Fall 2016, MATH 215 Calculus III, Exam 1

10/13/2016, 6:10-7:40pm (90 minutes)

- **Your name:** ______________________________

- **Circle your section and write your Lab time:**

Section	Time	Professor	GSI	Lab Time (e.g. Th 10-11)
10	8–9	Divakar Viswanath	Trevor Hyde	____________
20	8–9	Howard Levinson	Rob Silversmith	____________
30	9–10	Divakar Viswanath	Qingtang Su	____________
40	10–11	Daniel Burns	Harry Richman	____________
50	11–12	Yongbin Ruan	Montek Singh Gill	____________
60	12–1	Yongbin Ruan	Harry Lee	____________
70	1-2	Alejandro Uribe	Emanuel Reinecke	____________
80	2-3	Sema Gunturkun	John Kilgore	____________
90	3-4	Sema Gunturkun	Michael Newman	____________

Instructions:

- This examination booklet contains 8 problems on 12 sheets of paper including the front cover. The last page is left blank for your own use. The second from last page contains a list of formulas.
- This is a 90-minute exam.
- This is a closed book exam. Electronic devices, calculators and note-cards are not allowed.
- Show your work and explain clearly.

Your scores:

Problem	Your score	Maximum score
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
Total		80

1. Consider the planes $x + y + z = 1$ and $x - y + z = 2$.

   (a) (5 points) Recall that the angle between two planes is by definition equal to the angle between their normals. If θ is the angle between the two given planes, find $\cos\theta$.

   (b) (5 points) Find a plane that is orthogonal to the two given planes and which goes through the point $(2, 1, 3)$.

(a): The angle is given by $\cos\theta =$ ________________________________

(b): The equation of the plane is ________________________________

2. (10 points) Consider $f(x, y) = x^2 + \frac{y^2}{4}$. Let P be the point with $(x, y) = (1, 2)$. Find the gradient ∇f, the direction (unit vector) of fastest increase, the direction (unit vector) of fastest decrease, and the direction in which the function neither increases nor decreases at the point P ($(x, y) = (1, 2)$).

The gradient vector is ________________________________

The direction of fastest increase is ________________________________

The direction of fastest decrease is ________________________________

The direction in which the function neither increases nor decreases is ________________

3. (10 points) Let $f(x, y) = F(x^2 + y^2) + G(xy)$, where F and G are functions of a single variable. Assume $F(2) = 1$, $F'(2) = 2$ and $G(-1) = -1$, $G'(-1) = -2$. Find $f(x, y)$, $\partial f/\partial x$ and $\partial f/\partial y$ at $(x, y) = (1, -1)$.

$f(1, -1) =$ ____

$\frac{\partial f}{\partial x}(1, -1) =$ ____

$\frac{\partial f}{\partial y}(1, -1) =$ ____

4. Consider the plane $x + 2y + 2z = 4$.

   (a) (5 points) Find the distance from the plane to the origin.

   (b) (5 points) Find the point on the sphere $x^2 + y^2 + z^2 = 1$ which is closest to the plane.

(a): the distance is________________________________

(b): the closest point is ________________________________

5. (10 points) The pressure P, volume V, and temperature T of one mole of an ideal gas satisfy $PV = RT$, where R is a constant. Suppose R is measured using $R = PV/T$.

   (a) (5 points) Find the differential dR.

   (b) (5 points) If the percentage errors in the measurement of P, V, and T are 1%, 2%, and 3%, respectively, find the maximum percentage error in R.

$dR =$ ____

The maximum percentage error in R is ____________________________

6. Consider the surface $xyz = 1$ in $\mathbb{R}^3$. Let P be the point $(x, y, z) = (2, 1, \frac{1}{2})$.

   (a) (5 points) Find the normal to the surface at the point P.

   (b) (5 points) Find the tangent plane to the surface at the point P.

The normal vector is ____

The tangent plane is ____

7. An unknown line through the point $(1, 2, 3)$ intersects the given line $\mathbf{r}(t) = (t+3, 3-t, t+2)$ at right angles

   (a) (7 points) Find the point of intersection of the two lines.

   (b) (3 points) Find the parametric equation of the unknown line.

(a): The point of intersection is ________________________________

(b): The equation of the line is ________________________________

8. In this problem, take $g = 10\,\mathrm{m/s}^2$. A projectile is fired at an angle α, $0 < \alpha < 90^o$ on level ground at a speed of 120 m/s. Let $\mathbf{r}(t)$ be its position as a function of time.

   (a) (3 points) Take $\mathbf{r}(0) = (0, 0)$, $\mathbf{r}'(0) = 120(\cos\alpha, \sin\alpha)$, and $\mathbf{r}''(t) = (0, -g)$, and calculate $\mathbf{r}(t)$. Wind resistance is ignored.

   (b) (3 points) Express the range of the projectile as a function of α.

   (c) (4 points) If the range is 720 m, find the two possible values of α. You may use the trig identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$.

(a) The position $\mathbf{r}(t)$ is given by ______________________________

(b) The range is ______________________________

(c) The two values of α are ______________________________

Formula sheet: You may find some of the following formulas useful

- $\sin^2(x) + \cos^2(x) = 1$, $\cos(2x) = \cos^2(x) - \sin^2(x)$, $\sin(2x) = 2\sin(x)\cos(x)$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$, $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$, $\cos(\pi/4) = \sqrt{2}/2$, $\sin(\pi/4) = \sqrt{2}/2$, $\cos(\pi/6) = \sqrt{3}/2$, $\sin(\pi/6) = 1/2$, $\cos(0) = 1$, $\sin(0) = 0$.
- $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$.
- $\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|^2}\right)\mathbf{b}$, $\quad \text{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|}$
- $\langle a,b,c\rangle \times \langle d,e,f\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \langle bf - ce, -(af - cd), ae - bd\rangle.$
- $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$, $\quad \mathbf{a}\times(\alpha\mathbf{b}) = (\alpha\mathbf{a})\times\mathbf{b} = \alpha(\mathbf{a}\times\mathbf{b})$.
- Area of the parallelogram determined by the vectors $\mathbf{v_1} = \langle a,b,c\rangle$ and $\mathbf{v_2} = \langle d,e,f\rangle$ is $|\mathbf{v_1}\times\mathbf{v_2}|$.
- Volume of the parallelepiped determined by the vectors $\mathbf{v_1} = \langle a,b,c\rangle$, $\mathbf{v_2} = \langle d,e,f\rangle$, and $\mathbf{v_3} = \langle g,h,i\rangle$ is $|\mathbf{v_1}\cdot(\mathbf{v_2}\times\mathbf{v_3})| =$ absolute value of $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$
- Distance from a point (a,b,c) to a plane $Ax + By + Cz + D = 0$ is $\frac{|Aa+Bb+Cc+D|}{\sqrt{A^2+B^2+C^2}}$.
- Arc length function: Length of curve from $\mathbf{r}(\alpha) = (x(\alpha), y(\alpha), z(\alpha))$ to $\mathbf{r}(t) = (x(t), y(t), z(t))$ is $\int_\alpha^t |\mathbf{r}'(u)|du$.
- $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, $\quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$
- The circumference of a circle of radius a is $2\pi a$.
- The area of a disk of radius a is πa^2.
- The volume of a right circular cylinder of radius a and height h is $\pi a^2 h$.

Scratch paper.