# DIFFUSION OF INNOVATIONS ON STRONGLY REGULAR GRAPHS

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ABSTRACT. When a new product is introduced to a market and depends on word-of-mouth advertising, the structure of the underlying social network is important in determining the market share. Modeling network interactions as a coordination game and assuming the structure of a rook's graph, we determine the minimum number of initial adopters sufficient to achieve a monopoly and illustrate patterns of initial adopters that will reach this goal. The result for rook's graphs is extended to a theorem establishing a lower bound on the size of the initiating set for a monopoly in any strongly regular graph. In case the initial adopters do not achieve a monopoly, we show that treating diffusion of innovation as a repeated game can increase market share.

## 1. INTRODUCTION

Suppose Superior Company A is a producer of cell phones and develops a superior camera for a phone it wants to market in a community of users currently monopolized by phones with inferior cameras that are produced by Inferior Company B. Suppose further that a design incompatibility prevents users of different phones from sharing photos. In this scenario, an individual will weigh the benefit of enhanced picture quality against the benefit of being able to share photos with their friends. Clearly Superior Company A has an interest in understanding the decisions made by individuals as it assesses the extent to which the community can be a profitable market for its product. Prospects for Superior Company A range from attracting a small set of devoted but isolated users to supplanting Inferior Company B and achieving its own monopoly.

The scenario above is an example of a diffusion of innovations problem. As described in [1, 2], diffusion of innovations is the theory of the spread of new ideas, opinions, or products throughout a society. The rate and extent to which an innovation spreads depends on characteristics of the idea or product, characteristics of the individuals involved, and characteristics of the network of social connections among those individuals. Work by sociologists focused on individuals, each of whom fell into one of five categories ranging from innovators and early adopters through to laggards. An early mathematical model was provided by Bass [3] under the assumption that the probability of adopting by those who had not yet adopted is a linear function of the number that had previously adopted. Granovetter [4] developed models for situations involving a choice between two alternatives whose

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costs and/or benefits each depend on how many other members of the network choose which alternative. In [5], Morris localized network interactions so that costs and/or benefits depend only on the choices made by one's neighbors. An application to random graphs is provided by Watts [6]. More recently, Kleinberg [7] has discussed implications that these models have for so-called "viral marketing" in online communities.

The doctoral thesis of Wodlinger [8] gives an up-to-date account of progress on our problem and its variations. Chang and Lyuu [9] document known families of graphs for which the problem has been solved. Many of the results are established in contexts other than the diffusion of innovations, ranging from graph-coloring problems to computer science applications.

In this paper we employ game theory to understand interactions among decision processes of individuals and graph theory to model networks of friends within the community. We are concerned with how, given an initial set of users of A, the structure of the graph determines the extent to which the camera-phone A will saturate the market. Our focus is on a family of strongly regular graphs, the rook's graphs, for which we describe the minimal number and patterns of initial users of A that result eventually in use of A by every member of the community. Strongly regular graphs are amenable to this work because neighborhoods of distinct vertices have the same structure throughout the graph.

The plan of the paper is as follows. In section 2 we describe a coordination-game model for diffusion of innovations. The model posits a set of adopters that initiates a cascade as their neighbors begin to adopt A. The cascade may or may not result in a monopoly for A and in section 3 we characterize the obstruction to a monopoly. Section 4 defines what it means for a graph to be strongly regular and introduces the rook's graphs that are the focus of section 5. Our main theorem extends the work on rook's graphs to all strongly regular graphs and appears in section 6. In section 7 we discuss the how extended time horizons, using discounted future values in making decisions, can facilitate cascades by lowering thhe adoption threshold for players. Avenues for future work are described in section 8.

## 2. Description of the Model

We employ a model originally proposed by Morris [5], as described by Easley and Kleinberg [10] in their book *Networks, Crowds and Markets.* To begin, we imagine a population consisting of technology users. The incumbent technology, used by everyone at time $t = -1$, is B and an innovative technology A is to be introduced to a subset $S$ of the user population at time $t = 0$.

The marketplace for the innovation is a graph $G = G(V, E)$ with vertex set $V$ and edge set $E$. We will refer to the vertices of the graph as nodes. All and only those nodes (players) that share an edge (are neighbors) interact with each other using the technology. We use the notation $N(u)$ to indicate the neighborhood of the node $u$, the set consisting of all of its neighbors. The size of the neighborhood is indicated by $|N(u)|$. Time is discrete and to begin each discrete time step, each player selects the technology they will use, making a single choice that is used for interactions with each of its neighbors during that time step. Players have local knowledge only: they are aware of their neighbors' selections during the previous time step, but not the selections of their neighbors' neighbors.

Interactions are pairwise and each interaction may be described as a symmetric two-person, two-strategy coordination game whose payoffs are: $a$ to both players if both use innovation A, $b$ to both players if both use the incumbent technology B, and 0 if the two players fail to coordinate their technology selections. The payoff matrix is shown below:

| | A | B |
|---|---|---|
| A | $a\ a$ | 0 0 |
| B | 0 0 | $b\ b$ |

We imagine that use of the innovation confers some advantage so that $a > b > 0$. In this two-person game, coordinating on either strategy is a Nash equilibrium. There is also a mixed-strategy Nash equilibrium in which each player selects the innovation with probability $\frac{b}{a+b}$. The expected payoff for the mixed strategy Nash equilibrium for the two-person game is the harmonic mean $\frac{2ab}{a+b}$ , satisfying $a > \frac{2ab}{a+b} > b$ . The total payoff to a player in time step $t$ is the sum of payoffs from the interactions with each neighbor. If a node is currently using B, its payoff would be $b$ multiplied by the number of neighbors using B. The alternative, A, would give it a payoff of $a$ for each of its neighbors using A. Since the goal for each player is to maximize their expected payoff, the decision rule is to select A at time $t$ whenever the outcome from time $t-1$ satisfies

$$a * (\text{number of neighbors using A}) > b * (\text{number of neighbors using B}).$$

Employing further the language of game theory for our context, we say that A is a best response to the selections made by one's neighbors at time $t$ provided the payoff from B is less than or equal to the payoff from A given those selections. Our rule indicates a player will switch to A at time $t$ provided the use of A would have been a best response at time $t-1$. We posit that choosing A involves a commitment by the player to continue using A for a certain number of time steps. To simplify the model, we assume that once a player has selected A, they continue using A forever.

It is convenient to define the *individual threshold value* as the fraction of neighbors needed in order for the individual to adopt an innovation. The threshold value for adopting A is $\frac{b}{a+b}$. A node shifts to A if at time $t-1$

$$\frac{\#\text{ neighbors using A}}{\#\text{ neighbors in total}} \geq \frac{b}{a+b}$$

For example, if the payoffs of using A and B are equal, an individual in the network will need at least $\frac{1}{2}$ of its neighbors to be using A in order to adopt A. Otherwise it will stick with B as it will expect a higher payoff from using B.

## 3. Cascades and the Impenetrable Cluster Theorem

In a network, there might be a group consisting of closely connected individuals, think of close-knit friends in our camera-phone scenario. The corresponding subgraph in the network is called a cluster. It is hard for an outside innovation to penetrate a highly-connected cluster. Figure 1 is adopted from *Networks, Crowds and Markets* [2]. For this example, we suppose that the payoff of using technology A is 3 and that of B is 2. The individual threshold value is then $\frac{b}{a+b} = \frac{2}{5}$. At left, two initial adopters of innovation A, nodes 7 and 8, are highlighted. Nodes 5 and 10, each with half its neighbors using A, elect to switch to B at time $t = 1$. At

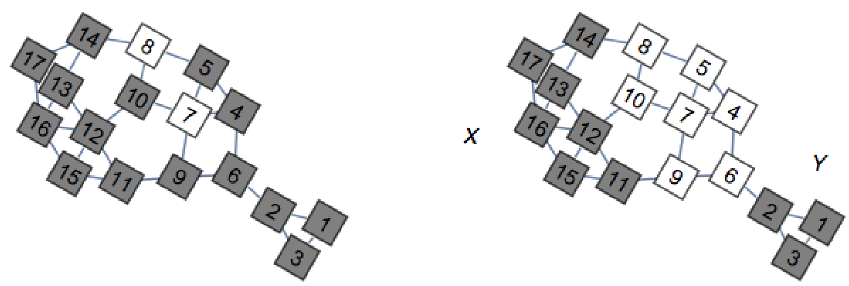



FIGURE 1. A network with two initial adopters of innovation A and a threshold of $\frac{2}{5}$. A complete cascade is prevented by clusters $X$ and $Y$.

time $t = 2$, nodes 4, 6, and 9 also adopt innovation A. The non-decreasing subsets of A-users initiated by the subset $S = \{7, 8\}$ is called a *cascade*. In this case, the cascade terminates with the subset $\{4, 5, 6, 7, 8, 9, 10\}$. Consider clusters $X$ and $Y$, marked at right in the figure. For each node in $X$ or $Y$, fewer than two-fifths of its neighbors use A in every time step. Therefore, no node within $X$ or $Y$ will ever adopt innovation A. We say that clusters $X$ and $Y$ prevent a *complete cascade* (a cascade in which every node eventually adopts the innovation).

We generalize from this example by making the following definition.

**Definition.** For a graph $G(V, E)$, a subset $X$ of nodes (alternatively $X$ may refer to the corresponding subgraph) is called a cluster. The *density* of a cluster is given by

$$\text{density of } X = \min_{x \in X} \frac{\text{number of neighbors of } x \text{ in } X}{\text{number of neighbors of } x}$$

.

In Figure 1, each of the subgraphs $X$ and $Y$ has density $\frac{2}{3}$, realized by nodes 2, 11, 12, and 14. The key to understanding cascades is the relationship between cluster density and threshold value.

**Theorem.** *(Impenetrable Cluster Theorem) Let $G(E, V)$ be a network of users of technology B. Let A be an innovation, and $S$ a subset of initial adopters of A. Suppose the respective payoffs are a and b so that the threshold for adopting innovation A is $q = \frac{b}{a+b}$. Then $S$ generates a complete cascade if and only if the complement of $S$ in $V$ contains no cluster with density greater than $1 - q$.*

*Proof.* The proof is by contradiction. Suppose there exists a cluster $X \subseteq V \setminus S$ of density greater than $1 - q$. Assume at time $t$, a node $x \in X$ becomes the first to adopt the new innovation A. Since $x$ is the first node in the cluster to adopt A, it must have made a decision at time $t - 1$ to switch. At that time, none of its neighbors inside the cluster were using A, since it is the first to make the switch. Hence, the more than $(1-q)$ fraction of its neighbors inside the cluster were using B. This means that less than $q$ fraction of its neighbors, all outside of $X$, were using A at time $t-1$. But if less than $q$ fraction of its neighbors were using A, the decision rule dictates that node $x$ would not have become an A-user at time $t$. Therefore, no node in $X$ ever converts to using A.

For the converse, suppose $S$ fails to generate a complete cascade and let $Y$ be the non-empty complement of the set of nodes that eventually switch to A. Every node $y \in Y$ continues to use B forever. This implies that each $y \in Y$ has fewer than $q$ fraction of its neighbors in $V \setminus Y$. Hence $Y$ is a cluster with density greater than $1-q$ lying in the complement of $S$. $\square$

The theorem says that not only are high-density clusters obstacles to complete cascades, they are the only obstacles.

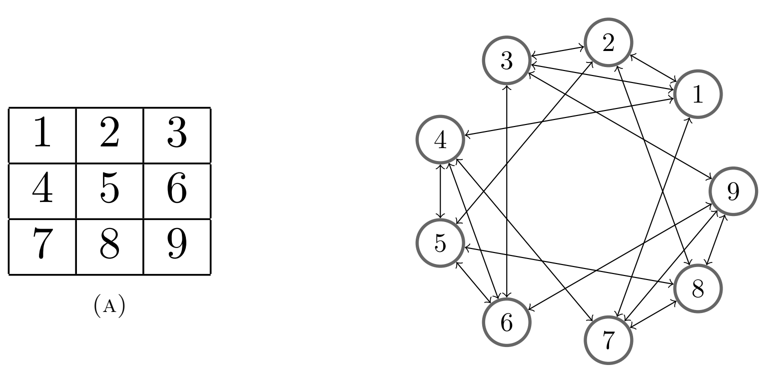
## 4. Application to Strongly Regular Graphs

A graph $G(V, E)$ with $v$ nodes is *$k$-regular* if every node has exactly $k$ neighbors. A $k$-regular graph is *strongly regular* provided two additional conditions are satisfied:

- pairs of neighboring nodes have $\lambda$ neighbors in common
- pairs of non-neighboring nodes have $\mu$ neighbors in common

The notation $SRG(v, k, \lambda, \mu)$ is used to indicate a strongly regular graph; the numbers $v, k, \lambda,$ and $\mu$ are called the parameters of the strongly regular graph. Cameron [11] provides a brief introduction to the subject. The notation is somewhat ambiguous since, for example, there are 32548 non-isomorphic strongly-regular graphs with parameters $(36, 15, 6, 6)$.

A family of strongly regular graphs that is easy to describe is the family of rook's graphs, constructed from chessboards. Each square on the chessboard represents a node. If we place a rook on a square, all the nodes that can be reached from that square via a legal chess move by the rook will be connected to it. The 3x3 rook's graph is illustrated below; $rook(3,3)$ is an $SRG(9,4,1,2)$.




(A) (B)

Figure 2. The 3x3 rook's graph $rook(3,3)$.

More generally, the graph $rook(n, n)$ is an $SRG(n^2, 2(n-1), n-2, 2)$. We will also refer to the rectangular rook's graphs $rook(p, q)$, which are $(p+q-2)$-regular but not strongly regular since connected nodes may share either $p-2$ or $q-2$ neighbors.

**Lemma.** *Consider $rook(p,q)$, $p,q \leq n$, embedded as a subgraph of $rook(n,n)$ and suppose a threshold value of $\frac{1}{2}$. If $(p-1)+(q-1) > n-1$, then $rook(p,q)$ is an impenetrable cluster.*

*Proof.* A node in the subgraph has $2(n-1)$ neighbors in $rook(n,n)$ of which $(p-1)+(q-1)$ are in $rook(p,q)$. The result follows. □

## 5. Subsets that initiate a complete cascade

We now address for $rook(n,n)$ the fundamental question concerning the size and pattern within a minimal set of initial adopters $S$ capable of initiating a complete cascade. As is customary [2], we consider this question for the case in which $a=b$ so that A has no real advantage over B and the individual threshold has value $\frac{1}{2}$. If $S$ initiates a complete cascade in this scenario, it will certainly initiate a complete cascade when A has an advantage since the threshold will decrease. An understanding of the rook's graphs will lead to the general result for strongly regular graphs that is proved in the next section.

**Definition.** The *complete cascade number* for a graph $G$ is defined as the minimum, taken over subsets $S$ of nodes in $G$ that initiate a complete cascade under the assumption of an individual threshold value of $\frac{1}{2}$, of the number of nodes in $S$.

The following lemma is useful.

**Lemma.** *If $0 < \mu < \frac{k}{2}$, then the complement of the closed neighborhood $\{u\} \cup N(u)$ of a node $u$ in $SRG(v,k,\lambda,\mu)$ constitutes an impenetrable cluster.*

*Proof.* Let $u$ be a fixed node in $SRG(v,k,\lambda,\mu)$ and $x$ a node in the complement of $\{u\} \cup N(u)$. Since $(u,x)$ is not an edge in the graph, $x$ shares exactly $\mu$ neighbors with $u$. Hence $x$ has $k-\mu$ neighbors in the complement of $\{u\} \cup N(u)$ and $\mu < \frac{k}{2}$ implies $k-\mu > \frac{k}{2}$. The threshold for $x$ to convert to A has not been met. This is true for every $x$ in the complement showing that the complement is impenetrable. □

For convenience we will denote the closed neighborhood $\{u\} \cup N(u)$ by the symbol $N_0(u)$.

**Corollary.** *Let $\{u_0, \ldots, u_{m-1}\}$ be a set of $m$ distinct nodes in $SRG(v,k,\lambda,\mu)$ and let $U = \cup_{i=0}^{m-1} N_0(u_i)$. If $0 < \mu < \frac{k}{2m}$ then the complement of $U$ constitutes an impenetrable cluster.*

*Proof.* Let $x$ be a node in the complement of $U$. For each $u_i$, $|N_0(x) \cap N_0(u_i)| = \mu$ so that $|N_0(x) \cap U| \leq m\mu$. Therefore the node $x$ has at least $k - m\mu$ neighbors in the complement of $U$. The inequality on $\mu$ implies $k - m\mu > \frac{k}{2}$. □

Rearranging the inequality, we may restate the corollary as follows:

**Corollary.** *Let $G$ be a strongly regular graph with parameters $(v,k,\lambda,\mu)$. Suppose the set $S$ of initial adopters of innovation $A$ is contained in a union $U$ of closed neighborhoods of $m$ distinct nodes. If $m < \frac{k}{2\mu}$ then the cascade initiated by $S$ cannot be complete. In this case, a superset $S^*$ of $S$ that initiates a complete cascade must contain at least $|S| + \frac{k}{2} - m\mu$ nodes.*

This observation suggests a greedy algorithm by which to iteratively construct a subset $S$ that initiates a complete cascade. Select a node $u_0$ and let $S_0$ be a set of $\frac{k}{2}$ nodes in $N(u_0)$. The nodes in $S_0$ suffice to convert $u_0$ to a user of A. Next, select a node $u_1$ in the complement of $N(u_0)$ such that $N(u_1)$ contains as many elements of $S_0$ as possible (at most $\mu$). Call this number $\mu_1$. Form $S_1$ by adding $\frac{k}{2} - \mu_1$ nodes from $N(u_1) \setminus N(u_0)$ so that both $u_0$ and $u_1$ are converted to using A at time $t = 1$, and so forth. The cascade resulting from the set thus constructed can be checked for completeness at each stage. For a finite graph the algorithm will terminate. The rook's graphs allow visualization of this algorithm, as illustrated here for $rook(5,5)$. The elements of $S$ are shown as stars and for this example $S_1 = S$ is sufficient to initiate a complete cascade.

| | | | | |
|---|---|---|---|---|
| $u_0$ | | | ★ | ★ |
| | | | | |
| | | | | |
| ★ | | | | |
| ★ | | | | |

$$\Longrightarrow$$

| | | | | |
|---|---|---|---|---|
| $u_0$ | | | ★ | ★ |
| | | | | ★ |
| | | | | |
| ★ | | | | |
| ★ | ★ | | | $u_1$ |

Recognizing patterns allows us to specify complete cascade initiating sets in the entire family of strongly regular rook's graphs. Here are two patterns that can be generalized to show how an initial set comprised of approximately one fourth of the nodes can initiate a complete cascade in $rook(n,n)$. In the figures, we mark the time at which a particular square in the grid converts to innovation A in order to indicate how the cascade progresses initially toward completion.

| | | | | | |
|---|---|---|---|---|---|
| ★ | 2 | ★ | 1 | ★ | 3 |
| 3 | ★ | 2 | ★ | 1 | |
| | 3 | ★ | 2 | ★ | |
| | | 3 | ★ | 2 | |
| | | | 3 | ★ | |
| | | | | 3 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| ★ | ★ | ★ | 4 | 3 | 2 | 1 |
| ★ | ★ | 4 | | 4 | 3 | 2 |
| ★ | 4 | | | | 4 | 3 |
| 4 | | | | | | 4 |
| 3 | 4 | | | | 4 | ★ |
| 2 | 3 | 4 | | 4 | ★ | ★ |
| 1 | 2 | 3 | 4 | ★ | ★ | ★ |

Our next theorem shows that these patterns realize smallest possible number of initial adopters able to initiate a complete cascade in $rook(n,n)$.

**Theorem.** *The complete cascade number for $rook(n,n)$ is*

$$\begin{cases} \frac{1}{4}(n+1)(n-1) & \textit{if } n \textit{ is odd} \\ \frac{1}{4}n^2 & \textit{if } n \textit{ is even} \end{cases}.$$

*Proof.* We employ the lemma to show that no smaller subset can initiate a complete cascade. To make the proof as clear as possible we refer to both the vertex set $V$ of the graph $G = rook(n,n)$ and to the squares in the underlying $n$-by-$n$ chessboard grid.

Suppose first that $n$ is odd and write $n = 2m+1$. Let $S$ be a subset of $\frac{1}{4}(n+1)(n-1) - 1 = m^2 + m - 1$ nodes, our set of initial adopters of innovation A. We will show that no matter how these nodes are placed in the underlying $n$-by-$n$ grid, the resulting cascade cannot be complete. To begin, note that in order for use of A to spread beyond our initial set, there must be a square $u_0$, an initial user of

B, whose row and column contain at least $n-1=2m$ initial adopters from $S$. Hence at most $(m^2+m-1)-2m$ initial adopters of A lie in the complement of $U_0=\{u_0\}\cup N(u_0)$.

If no initial user of B in $G\setminus U_0$ were to have at least $2m-2$ initial adopters of A as neighbors, the cascade initiated by $S$ would not penetrate $(G\setminus U_0)\setminus S$. Hence we may suppose there is a node $u_1\in(G\setminus U_0)\setminus S$ with $|N(u_1)\cap(S\setminus U_0)|\geq 2m-2$.

Set $U_1=U_0\cup\{u_1\}\cup N(u_1)$. Note that $G\setminus U_1$ corresponds to an $(n-2)$-by-$(n-2)$ grid, containing at most $(m^2+m-1)-2m-(2m-2)$ initial adopters of A. Repeating the above argument, there is a node $u_2\in(G\setminus U_1)\setminus S$ whose neighborhood contains at least $2m-4$ initial adopters of A, and we may continue to decrease the size of the grid under consideration.

Proceeding in this way leads eventually to the $(n-(m-1))$-by-$(n-(m-1))$ grid $G\setminus U_{m-2}$, containing at most one initial node according to the following count of initial nodes already placed:

$$\begin{aligned}
\#\text{ nodes of } S \text{ placed in } G\setminus U_{m-2}&\\
&=(m^2+m-1)-(2m+(2m-2)+\ldots+(2m-2(m-2)))\\
&=(m^2+m-1)-\sum_{j=0}^{m-2}2(m-j)\\
&=(m^2+m-1)-2m(m-1)+2\sum_{j=0}^{m-2}j\\
&=(m^2+m-1)-2m(m-1)+(m-2)(m-1)\\
&=m^2(1-2+1)+m(1+2-3)+(-1+2)=1.
\end{aligned}$$

Placement of the remaining initial node leaves a rectangular grid of initial users of B having size $(n-(m-1))$-by-$(n-m)$. In other words we have initial users of B filling a copy of $rook(p,q)\subset rook(n,n)$ with

$$(p-1)+(q-1)=(n-m)+(n-m-1)=2n-(2m+1)=n>n-1.$$

Hence the complement of $S$ contains an impenetrable cluster.

For $n=2m$ even with a set $S$ of initial adopters having size $m^2-1$, the argument is the same and eventually yields an impenetrable $rook(m+1,m+1)$ in the complement $S$. □

By the theorem, a subset of five initial nodes cannot initiate a complete cascade within $rook(5,5)$. We now show that not every subset of six initial nodes will initiate a complete cascade, so that the understanding of patterns is important. Among subsets that do not initiate a complete cascade, we find that there are, up to equivalence, exactly four subsets whose complement does not contain a subgraph isomorphic to $rook(p,q)$ with $p-1+q-1>n-1$. We also show that there are, again up to equivalence, exactly four subsets of six initial nodes that do initiate complete cascades. The fundamental tool for these considerations is the group of graph automorphisms of $rook(5,5)$.

**Definition.** A *graph automorphism* is a one-to-one and onto function from the vertex set of the given graph to itself that preserves all edge relationships. The set of graph automorphisms forms a group under composition of functions.

The group of graph automorphisms for $rook(n, n)$ consists of mappings that permute the rows of the underlying grid, reorder the columns, or interchange rows and columns. The group has size $2(n!)^2$; $28,800$ for $rook(5, 5)$. Meanwhile, there are $\binom{25}{6} = 177,100$ subsets of 6 nodes. A search using Mathematica followed by a careful application of Burnside's lemma [12] reveals that the action of the automorphism group partitions this collection of subsets into 39 disjoint orbits.

Here are the diagrams for those subsets of six adopters that do not initiate complete cascades and whose complements do not contain an impenetrable subgraph of type $rook(p, q)$. Only the subset at top left will yield an expanded set of A-users, adding two A-users in the top row before the cascade terminates.

| | | | | |
|---|---|---|---|---|
| ★ | ★ | | ★ | |
| | ★ | | | |
| | | ★ | | |
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|---|---|---|---|---|
| ★ | ★ | | | |
| ★ | ★ | | | |
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| ★ | ★ | | | |
| | ★ | | | |
| | | ★ | | |
| | | | ★ | |
| | | | | ★ |

In contrast, here are upper triangular representatives for the four equivalence classes of initial diagrams that do initiate complete cascades. The reader is encouraged to follow the cascades to completion (the initial subset at top left requires 4 time steps, the others 5).

| | | | | |
|---|---|---|---|---|
| ★ | ★ | | | 1 |
| | ★ | | | |
| | | | | |
| | 1 | | ★ | ★ |
| | | | | ★ |

| | | | | |
|---|---|---|---|---|
| ★ | ★ | | | 1 |
| | ★ | | | ★ |
| | | | | |
| | | | ★ | |
| | | | | ★ |

| | | | | |
|---|---|---|---|---|
| ★ | ★ | | | 1 |
| | ★ | | | |
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| | | | | |
|---|---|---|---|---|
| ★ | | ★ | | 1 |
| | ★ | | | |
| | | | | ★ |
| | | | ★ | |
| | | | | ★ |

In the course of writing this paper we became aware that our result in this section on the complete cascade number for rook's graphs appears in [13]. Their proof relies on the fact that $rook(n, n)$ is the line graph for the complete bipartite graph $K_{n,n}$ and does not make explicit use of the properties of strongly regular graphs.

## 6. A Lower Bound on the Cascade Number for $SRG(v, k, \lambda, \mu)$

The proof idea for the theorem on rook's graphs extends to every strongly regular graph.

**Theorem.** *Let $G$ be a connected strongly regular graph whose parameters are $(v, k, \lambda, \mu > 0)$ and let $N = \lceil \frac{k}{2\mu} - 1 \rceil$ be the least integer greater than $\frac{k}{2\mu} - 1$. With adoption threshold $\frac{1}{2}$, if $S$ is a subset of adopters of $A$ that initiates a complete cascade in $G$, then*

$$|S| \geq (\frac{N+1}{2})(k - N\mu)$$

*Proof.* We note first that the connectivity condition, together with $\mu > 0$ means only that $G$ is not the complete graph $K_v$ on $v$ vertices. A complete cascade for $K_v$ requires $|S| = \frac{k}{2}$.

Suppose $S$ is a subset of minimum size that initiates a complete cascade. Then there must be a node $u_0 \in G \setminus S$ that has at least $\frac{k}{2}$ neighbors in $S$ (if not, either $S$ is not minimal or the cascade initiated by $S$ does not extend beyond $S$). Let $U_0 = \{u_0\} \cup N(u_0)$ and suppose $G \setminus (S \cup U_0)$ is nonempty.

We claim there must be a node in $G \setminus (S \cup U_0)$ which has at least $\frac{k}{2} - \mu$ neighbors in $S \setminus U_0$. The proof is by contradiction. For $x \in G \setminus (S \cup U_0)$ with $\frac{k}{2} - \mu - \varepsilon$ neighbors ($\varepsilon > 0$) in $S \setminus U_0$ we have

$$\begin{aligned} |N(x) \cap G \setminus (S \cup U_0)| &= |N(x)| - |N(x) \cap (S \cup U_0)| \\ &= k - (\mu + (\frac{k}{2} - \mu - \varepsilon)) \\ &= \frac{k}{2} + \varepsilon \\ &> \frac{k}{2} \end{aligned}$$

Thus $x$ has more than half its neighbors in the complement of $S$. Since we assumed this property for every $x \in G \setminus (S \cup U_0)$, this would make $G \setminus (S \cup U_0)$ an impenetrable cluster, a contradiction.

Let $u_1$ be a node in $G \setminus (S \cup U_0)$ that has at least $\frac{k}{2} - \mu$ neighbors in $S \setminus U_0$ and let $U_1 = U_0 \cup u_1 \cup N(u_1)$. Note that $U_1$ contains at least $\frac{k}{2} + (\frac{k}{2} - \mu)$ distinct nodes of $S$.

If $G \setminus (S \cup U_1)$ is nonempty, we may apply the argument above to conclude that since $S$ initiates a complete cascade, there exists a node $u_2 \in G \setminus (S \cup U_1)$ such that $N(u_2)$ contains at least $\frac{k}{2} - 2\mu$ nodes in $S$ that have not been accounted for previously.

So long as the sets $G \setminus (S \cup U_j)$ are nonempty and $\frac{k}{2} - j\mu > 0$ the process continues, resulting in a set of nodes $\{u_0, u_1, \ldots, u_N\}$ and a total of

$$\begin{aligned} \sum_{j=0}^{N} (\frac{k}{2} - j\mu) &= \frac{k}{2}(N+1) - \mu \sum_{j=0}^{N} j \\ &= \frac{k}{2}(N+1) - \mu \frac{N(N+1)}{2} \\ &= \frac{N+1}{2}(k - N\mu) \end{aligned}$$

distinct nodes in the set $S$.

Finally, we note that the sets $G \setminus (S \cup U_j)$ are indeed nonempty: since $G$ is $k$-regular a set of nodes whose closed neighborhoods cover all of $G$ must contain at least $\frac{v}{2}$ elements in order to cover the $\frac{kv}{2}$ edges in $G$. Clearly $N < \frac{k}{2\mu} \leq \frac{k}{2} < \frac{v}{2}$. □

## 7. Recasting diffusion of innovation as a repeated game

We now return to the general setting in which the underlying graph may have any structure whatsoever. Our interest is in developing methods by which the marketer of innovation A may recruit users of B from inside an impenetrable cluster. We first show that by considering a longer time horizon rather than merely the payoff in the next time step, some players from within an impenetrable cluster may be convinced to adopt A.

In repeated games, players consider the discounted future value of their decisions. In our scenario, a player at vertex $x$ in the graph $G$ may convert to the innovation A even if the immediate payoff is small, provided the conversion facilitates a cascade that provides sufficient benefit to the player in the long run. The decision by a current user of B will be based on the relative advantage of innovation A, the speed at which their neighbors adopt A, and the value of their individual discount factor $\delta$.

Setting $t = 0$ at the time that A is adopted and using $V_{A,t}$ to denote the set of users of A at time $t$, the discounted future payoff to the player at vertex $x$ is

$$\pi(x) = \sum_{t=0}^{\infty} a|N(x) \cap V_{A,t}|\delta^t$$

and the player should adopt A if that sum is greater than $b|N(x) \cap V_{A,0}^C|\frac{1}{1-\delta}$.

**Example.**

Consider the initial configuration below left for which no cascade ensues. As shown below right, if the player at location $(3, 4)$ in the grid were to adopt A at $t = 0$ then all of their neighbors would adopt A by time $t = 2$.

| | | | | |
|---|---|---|---|---|
| ★ | ★ | | | |
| ★ | ★ | | | |
| | | ★ | | |
| | | | ★ | |
| | | | | |

| | | | | |
|---|---|---|---|---|
| ★ | ★ | 2 | 1 | |
| ★ | ★ | 2 | 1 | |
| 1 | 1 | ★ | 0 | 2 |
| | | | ★ | |
| | | | 2 | |

The payoff to the player at location $(3, 4)$ resulting from this cascade is

$$2a + 6a\delta + 8a\frac{\delta^2}{1-\delta}$$

and, after clearing denominators, we see that the player at location $(3, 4)$ should adopt A provided

$$(1 + \delta^2) > 3\frac{b}{a}.$$

Note that if the player's discount factor satisfies $\delta > \sqrt{3} - 1 \approx 0.7321$, the player should adopt A no matter how small the advantage A has over B.

Perhaps the most general result that's simple to state without knowledge of the particular cascade involved is the following:

**Theorem.** *Let $G$ be a graph and let $x$ be a vertex in $G$, considered as a player in a repeated diffusion of innovations coordination game with payoffs $a$, respectively $b$, for two players who coordinate on A, respectively B. Let $N(x) \cap V_{A,0}$ denote the set of neighbors of $x$ that initially adopt strategy A and let $\delta \in (0,1)$ be the discount factor assigned to future payoffs by $x$.*

*Suppose $x$ initially uses B, but anticipates that switching to A will induce all of their neighbors to adopt A at or before time $T$. Then $x$ should adopt A provided*

$$\frac{\#\ \textit{neighbors using A}}{\#\ \textit{neighbors in total}} \geq \frac{b - a\delta^T}{a + b - a\delta^T}$$

*Proof.* Independent of the course of the intervening cascade, the payoff to $x$ from choosing A is at least

$$a|N(x) \cap V_{A,0}|(1 + \delta + \delta^2 + \cdots \delta^{T-1} + a|N(x)|\frac{\delta^T}{1-\delta}$$

Similarly, the payoff to $x$ from choosing to employ B at every stage is (assuming none of the neighbors actually switch to A) at most

$$(|N(x)| - |N(x) \cap V_{A,0}|)b\frac{1}{1-\delta}$$

Vertex $x$ will surely adopt A if the least possible payoff from doing so is greater than or equal to the greatest possible payoff from choosing B. Using the formula for the partial sum of a geometric series and rearranging terms in the inequality between payoffs gives the result. □

The same reasoning yields the following generalization.

**Corollary.** *Under the same conditions as in the previous theorem, suppose $x$ initially uses B, but anticipates that switching to A will induce fraction $\tau$, $\frac{b}{a+b} \leq \tau \leq 1$, of their neighbors to adopt A at or before time $T$. Then $x$ should adopt A provided*

$$\frac{\#\ \textit{neighbors using A}}{\#\ \textit{neighbors in total}} \geq \frac{b - a\tau\delta^T}{a + b - a\delta^T}$$

We note that for every $\delta \in (0,1)$ and every $\tau \in (\frac{b}{a+b}, 1]$, the threshold value here is less than the threshold value of $\frac{b}{a+b}$ that governs the decision to adopt A without consideration of future values.

## 8. Summary and Future Directions

We have investigated the diffusion of innovations problem on graphs as modeled by a pairwise coordination game between a vertex and each of its neighbors. In the general setting we have shown that impenetrable clusters are the only obstruction to achieving a monopoly and that consideration of discounted future values in a repeated game can reduce the adoption threshold for individual players. A new direction for our research into how the number and size of impenetrable clusters can be reduced is to consider graphs whose structure may change over time. We envision here the addition of edges, at some cost, and the research would involve cost-benefit analysis and the problem of how costs would be equitably shared between the marketer of the innovation and the players who benefit by acquiring new neighbors (who may initiate the construction of edges without the participation of the marketer).

We specialized to strongly regular graphs in order to address the question of the minimum size of a set of initial adopters capable of generating a complete cascade. In our main theorem we obtained a universal lower bound on this size for all graphs of this type. Additionally, we showed that the family of strongly regular rook's graphs realize exactly this lower bound. We have begun research into other families of strongly regular graphs to determine whether our bound is realized within those families, making use of graph automorphisms to reduce the number of cases to be considered and to understand patterns for minimal complete cascade initiating subsets in those families.

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