ECE 313 Homework 4

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1 Problem 1

For each of the following cases, determine whether the random variable is discrete or continuous. For the discrete random variables: (i) Describe the probability mass function (pmf) by finding the set of values that the random variable might take and their respective probabilities.

1. T is the lifetime of a light bulb.
2. Let Y represent the difference between the number of heads and the number of tails obtained when a fair coin is tossed 3 times.
3. An amateur mind game player has accepted a challenge from a computer program. The challenge consists of three rounds (with different games for each round) and the player is awarded \$1,000 for each round that the player has won. In addition, the final winner (who wins more than two out of the three rounds) is awarded additional \$3,000. A preliminary analysis has shown that the possibility of the player winning the computer program is 0.55 for the first round, 0.7 for the second round and 0.4 for the third round. Let X be a random variable on the total award that the player can win. Assume that all three rounds are played regardless of the results.
4. Let A represent an analog signal received by an analog to digital converter.

Answer:

1. It's continuous variable.
2. It's discrete variable. $Y \in \{-3, -1, 1, 3\}$

H	$Y = 2H - 3$	$P(H)$
0	-3	$\binom{3}{0}(0.5)^3 = \frac{1}{8}$
1	-1	$\binom{3}{1}(0.5)^3 = \frac{3}{8}$
2	1	$\binom{3}{2}(0.5)^3 = \frac{3}{8}$
3	3	$\binom{3}{3}(0.5)^3 = \frac{1}{8}$

PMF:

$$P(Y=y)=\begin{cases}\frac{1}{8} & y=-3\\ \frac{3}{8} & y=-1\\ \frac{3}{8} & y=1\\ \frac{1}{8} & y=3\\ 0 & \text{oterwise}\end{cases} \tag{1}$$

3. It's discrete variable. $X \in \{0, 1000, 2000, 6000\}$ Let $W_i = 1$ be the event that he wins for the i-th round, and $W_i = 0$ be the event that he loses. Define $S = W_1 + W_2 + W_3$, then we have:

$$X = 1000S + 3000 \cdot 1_{\{S=3\}} \tag{2}$$

We need to calculate the probability mass function of S

$$\begin{aligned} P(S=0) =&(1-0.55)(1-0.7)(1-0.4) = 0.081\\ P(S=1) =&0.55(1-0.7)(1-0.4) + 0.4(1-0.55)(1-0.7)\\ &+ 0.7(1-0.55)(1-0.4) = 0.342\\ P(S=2) =&0.55 \cdot 0.7(1-0.4) + 0.55 \cdot 0.4(1-0.7)\\ &+ 0.4 \times 0.7(1-0.55) = 0.423\\ P(S=3) =&0.55 \cdot 0.7 \cdot 0.4 = 0.154 \end{aligned} \tag{3}$$

Therefore, the PMF of X is

$$P(X=x)=\begin{cases}0.081 & x=0\\ 0.342 & x=1000\\ 0.423 & x=2000\\ 0.154 & x=6000\\ 0 & \text{otherwise}\end{cases} \tag{4}$$

4. It's continuous variable.

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2 Problem 2

The length of time a teacher can write on the blackboard without breaking the chalk is the random variable T (measured in minutes) with the following pdf:

$$f_T(t)=\begin{cases}(0.05)e^{-0.05t} & t \geq 0\\ 0 & t < 0\end{cases} \tag{5}$$

1. Determine the CDF $F_T(t)$ for T.
2. If the teacher has not broken the chalk after 10 minutes, what is the probability that she will break the chalk in the next 20 minutes?

Answer:

1. For $t \geq 0$:

$$F_T(t) = \int_0^t 0.05e^{-0.05s}ds = 1 - e^{-0.05t} \tag{6}$$

For $t < 0$:

$$F_T(t) = 0 \tag{7}$$

Therefore, we have

$$F_T(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-0.05t} & t \geq 0 \end{cases} \tag{8}$$

$$P(T \leq 30 | T > 10) = \frac{P(10 < T \leq 30)}{P(T > 10)} = \frac{F_T(30) - F_T(10)}{1 - F_T(10)} \tag{9}$$

2. We have
   - $F_T(10) = 1 - e^{-0.05 \times 10} = 1 - e^{-0.5}$
   - $F_T(30) = 1 - e^{-0.05 \times 30} = 1 - e^{-1.5}$

   Therefore

$$P = \frac{e^{-0.5} - e^{-1.5}}{e^{-0.5}} = 1 - e^{-1} \tag{10}$$

   Actually, the exponential distribution has no memory, so $P(T \leq t + s | T > s) = P(T \leq t)$. Here, $t = 20$ and $\lambda = 0.05$, so $P = 1 - e^{-0.05 \times 20} = 1 - e^{-1}$

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3 Problem 3

Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares their number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find $P(X = i)$ for $i \in \{0, 1, 2, 3, 4\}$

Answer: We can consider whether the number of player 1 has the largest number among the first k players. $X \geq i$ if and only if the number of player 1 is the largest number among $\{a_1, a_2, \ldots, a_{i+1}\}$. Due to the symmetry property, for arbitrary m numbers, the probability of each position to become the largest number is $\frac{1}{m}$. Therefore

$$\begin{aligned} &P(X \geq 0) = 1 \\ &P(X \geq 1) = P(a_1 = \max\{a_1, a_2\}) = \frac{1}{2} \\ &P(X \geq 2) = P(a_1 = \max\{a_1, a_2, a_3\}) = \frac{1}{3} \\ &P(X \geq 3) = \frac{1}{4} \\ &P(X \geq 4) = \frac{1}{5} \end{aligned} \tag{11}$$

Then we have

$$
\begin{aligned}
P(X=0) &= P(X \geq 0) - P(X \geq 1) = \frac{1}{2} \\
P(X=1) &= P(X \geq 1) - P(X \geq 2) = \frac{1}{6} \\
P(X=2) &= P(X \geq 3) - P(X \geq 4) = \frac{1}{12} \\
P(X=3) &= P(X \geq 4) - P(X \geq 3) = \frac{1}{20} \\
P(X=4) &= P(X \geq 4) = \frac{1}{5}
\end{aligned} \tag{12}
$$

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4 Problem 4

The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$
f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leq 10 \end{cases} \tag{13}
$$

1. Find $P(X > 20)$.
2. What is the cumulative distribution function of X?
3. What is the probability that, of 5 such types of devices, at least 2 will function for at least 15 hours? What assumptions are you making?

Answer:

1.

$$
P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2} dx = \left[-\frac{10}{x} \right]_{20}^{\infty} = \frac{1}{2} \tag{14}
$$

2. For $x \leq 10$, we have $F(x) = 0$.

   For $x > 10$, we have

$$
F(x) = \int_{10}^{x} \frac{10}{t^2} dt = \left[-\frac{10}{t} \right]_{10}^{x} = 1 - \frac{10}{x} \tag{15}
$$

   So we have

$$
F(x) = \begin{cases} 0 & x \leq 10 \\ 1 - \frac{10}{x} & x > 10 \end{cases} \tag{16}
$$

3. We first calculate the probability that a single device can work for longer than 15 hours:

$$
P(X \geq 15) = 1 - F(15) = \frac{2}{3} \tag{17}
$$

   The assumption is that the working hour of each device is i.i.d (independent identical distribution), then $Y \sim \text{Bin}(n=5, p=\frac{2}{3})$, we need to compute $P(Y \geq 2) = 1 - P(Y=0) - P(Y=1)$

   - $P(Y=0) = (1 - \frac{2}{3})^5 = \frac{1}{243}$
   - $P(Y=1) = \binom{5}{1} \cdot \frac{2}{3} \cdot \left(\frac{1}{3}\right)^4 = \frac{10}{243}$

   Therefore, we have

$$P(Y \geq 2) = 1 - \frac{11}{243} = \frac{232}{243} \tag{18}$$

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5 Problem 5

Consider the following program segments. The function `randInt()` returns a random integer from 1 to 10, with all integers being equally likely.

```
// Segment i. (while loop)
B = randInt();
while (B >= 7) {
  execute S;
  B = randInt();
}

// Segment ii. (do-while loop)
do {
  execute S;
  B = randInt();
} while (B >= 7);
```

In both cases, let the random variable X be the number of times the statement-group S is executed. Assume each call to `randInt()` is independent.

1. Let p be the probability that the loop condition (`B>=7`) is true. Find the value of p.
2. For each program segment, determine the set of possible values (the support) for the random variable X.
3. For each program segment (i and ii), derive the probability mass function (pmf) of X in terms of p. Name the specific type of distribution for each case.

Hint: One of these has a geometric distribution and the other has a modified geometric distribution.

Answer:

1. $p = 0.4$
2. In Segment i, we firstly assign a value to B, if B<7, then S executes for 0 times. So $X \in \{0, 1, 2, 3, \ldots\} = \mathbb{N}$.

   In Segment ii, we firstly executes S, then go to the branch. So $X \in \{1, 2, 3, \ldots\} = \mathbb{N}^+$
3. For Segment i, it is a geometric distribution

$$P(X = k) = p^k(1 - p), \quad k = 0, 1, 2, \ldots \tag{19}$$

For Segment ii, it is a modified geometric distribution where k starts from 1,

$$P(X = k) = p^k(1 - p), \quad k = 1, 2, 3, \ldots \tag{20}$$