ECE313 Homework 1

- **Name**: Zhao Hongshuo
- **NetID**: hz108

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1 Problem 1

Describe one event from the sample space of the following experiment, and determine the cardinality of the sample space:

1. Suppose you have a fair die and a fair coin. First you roll the die and let X be the number shown. If X is even, you toss the coin three times. If X is odd, you toss the coin five times. The outcome of the experiment is the result of the die roll and the coin flips.
2. You pick a random positive integer less than 2025 that has exactly one digit 6.
3. An entire deck of 52 cards is dealt evenly to four people (each receives a hand of 13 cards, excluding the kings and queens).
4. 5-digit numbers are formed from the integers $1, 2, \ldots, 9$ and no digit can appear more than once.

Answers:

1. Event example: The die shows 2, and the coin flips are H, T, H. The cardinality of the sample space is:

$$6 \times (2^3 + 2^5) = 240 \tag{1}$$

2. Event example: 666. The cardinality of the sample space is:

$$[(9 + 10) \times 9 + 100] \times 2 + 2 = 544 \tag{2}$$

3. Event example: (It's hard to describe). The cardinality of the sample space is:

$$\binom{52}{13} \times \binom{39}{13} \times \binom{26}{13} \times \binom{13}{13} \tag{3}$$

4. Event example: 98765. The cardinality of the sample space is:

$$9 \times 9 \times 8 \times 7 \times 6 = 27216 \tag{4}$$

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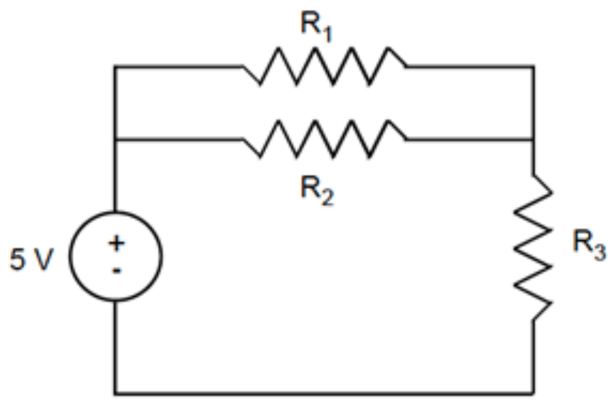
2 Problem 2

In this problem, you will construct a probability space $(\Omega, \mathcal{F}, P)$ for the following circuit. Each resistor in this circuit can fail independently, and we want to construct an experiment in which we examine the state of the circuit. When a resistor fails, it behaves as if it were an open circuit. The voltage source never fails.



1. What is a suitable sample space Ω for this experiment? List all elements in Ω.
2. Identify the event A corresponding to the statement "current flows through the circuit."
3. If each resistor fails with a probability p, what is the probability of event A (i.e., $P(A)$)?

Answers:

1. The sample space is all combinations of working/failed states for each resistor.
2. Event A is the subset of Ω where the circuit is closed. Specifically:

$$A = \{TTT, TFT, FTT\} \tag{5}$$

   where T means that the resistor is working, F means that the resistor is failed. The n-th letter represents the n-th resistor.
3.

$$P(A) = (1-p)^3 + 2p(1-p)^2 \tag{6}$$

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3 Problem 3

Consider sampling r items from a group of n objects, e.g., your pencil case contains n pens and pencils and you select r items from the pencil case. How many possible ways are there to sample r items from a group of n when sampling is done in the following ways:

1. Ordered and with replacement
2. Ordered and without replacement
3. Unordered and with replacement
4. Unordered and without replacement

Answers:

1. Each step has n choice, totally r steps, so the total possible ways are

$$n^r \tag{7}$$

2. By permutation definition, the total possible ways are

$$P(n,r) = \frac{n!}{(n-r)!} \tag{8}$$

3. It should be the total ways of ordered and with replacement divides by the permutation of r items, so the total possible ways are

$$\frac{n^r}{r!} \tag{9}$$

4. By combination definition, the total possible ways are

$$\binom{n}{r} \tag{10}$$

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4 Problem 4

A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be:

1. no complete pair.
2. Exactly one complete pair.

Answers:

1. There are totally 20 shoes, the cardinality of the sample space is $\binom{20}{8}$, all the possible ways that there is no complete pair is $\binom{10}{8} \times 2^8$. Therefore, the probability is

$$\frac{\binom{10}{8} \times 2^8}{\binom{20}{8}} \tag{11}$$

2. Also, the cardinality of the sample space is $\binom{20}{8}$, all the possible ways that there is exactly one complete pair is $\binom{10}{1} \times \binom{9}{6} \times 2^6$. Therefore, the probability is

$$\frac{\binom{10}{1} \times \binom{9}{6} \times 2^6}{\binom{20}{8}} \tag{12}$$

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5 Problem 5

Here we consider a simple game called the Prisoner's Dilemma. Two members of a criminal gang, A and B, are arrested and are under investigation. Police don't have enough evidence, which makes them long for the prisoner's confession. Criminals have a choice to confess (C) or to deny (D). To encourage the criminals to confess, police offers a bargain:

- If A and B both confess, each of them serves 2 years in prison
- if A betrays B (A confesses, B denies), then A will be set free while B will serve 3 years in prison (and vice versa).
- If A and B both remain silent, both of them serve 1 year in prison.

1. What is the sample space in the Prisoner's Dilemma?

A decides to confess and betray B. B is reluctant to betray A. Therefore, B is thinking of confessing with a probability of 0.3. (B does not know what A will choose and vice-versa).

2. What is the probability of A being set free, and B serving 3 years in prison?
3. What is the probability of both A and B serving 2 years in prison?

Answers:

1. The sample space is $\Omega = \{(C,C),(C,D),(D,C),(D,D)\}$
2. The probability of the event is equal to the probability that B decides to Deny. Therefore, the probability is

$$P(A \text{ Free}, B \text{ 3yrs}) = P(A=C, B=D) = 1 \times (1-0.3) = 0.7 \tag{13}$$

3. The probability of the event is equal to the probability that B decides to Confess. Therefore, the probability is

$$P(\text{Both 2 yrs}) = P(A=C, B=C) = 1 \times 0.3 = 0.3 \tag{14}$$

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6 Problem 6

Bear in mind that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Then use this to show:

1. If $A \subset B$, then $P(B-A) = P(B) - P(A)$. (Recall the definition of $B - A : B \cap A^c$)
2. $P(A) = P(A \cap B) + P(A - B)$. (Hint: Are $A \cap B$ and $A - B$ disjoint?)

Answers:

1. If $A \subset B$, then $B \cup A^c = \mathcal{F}$, meaning that $P(B \cup A^c) = P(\mathcal{F}) = 1$

$$\begin{aligned} P(B-A) &= P(B \cap A^c) = P(A^c \cap B) \\ &= P(A^c) + P(B) - P(A^c \cup B) \\ &= (1 - P(A)) + P(B) - 1 \\ &= P(B) - P(A) \end{aligned} \tag{15}$$

2. $(A-B) = (A \cap B^c) \subset B^c$, while $(A \cap B) \subset B$. Therefore, they are disjoint, their probability can be directly added to represent the event $(A-B) \cup (A \cap B)$. $A - B$ represents the part in A that does not belong to B, while $A \cap B$ represents the part in A that also belongs to B. Therefore, there union event is A.

$$\begin{aligned} P(A) &= P((A-B) \cup (A \cap B)) \\ &= P(A-B) + P(A \cap B) - \underbrace{P((A-B) \cap (A \cap B))}_{=0} \\ &= P(A-B) + P(A \cap B) \end{aligned} \tag{16}$$