How to get the general solution of the heat equation

Initial Conditions: $\dfrac{\partial u}{\partial t} = \beta \dfrac{\partial^2 u}{\partial x^2}, t > 0, 0 < x < L$

$u(0,t) = u(L,t) = 0, t > 0$

Let's assume that we can separate the variables in u(x,t):

$u(x,t) = X(x)T(t)$

Putting *X(x)* and *T(t)* back together, we get:

$\lambda = (\frac{n\pi}{L})^2, n = 1, 2, \ldots$

$X_n(x) = a_n \sin(\frac{n\pi x}{L})$

$T_n(t) = b_n e^{-\beta(\frac{n\pi}{L})^2 t}$

Putting *X(x)* and *T(t)* back together, we get:

$u_n(x,t) := X_n(x)T_n(t) = c_n e^{-\beta(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi x}{L})$

Finally, taking the linear combination, we get the general solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi x}{L})$$

More details

Since $\dfrac{\partial u}{\partial t} = \beta \dfrac{\partial^2 u}{\partial x^2}$

By separating the variables, we get:

$X''(x) + \lambda X(x) = 0$

where λ can be any constant

With process of elimination, we get:

$\lambda > 0$

The roots of the auxiliary equation are:

$\pm i\sqrt{\lambda}$

Thus, a general solution to $X''(x) + \lambda X(x) = 0$ is:

$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

The boundary conditions become:

$X(x) = X(L) = 0$

Solving $X''(x) + \lambda X(x) = 0, X(x) = X(L) = 0$

we get the system:

$C_1 = 0$

$C_1 \cos\sqrt{\lambda}L + C_2 \sin\sqrt{\lambda}L = 0$

the system reduces to solving

$C_2 \sin\sqrt{\lambda}L = 0$

which gives:

$\lambda = (\frac{n\pi}{L})^2$ (eigenvalue)

$X_n(x) = C_n \sin(\frac{n\pi x}{L})$ (eigenfunctions)

where the C_n's are arbitrary nonzero constants

Having determined that

$\lambda = (\frac{n\pi}{L})^2$

and that

$T'(t) = -\lambda\beta T(t)$

we get this linear first-order equation

$T'(t) + \beta(\frac{n\pi}{L})^2 T(t) = 0$

and a general solution is

$T_n(t) = b_n e^{-\beta(\frac{n\pi}{L})^2 t}$

for each *n=1, 2, 3,...*

More details

We assume that the variables can be separated:

$u(x,t) = X(x)T(t)$

$\dfrac{\partial u}{\partial t} = X(x)T'(t) \qquad \dfrac{\partial^2 u}{\partial x^2} = X''(x)T(t)$

Using substitution, we get:

$X(x)T'(t) = \beta X''(x)T(t)$

and separating the variables, we get:

$\dfrac{T'(t)}{\beta T(t)} = \dfrac{X''(x)}{X(x)}$

$\dfrac{T'(t)}{\beta T(t)}, \dfrac{X''(x)}{X(x)}$ both equal the constant -λ

$\dfrac{X''(x)}{X(x)} = -\lambda$ and $\dfrac{T'(t)}{\beta T(t)} = -\lambda$

$X''(x) = -\lambda X(x)$ and $T'(t) = -\lambda\beta T(t)$

We eliminated the options *λ < 0* and *λ = 0* because they wouldn't have given us any eigenvalue or eigenfunction, as shown here:

For *λ < 0* :
In this case, the roots of the auxiliary equation are $\pm\sqrt{\lambda}$ so a general solution to the differential equation is:

$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{\sqrt{-\lambda}x}$

To determine C_1 and C_2, we appeal to the boundary conditions:

$X(0) = C_1 + C_2 = 0$

$X(L) = C_1 e^{\sqrt{-\lambda}L} + C_2 e^{-\sqrt{-\lambda}L} = 0$

Therefore C_1, and hence C_2, is zero.
Thus, there is no nontrivial solution.

No eigenvalue
No eigenfunction

For *λ = 0* :
In this case, *r=0* is a repeated root of the auxiliary equation, and a general solution to the differential equation is:

$X(x) = C_1 + C_2 x$

The boundary conditions give:

C_1=0 and *C_1+ C_2 L=0*, which imply that *C_1= C_2 =0*

Thus, there is no nontrivial solution.

No eigenvalue
No eigenfunction

The auxiliary equation for $X''(x) + \lambda X(x) = 0$ is:

$r^2 + \lambda = 0$

Solving this, we get:

$r = \sqrt{-\lambda} = \pm i\sqrt{\lambda}$

For complex conjugate roots $\alpha \pm i\beta$

two solutions are $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$

and a general solution is $y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$

here $\alpha = 0, \beta = \sqrt{\lambda}$

We assume that we can separate the variables in u(x,t):

$u(x,t) = X(x)T(t)$

therefore, the boundary conditions $u(0,t) = u(L,t) = 0, t > 0$ give us:

$X(0)T(t) = 0, X(L)T(t) = 0, t \geq 0$

Ignoring the trivial solution T(t)=0 , we get

X(0) = X(L) = 0

When solving $X''(x) + \lambda X(x) = 0, X(0) = X(L) = 0$

Substituting *x=0* and *x=1* into our equation for *X(x)* :

$X(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 0$

and

$X(L) = c_1 \cos\sqrt{\lambda}L + c_2 \sin\sqrt{\lambda}L = 0$

Therefore, we get the system:

C_1=0

$C_1 \cos\sqrt{\lambda}L + C_2 \sin\sqrt{\lambda}L = 0$

When solving $C_2 \sin\sqrt{\lambda}L = 0$

Since $sin\theta = 0$ when $\theta = \pi n, n = 0, 1, 2, 3, \ldots$

Then: $\sqrt{\lambda}L = \pi n$

$\sqrt{\lambda} = \frac{\pi n}{L}$

$\lambda = (\frac{\pi n}{L})^2$

$X_n(x) = C_n \sin(\frac{n\pi x}{L})$

We assume that the variables can be separated:

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$X''(x) = -\lambda X(x)$ and $T'(t) = -\lambda\beta T(t)$