Reconstruction of the Acoustic Refractive Index from the Interior Transmission Eigenvalue Problem (ITEP)

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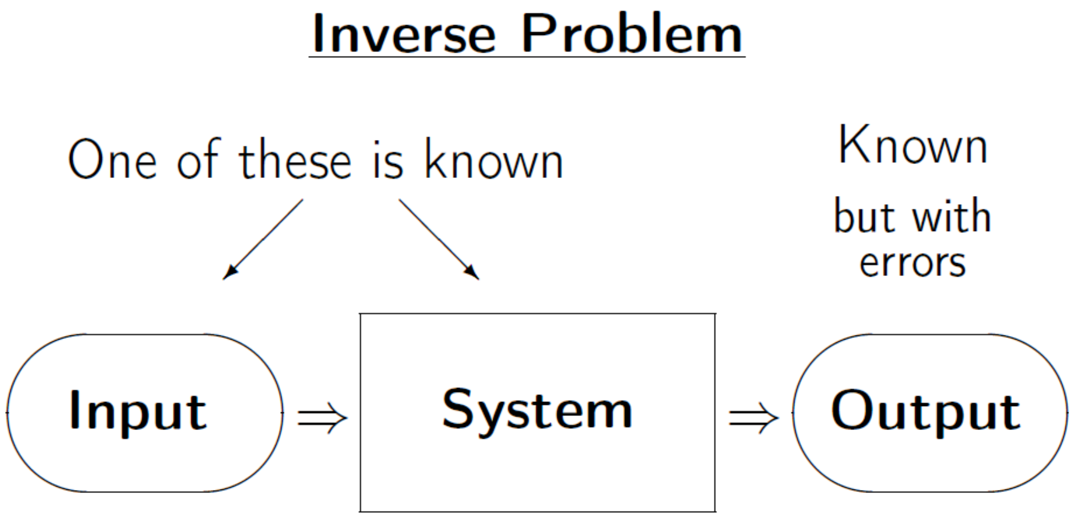
Introduction

Inverse Problems

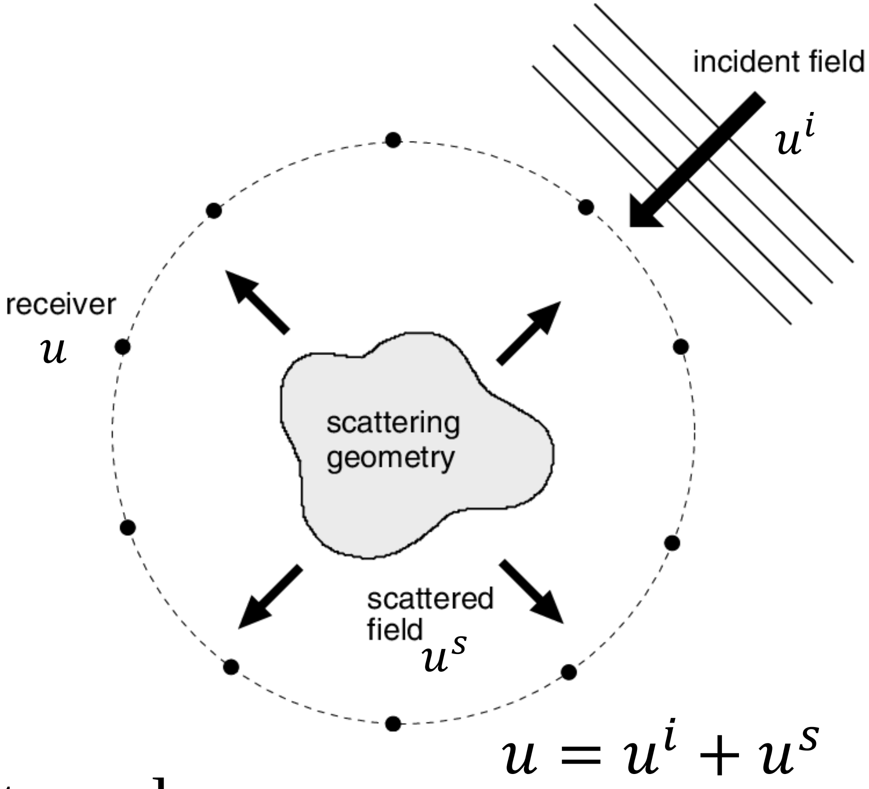


Forward Problem?

- **What are the applications?** Geophysical imaging, Computational Photography, Medical imaging, *Inverse Scattering*, …

- **Why are inverse problems difficult?** Ill-posedness (Existence, Uniqueness, and Stability) => Need to be Regularized

Inverse Scattering



- Scattering theory is concerned with the effect of an inhomogeneous medium has on an *<u>incident particle</u>* or *<u>wave</u>*.

- Inverse scattering problem for **acoustic** and electromagnetic waves can broadly be divided into two classes:

    - Inverse obstacle problem (IOP): Determine the shape (boundary) of an impenetrable obstacle from a scattered field.

    - Inverse medium problem (IMP): Determine medium property (<u>sound speed</u> or permittivity)

[Colton et al. 2019]

Inverse Medium Problem (IMP)

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in} \quad D,$$

$$\Delta u + k^2 u = 0 \qquad \text{in} \quad \mathbb{R}^3 \backslash \overline{D},$$

$$u(x) = e^{ikx\cdot d} + u^s(x),$$

Sommerfeld radiation condition $$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0.$$

where $\boxed{u^i(x) = e^{ikx\cdot d}}$ is the time-harmonic acoustic plane wave, $\boxed{k = \omega/c_0}$ the wave number, $\boxed{\omega}$ the angular frequency, $\boxed{c_0}$ the speed of sound in the homogeneous host medium, $\boxed{d}$ the direction of propagation, $\boxed{r = |x|}$, $\boxed{n = c_0^2/c^2}$ is the refractive index, $\boxed{c^2}$ the speed of sound in the in the inhomogeneous medium, $\boxed{D}$ the inhomogeneity.

[Colton et al. 2019]

Inverse Obstacle Problem (IOP)

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^3 \backslash \overline{D},$$

$$u(x) = e^{ikx\cdot d} + u^s(x),$$

$$u = 0 \quad \text{on} \quad \partial D,$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0.$$

where D is an impenetrable obstacle.

- Dirichlet boundary condition corresponds to a ***sound-soft*** obstacle.

[Colton et al. 2019]

Far-field Pattern

- Far-field pattern, or scattering amplitude, of the scattered acoustic wave is of interest. u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}, d) + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad r = |x| \to \infty$$

where $\hat{x} \coloneqq x/|x|$.

- Far-field equation:

$$(Fg)(\hat{x}) \coloneqq \Phi_\infty(\hat{x}, z)$$

where F is the far-field operator, g is a density function, and, $\Phi_\infty(\hat{x}, z)$ is the far field pattern corresponding to the fundamental solution $\Phi(x, z)$ oh the Helmholtz equation. (Regularization methods!).

[Colton et al. 2019]

Interior Transmission Eigenvalue Problem (ITEP)

- Are there any incident wave u^i such that scattered field u^s is identically zero?
  - ➢ The answer to this question leads to the interior transmission eigenvalue problem (ITEP).
- ITEP: find $k > 0$ and $v, w \in C^2(D)$ such that:

$$\Delta w + k^2 n(x) w = 0 \quad \text{in } D$$

$$\Delta v + k^2 v = 0 \quad \text{in } D$$

$$w = v \quad \text{on } \partial D$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D$$

The value of k for which there exist a non-trivial solution (v, w) for above equations are the interior transmission eigenvalues (ITE).

[Colton et al. 2019]

Inverse Spectral Problem (ISP)

- **Spectral Inversion**: Can one hear the shape of a drum? [Mark Kac 1966]
- Assume the shape of the scatterer is known => Determine the unknown refractive index $n(x)$ from transmission eigenvalues.
- ITEs is required => Inverse Spectral Problem for Transmission Eigenvalues
- Note:
  - ✓ Real transmission eigenvalue can be measured by scattering data
  - ✓ ITEs provide information about the material properties of scatterer ($n(x)$).

[Colton et al. 2019]

Project Overview

- **Main Steps of the project:**
  1) Extract the ITEs from far-field data
  2) Estimate the acoustic refractive index ($n(x)$) from the first ITE.

Methods

Linear sampling method (LSM)

- Consider the far-field operator F: $L^2(\Omega) \rightarrow L^2(\Omega)$ as:

$$(Fg)(\hat{x}) = \int_{\Omega} u_\infty(\hat{x}, d) g(d) \, \mathrm{d}s(d).$$

where $u_\infty(\hat{x}, d)$ is the far-field pattern corresponding to incident direction d and observation direction $\hat{x}$.

- The far-field equation can be written as a **linear inverse problem** of determining g from knowledge of

$$f = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot y}$$

where g and f are related by

$$Fg = f.$$

- If k is **not** a transmission eigenvalue, considering Ω as a unit circle/sphere:

$$\boxed{\|g\|_{L^2(\Omega)} \rightarrow \infty \text{ when } y \text{ approaches the boundary of the scatterer from the inside.}}$$

[Colton and Kirsch 1996] and [Tacchino et al. 2002]

LSM: Regularization

- $Fg = f$ is not in general solvable since f is <u>**not**</u> in the range (column space) of F (measurement noise, ...).
- Tikhonov regularization: find approximate solution g_α as solution to:

$$\min_g \left\{ \|F_\delta g_\alpha - f\|^2_{L^2(\Omega)} + \alpha \|g_\alpha\|^2_{L^2(\Omega)} \right\}$$

where F_δ is the far-field operator affected by measurement noise of the order of magnitude δ and α is the regularization parameter.

- By solving above equation, one can obtain:

$$g_\alpha = \left(F_\delta{}^T F_\delta + \alpha I\right)^{-1} F_\delta{}^T f$$

- **Choosing the regularization parameter:** <u>Morozov's discrepancy principle</u>, L-Curve method, Generalized Cross Validation, …

[Hansen, 2010]

LSM: Obtaining ITE

- Assume the shape of D found by LSM.
- LSM can be expected to fail when k is a ITE and in particular the norm of the (regularized) solution to

$$(Fg)(\hat{x}) = 1$$

should be large for such values of k.

- So, determining ITE from the far-field data

[Cakoni et al. 2007]

Estimation of n from the smallest real ITE

- ITEP (slide 11) can be transformed into a *fourth-order equation* for u: $= w - v \in H_0^2(D)$

$$(\Delta + k^2 n)\frac{1}{n-1}(\Delta + k^2)u = 0, \qquad \text{in } D$$

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0, \qquad \text{on } \partial D$$

- Weak form:

$$\int_D \frac{1}{n-1}(\Delta u + k^2 u)(\Delta\bar{\phi} + k^2 n\bar{\phi})\,\mathrm{d}x = 0, \quad \forall \phi \in H_0^2(D)$$

ϕ: test function

[Pallikarakis 2017]

Estimation of n from the first real ITE

- Use the Galerkin method to discretize the weak form!
- Assume $\{\varphi_i\}_{i=1}^{\infty}$ is a set of eigenfunctions for the problem (clamped plate problem):

$$L\varphi_i = \mu_i \varphi_i \quad \text{in } D$$

$$\phi_i = 0, \qquad \frac{\partial \phi_i}{\partial \nu} = 0 \quad \text{on } \partial D$$

where L is a forth order elliptic operator (Bilaplacian operator).

- By the Galerkin method, u_k is approximated as:

$$u_k^N = \sum_{i=1}^{(N)} c_i \phi_i$$

[Pallikarakis 2017]

Estimation of n from the first real ITE

- Put u_k^N in the weak form:

$$\left[A^{(N)} - \left(k^{(N)}\right)^2 B^{(N)} + \left(k^{(N)}\right)^4 C^{(N)}\right] \boldsymbol{c} = 0$$

a quadratic eigenvalue problem

where

$$A^{(N)} := \int_D \frac{1}{n(x)-1} \Delta\phi_i \Delta\bar{\phi}_j \, \mathrm{d}x$$

$$B^{(N)} := -\left(\int_D \frac{n(x)}{n(x)-1} \Delta\phi_i \bar{\phi}_j \, \mathrm{d}x + \int_D \frac{1}{n(x)-1} \phi_i \Delta\bar{\phi}_j \, \mathrm{d}x\right)$$

$$C^{(N)} := \int_D \frac{n(x)}{n(x)-1} \phi_i \bar{\phi}_j \, \mathrm{d}x$$

are $N \times N$ matrices and $\boldsymbol{c} = [c_1, c_2, \dots, c_N]^T, \quad i,j = 1, \dots, N.$

[Pallikarakis 2017]

Estimation of n from the first real ITE - Pseudocode

Given k_0; $\quad \% analytically\ or\ computationally$

$RI = [n_{min}: step: n_{max}]; \quad \% refractive\ indices$

for $idx = 1: \text{length}(RI)$

$n = RI(idx)$

Compute $\quad A^{(N)}, B^{(N)}, C^{(N)}$

Solve $\quad \left[A^{(N)} - \left(k^{(N)}\right)^2 B^{(N)} + \left(k^{(N)}\right)^4 C^{(N)}\right] \boldsymbol{c} = 0$

Find $\quad k_0^{(N)}$

Compute $\quad Err(idx) = norm(k_0 - k_0^n)$

end

$\left[:, idx_{opt}\right] = \min(Err)$

$n_{opt} = RI(idx_{opt})$

Results

LSM for Shape Reconstruction – 2D

- **Shape:** a unit circle centered at the origin.
- $k = 2\pi$
- **Far-field** patterns calculated:
  A. Analytically [Colton and Kirsch 1996]

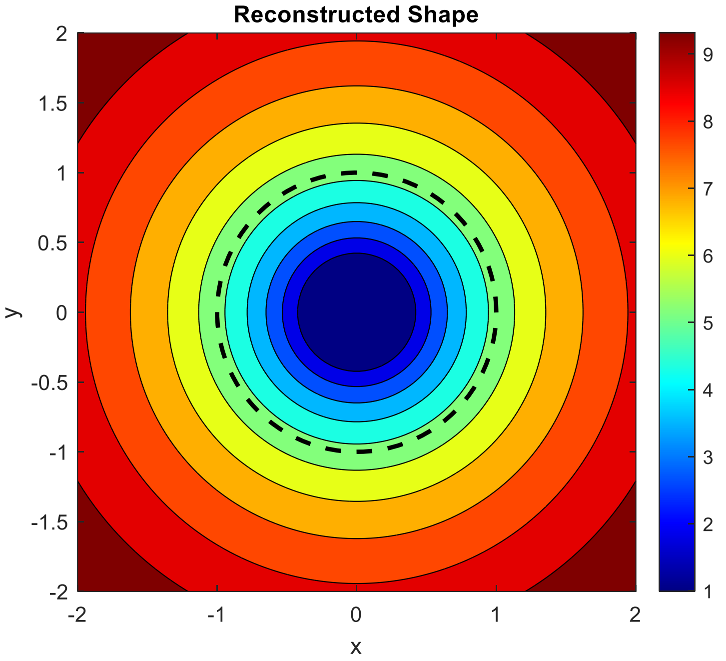
$$u_\infty(\phi;\theta) = -e^{-i\pi/4}\sqrt{2/\pi k}\sum_{n=-\infty}^{\infty}\frac{J_n(ka)}{H_n^1(ka)}e^{in(\phi-\theta)}$$

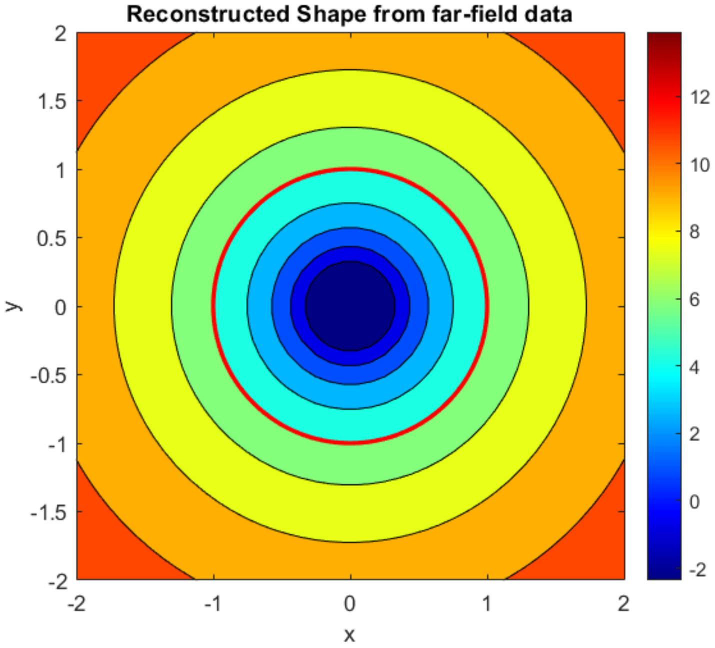
  B. Numerically by μ-diff toolbox [Thierry et al. 2015] in MATLAB
- **Tikhonov** Regularization

LSM for Shape Reconstruction – 2D

Number of incident angles = 180

Number of scattered angles = 180





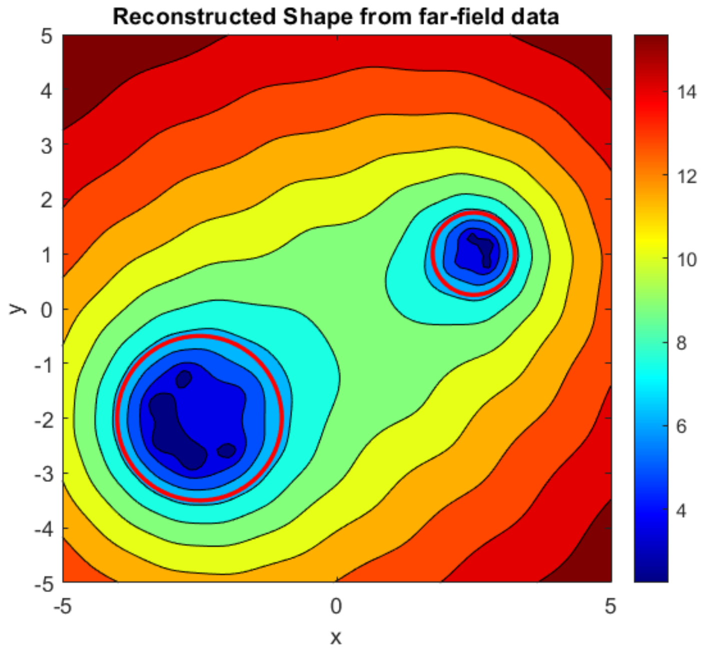
Impenetrable unit circle centered at the origin- **Analytical** far-field patterns.	Impenetrable unit circle centered at the origin ($\alpha = 10^{-5}$).

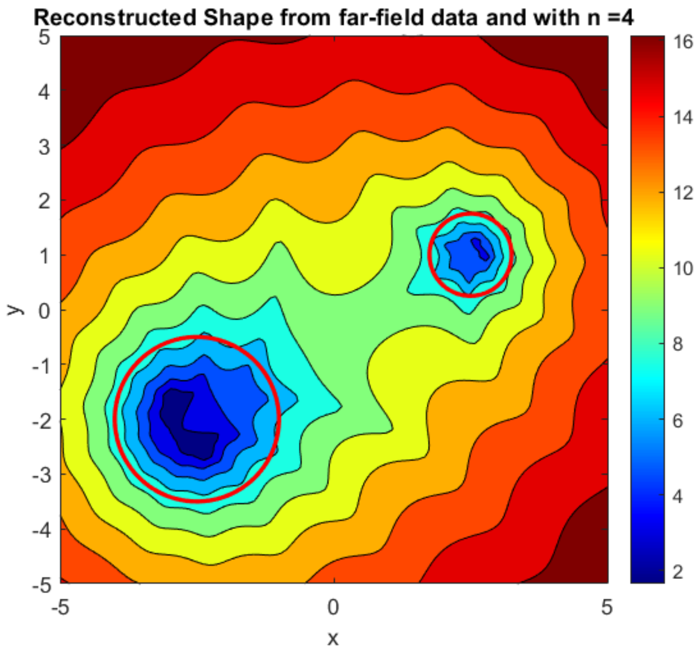
LSM for Shape Reconstruction – 2D

Shape: circles centered at (-2.5, 2.5) and (-2, 1) with a radius of 1.5 and 0.75, respectively

Far-field by μ-diff toolbox. **Tikhonov** Regularization

Number of incident angles = 180 Number of scattered angles = 180

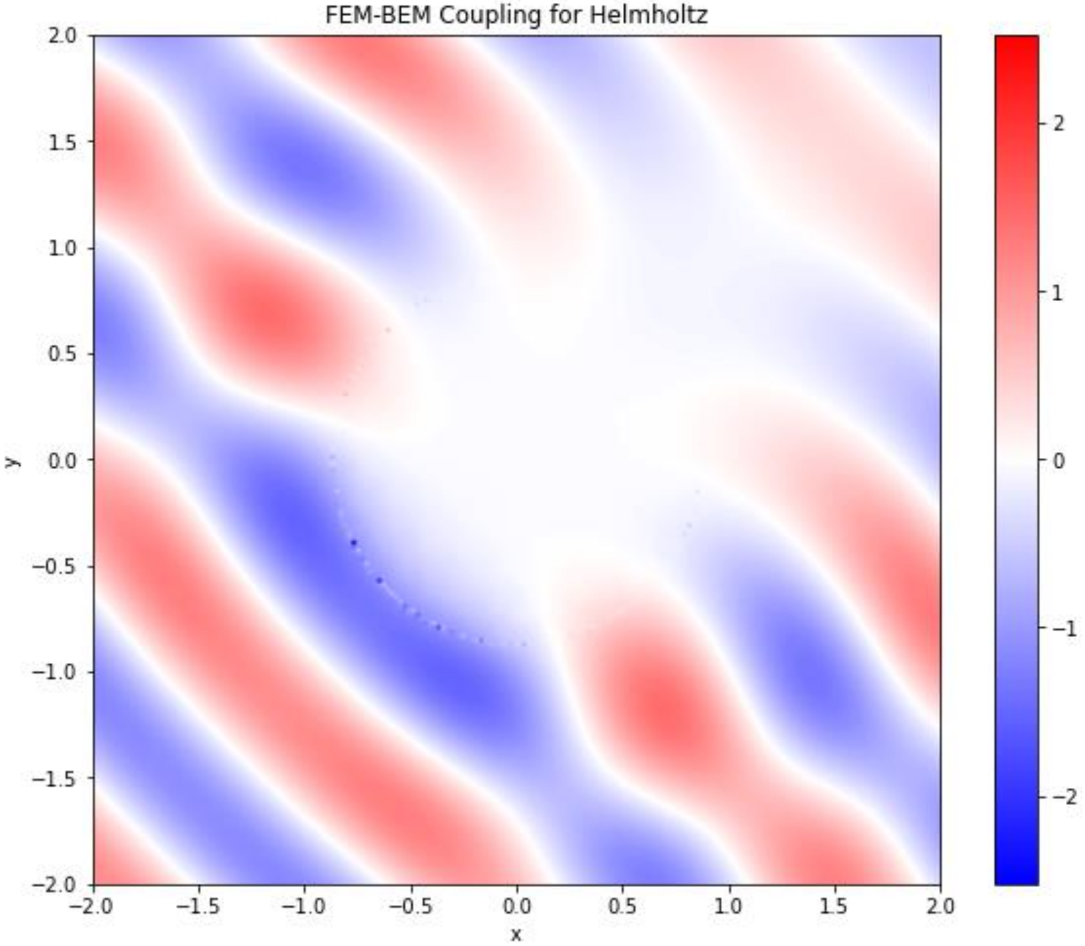




Impenetrable circles centered at (-2.5, 2.5) and (-2, 1) with a radius of 1.5 and 0.75, respectively. ($\alpha = 10^{-8}$).	Penetrable circles centered at (-2.5, 2.5) and (-2, 1) with a radius of 1.5 and 0.75, respectively and with $n = 4$ ($\alpha = 10^{-8}$).

LSM for Shape Reconstruction – 3D

- **Shape:** a unit sphere centered at the origin
- $k = 2\pi$
- **Far-field** patterns calculated:
  - By coupling the finite-element method (FEM) and the boundary-element method (BEM) in Python.
  - FEM was implemented by FEniCS [https://fenicsproject.org/].
  - BEM was implemented by Bempp [https://bempp.com/]



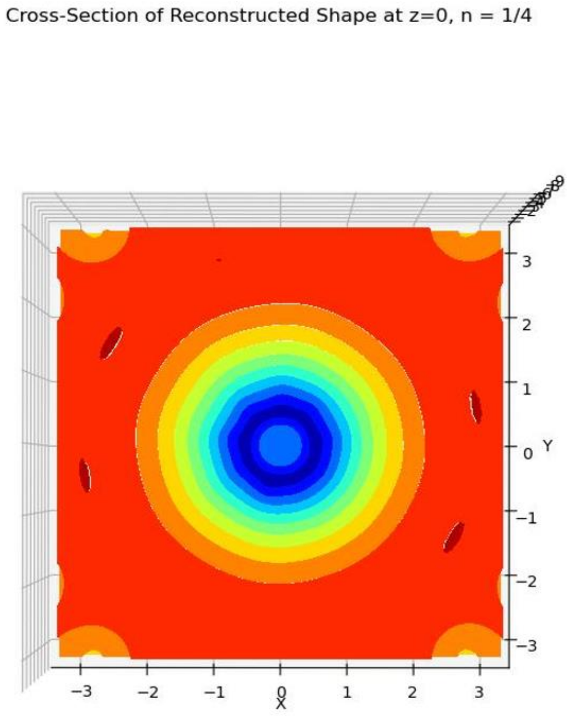
Total (near) field scattered from a unit sphere centered at the origin, cross section view at $z = 0$, $n = 1/4$

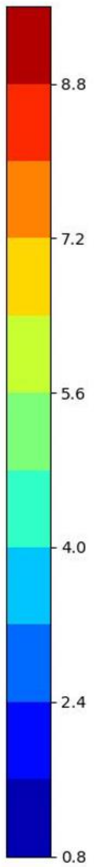
LSM for Shape Reconstruction – 3D

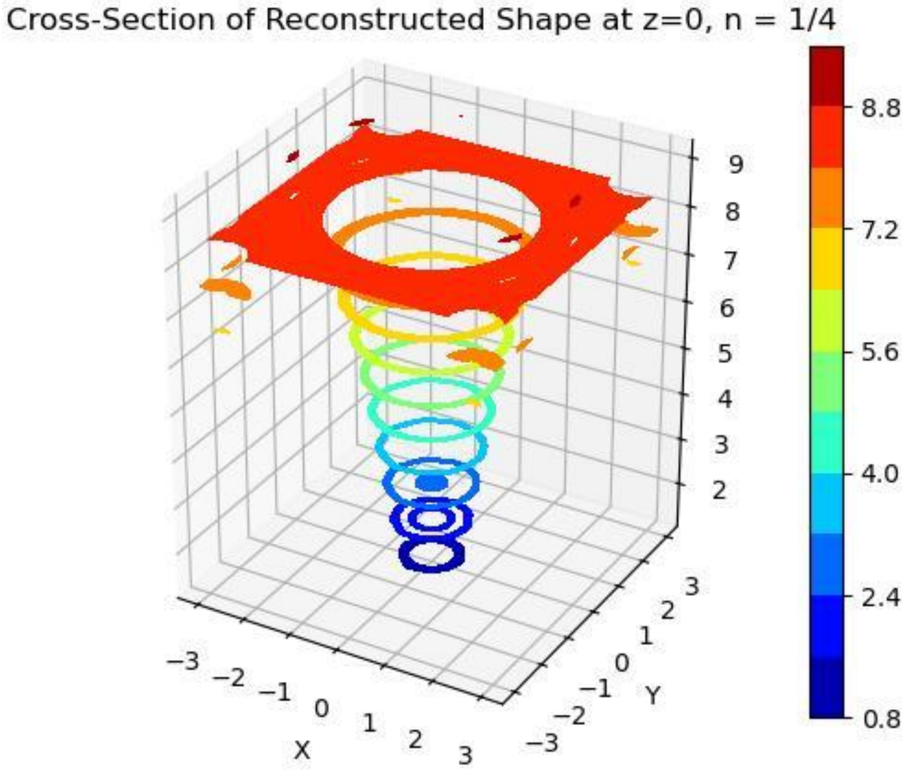
Shape: a unit sphere located at the origin with $n = 1/4$.

Far-field by FEM-BEM.   **Tikhonov** Regularization ($\alpha = 10^{-10}$)

Number of incident angles = 81   Number of scattered angles = 81







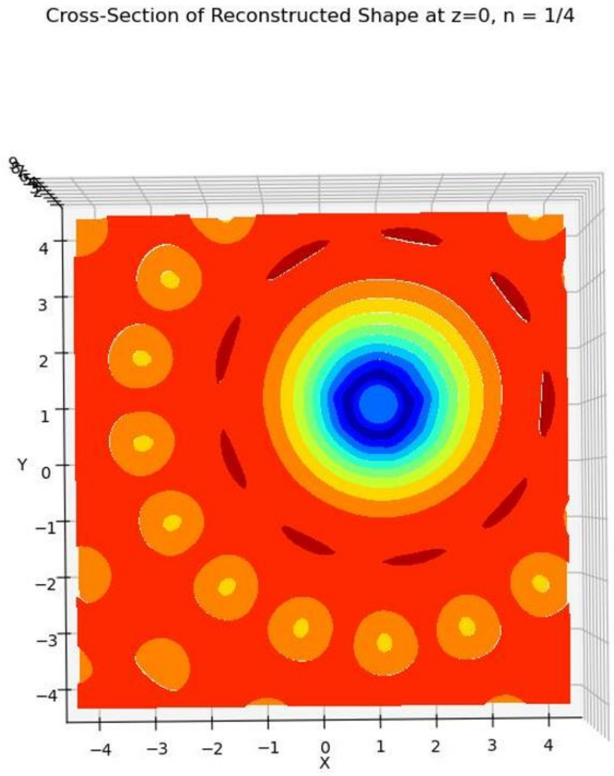
The figure shows a slice at $z = 0$.

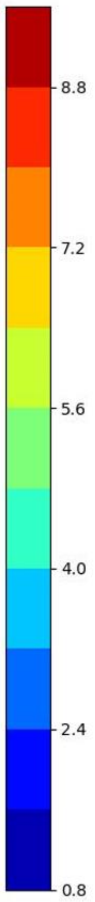
LSM for Shape Reconstruction – 3D

Shape: A unit sphere located at (1,1) with $n = 1/4$.

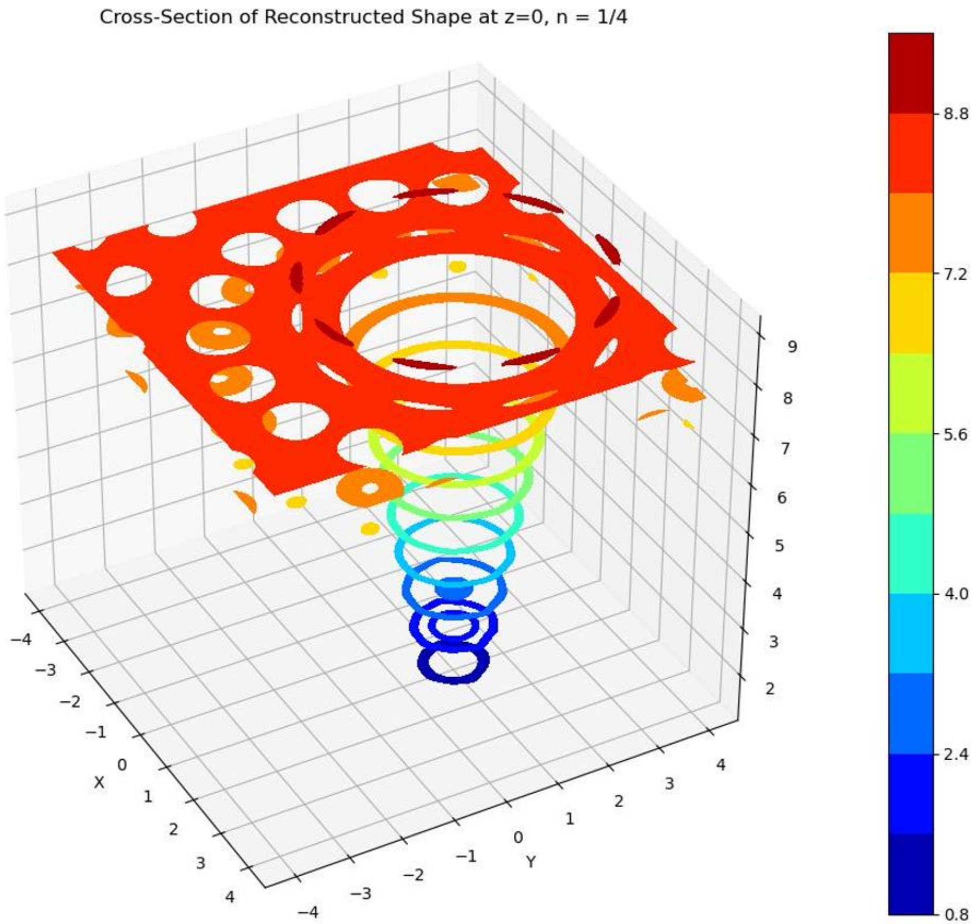
Far-field by FEM-BEM.   **Tikhonov** Regularization ($\alpha = 10^{-10}$)

Number of incident angles = 81   Number of scattered angles = 81





The figure shows a slice at z=0.

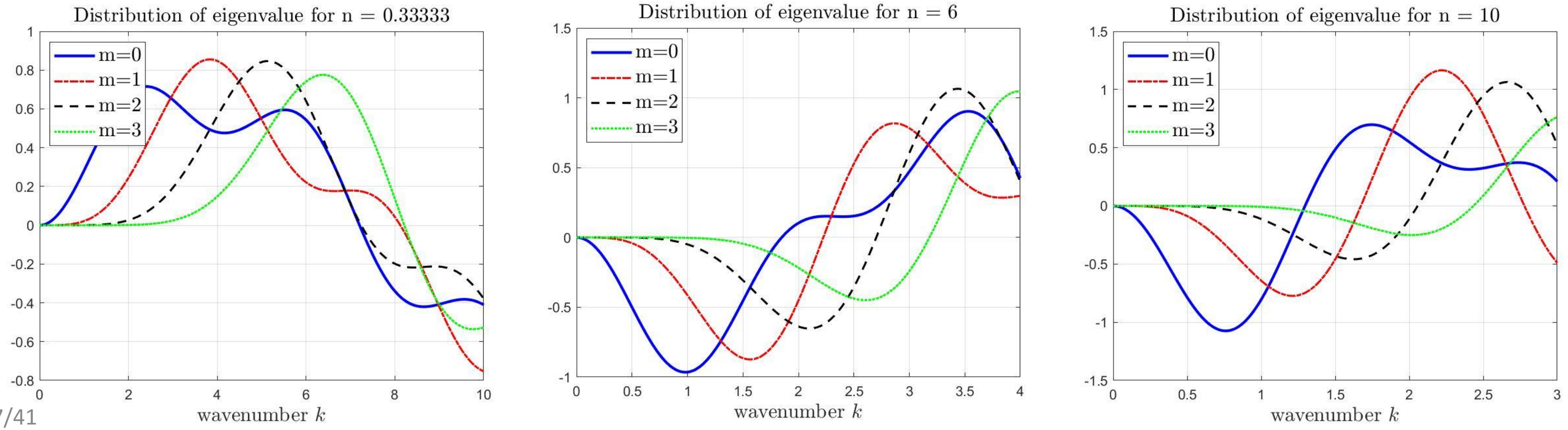


Extraction of the Smallest ITE from Far-field – 2D

- **Shape:** unit circle centered at the origin
- **Analytical solution:** for a circle with constant $n(x)$ and radius r, the first transmission eigenvalue is the lowest positive for which [Cakoni et al. 2007]:

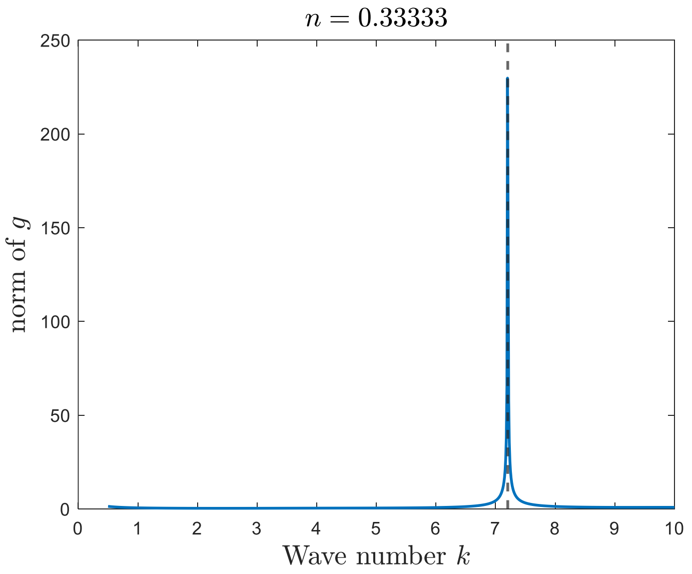
$$\det\begin{pmatrix} J_m(kr) & J_m(k\sqrt{n}r) \\ J_m'(kr) & J_m'(k\sqrt{n}r) \end{pmatrix} = 0, \qquad m = 0, 1, \ldots$$

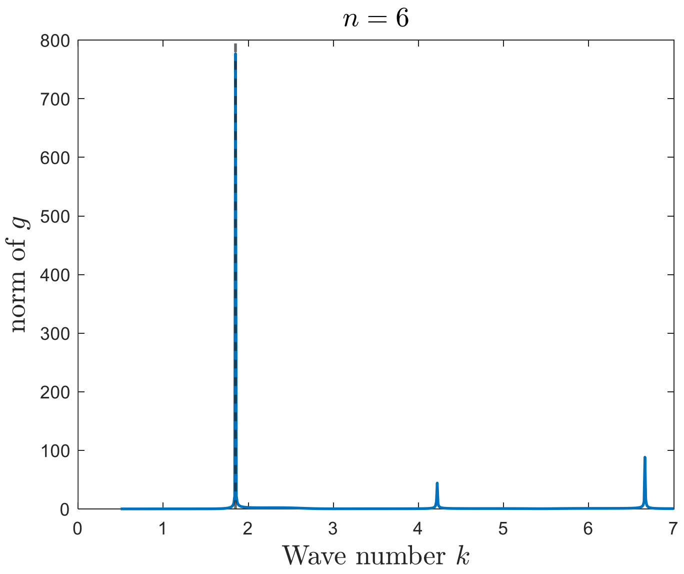
where J_m are the Bessel functions of the first kind.

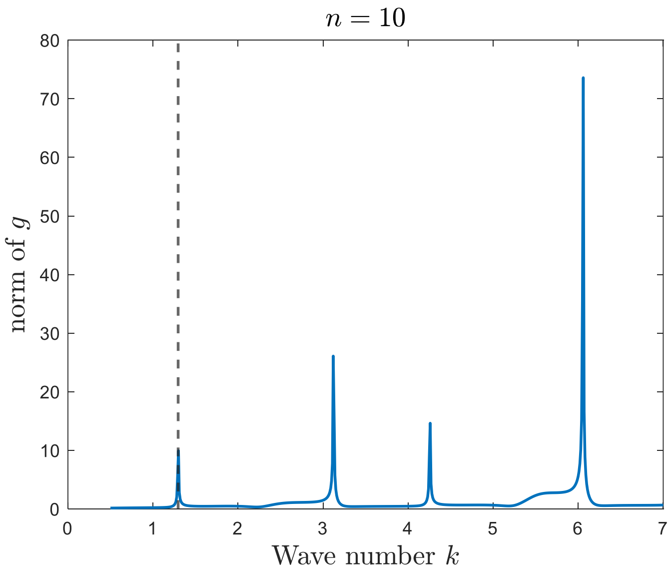


Extraction of the Smallest ITE from Far-field – 2D

- Using LSM (slide 17)
- **Far-field** by μ-diff toolbox.   **Tikhonov** Regularization ($\alpha = 10^{-10}$)
- Number of incident angles = 61;   Number of scattered angles = 61



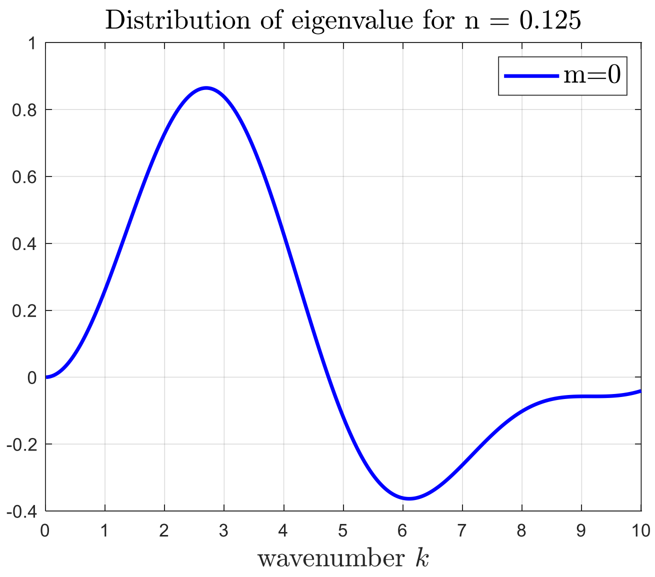


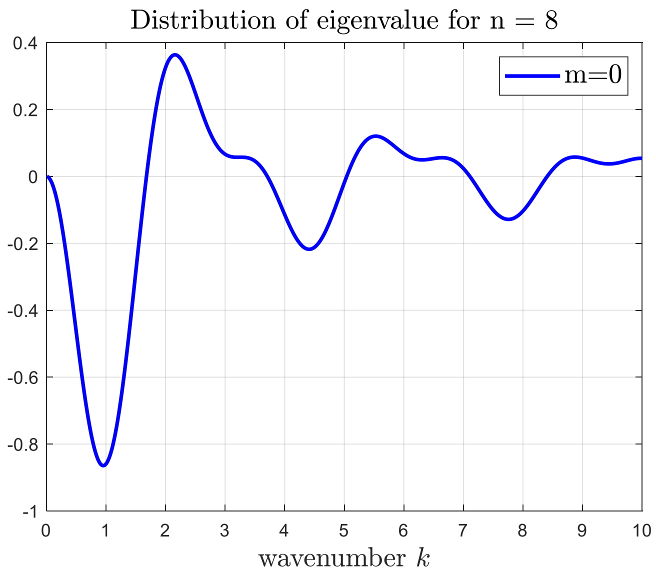


Extraction of the Smallest ITE from Far-field – 3D

- **Shape:** a unit sphere centered at the origin
- **Analytical solution:** for a sphere with constant $n(x)$ and radius r, the first transmission eigenvalue is obtained from [Leung and Colton 2012]:

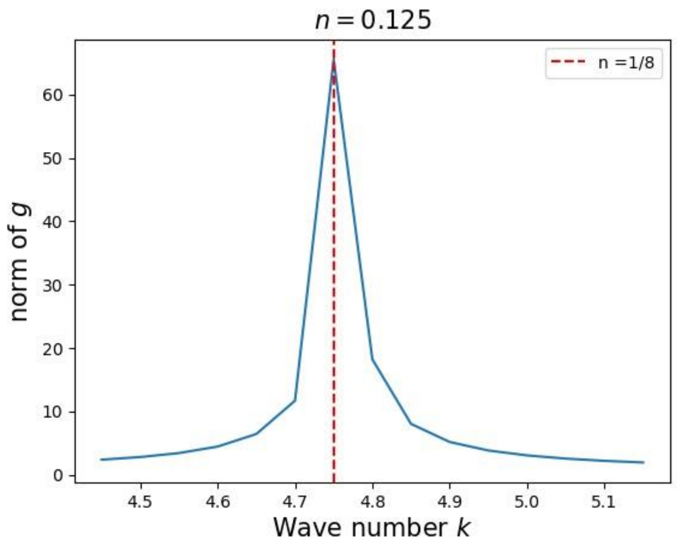
$$d(k) := \frac{\sqrt{n}\sin(k)\cos(\sqrt{n}k) - \cos(k)\sin(\sqrt{n}k)}{\sqrt{n}k} = 0$$

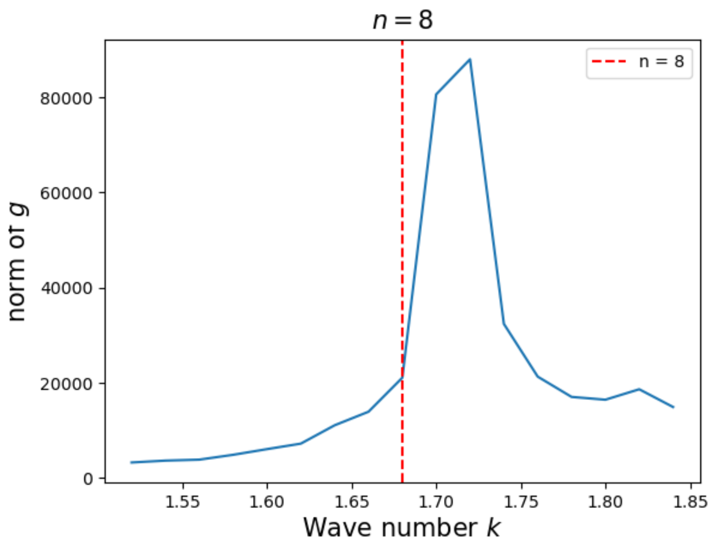




Extraction of the Smallest ITE from Far-field – 3D

- Using LSM (slide 17)
- **Far-field** by FEM-BEM. **Tikhonov** Regularization ($\lambda = 10^{-12}$)
- Number of incident angles = 81; Number of scattered angles = 81





Estimation of n from the smallest real ITE – 2D

- For unite circle, now we construct the basis $\{\varphi_i\}_{i=1}^{N}$ for the clamped-plate problem (slide 19) with $L = \Delta\Delta$ (bilaplacian operator). In polar coordinates [Chakraverty 2008]:

$$\varphi_i(r,\theta) = \left[a_i J_m\left(\mu_{m,j} r\right) + I_m\left(\mu_{m,j} r\right)\right] \cos(i\theta)$$

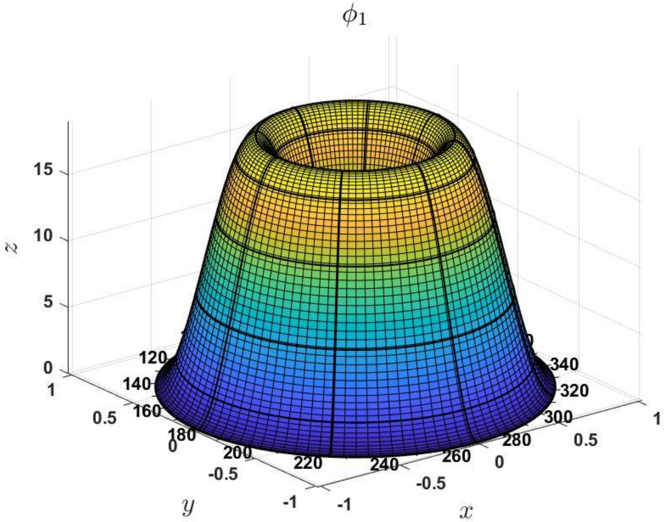
where I_m is modified (hyperbolic) Bessel function of first kind and eigenvalues $\mu_{m,j}$ can be computed from the relation:

$$\det\begin{pmatrix} J_m(\mu_{m,j} r) & J'_m(\mu_{m,j} r) \\ I_m(\mu_{m,j} r) & I'_m(\mu_{m,j} r) \end{pmatrix} = 0, \qquad m = 0, 1, \ldots$$

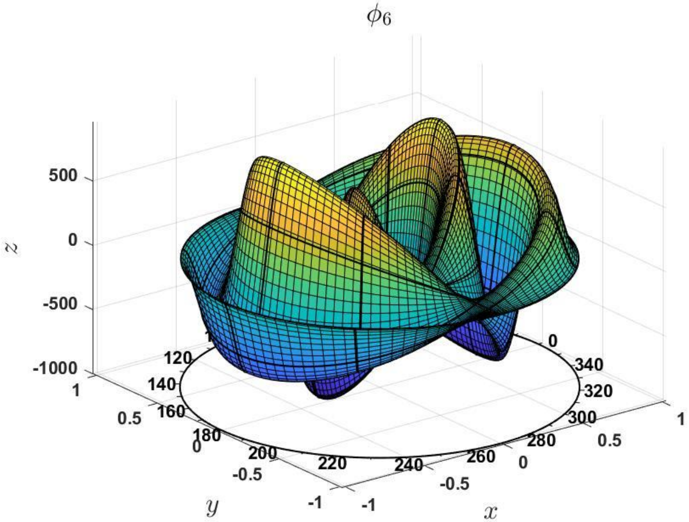
- **Note:** $\mu_{m,j}$ should be ordered and use the appropriate Bessel functions. j denotes j^{th} zero of above relation for each m.

Estimation of n from the smallest real ITE – 2D

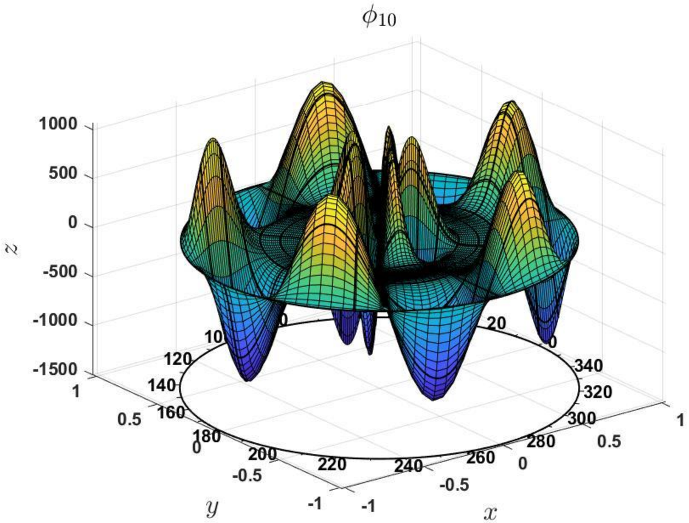
ϕ_1
$m = 0$
$\mu_{0,1}$



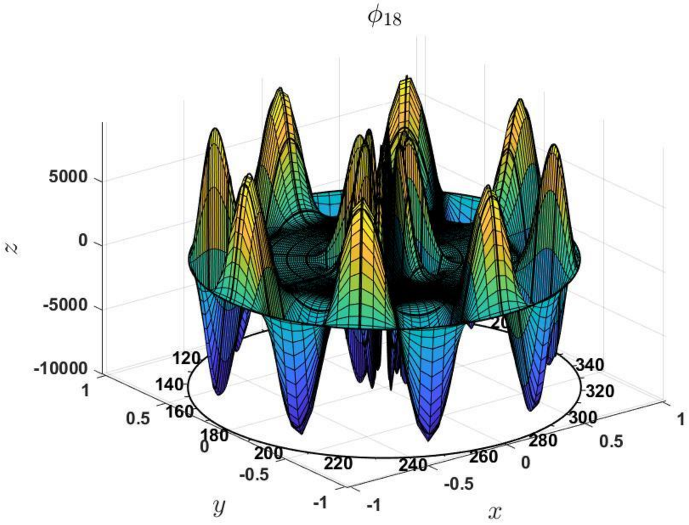
ϕ_6
$m = 1$
$\mu_{1,2}$



ϕ_{10}
$m = 5$
$\mu_{5,1}$

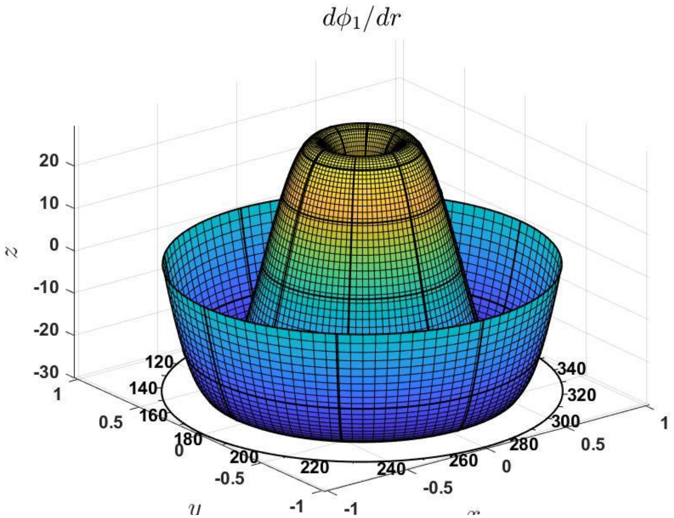


ϕ_{18}
$m = 8$
$\mu_{8,1}$

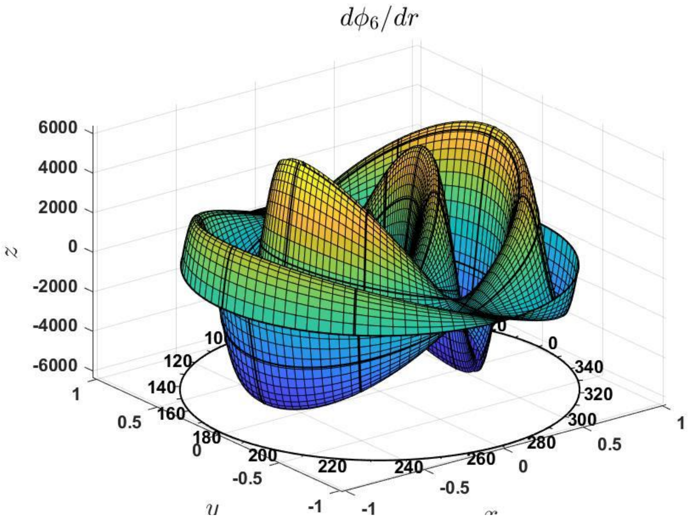


Estimation of n from the smallest real ITE – 2D

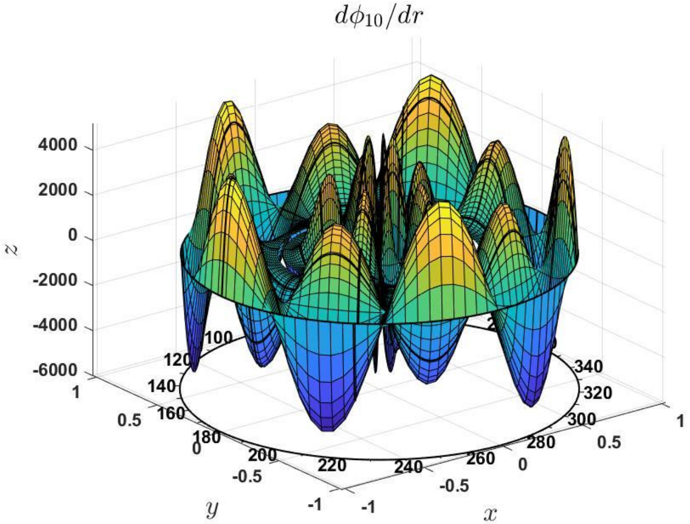
$d\phi_1/dr$
$m = 0$
$\mu_{0,1}$



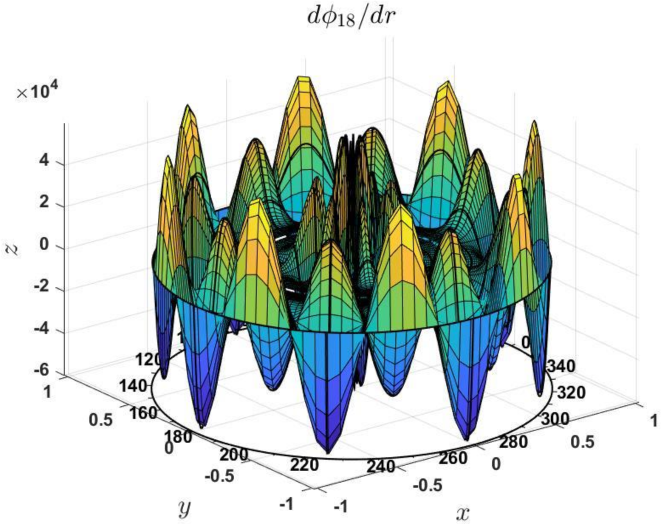
$d\phi_6/dr$
$m = 1$
$\mu_{1,2}$



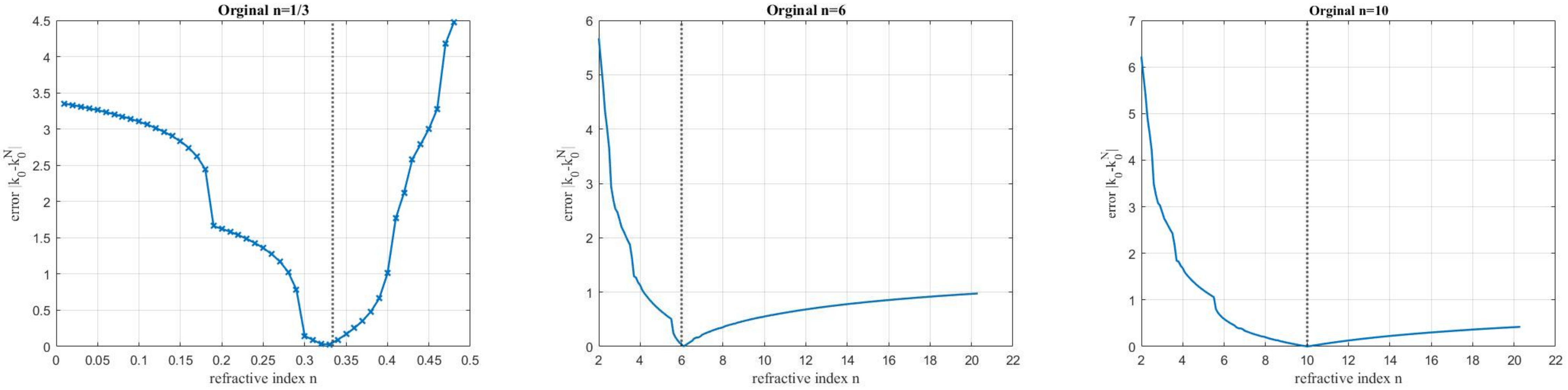
$d\phi_{10}/dr$
$m = 5$
$\mu_{5,1}$



$d\phi_{18}/dr$
$m = 8$
$\mu_{8,1}$



Estimation of n from the smallest real ITE – 2D



Estimation of n from the smallest real ITE – 3D

- For unite sphere, now we construct the basis $\{\varphi_i\}_{i=1}^{N}$ for the bilaplacian operator (slide 19) with $L = \Delta\Delta$. In spherical coordinates (assuming azimuthal symmetry):

$$\varphi_i(r,\theta) = \left[a_i j_m\left(\mu_{m,j} r\right) + i_m\left(\mu_{m,j} r\right)\right] P_m(\cos(\theta))$$

where j_m, i_m , and P_m are the spherical Bessel functions of the first kind, the Modified spherical Bessel functions of the first kind, and Legendre polynomials.

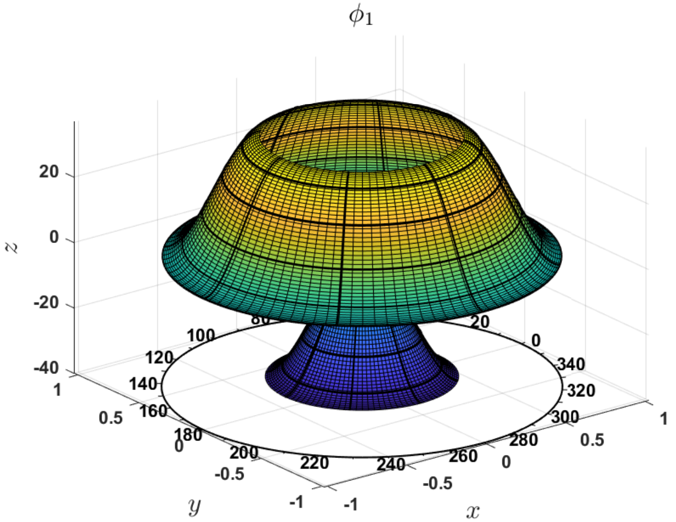
- Eigenvalues $\mu_{m,j}$ can be computed from the relation:

$$\det\begin{pmatrix} j_m(\mu_{m,j} r) & j_m'(\mu_{m,j} r) \\ i_m(\mu_{m,j} r) & i_m'(\mu_{m,j} r) \end{pmatrix} = 0, \qquad\qquad m = 0, 1, \ldots$$

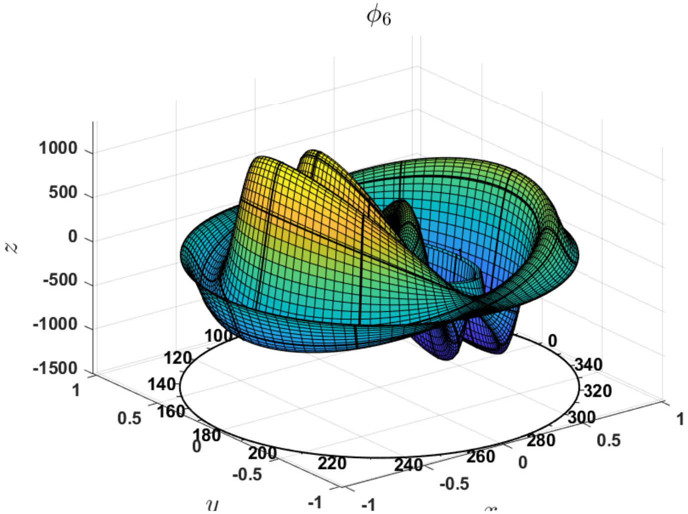
- **Note:** $\mu_{m,j}$ should be ordered and use the appropriate Bessel functions. j denotes j^{th} zero of above relation for each m.

Estimation of n from the smallest real ITE – 3D

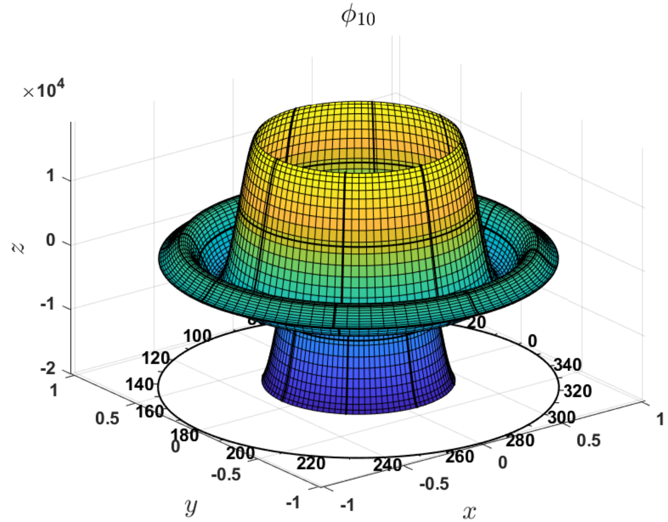
ϕ_1
$m = 0$
$\mu_{0,1}$



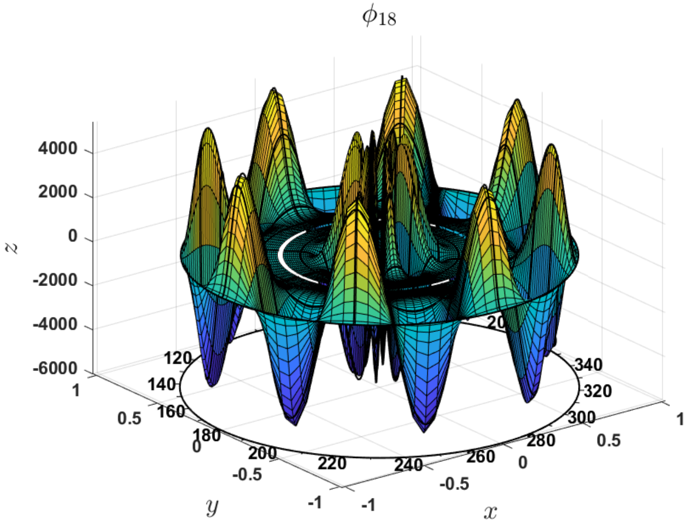
ϕ_6
$m = 1$
$\mu_{1,2}$



ϕ_{10}
$m = 5$
$\mu_{5,1}$

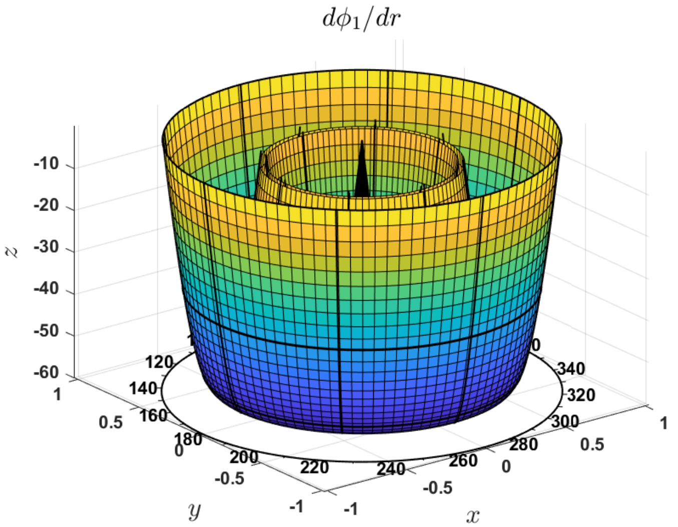


ϕ_{18}
$m = 8$
$\mu_{8,1}$

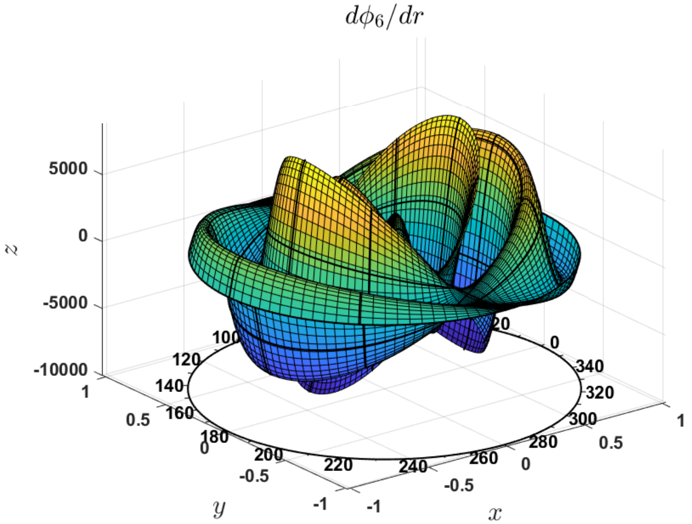


Estimation of n from the smallest real ITE – 3D

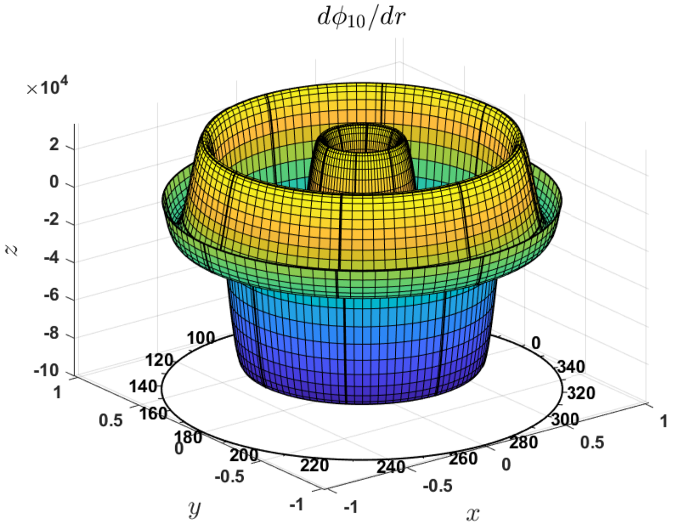
$d\phi_1/dr$
$m = 0$
$\mu_{0,1}$



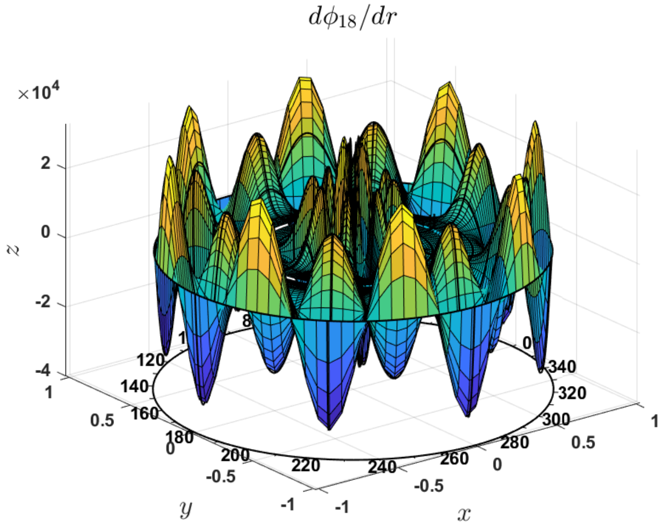
$d\phi_6/dr$
$m = 1$
$\mu_{1,2}$



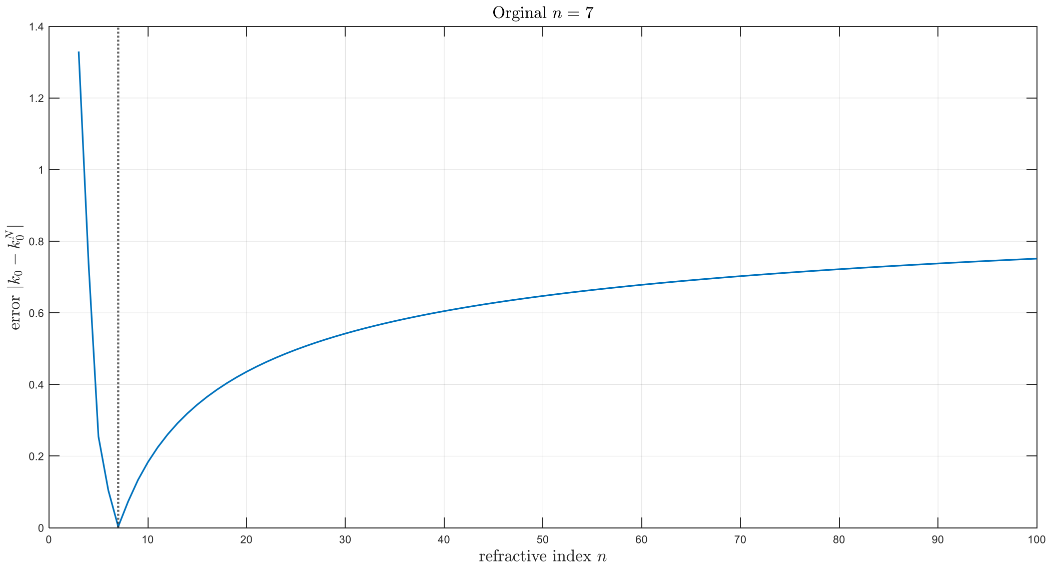
$d\phi_{10}/dr$
$m = 5$
$\mu_{5,1}$

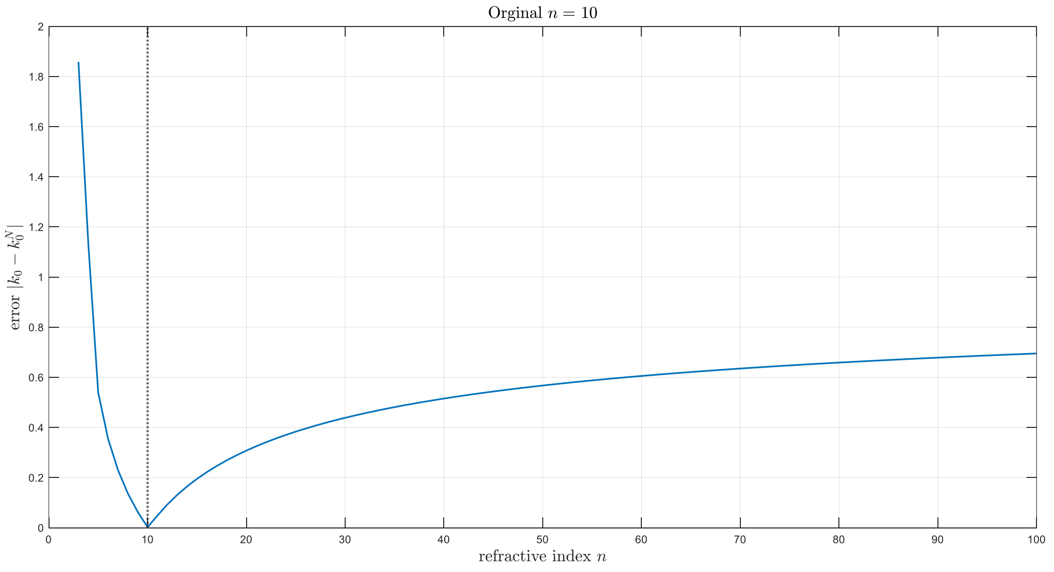


$d\phi_{18}/dr$
$m = 8$
$\mu_{8,1}$



Estimation of n from the smallest real ITE – 3D





Conclusion

Conclusion

- In this project we established the fundamental basis for the reconstruction of the acoustic refractive index from ITEP.

- The project has two main parts: A) Extraction of the ITEs from far-field data; and B) Estimation of the acoustic refractive index ($n(x)$) from the smallest ITE.

- 2D cases were the implementation of the previous works, **BUT** 3D cases were completely new and implemented during this project.

- For the first time, the FEM-BEM method has been used for this problem.

References

- Colton, David, and Andreas Kirsch. "A simple method for solving inverse scattering problems in the resonance region." Inverse problems 12.4 (1996): 383.
- Tacchino, Andrea, Joe Coyle, and Michele Piana. "Numerical validation of the linear sampling method." Inverse Problems 18.3 (2002): 511.
- Cakoni, Fioralba, David Colton, and Peter Monk. "On the use of transmission eigenvalues to estimate the index of refraction from far field data." Inverse Problems 23.2 (2007): 507.
- Chakraverty, Snehashish. Vibration of plates. CRC press, 2008.
- Hansen, Per Christian. Discrete inverse problems: insight and algorithms. Society for Industrial and Applied Mathematics, 2010.
- Logg, Anders, Kent-Andre Mardal, and Garth Wells, eds. Automated solution of differential equations by the finite element method: The FEniCS book. Vol. 84. Springer Science & Business Media, 2012.
- Leung, Yuk-J., and David Colton. "Complex transmission eigenvalues for spherically stratified media." Inverse Problems 28.7 (2012): 075005.
- Thierry, Bertrand, et al. "μ-diff: An open-source Matlab toolbox for computing multiple scattering problems by disks." Computer Physics Communications 192 (2015): 348-362.
- Pallikarakis, Nikolaos. The inverse spectral problem for the reconstruction of the refractive index from the interior transmission problem. Diss. 2017.
- Colton, David L., Rainer Kress, and Rainer Kress. Inverse acoustic and electromagnetic scattering theory. 4th Ed. Vol. 93. Berlin: Springer, 2019.