Entanglement transitions in random pure states

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In this paper, we use large-N perturbation theory to compute the entanglement negativity of random induced mixed states. Our result reproduces the two well-known limits: volume law states and separable states. We also find that the volume law states can be further divided into two categories in terms of subsystem-size scaling of the entanglement negativity: a linear scaling phase, where the state is maximally entangled and the logarithmic negativity is bound by the size of the smaller subsystem $\mathcal{E} \sim V_{A_1}$, and a saturated phase, where $\mathcal{E} \sim (V_A - V_B)/2$, which is independent of individual subsystem sizes V_{A_1} and V_{A_2}. In the latter case, the spectral density can be well-approximated by a semi-circle law. We show that the large-N perturbation theory results match with those of the random matrix simulations. Our finding indicates that the average logarithmic negativity behaves very similar to the 1/2-Rényi mutual information.

I. INTRODUCTION

Random pure states represent typical volume law entangled (thermal) pure states. Their virtue is that they are described by Wishart random matrix theory and hence, various well-established random matrix theory tools are available to carry out calculations on them.

In this paper, we use the large-N perturbation theory which was recently developed by one of us to compute the spectral density of partial transpose and characterize the reduced density matrix in several limits. The result is summarized in Fig. 1.

We briefly go over our setup. We would like to study the entanglement in random mixed states obtained by partial tracing random pure states. An ensemble of random induced mixed states $\{\rho\}$ corresponding to the Hilbert space $\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ is generated by reduced density matrices which are obtained by partial tracing random pure states (or Page states [1]) in a composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. It is more convenient to represent such a random pure state in a tensor product basis

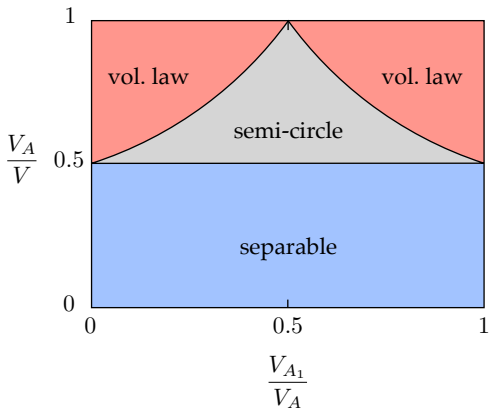


FIG. 1. Phase diagram of reduced density matrix obtained from random pure states (or Page states).

as in

$$|\Psi\rangle = \sum_{i=1}^{L_A} \sum_{\alpha=1}^{L_B} X_{i\alpha} \, |\Psi_A^{(i)}\rangle \otimes |\Psi_B^{(\alpha)}\rangle, \tag{1}$$

in terms of a $L_A \times L_B$ rectangular random matrix X whose elements $(X_{i\alpha})$ are independent random complex variables. Here, $L_A = L_{A_1} \times L_{A_2}$ and L_B denote the size of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Throughout this paper, we consider A and B systems to be comprised of qubits. In other words, $L_s = 2^{N_s}$ where $s = A_1, A_2, B$ and L_s is the number of qubits. This choice is not a necessary ingredient for our calculations and is mainly meant as a physical description of the system.

By definition, the joint probability density is given by

$$P\left(\{X_{i\alpha}\}\right) = \mathcal{Z}^{-1} \exp\left\{-L_A L_B \mathrm{Tr}(XX^\dagger)\right\}. \tag{2}$$

The random reduced density matrix of system A is then given by

$$\rho_A = \frac{XX^\dagger}{\mathrm{Tr}(XX^\dagger)}. \tag{3}$$

We note that ρ_A is a $L_A \times L_A$ square matrix, and the denominator (which is also a random variable) is there to enforce the normalization condition $\mathrm{Tr}\rho_A = 1$.

The eigenvalues of ρ_A (aka the entanglement spectrum) contains information about the entanglement between A and B. In the limit $L_A, L_B \to \infty$ while the ratio L_A/L_B is finite (which we will refer to as the large L limit from now on), the joint probability density function of these eigenvalues can be derived [2, 3]. In doing so, the crucial approximation is that the normalization factor in Eq. (3) is a random variable $\mathrm{Tr}(XX^\dagger) = 1+\delta$ whose fluctuations about its mean 1 is negligible to the leading order in $L_A L_B$. Hence, to the leading order, the denominator can be replaced by its mean value, and we may write

$$\rho_A \approx XX^\dagger. \tag{4}$$

The density matrix can be recast in terms of the matrix $W/L_A = XX^\dagger$ which is the celebrated Wishart-Laguerre ensemble [4] and is extensively studied in the random matrix theory literature. From this observation, one can infer several properties of ρ_A in the large L limit. Among all, the spectral density of the eigenvalues is given by an appropriately scaled Marcenko-Pastur (MP) function [4],

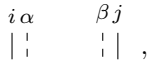
II. LARGE-N PERTURBATION THEORY

In this section, we use graphical representation of a partially transposed random mixed state to compute its momets and eventually derive the corresponding resolvent function and the spectral density.

A. Moments of partial transpose

We begin by reviewing the diagrammatic approach to random pure states [5, 6]. A matrix element of the density matrix associated with the random pure state (1) is denoted as

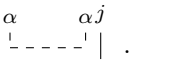
$$[|\Psi\rangle\langle\Psi|]_{i\alpha,j\beta} = X^*_{i\alpha}X_{j\beta} = \quad , \tag{5}$$



where $\mathcal{N}$ is the normalization factor, the left (right) pair of lines represents a bra (ket) state, and solid (dashed) lines correspond to subsystem A (B). Note that each line carries an index. The lower end of the diagrams are reserved for matrix manipulations such as tracing and multiplication, while the upper ends of the lines are used for ensemble averaging.

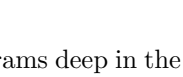
Therefore, a matrix element of the reduced density matrix is represented by

$$[\rho]_{i,j} = \sum_{\alpha=1}^{L_B} X^*_{i\alpha}X_{j\alpha} = \quad . \tag{6}$$



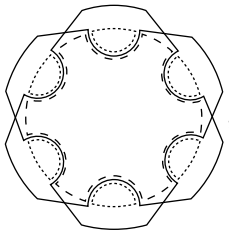
For brevity, from now on we drop the subscript A in ρ_A unless stated otherwise. Similarly, tracing over the subsystem A degrees of freedom leads to the following diagram,

$$\mathrm{Tr}\rho = \sum_{i=1}^{L_A}\sum_{\alpha=1}^{L_B} |X_{i\alpha}|^2 = \quad . \tag{7}$$

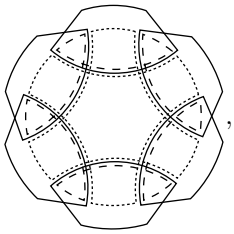


Let us look at the dominant diagrams deep in the NPT limit, $L_A \gg L_B$, when one subsystem (A_1 or A_2) is much larger than the other.

$$\langle \mathrm{Tr}\left(\rho^{T_2}\right)^{n_e}\rangle \approx \begin{cases} L_B^{1-n_e}L_{A_2}^{2-n_e} & L_{A_1} \gg L_{A_2} \\ L_B^{1-n_e}L_{A_1}^{2-n_e} & L_{A_1} \ll L_{A_2} \end{cases} \tag{8}$$



(9)



(10)

To sum up, This in turn implies that

$$\langle \mathcal{E}\rangle \approx \begin{cases} L_B^{1-n_e}L_{A_2}^{2-n_e} & L_{A_1} \gg L_{A_2} \\ L_B^{1-n_e}L_{A_1}^{2-n_e} & L_{A_1} \ll L_{A_2} \end{cases} \tag{11}$$

B. Resolvent function

Our goal in this section is to derive the spectral density of the partially transposed density matrix. To this end, we define a one-point Green function (or a resolvent function) as

$$G(z) = \frac{1}{L_A}\left\langle \mathrm{Tr}\left(\frac{1}{z-H}\right)\right\rangle, \tag{12}$$

where we use a particular normalization and define $H = L_B L_{A_2}(XX^\dagger)^{T_2}$ to carry out calculations systematically such that $1/N$ perturbative expansion makes sense. The actual normalization will be included via rescaling after the spectral density is evaluated. We can then compute the spectral density using the identity

$$P_\Gamma(\xi) = -\frac{L_A}{\pi}\mathrm{Im}\lim_{\epsilon\to 0} G(z)\big|_{z=\xi+i\epsilon}, \tag{13}$$

because

$$\lim_{\epsilon\to 0}\frac{1}{\lambda+i\epsilon} = \mathrm{PV}\,\frac{1}{\lambda} - i\pi\delta(\lambda). \tag{14}$$

The Feynman diagram approach follows by expanding $G(z)$ in inverse powers of z. In our graphical representation of a given term, we insert the diagram (**??**) for every power of ρ^{T_2}. The ensemble average is represented by triple lines with amplitude $1/L_{A_1}$ (given the above

normalization),

$$\langle H_{i_1 j_1 \alpha} H_{i_2 j_2 \beta} \rangle \equiv \quad = \frac{1}{L_{A_1}} \delta_{i_1 j_1} \delta_{i_2 j_2} \delta_{\alpha\beta}, \quad (15)$$

and as usual close loop of subsystem $s = A_1, A_2$, or B gives a factor of L_s.

We note that since there is an even/odd effect for the Rényi negativity, we need to consider two self-energy functions in the expansion of the resolvent function

$$= + \Sigma_o + \Sigma_e + \Sigma_o \Sigma_e + \Sigma_e \Sigma_o + \cdots = \frac{1}{z - \Sigma_o(z) - \Sigma_e(z)}, \quad (16)$$

where

$$\Sigma_o = + \cdots, \quad (17)$$

$$\Sigma_e = + \cdots, \quad (18)$$

corresponding to effectively crossing and non-crossing diagrams of order L_B/L_{A_1} and $L_B L_{A_2}/L_{A_1}$, respectively. To derive a Schwinger-Dyson equation we first define

$$\Sigma_o = , \quad (19)$$

and

$$\Sigma_e = , \quad (20)$$

which lead to the following algebraic relations,

$$\Sigma_o(z) = \alpha F_o(z), \quad (21)$$
$$\Sigma_e(z) = \beta F_e(z), \quad (22)$$

Here, the Hilbert space dimension ratios are given by

$$\alpha = \frac{L_B}{L_{A_1}}, \qquad \beta = \frac{L_B L_{A_2}}{L_{A_1}}. \quad (23)$$

Next, we write self-consistent conditions for F-functions as in

$$= + , \quad (24)$$

$$= , \quad (25)$$

which lead to the following algebraic relations

$$F_o(z) = 1 + F_e(z) G(z), \quad (26)$$
$$F_e(z) = F_o(z) G(z). \quad (27)$$

They can be solved in terms of $G(z)$ as in

$$F_2(z) = G(z) \cdot F_1(z) = \frac{G(z)}{1 - G^2(z)}. \quad (28)$$

Solving the self-energy equation for $G(z)$, we obtain the following cubic equation

$$z G^3(z) + (\beta - 1) G^2(z) + (\alpha - z) G(z) + 1 = 0. \quad (29)$$

The proper solution to the above equation can be written as

$$G(z) = \frac{2 e^{-i\theta} Q(z)}{(R(z) + \sqrt{D(z)})^{1/3}} - e^{i\theta} (R(z) + \sqrt{D(z)})^{1/3} + \frac{1 - \beta}{3} \quad (30)$$

where $\theta = \pi/3$.

$$Q(z) = \frac{3z(\alpha - z) + (\beta - 1)^2}{9z^2}, \quad (31)$$
$$R(z) = \frac{9z(\beta - 1)(\alpha - z) - 27z^2 - 2(\beta - 1)^3}{54 z^3}, \quad (32)$$
$$D(z) = Q^3(z) + R^2(z). \quad (33)$$

In the limit, $L_{A_1} \ll L_B L_{A_2}$, we have $\beta \gg 1$. Upon appropriate rescaling of variable $z \to y L_{A_2}$ which also implies $G(z) \to L_{A_2}^{-1} \tilde{G}(y)$ where $\tilde{G}(y) := G(y L_{A_2})$, we obtain

$$\frac{y}{L_{A_2}^2} \tilde{G}^3(y) + (r - \frac{1}{L_{A_2}^2}) \tilde{G}^2(y) + (r - y) \tilde{G}(y) + 1 = 0, \quad (34)$$

in which $r = L_B/L_A$. The $1/L_{A_2}$ terms are negligible and we arrive at

$$r \tilde{G}^2(y) + (r - y) \tilde{G}(y) + 1 = 0. \quad (35)$$

that is the semi-circle law.

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