. Math. vol. 34 (2016) pp. 113–134.

[27] Y. Yang, Y. Chen, Y. Huang, W. Yang, *Convergence analysis of Legendre-collocation methods for nonlinear Volterra type integral Equations*, Adv. Appl. Math. Mech. vol. 7 (2015) pp. 74–88.

[28] Y. Yang, Y. Chen, *Spectral collocation methods for nonlinear Volterra integro-differential equations with weakly singular kernels*, Bulletin Malaysian Math Sci Soc., 2017, in press, DOI 10.1007/s40840-017-0487-7.

[29] Y. Yang, *Jacobi spectral Galerkin methods for fractional integro-differential equations*, Calcolo vol. 52 (2015) pp. 519–542.

[30] Y. Yang, *Jacobi spectral Galerkin methods for Volterra integral equations with weakly singular kernel*, Bulletin Korean Math Soc vol. 53 (2016) pp. 247–262.

[31] C. Canuto, M. Hussaini, A. Quarteroni A, *Spectral Methods Funda-mentals in Single Domains*, Springer-Verlag, Berlin Heidelberg, 2006.

[32] P. Nevai, *Mean convergence of Lagrange interpolation: III*, Trans. Am. Math. Soc. vol. 282 (1984) pp. 669–698.

[33] G. Mastroianni, D. Occorsto, *Optimal systems of nodes for Lagrange interpolation on bounded intervals: A survey*, J. Comput. Appl. Math. vol. 134 (2001) pp. 325–341.

[34] D. Henry, *Geometric theory of semilinear parabolic equations*, Springer-Verlag, Berlin Heidelberg, 1989.

[35] D. Ragozin, *Polynomial approximation on compact manifolds and homogeneous spaces*, Trans. Am. Math. Soc. vol. 150 (1970) pp. 41–53.

[36] D. Ragozin, *Constructive polynomial approximation on spheres and projective spaces*, Trans. Am. Math. Soc. vol. 162 (1971) pp. 157–170.

[37] D. Colton, R. Kress, *Inverse coustic and electromagnetic scattering theory, applied mathematical sciences*, 2nd ed., Springer-Verlag, Heidelberg, 1998.

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