ECE 313: Problem Set 12: Problems

Due: Saturday, Dec 14 at 11:59:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.7 – 4.8

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: You must upload handwritten homework to BB. No typeset homework will be accepted. No late homework will be accepted. Please write on the top right corner of the first page:
NAME AS IT APPEARS ON BB
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings.

1. **[Correlation of continuous-type random variables]**
   Suppose X and Y are jointly distributed with the following joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} \frac{3}{2}(1-|u-1|-|v|), & \text{if } -1 \le v \le 1 \text{ and } |v| \le u \le 2-|v|, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Sketch the support of $f_{X,Y}$, and describe the shape of $f_{X,Y}$ over the support.

   (b) Find $\mathrm{Cov}(X,Y)$ and $\rho_{X,Y}$.

(c) Are X and Y uncorrelated? Are X and Y independent?

(d) Suppose $\begin{matrix} W \\ Z \end{matrix} = \begin{matrix} a & b \\ c & d \end{matrix} \begin{matrix} X \\ Y \end{matrix}$. Express $\rho_{W,Z}$ with respect to a, b, c, and d. (Hint: You may use the fact $\mathrm{Var}(X) = \mathrm{Var}(Y)$, which should be clear from your answer to part (a). You do not need to calculate $\mathrm{Var}(X)$.).

2. **[Some moments for a random rectangle]**
Let $A = XY$ denote the area and $L = 2(X + Y)$ the length of the perimeter, of a rectangle with length X and height Y, such that X and Y are independent, and uniformly distributed on the interval $[0, 1]$.

(a) Find $\mathbb{E}[A]$ and $\mathbb{E}[L]$.

(b) Find $\mathrm{Var}(A)$. (Hint: Find $\mathbb{E}[A^2]$ first.)

(c) Find $\mathrm{Var}(L)$.

(d) Find $\mathrm{Cov}(A, L)$. (Hint: Find $\mathbb{E}[AL]$ first.)

(e) Find the correlation coefficient, $\rho_{A,L}$. (Hint: Should be less than, but fairly close to, one. Why?)

3. **[Jointly Distributed Random Variables]**
Let the random variables X and Y be such that $\mathbb{E}[X] = 1$, $\mathbb{E}[Y] = 4$, $\mathrm{Var}(X) = 4$, $\mathrm{Var}(Y) = 9$, and $\rho = 0.1$. Let $W = 3X + Y$.

(a) Find $\mathbb{E}[W]$ and $\mathrm{Var}(W)$.

(b) Let $Z = X + Y$, find the joint PDF $f_{W,Z}$ in terms of the joint PDF $f_{X,Y}$.

4. **[Covariance I]**
Consider random variables X and Y on the same probability space.

(a) If $\mathrm{Var}(X + 2Y) = 40$ and $\mathrm{Var}(X - 2Y) = 20$, what is $\mathrm{Cov}(X, Y)$?

(b) In part (a), determine $\rho_{X,Y}$ if $\mathrm{Var}(X) = 2\,\mathrm{Var}(Y)$.

(c) If $\mathrm{Var}(X + 2Y) = \mathrm{Var}(X - 2Y)$, are X and Y uncorrelated?

(d) If $\mathrm{Var}(X) = \mathrm{Var}(Y)$, are X and Y uncorrelated?

5. **[Covariance II]**
Rewrite the expressions below in terms of $\text{Var}(X)$, $\text{Var}(Y)$, $\text{Var}(Z)$, and $\text{Cov}(X,Y)$.

(a) $\text{Cov}(3X+2, 5Y-1)$

(b) $\text{Cov}(2X+1, X+5Y-1)$

(c) $\text{Cov}(2X+3Z, Y+2Z)$ where Z is uncorrelated to both X and Y.

6. **[Covariance III]**
Random variables X_1 and X_2 represent two observations of a signal corrupted by noise. They have the same mean μ and variance σ^2. The signal-to-noise-ratio (SNR) of the observation X_1 or X_2 is defined as the ratio $\text{SNR}_X = \frac{\mu^2}{\sigma^2}$. A system designer chooses the averaging strategy, whereby she constructs a new random variable $S = \frac{X_1+X_2}{2}$.

(a) Show that the SNR of S is twice that of the individual observations, if X_1 and X_2 are uncorrelated.

(b) The system designer notices that the averaging strategy is giving $\text{SNR}_S = (1.5)\text{SNR}_X$. She correctly assumes that the observations X_1 and X_2 are correlated. Determine the value of the correlation coefficient ρ_{X_1,X_2}.

(c) Under what condition on ρ_{X_1,X_2} can the averaging strategy result in an SNR_S that is as high as possible?