

EFFECTIVE INSEPARABILITY AND SOME APPLICATIONS IN META-MATHEMATICS

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ABSTRACT. Effectively inseparable pairs and their properties play an important role in the meta-mathematics of arithmetic and incompleteness. Different notions are introduced and shown in the literature to be equivalent to effective inseparability. We give a much simpler proof of these equivalences using the strong double recursion theorem. Then we prove some results about the application of effective inseparability in meta-mathematics.

1. INTRODUCTION

Since Gödel, research on incompleteness has greatly deepened our understanding of the incompleteness phenomenon. In particular, Smullyan's work in [20, 21] provides a unique way to understand incompleteness in an abstract way via metamathematical research of formal systems. This paper is inspired by Smullyan's work.

Recursion-theoretic proofs of metamathematical results tend to rely on an effectively inseparable pair of recursively enumerable (RE) ¹ sets and its properties. Effectively inseparable sets arise naturally in the meta-mathematics of arithmetic. For example, the pair of Gödel numbers of provable and refutable sentences of **PA** is effectively inseparable. The motivation of this paper is to study uniform versions of incompleteness/undecidability via the notion of effective inseparability. Especially, we study effectively inseparable theories that exhibit similar behaviors connected to incompleteness/undecidability.

In this paper, a theory is an RE theory of classical first-order logic in finite signature, and we identify a theory with the set of sentences provable in it. We always assume the *arithmetization* of the base theory and we will usually work with a bijective Gödel numbering of the sentences. Under arithmetization, we equate a set of sentences with the set of Gödel's numbers of sentences. For any formula ϕ , we use $\ulcorner \phi \urcorner$ to denote the Gödel number of ϕ . We first introduce the standard notions of recursively inseparable and effectively inseparable pairs of RE sets.

Definition 1.1 ([13]). Let (A, B) be a disjoint pair of RE sets.

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¹Some authors use “computable” instead of “recursive” in the literature.

- (1) We say (A, B) is *recursively inseparable* (RI) if there exists no recursive superset of A which is disjoint from B .
- (2) We say (A, B) is *effectively inseparable* (EI) if there is a recursive function $f(x, y)$ such that for any i and j , if $A \subseteq W_i$ and $B \subseteq W_j$ with $W_i \cap W_j = \emptyset$, then $f(i, j) \notin W_i \cup W_j$.

Effective inseparability can be viewed as an effective version of recursive inseparability. For a disjoint pair (A, B) of RE sets, if (A, B) is RI, then for any i and j , if $A \subseteq W_i$, $B \subseteq W_j$ and $W_i \cap W_j = \emptyset$, then $W_i \cup W_j \neq \mathbb{N}$; if (A, B) is EI, then we can effectively pick an element not in $W_i \cup W_j$.

In Definition 1.2, we introduce the notions of recursively inseparable theories and effectively inseparable theories which are based on the nuclei of a theory.

Definition 1.2 (The nuclei of a theory, Smullyan). Let T be a consistent RE theory.

- (1) The pair (T_P, T_R) are called the *nuclei* of the theory T , where T_P is the set of Gödel numbers of sentences provable in T , and T_R is the set of Gödel numbers of sentences refutable in T . In other words, $T_P = \{\ulcorner \phi \urcorner : T \vdash \phi\}$, and $T_R = \{\ulcorner \phi \urcorner : T \vdash \neg \phi\}$.
- (2) We say T is RI if (T_P, T_R) is recursively inseparable.
- (3) We say T is EI if (T_P, T_R) is effectively inseparable.

The nuclei of a theory play an important role in meta-mathematical research on incompleteness and undecidability. The notion of an EI theory is stronger than that of a RI theory. A RI theory may not be an EI theory (see Section 3), and EI theories have nice properties. The notion of an EI theory is central in research on the incompleteness phenomenon (see [17, 19, 21]). For example, if T is a consistent EI theory, then there is a recursive function f such that if $T_P = W_i$ and $T_R = W_j$, then $f(i, j)$ converges and outputs the code of a sentence ϕ which is independent of T (i.e., $T \not\vdash \phi$ and $T \not\vdash \neg \phi$). We could view effective inseparability as the effective version of essential incompleteness.

Smullyan introduces different notions (see Definition 2.3) and essentially shows in [21] that all these notions are equivalent to the notion of effective inseparability. These equivalences reveal the central role of effective inseparability in the meta-mathematics of arithmetic. We will show in Section 2 that these equivalences can be proved in a much simpler and more efficient way using the strong double recursion theorem.

The structure of this paper is as follows. In Section 1, we introduce the notion of effectively inseparable theories and the motivation of this paper. In Section 2, we study alternative characterizations of effectively inseparable pairs and establish these equivalences in a much simpler and more efficient way than the proofs in [21] using the strong double recursion theorem. In Section 3, we discuss some applications of the notion of effective inseparability in the meta-mathematics of arithmetic.

One main result in Section 2 is that we give a simpler and more efficient proof of the equivalences of effective inseparability using the strong double recursion theorem. Even if these equivalences have been proved by Smullyan, the proof is very complex for us and only one direction uses the double recursion theory. We directly prove some directions which are indirect in Smullyan's proof using a new method (strong double recursion theory). Our proof diagram in Theorem 2.16 is much simpler than Smullyan's proof diagram in Theorem 2.10. Section 3 is partly (Section 3.1) an exposition of some of the literature but it also contains original results in Section 3.2-3.4.

2. ALTERNATIVE CHARACTERIZATIONS OF EFFECTIVELY INSEPARABLE PAIRS

Smullyan proved in [21] that many different notions (see Definition 2.3) of pair of RE sets are equivalent to the notion of effectively inseparable pair of RE sets. Smullyan's results are summarized in Theorem 2.10 which is not specifically stated in [21] as a theorem. In this section, we give a much simpler and more efficient proof of Theorem 2.10 to establish the equivalence of notions in Definition 2.3. The main tool we use is the strong double recursion theorem for which we refer to [21].

2.1. Basic definitions and facts. We first introduce some definitions. Our notations are standard.

Definition 2.1 (Basic notations).

- (1) For any set A and function $f(x)$, define $f^{-1}[A] = \{x : f(x) \in A\}$.
- (2) We denote the *recursively enumerable* set with index i by W_i . I.e., $W_i = \{x : \exists y T_1(i, x, y)\}$, where $T_1(z, x, y)$ is the Kleene predicate (cf.[9]). We denote the recursive function with index e by ϕ_e .

We list some examples of EI pairs of RE sets.

Example 2.2.

- (A, B) is EI, where $n \in A \Leftrightarrow \exists p[T_1((n)_0, n, p) \wedge \forall q \leq p \neg T_1((n)_1, n, q)]$ and $n \in B \Leftrightarrow \exists q[T_1((n)_1, n, q) \wedge \forall p \leq q \neg T_1((n)_0, n, p)]^2$ (see [9]).
- (A_i, A_j) is EI for $i \neq j$, where $A_i = \{e : \phi_e(e) = i\}$ (see [13]).
- If (A, B) is a disjoint pair of non-empty RE sets, then (X, Y) is EI where $X = \{e : \phi_e(e) \in A\}$ and $Y = \{e : \phi_e(e) \in B\}$.

Remark. We give some examples of RI pairs of RE sets which are not EI. Ershov [4] shows that there is a disjoint pair (A, B) of RE sets such that both A and B are creative, (A, B) is RI but is not EI. As a corollary of a theorem by Friedberg and Yates, there exist recursive functions σ_1 and σ_2 such that if W_e is non-recursive, then $(W_{\sigma_1(e)}, W_{\sigma_2(e)})^3$ is RI but is not EI (see [2]).

Notions in Definition 2.3 are introduced in [21]. Smullyan proved in [21] that these notions are equivalent to the notion of effectively inseparable pair of RE sets. We will give a much simpler and more efficient proof of these equivalences (see Theorem 2.16).

Definition 2.3 ([21]). Let (A, B) be a disjoint pair of RE sets.

- (1) We say (A, B) has a *separation function* (denoted by SF) if there is a recursive function $S(x, y, z)$ such that for any RE relations $M_1(x, y)$ and $M_2(x, y)$, there is $h \in \omega$ such that for any $x, y \in \omega$, we have:
 - (i) $M_1(x, y) \wedge \neg M_2(x, y) \Rightarrow S(h, x, y) \in A$;
 - (ii) $M_2(x, y) \wedge \neg M_1(x, y) \Rightarrow S(h, x, y) \in B$.
- (2) We call a recursive function $f(x, y)$ a *Kleene function* for (A, B) if for any x and y , we have:
 - (i) if $f(x, y) \in W_y - W_x$, then $f(x, y) \in A$;
 - (ii) if $f(x, y) \in W_x - W_y$, then $f(x, y) \in B$.

We say (A, B) is a *Kleene pair* (denoted by KP) if it has a Kleene function.

²If $n = \langle a, b \rangle$, then $(n)_0 = a$ and $(n)_1 = b$.

³In fact, $W_{\sigma_1(e)}$ and $W_{\sigma_2(e)}$ are Turing incomparable (see [2]).

- (3) We say (A, B) is *weakly doubly co-productive* (WDCP) if there is a recursive function $f(x, y)$ such that for any $i, j \in \omega$, we have:
 - (i) if $W_i = W_j = \emptyset$, then $f(i, j) \notin A \cup B$;
 - (ii) if $W_i = \emptyset$ and $W_j = \{f(i, j)\}$, then $f(i, j) \in B$;
 - (iii) $W_i = \{f(i, j)\}$ and $W_j = \emptyset$, then $f(i, j) \in A$.
- (4) We say (A, B) is *doubly co-productive* (DCP) if there is a recursive function $f(x, y)$ such that for any $i, j \in \omega$, if $W_i \cap W_j = \emptyset$ and $W_i \cap A = \emptyset$ and $W_j \cap B = \emptyset$, then $f(i, j) \notin A \cup B \cup W_i \cup W_j$.
- (5) We say (A, B) is *semi-DG* if there is a recursive function $f(x, y)$ such that for any $i, j \in \omega$, if $W_i \cap W_j = \emptyset$, then we have:
 - (i) if $f(i, j) \in W_i$, then $f(i, j) \in A$;
 - (ii) if $f(i, j) \in W_j$, then $f(i, j) \in B$.
- (6) We say (A, B) is *doubly generative* (DG) if there is a recursive function $f(x, y)$ such that for any $i, j \in \omega$, if $W_i \cap W_j = \emptyset$, then we have:
 - (i) $f(i, j) \in A$ iff $f(i, j) \in W_i$;
 - (ii) $f(i, j) \in B$ iff $f(i, j) \in W_j$.
- (7) We say (A, B) is *semi-reducible* to (C, D) if there is a recursive function $f(x)$ such that if $x \in A$, then $f(x) \in C$, and if $x \in B$, then $f(x) \in D$. We say (C, D) is *semi-doubly universal* (semi-DU) if any disjoint pair (A, B) of RE sets is semi-reducible to (C, D) .
- (8) We say (A, B) is *reducible* to (C, D) if there is a recursive function $f(x)$ such that $x \in A$ iff $f(x) \in C$ and $x \in B$ iff $f(x) \in D$. We say (C, D) is *doubly universal* (DU) if any disjoint pair (A, B) of RE sets is reducible to (C, D) .
- (9) We say (A, B) is *weakly effective inseparable* (WEI) if there is a recursive function $f(x, y)$ such that for any i and j , we have:
 - (i) if $W_i = A$ and $W_j = B$, then $f(i, j) \notin A \cup B$;
 - (ii) if $W_i = A$ and $W_j = B \cup \{f(i, j)\}$, then $f(i, j) \in A$;
 - (iii) if $W_i = A \cup \{f(i, j)\}$ and $W_j = B$, then $f(i, j) \in B$.
- (10) We say (A, B) is *completely effective inseparable* (CEI) if there is a recursive function $f(x, y)$ such that for any i and j , if $A \subseteq W_i$ and $B \subseteq W_j$, then $f(i, j) \in W_i \Leftrightarrow f(i, j) \in W_j$.

Fact 2.4 ([11, 18]). The following are equivalent:

- A disjoint pair (A, B) of RE sets is DU;
- Every disjoint pair of RE sets is reducible to (A, B) under a 1-1 function $g(x)$.

2.2. The reduction theorem. In this section, we will show in Theorem 2.7 that if (A, B) has any property P in Definition 2.3 and (A, B) is reducible to (C, D) , then (C, D) also has the property P . We call this fact the reduction theorem. However, it is not true that if (C, D) has any property P in Definition 2.3 and (A, B) is reducible to (C, D) , then (A, B) has the property P .

Proposition 2.5 (Smullyan).

- (1) If (A, B) is a Kleene pair and (A, B) is semi-reducible to (C, D) , then (C, D) is a Kleene pair (Lemma A₁, p.70, [21]).
- (2) If (A, B) is EI and (A, B) is semi-reducible to (C, D) , then (C, D) is EI (Proposition 1, p.220, [17]).

The main tool we use in Theorem 2.7 is the s-m-n theorem.

Theorem 2.6 (The s-m-n theorem, [13]). *For any $m, n \geq 1$, there exists a recursive function s_n^m of $m+1$ variables such that for all $x, y_1, \dots, y_m, z_1, \dots, z_n$, we have*

$$\phi_{s_n^m(x, y_1, \dots, y_m)}^n(z_1, \dots, z_n) = \phi_x^{m+n}(y_1, \dots, y_m, z_1, \dots, z_n).$$

Now, we prove the reduction theorem.

Theorem 2.7 (The reduction theorem). *Let (A, B) and (C, D) be disjoint pairs of RE sets.*

- (1) *If (A, B) is semi-DU and (A, B) is semi-reducible to (C, D) , then (C, D) is semi-DU.*
- (2) *If (A, B) is DU and (A, B) is reducible to (C, D) , then (C, D) is DU.*
- (3) *If (A, B) has a separation function and (A, B) is reducible to (C, D) , then (C, D) has a separation function.*
- (4) *If (A, B) is CEI and (A, B) is reducible to (C, D) , then (C, D) is CEI.*
- (5) *If (A, B) is DG and (A, B) is reducible to (C, D) , then (C, D) is DG.*
- (6) *If (A, B) is semi-DG and (A, B) is reducible to (C, D) , then (C, D) is semi-DG.*
- (7) *If (A, B) is DCP and (A, B) is reducible to (C, D) , then (C, D) is DCP.*
- (8) *If (A, B) is WEI and (A, B) is reducible to (C, D) , then (C, D) is WEI.*
- (9) *If (A, B) is WDCP and (A, B) is reducible to (C, D) , then (C, D) is WDCP.*

Proof. Items (1) and (2) are easy to check.

(3): Suppose (A, B) has a separation function $f(x, y, z)$ and (A, B) is reducible to (C, D) via the function $g(x)$. It is easy to check that $s(x, y, z) = g(f(x, y, z))$ is a separation function for (C, D) .

We show that (4)-(9) hold. Let P be any one of the following properties: CEI, DG, semi-DG, DCP, WEI, WDCP. Suppose (A, B) has the property P via the recursive function $f(x, y)$ and (A, B) is reducible to (C, D) via the recursive function $g(x)$. By the s-m-n theorem, there is a recursive function $h(x)$ such that $g^{-1}[W_i] = W_{h(i)}$ for any i . Define $s(i, j) = g(f(h(i), h(j)))$. Note that the function s is recursive. We show that (C, D) has the property P via the recursive function s .

(4): Let P be CEI. We show that (C, D) is CEI via the recursive function s : if $C \subseteq W_i$ and $D \subseteq W_j$, then $s(i, j) \in W_i$ if and only if $s(i, j) \in W_j$.

Suppose $C \subseteq W_i$ and $D \subseteq W_j$. Note that $A = g^{-1}[C] \subseteq g^{-1}[W_i] = W_{h(i)}$ and $B = g^{-1}[D] \subseteq g^{-1}[W_j] = W_{h(j)}$. Since (A, B) is CEI via the recursive function f , we have: $f(h(i), h(j)) \in W_{h(i)} \Leftrightarrow f(h(i), h(j)) \in W_{h(j)}$. Since $g^{-1}[W_i] = W_{h(i)}$, $g^{-1}[W_j] = W_{h(j)}$ and $s(i, j) = g(f(h(i), h(j)))$, we have $s(i, j) \in W_i \Leftrightarrow s(i, j) \in W_j$.

(5): Let P be DG. We show that (C, D) is DG via the recursive function s : if $W_i \cap W_j = \emptyset$, then $s(i, j) \in C \Leftrightarrow s(i, j) \in W_i$ and $s(i, j) \in D \Leftrightarrow s(i, j) \in W_j$.

Suppose $W_i \cap W_j = \emptyset$. Since $W_{h(i)} \cap W_{h(j)} = \emptyset$ and (A, B) is DG via the recursive function f , we have $f(h(i), h(j)) \in A \Leftrightarrow f(h(i), h(j)) \in W_{h(i)}$. Since $A = g^{-1}[C]$, we have

$$s(i, j) \in C \Leftrightarrow f(h(i), h(j)) \in A \Leftrightarrow f(h(i), h(j)) \in W_{h(i)} = g^{-1}[W_i] \Leftrightarrow s(i, j) \in W_i.$$

By a symmetric argument, we have $s(i, j) \in D \Leftrightarrow s(i, j) \in W_j$.

(6): The argument is similar to (5).

(7): Let P be DCP. We show that (C, D) is DCP via the recursive function s : if $W_i \cap W_j = \emptyset, W_i \cap C = \emptyset$ and $W_j \cap D = \emptyset$, then $s(i, j) \notin C \cup D \cup W_i \cup W_j$.

Note that $W_{h(i)} \cap W_{h(j)} = \emptyset, g^{-1}[W_i] \cap g^{-1}[C] = \emptyset, g^{-1}[W_j] \cap g^{-1}[D] = \emptyset, A = g^{-1}[C]$ and $B = g^{-1}[D]$. Since (A, B) is DCP via the recursive function f , $W_{h(i)} \cap A = \emptyset$ and $W_{h(j)} \cap B = \emptyset$, we have $f(h(i), h(j)) \notin A \cup B \cup W_{h(i)} \cup W_{h(j)}$. Thus, $s(i, j) \notin C \cup D \cup W_i \cup W_j$.

(8): Let P be WEI. We show that (C, D) is WEI via the recursive function s : (I) if $W_i = C$ and $W_j = D$, then $s(i, j) \notin C \cup D$; (II) if $W_i = C$ and $W_j = D \cup \{s(i, j)\}$, then $s(i, j) \in C$; (III) if $W_i = C \cup \{s(i, j)\}$ and $W_j = D$, then $s(i, j) \in D$.

(I) Suppose $W_i = C$ and $W_j = D$. Note that $A = g^{-1}[C] = g^{-1}[W_i] = W_{h(i)}$, $B = g^{-1}[D] = g^{-1}[W_j] = W_{h(j)}$. Since (A, B) is WEI via the recursive function f , $f(h(i), h(j)) \notin A \cup B$. Thus, $s(i, j) \notin C \cup D$.

(II) Suppose $W_i = C$ and $W_j = D \cup \{s(i, j)\}$. Note that $A = g^{-1}[W_i] = W_{h(i)}$, $B \cup g^{-1}[\{s(i, j)\}] = g^{-1}[W_j] = W_{h(j)}$. From [21], WEI implies DU. From (2), (C, D) is DU. By Fact 2.4, we can assume that g is injective, and thus $g^{-1}[\{s(i, j)\}] = \{f(h(i), h(j))\}$. Since (A, B) is WEI via f , $f(h(i), h(j)) \in A$. Thus, $s(i, j) \in C$.

(III) Follows by a symmetric argument as for (II).

(9): Let P be WDCP. We show that (C, D) is WDCP via the recursive function s : (I) if $W_i = W_j = \emptyset$, then $s(i, j) \notin C \cup D$; (II) if $W_i = \emptyset$ and $W_j = \{s(i, j)\}$, then $f(i, j) \in D$; (III) if $W_i = \{s(i, j)\}$ and $W_j = \emptyset$, then $f(i, j) \in C$.

(I) Suppose $W_i = W_j = \emptyset$. Since $W_{h(i)} = W_{h(j)} = \emptyset$ and (A, B) is WDCP via the recursive function f , $f(h(i), h(j)) \notin A \cup B$. Thus, $s(i, j) \notin C \cup D$.

(II) Suppose $W_i = \emptyset$ and $W_j = \{s(i, j)\}$. From [21], WDCP implies DU. From (2), (C, D) is DU. By Fact 2.4, we can assume that g is injective. Then $W_{h(i)} = \emptyset$ and $W_{h(j)} = g^{-1}[W_j] = g^{-1}[\{s(i, j)\}] = \{f(h(i), h(j))\}$. Since (A, B) is WDCP via the recursive function f , $f(h(i), h(j)) \in B$. Thus, $s(i, j) \in D$.

(III) Follows by a symmetric argument as for (II).

□

It is easy to see that for a disjoint pair (A, B) of RE sets, (A, B) is DU (semi-DU, has separation function) if and only if (B, A) is DU (semi-DU, has separation function). The following proposition is an easy observation.

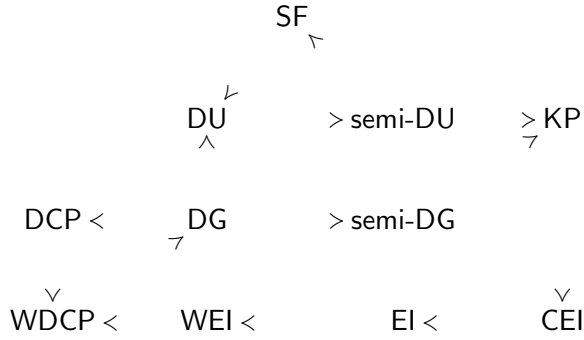
Proposition 2.8. *For any disjoint pair (A, B) of RE sets, if (A, B) is CEI (EI, WEI, DG, Semi-DG, DCP, WDCP, Kleene pair) via the recursive function $f(i, j)$, then (B, A) is CEI (EI, WEI, DG, Semi-DG, DCP, WDCP, Kleene pair) via the recursive function $g(i, j) = f(j, i)$.*

Corollary 2.9. *Let P be any property in Definition 2.3 and (A, B) be any disjoint pair of RE sets. Then (A, B) has the property P iff (B, A) has the property P .*

2.3. A simpler proof of Smullyan's theorem. Theorem 2.10 is not specifically stated as a theorem in [21], but results in [21] in fact imply Theorem 2.10 from which notions in Definition 2.3 are equivalent to the notion of effective inseparability. In this section, we give a much simpler proof of Theorem 2.10 via the strong double recursion theorem.

Theorem 2.10 (Smullyan, [21]). *Let (A, B) be a disjoint pair of RE sets, and P be any one of the following properties: Kleene pair (KP), having a separation function (SF), WEI, CEI, semi-DG, DG, semi-DU, DU, WDCP and DCP. Then (A, B) has the property P if and only if (A, B) is EI.*

Proof. Smullyan proved the following diagram in [21]. All notions in this diagram⁴ are equivalent.



- (1) $\text{SF} \Rightarrow \text{DU}$: Theorem 2 in p.91, [21].
- (2) $\text{DU} \Rightarrow \text{semi-DU}$: by definition.
- (3) $\text{semi-DU} \Rightarrow \text{SF}$: Proposition 1 in p.90, [21].
- (4) $\text{DG} \Rightarrow \text{DU}$: Theorem 2 in p.86, [21].
- (5) $\text{semi-DU} \Rightarrow \text{KP}$: Theorem 9 in p.75, [21].
- (6) $\text{DG} \Rightarrow \text{semi-DG}$: by definition.
- (7) $\text{semi-DG} \Rightarrow \text{KP}$: Theorem 13 in p.79, [21].
- (8) $\text{KP} \Rightarrow \text{CEI}$: Proposition 2 in p.68, [21].
- (9) $\text{CEI} \Rightarrow \text{EI}$: by definition.
- (10) $\text{EI} \Rightarrow \text{WEI}$: by definition.
- (11) $\text{WEI} \Rightarrow \text{WDCP}$: Theorem 3 in p.126, [21].
- (12) $\text{WDCP} \Rightarrow \text{DG}$: Theorem 2 in p.123, [21].
- (13) $\text{DG} \Rightarrow \text{DCP}$: by definition (easy to check).
- (14) $\text{DCP} \Rightarrow \text{WDCP}$: by definition.

□

For the above proofs in [21], only the proof of “WDCP \Rightarrow DG” uses the double recursion theorem. In the following, we will give a much simpler and more direct proof of Theorem 2.10 using the **strong** double recursion theorem (see Theorem 2.16). Our main observation is: using the **strong** double recursion theorem, we can directly prove many implication relations among the notions in Theorem 2.10. We first introduce the strong double recursion theorem in [21].

Theorem 2.11 (The strong double recursion theorem, Theorem 2 in p.107, [21]). *For any RE relations $R_1(x, y_1, y_2, z_1, z_2)$ and $R_2(x, y_1, y_2, z_1, z_2)$, there are recursive functions $t_1(y_1, y_2)$ and $t_2(y_1, y_2)$ such that for all i and j , we have:*

- (1) $W_{t_1(i, j)} = \{x : R_1(x, i, j, t_1(i, j), t_2(i, j))\};$

⁴I would like to thank my student for the help to draw the pictures.

$$(2) W_{t_2(i,j)} = \{x : R_2(x, i, j, t_1(i, j), t_2(i, j))\}.$$

Lemma 2.12. *For any RE relation $M_1(x, y, z_1, z_2)$ and $M_2(x, y, z_1, z_2)$, there are recursive functions $t_1(y_1, y_2)$ and $t_2(y_1, y_2)$ such that*

$$(1) W_{t_1(y_1, y_2)} = \{x : M_1(x, y_2, t_1(y_1, y_2), t_2(y_1, y_2))\};$$

$$(2) W_{t_2(y_1, y_2)} = \{x : M_2(x, y_1, t_1(y_1, y_2), t_2(y_1, y_2))\}.$$

Proof. Apply Theorem 2.11 to $R_1(x, y_1, y_2, z_1, z_2) = M_1(x, y_2, z_1, z_2)$ and $R_2(x, y_1, y_2, z_1, z_2) = M_2(x, y_1, z_1, z_2)$. \square

Remark. Lemma 2.12 has different variants. For instance, we also have that for any RE relation $M_1(x, y, z_1, z_2)$ and $M_2(x, y, z_1, z_2)$, there are recursive functions $t_1(y_1, y_2)$ and $t_2(y_1, y_2)$ such that $W_{t_1(y_1, y_2)} = \{x : M_1(x, y_1, t_1(y_1, y_2), t_2(y_1, y_2))\}$ and $W_{t_2(y_1, y_2)} = \{x : M_2(x, y_2, t_1(y_1, y_2), t_2(y_1, y_2))\}$. But in the proof of Theorem 2.13, what we need is the form as in Lemma 2.12.

Now, we directly prove that WEI implies DG. The main tool we use is Lemma 2.12 which follows from the strong double recursion theorem.

Theorem 2.13. *If (A, B) is WEI, then (A, B) is DG.*

Proof. Suppose (A, B) is WEI via the recursive function g :

- (I) if $W_i = A$ and $W_j = B$, then $g(i, j) \notin A \cup B$;
- (II) if $W_i = A$ and $W_j = B \cup \{g(i, j)\}$, then $g(i, j) \in A$;
- (III) if $W_i = A \cup \{g(i, j)\}$ and $W_j = B$, then $g(i, j) \in B$.

Define:

$$M_1(x, y, z_1, z_2) : x \in A \vee (x = g(z_1, z_2) \wedge x \in W_y);$$

$$M_2(x, y, z_1, z_2) : x \in B \vee (x = g(z_1, z_2) \wedge x \in W_y).$$

Apply Lemma 2.12 to RE relations $M_1(x, y, z_1, z_2)$ and $M_2(x, y, z_1, z_2)$. Then there are recursive functions $t_1(y_1, y_2)$ and $t_2(y_1, y_2)$ such that

$$(1) W_{t_1(i,j)} = \{x : M_1(x, j, t_1(i, j), t_2(i, j))\} = A \cup (W_j \cap \{g(t_1(i, j), t_2(i, j))\});$$

$$(2) W_{t_2(i,j)} = \{x : M_2(x, i, t_1(i, j), t_2(i, j))\} = B \cup (W_i \cap \{g(t_1(i, j), t_2(i, j))\}).$$

Define $f(i, j) = g(t_1(i, j), t_2(i, j))$. Clearly, f is recursive. Note that

$$W_{t_1(i,j)} = A \cup (W_j \cap \{f(i, j)\})$$

and

$$W_{t_2(i,j)} = B \cup (W_i \cap \{f(i, j)\}).$$

We show that (A, B) is DG via f : if $W_i \cap W_j = \emptyset$, then $f(i, j) \in W_i \Leftrightarrow f(i, j) \in A$ and $f(i, j) \in W_j \Leftrightarrow f(i, j) \in B$.

Suppose $W_i \cap W_j \neq \emptyset$.

- (1) If $f(i, j) \in W_i$, then $W_{t_1(i,j)} = A$ and $W_{t_2(i,j)} = B \cup \{f(i, j)\}$. By condition (II), we have $f(i, j) \in A$.
- (2) If $f(i, j) \in W_j$, then $W_{t_1(i,j)} = A \cup \{f(i, j)\}$ and $W_{t_2(i,j)} = B$. By condition (III), we have $f(i, j) \in B$.
- (3) If $f(i, j) \notin W_i \cup W_j$, then $W_{t_1(i,j)} = A$ and $W_{t_2(i,j)} = B$. By condition (I), we have $f(i, j) \notin A \cup B$.

Thus, $f(i, j) \in W_i \Leftrightarrow f(i, j) \in A$ and $f(i, j) \in W_j \Leftrightarrow f(i, j) \in B$. \square

Theorem 2.14 is a corollary of Theorem 2.11, the strong double recursion theorem (see pp.107-108 in [21]).

Theorem 2.14 (Theorem 2.4 in p.108, [21]). *For any two RE relations $M_1(x, y, z)$ and $M_2(x, y, z)$ and any recursive function $g(x, y)$, there are recursive functions $f_1(y)$ and $f_2(y)$ such that for any y ,*

- (1) $W_{f_1(y)} = \{x : M_1(x, y, g(f_1(y), f_2(y)))\};$
- (2) $W_{f_2(y)} = \{x : M_2(x, y, g(f_1(y), f_2(y)))\}.$

Now, we directly prove that WDCP implies DU. The main tool we use is Theorem 2.14, a corollary of the strong double recursion theorem.

Theorem 2.15. *If (C, D) is WDCP, then (C, D) is DU.*

Proof. Suppose (C, D) is WDCP via the recursive function g :

- (I) if $W_i = \{g(i, j)\}$ and $W_j = \emptyset$, then $g(i, j) \in C$;
- (II) if $W_i = \emptyset$ and $W_j = \{g(i, j)\}$, then $g(i, j) \in D$;
- (III) If $W_i = W_j = \emptyset$, then $f(i, j) \notin C \cup D$.

We show that (C, D) is DU: any disjoint pair of RE sets is reducible to (C, D) . Let (A, B) be a disjoint pair of RE sets. Define

$$M_1(x, y, z) : y \in A \wedge x = z$$

and

$$M_2(x, y, z) : y \in B \wedge x = z.$$

Apply Theorem 2.14 to $M_1(x, y, z)$ and $M_2(x, y, z)$. Then there are recursive functions $f_1(y)$ and $f_2(y)$ such that for any y :

$$W_{f_1(y)} = \{x : y \in A \wedge x = g(f_1(y), f_2(y))\}$$

and

$$W_{f_2(y)} = \{x : y \in B \wedge x = g(f_1(y), f_2(y))\}.$$

Define $h(y) = g(f_1(y), f_2(y))$.

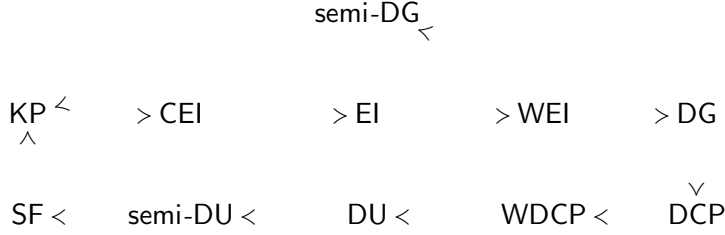
We show that (A, B) is reducible to (C, D) via the recursive function h .

- (1) Suppose $y \in A$. Then $W_{f_1(y)} = \{g(f_1(y), f_2(y))\}$ and $W_{f_2(y)} = \emptyset$. Thus, by condition (I), $h(y) \in C$.
- (2) Suppose $y \in B$. Then $W_{f_1(y)} = \emptyset$ and $W_{f_2(y)} = \{g(f_1(y), f_2(y))\}$. Thus, by condition (II), $h(y) \in D$.
- (3) Suppose $y \notin A \cup B$. Then $W_{f_1(y)} = W_{f_2(y)} = \emptyset$. Thus, by condition (III), $h(y) \notin C \cup D$.

Thus, $y \in A \Leftrightarrow h(y) \in C$ and $y \in B \Leftrightarrow h(y) \in D$. □

Now, we give a much simpler and more efficient proof of Theorem 2.10 via the strong double recursion theorem.

Theorem 2.16. *Theorem 2.10 can be proved via the strong double recursion theorem as in the following picture:*



Proof. (1) $\text{KP} \Rightarrow \text{CEI}$: Proposition 2 in p.68, [21].

(2) $\text{CEI} \Rightarrow \text{EI}$: by definition.

(3) $\text{EI} \Rightarrow \text{WEI}$: by definition.

(4) $\text{WEI} \Rightarrow \text{DG}$: Theorem 2.13.

(5) $\text{DG} \Rightarrow \text{DCP}$: by definition (easy to check).

(6) $\text{DCP} \Rightarrow \text{WDCP}$: by definition.

(7) $\text{WDCP} \Rightarrow \text{DU}$: Theorem 2.15.

(8) $\text{DU} \Rightarrow \text{semi-DU}$: by definition.

(9) $\text{semi-DU} \Rightarrow \text{SF}$: Proposition 1 in p.90, [21].

(10) $\text{SF} \Rightarrow \text{KP}$: Suppose (A, B) has a separation function. By Theorem 2 of [21] in p.91, (A, B) is DU (i.e., any disjoint pair of RE sets is reducible to (A, B)). By Theorem 1 of [21] in p.68, there exists a Kleene pair (K_1, K_2) . Since (K_1, K_2) is reducible to (A, B) , by Proposition 2.5, (A, B) is a Kleene pair.

(11) $\text{DG} \Rightarrow \text{semi-DG}$: by definition.

(12) $\text{semi-DG} \Rightarrow \text{KP}$: Theorem 13 in p.79, [21].

□

Our proof of Theorem 2.16 via the strong double recursion theorem is much simpler than the corresponding proof in [21]. Among notions in Definition 2.3, five key notions are DG, WEI, DU, CEI, EI. Smullyan has shown in [21] that these notions are equivalent in a complex way. In the rest of this section, we prove the equivalence of these notions via a much simpler way (Theorem 2.21) than those proofs in [21] using the strong double recursion theorem.

Theorem 2.17 (Theorem 4, p.57, [21]). *There is a recursive function $\sigma(x, y)$ such that for all i and j ,*

- (1) $W_{\sigma(i,j)}$ and $W_{\sigma(j,i)}$ are disjoint supersets of $W_i - W_j$ and $W_j - W_i$ respectively;
- (2) If W_i and W_j are disjoint, then $W_i = W_{\sigma(i,j)}$ and $W_j = W_{\sigma(j,i)}$.

We directly prove that DG implies WEI. The main tool we use is Theorem 2.17.

Proposition 2.18. *If (C, D) is DG, then it is WEI.*

Proof. The idea of this proof is essentially from [21]. Suppose (C, D) is DG via the recursive function $k(x, y)$. Define $g(i, j) = k(\sigma(j, i), \sigma(i, j))$. We show that (C, D) is WEI via g .

- (1) Suppose $W_i = C$ and $W_j = D$. We show that $g(i, j) \notin C \cup D$.

Since $W_i \cap W_j = \emptyset$, by Theorem 2.17, $W_i = W_{\sigma(i,j)}$ and $W_j = W_{\sigma(j,i)}$. Since (C, D) is DG via the recursive function k , we have $g(i, j) \in C \Leftrightarrow g(i, j) \in W_j$ and $g(i, j) \in D \Leftrightarrow g(i, j) \in W_i$. Thus, $g(i, j) \notin C \cup D$.

- (2) Suppose $W_i = C$ and $W_j = D \cup \{g(i, j)\}$. We show that $g(i, j) \in C$.
 Suppose $g(i, j) \notin C$. Then $W_i \cap W_j = \emptyset$. Then $g(i, j) \in C \Leftrightarrow g(i, j) \in W_j$. So $g(i, j) \notin W_j$, which leads to a contradiction.
- (3) Suppose $W_i = C \cup \{g(i, j)\}$ and $W_j = D$. By a symmetric argument as (2), we can show that $g(i, j) \in D$.

Thus, we have shown that (C, D) is WEI via the recursive function g . \square

The main tool we use in the proof of Lemma 2.19 is Theorem 2.14, a corollary of the strong double recursion theorem.

Lemma 2.19. *For any disjoint pair of RE sets A and B , for any RE sets C and D , and any recursive function $g(x, y)$, there are recursive functions $f_1(y)$ and $f_2(y)$ such that for any y ,*

- (I) if $y \in A$, then $W_{f_1(y)} = C$ and $W_{f_2(y)} = D \cup \{g(f_1(y), f_2(y))\}$;
- (II) if $y \in B$, then $W_{f_1(y)} = C \cup \{g(f_1(y), f_2(y))\}$ and $W_{f_2(y)} = D$;
- (III) if $y \notin A \cup B$, then $W_{f_1(y)} = C$ and $W_{f_2(y)} = D$.

Proof. Define

$$M_1(x, y, z) : x \in C \vee (y \in B \wedge x = z)$$

and

$$M_2(x, y, z) : x \in D \vee (y \in A \wedge x = z).$$

Apply Theorem 2.14 to $M_1(x, y, z)$ and $M_2(x, y, z)$. Then there are recursive functions $f_1(y)$ and $f_2(y)$ such that for any y :

- (1) $W_{f_1(y)} = \{x : M_1(x, y, g(f_1(y), f_2(y)))\} = \{x : x \in C \vee (y \in B \wedge x = g(f_1(y), f_2(y)))\}$;
- (2) $W_{f_2(y)} = \{x : M_2(x, y, g(f_1(y), f_2(y)))\} = \{x : x \in D \vee (y \in A \wedge x = g(f_1(y), f_2(y)))\}$.

It is easy to check that (I)-(III) hold. \square

It is an exercise in [21] (p. 126) that WEI implies DU. We give proof details of it here. The main tool we use is Lemma 2.19 which follows from the strong double recursion theorem.

Theorem 2.20 (Exercise 1 in p. 126, [21]). *If (C, D) is WEI, then it is DU.*

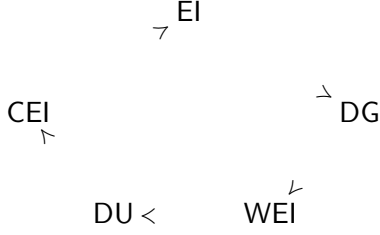
Proof. Suppose (C, D) is WEI via the recursive function g . Let (A, B) be a disjoint pair of RE sets. Apply Lemma 2.19 to A, B, C, D and g . Then there are recursive functions f_1 and f_2 such that conditions (I)-(III) in Lemma 2.19 hold.

We show that (A, B) is reducible to (C, D) via $h(y) = g(f_1(y), f_2(y))$.

- (1) We show that if $y \in A$, then $h(y) \in C$. Suppose $y \in A$. Then, by (I) in Lemma 2.19, $W_{f_1(y)} = C$ and $W_{f_2(y)} = D \cup \{g(f_1(y), f_2(y))\}$. Since (C, D) is WEI via g , we have $h(y) = g(f_1(y), f_2(y)) \in C$.
- (2) We show that if $y \in B$, then $h(y) \in D$. Suppose $y \in B$. Then, by (II) in Lemma 2.19, $W_{f_1(y)} = C \cup \{g(f_1(y), f_2(y))\}$ and $W_{f_2(y)} = D$. Since (C, D) is WEI via g , we have $h(y) \in D$.
- (3) We show that if $y \notin A \cup B$, then $h(y) \notin C \cup D$. Suppose $y \notin A \cup B$. Then, by (III) in Lemma 2.19, $W_{f_1(y)} = C$ and $W_{f_2(y)} = D$. Since (C, D) is WEI via g , we have $h(y) \notin C \cup D$.

\square

Theorem 2.21. *The equivalence of DG, WEI, DU, CEI and EI can be proved as in the following picture:*



Proof. (1) $\text{DG} \Rightarrow \text{WEI}$: Proposition 2.18.

(2) $\text{WEI} \Rightarrow \text{DU}$: Theorem 2.20.

(3) $\text{DU} \Rightarrow \text{CEI}$: Since there exists a Kleene pair, by Proposition 2.5(1), if (A, B) is DU, then (A, B) is a Kleene pair. By Proposition 2 of [21] in p.68, (A, B) is CEI.

(4) $\text{CEI} \Rightarrow \text{EI}$: By definition.

(5) $\text{EI} \Rightarrow \text{DG}$: Follows from Theorem 2.13 since EI implies WEI by definition. \square

Remark. The proof of “ $\text{DG} \Rightarrow \text{DU}$ ” in [21] is complex and does not use any version of recursion theorems. From Theorem 2.21, we also give a simpler proof of “ $\text{DG} \Rightarrow \text{DU}$ ” via the strong double recursion theorem.

3. SOME APPLICATIONS IN META-MATHEMATICS

In this section, we discuss some applications of effective inseparability in the meta-mathematics of arithmetic. This section is partly an exposition of some of the literature, but it also contains original results. In Section 3.1, we examine some important meta-mathematical properties of theories and the relationship among them. In Section 3.2, we examine Smoryński’s theorem about effective inseparability and its application. In Section 3.3, we examine Shoenfield’s theorems and their applications to recursively inseparable theories and effective inseparable theories. In Section 3.4, we show that there are many EI theories weaker than the theory **R**.

3.1. Some meta-mathematical properties. In Definition 3.1, we list some important meta-mathematical properties of theories. Then we examine the relationship among these properties. This section is mainly an exposition of some of the literature.

Definition 3.1 (Essentially undecidable, creative theory, Rosser theory, Exact Rosser theory). Let T be a consistent RE theory and (A, B) be a disjoint pair of RE sets.

- (1) We say T is *essentially undecidable* (EU) if any consistent RE extension of T is undecidable.
- (2) We say $A \subseteq \mathbb{N}$ is *productive* if there exists a recursive function $f(x)$ (called a productive function for x) such that for every number i , if $W_i \subseteq A$, then $f(i) \in A - W_i$.
- (3) We say $A \subseteq \mathbb{N}$ is *creative* if A is RE and the complement of A is productive.
- (4) We say T is *creative* if T_P is creative.
- (5) We say (A, B) is *separable* in T if there is a formula $\phi(x)$ with only one free variable such that if $n \in A$, then $T \vdash \phi(\bar{n})$, and if $n \in B$, then $T \vdash \neg\phi(\bar{n})$.

- (6) We say (A, B) is *exactly separable* in T if there is a formula $\phi(x)$ with only one free variable such that $n \in A \Leftrightarrow T \vdash \phi(\bar{n})$, and $n \in B \Leftrightarrow T \vdash \neg\phi(\bar{n})$.
- (7) We say T is a *Rosser theory* if any disjoint pair of RE sets is separable in T .
- (8) We say T is an *exact Rosser theory* if any disjoint pair of RE sets is exactly separable in T .

Theorem 3.2 (Smullyan, Theorem 2, p.221, [17]). *For a consistent RE theory T , if T is Rosser, then T is El.*

Proof. Smullyan proves this fact via Proposition 2.5(2). Here we give a short proof via Theorem 2.10. Since T is Rosser, (T_P, T_R) is semi-DU. Thus, by Theorem 2.10, T is El. \square

The theory \mathbf{R} was introduced by Tarski, Mostowski and R. Robinson in [22] which is an important base theory in the study of incompleteness and undecidability, and has many nice meta-mathematical properties.

Definition 3.3. Let \mathbf{R} be the theory consisting of schemes Ax1-Ax5 with $L(\mathbf{R}) = \{\mathbf{0}, \mathbf{S}, +, \cdot, \leq\}$ where we define $x \leq y$ as $\exists z(z + x = y)$, and $\bar{n} = \mathbf{S}^n \mathbf{0}$ for $n \in \mathbb{N}$.

- Ax1: $\bar{m} + \bar{n} = \overline{m + n}$;
- Ax2: $\bar{m} \cdot \bar{n} = \overline{m \cdot n}$;
- Ax3: $\bar{m} \neq \bar{n}$, if $m \neq n$;
- Ax4: $\forall x(x \leq \bar{n} \rightarrow x = \bar{0} \vee \dots \vee x = \bar{n})$;
- Ax5: $\forall x(x \leq \bar{n} \vee \bar{n} \leq x)$.

Definition 3.4 (Definable, strongly representable, weakly representable). Let T be a consistent RE theory.

- (1) We say a total n -ary function f on \mathbb{N} is *definable* in T if there exists a $L(T)$ -formula $\phi(x_1, \dots, x_n, y)$ such that for any $\langle a_1, \dots, a_n \rangle \in \mathbb{N}^n$,

$$T \vdash \forall y[\phi(\bar{a}_1, \dots, \bar{a}_n, y) \leftrightarrow y = \overline{f(a_1, \dots, a_n)}].$$

We say $\phi(x_1, \dots, x_n, y)$ *defines* f in T .

- (2) We say an n -ary relation R on \mathbb{N} is *strongly representable* in T if there exists a $L(T)$ -formula $\phi(x_1, \dots, x_n)$ such that for any $\langle a_1, \dots, a_n \rangle \in \mathbb{N}^n$, if $R(a_1, \dots, a_n)$ holds, then $T \vdash \phi(\bar{a}_1, \dots, \bar{a}_n)$, and if $R(a_1, \dots, a_n)$ does not hold, then $T \vdash \neg\phi(\bar{a}_1, \dots, \bar{a}_n)$. We say $\phi(x_1, \dots, x_n)$ *strongly represents* the relation R .
- (3) We say an n -ary relation R on \mathbb{N} is *weakly representable* in T if there exists a $L(T)$ -formula $\phi(x_1, \dots, x_n)$ such that for any $\langle a_1, \dots, a_n \rangle \in \mathbb{N}^n$, $R(a_1, \dots, a_n)$ holds if and only if $T \vdash \phi(\bar{a}_1, \dots, \bar{a}_n)$. We say $\phi(x_1, \dots, x_n)$ *weakly represents* the relation R .

In this paper, we use the following nice properties of \mathbf{R} :

Fact 3.5 ([10, 22]).

- All recursive functions are definable in \mathbf{R} .
- Any disjoint pair of RE sets is separable (in fact exactly separable) in \mathbf{R} . Thus, \mathbf{R} is a Rosser theory (in fact an exact Rosser theory).

As a corollary, if T is a consistent RE extension of \mathbf{R} , then T is a Rosser theory, and hence is El. Fact 3.6 provides us with some sufficient conditions to show that a theory is essentially undecidable and recursively inseparable.

Fact 3.6 ([15, 19]). Let T be a consistent RE theory.

- (1) If T satisfies any one of the following conditions⁵, then T is essentially undecidable:
 - (A) All recursive functions are definable in T .
 - (B) All recursive sets are strongly representable in T .
 - (C) All recursive sets are weakly representable in T .
- (2) If all recursive sets are strongly representable in T , then T is RI.

Remark. It is easy to check that: (1) if T is EI, then T is RI; (2) if T is RI, then T is EU; (3) if T is EI, then T is creative (for any disjoint pair (A, B) of RE sets, if (A, B) is EI, then both A and B are creative).

It has been shown that if T satisfies Condition (A) in Fact 3.6 plus some natural additional condition, then T is creative (e.g., Theorem 3.7). Theorem 3.7-3.10 provide us with some sufficient conditions to show that a theory is creative, EI and exact Rosser.

Theorem 3.7 (Ehrenfeucht and Feferman, [3]). *Suppose T is a consistent RE theory and has a formula $x \leq y$ with two free variables x and y , satisfying the following conditions⁶:*

- (1) *For each $n \in \mathbb{N}$, $T \vdash \forall x(x \leq \bar{n} \leftrightarrow x = \bar{0} \vee x = \bar{1} \vee \cdots \vee x = \bar{n})$.*
- (2) *For each $n \in \mathbb{N}$, $T \vdash \forall x(x \leq \bar{n} \vee \bar{n} \leq x)$.*
- (3) *All recursive functions are definable in T .*

Then any RE set is weakly representable in T and hence T is creative.

Definition 3.8. Let T be a consistent RE theory.

- (1) We say that all recursive sets are *uniformly weakly representable* in T if there is a recursive function $f(x)$ such that for every number i , if W_i is recursive, then $f(i)$ is the Gödel number of a formula of T which weakly represents W_i .
- (2) We say all recursive sets are *uniformly strongly representable* in T if there is a recursive function $g(x, y)$ such that for all numbers i and j , if W_j is the complement of W_i , then $g(i, j)$ is the Gödel number of a formula which strongly represents W_i in T .

Theorem 3.9 (Smullyan, [18]). *Let T be a consistent RE theory.*

- (1) *If all recursive sets are uniformly weakly representable in T , then T is creative.*
- (2) *If T is a consistent theory in which all recursive sets are uniformly strongly representable, then T is EI.*

Theorem 3.10 (Putnam and Smullyan, [12]). *Let T be a consistent RE theory.*

- (1) *If all recursive functions are definable in T and some EI pair of RE sets is separable in T , then T is an exact Rosser theory (see Theorem 3, [12]).*
- (2) *If T is a Rosser theory in which all recursive functions are definable, then T is an exact Rosser theory (see Theorem 4, [12]).*

A natural question is: if a consistent RE theory T satisfies (A), (B) or (C) in Fact 3.6, is it creative? Shoenfield answers this question negatively (see Theorem 3.11).

⁵Condition A is stronger than condition B, and condition B is stronger than condition C.

⁶Shoenfield shows in [14] that conditions (1)-(3) can be replaced by weaker ones.

Theorem 3.11 (Shoenfield, [15]). *There is a theory T in which any recursive function is definable but T is not creative, and no non-recursive set is weakly representable. In fact, T has Turing degree $< 0'$.*

Definition 3.12. Given theories S and T , we say S is *interpretable* in T if there is a mapping from formulas in the language of S to formulas in the language of T such that axioms of S are provable in T under this mapping (for the precise definition, see [7]). We demand that this mapping commutes with the propositional connectives.

From Theorem 3.11, we have some important corollaries:

Corollary 3.13. *The theory T in Theorem 3.11 has the following properties:*

- *It is not true that any RE set is weakly representable in T ;*
- *T is RI, but is not EI;*
- *\mathbf{R} is not interpretable in T .*

Moreover, we have:

- *“Any recursive function is definable in T ” does not imply “ T is creative”.*
- *“Any recursive function is definable in T ” does not imply “any RE set is weakly representable in T ”.*
- *“ T is RI” does not imply “ T is creative”.*
- *“ T is RI” does not imply “ T is EI”.*
- *“Any recursive function is definable in T ” does not imply “ \mathbf{R} is interpretable in T ”.*

Remark. It was an open question that “whether any recursive function is definable in T implies that \mathbf{R} is interpretable in T ”. Emil Jeřábek answered this question negatively and gave a counterexample in [5]. In fact, Theorem 3.11 provides us with a new counterexample.

3.2. Smoryński’s theorem and its application. In this section, we examine Smoryński’s theorem about effective inseparability and its application.

Theorem 3.14 (Smoryński, [16]). *Let (A, C) and (B, D) be pairs of effective inseparable RE sets with $A \subseteq B$ and $C \subseteq D$. Then there is a recursive function f such that for any x , we have:*

$$\begin{aligned} x \in A &\Leftrightarrow f(x) \in A \Leftrightarrow f(x) \in B; \\ x \in C &\Leftrightarrow f(x) \in C \Leftrightarrow f(x) \in D. \end{aligned}$$

We show that the condition “ $A \subseteq B$ and $C \subseteq D$ ” in Theorem 3.14 is necessary. Before the proof, we first introduce some definitions and facts.

Definition 3.15. We say $\mathbf{Prov}_T(x)$ is a standard provability predicate if it satisfies the following properties:

- D1:** If $T \vdash \phi$, then $T \vdash \mathbf{Prov}_T(\overline{\Gamma\phi\overline{}})$.
- D2:** $T \vdash \mathbf{Prov}_T(\overline{\Gamma\phi \rightarrow \psi\overline{}}) \rightarrow (\mathbf{Prov}_T(\overline{\Gamma\phi\overline{}}) \rightarrow \mathbf{Prov}_T(\overline{\Gamma\psi\overline{}}))$.
- D3:** $T \vdash \mathbf{Prov}_T(\overline{\Gamma\phi\overline{}}) \rightarrow \mathbf{Prov}_T(\overline{\Gamma\mathbf{Prov}_T(\overline{\Gamma\phi\overline{}})\overline{}})$.

Theorem 3.16 (Löb’s theorem). *Let T be consistent RE theory. For any sentence ϕ and standard provability predicate $\mathbf{Prov}_T(x)$, we have $T \vdash \mathbf{Prov}_T(\overline{\Gamma\phi\overline{}}) \rightarrow \phi$ if and only if $T \vdash \phi$.*

Fact 3.17 (Folklore). For any formula $\theta(x)$, the pair $(\text{Fix}_T(\theta), \text{Fix}_T(\neg\theta))$ is **El** where $\text{Fix}_T(\theta) = \{\phi : T \vdash \phi \leftrightarrow \theta(\overline{\phi})\}$.

Theorem 3.18. *The condition “ $A \subseteq B$ and $C \subseteq D$ ” in Theorem 3.14 cannot be dropped.*

Proof. Suppose that the condition “ $A \subseteq B$ and $C \subseteq D$ ” in Theorem 3.14 can be dropped. Let T be a consistent RE extension of **R**. Note that (T_P, T_R) is **El**, and $(\text{Fix}_T(\theta), \text{Fix}_T(\neg\theta))$ is **El** for any formula $\theta(x)$ (by Fact 3.17). Apply Theorem 3.14 to (T_P, T_R) and $(\text{Fix}_T(\theta), \text{Fix}_T(\neg\theta))$ where $\theta = \neg\mathbf{Prov}_T(x)$. Then there is a recursive function f such that for any sentence ϕ , we have:

- (1) $T \vdash \phi \Leftrightarrow T \vdash f(\phi) \Leftrightarrow T \vdash f(\phi) \leftrightarrow \neg\mathbf{Prov}_T(\overline{f(\phi)});$
- (2) $T \vdash \neg\phi \Leftrightarrow T \vdash \neg f(\phi) \Leftrightarrow T \vdash f(\phi) \leftrightarrow \mathbf{Prov}_T(\overline{f(\phi)}).$

By Löb’s theorem, we have $T \vdash f(\phi) \leftrightarrow \mathbf{Prov}_T(\overline{f(\phi)})$ if and only if $T \vdash f(\phi)$. By (2), we have $T \vdash \neg\phi \Leftrightarrow T \vdash f(\phi)$. By (1) and (2), we have $T \vdash \phi \Leftrightarrow T \vdash \neg\phi$ for any sentence ϕ , which leads to a contradiction. \square

Theorem 3.19. *Let T be a Rosser theory. Let (A, B) be any **El** pair of RE sets. Suppose (A, B) is separable in T by the formula $\phi(x)$. Define $f : n \mapsto \ulcorner \phi(\overline{n}) \urcorner$. Then there is a recursive function g such that for any $n \in \omega$:*

- (1) $n \in f[A] \Leftrightarrow g(n) \in f[A] \Leftrightarrow g(n) \in T_P;$
- (2) $n \in f[B] \Leftrightarrow g(n) \in f[B] \Leftrightarrow g(n) \in T_R.$

Proof. Note that f is recursive, and if $n \in A$, then $f(n) \in T_P$, and if $n \in B$, then $f(n) \in T_R$. Since (A, B) is semi-reducible to $(f[A], f[B])$ and (A, B) is **El**, by Proposition 2.5, $(f[A], f[B])$ is **El**. Since T is a Rosser theory, (T_P, T_R) is **El**. Note that $f[A] \subseteq T_P$ and $f[B] \subseteq T_R$. Apply Theorem 3.14 to $(f[A], f[B])$ and (T_P, T_R) . Then there is a recursive function g such that for any $n \in \omega$, we have:

$$\begin{aligned} n \in f[A] &\Leftrightarrow g(n) \in f[A] \Leftrightarrow g(n) \in T_P; \\ n \in f[B] &\Leftrightarrow g(n) \in f[B] \Leftrightarrow g(n) \in T_R. \end{aligned}$$

\square

3.3. Shoenfield’s theorems and their applications. Shoenfield’s theorems in [15] are an important tool in the proof of Theorem 3.25. In Theorem 3.20 and Theorem 3.21, we give detailed proofs of Shoenfield’s theorems. We say that a set D separates B and C if $B \subseteq D$ and $D \cap C = \emptyset$.

Theorem 3.20 (Shoenfield, [14]). *For any RE set A , there is a disjoint pair (B, C) of RE sets such that $B, C \leq_T A$, and for any RE set D that separates B and C , we have $A \leq_T D$. If A is a non-recursive RE set, then (B, C) is recursively inseparable.*

Proof. Suppose $A = W_e$. Define

$$x \in B \Leftrightarrow \exists y [T_1(e, (x)_0, y) \wedge \forall z \leq y \neg T_1((x)_1, x, z)]$$

and

$$x \in C \Leftrightarrow \exists y [T_1(e, (x)_0, y) \wedge \exists z \leq y T_1((x)_1, x, z)].$$

We show that (B, C) has the stated properties.

Note that B and C are disjoint RE sets. If $(x)_0 \notin A$, then $x \notin B$ and $x \notin C$; if $(x)_0 \in A$, then we can decide either $x \in B$ or $x \in C$. Thus, we have $B, C \leq_T A$.

Suppose D is a RE set with index n , and $B \subseteq D$ and $D \cap C = \emptyset$. We show that $A \leq_T D$.

Claim. $x \in A \Leftrightarrow \exists z(T_1(n, \langle x, n \rangle, z) \wedge \exists y < z T_1(e, x, y))$.

Proof. The right-to-left direction is obvious. Now we show the left-to-right direction. Suppose $x \in A$. Then either $\langle x, n \rangle \in B$ or $\langle x, n \rangle \in C$. Suppose $\langle x, n \rangle \in C$. Let y be the unique witness such that $T_1(e, x, y)$ holds. Then there exists $z \leq y$ such that $T_1(n, \langle x, n \rangle, z)$. Then $\langle x, n \rangle \in D$, which contradicts that $D \cap C = \emptyset$. Thus we have $\langle x, n \rangle \in B$.

Let y be the unique witness such that $T_1(e, x, y)$ holds. Since $\langle x, n \rangle \in D$, we have $T_1(n, \langle x, n \rangle, z)$ holds for some z . Since for all $z \leq y$, $\neg T_1(n, \langle x, n \rangle, z)$ holds, we have $z > y$. Thus, we have $\exists z(T_1(n, \langle x, n \rangle, z) \wedge \exists y < z T_1(e, x, y))$. This ends the proof of the claim. \square

Now we show that $A \leq_T D$. If $\langle x, n \rangle \notin D$, then $x \notin A$. If $\langle x, n \rangle \in D$, from the above claim, we can effectively decide whether $x \in A$.

We show if A is non-recursive, then (B, C) is recursively inseparable: if X is a recursive set separating B and C , then $A \leq_T X$ and hence A is recursive, that leads to a contradiction. \square

Theorem 3.21 (Shoenfield, [14]). *For any RE set A , there is a consistent RE theory T having one non-logical symbol that has the same Turing degree as A , and T is essentially undecidable if A is not recursive.*

Proof. Pick the pair $\langle B, C \rangle$ of RE sets from A as constructed in Theorem 3.20. Now we define the theory $T_{(B,C)}$ with $L(T_{(B,C)}) = \{R\}$ where R is a binary relation symbol.

Let Φ_n be the statement that there exists an equivalence class of R of size precisely $n + 1$. Let Ψ_n be the statement that there is at most one equivalence class of R of size precisely n . Let Υ_n be the statement that there are at least n equivalence class of R with at least n elements⁷.

We denote the theory $T_{(B,C)}$ by T , which contains the following axioms:

- (1) the axiom asserting that R is an equivalence relation;
- (2) Ψ_n for each $n \in \omega$;
- (3) Υ_n for each $n \in \omega$;
- (4) Φ_n for all $n \in B$;
- (5) $\neg\Phi_n$ for all $n \in C$.

Clearly, T is a consistent RE theory. Since Φ_n is provable iff $n \in B$, and $\neg\Phi_n$ is provable iff $n \in C$, we have B and C are recursive in T .

Lemma 3.22 is a reformulation of Janiczak's Lemma 2 in [8] in the context of the theory T . Janiczak's Lemma is proved by means of a method known as the elimination of quantifiers.

⁷As Visser pointed out, we include this axiom just to make the proof of Lemma 3.22 more easy.

Lemma 3.22 (Janiczak, Lemma 2 in [8]). *Over T , every sentence is equivalent to a boolean combination of the Φ_n , and this boolean combination can be found explicitly from the given sentence.*

By Lemma 3.22, T is recursive in B and C . Hence T is recursive in A since $B, C \leq_T A$. Since B separates B and C , by Theorem 3.20, $A \leq_T B$. Thus, since B is recursive in T , T has the same Turing degree as A .

By a standard argument, we can show that T is essentially undecidable if A is not recursive. \square

Remark. The proof of Theorem 3.21 essentially shows that: given a disjoint pair (B, C) of RE sets, there is a RE theory $T_{(B,C)}$ such that $T_{(B,C)} \equiv_T (B, C)$ and if (B, C) is recursively inseparable, then $T_{(B,C)}$ is essentially undecidable.

Definition 3.23 ([13]).

- (1) Given $A, B \subseteq \mathbb{N}$, we say A is m -reducible to B (denoted by $A \leq_m B$) if there is a recursive function f such that $n \in A \Leftrightarrow f(n) \in B$.
- (2) We say $X \subseteq \mathbb{N}$ is Σ_n^0 -complete if X is a Σ_n^0 set and for any Σ_n^0 set Y , $Y \leq_m X$. Similarly for Π_n^0 -complete.

Now, we discuss some applications of Theorem 3.20 and Theorem 3.21. We first show that the set of indexes of recursively inseparable theories is Π_3^0 -complete. We use the following result from [13].

Fact 3.24 (Theorem XVI, p. 327, [13]). The set $\{e : W_e \text{ is recursive}\}$ is Σ_3^0 -complete.

Theorem 3.25. *The set $\{e : W_e \text{ is a RI theory}\}$ is Π_3^0 -complete.*

Proof. Define $U_0 = \{e : W_e \text{ is undecidable}\}$ and $U_1 = \{e : W_e \text{ is a RI theory}\}$. We show that there is a recursive function s such that $U_0 \leq_m U_1$ via the function s .

Note that the construction of the pair (B, C) from a RE set A in Theorem 3.20 is effective: there are recursive functions f_0 and f_1 such that if $A = W_e$, then $B = W_{f_0(e)}$ and $C = W_{f_1(e)}$ such that (B, C) has the properties as in Theorem 3.20. Given a disjoint pair (B, C) of RE sets, note that the construction of $T_{(B,C)}$ in Theorem 3.21 is also effective: there is a recursive function $h(x, y)$ such that if $B = W_n$ and $C = W_m$ and $B \cap C = \emptyset$, then $T_{(B,C)} = W_{h(n,m)}$ such that $T_{(B,C)}$ has the same Turing degree as A .

Define $s(n) = h(f_0(n), f_1(n))$. Note that s is recursive. Suppose $A = W_e$ and $B = W_{f_0(e)}$ and $C = W_{f_1(e)}$ are constructed from A as in Theorem 3.20. Then $s(e)$ is the index of the theory $T_{(B,C)}$ constructed from (B, C) as in Theorem 3.21. We show that $e \in U_0 \Leftrightarrow s(e) \in U_1$.

Case one: $e \in U_0$. Note that (B, C) is recursively inseparable since $A = W_e$ is not recursive. We denote the theory $T_{(B,C)}$ by T . Define $g : n \mapsto \ulcorner \Phi_n \urcorner$. Note that g is recursive, if $n \in B$, then $g(n) \in T_P$, and if $n \in C$, then $g(n) \in T_R$. We show that T is RI. Suppose T is not RI. I.e., there is a recursive set X such that $T_P \subseteq X$ and $X \cap T_R = \emptyset$. Note that $B \subseteq g^{-1}[T_P] \subseteq g^{-1}[X]$ and $C \subseteq g^{-1}[T_R] \subseteq g^{-1}[\overline{X}] = \overline{g^{-1}[X]}$. Since g and X are recursive, $g^{-1}[X]$ is recursive. Thus, $g^{-1}[X]$ is a recursive set separating B and C , which contradicts that (B, C) is RI. Thus, $s(e) \in U_1$.

Case two: $e \notin U_0$. Since $T_{(B,C)}$ has the same Turing degree as $A = W_e$, $T_{(B,C)}$ is recursive. Thus, $s(e) \notin U_1$. Hence, $U_0 \leq_m U_1$ via the recursive function s . By Fact 3.24, U_0 is Π_3^0 -complete. It is easy to check that U_1 is Π_3^0 . Hence, U_1 is Π_3^0 -complete. \square

As a corollary of Theorem 3.25, the set $\{e : (W_{e_0}, W_{e_1}) \text{ is RI}\}$ is Π_3^0 -complete.

A natural question is whether Theorem 3.20 and Theorem 3.21 can be generalized in the following sense: given a non-recursive RE set A , is there an EI pair (B, C) such that A, B and C have the same Turing degree? or is there an EI theory T such that T has the same Turing degree as A ? Let A be a non-recursive RE set. If (B, C) is EI, then both B and C have the Turing degree $0'$. Thus, if A has the Turing degree less than $0'$, then there is no such an EI pair (and there is no such an EI theory) with the same Turing degree as A . If A has the Turing degree $0'$, then any EI pair (and any EI theory) has the same Turing degree as A .

Now, we show that the set of indexes of effectively inseparable theories is Σ_3^0 using $\text{EI} \Leftrightarrow \text{DG}$.

Theorem 3.26. *The set $\{e : W_e \text{ is an EI theory}\}$ is Σ_3^0 .*

Proof. Define $U = \{e : W_e \text{ is an EI theory}\}$. A direct computation from the definition of EI shows that U is Σ_4^0 . Using that $\text{EI} \Leftrightarrow \text{DG}$, we could show that U in fact is Σ_3^0 .

By the s-m-n theorem, there is a recursive function h such that if $T = W_e$, then $W_{h(e)} = T_R$. Note that “ $e \in U$ ” is equivalent to the following formula:

$$\exists n \forall i \forall j [W_i \cap W_j = \emptyset \rightarrow ((\phi_n(i, j) \in W_e \leftrightarrow \phi_n(i, j) \in W_i) \wedge (\phi_n(i, j) \in W_{h(e)} \leftrightarrow \phi_n(i, j) \in W_j))].$$

Since $e \in U$ can be written in the form $\exists \forall \forall [\Pi_1^0 \rightarrow ((\Sigma_1^0 \rightarrow \Sigma_1^0) \wedge (\Sigma_1^0 \rightarrow \Sigma_1^0))]$, U is Σ_3^0 . \square

Remark. We know that $\{e : W_e \text{ is creative}\}$ is Σ_3^0 -complete (see [13]). We conjecture that $\{e : W_e \text{ is an EI theory}\}$ is Σ_3^0 -complete.

3.4. There are many EI theories weaker than the theory \mathbf{R} . In this section, we show that there are many EI theories weaker than the theory \mathbf{R} .

Definition 3.27. For RE theories S and T , we use $S \trianglelefteq T$ to denote that S is interpretable in T , and $S \triangleleft T$ to denote that S is interpretable in T but T is not interpretable in S .

Theorem 3.28 (Smullyan, Theorem 4, [17]). *For consistent theories T_1 and T_2 , if $T_1 \trianglelefteq T_2$ and T_1 is $\text{RI}(\text{EI})$, then T_2 is $\text{RI}(\text{EI})$.*

Corollary 3.29. *If the theory \mathbf{R} is interpretable in T , then T is EI.*

A natural question is: if T is $\text{RI}(\text{EI})$, is the theory \mathbf{R} interpretable in T ? We answer these questions negatively.

Definition 3.30. Given any disjoint pair (A, B) of RE sets, we construct the theory $T_{(A,B)}$ as follows. Let $L(T_{(A,B)}) = \{\mathbf{0}, \mathbf{S}, P\}$. The axioms of $T_{(A,B)}$ consist of:

- (1) $\overline{m} \neq \overline{n}$ if $m \neq n$;
- (2) $P(\overline{n})$ if $n \in A$;
- (3) $\neg P(\overline{n})$ if $n \in B$.

Theorem 3.31 (Cheng, [1]). *If (A, B) is RI, then $T_{(A,B)}$ is essentially undecidable and $T_{(A,B)} \triangleleft \mathbf{R}$.*

Theorem 3.32 shows that from any RI pair, we can effectively construct a RI theory which is strictly weaker than the theory \mathbf{R} w.r.t. interpretation.

Theorem 3.32. *If (A, B) is RI, then there is a RE theory $T_{(A,B)}$ such that $T_{(A,B)}$ is RI and $T_{(A,B)} \triangleleft \mathbf{R}$.*

Proof. Let $T_{(A,B)}$ be the theory constructed as in Definition 3.30, and we denote it by T . Define $f : n \mapsto \ulcorner P(\bar{n}) \urcorner$. Note that f is recursive, if $n \in A$, then $f(n) \in T_P$, and if $n \in B$, then $f(n) \in T_R$.

By Theorem 3.31, $T \triangleleft \mathbf{R}$. By a similar argument as in Theorem 3.25, we can show that T is RI: if X is a recursive set separating T_P and T_R , then $f^{-1}[X]$ is a recursive set separating A and B . □

Theorem 3.33 shows that from any EI pair, we can effectively construct an EI theory which is strictly weaker than the theory \mathbf{R} w.r.t. interpretation.

Theorem 3.33. *If (A, B) is EI, then there is a RE theory $T_{(A,B)}$ such that $T_{(A,B)}$ is EI and $T_{(A,B)} \triangleleft \mathbf{R}$.*

Proof. Let $T_{(A,B)}$ be the theory constructed as in Definition 3.30, and we denote it by T . Define the recursive function $f : n \mapsto \ulcorner P(\bar{n}) \urcorner$.

By Theorem 3.31, $T \triangleleft \mathbf{R}$. Now, we show that T is EI. Suppose (A, B) is EI via the recursive function h . By the s-m-n theorem, there is a recursive function g such that for any $i \in \omega$, $f^{-1}[W_i] = W_{g(i)}$.

Define $s(i, j) = f(h(g(i), g(j)))$. We show that T is EI via the recursive function s .

Suppose $T_P \subseteq W_i$, $T_R \subseteq W_j$ and $W_i \cap W_j = \emptyset$. Note that $A \subseteq f^{-1}[T_P] \subseteq f^{-1}[W_i] = W_{g(i)}$, $B \subseteq f^{-1}[T_R] \subseteq f^{-1}[W_j] = W_{g(j)}$, and $W_{g(i)} \cap W_{g(j)} = \emptyset$. Since (A, B) is EI via the recursive function h , $h(g(i), g(j)) \notin W_{g(i)} \cup W_{g(j)} = f^{-1}[W_i] \cup f^{-1}[W_j]$. Thus, $s(i, j) \notin W_i \cup W_j$. I.e. T is EI via the recursive function s . □

We conclude the paper with one question for future research, which we did not explore.

Question 3.34. Let T be a consistent RE theory. If T is a Rosser theory, is T an exact Rosser theory?

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