14 Partial Derivatives

14.1 Functions of Several Variables

Context

- Functions of Two Variables
- Graphs
- Level Curves and Contour Maps
- Functions of Three or More Variables

Functions of Several Variables (1 of 1)

In this section we study functions of two or more variables from four points of view:

- verbally (by a description in words)
- numerically (by a table of values)
- algebraically (by an explicit formula)
- visually (by a graph or level curves)

Functions of Two Variables

Functions of Two Variables (1 of 2)

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point. We can think of T as being a function of the two variables x and y, or as a function of the pair (x, y). We indicate this functional dependence by writing $T = f(x, y)$.

The volume V of a circular cylinder depends on its radius r and its height h. In fact, we know that $V = \pi r^2 h$. We say that V is a function of r and h, and we can write $v(r,h) = \pi r^2 h$.

Functions of Two Variables (2 of 2)

Definition **A function *f* of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x,y) \mid (x,y) \in D\}$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable**. [Compare this with the notation $y = f(x)$ for functions of a single variable.]

Example 3 (1 of 3)

In regions with severe winter weather, the *wind-chill index* is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the actual temperature T and the wind speed v.

So W is a function of T and v, and we can write $W = f(T, v)$.

Example 3 (2 of 3)

Table 1 records values of W compiled by the US National Weather Service and the Meteorological Service of Canada.

Wind speed (km/h)

Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

Wind-chill index as a function of air temperature and wind speed

Table 1

Example 3 (3 of 3)

For instance, the table shows that if the actual temperature is −5°C and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about −15°C with no wind.

So

$$f(-5, 50) = -15$$

Example 4 (1 of 6)

In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899−1922.

They considered a simplified view of the economy in which **production output** is determined by the **amount of labor involved** and the **amount of capital invested**.

While many other factors affect economic performance, their model proved to be remarkably accurate.

Example 4 (2 of 6)

function Cobb and Douglas used to model production was of the form

$$P(L,\ K) = bL^{\alpha}K^{1-\alpha} \qquad \mathbf{1}$$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings).

Example 4 (3 of 6)

Cobb and Douglas used economic data published by the government to obtain Table 2.

Year	*P*	*L*	*K*
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208

Year	*P*	*L*	*K*
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

Table 2

Example 4 (4 of 6)

They took the year 1899 as a baseline and *P*, *L*, and *K* for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 values.

Cobb and Douglas used the method of least squares to fit the data of Table 2 to the function

$$P(L, K) = 1.01 L^{0.75} K^{0.25} \tag{2}$$

Example 4 (5 of 6)

If we use the model given by the function in Equation 2 to compute the production in the years 1910 and 1920, we get the values

$$P(147,\ 208) = 1.01(147)^{0.75}(208)^{0.25} \approx 161.9$$

$$P(194,\ 407) = 1.01(194)^{0.75}(407)^{0.25} \approx 235.8$$

which are quite close to the actual values, 159 and 231.

The production function (1) has subsequently been used in many settings, ranging from individual firms to global economics. It has become known as the **Cobb-Douglas production function**.

Example 4 (6 of 6)

Its domain is $\{(L,k) \mid L \geq 0, k \geq 0\}$ because L and K represent labor and capital and are therefore never negative.

单选题 1分

设置

MCQ: We have known wind-chill $W = f(\mathrm{T}, \mathrm{v})$,
cylinder volume $V(\mathrm{r}, \mathrm{h}) = \pi r^2 h$,
Cobb–Douglas production $P(\mathrm{L}, \mathrm{K}) = 1.01\, L^{0.75} K^{0.25}$.
Which statement is **correct**?

A The domain of $V(\mathrm{r}, \mathrm{h})$ is all $(\mathrm{r}, \mathrm{h}) \in \mathbb{R}^2$

B For a fixed volume $V(\mathrm{r}, \mathrm{h}) = c > 0$ (r, h) satisifies $h = \frac{c}{\pi r^2}$

C The statement $f(-5{,}50) = -15$ means: “When the actual temperature is $-15°C$ and wind speed is 50 *km/h*, it feels like $-5°C$ with no wind.”

D Increasing both L and K by 10% increases P by 20%

提交

Example

Solution:

- $V = \pi r^2 h \Rightarrow h = c/(\pi r^2)$,which is exactly a volume level curve (B true).
- Physical constraints give $r \geq 0,\ h \geq 0$ not all $\mathbb{R}^2$;(A is false).
- $f(-, 550) = -15$actually means: at $-5°\,C$with wind $50\ km/h$, it feels like $-\ 15°\,C$in calm air; C reverses the meaning.
- In Cobb–Douglas, exponents sum to 1 (constant returns), so +10% in both inputs ⇒ P+10%, not 20%; D is false.

Graphs

Graphs (1 of 3)

Another way of visualizing the behavior of a function of two variables is to consider its graph.

Definition If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

The graph of a function f of two variables is a surface S with equation $z = f(x, y)$.

Graphs (2 of 3)

We can visualize the graph *S* of *f* as lying directly above or below its domain *D* in the *xy*-plane (see Figure 5).

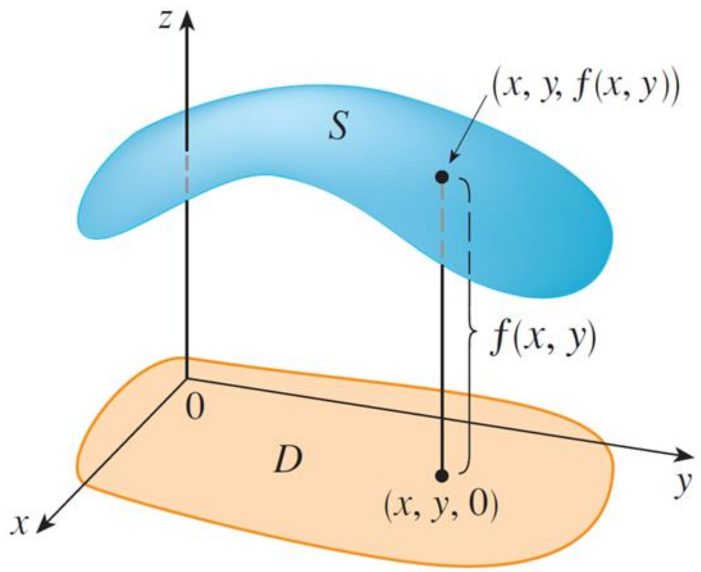


Figure 5

Graphs (3 of 3)

The function $f(x, y) = ax + by + c$ is called as a **linear function**.

The graph of such a function has the equation

$$z = ax + by + c \qquad \text{or} \qquad ax + by - z + c = 0$$

so it is a plane. In much the same way that linear functions of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

Example 6

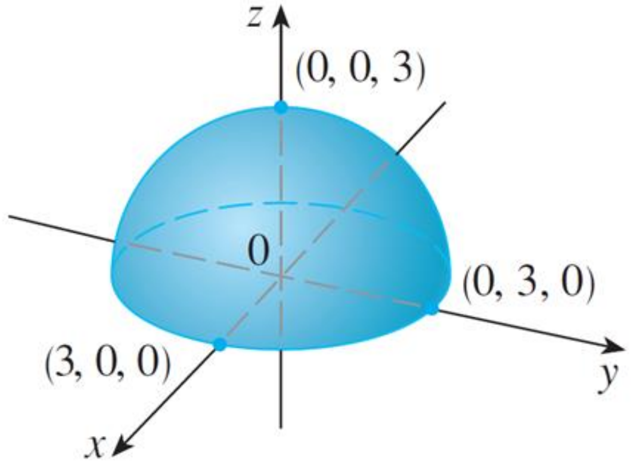
Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution:

In Example 2 we found that the domain of g is the disk with center (0, 0) and radius 3. The graph of t has equation $z = \sqrt{9 - x^2 - y^2}$.

We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + x^2 = 9$, which we recognize as an equation sphere with center the origin and radius 3.

But, since $z \geq 0$, the graph of g is just the top half of this sphere (see Figure 7).



Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Figure 7

Level Curves and Contour Maps

Level Curves and Contour Maps (1 of 10)

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour curves*, or *level curves*.

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it is a curve in the xy-plane that shows where the graph of f has height k (above or below the xy-plane). A collection of level curves is called a **contour map**.

Level Curves and Contour Maps (2 of 10)

Contour maps are most descriptive when the level curves $f(x, y) = k$ are drawn for equally spaced values of k, and we assume that this is the case unless indicated otherwise. You can see from Figure 11 the relation between level curves and horizontal traces.

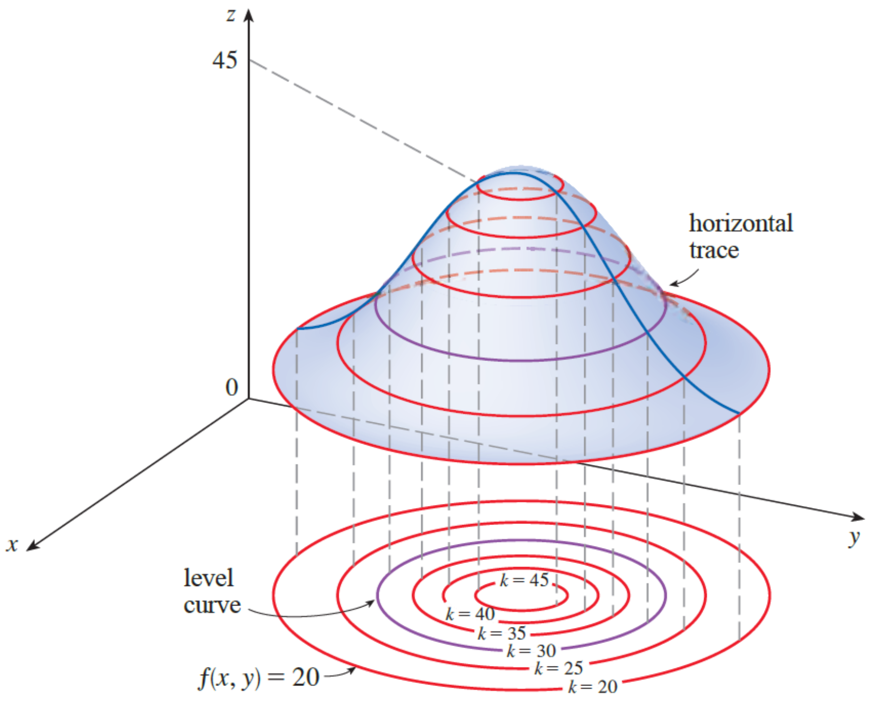


Figure 11

Level Curves and Contour Maps (3 of 10)

The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane. So if you draw a contour map of a function and visualize the level curves being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph.

The surface is steeper where the level curves are close together and somewhat flatter where they are farther apart.

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in Figure 12.

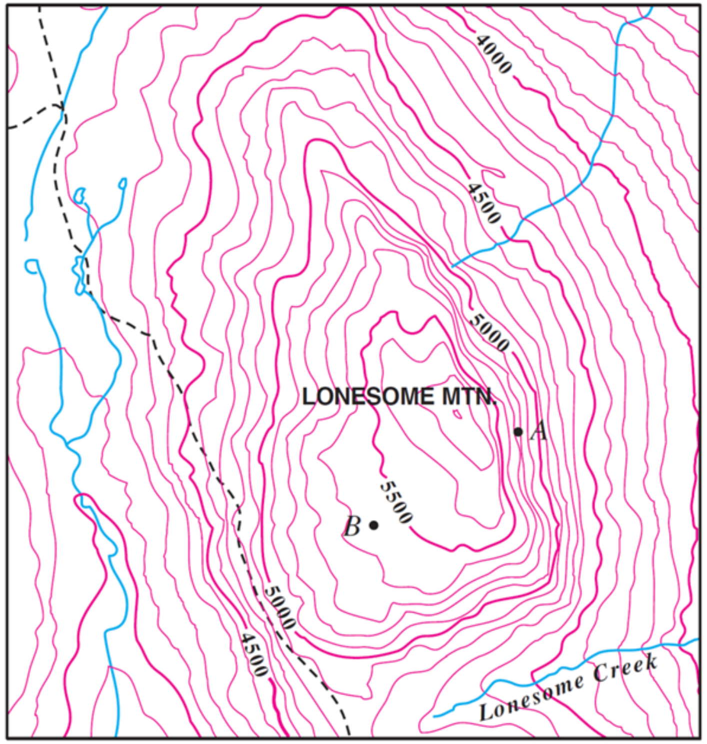


Figure 12

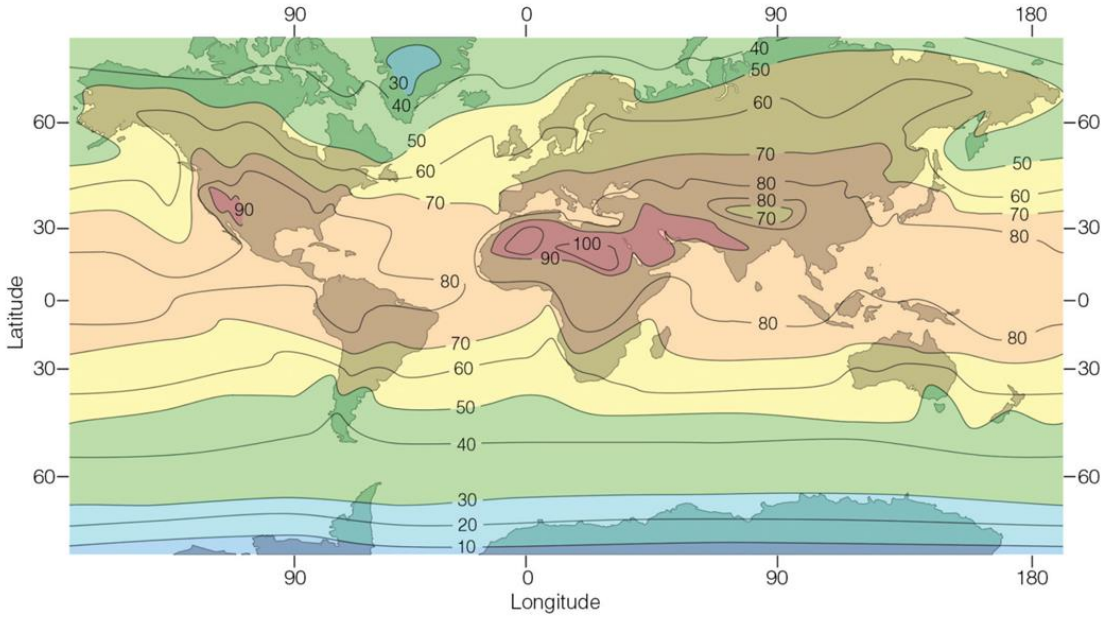
Level Curves and Contour Maps (5 of 10)

The level curves are curves of constant elevation above sea level. If you walk along one of these contour lines, you neither ascend nor descend.

Another common example is the temperature function introduced in the opening paragraph of this section. Here the level curves are called **isothermals;** they join locations with the same temperature.

Level Curves and Contour Maps (6 of 10)

Figure 13 shows a weather map of the world indicating the average July temperatures. The isothermals are the curves that separate the colored bands.



Average air temperature near sea level in July (°F)

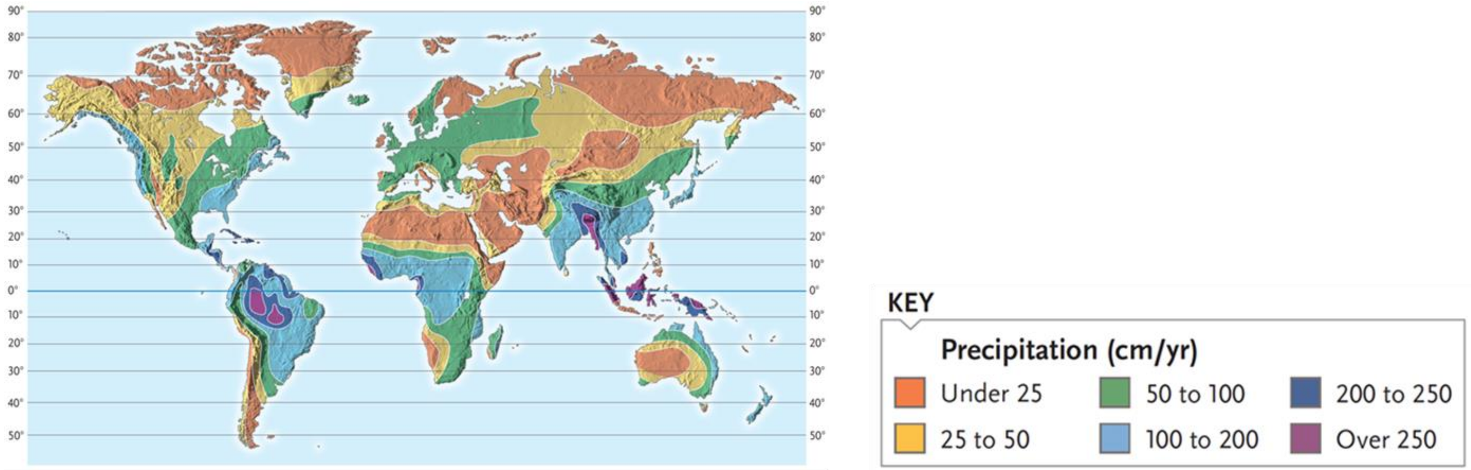
Figure 13

Level Curves and Contour Maps (7 of 10)

In weather maps of atmospheric pressure at a given time as a function of longitude and latitude, the level curves are called **isobars;** they join locations with the same pressure.

Surface winds tend to flow from areas of high pressure across the isobars toward areas of low pressure and are strongest where the isobars are tightly packed.

A contour map of world-wide precipitation is shown in Figure 14.



Precipitation

Figure 14

Here the level curves are not labeled but they separate the colored regions and the amount of precipitation in each region is indicated in the color key.

Example 9

A contour map for a function f is shown in Figure 15. Use it to estimate the values of $f(1, 3)$ and $f(4, 5)$.

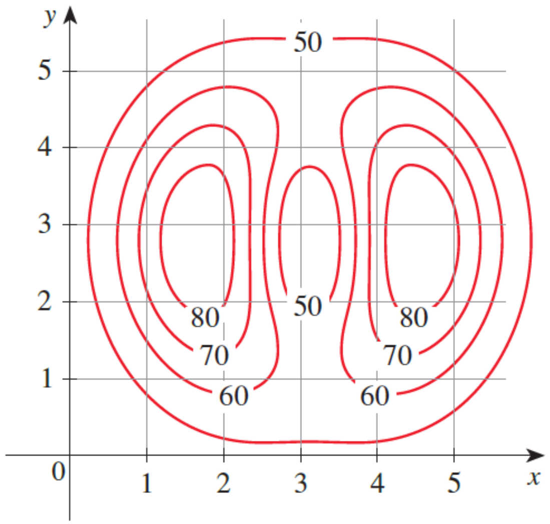


Figure 15

Example 9 – Solution

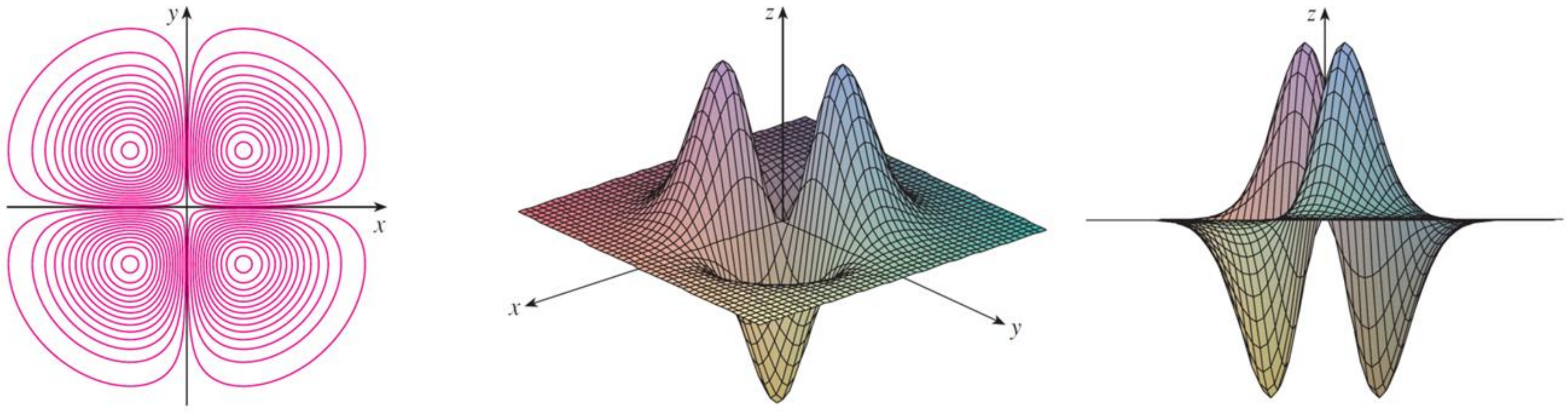
The point (1, 3) lies partway between the level curves with z-values 70 and 80. We estimate that

$$f(1, 3) \approx 73$$

Similarly, we estimate that $f(4, 5) \approx 56$

Level Curves and Contour Maps (9 of 10)

For some purposes, a contour map is more useful than a graph. It is also true in estimating function values. Figure 20 shows some computer-generated level curves together with the corresponding computer-generated graphs.

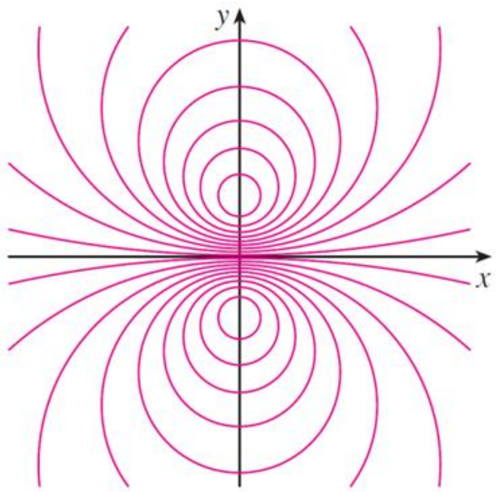


(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$

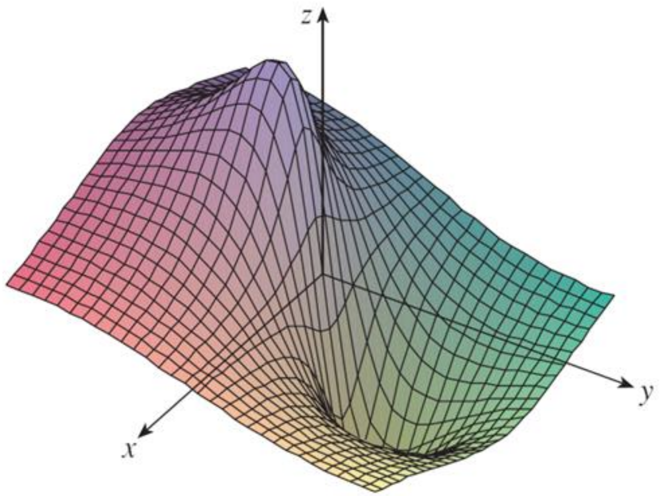
(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$

Figure 20

Notice that the level curves in part (c) crowd together near the origin. That corresponds to the fact that the graph in part (d) is very steep near the origin.



(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$



(d) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

Figure 20

Functions of Three or More Variables

Functions of Three or More Variables (1 of 5)

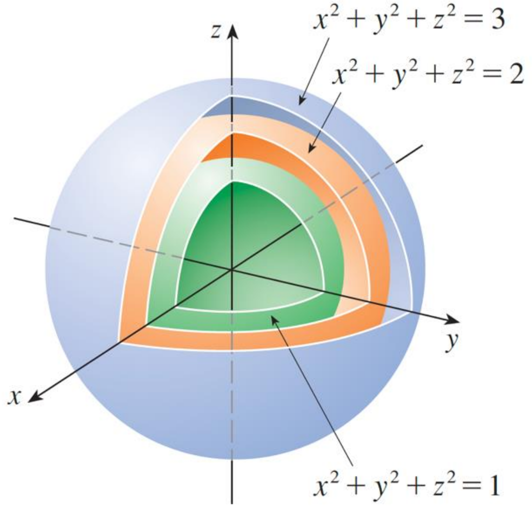
A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

Functions of Three or More Variables (2 of 5)

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

Functions of any number of variables can be considered. A **function of *n* variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers.



Functions of Three or More Variables (3 of 5)

For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the ith ingredient, and x_i units of the i th ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \ldots, x_n$:

$$C = f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{3}$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$.

Functions of Three or More Variables (4 of 5)

Sometimes we use vector notation to write such functions more compactly:

If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $\boldsymbol{f}(x_1, x_2, \ldots, x_n)$.

With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \ldots c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n.

Functions of Three or More Variables (5 of 5)

In view of the one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$: and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in V_n, we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \ldots, x_n$

2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$

3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$

多选题 1分

设置

MAQ: Let $f(x, y, z) = \ln(z - y) + xy\sin z$

And $g(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x}$ with $\boldsymbol{c} \in \mathbb{R}^n \backslash \{\boldsymbol{0}\}$,Decide which statements are true:

A The domain of f is the open half-space $D = \{z > y\}$

B For any integer m and real k,$S_k := \{f = k\}$ satisfies $S_k \cap \{z = m\pi\} = \{(x, m\pi - e^k, m\pi): x \in \mathbb{R}\}$

C The distance between the hyperplanes $\{\boldsymbol{c} \cdot \boldsymbol{x} = k_1\}$ and $\{\boldsymbol{c} \cdot \boldsymbol{x} = k_2\}$ equals $\frac{|k_1 - k_2|}{|\boldsymbol{c}|}$

D The set $\{x: \boldsymbol{c} \cdot \boldsymbol{x} \leq k\}$ is a closed convex half-space whose boundary is hyperplane $\boldsymbol{c} \cdot \boldsymbol{x} = k$ with normal vector parallel to $\mathbf{c}$

提交

Example

Solution:

For option A, ln(z−y) is defined iff $z - y > 0$. And strict inequality $\Rightarrow$ open

For option B, On the plane $z = m\pi$, we have , so the equation $f = k$ becomes ln(mπ−y)=k and there is no constraint on x. It's correct.

For C, they are parallel hyperplanes with normal vector c.

So the distance is $\frac{|k_1 - k_2|}{|c|}$

For option D, it is the inverse image of $(-\infty, k]$ under the continuous map $\boldsymbol{x} \to \boldsymbol{c} \cdot \boldsymbol{x}$, hence **closed**. And it is **convex** because a linear inequality defines a convex set.

Recap

- Functions of Two Variables
- Graphs
- Level Curves and Contour Maps
- Functions of Three or More Variables