13 Vector Functions

13.1 Vector Functions and Space Curves

Context

- Vector-Valued Functions
- Limits and Continuity
- Space Curves
- Use Technology to Draw Space Curves

Vector-Valued Functions

Vector-Valued Functions (1 of 2)

In general, a function is a rule that assigns to each element in the domain an element in the range.

A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors.

We are most interested in vector functions **r** whose values are three-dimensional vectors.

This means that for every number t in the domain of **r** there is a unique vector in V_3 denoted by $\mathbf{r}(t)$.

Vector-Valued Functions (2 of 2)

If *f*(*t*), *g*(*t*), and *h*(*t*) are the components of the vector **r**(*t*), then *f*, *g*, and *h* are real-valued functions called the **component functions** of **r** and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We use the letter *t* to denote the independent variable because it represents time in most applications of vector functions.

Example 1

If $\mathbf{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$

then the component functions are

$$f(t) = t^3 \quad g(t) = \ln(3-t) \quad h(t) = \sqrt{t}$$

By our usual convention, the domain of **r** consists of all values of *t* for which the expression for **r**(*t*) is defined.

The expressions t^3, ln(3 − *t*), and $\sqrt{t}$ are all defined when $3 - t > 0$ and $t \geq 0$.

Therefore the domain of **r** is the interval [0, 3).

Limits and Continuity

Limits and Continuity (1 of 2)

The **limit** of a vector function **r** is defined by taking the limits of its component functions as follows.

1 If $\mathbf{r}(t) = \langle f(t),\ g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t),\ \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Limits of vector functions obey the same rules as limits of real-valued functions.

单选题 1分

设置

MCQ: Find $\lim_{t\to 0} \mathbf{r}(t)$, where $\mathbf{r}(t) = (1+t^3)\mathbf{i} + te^{-t}\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$

A $\mathbf{i}+\mathbf{j}+\mathbf{k}$

B $\mathbf{i}+\mathbf{k}$

C $\mathbf{i}+\mathbf{j}$

D $\mathbf{i}$

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Example 2

Find $\lim_{t \to 0} \mathbf{r}(t)$, where $\mathbf{r}(t) = (1+t^3)\mathbf{i} + te^{-t}\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$.

Solution:

According to Definition 1, the limit of **r** is the vector whose components are the limits of the component functions of **r**:

$$\lim_{t \to 0} \mathbf{r}(t) = \left[\lim_{t \to 0}(1+t^3)\right]\mathbf{i} + \left[\lim_{t \to 0} te^{-t}\right]\mathbf{j} + \left[\lim_{t \to 0} \frac{\sin t}{t}\right]\mathbf{k}$$

$$= \mathbf{i} + \mathbf{k}$$

Limits and Continuity (2 of 2)

A vector function **r** is **continuous at *a*** if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

In view of Definition 1, we see that **r** is continuous at *a* if and only if its component functions *f*, *g*, and *h* are continuous at *a*.

多选题 1分

设置

MAQ:

Let $\boldsymbol{r}(t) = \left(t, t^2 sin\frac{1}{t}\right), \boldsymbol{u}(t) = \frac{\boldsymbol{r}(t)}{|\boldsymbol{r}(t)|}$ $(t \neq 0)$

Decide, as $t \to 0$:

A $\lim_{t\to 0} \boldsymbol{r}(t) = \mathbf{0}$

B For any sequence $t_k \downarrow 0$(approaching 0 from the right), $\boldsymbol{u}(t_k) \to (1,0)$

C For any sequence $t_k \uparrow 0$(approaching 0 from the left), $\boldsymbol{u}(t_k) \to (-1,0)$

D $\lim_{t\to 0} \boldsymbol{u}(t)$ does not exist

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Example

Solution:

$$|\boldsymbol{r}(t)| = |t|\sqrt{1+t^2(\sin\frac{1}{t})^2}$$

$$\boldsymbol{u}(t) = \left(\frac{\operatorname{sgn}(t)}{\sqrt{1+t^2\left(\sin\frac{1}{t}\right)^2}}, \frac{|t|sin\frac{1}{t}}{\sqrt{1+t^2\left(\sin\frac{1}{t}\right)^2}}\right)$$

For option A $\lim\limits_{t\to 0} t^2 sin\frac{1}{t} \leq \lim\limits_{t\to 0} t^2 = 0$, it's true

For option B,C, $\boldsymbol{u}(t)$, from $\boldsymbol{u}(t)$ above it's true

For option D, from option B, C, has distinct one-sided limits, it's true

Space Curves

Space Curves (1 of 4)

There is a close connection between continuous vector functions and space curves.

Suppose that *f*, *g*, and *h* are continuous real-valued functions on an interval *I*.

Then the set *C* of all points (*x*, *y*, *z*) in space, where

$$\mathbf{2} \quad x = f(t) \quad y = g(t) \quad z = h(t)$$

and *t* varies throughout the interval *I*, is called a **space curve**.

Space Curves (2 of 4)

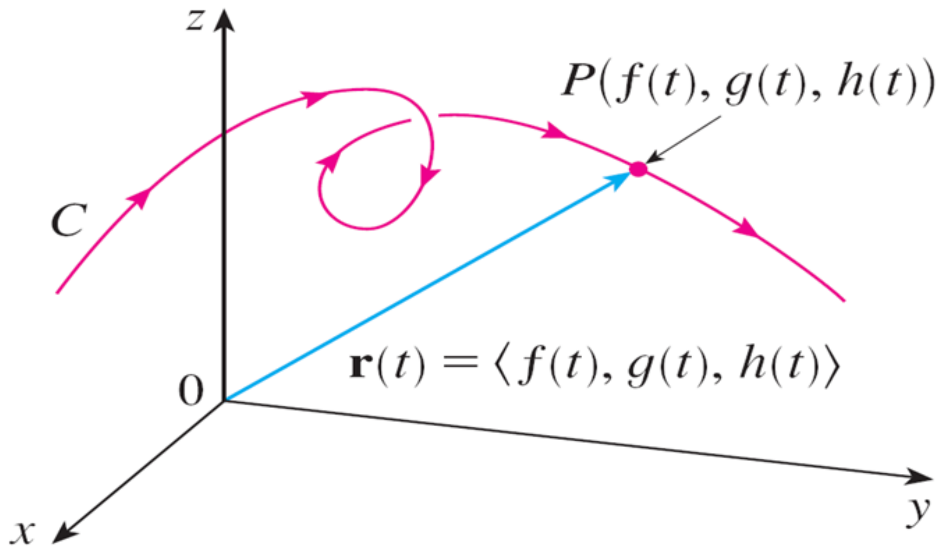
The equations in (2) are called **parametric equations of *C*** and *t* is called a **parameter**.

We can think of *C* as being traced out by a moving particle whose position at time *t* is $(f(t), g(t), h(t))$.

If we now consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on *C*.

Space Curves (3 of 4)

Thus any continuous vector function **r** defines a space curve *C* that is traced out by the tip of the moving vector **r**(*t*), as shown in Figure 1.



C is traced out by the tip of a moving position vector **r**(*t*).

Figure 1

Example 4

Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\boldsymbol{k}$$

Solution:

The parametric equations for this curve are

$$x = \cos t \quad y = \sin t \quad z = t$$

Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the curve must lie on the circular cylinder $x^2 + y^2 = 1$.

The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy-plane.

Example 4 – Solution

(The projection of the curve onto the *xy*-plane has vector equation $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$.) Since $z = t$, the curve spirals upward around the cylinder as t increases. The curve, shown in Figure 2, is called a **helix**.

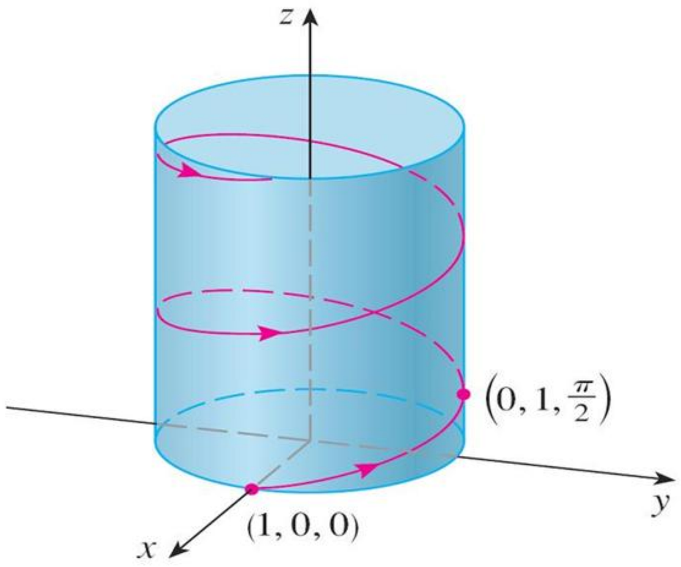


Figure 2

Space Curves (4 of 4)

The corkscrew shape of the helix in Example 4 is familiar from its occurrence in coiled springs.

It also occurs in the model of DNA (deoxyribonucleic acid, the genetic material of living cells).

In 1953 James Watson and Francis Crick showed that the structure of the DNA molecule is that of two linked, parallel helixes that are intertwined as in Figure 3.

A double helix

Figure 3

MAQ: Let $\mathbf{r}(t) = \langle \cos(t), \sin(t), \sin(2t) \rangle$, $t \in \mathbb{R}$ .Which statements are true?

A The trajectory is closed

B $\mathbf{r}(\mathrm{t})$ is **injective** (different t give different points)

C The projection onto the xz-plane is $z = 2x\sqrt{1 - x^2}$

D The curve has no intersection with the z-axis

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Example

Solution:

For option A, $\boldsymbol{r}(t) = \boldsymbol{r}(t + 2\pi)$, it's true, option B is flase.
For option C, it's $z^2 = 4x^2(1 - x^2)$ $z = -2x\sqrt{1 - x^2}$ can not be ignored, false.
For option D, points on the z-axis have the form (0, 0, z), it's impossible here.

Using Technology to Draw Space Curves

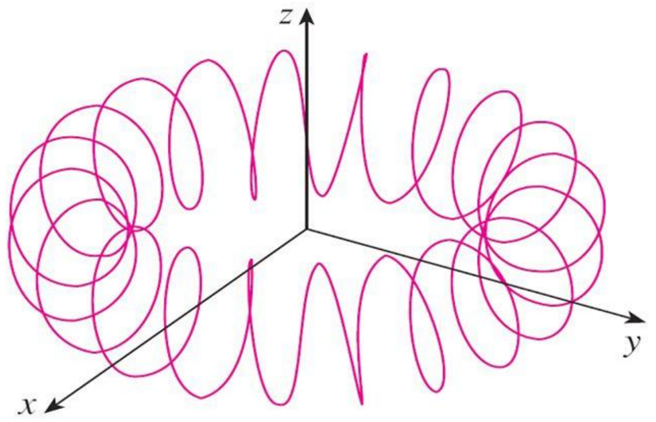
Using Technology to Draw Space Curves (1 of 7)

Space curves are inherently more difficult to draw by hand than plane curves; for an accurate representation we need to use technology.

For instance, Figure 8 shows a computer-generated graph of the curve with parametric equations

$$x = (4 + \sin 20t)\cos t$$
$$y = (4 + \sin 20t)\sin t$$
$$z = \cos 20t$$

It's called a **toroidal spiral** because it lies on a torus.



A toroidal spiral

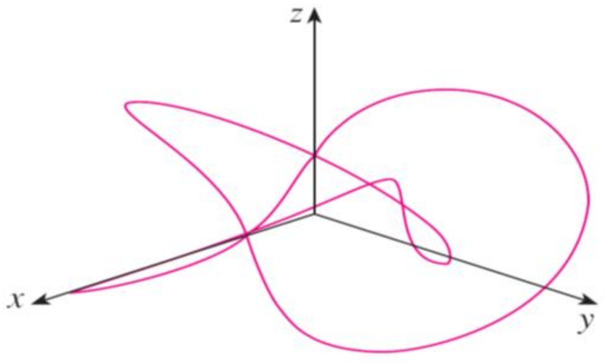
Figure 8

Using Technology to Draw Space Curves (2 of 7)

Another interesting curve, the **trefoil knot**, with equations

$$
\begin{aligned}
x &= (2 + \cos 1.5t) \cos t \\
y &= (2 + \cos 1.5t) \sin t \\
z &= \sin 1.5t
\end{aligned}
$$

is graphed in Figure 9. It wouldn't be easy to plot either of these curves by hand.



A trefoil knot

Figure 9

Using Technology to Draw Space Curves (3 of 7)

Even when a computer is used to draw a space curve, optical illusions make it difficult to get a good impression of what the curve really looks like. (This is especially true in Figure 9.)

The next example shows how to cope with this problem.

Example 8

Use a computer to draw the curve with vector equation $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$.

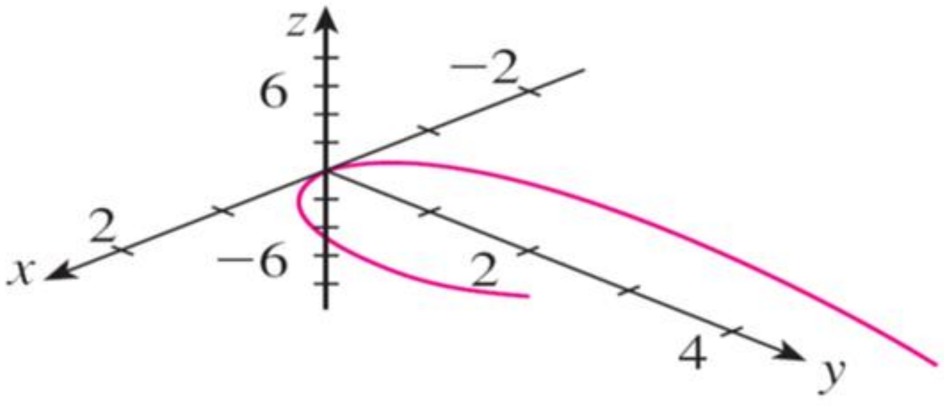
This curve is called a **twisted cubic**.

Solution:

We start by using the computer to plot the curve with parametric equations

$$x = t,\ y = t^2,\ z = t^3 \text{ for } -2 \le t \le 2.$$

The result is shown in Figure 10(a), but it's hard to see the true nature of the curve from that graph alone.



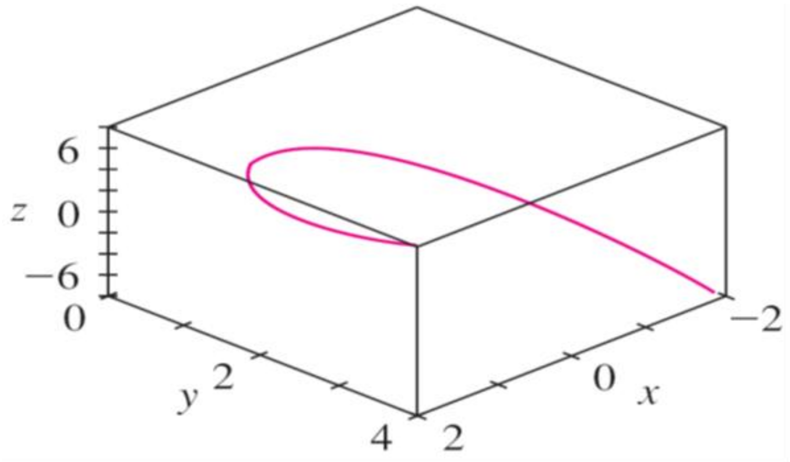
View of the twisted cubic

Figure 10(a)

Example 8 – Solution (1 of 4)

Some three-dimensional computer graphing programs allow the user to enclose a curve or surface in a box instead of displaying the coordinate axes.

When we look at the same curve in a box in Figure 10(b), we have a much clearer picture of the curve.



View of the twisted cubic

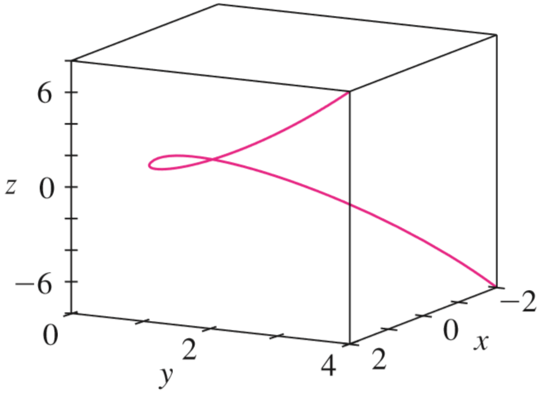
Figure 10(b)

Example 8 – Solution (2 of 4)

We can see that it climbs from a lower corner of the box to the upper corner nearest us, and it twists as it climbs.

We get an even better idea of the curve when we view it from different vantage points.

Figure 10(c) shows the result of rotating the box to give another viewpoint.



View of the twisted cubic

Figure 10(c)

Example 8 – Solution (3 of 4)

Figures 10(d), 10(e), and 10(f) show the views we get when we look directly at a face of the box.

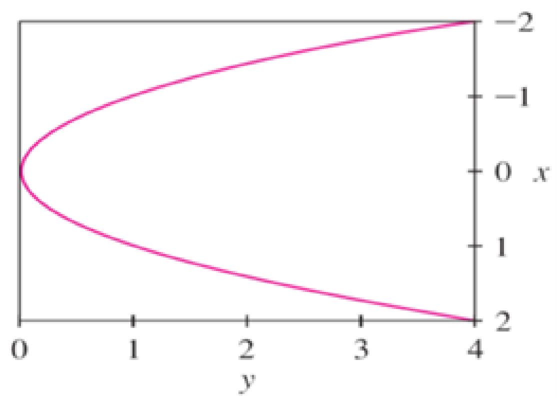


Figure 10(d)

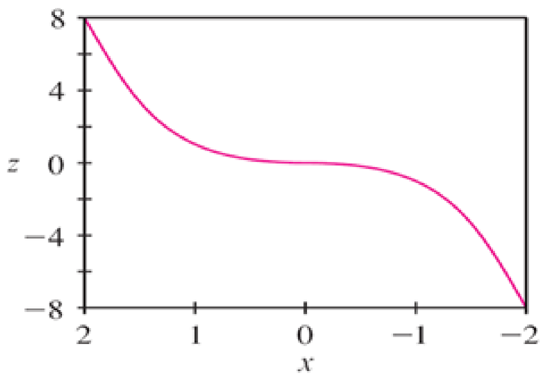


Figure 10(e)

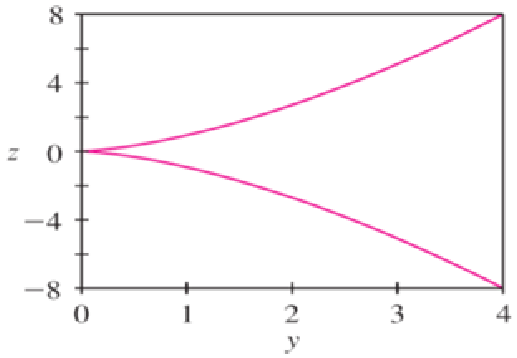


Figure 10(f)

Views of the twisted cubic

In particular, Figure 10(d) shows the view from directly above the box.

Example 8 – Solution (4 of 4)

It is the projection of the curve onto the *xy*-plane, namely, the parabola $y = x^2$.

Figure 10(e) shows the projection on the *xz*-plane, the cubic curve $z = x^3$.

It's now obvious why the given curve is called a twisted cubic.

Using Technology to Draw Space Curves (4 of 7)

Another method of visualizing a space curve is to draw it on a surface.

For instance, the twisted cubic in Example 8 lies on the parabolic cylinder $y = x^2$.

(Eliminate the parameter from the first two parametric equations, $x = t$ and $y = t^2$.)

Figure 11 shows both the cylinder and the twisted cubic, and we see that the curve moves upward from the origin along the surface of the cylinder.

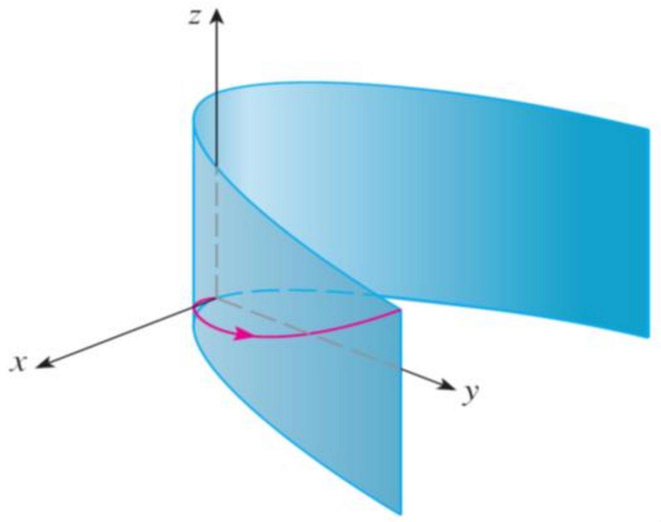


Figure 11

Using Technology to Draw Space Curves (5 of 7)

We also used this method in Example 4 to visualize the helix lying on the circular cylinder.

A third method for visualizing the twisted cubic is to realize that it also lies on the cylinder $z = x^3$.

So it can be viewed as the curve of intersection of the cylinders

$y = x^2$ and $z = x^3$. (See Figure 12.)

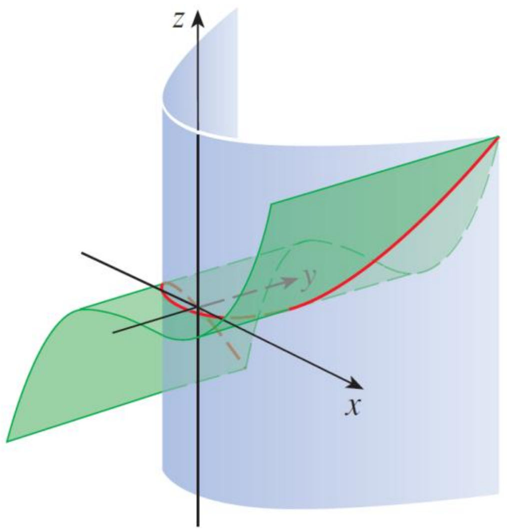


Figure 12

Recap

- Vector-Valued Functions
- Limits and Continuity
- Space Curves
- Use Technology to Draw Space Curves