How to get the solution in real form from the solution in complex form

Initial conditions:

$$ay'' + by' + cy = 0$$
$$ar^2 + br + c = 0$$
$$b^2 - 4ac < 0$$
$$\Delta < 0$$
$$r_1 = \alpha + i\beta \quad \text{and} \quad r_2 = \alpha - i\beta$$

The solution in complex form is: $e^{r_1 t}$ and $e^{r_2 t}$

$$y(t) = c_1 e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t}$$

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{(\alpha+i\beta)t} = e^{\alpha t + i\beta t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}(\cos\beta t + i\sin\beta t) = u(t) + iv(t)$$

if $u(t) + iv(t)$ is a solution to $ay'' + by' + cy = 0$
Then $u(t)$ and $v(t)$ are also solutions.

The solution in real form is: $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$

$$y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$$

More details

Proof:

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2} + ... + \frac{(i\theta)^n}{n} + ... \\ &= 1 + (i\theta) - \frac{\theta^2}{2} - \frac{\theta^3}{3} + \frac{\theta^4}{4} + \frac{\theta^5}{5} + ... \\ &= 1 + (i\theta) - \frac{\theta^2}{2} - \frac{\theta^3}{3} + \frac{\theta^4}{4} + \frac{\theta^5}{5} + ... \\ &= (1 - \frac{\theta^2}{2} + \frac{\theta^4}{4} + ...) + i(\theta - \frac{\theta^3}{3} + \frac{\theta^5}{5}) + ... \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Proof:

Assume that $z = u + iv$, $az'' + bz' + cz = 0$
and hence

$$a(u'' + iv'') + b(u' + iv') + c(u + iv) = 0$$
$$(au'' + bu' + cu) + i(av'' + bv' + cv) = 0$$

But a complex number is zero if and only if its real and imaginary parts are both zero . Thus we must have: $au'' + bu' + cu = 0$ and $av'' + bv' + cv) = 0$

so u(t) and v(t) are both solutions with real values.

More details

Recall:

$$i^2 = -1$$
$$i = \sqrt{(-1)}$$
$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2} + ... + \frac{(i\theta)^n}{n} + ...$$
$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4} + ... + ...$$
$$\sin\theta = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} + ...$$

Recall:

If z is a complex number,
then $z = u + iv$
with *u*: the real part
v: the imaginary part