MSM3F1/4F1a - COMPLEX VARIABLE THEORY FOR PHYSICISTS

1. Introduction

1.1. A review of complex numbers.

1. Complex number. A complex number z is defined as $z = x + iy$, where x and y are real numbers and $i^2 = -1$. x is called the real part of z and y is the imaginary part of z. Notation: $x = \Re(z)$, $iy = \Im(z)$), so that $z = \Re(z) + i\Im(z)$.

2. Equality. Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$. Note that for complex numbers we don't have the notions of 'smaller' and 'greater', only equal or not.

3. Modulus. The real number $r = \sqrt{x^2 + y^2}$ is known as the modulus of the complex number $z = x + iy$. In mathematical notation: $r \equiv |z| = \sqrt{x^2 + y^2}$.

4. Argand plane. Geometrically, complex numbers can be represented using a Cartesian coordinate plane (in the theory of complex numbers known as 'Argand plane'), where each point with coordinates (x, y) is regarded as an image of the complex number $z = x + iy$.

5. Numbers as vectors. Given the definition of the modulus, it is convenient to think of a complex number as a *vector* in the Argand plane pointing from the origin to the point with coordinates (x, y). Then, $|z|$ is simply the modulus of this vector. (Note that this representation of complex numbers as vectors should not be taken to the extreme: although complex numbers treated as vectors do form a linear vector field, they are more than vectors; they are *numbers* and, unlike with vectors, we can divide by complex numbers, define various functions of them, etc.)

6. Argument. The geometric representation of complex numbers as vectors in the complex plane allows us, for $z \neq 0$, to introduce, alongside the modulus of z, its argument $\theta \equiv \arg(z)$ by

$$\tan(\arg z) = \frac{y}{x}.$$

Geometrically, $\arg(z)$ is the angle between the real semi-axis and the vector z. Given that the above equation determines the argument only up to integer multiplies of 2π, it is convenient to define the principal argument, $\mathrm{Arg}\,(z)$, as the argument lying between $-\pi$ and π (inclusive). In other words,

$$\mathrm{Arg}\,(z) = \arctan\frac{y}{x},\ (x > 0); \quad \mathrm{Arg}\,(z) = \mathrm{sign}\,(y)\frac{\pi}{2},\ (x = 0);$$

$$\mathrm{Arg}\,(z) = \arctan\frac{y}{x} + \pi,\ (x < 0,\ y \geqslant 0); \quad \mathrm{Arg}\,(z) = \arctan\frac{y}{x} - \pi,\ (x < 0,\ y < 0).$$

7. r-θ **representation.** From the geometric representation of $z = x + iy$ we have $x = r\cos\theta$, $y = r\sin\theta$, so that

$$z = x + iy = \underbrace{\sqrt{x^2+y^2}}_{=r}(\underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{=\cos\theta} + i\underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{=\sin\theta}) = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

This is Euler's formula from 1748.

8. Conjugate. The complex conjugate of a complex number $z = x + iy$ is the number $\overline{z} = x - iy$. In terms of the modulus and the argument, we have $\overline{z} = re^{-i\theta}$.

9. Properties 1. By direct calculation and the use of the representation of complex numbers in terms of modulus and argument, we can easily prove the following statements:

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
(ii) $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$
(iii) $z\overline{z} = |z|^2$
(iv) $|z_1 z_2| = |z_1||z_2|$
(v) $|z_1 + z_2| \leqslant |z_1| + |z_2|$ (triangular rule).

10. Properties 2. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, we can show that

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2),$$
$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta),$$

which is the familiar De Moivre's formula.

Finally, for $\alpha \in \mathbb{R}$, we have

$$z^\alpha = r^\alpha e^{i\alpha(\theta + 2\pi n)}.$$

1.2. **Functions of complex variables.**

We will be interested in functions $f(z)$ which map a set of complex numbers to another set of complex numbers, i.e. $w = f(z)$, where $z, w \in \mathbb{C}$. In other words,

$$f(z) = u(x,y) + iv(x,y),$$

where $u(x,y)$ and $v(x,y)$ are real functions of real variables.

Examples:

(a) Let $f(x) = z^2$. Then,

$$f(z) = (x+iy)^2 = x^2 - y^2 + i2xy,$$

so that $u(x,y) = x^2 - y^2)$, $v(x,y) = 2xy$. Using the representation $z = re^{i\theta}$, we have

$$f(z) = \left(re^{i\theta}\right)^2 = r^2 e^{i2\theta},$$

i.e. this function squares the modulus of z and doubles its argument.

(b) Let $f(z) = e^{i\alpha} z$, where $\alpha \in \mathbb{R}$. This function rotates z by the agnle α.

(c) Let $f(z) = \overline{z}$. This function reflects z with respect to the real axis.

(d) Let $f(z) = \dfrac{1}{z}$. This function, defined for $z \neq 0$, reflects z with respect to the circumference of the unit radius *and* with respect to the real axis as

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{\overline{z}}{|z|^2}.$$

1.3. **Limits of functions.**

The notion of a limit for a function of the complex variable $\lim_{z\to z_0} f(z)$ is analogous to the corresponding notion for a real function. Therefore, it is advisable to refresh what you know about the limits of real-valued functions of real variables.

For real numbers we have the notions of 'greater' and 'smaller', which we do not have for complex numbers. However, given the notion of the real-valued 'modulus' introduced earlier, we can define the distance between complex numbers z_1 and z_2 as $|z_1 - z_2|$, which is the direct generalization of the notion of absolute value in the real analysis. Then, we can follow the same route as in the real analysis and introduce the notion of the ϵ-vicinity of a complex number, saying that z is in the ϵ-vicinity of z_0 if the distance between z and z_0 is less than ϵ. 'In other words', $z \in V_\epsilon(z_0) \equiv |z - z_0| < \epsilon$.

Definition 1.1. A complex number z_0 is a limit point of a set S if in every vicinity of this number there is a point belonging to S and different from z_0, i.e. z_0 is a limit point if $\forall \epsilon > 0$ $\exists z \in S$, $z \neq z_0$: $|z - z_0| < \epsilon$. □

Definition 1.2. If we have a complex-valued function $f(z)$ of a complex variable z defined on a set S, i.e. $f : S \mapsto \mathbb{C}$, and z_0 is a limit point in S, then the limit of the function f as z tends to z_0 is a number W such that for an arbitrary positive real number ϵ there exists a positive real number δ such that for all $z \in S$ such that $|z - z_0| < \delta$ we have that $|f(z) - W| < \epsilon$.

'In other words', $W = \lim_{z\to z_0} f(z) \equiv \forall \epsilon > 0\ \exists \delta > 0$: $\forall z \in S$, $z \neq z_0$, $|z - z_0| < \delta$, $|f(z) - W| < \epsilon$.

1.4. **Continuity.**

Definition 1.3. A function $f(z)$ is continuous at z_0 if

(a) $f(z_0)$ is defined,
(b) $\lim_{z\to z_0} f(z) = f(z_0)$.

If $f(z)$ is continuous at every point of a set S, it is said to be continuous on S. □

Examples of continuous functions include

(a) Polynomials $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, where a_i, $(i = 0, 1, \ldots, n)$ are complex numbers, which are defined and continuous at every point of the complex plane,
(b) Rational functions of polynomials $f(z) = P(z)/R(z)$, where P and R are polynomials; these functions are defined and continuous at all points where $R(z) \neq 0$.

2. Differential, Derivatives and Analytic Functions

2.1. **Differential.** The term 'differential' is familiar, but we need to properly define it.

Definition 2.1. Let in the vicinity of z_0 an increment of a function $f(z)$, i.e. $\triangle f = f(z) - f(z_0)$ can be representable in terms of the increment $\triangle z = z - z_0$ of the independent variable as

$$\triangle f = \underbrace{A \overbrace{\triangle z}^{=dz}}_{=df} + o(\triangle z), \quad \text{as } \triangle z \to 0,$$

where $o(\triangle z)$ is a function that tends to 0 as $\triangle z \to 0$ *faster* than $\triangle z$, i.e. 0

$$\lim_{\triangle z \to 0} \frac{o(\triangle z)}{\triangle z} = 0.$$

(One can represent $o(\triangle z) = \epsilon(\triangle z) \cdot \triangle z$, where $\epsilon(\triangle z)$ is some function going to 0 as $\triangle z \to 0$.)

Then, the function $f(z)$ representable in the vicinity of z_0 in this form (i.e. leading order linearly) is called <u>differentiable</u>; the increment $\triangle z$ of the independent variable is called its <u>differential</u>, $\triangle z \equiv dz$; the *linear part* of the increment of the function is called the <u>differential</u> of this function, $df = A\,dz$. $\square$

2.2. Derivative.

Definition 2.2. Let $f(z)$ be defined in some neighbourhood of a point z_0. Then, the <u>derivative</u> of f at z_0 is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{1}$$

if this limit exists. If we use the notation $h = z - z_0$, where, obviously, $h \in \mathbb{C}$, then (1) can be written as

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}. \tag{2}$$

$\square$

It can be easily shown that — for a function of one variable — the terms 'differentiable' and 'having a derivative' are equivalent: if the function is differentiable, it has a derivative and vice versa. We also have that A in the definition of the differentiability is equal to f', so that, finally,

$$df = f'(z)\,dz$$

or

$$f'(z) = \frac{df}{dz}.$$

In other words, the derivative is simply a ratio of differentials.

Note that we can say nothing of the sort about a partial derivative as $\frac{\partial}{\partial}$ is essentially one symbol, not a ratio of anything. Furthermore, for a function of more than one variable 'differentiability' and 'having derivatives' are not equivalent: the function can have derivatives but not being differentiable (not even continuous); but differentiability, as a stronger condition, ensures the existence of derivatives.

Example: Let $f(z) = z^2$ and find $f'(z_0)$. Using the definition, we have

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{(z_0 + h)^2 - z_0^2}{h} = \lim_{h \to 0} \frac{(z_0 + h)^2 - z_0^2}{h}$$

$$= \lim_{h\to 0} \frac{z_0^2 + 2z_0h + h^2 - z_0^2}{h} = \lim_{h\to 0} \frac{2z_0h}{h} + \lim_{h\to 0} \frac{h^2}{h} = 2z_0.$$

Note that all derivatives of the elementary functions that we take for granted as 'rules' have been calculated from the definition of the derivative by taking the corresponding limit. □

2.3. Rules of differentiation.

All the rules of differentiation we know for functions of one real variable apply to the function of one complex variable, namely

(i) $\dfrac{d}{dz}[c_1f_1(z)+c_2f_2(z)] = c_1\dfrac{df_1(z)}{dz}+c_2\dfrac{df_2(z)}{dz}$, (here c_1 and c_2 are complex constants),

(ii) $\dfrac{d}{dz}[f_1(z)f_2(z)] = \dfrac{df_1(z)}{dz}f_2(z) + f_1(z)\dfrac{df_2(z)}{dz}$,

(iii) $\dfrac{d}{dz}\left[\dfrac{f_1(z)}{f_2(z)}\right] = \dfrac{\dfrac{df_1(z)}{dz}f_2(z) - f_1(z)\dfrac{df_2(z)}{dz}}{f_2^2(z)}$,

The chain rule also remains intact:

$$\frac{d}{dz}f(w(z)) = \frac{df}{dw}\frac{dw}{dz}.$$

2.4. Conditions of Differentiability: The Cauchy-Riemann equations.

Differentiability of function $f(z)$ at a point z_0 means that the limit in (1) is the *same* no matter along what path z approaches z_0. This is a very strong requirement which puts certain restrictions on the real and imaginary parts of f. Let us derive these requirements by considering two directions from which to approach z_0.

Let $f(x+iy) = u(x,y) + iv(x,y)$ and consider

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z)}{h},$$

where $h = t \in \mathbb{R}$. Then,

$$f'(z_0) = \lim_{t\to 0} \left\{ \left[\frac{u(x_0+t,y_0) - u(x_0,y_0)}{t}\right] + i\left[\frac{v(x_0+t,y_0) - v(x_0,y_0)}{t}\right]\right\}$$

$$= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0).$$

Now, if we consider $f'(z_0) = \lim_{h\to 0} \dfrac{f(z_0+h) - f(z)}{h}$ where we take $h = it$, $t \in \mathbb{C}$, then

$$f'(z_0) = \lim_{t\to 0} \left\{ \left[\frac{u(x_0,y_0+t) - u(x_0,y_0)}{it}\right] + i\left[\frac{v(x_0,y_0+t) - v(x_0,y_0)}{it}\right]\right\}$$

$$= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0).$$

Since both of the above expressions represent $f'(z_0)$, they must be the same and we can equate the real and imaginary parts of the right-hand sides, so that at z_0 where $f(z)$ is differentiable we have

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

These equations are known as the **Cauchy-Riemann equations**.[1]

Examples.

(a) Let us show that the Cauchy-Riemann equations are satisfied by the function $f(z) = z^2$ at every point. Indeed, as we already know, for this function $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$. Then, by calculating the derivatives of u and v and substituting them into the Cauchy-Riemann equations, we immediately see that these equations are satisfied.

(b) Let us show that for the function $f(z) = \Re(z)$ the Cauchy-Riemann equations are <u>not</u> satisfied. Indeed, for this function we have $f(z) = \Re(z) = x + i0$, so that we immediately see that

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 0,$$

and hence $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$.

2.5. **Analytic functions.** Now, we will introduce the central concept of the whole course.

Definition 2.3. A function $f(z)$ is said to be <u>**analytic**</u> (or <u>holomorphic</u>[2]) at a point z_0 if it is differentiable at this point. If $f(z)$ is differentiable at every point of a domain D, then it is said to be analytic in D. Notation: $f \in A(z_0)$; $f \in A(D)$. □

We will consider the conditions of analyticity but before we need to recap the notion of differentiability of real-valued functions of several (in our case, two) real variables.

Definition 2.4. A real-valued function $u(x, y)$ of two real variables x and y is called <u>differentiable</u> at a point (x_0, y_0) if in the neighbourhood of this point the increment of this function $\triangle u = u(x, y) - u(x_0, y_0)$ is represented in terms of the increments of the independent variables $\triangle x = x - x_0$, $\triangle y = y - y_0$ in the form

$$\triangle u = \underbrace{a \overbrace{\triangle x}^{=dx} + b \overbrace{\triangle y}^{=dy}}_{=du} + o(\triangle x, \triangle y),$$

where $o(\triangle x, \triangle y)$ is a function going to zero faster than linearly when $\triangle x$, $\triangle y \to 0$. □

Remark; As one can easily verify, differentiability of u guarantees the existence of its partial derivatives with respect to x, and y (which are equal to a and b, respectively), <u>but</u> the converse statement is not true: the existence of partial derivatives does not mean that the function is differentiable or even continuous. Here we have an important difference between the functions of several real variables and one real variable; for the latter differentiability and existence of the derivative are equivalent. □

[1] This historic label is more than unfair as these equations have been known to, and intensively studied by, D'Alembert and Euler in various applications before Cauchy and Riemann have even been born.

[2] This term, which is becoming increasingly popular, has been introduced by Cauchy's followers who used it to emphasize that these functions, as polynomials, have the same properties in the entire plane. The name comes from the Greek words $o\lambda o\varsigma$ meaning 'entire' and $\mu o\rho\varphi\varsigma$ meaning 'form', 'shape'.

Theorem 2.5. A complex-valued function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, is analytic as a function of the complex variable z at a point z_0 of a domain D if and only if **(a)** $u(x, y)$ and $v(x, y)$ as real-valued functions of real variables x and y are differentiable and **(b)** their derivatives satisfy the Cauchy-Riemann equations at this point. Or, to put it mathematically,

$$f(z = x+iy) \equiv u(x,y)+iv(x,y) \in A(z_0) \quad \Leftrightarrow \quad \begin{cases} u(x,y), v(x,y) \in D(x_0, y_0) \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \text{ at } x = x_0, y = y_0 \end{cases}$$

□

Examples:

(a) The polynomial $P_n(z) = a_0 z^n + a_1 z^{n-1} = \cdots + a_n$ is obviously an analytic function.

(b) The rational function $f(z) = \dfrac{c}{(z - z_0)^m}$, where $m \in \mathbb{N}$, $c, z_0 \in \mathbb{C}$ is also an analytic function for $z \neq z_0$.

(c) Let $f(z) = e^x(\cos y + i \sin y)$. Then, $u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$ and we have

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x},$$

so that the Cauchy-Riemann equations are satisfied and the function is analytic.

Remark. In some cases it is necessary to have conditions of analyticity of a function $f(z) = u + iv$ expressed in terms of the modulus and the argument of z, i.e. in terms of r and θ (applicable if $z \neq 0$). These conditions can be obtained by the standard change of variables procedure and are given by:

(a) u and v are differentiable functions of r and θ,

(b) $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}, \quad \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$. □

2.6. **Laplace's equation.** One of the reasons for the importance of (complex-valued) analytic functions is that they are closely related to harmonic function, i.e. solutions to Laplace's equation, which appears in many areas of physics. Indeed, let $u(x, y)$ and $v(x, y)$ be real-valued functions with continuous second derivatives in a domain D. Then, if $f(x + iy) = u(x, y) + iv(x, y)$ is analytic as a function of the complex variable $z = x + iy$, then u and v satisfy the Cauchy-Riemann's equations

$$u_x = v_y, \qquad u_y = -v_x.$$

By eliminating v from this set of equations (i.e. differentiating the first one with respect to x and the second one with respect to y and adding them up), we arrive at

$$u_{xx} + u_{yy} = 0,$$

which is Laplace's equation. Similarly, by eliminating u, we obtain

$$v_{xx} + v_{yy} = 0.$$

Hence both $u(x, y)$ and $v(x, y)$ are harmonic functions.

Definition 2.6. Harmonic functions $u(x, y)$ and $v(x, y)$ related by the Cauchy-Riemann equations are call harmonic conjugate. □

2.7. **Exponential function.**

Definition 2.7. We define the exponential function by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

which is obviously analytic for all $z \in \mathbb{C}$. □

Properties of exponentials

(i) $\frac{d}{dz} e^z = e^z$,
(ii) $e^0 = 1$,
(iii) $e^{z_1+z_2} = e^{z_1} e^{z_2}$, for all $z_1, z_2 \in \mathbb{C}$,
(iv) $e^z \neq 0$, for all $z \in \mathbb{C}$,
(v) $e^z = e^x(\cos y + i \sin y)$, where $z = x + iy$,
(vi) $e^{z+2\pi i} = e^z$, (periodicity with period $2\pi i$).

Note the difference between e^z and its real-valued counterpart e^x: the former is a periodic function whereas the latter is not.

Example of a simple problem whose solution we will need. Find all solutions to the equation $e^z = 3 + 4i$, where $z = x + iy$.

SOLUTION.

$$|e^z| = e^x = \sqrt{3^2 + 4^2} = 5, \quad \text{hence } x = \ln(5) \approx 1.609.$$

Then, since

$$e^z = 5(\cos y + i \sin y) = 3 + 4i,$$

we have $\cos y = \frac{3}{5}$ and $\sin y = \frac{4}{5}$, so that $y = \arcsin(\frac{4}{5}) + 2\pi k \approx 0.927 + 2\pi k$, $k \in \mathbb{Z}$. Hence

$$z = \ln(5) + i \arcsin(\tfrac{4}{5}) + 2\pi k i \approx 1.609 + 0.927i + 2\pi k i, \qquad k \in \mathbb{Z}.$$

2.8. **Trigonometric function.** We define two analytic functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

2.9. **Hyperbolic functions.** We define two more analytic functions

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.$$

Relationship between complex trigonometric and hyperbolic functions. Complex trigonometric and hyperbolic functions are obviously related:

$$\cosh(iz) = \cos(z), \qquad \cos(iz) = \cosh(z),$$

$$\sinh(iz) = i \sin(z), \qquad \sin(iz) = i \sinh(z).$$

2.10. **Logarithm.** The exponential function is periodic ($e^z = e^{z+2\pi ni}$, $n \in \mathbb{Z}$) so that it is not one-to-one mapping of $\mathbb{C}$ onto $\mathbb{C}$. This becomes a problem when we need to find the inverse of this function.

Definition 2.8. Given a non-zero complex number z, a complex number w such that $e^w = z$ is called a logarithm of z. Notation: we write $w = \log(z)$ or $w = \ln(z)$. □

Let $w = u + iv$. Then, from $z \equiv |z|e^{i\arg(z)} = e^w = e^{u+iv} = e^u e^{iv}$ we have $|z| = e^u$ and $v = \arg(z)$. Then,

$$u = \ln|z|$$

and $v = \arg(z) = \mathrm{Arg}\,(z) + 2\pi k$, $k \in \mathbb{Z}$. Therefore, if $z \neq 0$, we have

$$\boxed{\log(z) = \ln|z| + i[\mathrm{Arg}\,(z) + 2\pi k], \quad k \in \mathbb{Z}}$$

This does not define a function as it is multi-valued.

We define the principal logarithm denoted as Log or Ln) to be

$$\boxed{\mathrm{Log}\,(z) = \ln|z| + i\mathrm{Arg}\,(z), \qquad -\pi < \mathrm{Arg}\,(z) \leqslant \pi}$$

which is now a function with a cut along the negative semi-axis (as the Arg we introduced does not allow us to cross this semi-axis).

Examples.

(a) $\log(4) = 1.386\cdots + 2n\pi i$,
(b) $\mathrm{Log}\,(4) = 1.386\ldots$,
(c) $\log(-4) = 1.386\cdots + (2n+1)\pi i$,
(d) $\mathrm{Log}\,(-4) = 1.386\cdots + \pi i$,
(e) $\mathrm{Log}\,(4i) = 1.386\cdots + \dfrac{\pi i}{2}$.

Properties of logarithm.

(i) $\log(z_1 z_2) = \log(z_1) + \log(z_2)$,
(ii) $\log(z_1/z_2) = \log(z_1) - \log(z_2)$,
(iii) $\mathrm{Log}\,(z)$ is an analytic function whose derivative is $\dfrac{1}{z}$.

2.11. **The general exponent.**

We define for $z \neq 0$

$$\boxed{z^a = e^{a\log(z)}}$$

where z and a are complex numbers. In general, z^a is multi-valued. The principal value of z^a is

$$\boxed{z^a = e^{a\mathrm{Log}\,(z)}}$$

Example (Euler 1746): Determine i^i.

Solution. We have

$$i^i = e^{i\log(i)} = e^{i\{\ln|i|+i(\pi/2+2k\pi)\}} = e^{-(\pi/2+2k\pi)}.$$

The principal value is $e^{-\pi/2}$.

2.12. **Branch points.** Branch points are one type of singularity which we will encounter. The function z^a has a branch point at $z = 0$ and another at infinity, where the function is not analytic. The function z^a is multi-valued as we wind around these branch points, and this will be the basis of the formal definition that we'll give later, when we consider classification of singular points. This problem is resolved by introducing a *branch cut*, i.e. a boundary joining both branch points and hence forbidding winding around each of them.

Example. The negative real semi-axis is the branch cut for $\mathrm{Log}\,(z)$. □

2.13. **Riemann surface.**
For the function $w = \sqrt{z}$ (and some other functions) we can glue sheets of the complex z-plane together along the cut line to produce a surface with several sheets. We have

$$w = \sqrt{|z|}\left(\cos\frac{\arg(z)}{2} + i\sin\frac{\arg(z)}{2}\right).$$

Consider what happens if we go the full circle about the origin in the z-plane. If the argument varies from 0 to 2π, we will have one set of values for w, whereas if we go around the same circle in the z-plane with the argument of z varying from 2π to 4π, we will be going through *different* values of w. If we continue, and the argument of z increases above 4π, we will find ourselves again with the same values of w as in our first circle. In other words, our function is double-valued, so that after one circle we climb to the 'second sheet' (or leaf) of the surface where the values of w lie, and, after going along the second circle in the z-plane and the corresponding curve in the w-plane, we return to the first sheet (leaf). Thus, instead of having a z-plane with a cut introduced to avoid multi-valuedness, we can have a double-leaved surface where, by continuing the circular motion around the origin, we go from the first leaf to the second and then to the first one again. This construction is known as the Riemann surface. It can be imagined as produced when, instead of considering single-valued 'leaves', which we make single-valued by introducing branch cuts, we glue these leaves along these cuts.

Note: what we need to take from this is that, when we start considering contour integrals, we should remember that the domain of integration should be such that there is no multi-valuedness of the integrand and to ensure this we will need to introduce branch cuts as parts of the boundary of these domains.

2.14. **Mappings.**
It is not always straightforward to visualize complex functions. For a complex function

$$w = f(z), \qquad \text{where } w = u + iv,\ z = x + iy,$$

we need two planes: the z-plane and the w-plane. If we plot single points and their images, then we would not gain much insight into the properties of a given function. Therefore, it is much more informative to consider the examples where regions and families of curves in the z-plane are mapped onto the w-plane.

Example 1: $\boxed{w = z + b}$ (translation).

Example 2: $\boxed{w = az}$ (expansion/contractio + rotation).

(a) If $|a| = 1$, then $a = e^{i\alpha}$ and the mapping is simply a rotation about the origin by angle α.

(b) If a is real, then the mapping is an expansion (contraction) along the line connecting the origin and the given point.

(c) In a general case of a complex a, the mapping is an extension *and* a rotation.

Example 3: $\boxed{w = az + b}$

It is convenient to consider this mapping as a sequence: $w_1 = az$ (which is an expansion/contraction combined with rotation), $w = w_1 + b$ (which is a translation). Thus, our original mapping is the expansion/contraction combined with rotation + a translation.

Example 4: $\boxed{w = \dfrac{1}{z}}$ (inversion w.r.t. a unit circumference + reflection w.r.t. x-axis).

Using the polar form $z = re^{i\theta}$ and $w = \rho e^{i\varphi}$, we have

$$\rho e^{i\varphi} = \frac{1}{r} e^{-i\theta}.$$

Hence $\rho = \dfrac{1}{r}$, which means that the point in the interior of the unit circle in the z-plane are mapped to the exterior of the unit circle in the w-plane and vice versa. The $\varphi = -\theta$ is a reflection with respect to the x-axis.

Example 5: $\boxed{w = e^z}$

(a) Consider the image of a vertical line (i.e. a line parallel to the y-axis. Points of this line have the same value of $x = a$ whereas y varies from $-\infty$ to ∞. Hence, $w = e^{a+iy} = e^a(\cos y + i \sin y)$, so that, as y varies, we go in a circle of radius e^a.

(b) Consider the image of horizontal lines, $z = x + ib$, $-\infty < x < \infty$. Then $w = e^{x+ib} = e^x e^{ib}$, i.e. we have rays going from the origin to infinity with the argument ib.

(c) Now consider the image of a horizontal strip $z = x + iy$, where $-\infty < x < \infty$, $0 \leqslant y \leqslant \pi$. The above results give that the image is the upper half-plane.

Example 6: $\boxed{w = \mathrm{Log}\,(z)}$

If we have a one-to-one correspondence between the points we map and their images (the so called 'bijection'), then we can introduce the following principle. The properties of the inverse of the mapping $w = f(z)$ can be obtained from those of the mapping itself by interchanging the roles of the z-plane and the w-plane. As $w = \mathrm{Log}\,(z)$, $z \neq 0$, is the inverse

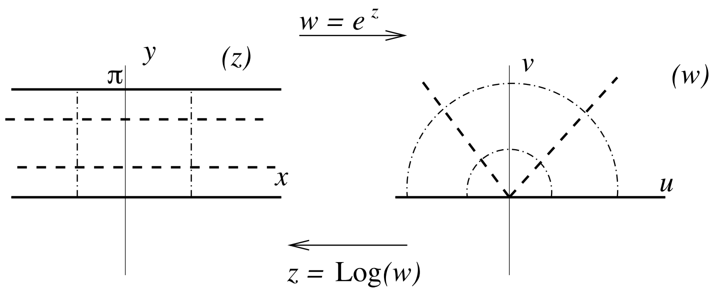

FIGURE 1. Mapping of a strip by the exponential function.

of the exponential function $z = e^w$, we know that the image of the upper half-plane is a strip $0 \leqslant y \leqslant \pi$.

Example 7: $\boxed{w = \sin(z)}$

We consider

$$w = u + iv = \sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y),$$

so that

$$u = \sin(x)\cosh(y), \quad v = \cos(x)\sinh(y).$$

(a) If we take the x-axis, i.e. $y = 0$ and $-\infty < x < \infty$, then the image is $u = \sin(x)$, $v = 0$. Hence, the x-axis maps onto a segment $-1 \leqslant u \leqslant 1$, $v = 0$ of the w-plane.

(b) If $y = b \neq 0$ and x varies (a horizontal line in the z-plane), then, using that $\sin^2 x + \cos^2 x = 1$, we find that

$$\frac{u^2}{\cosh^2(b)} + \frac{v^2}{\sinh^2(b)} = 1. \tag{3}$$

These are ellipses with semi-axes $\cosh(b)$ and $\sinh(b)$.

(c) If $x = a$ and y varies (a vertical line in the z-plane), then, using that $\cosh^2(y) - \sinh^2(y) = 1$ we have after eliminating y

$$\frac{u^2}{\sin^2(a)} - \frac{v^2}{\cos^2(a)} = 1.$$

These are hyperbolas intersecting the ellipses (3) at the right angles.

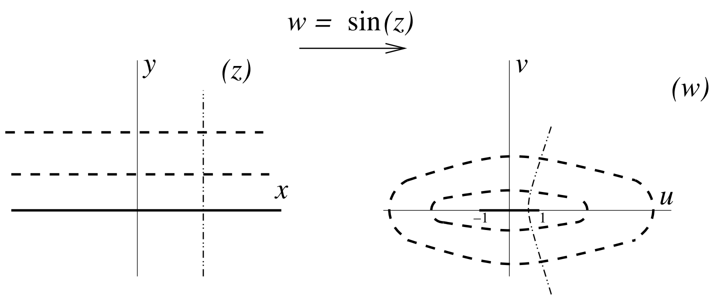

FIGURE 2. Mapping of a strip by the function $w = \sin(z)$.

3. Conformal Mapping

3.1. Introduction.

Definition 3.1. A mapping $w = f(z)$ that uses a continuous function f and preserves the angles between any two curves passing through a point z_0 is called conformal at this point. If the direction in which the angles are measured is also preserved, this is conformal mapping of the first kind; it it changes to the opposite, it is the conformal mapping of the second kind. If the mapping is conformal everywhere in a domain D with the exception of some critical points, it is called conformal in this domain. □

Theorem 3.2. If a complex-valued function $f(z)$ is analytic, then the mapping $w = f(z)$ is conformal, except at critical points $f'(0) = 0$. □

Gist of the proof.

$$\arg(df) = \arg(f'(z_0)dz) = \arg(f'(z_0)) + \arg(dz).$$

Hence all elementary segments dz are rotated by the same angle, and therefore the angle between any two of such segments is preserved. □

The above gist of the proof gives use the following follows:

> Geometric meaning of the **argument** of a derivative is that $\arg f'(z_0)$ is the angle by which the mapping $w = f(z)$ rotates all curves passing in the z-plane through the point z_0 about this point. (Once this fact is proven, the theorem follows immediately as the angle by which all curves are rotated about z_0 is the same, and hence the angles between these curves are preserved.)

The theorem above allows us to show that mappings are conformal simply by calculating the derivatives of the corresponding functions and showing that these derivatives are not equal to zero.

Examples.

(a) Show that $w = z^n$, $n = 2, 3, \ldots$, is conformal everywhere except $z = 0$.

Solution: The function is obviously analytic, so that we need to find only what the critical points are.

$$\frac{dw}{dz} = nz^{n-1} = 0 \quad \text{if } z = 0.$$

(b) Show that $w = e^z$ is conformal for all z.

> Geometric meaning of the **modulus** of a derivative $|f'(z_0)|$ is that it is the magnification ratio by which elementary segments $[z_0, z_0 + dz]$ are stretch (or shrunk) by the conformal mapping $w = f(z)$. This fact follows immediately from
> $$|f'(z_0)| = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}.$$

Remark. As we have seen in the examples of mappings, the fact that the conformal mapping preserves the angles does not mean that the shapes of the regions after mapping will be the same. Elementary segments are stretched differently at different points given that, in general, $|f(z)|$ varies from point to point. □

3.2. **Riemann's Theorem.** The following theoretical facts, which we will have without proofs, give the framework for conformal mapping. First, we will give two definitions.

Definition 3.3. A curve, i.e. the image of a segment $\gamma : [a, b] \mapsto \mathbb{C}$ where γ is continuous, is called simple (or Jordan's) if all its points are different, i.e. it has no self-intersections or self-touching, or, to put it mathematically, if $\forall\ t_1, t_2 \neq a, b$, such that $t_1 \neq t_2$, one has that $\gamma(t_1) \neq \gamma(t_2)$). A curve is called closed if $\gamma(a) = \gamma(b)$. $\square$

Definition 3.4. A domain D is called simply connected if for any closed simple curve passing through points of D the area enclosed by this curve also belongs to D. $\square$

Theorem 3.5. (Riemann, 1851). Any simply-connected domain D, whose boundary consists of more than one point, can be conformally mapped onto a circle (say, a unit circle), and there are infinitely many such mappings. $\square$

Corollary: For every two simply-connected domains, D in the z-plane and G in the w-plane, each having more than one boundary points, there exists infinitely many conformal mappings that map one domain onto the other. $\square$

Note that the condition that the boundary consists of more than one point is essential. Otherwise, if there is just one boundary point, without loss of generality, one can move it to infinity and hence have the mapping of $\mathbb{C}$ onto a unit disc by an analytic function. However, there is an important Liouville's theorem stating that a bounded (i.e. with a bounded modulus) function analytic in the whole complex plan is a constant, and hence we arrive at a contradiction with the statement of Riemann's theorem, as a constant doesn't map $\mathbb{C}$ onto a circle.

One can have different conditions *additional* to the conditions of Riemann's theorem that the function $w = f(z)$ needs to satisfy to ensure uniqueness of the mapping. To ensure uniqueness, we need **one** of the following:

(a) One chosen inner point $z_0 \in D$ is mapped onto one chosen inner point $w_0 \in G$ **and** at this point the argument of the derivative, $\arg f'(z_0) = \varphi$, is prescribed. (Poincarè's theorem, 1886).

OR

(b) One chosen inner point $z_0 \in D$ is mapped onto one chosen inner point $w_0 \in G$ **and** one chosen boundary point z_1 is mapped onto one chosen boundary point w_1.

OR

(c) Three chosen boundary points z_1, z_2, z_3 are mapped onto three chosen boundary points w_1, w_2, w_3.

The last condition of uniqueness is important for us in what follows as it shows that we should not try to map more than three points on the boundary to the corresponding points in the image when we are constructing a particular mapping.

3.3. **Linear fractional transformation.** In physical applications, we often need to map a complicated domain onto a simple one. Therefore, we have to study how to find or construct the appropriate function.

Definition 3.6. The linear fractional, or Möbius, transformations are mappings

$$w = \frac{az+b}{cz+d},$$

where a, b, c, d, are complex numbers, and $ad - bc \neq 0$. □

Notes:

1). We differentiate using the quotient rule

$$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-cb}{(cz+d)^2}.$$

The condition $ad - bc \neq 0$ guarantees that $w' \neq 0$, and hence the mapping is conformal.

2). The inverse mapping is determined by solving for z in terms of w,

$$z = \frac{b-dw}{cw-a},$$

which is also a linear fractional transformation.

Theorem 3.7. Every linear fractional transformation maps circles and straight lines in the z-plane on circles and straight lines in the w-plane (but not necessarily circles onto circles and straight lines onto straight lines). □

Reminder: It is worth reminding that both a circle and a straight line can be defined as a set of points for which the ratio of distances to two fixed points is a constant, i.e. in terms of complex numbers

$$\frac{|z-z_0|}{|z-z_1|} = k, \quad k \neq 0, \tag{4}$$

where $k = 1$ corresponds to a straight line and $k \neq 1$ corresponds to a circle. (Those who can't remember this fact can easily prove it either using analytic geometry or — and this is more fun — by using the school-level coordinate-free Euclidean geometry.)

After this reminder the proof of the theorem becomes obvious: in (4), which states that z is on a circle or a line, we express z in terms of w from the fractional linear transform and end up with an equation of the form of (4) for w, i.e. which is again a circle or a line.

Definition 3.8. The extended complex plane is the complex plane together with the point ∞ (infinity). □

Remarks:

1). We assign to the value $z = \infty$ in the linear transformation the value $w = a/c$ if $c \neq 0$ and $w = \infty$ if $c = 0$. Then, if $c \neq 0$, the value of $w = \infty$ in the inverse mapping transforms to $z = -d/c$. Thus, every linear fractional transformation maps the extended complex plane in a one-to-one and conformal manner onto itself.

2). The four numbers a, b, c, d determine a unique linear fractional mapping, but, as we'll see later, only three constants are independent.

Theorem 3.9. Three given distinct points can always be mapped onto three prescribed distinct points by one and only one linear fractional transformation. This transformation is given implicitly by the equation

$$\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1}=\frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1}.$$

Example: Mapping of a half-plane onto a disc. Let us map the upper half-plane onto a unit disc. This means that the x-axis must be mapped onto a unit circle. We find the linear fractional transformation which maps $z_1=-1$, $z_2=0$, $z_3=1$ onto $w_1=-1$, $w_2=-i$, $w_3=1$, respectively.

$$\frac{w-(-1)}{w-1}\cdot\frac{-i-1}{-i-(-1)}=\frac{z-(-1)}{z-1}\cdot\frac{0-1}{0-(-1)}.$$

After some algebra we obtain

$$w=\frac{z-i}{-iz+1}.$$

We have mapped the boundary but need to check whether the upper half-plane has been mapped onto the interior or the exterior of the unit circle. By taking an inner point $z=i$, we see that its image is $w=0$, so that we mapped onto the interior as planned.

Example: Mapping of a unit disc onto the right half-plane. We can use the linear fractional transformation to map $z_1=-1$, $z_2=i$, $z_3=1$ onto $w_1=0$, $w_2=i$, $w_3=\infty$, respectively. since we are ‘constructing’ the mapping, we can do this formally and then check the result:

$$\frac{w}{w-\infty}\cdot\frac{i-\infty}{i}=\frac{z+1}{z-1}\cdot\frac{i-1}{i+1},$$

and we, formally, cancel out the infinities on the left-hand side and after some algebra end up with

$$w=-\frac{z+1}{z-1}. \tag{5}$$

This is a linear fractional transformation, and we can see that it does map z_1, z_2, z_3 onto w_1, w_2, w_3. Now, we only need to check that we mapped the disc onto the right half-plane and not on the left. If we take an inner point $z=0$, we see that it maps onto $w=1$.

Example: Mapping of a sector onto a unit disc. We will consider a particular case where this mapping can be achieved by using a linear fractional transformation and a mapping of the form $w=z^n$, where n is an integer. As we know from Riemann's Theorem, there are many other ways of constructing the required mapping.

Let the sector be given by $-\pi/6\leqslant\arg(z)\leqslant\pi/6$ and the disc is $|w|\leqslant 1$. The mapping

$$w_1=z^3$$

maps the sector onto the right half of w_1-plane. We then apply a linear fractional transformation that maps this right half-plane onto the unit disc. To do so, we use the inverse of (5):

$$w=\frac{w_1-1}{w_1+1}.$$

Now, by substituting $w_1 = z^3$, we obtain

$$w = \frac{z^3 - 1}{z^3 + 1}.$$

Example: Mapping of discs onto discs. To map the unit disc in the z-plane onto the unit disc in the w-plane (both with the centre at the origin) we use

$$w = \frac{z - z_0}{\overline{z}_0 z - 1}, \qquad |z_0| < 1, \tag{6}$$

which also maps $z = z_0$ onto the centre $w = 0$. (Show as an exercise that for $|z| = 1$ one has $|w| = 1$, i.e. the circumference maps onto the circumference.)

3.4. **Applications of conformal mapping.**

One of the main applications of conformal mapping is to solve Laplace's equation which is encountered in many areas of physics. The idea is to map a domain D in the z-plane, where solving Laplace's equation may not be easy, onto a domain D^* in the w-plane, where solving Laplace's equation is easier, and use the fact that harmonic functions (i.e. solutions to Laplace's equation) remain harmonic under conformal mapping.

Theorem 3.10. If $\varphi^*(u, v)$ is harmonic in a domain D^*, i.e. $\varphi^*_{uu} + \varphi^*_{vv} = 0$ at every inner point of D^*, and we have a function that maps the domain D of the z-plain ($z = x + iy$) onto D^* of the w-plane ($w = u + iv$) conformally, then

$$\varphi(x, y) = \varphi^*(u(x, y), v(x, y))$$

is harmonic in D. □

Proof.

$$\begin{aligned}
\varphi_{xx} + \varphi_{yy} &= \varphi^*_{uu} u_x^2 + 2\varphi^*_{uv} u_x v_x + \varphi^*_{vv} v_x^2 + \varphi^*_u u_{xx} + \varphi^*_v v_{xx} \\
&\quad + \varphi^*_{uu} \underbrace{u_y^2}_{=v_x^2} + 2\varphi^*_{uv} u_y v_y + \varphi^*_{vv} \underbrace{v_y^2}_{u_x^2} + \varphi^*_u u_{yy} + \varphi^*_v v_{yy} \\
&= u_x^2(\varphi^*_{uu} + \varphi^*_{vv}) + 2\varphi^*_{uv}(u_x v_x + \underbrace{u_y}_{=-v_x} \underbrace{v_y}_{u_x}) + v_x^2(\varphi^*_{uu} + \varphi^*_{vv}) \\
&\quad + \varphi^*_u(u_{xx} + u_{yy}) + \varphi^*_v(v_{xx} + v_{yy}) \\
&= 0.
\end{aligned}$$

We have used here that u, v and φ^* are harmonic functions of their arguments and that u and v are harmonic conjugates, i.e. satisfy the Cauchy-Riemann equations. □

Illustration: Electrostatic field. In electrostatics, Faraday's law states that $\nabla \times \mathbf{E} = 0$, so that the electric field density $\mathbf{E}$ has a potential φ: $\mathbf{E} = \nabla\varphi$. Then, in the absence of electric charges we also have $\nabla \cdot \mathbf{E} = 0$, and hence

$$0 = \nabla \cdot \mathbf{E} = \nabla \cdot \nabla\varphi = \nabla^2\varphi.$$

If the potential is independent of one of the spatial (Cartesian) coordinates, then in the plane perpendicular to this coordinate we have

$$\nabla^2 = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0,$$

or, if we introduce polar coordinates by $x = r\cos\theta$, $y = r\sin\theta$,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = 0.$$

Consider how conformal mapping allows one to reduce a ‘difficult’ problem for Laplace’s equation to an easy problem whose solution is known.

Example (of an easy problem). Find the electric potential between two coaxial conducting cylinders extended to infinity on both ends.

Solution. Let the outer cylinder have the radius r_1 and the inner r_2 ($r_1 > r_2$). For convenience, we will place the origin of the coordinate system at the centre of the cylinders. Due to symmetry, φ is independent of θ, so that

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) = 0, \qquad \text{for } r_2 < r < r_1), \tag{7}$$

and the boundary conditions are $\varphi(r_1) = U_1$, $\varphi(r_2) = U_2$.

After integrating the ODE (7), we have $\varphi = A\ln r + B$ and the boundary conditions give two linear algebraic equations determining A and B:

$$U_1 = A\ln r_1 + B, \qquad U_2 = A\ln r_2 + B.$$

This is a system of linear algebraic equations for A and B.

Example (of a ‘difficult’ problem, which we will reduce to the easy one that we have already solved). Find the electric potential between two **non**-coaxial conducting cylinders C_1, given by $x^2 + y^2 = 1$, and C_2, given by $(x - 2/5)^2 + y^2 = (2/5)^2$, extended to infinity on both ends. Let on C_1 $\varphi = 0$ and on C_2 $\varphi = 110$.

Solution.

Step 1. We map the unit disc $|Z| = 1$ onto the unit disc $|w| = 1$ in such a way that C_2 is mapped onto a coaxial disc C_2^*, $|w| = r_0$. We have the linear fractional transformation above) that maps disc to disc (see (6)

$$w = \frac{z - b}{bz - 1},$$

where we have chosen $z_0 = b$ real. This mapping obviously maps a unit disc onto a unit disc for all b, so that we need to find b, to ensure that the image of C_2 is a coaxial disc C_2^*. (We note that in a general case $b \neq 2/5$.) We have the unknowns b and r_0 as we wish C_2^* to be coaxial but do not specify its radius. To determine these unknowns we need two conditions.

To find these two unknowns we map 2 point on the boundary of C_2 onto two points on the boundary of C_2^*. For simplicity we map $z = 0$ onto $w = r_0$ and $z = 4/5$ onto $w = -r_0$:

$$w(0) = \frac{-b}{-1} = r_0 \qquad \Rightarrow \qquad b = r_0,$$

$$w(4/5) = \frac{4/5 - b}{4b/5 - 1} = -r_0 \quad \Rightarrow \quad (\text{using that } r_0 = b) \quad \frac{4/5 - b}{4b/5 - 1} = -b,$$

which gives us a quadratic equation for b. By solving this equation, we find $b_1 = 2$, $b_2 = \frac{1}{2}$, and the only physical solution is $b = \frac{1}{2}$ as we require $b = r_0 < 1$ for C_2^* to be inside C_1^*.

The required mapping therefore is

$$w = \frac{2z - 1}{z - 2}.$$

Step 2. We know from the previous example that in the w-plane

$$\varphi^*(u, v) = A\ln|w| + B,$$

where

$$0 = A\ln(1) + B \qquad \Rightarrow \qquad B = 0,$$

$$110 = A\ln(1/2) + B \qquad \Rightarrow \qquad A = \frac{110}{\ln(1/2)}.$$

Thus, $\varphi^*(u,v) = A\ln|w|$ and, using the theorem, we have

$$\varphi(x,y) = a\ln\left|\frac{2z-1}{z-2}\right|,$$

or

$$\varphi(x,y) = A\ln\left|\frac{[(2x-1)(x-2)+2y^2]^2+9y^2]^{1/2}}{(x-2)^2+y^2}\right|.$$

In terms of a problem for the real-valued function which satisfies Laplace's equation, the above theorem allows one to change the variables and hence simplify the shape of the domain where Laplace's equation has to be solved.

In applications, it is often more convenient not to mix real-valued functions for which Laplace's equation needs to be solved and complex-valued functions used to map domains and hence simplify the problem but to introduce another complex-valued function for which the above real-valued function would be the real (or imaginary) part and hence use the techniques of handling analytic functions throughout.

Definition 3.11. If we have a harmonic function $\varphi(x,y)$ playing the role of a 'potential' in some application, then we can take the conjugate harmonic function $\psi(x,y)$ (i.e. the function related to φ by the Cauchy-Riemann equations) and form an analytic function

$$F(z) = \varphi + i\psi,$$

where $z = x+iy$, which will be called a complex potential. □

Note: The point of having F is that a function of one (albeit complex) variable is much easier to handle than a function of two variables for which we have a problem of solving a PDE. The lines $\varphi = \text{const}$ and $\psi = \text{const}$ will intersect at right angles except where $F'(z) = 0$.

Example. Find the electric potential between two semi-circular plates P_1: $r=1$, $-\pi < \theta < 0$ with potential $\varphi_1 = -3000$ and and P_2: $r=1$, $\pi < \theta < 0$ with potential $\varphi_2 = 3000$.

SOLUTION.

Step 1. We map the unit disc in the z-plane onto the right half-plane in the w-plane by using the linear fractional transformation we had earlier

$$w = f(z) = \frac{1+z}{1-z}.$$

The upper semi-circle in the z-plane is mapped onto the positive v-semiaxis in the w-plane and the lower semi-circle is mapped onto the negative v-semiaxis.

Step 2. We determine the potential φ^* in the w-plane, noting that it can depend only on the argument θ (in the w-plane) and not on the modulus. Then, Laplace's equation for φ^* turns into a simple ODE

$$\frac{d^2\varphi^*}{d\theta^2} = 0,$$

so that $\varphi^* = A\theta + B$, whist the boundary conditions $\varphi^*(\pi/2) = 3000$, $\varphi^*(-\pi/2) = -3000$ yield

$$A = \frac{6000}{\pi}, \qquad B = 0.$$

Thus,

$$\varphi^* = \frac{6000}{\pi}\theta = \frac{6000}{\pi}\arctan\left(\frac{v}{u}\right).$$

In order to use the conformal mapping, we note that φ^* is the real part of a complex potential

$$F^*(w) = -i\frac{6000}{\pi}\mathrm{Log}\,(w).$$

Then

$$F(z) = F^*(f(z)) = -\frac{6000i}{\pi}\mathrm{Log}\left(\frac{1+z}{1-z}\right).$$

The real part of $F(z)$ is the potential we regquire

$$\begin{aligned}\varphi(x,y) = \Re(F(z)) &= \frac{6000}{\pi}\Im\left[\mathrm{Log}\left(\frac{1+z}{1-z}\right)\right]\\ &= \frac{6000}{\pi}\mathrm{Arg}\left(\frac{1+z}{1-z}\right)\\ &= \frac{6000}{\pi}\arctan\left(\frac{2y}{1-x^2-y^2}\right).\end{aligned}$$

The lines $\varphi = \text{const}$ are 'equipotential lines' and the imaginary part of $F(x)$ gives us the 'lines of force':

$$\psi(x,y) = \Im(F(z)) = -\frac{6000}{\pi}\Re\left[\mathrm{Log}\left(\frac{1+z}{1-z}\right)\right]$$

$$= -\frac{6000}{\pi}\left|\frac{1+z}{1-z}\right| = -\frac{6000}{\pi}\frac{\sqrt{(1-x^2-y^2)^2+4y^2}}{(1-x)^2+y^2}.$$

Remark: The concepts we used here apply more widely than just in electrostatics. In particular, the technique we touched upon was behind the development of subsonic aerodynamics.

Definition 3.12. The transformation

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

is known as the Joukowski transformation. $\square$

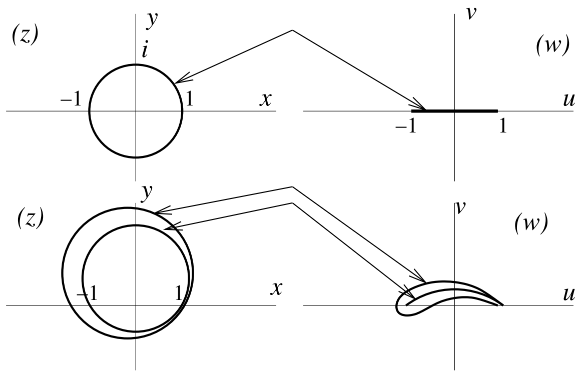

FIGURE 3. The Joukowski transformation maps a unit circle onto a segment $[-1,1]$ (above), so that a circle passing through points $z_1 = -1$, $z_2 = 1$ is mapped onto an arc and a large circle touching this one at $z = 1$ produces what is known as the Joukowski airfoil (below).

The Joukowski transformation maps the outside (as well as the inside) of the unit circle onto a complex plane such that the boundary (i.e. the circumference) is mapped onto the segment $[-1,1]$ of the real axis (Fig. 3). This transformation plays a very important role in subsonic airdynamics, it is used to design the so-called Joukowski airfoil, etc.

A glimpse of the airfoil theory (for information only)

In subsonic airdynamics, with a good approximation the gas can be regarded as an incompressible fluid, so that the conservation of mass equation takes the form

$$\nabla \cdot \mathbf{V} = 0, \tag{8}$$

where $\mathbf{V}$ is the velocity field. In airdynamics we can often neglect the gas viscosity, and for the flows of an inviscid fluid it can be shown that the velocity field is potential, i.e. there exists a function φ such that $\mathbf{V} = \nabla\varphi$. Using this in (8), we have that φ satisfies Laplace's equation $\nabla^2\varphi = 0$, i.e. φ is harmonic. The harmonic conjugate to φ is known as the streamfunction ψ, and we can form a complex potential

$$F(z) = \varphi + i\psi,$$

such that

$$\frac{dF}{dz} = \frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial y} = V_x - iV_y = \overline{V},$$

where $\overline{V}$ is the complex conjugate of the analytic function V, which is the complex velocity.

Then, to find how a gas with a uniform velocity V_∞ at infinity flows past a body — and, importantly, what lift it produces — we need to find $F(z)$ satisfying the boundary conditions

(i) $F \to \overline{V}_\infty z$ as $z \to \infty$,

(ii) $\dfrac{\partial\varphi}{\partial\mathbf{n}} = 0$ on the surface of the body (the impermeability conditions) which is equivalent to $\psi = \text{const}$ on the surface.

This problem can be easily solved for a simple body profile, like a disc of radius a, and then, using conformal mapping, we can obtain the flow past a more elaborate profile, like an airfoil. For a cylinder of radius a, the general solution is given by

$$F(z) = \overline{V}_\infty z + \frac{V_\infty a^2}{z} + \frac{\Gamma}{2\pi i}\operatorname{Log} z, \tag{9}$$

where Γ is a real constant known as the circulation. The first two terms on the right-hand side correspond to a 'straight' flow past a cylinder (with the stagnation points for the velocity field on the opposite sides of the cylinder) and the third term describes a rotational flow around a cylinder with Γ characterizing its intensity. In the solution (9) Γ is arbitrary and, when we find the pressure on the body using equations of fluid mechanics and, by integrating it over the contour of the body, find the total force acting on the body, we find that it is the value of Γ that determines this force. If in the above $z = z(w)$, we have the general solution for a flow past a body defined by the mapping $w = w(z)$.

Now, we need the Joukowski transformation. We know that it maps a unit circumference onto a segment $[-1,1]$. It can be easily shown that a (non-unit) circumference K passing through points $z=-1$ and $z=1$ will be mapped onto an arc connecting $w=-1$ and $w=1$. The image of a circumference K_1 that embraces K and touches it at the point $z=1$ is known as the Joukowski profile. This profile has a sharp edge downsteam and it is the kind of profile you observe when, to relieve boredom, you look in the aircraft window.

Now, knowing the flow past a disc and the mapping of this disc onto the Joukowski airfoil (which is actually a two parametric family of airfoils, with two parameters determining its bent and thickness), we can have the complex potential of the flow past the airfoil

$$F^*(w) = F(z(w)).$$

The complex velocity (in the w-plane) is then given by

$$\overline{V}_{(w)} = \frac{dF^*}{dw} = \frac{dF}{dz}\frac{dz}{dw}.$$

We note that, in general, this velocity is infinite at the sharp edge of the airfoil as there the conformity of our mapping is obviously lost and $dz/dw = \infty$. However, in our solution we had an arbitrary constant Γ which we can now chose to make dF/dz at the sharp edge zero and hence the velocity finite. This will determine Γ and hence the lift force on the airfoil. This requirement of finiteness of velocity at the sharp edge, which determines the value of the circulation and the lift force, is known as Joukowski theorem.

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4. Contour integration and Cauchy Theorem

4.1. **Contours.**

Note: The main vehicle in the study of complex integrals is the so-called rectifiable curve, i.e. a curve for which the supremum ('upper limit') of the lengths of the inscribed piece-wise linear curves is finite (this supremum is called the length of this curve). In our course we will use a simpler notion of a piece-wise smooth *contour*, whose length always exists. This simplification covers all applications we are interested in.

Definition 4.1. A curve or a path is a continuous function $\gamma : [a, b] \mapsto \mathbb{C}$. We generally identify the curve (path) with its image

$$\Gamma = \{z \in \mathbb{C} \mid z = \gamma(t),\ \forall t \in [a, b],\ \gamma \in C[a, b]\}\,.$$

The initial point of the curve is $\gamma(a)$ and the termination point is $\gamma(b)$. An important addition to the definition of a curve we had before is that the curve has an orientation, i.e. the direction that one moves along it as t varies from a to b. $\Box$

Example. $\gamma(t) = (1 + 2t) + it^3$ is a path. $\Box$

Definition 4.2. Let $\gamma(t)$ be a path for $t \in [a, b]$. Then,

(1) γ is closed if it ends where it begins, i.e. $\gamma(b) = \gamma(a)$.
(2) γ is smooth if $\gamma'(t)$ is continuous **and** nonzero, i.e. $\gamma' \in C[a, b]$, $\gamma'(t) \neq 0,\ \forall t \in (a, b)$. (The condition $\gamma'(t) \neq 0$ is essential: for the curve $\gamma(t) = t^3 + it^2$, $-1 \leqslant t \leqslant 1$ one has $\gamma' \in C(-1, 1)$ but $\gamma'(0) = 0$ and, as one can easily verify, the curve is not smooth.)
(3) γ is a contour if it is piecewise smooth, i.e. the number of points where it is not smooth is finite.
(4) γ is simple (or Jordan's) if the curve Γ does not cross or touch itself, i.e. $\forall t_1, t_2 \in (a, b),\ t_1 \neq t_2,\ \gamma(t_1) \neq \gamma(t_2)$.

Examples.

(a) $\gamma_1 : [-1, 1] \mapsto \mathbb{C}$ given by $\gamma_1(t) = t$ is a contour.
(b) $\gamma_2 : [0, \pi] \mapsto \mathbb{C}$ given by $\gamma_2(t) = e^{it} = \cos t + i \sin t$ is a contour.
(c) $\Gamma_1 \cup \Gamma_2$ is a closed contour.

4.2. **Contour integrals.**

Definition 4.3. Let $f(z)$ be a function of a complex variable z defined in $D \subset \mathbb{C}$ and $\gamma : [a, b] \mapsto D$ be a contour. Let $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ be a set of points in $[a, b]$, $z_k = \gamma(t_k)$, $k = 0, \ldots, n$ and $\zeta_k = \gamma(\tau_k)$, where τ_k is an arbitrary point in $[t_{k-1}, t_k]$. Then, the sum

$$\sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1})$$

is called an integral sum. Let $\delta_T = \max\{t_1 - t_0, t_2 - t_1, \ldots, t_n - t_{n-1}\}$. Then, the limit

$$\lim_{\substack{\delta_T \to 0 \\ n \to \infty}} \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}) = \int_{\Gamma} f(z)\, dz$$

is called the integral of $f(z)$ along Γ. $\square$

Note that this definition is completely analogous to the definition of the definite integral you had (or should have had?) in the course on integral calculus.

Properties that follow from the definition:
Let Γ, Γ_1 and Γ_2 be curves.

(i) $\int_{\Gamma} f(z)\, dz = -\int_{-\Gamma} f(z)\, dz$.

(ii) If $\Gamma = \Gamma_1 \cup \Gamma_2$, then $\int_{\Gamma} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz + \int_{\Gamma_2} f(z)\, dz$.

(iii) The integral does not depend on the choice of parametrization of the curve.

Definition 4.4. Let us define the length of the contour γ by

$$L = \int_{t=a}^{b} |\gamma'(t)|\, dt.$$

$\square$

Then, we have the following

Estimation Lemma. If $f(z)$ is continuous, γ is a contour of length L and $|f(z)| \leqslant M$ for all z on γ, then

$$\left| \int_{\Gamma} f(z)\, dz \right| \leqslant ML.$$

$\square$

Note: To calculate integrals directly using the above definition is possible (and should we have more time, we would have done a few exercises) but not very convenient, and we will consider the theorems that have been proven to simplify the task. However, in applications, when integrals are calculated numerically, it is some times necessary to calculate the integrals using the definition. $\square$.

The first simplification that we will consider is the reduction of the complex integral to the simple integral with a real dummy variable:

$$\int_{\Gamma} f(z)\, dz = \int_{t=a}^{t=b} f(\gamma(t))\gamma'(t)\, dt.$$

This is already a 'workable' formula, though, as we will see, much more significant simplifications are possible.

Examples.

(a) Suppose that $f(z) = z^n$, where $n \in \mathbb{Z}$ and $n \neq -1$. Let $\gamma : [0, 2\pi] \mapsto \mathbb{C}$ be defined by $\gamma = re^{it}$, $r > 0$. Then,

$$\int_\Gamma f(z)\,dz = \int_{t=0}^{2\pi} f(\gamma(t))\gamma'(t)\,dt = r\int_{t=0}^{2\pi} f(re^{it})ie^{it}\,dt$$

$$= r^{n+1}\int_{t=0}^{2\pi} e^{int}ie^{it}\,dt = r^{n+1}\int_{t=0}^{2\pi} ie^{i(n+1)t}\,dt = r^{n+1}\left[\frac{1}{n+1}e^{i(n+1)t}\right]_{t=0}^{2\pi} = 0.$$

(b) Suppose that $D = \mathbb{C}\backslash\{0\}$, $f(z) = 1/z$ and γ as in (a). Then,

$$\int_\Gamma f(z)\,dz = \int_{t=0}^{2\pi} \frac{1}{r}e^{-it}ire^{it}\,dt = 2\pi i.$$

4.3. **Cauchy's Integral Theorem - 3 in 1.**

The central theorem of the theory of analytic functions has been proven by A. Cauchy in 1825.

Theorem 4.5. (Cauchy's Integral Theorem 1)

Let $D \in \mathbb{C}$ be a simply connected domain and $f(z)$ be a (single-valued) analytical function in D, then for every rectifiable closed curve $\Gamma \subset D$

$$\oint_\Gamma f(z)\,dz = 0.$$

Note that an immediate consequence of Cauchy's Theorem is that, under the conditions of the theorem, an integral from $z = z_1$ to $z = z_2$ is independent of the path connecting z_1 and z_2.

Examples where Cauchy's Theorem gives us the solutions immediately.

(a) $\int_\Gamma e^z\,dz = 0$ for any closed contour Γ.

(b) $\displaystyle\int_{|z|=1} \frac{dz}{z^2+4} = 0$. The function $1/(z^2+4)$ is not analytic at $z = \pm 2i$ but these points lie outside the unit circle.

The principle of deformation. One of the main consequences of Cauchy's Integral Theorem is that, if we need to take an integral of an analytic function $f(z)$ along a curve between two points, the integral will remain the same if we continuously deform the curve keeping the end points fixed *provided* that, as we deform the curve, it doesn't pass over a point where $f(z)$ is not analytic. Thus, Cauchy's theorem allows us to replace 'inconvenient' contours of integration with the 'convenient' ones.

Theorem 4.6. (Morera's theorem)
If $f(z)$ is *continuous* in a simply connected domain and if

$$\int_{\Gamma} f(z)\,dz = 0$$

for every simple closed contour Γ, then $f(z)$ is analytic.

In other words; $f \in C(D)$, D simply connected, $\forall\Gamma$, Γ closed and simply connected $\oint_{\Gamma} f(z)\,dz = 0 \Rightarrow f \in A(D)$.
□

This theorem is the converse of Cauchy's Integral Theorem but in practice it is much less useful: there are easier ways of investigating the function with regards its analyticity, though for some proofs Morera's theorem could be useful. Closely related to Morera's theorem is the following

Theorem 4.7. Let $f(z)$ be *continuous* in a domain D such that the integrals of $f(z)$ along every rectifiable curve in D depend only on the starting and the end points of the curves. Then, the function

$$F(z) = \int_{z_0}^{z} f(z)\,dz$$

is analytic in D and $F'(z) = f(z)$. □

Theorem 4.8. (Fundamental Theorem of the Calculus) If $f : D \mapsto \mathbb{C}$ and $f(z)$ has an analytic antiderivative $F(z)$, then for any curve Γ connecting points z_1 and z_2 one has

$$\int_{\Gamma} f(z)\,dz = F(z_2) - F(z_1).$$

□

This theorem is not as widely used as the corresponding theorem in the real analysis but in some cases it can be useful.

Examples
(a)

$$\int_{z=0}^{1+i} z^2\,dz = \left[\frac{1}{3}z^3\right]_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{1}{3}(\sqrt{2})^3 e^{3i\pi/4} = \frac{2}{3}(-1+i).$$

(b) $\displaystyle\int_{z=-i}^{i} \frac{1}{z}\,dz$. In this example, D is the complex plane without zero <u>and</u> the negative real semi-axis. It is simply connected and

$$\int_{z=-i}^{i} \frac{dz}{z} = [\mathrm{Ln}\,(z)]_{z=-i}^{i} = \mathrm{Log}\,(i) - \mathrm{Log}\,(-i) = i\tfrac{\pi}{2} - (-i\tfrac{\pi}{2}) = i\pi.$$

The most important consequence of Cauchy's Integral Theorem is the following Cauchy's Integral formula.

Theorem 4.9. (Cauchy's Integral Theorem 2 also known as Cauchy's Integral Formula)
Let $f(z)$ be analytic in a simply connected domain D. Then, for any point $z_0 \in D$ and any simple closed contour Γ in D that encloses z_0

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(z)}{z - z_0}\, dz.$$

Proof. To prove this theorem, we use the deformation principle and deform Γ into a circle γ_ρ of radius ρ around z_0. Then, we need to show that

$$\int\limits_{\gamma_\rho} \frac{f(z)}{z - z_0}\, dz - 2\pi i f(z_0) = 0.$$

Using the results we had about the integration along a circle of $1/z$, we have

$$\int\limits_{\gamma_\rho} \frac{f(z)}{z - z_0}\, dz - 2\pi i f(z_0) = \int\limits_{\gamma_\rho} \frac{f(z)}{z - z_0}\, dz - f(z_0) \int\limits_{\gamma_\rho} \frac{dz}{z - z_0} = \int\limits_{\gamma_\rho} \frac{f(z) - f(z_0)}{z - z_0}\, dz. \tag{10}$$

Since $f \in C(z_0)$, we have that $\forall \epsilon > 0\ \exists \delta(\epsilon, z_0)$ such that for $\rho < \delta$

$$|f(z) - f(z_0)| < \epsilon \qquad (z \in \gamma_\rho).$$

Therefore,

$$\left| \int\limits_{\gamma_\rho} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| < \frac{\epsilon}{\rho} 2\pi\rho = 2\pi\epsilon.$$

Thus, we have (by definition) that

$$\lim_{\rho \to 0} \int\limits_{\gamma_\rho} \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0$$

However, the integral under the limit is the same as the last term in (10) and it is independent of ρ (remember the deformation principle). Therefore, this integral is equal to zero and hence we have proven the theorem. □

Remark. Cauchy's integral formula essentially states that the values of an analytic function inside a contour are completely determined by the values of this function on the contour. (Given the close connection between analytic and harmonic functions this is not surprising but, nevertheless should be noted.) □

Examples.

(a) Let Γ be a circle $|z| = 4$. Then,

$$\int\limits_{|z|=4} \frac{e^z}{z - 2}\, dz = 2\pi i e^2.$$

(b) Suppose that Γ is a circle $|z| = 2$. Evaluate the integral

$$\int\limits_{\Gamma} \frac{\cos(z)\, dz}{1 + z^2}.$$

Solution. We note that $\cos(z)$ is entire and the denominator has zeros at $z = \pm i$. Using partial fractions, we obtain

$$\frac{\cos(z)}{1+z^2} = \frac{\cos(z)}{(z-i)(z+i)} = \frac{\cos(z)}{2i(z-i)} - \frac{\cos(z)}{2i(z+i)}.$$

Using the Cauchy Integral Formula and that $\cos(iz) = \cosh(z)$, we have

$$\int_\Gamma \frac{\cos(z)}{z-i}\,dz = 2\pi i \cos(i) = 2\pi i \cosh(1)$$

and

$$\int_\Gamma \frac{\cos(z)}{z+i}\,dz = 2\pi i \cos(-i) = 2\pi i \cosh(1).$$

Thus,

$$\int_\Gamma \frac{\cos(z)}{1+z^2}\,dz = \frac{2\pi i}{2i}[\cosh(1) - \cosh(1)] = 0.$$

(c) Let in the example above Γ be a *semi*-circle, i.e. $\Gamma = \{z : |z| = 2, \Im(z) > 0\} \cup \{z : \Im(z) = 0, -2 \leqslant \Re(z) \leqslant 2\}$.

Solution. Then, Γ will enclose only one of the poles and the other partial fraction will be analytic inside Γ (so that, according to Cauchy's theorem, its integral will be zero). As a result, the integral will be equal to $\pi\cosh(1)$. □

Theorem 4.10. (Cauchy's Integral Theorem 3 — Integral Formula for higher derivatives)

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D. The values of these derivatives at a point $z_0 \in D$ are given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \tag{11}$$

where Γ is any simple closed contour in D that encloses z_0. □

For the time being we will leave this theorem without a proof. The proof will be given later when we turn to power series.

As with Cauchy's Integral Theorem for the function, this theorem can be used 'in reverse', i.e. not for calculating the values of functions but for evaluating the integrals.

Example. Evaluate

$$\int_{|z|=1/3} \frac{\cos(z)}{z^2(z-1)}\,dz.$$

Solution. Inside Γ, which is the circle of radius $1/3$ we have a singularity at $z = 0$, so that we have the integral for the form

$$\int_\Gamma \frac{f(z)}{z^2}\,dx, \qquad \text{where} \quad f(z) = \frac{\cos(z)}{z-1}.$$

The function $f(z)$ is obviously analytic inside Γ. Then,

$$\int_{\Gamma} \frac{f(z)}{z^2} = \frac{2\pi i}{1!} f'(0).$$

All what we need is calculate is $f'(z)$:

$$f'(z) = \frac{-(z-1)\sin(z) - \cos(z)}{(z-1)^2}$$

Then, $f'(0) = -\cos(0) = -1$, and hence

$$\int_{|z|=1/3} \frac{\cos(z)}{z^2(z-1)}\, dz = -2\pi i.$$

Alternatively, we can again use partial fractions

$$\frac{1}{z^2(z-1)} = \frac{1}{z-1} - \frac{z+1}{z^2},$$

so that

$$\int_{|z|=1/3} \frac{\cos(z)}{z^2(z-1)}\, dz = \underbrace{\int_{|z|=1/3} \frac{\cos(z)}{z-1}\, dz}_{=0} - \int_{|z|=1/3} \frac{(z+1)\cos(z)}{z^2}\, dz = -2\pi i \left.\frac{d[(z+1)\cos(z)]}{dz}\right|_{z=0}$$

$$= -2\pi i.$$

5. Power Taylor and Laurent series

5.1. **Power series.**

Definition 5.1. A series $\sum_{n=0}^{\infty} w_n$, where $w_n \in \mathbb{C}$ for $n = 0, 1, 2, \ldots$, is called <u>convergent</u> if there exists the following limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} w_n = W.$$

This is obviously equivalent to the following real-valued limits

$$\lim_{N\to\infty} \sum_{n=0}^{N} \Re(w_n) = \Re(W) \quad \text{and} \quad \lim_{N\to\infty} \sum_{n=0}^{N} \Im(w_n) = \Im(W).$$

Definition 5.2. The series $\sum_{n=0}^{\infty} w_n$ is called <u>absolutely convergent</u> if the series made of moduli of the terms, i.e. $\sum_{n=0}^{\infty} |w_n|$ converges. $\Box$

If we use the notation $w_n = u_n + iv_n$, then from the inequalities

$$\left.\begin{array}{l}|u_n|\\|v_n|\end{array}\right\} \leqslant |w_n| \leqslant |u_n| + |v_n|$$

we have that absolute convergence of $\sum w_n$ is equivalent to simultaneous absolute convergence of $\sum u_n$ and $\sum v_n$. Then, as for absolutely convergent real series, the sum of this absolutely convergent series is independent of the order of the terms.

Definition 5.3. A power series in powers of $(z - z_0)$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \equiv a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots,$$

where z is a variable, a_0, a_1, ... are constants, and z_0 is a constant called the centre of the series. □

Definition 5.4. (Reminder?) The supremum (or the least upper bound) of a number set S is the smallest number A that is greater or equal to all numbers in the set. In other words, $A = \sup S \equiv$ (a) $\forall s \in S$, $s \leqslant A$, (b) $\forall \epsilon > 0$, $\exists s \in S : s > A - \epsilon$. □

Note that, firstly, the *supremum* shouldn't be confused with a *maximum*. For example, the set of numbers formin an interval $X = \{x \,!\, a < x < b\}$ has b is the supremum but there is no maximum, i.e. a number *belonging to* X and such that it is greater than the rest (and, possibly, equal to some of the rest). Secondly, it is insufficient just to say 'least upper bound' or 'greater or equal than the rest' as the definition: the definition is what we 'say' in terms of ϵ as it is this definition that gives us the way to actually *use* it.

Definition 5.5. The radius of convergence of a power series is defined to be

$$R = \sup\{|z - z_0| \;:\; \sum_{n=0}^{\infty} |a_n(z - z_0)^n| \;\text{ converges}\}.$$

In other words, the radius of convergence is the supremum of all values of $|z - z_0|$ for which the series converges absolutely. We write $R = \infty$ if $\sum_{n=0}^{\infty} |a_n(z - z_0)^n|$ converges for arbitrary large $|z - z_0|$. □

Definition 5.6. The number A is called the limit superior (or, sometimes, the upper limit; notation lim sup or $\overline{\lim}$) of a sequence $\{a_n\}_0^\infty$ if it is the supremum for all subsequences. More specifically, $A = \lim\limits_{n\to\infty} \sup a_n$ if $\forall \epsilon > 0$,

(a) $\exists N_+ : \forall n > N_+$, $a_n < A + \epsilon$,
(b) $\forall N_-$, $\exists n > N_- : a_n > A - \epsilon$.

Theorem 5.7. (The Cauchy-Hadamard Theorem) Let

$$\sum_0^{\infty} a_n(z - z_0)^n$$

be a power series with complex coefficients a_n ($n = 0, \dots, \infty$), then for its radius of convergence R one has

$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n},$$

that is

(a) if $1/R = 0$, the series converges everywhere,
(b) if $1/R = \infty$, the series converges only at $z = z_0$,
(c) if $0 < 1/R < \infty$, the series converges absolutely inside the circle $|z - z_0| < R$. and diverges outside it.

□

Properties:

(a) Termwise addition or subtraction of two power series with the same center and with the radii of convergence R_1 and R_2 yields a power series with radius of convergence given by the minimum of R_1 and R_2: if

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad |z - z_0| < R_1, \qquad \sum_{n=0}^{\infty} b_n(z - z_0)^n, \quad |z - z_0| < R_2,$$

then

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \pm \sum_{n=0}^{\infty} b_n(z - z_0)^n = \sum_{n=0}^{\infty}(a_n \pm b_n)(z - z_0)^n$$

and it converges in

$$|z - z_0| < R = \min(R_1, R_2),$$

(b) Termwise multiplication of two power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} b_n(z - z_0)^n$$

is given by

$$\sum_{n=0}^{\infty}(a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0)(z - z_0)^n.$$

(c) Termwise differentiation and integration of power series produce power series with the same radius of convergence.
(d) The sum of a power series is an analytic function at every point within its radius of convergence.

5.2. **Taylor series.**

We have stated (sadly, without a proof) as one of the properties of power series that "the sum of a power series is an analytic function at every point within its radius of convergence". In other words,

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = f(z) \in A(|z - z_0| < R),$$

where R is the radius of convergence of the series. Using differentiability of the series, it is easy to express the coefficients of the series in terms of the derivatives of $f(z)$: $a_n = f^{(n)}(z_0)/n!$ for $n = 0, 1, 2, \dots$. Thus, we can write the series down as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = f(z). \tag{12}$$

The series on the left-hand side is called the Taylor series. If $z_0 = 0$ the series is called the Maclaurin series.

Above, we expressed the coefficients of a power series in terms of the derivatives of its sum which is an analytic function. Now we need to consider whether an analytic function is expandable into such a series. The answer is given by the following

Theorem 5.8. Let $f(z)$ be a function analytic in a domain D. Then, if $z_0 \in D$ and r is the distance from z_0 to the boundary of D, then (a) $f(z)$ is infinitely differentiable and (b) in the circle $|z - z_0| < r$ it is expandable in a power series in $z - z_0$. $\square$

Proof. Let z be an arbitrary point in $|z - z_0| < r$. We will consider a circle of radius ρ, $0 < \rho < r$ with the centre at z_0 such that it contains z as an inner point (i.e. $|z - z_0| < \rho$). If γ_ρ is the circumference bounding this circle, then Cauchy's Integral Formula states that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)\, d\zeta}{\zeta - z}.$$

We can expand $\dfrac{1}{\zeta - z}$ as follows:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}},$$

so that

$$\frac{f(\zeta)}{\zeta - z} = \sum_{n=0}^{\infty} f(\zeta) \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}. \tag{13}$$

For a fixed z the series converges uniformly with regard to $\zeta \in \gamma_\rho$ since

$$\left| f(\zeta) \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \right| < \max_{\gamma_\rho} |f(\zeta)| \frac{|z - z_0|^n}{\rho^{n+1}}$$

and the number series $\displaystyle\sum_{n=0}^{\infty} \max_{\gamma_\rho} |f(\zeta)| \frac{|z - z_0|^n}{\rho^{n+1}}$ converges as a geometric progression with the common ratio $\dfrac{|z - z_0|}{\rho} < 1$.

Then (i.e. because the series converges uniformly) the series (13) can be integrated term-wise:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)\, d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{\gamma_\rho} \sum_{n=0}^{\infty} f(\zeta) \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}} \cdot (z - z_0)^n = \sum_{n=0}^{n} a_n (z - z_0)^n.$$

Here

$$a_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}, \qquad (n = 0, 1, 2, \dots). \tag{14}$$

Since z was chosen arbitrarily in $|z - z_0| < r$, we have shown that the expansion $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is valid for all z. This completes the proof of our theorem.

Now, given that the coefficients in the power expansions are also given by $a_n = \frac{f^{(n)}(z_0)}{n!}$, from (14) we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int\limits_{\gamma_\rho} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}, \qquad (n = 0, 1, 2, \dots),$$

where, by the deformation principle, we can deform γ_ρ into any Γ within the area of analyticity of $f(z)$. This proves Cauchy's Integral Formula for Higher Derivatives (11) which we had earlier without a proof. □

Thus, if we take a point z_0 where $f(z)$ is analytic and formally construct a Taylor series (12), i.e. set a correspondence

$$f(z) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,$$

then this series will converge to $f(z)$ in the largest open disc with centre z_0 contained in D.

Example. Find the radius of convergence R for the function

$$f(z) = \frac{1 + z + z^2}{(1 + z^2)(3 + z^2)}$$

expanded about the origin.

SOLUTION. The function $f(z)$ has singularities at $z = \pm i$ and $z = \pm\sqrt{3}i$. Hence $f(z)$ is analytic within the unit disc and $R = 1$. □

Special Taylor (Maclaurin) series

(i) $(1 + z)^m = 1 + mz + \cdots = 1 + \sum_{n=1}^{\infty} \frac{m(m-1)\dots(m-n+1)}{n!} z^n, \quad |z| < 1 = R,$

The most important for us particular case of the above is

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1 = R.$$

(ii) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad R = \infty.$

(iii) $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad R = \infty.$

(iv) $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \qquad R = \infty.$

(v) $\mathrm{Ln}\,(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}, \qquad |z| < 1 = R.$

To expand a function in a Taylor series, we can:

a). Use the definition and simply calculate the coefficients (this should be the last resort!),
b). Use the known Maclaurin series.

c). Use properties of the Taylor series, namely that it can be termwise differentiated and integrated.

Examples.

(a) Find the Maclaurin series of $f(z) = \dfrac{1}{1+z^2}$.

$$f(z) = \frac{1}{1-(-z^2)} = 1 + (-z^2) + (-z^2)^2 + (-z^2)^3 + \cdots = \sum_{n=0}^{\infty} (-z^2)^n.$$

(b) Find the Maclaurin series of $f(z) = \arctan(z)$.

SOLUTION. This function is not in the class of elementary Maclaurin series that we had, but its derivative is, and we can use this and the fact that power series can be termwise integrated.

$$f'(z) = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n.$$

Then, after termwise integration, we have

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}, \qquad |z| < 1.$$

(c) Find the first two (nonzero) terms in the Maclaurin series of $f(z) = \tan(z)$.

SOLUTION.

There is always a way to find the coefficients in a series by 'brutal force', i.e. by calculating $f'(0)$, $f''(0)$, etc. This should always be the last resort, but here we will use this method first.

$$f'(z) = \frac{1}{\cos^2(z)} = 1 + f^2(z), \quad f''(z) = 2ff', \qquad f^{(3)} = 2(f')^2 + 2ff'', \quad \text{etc.}$$

Then, $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = 2$, etc, and

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = z + \frac{2}{3!} z^3 + \dots.$$

The preferred way is to use that

$$f(z) = \tan(z) = \sin(z) \frac{1}{\cos(z)} = \sin(z) \frac{1}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots}$$

$$= \left(z - \frac{z^3}{3!} + \dots\right) \left[1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \dots\right) + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \dots\right)^2 + \dots\right].$$

We can now simply multiply the series and, keeping track of the same powers, obtain the same result.

(d) Find the Maclaurin series of $f(z) = \sin^2(z)$.

SOLUTION. We have

$$f'(z) = 2\sin(z)\cos(z) = \sin(2z) = \sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n+1}}{(2n+1)!}, \qquad \text{for all } z \in \mathbb{C}.$$

Then, after termwise integration, we obtain

$$f(z) = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} \frac{z^{2n+2}}{(2n+2)!}, \qquad \text{for all } z \in \mathbb{C}.$$

5.3. **Laurent series.**
A series closest to the Taylor series we considered is the one with negative powers

$$a_0 + a_1(z-z_0)^{-1} + a_2(z-z_0)^{-2} + \cdots \equiv \sum_{n=0}^{\infty} a_n (z-z_0)^{-n}. \tag{15}$$

A substitution $\zeta = \dfrac{1}{z-z_0}$ turns this series into the usual power series

$$a_0 + a_1\zeta + a_2\zeta^2 + \cdots \equiv \sum_{n=0}^{\infty} a_n \zeta^n,$$

which converges inside a disc with a certain radius of convergence $R > 0$. This means that the original series (15) converges *outside* a disc of radius $R_1 = 1/R$ with the centre at z_0. (This radius of convergence can be expressed in terms of the limit of a certain sequence made of the coefficients a_n, but we will not go into these details here.)

Thus, we come to a natural generalization of the Taylor series given by the following definition.

Definition 5.9. A series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

which is a sum of two series, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$, is known as the Laurent series.

□

The main result that we need in the theory of power series is the following theorem.

Theorem 5.10. (Laurent's Theorem, 1843)
Let $f(z)$ be a function analytic in an annulus with centre z_0, $D = \{z \mid R_1 < |z-z_0| < R_2\}$, where R_1 can be equal to zero and R_2 can be equal to infinity, and let γ_ρ be the contour $\{z : |z-z_0| = \rho\}$, where $R_1 < \rho < R_2$. Then,

(1) $\forall z \in D$ the function $f(z)$ is represented by its Laurent series, i.e.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

(2) the coefficient of this series are given by

$$a_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}. \tag{16}$$

By the deformation principle, we can replace γ_ρ in the above integral by any curve into which γ_ρ can be continuously deformed without cross boundaries or points of non-analyticity. $\square$

Note that (16) has the same form as the expression for the coefficients in a Taylor series but not it applies also to $n < 0$. For a function analytic inside γ_ρ the coefficients with $n < 0$ will obviously be zero.

Uniqueness. The Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique. However, $f(z)$ may have different Laurent series for different annuli even with the same centre.

Example. Find all possible Laurent series for

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

about the centre $z_0 = 0$.

SOLUTION. We note that $f(z)$ has singularities at $z = 0$ and $z = 1$.

Case 1: $0 < |z| < 1$

We note that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

is an infinite geometric progression which is convergent in $|z| < 1$. Thus, we have the Laurent series

$$f(z) = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n = -\sum_{n=-1}^{\infty} z^n,$$

which is convergent for $0 < |z| < 1$.

Case 2: $|z| > 1$

$$\frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=1}^{\infty}\frac{1}{z^n}.$$

is an infinite geometric progression which is convergent for $\left|\frac{1}{z}\right| < 1$. Thus, we have the Laurent series

$$f(z) = -\frac{1}{z} + \sum_{n=1}^{\infty}\frac{1}{z^n} = \sum_{n=2}^{\infty}\frac{1}{z^n},$$

which converges for $|z| > 1$.

Example. Find all Taylor and Laurent series of

$$f(z) = \frac{3-2z}{(z-1)(z-2)} = -\frac{1}{z-1} - \frac{1}{z-2}$$

with centre $z_0 = 0$.

SOLUTION. The last example dealt with the first fraction. For the second fraction we have two cases.

Case 1: $|z| < 2$

Inside the circle $|z| = 2$ the second fraction has no singularities, so we have a Taylor expansion

$$\frac{1}{z-2} = \frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n,$$

which is a geometric progression convergent for $|z| < 2$.

Case 2: $|z| > 2$

$$\frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\frac{2}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n,$$

which is a geometric progression convergent for $|z| > 2$.

Thus, the complex plane splits into three regions: (i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$, where we have to combine the corresponding expansions for $-\dfrac{1}{z-1}$ and $-\dfrac{1}{z-2}$.

Region (i): $|z| < 1$

$$f(z) = \sum_{n=0}^{\infty} z^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n.$$

In Region (i) the function expands in a Taylor series.

Region (ii): $1 < |z| < 2$

$$f(z) = -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n.$$

As we see, Region (ii) requires a Laurent series.

Region (iii): $|z| > 2$

$$f(z) = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = -\sum_{n=0}^{\infty} (1 + 2^n) \frac{1}{z^{n+1}}.$$

Region (iii) requires series in terms of $\dfrac{1}{z}$ for both $-\dfrac{1}{z-1}$ and $-\dfrac{1}{z-2}$.

5.4. **Zeros and singularities.**

Definition 5.11. A point z_0 is said to be an <u>isolated singularity</u> of a function $f(z)$ if $f(z)$ is analytic in a punctured neighbourhood of z_0, i.e. $\exists r > 0$: $f \in A(z)\ \forall z \in \{z \mid 0 < |z - z_0| < r\}$. $\square$

Examples.

(a) The function $f(z) = \dfrac{1 + z + z^2}{(1 + z^2)(3 + z^2)}$ has isolated singularities at $z = z \pm i$ and $z = \pm\sqrt{3}i$.

(b) The function $f(z) = \mathrm{Log}\,(z)$ has a singular point at $z = 0$ but it is not isolated: the negative real semi-axis is made of points where $f(z)$ is not analytic.

Similarly to isolated singularities, we define the notion of isolated zeros.

Definition 5.12. A point z_0 is said to be an isolated zero of a function $f(z)$ if $f(z)$ is nonzero in a punctured neighbourhood of z_0, i.e. $\exists r$: $f(z) \neq 0\ \forall z \in \{z \mid 0 < |z - z_0| < r\}$. $\square$

Definition 5.13. A function $f(z)$ analytic in D has a zero of order n at $z = z_0$ if $f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0$ and $f^{(n)}(z_0) \neq 0$. If $n = 1$, i.e. $f(z_0) = 0$ and $f'(z_0) \neq 0$, this zero of order one is usually called a *simple* zero. $\square$

Examples.

(a) The function $f(z) = 1 + z^2 = (z - i)(z + i)$ has simple zeros at $z = i$ and $z = -i$.
(b) The function $f(z) = 1 - \cos(z)$ has a second order zero at $z = 0$ (and, of course, such zeros at $z = 2\pi k$, $k = \pm 1, \pm 2, \ldots$).
(c) If $f(z)$ has an nth-order zero at $z = z_0$, its Taylor series takes the form

$$f(z) = (z - z_0)^n [a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots], \qquad a_0 \neq 0.$$

Theorem 5.14. All zeros of an analytic function $f(z)$ ($\not\equiv 0$) are isolated, i.e., each of them has a neighbourhood that contains no further zeros of $f(z)$. $\square$

Definition 5.15. Suppose that $f(z)$ has a singularity at $z = z_0$ and the Laurent's expansion of $f(z)$ about z_0 has the form

$$f(z) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $0 < |z - z_0| < R$. The first sum on the right-hand side is called the principal part of the Laurent expansion. $\square$

There are the following three possibilities here.

Definition 5.16.

(i) If the principal part of the expansion of $f(z)$ vanishes (i.e. $a_n = 0,\ n \leqslant 1$), then $f(z)$ has **a removable singularity** at z_0. (Actually, the term 'singularity' could be misleading here, as the only thing singular about this point is that $f(z)$ might not be defined at z_0, and once we define it there for f to be continuous, the 'singularity' disappears.)
(ii) If the principal part of the expansion of $f(z)$ has only finitely many non-zero terms, i.e.

$$\sum_{n=-1}^{-M} a_n (z - z_0)^n = \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \ldots + \frac{a_{-M}}{(z - z_0)^M},$$

then the singularity at z_0 is **a pole of order** M. Poles of order one are called *simple* poles.
(iii) If the principal part has infinitely many non-zero terms, then $f(z)$ has **an essential singularity** at z_0.

Theorem 5.17.

(i) $f(z)$ has a removable singularity at $z_0 \Leftrightarrow |f(z)|$ is bounded in the vicinity of z_0.
(ii) $f(z)$ has a pole at $z_0 \Leftrightarrow \lim_{z \to z_0} f(z) = \infty$.

(iii) $f(z)$ has an essential singularity at $z_0 \Leftrightarrow \lim_{z\to z_0} f(z)$ does not exist. $\square$

Sometimes this theorem is used as a definition of singular points in which case the definition we gave becomes a theorem.

Examples.

(a) $f(z) = \dfrac{1}{z}$ has a simple pole at $z = 0$.

(b) $f(z) = \dfrac{z-3}{(z-4)^4}$ has a pole of order 4 at $z = 4$ because

$$f(z) = \frac{z-3}{(z-4)^4} = \frac{z-4}{(z-4)^4} + \frac{1}{(z-4)^4} = \frac{1}{(z-4)^3} + \frac{1}{(z-4)^4}.$$

(c) The expansion of $\sin(1/z)$ about $z = 0$ is given by

$$\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{1}{z^{2n+1}},$$

so that $z = 0$ is an essential singularity.

(d) The function $f(z) = \dfrac{\sin(z)}{z}$ is not defined at $z = 0$ but its Laurent expansion is

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots,$$

i.e. the principal part of the expansion is zero and hence $z = 0$ is a removable singularity, and the series does converge at $z = 0$. The function

$$f(z) = \begin{cases} \dfrac{\sin(z)}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

is entire.

Poles are caused by zeros in the denominator; this is the major reason for the importance of zeros.

Theorem 5.18. If $h(z)$ and $f(z)$ are analytic at $z = z_0$ and $h(z_0) \neq 0$ whereas $f(z)$ has a zero of order n at z_0, then the function $h(z)/f(z)$ has a pole of order n at z_0. $\square$

Definition 5.19. Suppose that $f(z)$ is analytic in $\{z : |z| > R\}$, then we can define

$$g(\zeta) = f\left(\frac{1}{z}\right), \quad \text{where } \zeta = \frac{1}{z}.$$

We say that $f(z)$ has a removable singularity, pole of order M or isolated essential singularity **at** $z = \infty$ if and only if $g(z)$ has a corresponding singularity at zero. $\square$

Branch points are another type of singularity. Let us take a function $f(z) = \sqrt{z}$ and look at its value as we go around the point $z = 0$ along a circumference of a small radius. If $\Gamma_\epsilon = \{z = \epsilon e^{i\theta} \,|\, \epsilon > 0,\, 0 \leqslant \theta \leqslant 2\pi\}$ and we start from the point $z = \epsilon$ where $f(z) = \sqrt{\epsilon}$, then, after going the full circle, we arrive at the same point in the z-plane but the value of f there will be $f(\epsilon e^{2\pi i}) = \sqrt{\epsilon}e^{i\pi} = -\sqrt{\epsilon} \neq \sqrt{\epsilon}$. After going along the same line once again, we will again arrive at the same point and $f(z)$ will be $\sqrt{\epsilon}$, the value we started from.

Definition 5.20. For a function $f(z)$, a point in the z-plane such that, after going around it along a sufficiently small simple closed curve, we arrive at a different value of $f(z)$ is called a branch point of $f(z)$. If, after going around this point n times, we eventually arrive at the value of $f(z)$ we started from, the branch point is called algebraic of order $n-1$. □

Example. For the function $f(z) = \sqrt{z}$ the point $z = 0$ is an algebraic branch point of order 1. □.

Definition 5.21. In order to deal only with single-valued functions (i.e., strictly speaking, with functions), we introduce artificial boundaries known as brunch cuts. □

Example. Consider a choice for branch cut for the function

$$f(z) = (z+1)^{1/2}(z-1)^{1/2}.$$

This function has branch points at $z = \pm 1$. To ensure single-valuedness of $f(z)$, we need to introduce a branch cut, i.e. a boundary of the domain where $f(z)$ is defined. If we introduce a branch cut along the x-axis for $-1 \leqslant x \leqslant 1$ (or along any simple curve connecting the branch points), our function will become single-valued. Alternatively, $x \leqslant -1$ together with $x \geqslant 1$ can be another branch cut that achieves the same purpose (see Fig. 4).

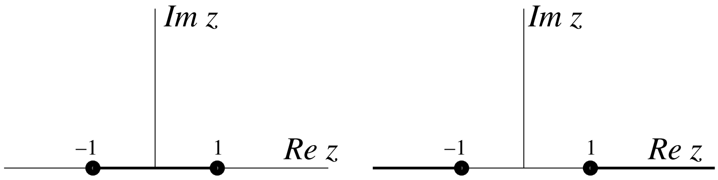

FIGURE 4. Possible cuts to avoid multivaluedness of $f(z) = (z+1)^{1/2}(z-1)^{1/2}$.

6. Calculus of residues

6.1. Residues.

We will begin by defining the main concepts.

Definition 6.1. Let $f(z)$ be defined on D and let z_0 be an isolated singularity of $f(z)$. Suppose that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

is the Laurent expansion of f centred at z_0. The coefficient a_{-1} is called the residue of f at z_0 and denoted by $\operatorname{Res}(f, z_0)$. □

We deduce from Laurent's Theorem that

$$\operatorname{Res}(f, z_0) \equiv a_{-1} = \frac{1}{2\pi i} \int_\Gamma f(z)\, dz,$$

where Γ is any closed contour (anticlockwise) enclosing the only singularity z_0. The following rules are useful for evaluating residues:

Rule 1. If $f(z)$ has a simple pole at z_0, then

$$f(z) = \frac{a_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \dots,$$

and hence

$$\boxed{\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)}$$

Rule 2. Suppose that $f(z)$ has a pole of order m at z_0, i.e.

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-(m-1)}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \ldots.$$

Then,

$$\boxed{\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]}$$

Rule 3. Sometimes it is convenient simply to calculate a few terms in the Laurent expansion and hence find the coefficient a_{-1}. For example, if $h(z)$ and $g(z)$ are analytic on D and $f(z) = g(z)/h(z)$ or $f(z) = g(z)h(z)$ has an isolated singularity at z_0, then the residue can be found directly, by calculating the first few terms in the Taylor expansion of $g(z)$ and $h(z)$ and finding the first few terms in the Laurent expansion of $f(z)$.

Rule 4 (a corollary of Rule 1). If $f(z) = g(z)/h(z)$ has a simple pole at z_0 where $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) \neq 0$, then

$$\operatorname{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

Examples.

(a) Find the Res $(f, 0)$ for $f(z) = \dfrac{\sin(z)}{z^4}$.

Solution. Here the easiest is to apply Rule 3:

$$f(z) = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right]$$

so that at $z = 0$ we have a pole of order 3. The Laurent series is

$$f(z) = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}z - \ldots$$

and hence Res $(f, 0) = -1/3! = -1/6$.

(b) Find Res $(f, 1)$ for $f(z) = \dfrac{z+1}{(z-1)^3(z+3)}$.

Solution. $f(z)$ has a pole of order 3 at $z = 1$. We apply Rule 2 and take the second derivative of the following function

$$g(z) = (z-1)^3 f(z) = \frac{z+1}{z+3}.$$

Then,

$$g'(z) = \frac{(z+3) - (z+1)}{(z+3)^2} = \frac{2}{(z+3)^2}, \qquad g''(z) = -\frac{4}{(z+3)^3}.$$

Therefore,

$$\operatorname{Res}(f, 1) = \frac{1}{2!} g''(1) = \frac{1}{2}\left(-\frac{4}{4^3}\right) = -\frac{1}{32}.$$

(c) Find the Res $(f, 0)$ for $f(z) = z^2 \sin(1/z)$.

SOLUTION. We apply Rule 3,

$$f(z) = z^2 \left[\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \dots \right],$$

and obtain Res $(f, 0) = -1/6$.

(d) Find the residue Res (f, i) for $f(z) = \dfrac{9z + i}{z(z^2 + 1)}$.

SOLUTION. We can see immediately that this is the situation where we can apply Rule 4. Representing $f(z)$ as $f(z) = g(z)/h(z)$, where $g(z) = 9z+i$, $h(z) = z(z^2+1)$, we differentiate $h(z)$ to find

$$h'(z) = 3z^2 + 1, \quad \text{so that} \quad h'(i) = -2.$$

Then, given that $g(i) = 9i + i = 10i$, we have

$$\text{Res}\,(f, i) = \frac{g(i)}{h'(i)} = \frac{10i}{(-2)} = -5i.$$

(e) Find Res $(f, 1)$ for $f(z) = \dfrac{\cosh(\pi z)}{z^4 - 1}$.

SOLUTION. We remember that the equation $z^4 = 1$ does not have 'root of multiplicity 4 at $z = 1$'; it has simple roots at ± 1, $\pm i$. Therefore, at $z = 1$ we have a simple pole and hence can apply Rule 4. Let $f(z) = g(z)/h(z)$, where $g(z) = \cosh(\pi z)$ and $h(z) = z^4 - 1$. Then, $h'(z) = 4z^3$ and $h'(1) = 4$. Therefore,

$$\text{Res}\,(f, 1) = \frac{g(1)}{h'(1)} = \frac{\cosh(\pi)}{4}.$$

Theorem 6.2. (Cauchy's Residue Theorem)

Suppose that Γ is a simple closed contour in a domain D and that $f(z)$ is analytic inside Γ except for a finite number of isolated singularities z_1, z_2, $\dots$, z_n of the single-valued nature (i.e. excluding branch points). Then,

$$\boxed{\oint_{\Gamma} f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res}\,(f, z_k).}$$

□

Proof.

[Note that in the formulation of the theorem it would be sufficient to require just that the singularities are isolated as one can prove that there can only be a finite number of such singularities in a bounded domain. We leave this bit without a proof and hence in the formulation of the theorem require a *finite* number of singularities.]

Firstly, we note that

$$\oint_{|z-z_k|=r_k} (z - z_k)^m\, dz = \oint_{|z-z_k|=r_k} (z - z_k)^m\, d(z - z_k) = \int_{\theta=0}^{2\pi} r_k^m e^{im\theta} \underbrace{ir_k e^{i\theta}\, d\theta}_{=d(z-z_0)}$$

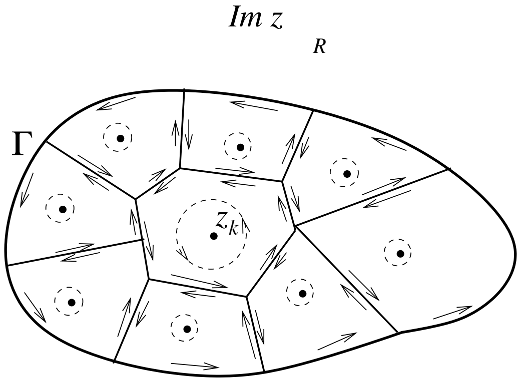

FIGURE 5. An illustrative sketch for the proof of Cauchy's Residue Theorem. The arrows show the direction of integration along each contour.

$$= i r_k^{m+1} \int_0^{2\pi} e^{i(m+1)\theta}\, d\theta = \begin{cases} 2\pi i, & m = -1 \\ 0, & m \neq -1 \end{cases}$$

Now, we can expand $f(z)$ in the Laurent series about each of the singular points

$$f(z) = \sum_{m=-\infty}^{\infty} a_m^{(k)} (z - z_k)^m, \qquad (k = 1, \dots, n).$$

For each of z_k, this representation is valid in a certain annulus with the center at z_k. We can expand the outer boundary of the annulus until it hits the nearest singularity, so that in the sufficiently small vicinity of z_k the above representation is certainly valid.

The region confined by Γ can be split into n regions G_k $(k = 1, \dots, n)$ each containing one singularity z_k. The boundary Γ_k of G_k have the 'inner' parts contained inside the region confined by Γ (each of these parts separates two regions G_k, $k = 1, \dots, n$) and, possibly, also parts of Γ itself (Fig. 5). Then,

$$\oint_\Gamma f(z)\, dz = \sum_{k=1}^{n} \oint_{\Gamma_k} f(z)\, dz,$$

since the integration along the 'inner' parts will be in both directions for the integrals over the contours sharing this part of the boundary (Fig. 5) and hence give no net contribution.

Now, if we take a small circle $|z - z_k| = r_k$ around each singular point, there will be no singularities between this circle and Γ_k, so that, by the deformation principle (based, as we remember, on Cauchy's Integral Theorem),

$$\oint_{\Gamma_k} f(z)\, dz = \oint_{|z - z_k| = r_k} f(z)\, dz.$$

Now, we have all the ingredients and can prove the theorem 'in one line'

$$\oint_\Gamma f(z)\, dz = \sum_{k=1}^{n} \oint_{\Gamma_k} f(z)\, dz = \sum_{k=1}^{n} \oint_{|z - z_k| = r_k} f(z)\, dz$$

$$= \sum_{k=1}^{n} \oint_{|z-z_k|=r_k} \underbrace{\sum_{m=-\infty}^{\infty} a_m^{(k)}(z-z_k)^m}_{=f(z) \text{ around } z_k} dz = \sum_{k=1}^{n} \sum_{m=-\infty}^{\infty} a_m^{(k)} \oint_{|z-z_k|=r_k} (z-z_k)^m \, dz = 2\pi i \sum_{k=1}^{n} a_{-1}^{(k)}$$

$$\equiv 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

In the above, we have taken integration under the sum since the Laurant series converges uniformly and hence can be integrated term-wise. In the last step, we also used the notation $a_{-1}^{(k)} \equiv \operatorname{Res}(f, z_k)$. □

Example. Evaluate the following integral around a circle of radius 2 centred at the origin:

$$\int_{\Gamma} f(z)\, dz = \int_{|z|=2} \frac{4-3z}{z(z-1)}\, dz.$$

Solution. We have two simple poles inside the circle, at $z = 0$ and $z = 1$.

$$\operatorname{Res}(f, 0) = \lim_{z \to 0} \frac{4-3z}{z-1} = -4 \quad \text{(Rule 1)}.$$

$$\operatorname{Res}(f, 1) = \lim_{z \to 1} \frac{4-3z}{z} = 1 \quad \text{(Rule 1)}.$$

By applying Cauchy's Residue Theorem, we obtain

$$\int_{\gamma} f(z)\, dz = 2\pi i(-4+1) = -6\pi i.$$

7. Evaluation of real integrals

The main use of Cauchy's Residue Theorem is evaluating real integrals.

Type I: Integrals of the form $\displaystyle\int_{\theta=0}^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$, where f is finite for all values of θ.

Let $z = e^{i\theta}$, so that

$$d\theta = -i\frac{dz}{z}, \quad \cos\theta = \frac{z+z^{-1}}{2}, \quad \sin\theta = \frac{z-z^{-1}}{2i}.$$

Then,

$$\int_{\theta=0}^{2\pi} f(\cos\theta, \sin\theta), d\theta = \int_{|z|=1} f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\left(-\frac{i}{z}\right) dz.$$

Example. Evaluate

$$I = \int_{0}^{2\pi} \frac{d\theta}{5-4\cos\theta}.$$

Solution.

$$I = \int\limits_{|z|=1} \frac{1}{5-2(z+z^{-1})}\left(-\frac{i}{z}\right) dz = -i \int\limits_{|z|=1} \frac{dz}{5z-2(z^2+1)}$$

$$= i \int\limits_{|z|=1} \frac{dz}{2z^2-5z+2} = i \int\limits_{|z|=1} \frac{dz}{(2z-1)(z-2)}.$$

The function under the integral has simple poles at $z=\frac{1}{2}$ and $z=2$, and only $z=\frac{1}{2}$ is inside the unit circle.

$$\operatorname{Res}\left(f(z), \tfrac{1}{2}\right) = \lim_{z\to 1/2}(z-\tfrac{1}{2})\frac{1}{2(z-\frac{1}{2})(z-2)} = -\frac{1}{3}.$$

Then,

$$I = i\,2\pi i\left(-\frac{1}{3}\right) = \frac{2\pi}{3}.$$

□

Type II: Integrals of the form $\int\limits_{x=-\infty}^{\infty} f(x)\,dx$, where

(a) $f(z)$ is analytic in the upper half-plane except for a finite number of poles,
(b) $|f(z)|$ vanishes as $O(1/z^k)$ $(k>1)$, i.e. $|f(z)| \leqslant C|z|^{-k}$ as $|z|\to\infty$, $0\leqslant \arg(z) \leqslant \pi$.

Definition 7.1. We define the improper integral to be the Cauchy principal value

$$\int\limits_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int\limits_{-R}^{R} f(x)\,dx.$$

□

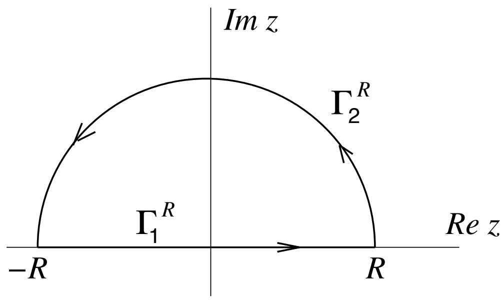

Figure 6. The contour used in Type II and Type III integrals.

In order to find the value of our integral, we integrate $f(z)$ along a contour $\Gamma^R = \Gamma_1^R + \Gamma_2^R$, where Γ_1^R is a segment $[-R,R]$ along the real axis and $\Gamma_2^2 = \{z\,:\,|z|=R, 0\leqslant \arg(z)\leqslant\pi\}$

is a semicircle of radius R connecting the ends of the segment across the upper half-plane. We note that

$$\left| \int_{\Gamma_2^R} f(z)\,dz \right| \leqslant CR^{-k} \cdot \pi R \to 0, \quad \text{as } R \to \infty$$

by the Estimation Lemma.

Hence

$$\lim_{R\to\infty} \int_{\Gamma^R} f(z)\,dz = \lim_{R\to\infty} \left\{ \int_{\Gamma_1^R} f(z)\,dz + \int_{\Gamma_2^R} f(z)\,dz \right\} = \int_{x=-\infty}^{\infty} f(x)\,dx.$$

$$= 2\pi i (\text{The sum of residues in the upper half-plane}).$$

Example. Evaluate

$$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 5x^2 + 4}\,dx.$$

Solution. We consider

$$f(z) = \frac{z^2}{z^4 + 5z^2 + 4} = \frac{z^2}{(z^2+1)(z^2+4)}.$$

(i) $f(z)$ has four simple poles at $\pm i$, $\pm 2i$ of which two are in the upper half-plane,
(ii) $|f(z)|$ vanishes faster than $1/|z|$ as $|z| \to \infty$.

All what we have to do is to calculate the residues at $z = i$ and $z = 2i$.

$$\operatorname{Res}(f, i) = \lim_{z\to i}(z - i)\frac{z^2}{(z^2+1)(z^2+4)} = -\frac{1}{6i}.$$

$$\operatorname{Res}(f, 2i) = \lim_{z\to 2i}(z - 2i)\frac{z^2}{(z^2+1)(z^2+4)} = \frac{1}{3i}.$$

Thus,

$$\int_{x=-\infty}^{\infty} \frac{x^2}{x^4 + 5x^2 + 4}\,dx = 2\pi i \left(-\frac{1}{6i} + \frac{1}{3i} \right) = \frac{\pi}{3}.$$

□

Example. Evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}.$$

Solution. We consider

$$f(z) = \frac{1}{(x^2+a^2)(x^2+b^2)}.$$

(i) $f(z)$ has four simple poles at $\pm ia$ and $\pm ib$, of which two are in the upper half-plane,
(ii) $|f(z)| = O(1/z^4)$ as $z \to \infty$.

We only require the residues:

$$\operatorname{Res}(f, ia) = \lim_{z\to ia}(z-ia)\frac{1}{(z^2+a^2)(z^2+b^2)} = \frac{1}{2ia(b^2-a^2)}.$$

$$\operatorname{Res}(f, ib) = \lim_{z\to ib}(z-ib)\frac{1}{(z^2+a^2)(z^2+b^2)} = \frac{1}{2ib(a^2-b^2)}.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i\left\{\frac{1}{2i(b^2-a^2)}\left(\frac{1}{a}-\frac{1}{b}\right)\right\} = \frac{\pi}{ab(a+b)}.$$

□

Jordan's Lemma. Let $f(z)$ be a function defined in the upper half-plane and Γ_R be a semi-circle of radius R in the upper half-plane. If $|f(z)| \to 0$ as $z \to \infty$, i.e. $\lim_{R\to\infty} \sup_{z\in\Gamma_R} |f(z)| = 0$, then for any $\alpha > 0$

$$\lim_{R\to\infty} \int_{\Gamma_R} e^{i\alpha z} f(z)\, dz = 0.$$

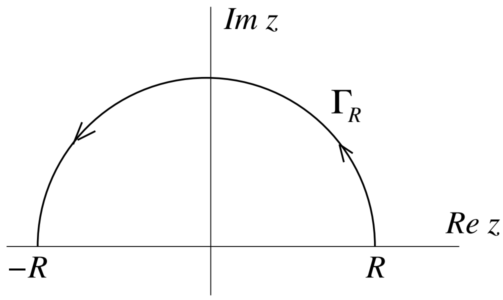

FIGURE 7. An illustrative sketch for Jordan's Lemma.

Proof. We define

$$M(R) = \sup_{z\in\Gamma_R} |f(z)|, \quad \text{and} \quad I_R = \int_{\Gamma_R} e^{i\alpha z} f(z)\, dz.$$

Let $z = Re^{i\theta}$, so that $dz = iRe^{i\theta}\, d\theta$. Then,

$$|I_R| = \left|\int_{\theta=0}^{\pi} e^{i\alpha R\cos\theta - \alpha R\sin\theta} f(Re^{i\theta}) iRe^{i\theta}\, d\theta\right| \leqslant \int_{\theta=0}^{\pi} \left|e^{i\alpha R\cos\theta - \alpha R\sin\theta} f(Re^{i\theta}) iRe^{i\theta}\right| d\theta.$$

We know that $\left|e^{i\alpha R\cos\theta} ie^{i\theta}\right| = 1$ and $|f(Re^{i\theta})| \leqslant M(R)$. Therefore,

$$|I_R| \leqslant R\,M(R) \int_{\theta=0}^{\pi} e^{-\alpha R\sin\theta}\, d\theta = 2R\,M(R) \int_{\theta=0}^{\pi/2} e^{-\alpha R\sin\theta}\, d\theta.$$

We observe that, for $0 \leqslant \theta \leqslant \pi/2$, we have $\sin\theta \geqslant \dfrac{2\theta}{\pi}$ (Jordan's inequality) and hence

$$e^{-\alpha R\sin\theta} \leqslant e^{-2\theta\alpha R/\pi}.$$

Thus,

$$|I_R| \leqslant 2R\,M(R) \int\limits_{\theta=0}^{\pi/2} e^{-2\theta\alpha R/\pi}\,d\theta = \frac{M(R)\pi}{\alpha}(1-e^{-\alpha R}) \leqslant \frac{M(R)\pi}{\alpha}.$$

Since $\lim\limits_{R\to\infty} M(R) = 0$, we finally have $\lim\limits_{R\to\infty} I_R = 0$. $\square$

Type III: Integrals of the form

$$\int\limits_{x=-\infty}^{\infty} f(x)\cos(\alpha x)\,dx \quad \text{and} \quad \int\limits_{x=-\infty}^{\infty} f(x)\sin(\alpha x)\,dx,$$

where

(i) $f(z)$ is analytic in the upper half-plane <u>except for a finite number of poles</u>,
(ii) $\lim\limits_{z\to\infty} |f(z)| = 0$, $0 \leqslant \arg(z) \leqslant \pi$.

We use the same contour as for Type II, and the application of the residue theorem is similar, except that Jordan's Lemma is used.

$$\int\limits_{\Gamma_1^R\cup\Gamma_2^R} f(z)e^{i\alpha z}\,dz = 2\pi i\left(\sum \text{ of residues of } f(z)e^{i\alpha z} \text{ inside } \Gamma_1^R \cup \Gamma_2^r\right),$$

$$\lim_{R\to\infty}\int\limits_{\Gamma_2^R} f(z)e^{i\alpha z}\,dz = 0 \quad \text{by Jordan's Lemma}),$$

$$\lim_{R\to\infty}\int\limits_{\Gamma_1^R} f(z)e^{i\alpha z}\,dz = \int\limits_{-\infty}^{\infty} f(z)e^{i\alpha z}\,dz.$$

Thus, we have

$$\int\limits_{x=-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = 2\pi i\left(\sum \text{ of residues in the upper half-plane}\right).$$

Equating the real and imaginary parts, we obtain

$$\int\limits_{x=-\infty}^{\infty} f(x)\cos(\alpha x)\,dx = \Re\left\{2\pi i\left(\sum \text{ of residues in the upper half-plane}\right)\right\},$$

$$\int\limits_{x=-\infty}^{\infty} f(x)\sin(\alpha x)\,dx = \Im\left\{2\pi i\left(\sum \text{ of residues in the upper half-plane}\right)\right\}.$$

Note that we can't turn, say, $f(x)\cos(\alpha x)$ into $f(z)\cos(\alpha z)$ and treat our integral as Type II integral: we remember that in the complex plane $\cos(\alpha z)$ is unbounded.

Example. Evaluate $I = \int\limits_{x=-\infty}^{\infty} \dfrac{x\sin(x)}{x^2+a^2}\,dx$ for $a > 0$.

Solution. We consider the complex integral

$$I_R = \int\limits_{\Gamma_R} g(z)\, dz,$$

where

$$g(z) = \frac{ze^{iz}}{z^2 + a^2}$$

and $\Gamma_R = \Gamma_R^1 \cup \Gamma_R^2$; $\Gamma_R^1 = [-R, R]$ and Γ_R^2 is a semi-circle of radius R in the upper half-plane.

The integrand has two simple poles at $z = \pm ia$.

$$\mathrm{Res}\,(g, ia) = \lim_{z \to ia} (z - ia) g(z) = \lim_{z \to ia} \frac{ze^{iz}}{z + ia} = \frac{iae^{-a}}{2ia} = \frac{e^{-a}}{2}.$$

Hence $I_R = \pi i e^{-a}$.

$$\begin{aligned}\lim_{R \to \infty} I_R &= \lim_{R \to \infty} \int\limits_{\Gamma_R^1} \frac{ze^{iz}}{z^2 + a^2}\, dz + \lim_{R \to \infty} \int\limits_{\Gamma_R^2} \frac{ze^{iz}}{z^2 + a^2}\, dz \\ &= \int\limits_{x=-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2}\, dx = \pi i e^{-a}.\end{aligned}$$

Equating imaginary parts, we obtain

$$\int\limits_{x=-\infty}^{\infty} \frac{x \sin(x)}{x^2 + a^2}\, dx = \pi e^{-a}.$$

□

Definition 7.2. Another kind of improper integral is a definite integral

$$\int\limits_{x=A}^{B} f(x)\, dx$$

whose integrand becomes infinite at a point $a \in (A, B)$. We define the Cauchy principal value to be

$$P \int\limits_{x=A}^{B} f(x)\, dx = \lim_{\epsilon \to 0} \left(\int\limits_{A}^{a-\epsilon} f(x)\, dx + \int\limits_{x+\epsilon}^{B} f(x)\, dx \right).$$

Type IV: We will consider the case where the integrand has a pole on the real axis.

Lemma 7.3. (Indentation Lemma) If $f(z)$ has a simple pole at $z = a$ on the real axis and $\Gamma^\epsilon = \{z = a + \epsilon e^{i\theta} \,|\, 0 \leqslant \theta \leqslant \pi\}$ is a semi-circle 'by-passing' it *in the anti-clockwise direction*, then

$$\lim_{\epsilon \to 0} \int\limits_{\Gamma^\epsilon} f(z)\, dz = \pi i \mathrm{Res}\,(f(z), a).$$

More generally, if $\Gamma^\epsilon(\theta_1, \theta_2) = \{z = a + \epsilon e^{i\theta} \,|\, 0 \leqslant \theta_1 \leqslant \theta \leqslant \theta_2 \leqslant 2\pi\}$, then

$$\lim_{\epsilon \to 0} \int\limits_{\Gamma^\epsilon(\theta_1,\theta_2)} f(z)\,dz = (\theta_2 - \theta_1) i \mathrm{Res}\,(f(z), a).$$

Proof. By the definition of a simple pole, the integrand $f(z)$ at $z = a$ has the Laurent series

$$f(z) = \frac{\mathrm{Res}\,(f(z), a)}{z - a} + g(z),$$

where $g(z)$ is analytic on Γ^ϵ. Then,

$$\begin{aligned}\int\limits_{\Gamma^\epsilon} f(z)\,dz &= \int\limits_{\theta=0}^{\pi} \frac{\mathrm{Res}\,(f(z), a)}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\,d\theta + \int\limits_{\Gamma^\epsilon} g(z)\,dz \\ &= \pi i \mathrm{Res}\,(f(z), a) + \int\limits_{\Gamma^\epsilon} g(z)\,dz.\end{aligned}$$

By the Estimation Lemma,

$$\lim_{\epsilon \to 0} \int\limits_{\Gamma^\epsilon} g(z)\,dz = 0.$$

Thus,

$$\lim_{\epsilon \to 0} \int\limits_{\Gamma^\epsilon} f(z)\,dz = \pi i \mathrm{Res}\,(f(z), a),$$

□

Example: Evaluate $\displaystyle\int\limits_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$.

Consider the imaginary part of

$$P \int\limits_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz.$$

We choose the following contour to avoid the pole

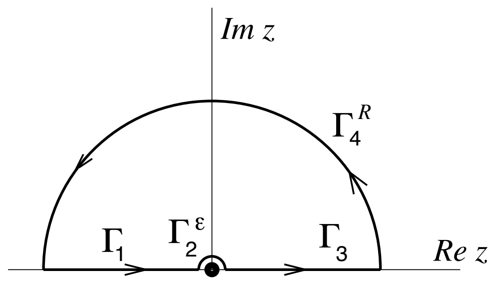

FIGURE 8. A contour with an indentation to avoid a pole.

Analytically it is given as $\Gamma = \Gamma_1 \cup \Gamma_2^\epsilon \cup \Gamma_3 \cup \Gamma_4^R$, where
$\Gamma_1 = \{z = x + iy \,|\, -R < x < -\epsilon,\ y = 0\}$,

$\Gamma_2^\epsilon = \{z = \epsilon e^{i\theta} \,|\, 0 < \theta < \pi\}$,
$\Gamma_3 = \{z = x + iy \,|\, \epsilon < x < R,\ y = 0\}$,
$\Gamma_4^R = \{z = Re^{i\theta} \,|\, 0 < \theta < \pi\}$.

The simple pole at $z = 0$ has the residue

$$\mathrm{Res}\left(\frac{e^{iz}}{z}, 0\right) = \lim_{z\to 0} e^{iz} = 1.$$

By Cauchy's Integral Theorem, for the contour Γ we have

$$\oint_\Gamma \frac{e^{iz}}{z}\,dz = \int_{\Gamma_1} \frac{e^{iz}}{z}\,dz + \int_{\Gamma_2^\epsilon} \frac{e^{iz}}{z}\,dz + \int_{\Gamma_3} \frac{e^{iz}}{z}\,dz + \int_{\Gamma_4^R} \frac{e^{iz}}{z}\,dz = 0.$$

In the limit $R \to \infty$, by Jordan's lemma we obtain

$$\lim_{R\to\infty} \int_{\Gamma_4^R} \frac{e^{iz}}{z}\,dz, = 0$$

and, obviously,

$$\lim_{R\to\infty} \int_{\Gamma_1 \cup \Gamma_3} \frac{e^{iz}}{z}\,dz = P\int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz,$$

whilst, by the Indentation Lemma,

$$\lim_{\epsilon\to\infty} \int_{\Gamma_2^\epsilon} \frac{e^{iz}}{z}\,dz = -\pi i,$$

where we noted, of course, that the integration here is in the *clock*wise direction, and hence we have a minus.

Thus,

$$P\int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz = \pi i$$

and

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \pi.$$

Examples. Consider the following integral

$$I = \int_{-\infty}^{\infty} \frac{x\cos x}{x^2 - a^2}\,dx.$$

The contour needs more than one semicircle on the axis. We have the contour

$$\Gamma_1 = \Gamma_{R,+} \cup (-R, -a-\epsilon) \cup \Gamma_{-a,+} \cup (-a+\epsilon, a-\epsilon) \cup \Gamma_{a,+} \cup (a+\epsilon, R),$$

where

$\Gamma_{R,+} = \{z = Re^{i\theta} \,|\, 0 \leqslant \theta \leqslant \pi\}$,
$\Gamma_{-a,+} = \{z = -a + \epsilon e^{i\theta} \,|\, 0 \leqslant \theta \leqslant \pi\}$,
$\Gamma_{a,+} = \{z = a + \epsilon e^{i\theta} \,|\, 0 \leqslant \theta \leqslant \pi\}$.

Type V. Integrals of the form

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{\varphi(e^x)}\,dx.$$

For integrals of this type, we require a rectangular contour:

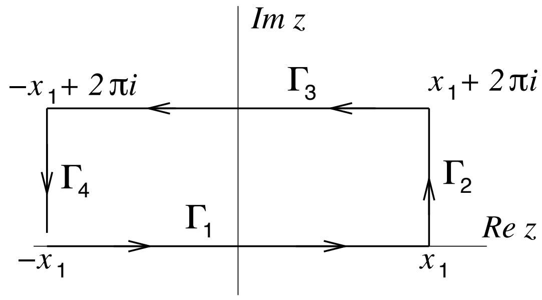

FIGURE 9. The contour used in Type V integrals.

$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where
$\Gamma_1 = \{z = x \,|\, -x_1 \leqslant x \leqslant x_1\}$,
$\Gamma_2 = \{z = x_1 + iy \,|\, 0 \leqslant y \leqslant 2\pi\}$,
$\Gamma_3 = \{z = x + 2\pi i \,|\, -x_1 \leqslant x \leqslant x_1\}$, $\Gamma_4 = \{z = -x_1 + iy \,|\, 0 \leqslant y \leqslant 2\pi\}$.
We require φ to be such that

$$\int_{\Gamma_j} \frac{e^{az}}{\varphi(e^z)}\,dz \to 0 \quad \text{as } x_1 \to \infty \quad \text{for } j = 2, 4.$$

We parametrise Γ_3 by $z = x + 2\pi i$ starting from $x = x_1$ and ending at $x = -x_1$.

$$\int_{\Gamma_3} \frac{e^{az}}{\varphi(e^z)}\,dz = \int_{x=x_1}^{-x_1} \frac{e^{ax}e^{2\pi ia}}{\varphi(e^x)}\,dx = -e^{2\pi ia} \int_{x=-x_1}^{x_1} \frac{e^{ax}}{\varphi(e^x)}\,dx = -e^{2\pi ia} I.$$

Thus,

$$(1 - e^{2\pi ia}) I = 2\pi i \left(\sum \text{ of residues between } \Im(z) = 0 \text{ and } \Im(z) = 2\pi\right),$$

and hence

$$I = \frac{2\pi i}{1 - e^{2\pi ia}} \left(\sum \text{ of residues between } \Im(z) = 0 \text{ and } \Im(z) = 2\pi\right).$$

Example. Evaluate $I = \displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1}\,dx$ for $0 < a < 1$.

The denominator has zeros when $e^z = -1$ or $z = i\pi(2k+1)$ where $k \in \mathbb{Z}$. Only $z = i\pi$ is in the region of interest. We expand the denominator in terms of $\zeta = z - i\pi$:

$$e^z + 1 = 1 + e^{z - i\pi} e^{i\pi} = 1 - e^{z - i\pi} = 1 - \left[1 + (z - i\pi) + \frac{(z - i\pi)^2}{2!} + \dots\right]$$

$$= -(z - i\pi) - \frac{(z - i\pi)^2}{2!} - \dots.$$

Hence $z = i\pi$ is a simple pole with

$$\text{Res}\left(\frac{e^{az}}{e^z + 1}, i\pi\right) = \lim_{z \to i\pi}(z - i\pi)\frac{e^{az}}{e^z + 1} = -e^{i\pi a}.$$

Therefore

$$I = \frac{-2\pi i e^{i\pi a}}{1 - e^{2\pi i a}} = \pi\left(\frac{2i}{e^{i\pi a} - e^{-i\pi a}}\right) = \frac{\pi}{\sin(\pi a)}.$$

□

7.1. **Integrals involving branch cuts.** In many cases, the functions we have to integrate involve branch points which makes it necessary to introduce branch cuts as boundaries having the branch points as their ends. Then, in the resulting domain we 'move' continuously to ensure that we don't cross the branch cuts (see the move from Γ_4 to Γ_2 or from one singular point to the next in the example below).

Example. Evaluate $I = \displaystyle\int_0^\infty \frac{\sqrt{x}}{x^3 + 1}\, dx.$

SOLUTION. The function $\dfrac{1}{z^3 + 1}$ has simple poles at $e^{i\pi/3}$, $e^{i\pi}$ and $e^{i5\pi/3}$ (note that the last two points are at $e^{i\pi}$, $e^{i5\pi/3}$ but not at $e^{-i\pi}$, $e^{-i\pi/3}$).

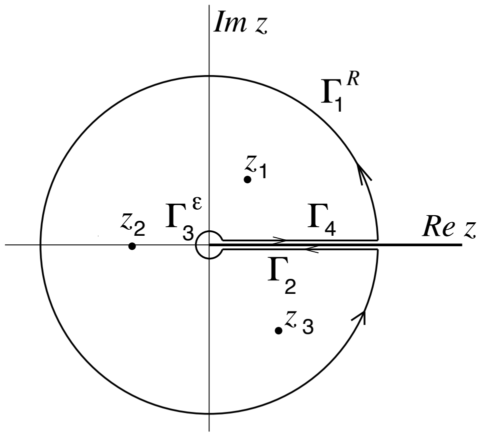

FIGURE 10. A 'keyhole' contour with a branch cut along the positive real axis.

We adopt a branch cut along the positive real axis and take a closed 'keyhole' contour $\Gamma = \Gamma_1^R \cup \Gamma_2 \cup \Gamma_3^\epsilon \cup \Gamma_6$, where

$\Gamma_1^R = \{z = Re^{i\theta} \,|\, R > 0,\, 0 < \theta < 2\pi\}$,
$\Gamma_2 = \{z = re^{i2\pi} \,|\, \epsilon < r < R\}$
$\Gamma_3^\epsilon = \{z = \epsilon e^{i\theta} \,|\, 0 < \theta < 2\pi\}$,
$\Gamma_4 = \{z = re^{i0} \,|\, \epsilon < r < R\}$.

Then,

$$\int_{\Gamma_1^R} \frac{z^{1/2}}{z^3+1}\,dz \to 0 \quad \text{as } R\to\infty \quad \text{and} \quad \int_{\Gamma_3^\epsilon} \frac{z^{1/2}}{z^3+1}\,dz \to 0 \quad \text{as } \epsilon\to 0$$

from the Estimation Lemma.

On Γ_4, we have $z = xe^{i0} = x$ and hence

$$\int_{\Gamma_4} \frac{z^{1/2}}{z^3+1}\,dz = \int_{x=\epsilon}^{R} \frac{x^{1/2}}{x^3+1}\,dx \to I \quad \text{as } \epsilon\to 0,\ R\to\infty.$$

On Γ_2, we have $z = xe^{2\pi i}$ and hence $z^{1/2} = x^{1/2}e^{i\pi} = -x^{1/2}$. Then,

$$\int_{\Gamma_2} \frac{z^{1/2}}{z^3+1}\,dz = \int_{x=R}^{\epsilon} \frac{(-x^{1/2})}{x^3+1}\,dx = \int_{x=\epsilon}^{R} \frac{x^{1/2}}{x^3+1}\,dx \to I \quad \text{as } \epsilon\to 0,\ R\to\infty.$$

By Cauchy's Residue Theorem, we have $2I = 2\pi i$(sum of residues), so that all what we need to do is to calculate the residues. Let $f(z) = g(z)/h(z)$, where $g(z) = z^{1/2}$ and $h(z) = 1+z^3$. The residue at each of the three simple poles z_i, z_2, z_3 is calculated using Rule 4:

$$\mathrm{Res}\,(f, z_k) = \frac{z_k^{1/2}}{3z_k^2} = \frac{z_k^{-3/2}}{3}, \qquad k = 1, 2, 3.$$

Hence

$$I = \pi i \left\{ \mathrm{Res}\,(f, e^{i\pi/3}) + \mathrm{Res}\,(f, e^{i\pi}) + \mathrm{Res}\,(f, e^{i\pi 5/3}) \right\}$$
$$= \frac{\pi i}{3}\left\{ e^{-i\pi/2} + e^{-i\pi 3/2} + e^{-i\pi 5/2} \right\} = \frac{\pi i}{3}(-i+i-i) = \frac{\pi}{3.}$$

8. Evaluation of complex integrals

The principles of taking these integrals are the same as before: (a) if we already have a closed contour, then Cauchy's Residue Theorem allows us to take it by calculating the residues at the singular points; (b) if the contour is not closed, we make it a part of a closed contour, either directly or in a certain limit (like $\epsilon \to 0$, $R \to \infty$, etc) choosing the parts of the closed contour that we are free to choose in such a way that it is easy to take the integrals along these parts and taking the integral over the closed contour as in (a).

Example ('cheesecake' contour). Evaluate $I = \int_{x=0}^{\infty} e^{itx^2}\,dx$ for $t > 0$.

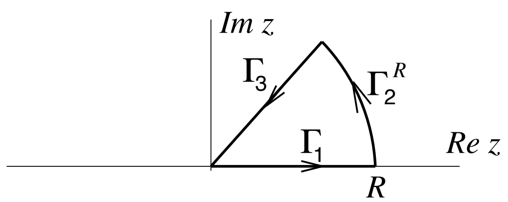

Figure 11. A cheesecake contour.

Solution. To evaluate the integral, we use a 'cheesecake' contour $\Gamma = \Gamma_1 \cup \Gamma_2^R \cup \Gamma_3$, where

$\Gamma_1 = \{z = x + i0 \,|\, 0 < x < R\}$,
$\Gamma_2^R = \{z = Re^{i\theta} \,|\, 0 < \theta < \pi/4\}$,
$\Gamma_3 = \{z = re^{i\pi/4} \,|\, 0 < r < R\}$.
By Cauchy's Residue Theorem, we have

$$\oint\limits_{\Gamma} e^{itz^2}\, dz = 0.$$

Firstly, we estimate the integral on Γ_2^R:

$$\left| \int\limits_{\Gamma_2^R} e^{itz^2}\, dz \right| = \left| \int\limits_{\theta=0}^{\pi/4} e^{itR^2[\cos(2\theta)+i\sin(2\theta)]} iRe^{i\theta}\, d\theta \right|$$

$$\leqslant R \int\limits_{\theta=0}^{\pi/4} e^{-tR^2\sin(2\theta)}\, d\theta \leqslant R \int\limits_{\theta=0}^{\pi/4} e^{-tR^2\, 4\theta/\pi}\, d\theta$$

(using Jordan's inequality $\sin(\varphi) \geqslant 2\varphi/\pi$, $0 \leqslant \varphi \leqslant \pi/2$)

$$\leqslant \left[\frac{-\pi}{4tR} e^{-tR^2\, 4\theta/\pi} \right]^{\pi/4} = \frac{\pi}{4tR}\left[1 - e^{-tR^2}\right] \to 0 \quad \text{as } R \to \infty.$$

Secondly, we consider $\Gamma_3 : \; z = re^{i\pi/4}$.

$$\int\limits_{\Gamma_3} e^{itz^2}\, dz = \int\limits_{r=R}^{0} e^{-tr^2} e^{i\pi/4}\, dr = -e^{i\pi/4} \int\limits_{r=0}^{R} e^{-tr^2}\, dr$$

Finally, we state the result on $\Gamma_1 : \; z = x$.

$$\int\limits_{\Gamma_1} e^{itz^2}\, dz = \int\limits_{x=0}^{R} e^{itx^2}\, dx.$$

Thus, we combine these results to obtain

$$\int\limits_{x=0}^{\infty} e^{itx^2}\, dx = -e^{i\pi/4} \int\limits_{r=0}^{\infty} e^{-tr^2}\, dr$$

(or, after using the substitution $u = \sqrt{t}r$)

$$e^{i\pi/4} \int\limits_{u=0}^{\infty} e^{-u^2} \frac{du}{\sqrt{t}} = e^{i\pi/4} \frac{1}{2}\sqrt{\frac{\pi}{t}},$$

where we used that $\int_0^\infty e^{-u^2}\, du = \sqrt{\pi}/2$.

9. Fourier Transforms

9.1. **Preliminaries.** Fourier's integral formula which appears in the course on functional analysis (and is given here without proof),

$$f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} dk \int\limits_{-\infty}^{\infty} f(t) e^{-ik(t-x)} dt,$$

that is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} f(t) e^{-ikt} dt \right)}_{=F(k)} e^{ikx} \, dk,$$

allows one to introduce the Fourier transform and inverse Fourier transform in the following way.

Definition 9.1. If $f(x)$ is a function integrable (here we deal with the Lebesgue integral) on $-\infty < x < \infty$ [i.e. $f \in L(-\infty, \infty)$], then

$$F[f] \equiv F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{17}$$

is called the Fourier transform of function f. (The alternative notation for the Fourier transform of f, g etc are $F[f]$, $F[g]$ etc or $\hat{f}(k)$, $\hat{g}(k)$ etc.)

The inverse transform is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx} \, dk. \tag{18}$$

□

A few important properties of the Fourier transform are as follows.

	Function	Fourier transform		Function	Fourier transform
(i)	$\frac{df}{dx}$	$ikF(k)$	(iv)	$e^{iax} f(x)$	$F(k-a)$
(ii)	$xf(x)$	$i\frac{dF}{dk}$	(v)	$af(x) + bg(x)$	$aF(k) + bG(k)$
(iii)	$f(x-a)$	$e^{-iak} F(k)$	(vi)	$f(ax)$	$\frac{1}{a} F\left(\frac{k}{a}\right)$

Definition 9.2. The integral

$$\int_{-\infty}^{\infty} f(y) g(x-y) \, dy = f * g$$

is called the convolution of functions f and g. Clearly, $f * g$ is a function of x. □

An important property of this operation is given by

Theorem 9.3. (Convolution Theorem)

$$F[f * g] = F[f] F[g].$$

□

Proof: Prove it as an exercise.
Some useful transforms:

	$f(x)$	$F(k)$
Delta function	$\delta(x)$	1
Square pulse	$H(a-\|x\|)$	$\frac{2}{k}\sin ak$
Exponential	$e^{-a\|x\|}$	$\frac{2a}{a^2+k^2}$
Heaviside function	$H(x)$	$\pi\delta(k)+\frac{1}{ik}$
Sign	$H(x)-H(-x)$	$\frac{2}{ik}$
Constant	1	$2\pi\delta(k)$
Gaussian	$e^{-x^2/2}$	$\sqrt{2\pi}e^{-k^2/2}$

9.2. **Solving PDEs.** The key thing in the application of the Fourier transform to PDEs is the property listed as (i) above, namely the fact that the Fourier transform 'kills' derivatives thus turning PDEs into ODEs or simpler PDEs.

Example. Consider the following initial-value problem

$$
\begin{aligned}
u_t &= Du_{xx}, \qquad (-\infty < x < \infty,\ t>0) && (19)\\
u(x,0) &= u_0(x). && (20)
\end{aligned}
$$

Solution. Taking the Fourier transform of (19) with respect to x, we get

$$F[u_{xx}] = ikF[u_x] = -k^2F[u] = -k^2\tilde{u},$$

where for brevity we used the notation $\tilde{u} \equiv F[u]$.

Thus we have

$$\frac{d\tilde{u}}{dt} = -Dk^2\tilde{u} \tag{21}$$

The Fourier tranform of the initial condition (20) gives

$$\tilde{u}(k,0) = F[u_0] = \tilde{u}_0(k). \tag{22}$$

Equation (21) can be integrated immediately giving

$$\tilde{u} = A(k)e^{-Dk^2t}.$$

It is <u>important</u> that the constant of integration is actually a function of k, the transform variable.

Using condition (22) to determine $A(k)$ we find that $A(k) = \tilde{u}_0(k)$. Thus, we have arrived at the *'solution in terms of the Fourier transforms'*

$$\tilde{u}(k,t) = \tilde{u}_0(k)e^{-Dk^2 t}. \tag{23}$$

The remaining step is to invert the transform. The standard way of doing this is (a) either to consult the corresponding entry in the table of transforms and find the answer (if you are lucky), or (b) to use the residue calculus to take the corresponding integral (usually doable when the solution in terms of transforms is expressed via elementary functions), or — and this is much more tricky — (c) invert the transform numerically.

In our case, the original problem is so simple that the inverse transform can be found in an elementary way and in a general case.

We can verify this by taking the corresponding integral (see the example above with a 'cheesecake' contour) that

$$F^{-1}[e^{-Dk^2 t}] = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}, \quad \text{i.e.} \quad e^{-Dk^2 t} = F\left[\frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}\right].$$

Thus, solution (23) has the form

$$\tilde{u}(k,t) = F[u_0(x)]F\left[\frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}\right] = F\left[u_0(x) * \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}\right],$$

where in the last equality we have used the convolution theorem. Then, we have

$$u(x,t) = u_0(x) * \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}},$$

that is

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} u_0(s)e^{-(x-s)^2/4Dt}\, ds,$$

which is the desired solution of the original PDE problem.

□

10. Laplace transforms

This transform is useful for some linear equations in the case where one of the arguments, usually the time, varies from 0 to $+\infty$.

10.1. Preliminaries.

Definition 10.1. If $g(t)$ is a function of a real variable t, $0 \leqslant t < +\infty$, integrable on an arbitrary interval $(0, A)$ and $s = \alpha + ik$ is a complex variable, then

$$L[g] \equiv G(s) = \int_0^{\infty} e^{-st} g(t)\, dt \tag{24}$$

is called the Laplace transform of the function g. A sufficient condition for convergence of the above integral which we will use here is that $|g|$ is limited from above by an exponential function, that is $\exists \alpha,\ B$: $|g(t)| < Be^{\alpha t}$ as $t \to +\infty$. Obviously, if the integral in (24) converges for $s = s_0$, then it converges for all s, $\Re(s) > \Re(s_0)$. $\square$

10.2. **Inverse transform.** The inversion formula can be derived from the corresponding formula for the Fourier transform in the following way. Let us consider a function

$$h(t) = \begin{cases} e^{-\alpha t} g(t), & t \geqslant 0 \\ 0, & t < 0 \end{cases},$$

where α is as above with $G(s)$ defined for $\Re(s) > \alpha$. Since $g(t)$ is not exponentially large as $t \to \infty$, we can take the Fourier transform of $h(t)$,

$$H(k) = \int_{-\infty}^{\infty} e^{-ikT} h(T)\, dT,$$

and consider the inverse transform

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikt} H(k)\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikt} \underbrace{\int_{-\infty}^{\infty} e^{-ikT} \overbrace{h(T)}^{=e^{-\alpha T} g(T)}\ dT}_{=H(k)}\ dk$$

In terms of $g(t)$ this gives

$$e^{-\alpha t} g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikt} \int_{0}^{\infty} e^{-(\alpha+ik)T} g(T)\, dT\, dk$$

that is

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\alpha+ik)t} \underbrace{\int_{0}^{\infty} e^{-(\alpha+ik)T} g(T)\, dT}_{=G(s)}\ dk.$$

Now let $s = \alpha + ik$ and $ds = i\, dk$. Then, the above equality becomes

$$g(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{st} G(s)\, ds, \tag{25}$$

which is known as the <u>Bromwich inversion integral</u> for the Laplace transform. Note that now we have to integrate along a <u>vertical</u> line in the complex s-plane (the <u>Bromwich path</u>) which lies to the right of any poles of the integrand. (Remember that $G(s)$ exists provided $\Re(s) > \alpha$.)

As for Fourier transforms we can evaluate the integral by closing the contour with a large semi-circle and using residue calculus. For $t < 0$ we need to close the contour to the right so that $\Re(st) < 0$ for $|s| \gg 1 \implies$ no poles enclosed $\implies g(t) \equiv 0$ for $t < 0$. For $t > 0$ the contour is to the left and hence we get contribution from poles.

The most important (for our purposes) properties of Laplace's transform are:

	Function	Laplace transform
(i)	$af(t) + bg(t)$	$aF(s) + bG(s)$
(ii)	$\dfrac{df}{dt}$	$sF(s) - f(0)$
(iii)	$\dfrac{d^2 f}{dt^2}$	$s^2F(s) - sf(0) - f'(0)$
(...)	$\dfrac{d^n f}{dt^n}$	$s^nF(s) - \sum\limits_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$

If we re-define convolution of two functions by

$$f * g = \int_0^\infty f(\tau)g(t-\tau)\, d\tau,$$

then we will have the familiar

Theorem 10.2. (Convolution Theorem) $L[f * g] = L[f]L[g]$. $\square$

Some other useful properties of the Laplace transform are listed below.

	Function	Laplace transform
(iv)	$e^{bt}f(t)$	$F(s-b)$
(v)	$\dfrac{f(t)}{t}$	$\int_s^\infty F(z)\, dz$
(vi)	$tf(t)$	$-\dfrac{dF}{ds}$
(vii)	$H(t-b)f(t-b)$	$e^{-bs}F(s)$
(viii)	$f(cx)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$

Example. Consider $G(s) = \dfrac{1}{s-a}$, where a is real. We have a simple pole at $s = a$ and can evaluate the Bromwich integral by considering an expanding semi-circle

$$\Gamma = \{s = \alpha + iy \mid -R \leqslant y \leqslant R\} \cup \{s = x + iy \mid x^2 + y^2 = R^2 + \alpha^2,\, x < \alpha\}, \quad R \to +\infty.$$

Then

$$g(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{e^{st}}{s-a}\, ds = \sum \text{residues of } \frac{e^{st}}{s-a} = e^{at}.$$

Here we can see a link between simple poles and exponential function.

Useful info. The following table gives some useful Laplace transforms:

$f(t)$	$F(s)$	$f(t)$	$F(s)$
e^{at}	$\frac{1}{s-a}$	t^k	$\frac{k!}{s^{k+1}}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$H(t-b)$	$\frac{1}{s}e^{-bs}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\delta(t-b)$	e^{-bs}
$\cosh at$	$\frac{s}{s^2-a^2}$	$a(4\pi t^3)^{-1/2}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$(\pi t)^{-1/2}e^{-a^2/4t}$	$\frac{1}{\sqrt{s}}e^{-a\sqrt{s}}$

10.3. **Solving PDEs.** The idea of using the Laplace transform for solving PDEs is the same as for the Fourier transform. We will illustrate this idea with the following

Example. Let us consider a string, initially at rest, driven by a given displacement at the end with no disturbances coming from infinity.

$$u_{xx} - \frac{1}{c^2}u_{tt} = 0, \tag{26}$$
$$u(x,0) = u_t(x,0) = 0, \tag{27}$$
$$u(0,t) = g(t), \tag{28}$$
$$u(\infty,t) = 0. \tag{29}$$

The use of the Laplace transform is appropriate since we need a solution for $t > 0$ and the structure of equations and auxiliary conditions is likely to lead to simplifications.

The Laplace transform of (26) is given by

$$U_{xx} - \frac{1}{c^2}\left[s^2U - su(x,0) - u_t(x,0)\right] = 0.$$

Note that, unlike what we had with the Fourier transform, the initial conditions become part of the equations. In our case, initial conditions (27) are such that the corresponding terms vanish and we have an ODE

$$U_{xx} - \frac{s^2}{c^2}U = 0.$$

After integration we have

$$U(x,s) = A(s)e^{sx/c} + B(s)e^{-sx/c}. \tag{30}$$

Condition (29) (= no incoming waves) gives $A(s) = 0$. The Laplace transform of boundary condition (28) gives

$$L[u(0,t)] = U(0,s) = L[g(t)] = G(s).$$

From (30), with $A(s) = 0$, we have that $U(0, s) = B(s)$ Thus, $B(s) = G(s)$ and we arrive at the solution in terms of transforms

$$U(x, s) = G(s)e^{-sx/c},$$

Inversion of this solution is simple because we do not need to actually take the Bromwich integral. Indeed,

$$u(x,t) = \frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty} e^{st}G(s)e^{-sx/c}\,ds = \frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty} e^{s(t-x/c)}G(s)\,ds = g(t-x/c).$$

In other words, the imposed disturbances propagate at speed c.

Remark. If we consider a similar problem for the diffusion equation (there we will have of course only one initial condition), then the ODE for U will be the same with the only "minor" difference that there will be s instead of s^2. As a result, we will have a square root in the solution and will have to consider the branch point, take a contour of integration with a branch cut, etc. Note that our branch cut should not cross the Bromwich path, so that, for example, if we have a branch point, for simplicity, at $s = 0$, the contour of integration take the form (Fig. 12):

$$\Gamma = B \cup \Gamma^R_+ \cup C_+ \cup \Gamma^\epsilon \cup C_- \cup \Gamma^R_-,$$

where

$$\begin{aligned}
B &= \{s = \alpha + iy \mid -\sqrt{R^2-\alpha^2} < y < \sqrt{R^2+\alpha^2}\} \quad \text{(a part of the Bromwich path)},\\
\Gamma^R_+ &= \{s = R^{i\theta} \mid \arccos(\alpha/R) \leqslant \theta \leqslant \pi\} \quad \text{(the upper half of a large semi-circle)},\\
C_+ &= \{s = re^{i\pi} \mid \epsilon \geqslant r \leqslant R\} \quad \text{(the upper side of a cut along the negative } x\text{-axis)},\\
\Gamma^\epsilon &= \{s = \epsilon e^{i\theta} \mid -\pi \leqslant \theta \leqslant \pi\} \quad \text{(a small circle around the branch point)},\\
C_- &= \{s = re^{-i\pi} \mid \epsilon \geqslant r \leqslant R\} \quad \text{(the lower side of the cut along the negative } x\text{-axis)},\\
\Gamma^R_- &= \{s = Re^{i\theta} \mid -\pi \leqslant \theta \leqslant -\arccos(\alpha/R)\} \quad \text{(the lower half of the large semi-circle)},
\end{aligned}$$

As $R \to \infty$, (i) the integral over $B_{\mp R}$ will tend to the Bromwich integral, and (ii) the integral over $K_{R+} \cup K_{R-}$ vanish. Besides, we also take the limit $\epsilon \to 0$, which allows us (iii) to evaluate the integral over K_ϵ and (iv) take the integral along the cut, which is now from $-\infty$ to 0.

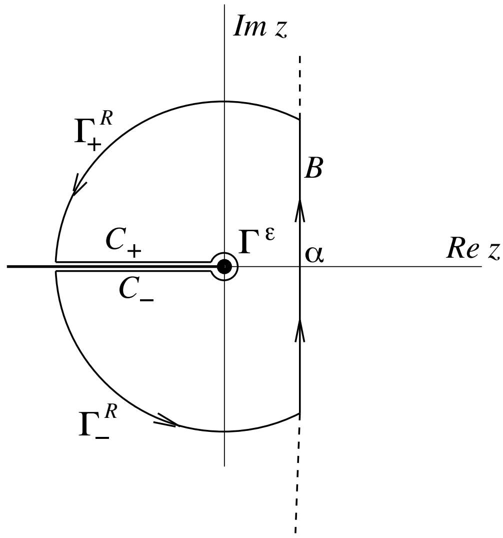

FIGURE 12. A contour for the inverse Laplace transform in the case of a branch point at the origin.