Math 241 Calculus III (Prof. Honold) **Midterm Exam 1** 2022/03/16

Name: ____________________ Student No.: ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. The volume of the pyramid ("tetrahedron") with vertices $(b,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$ is equal to 1 for

   ☐ $b=\frac{3}{2}$ ☐ $b=\frac{5}{2}$ ■ $b=-\frac{3}{2}$ ☐ $b=-\frac{1}{2}$ ☐ $b=\frac{1}{2}$

2. The distance from the point $(1,0,0)$ to the line connecting the points $(0,0,1)$ and $(1,2,3)$ is

   ☐ $\frac{1}{3}\sqrt{21}$ ☐ 3 ■ $\frac{1}{3}\sqrt{17}$ ☐ $\frac{17}{9}$ ☐ $\frac{21}{9}$

3. The inverse matrix of $\begin{pmatrix} 2 & 3 & 5 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$ has the form $\begin{pmatrix} * & * & * \\ * & c & * \\ * & * & * \end{pmatrix}$ with

   ☐ $c=0$ ☐ $c=1$ ☐ $c=2$ ☐ $c=3$ ■ $c=4$

4. The reflection of $\mathbb{R}^2$ at the line $\sqrt{3}x+y=0$ is afforded by the matrix

   ☐ $\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$ ■ $\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$ ☐ $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

   ☐ $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$ ☐ $\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$

5. The maximum rank of $\mathbf{A}\in\mathbb{R}^{3\times 4}$ with all row sums and all columns sums equal to zero is

   ☐ 0 ☐ 1 ■ 2 ☐ 3 ☐ 4

6. The linear system $2x_1-x_2=x_1+ax_2+x_3=x_1-x_2+2x_3=0$ has a nonzero solution for

   ☐ $a=-\frac{1}{4}$ ■ $a=-\frac{3}{4}$ ☐ $a=\frac{1}{4}$ ☐ $a=\frac{3}{4}$ ☐ $a=-\frac{5}{4}$

7. If $f\colon[0,\pi]\to\mathbb{R}^3$ satisfies $f(0)=(0,0,1)$ and $f'(t)=(2t,\sin t,\cos t)$ then the point $f(\pi)$ is equal to

   ☐ $(\pi,0,0)$ ☐ $(\pi^2,-2,1)$ ■ $(\pi^2,2,1)$ ☐ $(\pi^2,2,0)$ ☐ $(\pi^2,-2,0)$

8. The twisted cubic $f(t)=(t,t^2,t^3)$, $t\in\mathbb{R}$ intersects the plane $3x-y+2z=4$ in an angle of

   ☐ 0° ■ 30° ☐ 45° ☐ 60° ☐ 90°

9. The arc length of the curve $g(t)=\left(t^3+3t+1,\sqrt{3}t^2,4t-2\right)$, $t\in[0,2]$ is

   ☐ 36 ☐ 84 ☐ 17 ☐ 12 ■ 18

10. For a differentiable curve $\gamma=\gamma(t)$ in $\mathbb{R}^3$ and a (constant) vector $\mathbf{u}\in\mathbb{R}^3$ with $|\mathbf{u}|=1$ the derivative $\frac{\mathrm{d}}{\mathrm{d}t}|\gamma-(\gamma\cdot\mathbf{u})\mathbf{u}|^2$ is equal to

    ☐ $2\,|\gamma-(\gamma\cdot\mathbf{u})\mathbf{u}|$ ☐ 2 ☐ $2\,|\gamma-(\gamma\cdot\mathbf{u})\mathbf{u}|\,(\gamma'-(\gamma'\cdot\mathbf{u})\mathbf{u})$

    ■ $2\gamma\cdot\gamma'-2(\gamma\cdot\mathbf{u})(\gamma'\cdot\mathbf{u})$ ☐ $2\,|\gamma'-(\gamma'\cdot\mathbf{u})\mathbf{u}|$

Time allowed: 40 min CLOSED BOOK ***Good luck!***

Notes

1 Since the volume of the pyramid is $1/6$ times the volume of the parallelepiped spanned by, say,

$$\begin{aligned}(1,-1,-1)-(b,1,1)&=(1-b,-2,-2),\\(-1,1,-1)-(b,1,1)&=(-1-b,0,-2),\\(-1,-1,1)-(b,1,1)&=(-1-b,-2,0),\end{aligned}$$

the condition translates into

$$\pm 6=\begin{vmatrix}1-b&-2&-2\\-1-b&0&-2\\-1-b&-2&0\end{vmatrix}=\begin{vmatrix}1-b&-2&-2\\-2&2&0\\-2&0&2\end{vmatrix}=4(1-b)+0+0-8-0-8=-4b-12.$$

The solutions are $b_1=-\frac{9}{2}$, $b_2=-\frac{3}{2}$, of which only b_2 is offered.

2 The line is $(0,0,1)+\mathbb{R}\mathbf{u}$ with $\mathbf{u}=(1,2,3)-(0,0,1)=(1,2,2)$, and the desired distance d is $|\mathbf{b}-\operatorname{proj}_{\mathbf{u}}(\mathbf{b})|=\left|\mathbf{b}-\frac{\mathbf{b}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right|$ with $\mathbf{b}=(1,0,0)-(0,0,1)$. The computation gives

$$\begin{aligned}d&=\left|\begin{pmatrix}1\\0\\-1\end{pmatrix}-\frac{(1,0,-1)\cdot(1,2,2)}{(1,2,2)\cdot(1,2,2)}\begin{pmatrix}1\\2\\2\end{pmatrix}\right|=\left|\begin{pmatrix}1\\0\\-1\end{pmatrix}+\frac{1}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}\right|\\&=\left|\begin{pmatrix}10/9\\2/9\\-7,9\end{pmatrix}\right|=\frac{1}{9}\left|\begin{pmatrix}10\\2\\-7\end{pmatrix}\right|=\frac{\sqrt{153}}{9}=\frac{1}{3}\sqrt{17}.\end{aligned}$$

3 Use the standard algorithm to determine the inverse matrix from the lecture as $\left(\begin{smallmatrix}1&-1&-1\\-2&4&-1\\1&-2&1\end{smallmatrix}\right)$ and read off $c=4$.
Smart solution: The middle column of the inverse matrix solves the linear system with extended coefficient matrix

$$\left[\begin{array}{ccc|c}2&3&5&0\\1&2&3&1\\0&1&2&0\end{array}\right]\to\left[\begin{array}{ccc|c}1&2&3&1\\0&1&2&0\\0&-1&-1&-2\end{array}\right]\to\left[\begin{array}{ccc|c}1&2&3&1\\0&1&2&0\\0&0&1&-2\end{array}\right].$$

Hence $x_3=-2$ and $c=x_2=-2x_3=4$.

4 The line, viz. $y=-\sqrt{3}x$, has polar coordinate representation $\phi=-\pi/3\triangleq-60^\circ$ and is the axis of

$$S(2\phi)=S(4\pi/3)=\begin{pmatrix}\cos(4\pi/3)&\sin(4\pi/3)\\\sin(4\pi/3)&-\cos(4\pi/3)\end{pmatrix}=\begin{pmatrix}-1/2&-\sqrt{3}/2\\-\sqrt{3}/2&1/2\end{pmatrix}.$$

5 Since the rows $\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3$ of $\mathbf{A}$ satisfy $\mathbf{r}_3=-\mathbf{r}_1-\mathbf{r}_2$, the rank of $\mathbf{A}$ (equal to the dimension of the row space) can be at most 2. Since $\mathbf{r}_1,\mathbf{r}_2$ can be chosen as linearly independent, the maximum rank is 2.

6 Since the system is homogeneous, it has a nonzero solution iff the columns of the coefficient matrix are linearly independent or, equivalently its rank is <3.

$$\begin{bmatrix}2&-1&0\\1&a&1\\1&-1&2\end{bmatrix}\to\begin{bmatrix}1&-1&2\\2&-1&0\\1&a&1\end{bmatrix}\to\begin{bmatrix}1&-1&2\\0&1&-4\\0&a+1&-1\end{bmatrix}\to\begin{bmatrix}1&-1&2\\0&1&-4\\0&0&4a+3\end{bmatrix}\to$$

Hence the answer is $a=-\frac{3}{4}$.

7 $f(\pi) = f(0) + \int_0^\pi f'(t)\,\mathrm{d}t = (0,0,1) + \int_0^\pi (2t, \sin t, \cos t)\,\mathrm{d}t = (0,0,1) + \big[(t^2, -\cos t, \sin t)\big]_0^\pi = (0,0,1) + (\pi^2, 2, 0) = (\pi^2, 2, 1)$

8 The unique intersection point is $(1,1,1) = f(1)$. With $\mathbf{n} = (3,-1,2)$ (a normal vector of the plane), the angle of intersection $\phi \in [0, \pi/2]$ satisfies

$$\cos(90^\circ - \phi) = \pm\frac{f'(1)\cdot\mathbf{n}}{|f'(1)|\,|\mathbf{n}|} = \pm\frac{(1,2,3)\cdot(3,-1,2)}{\sqrt{14}^2} = \frac{7}{14} = \frac{1}{2},$$

so that $\phi = 30^\circ$. (The minus sign needs to be chosen if the dot product is negative, which isn't the case here.)

9 We have $g'(t) = (3t^2 + 3, 2\sqrt{3}t, 4)$ and hence

$$\mathrm{L}(g) = \int_0^2 \sqrt{(3t^2+3)^2 + 12t^2 + 16}\,\mathrm{d}t = \int_0^2 \sqrt{9t^4 + 30t^2 + 25}\,\mathrm{d}t = \int_0^2 3t^2 + 5\,\mathrm{d}t = \big[t^3 + 5t\big]_0^2 = 18.$$

10 We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,|\gamma - (\gamma\cdot\mathbf{u})\mathbf{u}|^2 &= 2\,(\gamma - (\gamma\cdot\mathbf{u})\mathbf{u})\cdot\big(\gamma' - (\gamma'\cdot\mathbf{u})\mathbf{u}\big) \\
&= 2\left[\gamma\cdot\gamma' - (\gamma\cdot\mathbf{u})(\mathbf{u}\cdot\gamma') - (\gamma'\cdot\mathbf{u})(\gamma\cdot\mathbf{u}) + (\gamma\cdot\mathbf{u})(\gamma'\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{u})\right] \\
&= 2\left[\gamma\cdot\gamma' - (\gamma\cdot\mathbf{u})(\gamma'\cdot\mathbf{u})\right],
\end{aligned}$$

since $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2 = 1$.