Problem 1: Bias-Variance Decomposition

Part 1:

Let $y(x) = f(x) + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2)$ is the noise. The model's prediction is $\hat{y}(x) = g(x)$. The mean squared error (MSE) over test instances x_i is:

$$MSE = \frac{1}{t} \sum_{i=1}^{t} (f(x_i) + \epsilon_i - g(x_i))^2$$

Expanding:

$$MSE = \mathbb{E}\left[(f(x) + \epsilon - g(x))^2\right]$$

This decomposes as:

$$\mathbb{E}[\text{MSE}] = \underbrace{(\mathbb{E}[g(x)] - f(x))^2}_{\text{Bias}^2} + \underbrace{\mathbb{E}[(g(x) - \mathbb{E}[g(x)])^2]}_{\text{Variance}} + \underbrace{\mathbb{E}[\epsilon^2]}_{\text{Noise}}$$

Thus:

$$\mathbb{E}[\text{MSE}] = \text{Bias}^2 + \text{Variance} + \sigma^2$$

Part 2:

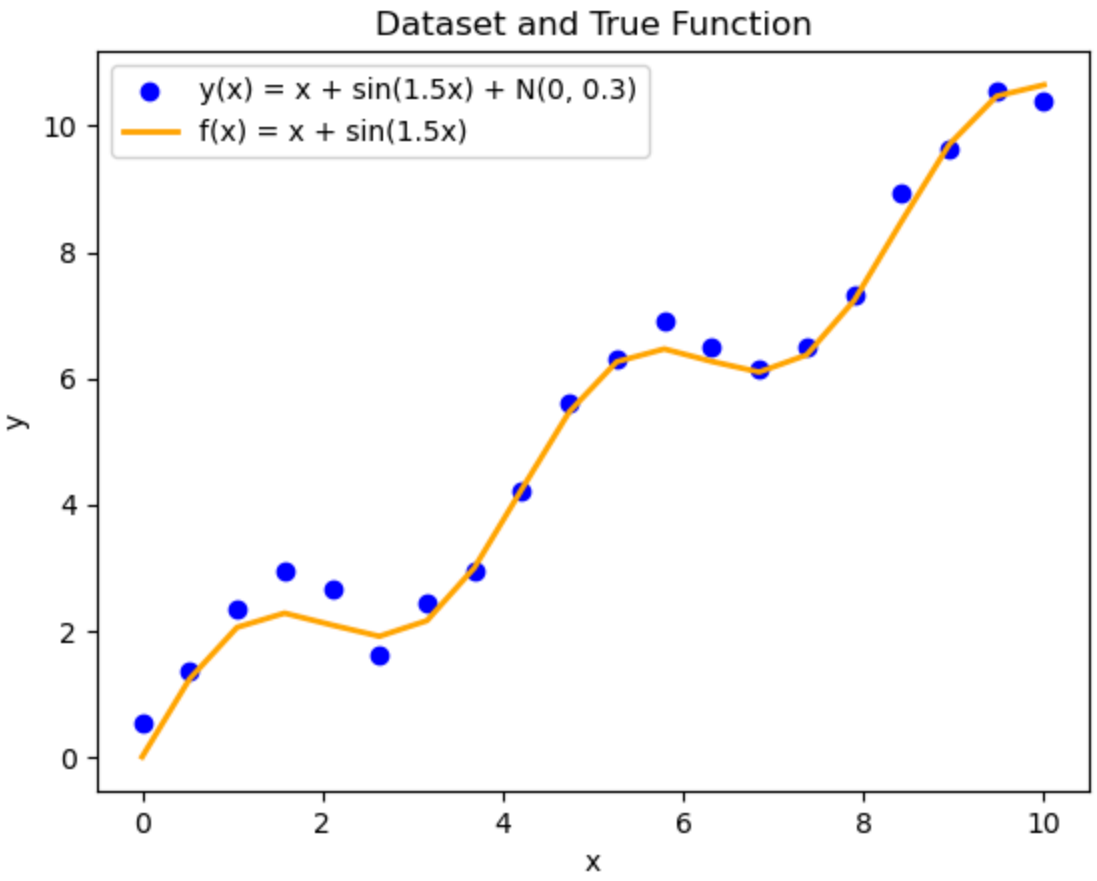
Given

$$y(x) = x + \sin(1.5x) + \mathcal{N}(0, 0.3)$$

Generate 20 points from y and display the dataset and f(x)

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt

# Set random seed for reproducibility
np.random.seed(0)

# Generate x values in the range [0, 10]
x = np.linspace(0, 10, 20)

# Generate y values with noise
y = x + np.sin(1.5 * x) + np.random.normal(0, 0.3, size=x.shape)

# Define the true function f(x)
f_x = x + np.sin(1.5 * x)
```

```
# Plot y(x) as a scatter plot and f(x) as a smooth line
plt.scatter(x, y, label="y(x) = x + sin(1.5x) + N(0, 0.3)", color='blue')
plt.plot(x, f_x, label="f(x) = x + sin(1.5x)", color='orange', linewidth=2)
plt.xlabel("x")
plt.ylabel("y")
plt.legend()
plt.title("Dataset and True Function")
plt.show()
```



Part 3:

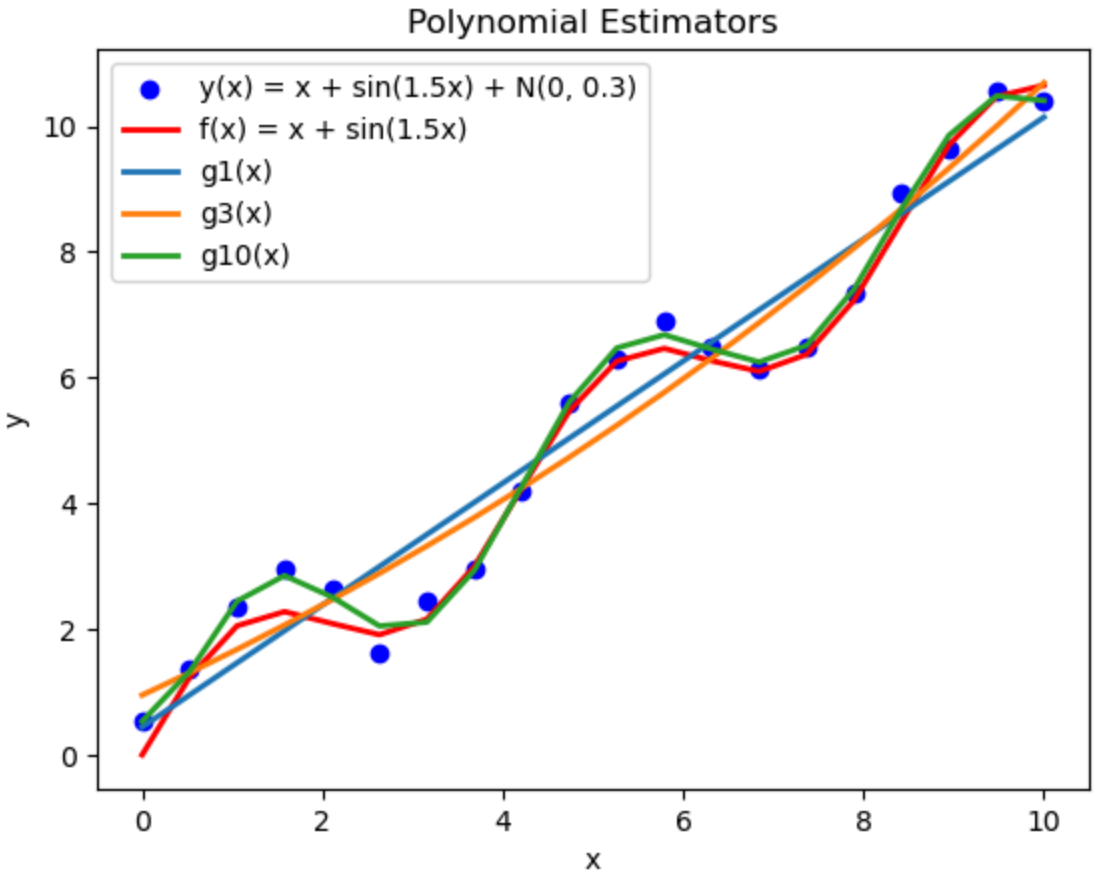
Use a weighted sum of polynomials as an estimator function for (f(x)). In particular, let the form of the estimator function be:

$$g_n(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \ldots + \beta_n x^n$$

Consider three candidate estimators, (g_1), (g_3), and (g_{10}). Estimate the coefficients of each of the three estimators using the sampled dataset and plot (y(x)), (f(x)), (g_1(x)), (g_3(x)), and (g_{10}(x)). Which estimator is underfitting? Which one is overfitting?

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt

from sklearn.preprocessing import PolynomialFeatures
```

```
from sklearn.linear_model import LinearRegression
from sklearn.pipeline import make_pipeline

degrees = [1, 3, 10]
np.random.seed(0)
x = np.linspace(0, 10, 20)
x = x[:, np.newaxis]
y = x + np.sin(1.5 * x) + np.random.normal(0, 0.3, size=x.shape)
f_x = x + np.sin(1.5 * x)

plt.scatter(x, y, label="y(x) = x + sin(1.5x) + N(0, 0.3)", color='blue')
plt.plot(x, f_x, label="f(x) = x + sin(1.5x)", color='red', linewidth=2)


for degree in degrees:
    model = make_pipeline(PolynomialFeatures(degree), LinearRegression())
    model.fit(x, y)
    y_pred = model.predict(x[:])
    plt.plot(x, y_pred, label=f'g{degree}(x)', linewidth=2)

plt.xlabel("x")
plt.ylabel("y")
plt.legend()
plt.title("Polynomial Estimators")
plt.show()
```

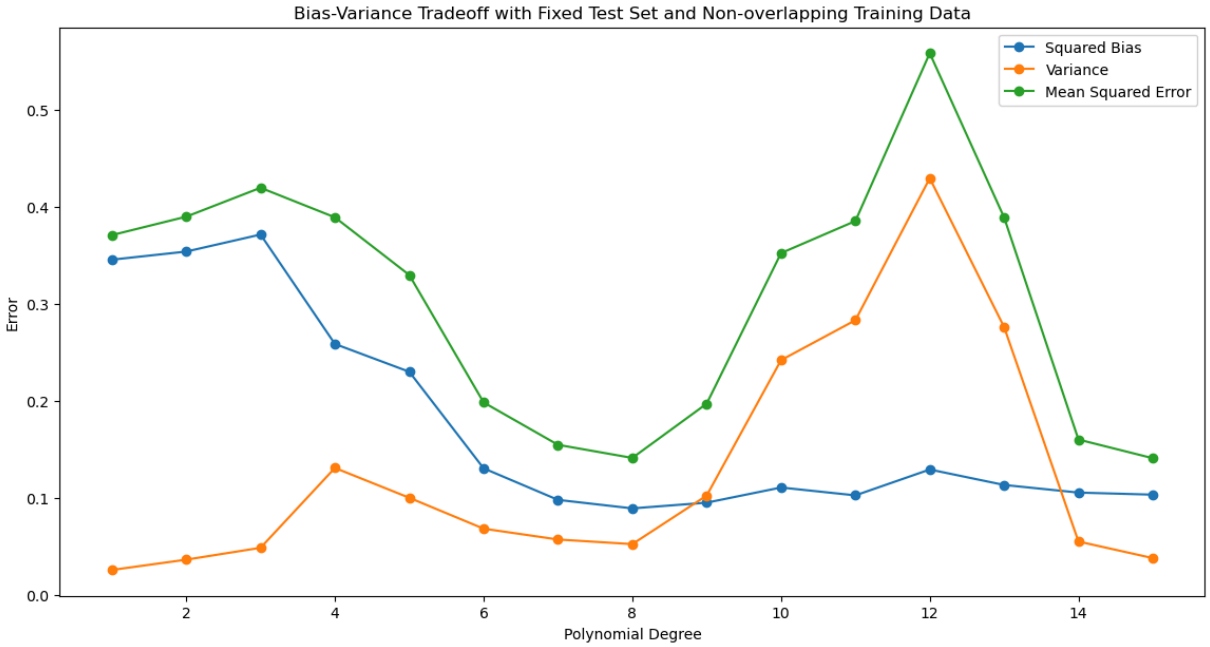


The estimators g_1(x) and g_3(x) are severely underfitting the data, while g_10(x) is overfitting the data.

Part 4: Bias-Variance Tradeoff

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from sklearn.pipeline import make_pipeline
from sklearn.preprocessing import PolynomialFeatures
from sklearn.linear_model import LinearRegression
from sklearn.metrics import mean_squared_error

# Function to generate dataset excluding test samples
def generate_training_dataset(test_set=None, n_train_samples=40):
    x_all = np.linspace(0, 10, 1000)
    y_all = x_all + np.sin(1.5 * x_all) + np.random.normal(0, 0.3, size=x_al

    if test_set is not None:
        mask = np.isin(x_all, test_set.squeeze(), invert=True)
        x_train_pool = x_all[mask]
        y_train_pool = y_all[mask]
    else:
        x_train_pool = x_all
        y_train_pool = y_all

    # Randomly sample for training
    idx = np.random.choice(len(x_train_pool), n_train_samples, replace=False
    return x_train_pool[idx][:, np.newaxis], y_train_pool[idx]  # Return as

x_test, y_test = generate_training_dataset(None, n_train_samples=10)
np.random.seed(0)
n_datasets = 100
n_samples = 40
degrees = range(1, 16)

# Arrays to store results
mse_test = np.zeros((n_datasets, len(degrees)))
predictions = np.zeros((n_datasets, len(degrees), len(y_test)))


for i in range(n_datasets):
    x_train, y_train = generate_training_dataset(x_test, n_train_samples=n_s
    for j, degree in enumerate(degrees):
        model = make_pipeline(PolynomialFeatures(degree), LinearRegression()
        model.fit(x_train, y_train)
        y_test_pred = model.predict(x_test)

        predictions[i, j, :] = y_test_pred
        mse_test[i, j] = mean_squared_error(y_test, y_test_pred)

# Calculate mean and variance of predictions across datasets
mean_predictions = np.mean(predictions, axis=0)
variance_predictions = np.var(predictions, axis=0)
bias_squared = np.mean((mean_predictions - y_test) ** 2, axis=1)
mse_test_mean = np.mean(mse_test, axis=0)

plt.figure(figsize=(14, 7))
```

```
plt.plot(degrees, bias_squared, label='Squared Bias', marker='o')
plt.plot(degrees, variance_predictions.mean(axis=1), label='Variance', marke
plt.plot(degrees, mse_test_mean, label='Mean Squared Error', marker='o')
plt.xlabel('Polynomial Degree')
plt.ylabel('Error')
plt.title('Bias-Variance Tradeoff with Fixed Test Set and Non-overlapping Tr
plt.legend()
plt.show()
```



The best model is at degree 8.

Part 5: L2 Regularization

In [ ]:

```
import numpy as np
from sklearn.pipeline import make_pipeline
from sklearn.preprocessing import PolynomialFeatures
from sklearn.linear_model import Ridge
from sklearn.metrics import mean_squared_error


def generate_training_dataset(test_set=None, n_train_samples=40):
    x_all = np.linspace(0, 10, 1000)  # Generate a large pool of data points
    y_all = x_all + np.sin(1.5 * x_all) + np.random.normal(0, 0.3, size=x_al


    if test_set is not None:
        mask = np.isin(x_all, test_set.squeeze(), invert=True)
        x_train_pool = x_all[mask]  # Exclude test samples from the x pool
        y_train_pool = y_all[mask]
    else:
        x_train_pool = x_all
        y_train_pool = y_all

    # Randomly sample from the remaining points for train
```

```
    idx = np.random.choice(len(x_train_pool), n_train_samples, replace=False
    return x_train_pool[idx][:, np.newaxis], y_train_pool[idx]

#fixed test set (size 10)
x_test, y_test = generate_training_dataset(None, n_train_samples=10)
degree = 10
alpha = 1.0

n_datasets = 100
n_samples = 40

# Arrays to store results
mse_test_ridge = np.zeros(n_datasets)
predictions_ridge = np.zeros((n_datasets, len(y_test)))


for i in range(n_datasets):

    x_train, y_train = generate_training_dataset(x_test, n_train_samples=n_s

    #Ridge regression
    model_ridge = make_pipeline(PolynomialFeatures(degree), Ridge(alpha=alph
    model_ridge.fit(x_train, y_train)
    y_test_pred_ridge = model_ridge.predict(x_test)

    mse_test_ridge[i] = mean_squared_error(y_test, y_test_pred_ridge)
    predictions_ridge[i, :] = y_test_pred_ridge


mean_predictions_ridge = np.mean(predictions_ridge, axis=0)
variance_predictions_ridge = np.var(predictions_ridge, axis=0)
bias_squared_ridge = np.mean((mean_predictions_ridge - y_test) ** 2)
mse_test_mean_ridge = np.mean(mse_test_ridge)


print(f"Unregularized Model (Degree {degree}):")
print(f"  Squared Bias: {bias_squared[degree-1]:.4f}")
print(f"  Variance: {variance_predictions[degree-1].mean():.4f}")
print(f"  Mean Squared Error: {mse_test_mean[degree-1]:.4f}")

print(f"Regularized Model (Degree {degree}, Alpha {alpha}):")
print(f"  Squared Bias: {bias_squared_ridge:.4f}")
print(f"  Variance: {variance_predictions_ridge.mean():.4f}")
print(f"  Mean Squared Error: {mse_test_mean_ridge:.4f}")
```

```
Unregularized Model (Degree 10):
  Squared Bias: 0.1104
  Variance: 0.2418
  Mean Squared Error: 0.3522
Regularized Model (Degree 10, Alpha 1.0):
  Squared Bias: 0.2116
  Variance: 0.0381
  Mean Squared Error: 0.2497
```

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/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.74457e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.11292e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=1.37722e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.65029e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.14654e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.35538e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.16829e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.84292e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=1.08856e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.5261e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.91355e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.8503e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=7.10461e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=7.96422e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
```

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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.8228e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=1.21069e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.72526e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.89184e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.58541e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=1.98561e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.2683e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.5423e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.67567e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.27761e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.51958e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=6.14133e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.02724e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=8.08975e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
```

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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=1.5753e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.54163e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.01958e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=8.40025e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=9.41755e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.1765e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.56372e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=6.19589e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.8962e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.8271e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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(rcond=1.64109e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.20173e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.44277e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.80385e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
```

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/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=6.88431e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=8.1635e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.2025e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.66808e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=1.01068e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=6.44486e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.52326e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=1.71974e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.15387e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=8.87974e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.4305e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=7.14242e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.72211e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=1.09692e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
```

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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.90181e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.48205e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.15675e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.43158e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.67297e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.20322e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=1.24456e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=8.96587e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.0855e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.15014e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=8.81863e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=6.32581e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.42182e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=8.73332e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
```

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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.03142e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=8.25931e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=6.70856e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.62506e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.19823e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.58836e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=1.38099e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.07996e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.6614e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.10873e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.22142e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.09759e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.32638e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.89028e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
```

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es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.11701e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.47675e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.52452e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=6.23332e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.07605e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=7.6816e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.58546e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=5.24255e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=1.07676e-20): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.47348e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=6.42758e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.0625e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.54347e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=3.52508e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
```

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/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=4.74343e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
/Users/bytedance/Documents/anaconda3/envs/COMS6998/lib/python3.9/site-packag
es/sklearn/linear_model/_ridge.py:216: LinAlgWarning: Ill-conditioned matrix
(rcond=2.88995e-21): result may not be accurate.
  return linalg.solve(A, Xy, assume_a="pos", overwrite_a=True).T
```

Conclusion

Bias:

- **Unregularized Model:** Squared bias is 0.1104.
- **Regularized Model:** Squared bias is 0.2116.
- **Ans:** The regularized model has a higher bias, as expected, due to reduced model flexibility.

Variance:

- **Unregularized Model:** Variance is 0.2418.
- **Regularized Model:** Variance is 0.0381.
- **Ans:** The regularized model has much lower variance, which is expected as regularization reduces sensitivity to training data.

Mean Squared Error (MSE):

- **Unregularized Model:** MSE is 0.3522.
- **Regularized Model:** MSE is 0.2497.
- **Ans:** The regularized model has a lower MSE, indicating better overall performance.

Answer: The regularized model has higher bias, but achieves much lower variance and MSE. This shows better generalization and makes it the better model.