Planetary orbits

Let us now see whether we can use Newton's universal laws of motion to derive Kepler's laws of planetary motion. Consider a planet orbiting around the Sun. It is convenient to specify the planet's instantaneous position, with respect to the Sun, in terms of the *polar coordinates* r and θ. As illustrated in Fig. 105, r is the radial distance between the planet and the Sun, whereas θ is the angular bearing of the planet, from the Sun, measured with respect to some arbitrarily chosen direction.

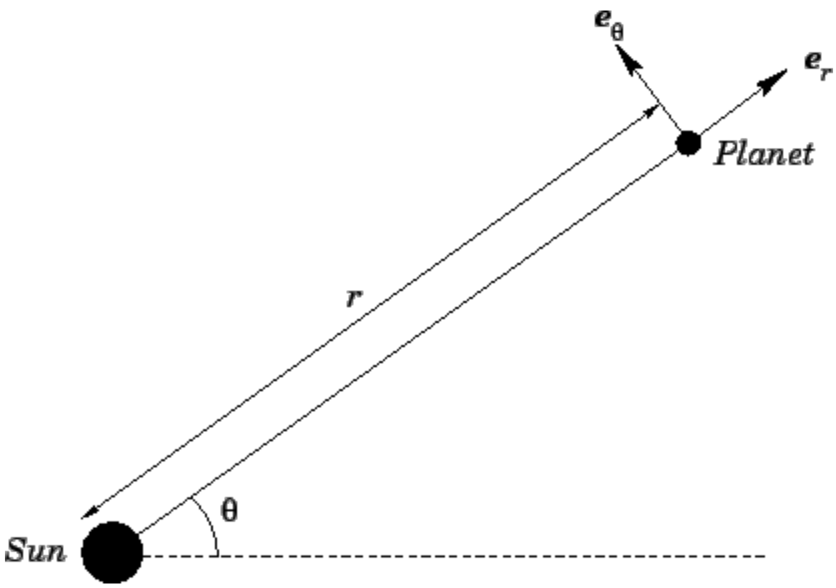

Figure 105: *A planetary orbit.*

Let us define two unit vectors, $\mathbf{e}_r$ and $\mathbf{e}_\theta$. (A unit vector is simply a vector whose length is unity.) As shown in Fig. 105, the *radial* unit vector $\mathbf{e}_r$ always points from the Sun towards the instantaneous position of the planet. Moreover, the *tangential* unit vector $\mathbf{e}_\theta$ is always normal to $\mathbf{e}_r$, in the direction of increasing θ. In Sect. 7.5, we demonstrated that when acceleration is written in terms of polar coordinates, it takes the form

$$\mathbf{a} = a_r\,\mathbf{e}_r + a_\theta\,\mathbf{e}_\theta, \tag{562}$$

where

$$a_r = \ddot{r} - r\,\dot{\theta}^2, \tag{563}$$

$$a_\theta = r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}. \tag{564}$$

These expressions are more complicated that the corresponding cartesian expressions because the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ *change direction* as the planet changes position.

Now, the planet is subject to a single force: *i.e.*, the force of gravitational attraction exerted by the Sun. In polar coordinates, this force takes a particularly simple form (which is why we are using polar coordinates):

$$\mathbf{f} = -\frac{G\,M_\odot\,m}{r^2}\,\mathbf{e}_r. \tag{565}$$

The minus sign indicates that the force is directed towards, rather than away from, the Sun.

According to Newton's second law, the planet's equation of motion is written

$$m\,\mathbf{a} = \mathbf{f}. \tag{566}$$

The above four equations yield

$$\ddot{r} - r\,\dot{\theta}^2 = -\frac{G\,M_\odot}{r^2}, \tag{567}$$

$$r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} = 0. \tag{568}$$

Equation (568) reduces to

$$\frac{d}{dt}\,(r^2\,\dot{\theta}) = 0, \tag{569}$$

or

$$r^2\,\dot{\theta} = h, \tag{570}$$

where h is a *constant of the motion*. What is the physical interpretation of h? Recall, from Sect. 9.2, that the angular momentum vector of a point particle can be written

$$\mathbf{l} = m\,\mathbf{r}\times\mathbf{v}. \tag{571}$$

For the case in hand, $\mathbf{r} = r\,\mathbf{e}_r$ and $\mathbf{v} = \dot{r}\,\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta$ [see Sect. 7.5]. Hence,

$$l = m\,r\,v_\theta = m\,r^2\,\dot{\theta}, \tag{572}$$

yielding

$$h = \frac{l}{m}. \tag{573}$$

Clearly, h represents the *angular momentum* (per unit mass) of our planet around the Sun. Angular momentum is conserved (*i.e.*, h is constant) because the force of gravitational attraction between the planet and the Sun exerts *zero torque* on the planet. (Recall, from Sect. 9, that torque is the rate of change of angular momentum.) The torque is zero because the gravitational force is *radial* in nature: *i.e.*, its line of action passes through the Sun, and so its associated lever arm is of length zero.

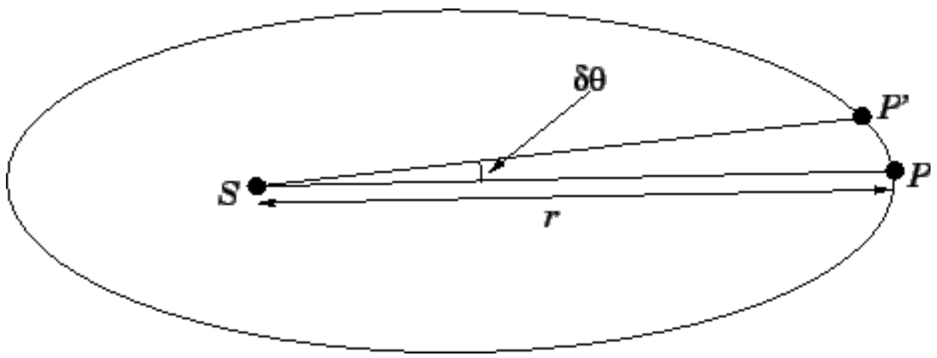

Figure 106: *The origin of Kepler's second law.*

The quantity h has another physical interpretation. Consider Fig. 106. Suppose that our planet moves from P to P' in the short time interval δt. Here, S represents the position of the Sun. The lines SP and SP' are both approximately of length r. Moreover, using simple trigonometry, the line PP' is of length $r\,\delta\theta$, where $\delta\theta$ is the small angle through which the line joining the Sun and the planet rotates in the time interval δt. The area of the triangle PSP' is approximately

$$\delta A = \frac{1}{2}\times r\,\delta\theta\times r\,: \tag{574}$$

i.e., half its base times its height. Of course, this area represents the area swept out by the line joining the Sun and the planet in the time interval δt. Hence, the rate at which this area is swept is given by

$$\lim_{\delta t\rightarrow 0}\frac{\delta A}{\delta t} = \frac{1}{2}\,r^2\,\lim_{\delta t\rightarrow 0}\frac{\delta\theta}{\delta t} = \frac{r^2\,\dot{\theta}}{2} = \frac{h}{2}. \tag{575}$$

Clearly, the fact that h is a constant of the motion implies that the line joining the planet and the Sun sweeps out area at a *constant rate*: *i.e.*, the line sweeps equal areas in equal time intervals. But, this is just Kepler's second law. We conclude that Kepler's second law of planetary motion is a direct manifestation of *angular momentum conservation*.

Let

$$r = \frac{1}{u}, \tag{576}$$

where $u(t) \equiv u(\theta)$ is a new radial variable. Differentiating with respect to t, we obtain

$$\dot{r} = -\frac{\dot{u}}{u^2} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta} = -h\,\frac{du}{d\theta}. \tag{577}$$

The last step follows from the fact that $\dot{\theta} = h\,u^2$. Differentiating a second time with respect to t, we obtain

$$\ddot{r} = -h\,\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\,\dot{\theta}\,\frac{d^2u}{d\theta^2} = -h^2\,u^2\,\frac{d^2u}{d\theta^2}. \tag{578}$$

Equations (567) and (578) can be combined to give

$$\frac{d^2u}{d\theta^2} + u = \frac{G\,M_\odot}{h^2}. \tag{579}$$

This equation possesses the fairly obvious general solution

$$u = A\,\cos(\theta-\theta_0) + \frac{G\,M_\odot}{h^2}, \tag{580}$$

where A and θ_0 are arbitrary constants.

The above formula can be inverted to give the following simple orbit equation for our planet:

$$r = \frac{1}{A\cos(\theta-\theta_0) + G\,M_\odot/h^2}. \tag{581}$$

The constant θ_0 merely determines the orientation of the orbit. Since we are only interested in the orbit's *shape*, we can set this quantity to zero without loss of generality. Hence, our orbit equation reduces to

$$r = r_0\,\frac{1+e}{1+e\,\cos\theta}, \tag{582}$$

where

$$e = \frac{A\,h^2}{G\,M_\odot}, \tag{583}$$

and

$$r_0 = \frac{h^2}{G\,M_\odot\,(1+e)}. \tag{584}$$

Formula (582) is the standard equation of an *ellipse* (assuming $e<1$), with the origin at a focus. Hence, we have now proved Kepler's first law of planetary motion. It is clear that r_0 is the radial distance at $\theta=0$. The radial distance at $\theta=\pi$ is written

$$r_1 = r_0\,\frac{1+e}{1-e}. \tag{585}$$

Here, r_0 is termed the *perihelion* distance (*i.e.*, the closest distance to the Sun) and r_1 is termed the *aphelion* distance (*i.e.*, the furthest distance from the Sun). The quantity

$$e = \frac{r_1-r_0}{r_1+r_0} \tag{586}$$

is termed the *eccentricity* of the orbit, and is a measure of its departure from circularity. Thus, $e=0$ corresponds to a purely circular orbit, whereas $e\rightarrow 1$ corresponds to a highly elongated orbit. As specified in Tab. 7, the orbital eccentricities of all of the planets (except Mercury) are fairly small.

Table 7: *The orbital eccentricities of various planets in the Solar System.*

Planet	e
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.056

According to Eq. (575), a line joining the Sun and an orbiting planet sweeps area at the constant rate $h/2$. Let T be the planet's orbital period. We expect the line to sweep out the *whole area* of the ellipse enclosed by the planet's orbit in the time interval T. Since the area of an ellipse is $\pi\,a\,b$, where a and b are the *semi-major* and *semi-minor* axes, we can write

$$T=\frac{\pi\,a\,b}{h/2}. \tag{587}$$

Incidentally, Fig. 107 illustrates the relationship between the aphelion distance, the perihelion distance, and the semi-major and semi-minor axes of a planetary orbit. It is clear, from the figure, that the semi-major axis is just the mean of the aphelion and perihelion distances: *i.e.*,

$$a=\frac{r_0+r_1}{2}. \tag{588}$$

Thus, a is essentially the planet's mean distance from the Sun. Finally, the relationship between a, b, and the eccentricity, e, is given by the well-known formula

$$\frac{b}{a} = \sqrt{1-e^2}. \tag{589}$$

This formula can easily be obtained from Eq. (582).

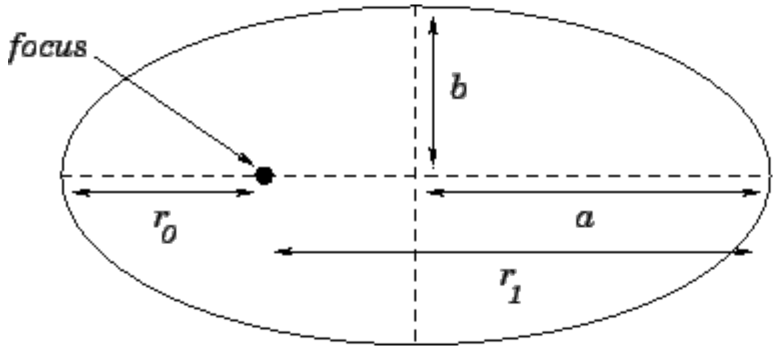

Figure 107: *Anatomy of a planetary orbit.*

Equations (584), (585), and (588) can be combined to give

$$a = \frac{h^2}{2\,G\,M_\odot}\left(\frac{1}{1+e}+\frac{1}{1-e}\right) = \frac{h^2}{G\,M_\odot\,(1-e^2)}. \tag{590}$$

It follows, from Eqs. (587), (589), and (590), that the orbital period can be written

$$T = \frac{2\pi}{\sqrt{G\,M_\odot}}\,a^{3/2}. \tag{591}$$

Thus, the orbital period of a planet is proportional to its mean distance from the Sun to the power $3/2$--the constant of proportionality being the *same* for all planets. Of course, this is just Kepler's third law of planetary motion.

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