ZJU-UIUC INSTITUTE

Online Final Examination

For Students, please read and sign the honor statement on a sheet of paper with your name, student ID number, date, and read any instructions below before starting your exam.

(For instructors, please complete the form below)

Course Code: Math 231	Semester: Spring 2020	Instructor: Cui Zhou
Exam Type: Closed-book ■ Open-book ☐ Partly Open-book ☐ Take Home ☐		
Exam Date: May 25, 2020	Start Time: 14:00	End Time: 17:00
Duration: 3 hrs	Total pages: 7	Total questions: 14+1

Specific requirements and instructions to students:

Please sign honor statement at the start of the exam.

This is a closed book exam. No books, no lecture notes, no calculators, or any aided tools can be used during the exam. Do all problems and show your work for full credits.

Academic Integrity

Academic integrity is essential for maintaining the quality of scholarship in the Institute and for protecting those who depend on the results of research work performed by faculty and students in the Institute. The faculty of the Zhejiang University/University of Illinois at Urbana-Champaign Institute expect all students to maintain academic integrity at all times in the classroom and the research laboratory and to conduct their academic work in accordance with the highest ethical standards of the engineering profession. Students are expected to maintain academic integrity by refraining from academic dishonesty, and by refraining from conduct which aids others in academic dishonesty, or which leads to suspicion of academic dishonesty. Violations of academic integrity will result in disciplinary actions ranging from failing grades on assignments and courses to probation, suspension or dismissal from the Institute.

Honor Statement

I have read the announcement concerning exam administration, the exam instructions, the academic integrity statement repeated from the course syllabus, and provided my reflection statement on why integrity and honesty are important. I promise to abide by the exam rules and regulations and agree to comport myself during the remotely administered exam in the same manner as if I were in a proctored examination room.

Please write "**I have read and will follow the Honor Statement**" on a sheet of paper and include the following information, then submit along with your answer sheet.

Signature:

Student ID number:

Date:

(Please go on to the next page for questions)

(Please go on to the next page for questions)

1. (5 pts) Evaluate the integral

$$\int x^2(lnx)^2dx$$

Solution: $\frac{1}{3}x^3\ln^2 x-\frac{2}{9}x^3\ln x+\frac{2}{27}x^3+C$

2. (5 pts) Evaluate the integral

$$\int_0^1 x^3\sqrt{1-x^2}dx$$

Solution: $\frac{2}{15}$

3. (5 pts) Evaluate the integral

$$\int_0^{\pi/2} 16x\cos^2 x\,dx$$

Solution: π^2-4

4. (5 pts) Evaluate the integral or show that it is divergent

$$\int_1^\infty \frac{\ln x}{x^4}dx$$

Solution: 1

5. (5 pts) Power series representation of the function $f(x)=\frac{x}{(1+x)^2}$ has the form $f(x)=a_o+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+O(x^6)$. Determine a_5

Solution: 5

6. (6 pts) Find the interval where the following power series is convergent:

$$\sum_{n=2}^{\infty}\frac{2^{-n}}{\ln n}(x-1)^n$$

Solution: $[-1,3)$

7. (5 pts). Find the area enclosed by the curve $x = a\cos t\,, y = b\sin t$ using parametric equations

Solution: πab

8. (7 pts) Prove that the following sequence converges and find its limit

$$a_{n+1} = \sqrt{1 + a_n}, a_1 = 1$$

Solution: First, we prove the sequence is monotonically increasing and bounded. There are multiple ways to do so. For example, note, $a_1 = 1, a_2 = \sqrt{2}$, etc. All the terms have to be positive. We have then $a_{n+1}^2 - a_n^2 = a_n - a_{n-1}$ which implies

$$a_{n+1} - a_n = \frac{a_n - a_{n-1}}{a_{n+1} + a_n}.$$

Thus, $a_{n+1} > a_n$ if $a_n > a_{n-1} > 0$. By induction on n, we conclude that a_n is monotonically increasing. It is also bounded since, if $a_n \geq 3$ then $a_{n+1} \leq a_n$.

By monotonic convergence theorem, the sequence converges $a_n \to a$. The limit must satisfy $a = \sqrt{1+a}$ or equivalently $a^2 - a - 1 = 0$. Thus, we have

$$a = (1 \pm \sqrt{5})/2.$$

The minus sign leads to negative values. Therefore the limit is $(1 + \sqrt{5})/2$.

Grading criteria: Full solution is 5 points. Proving monotonicity 2 points. Proving convergence 2 points. Correct limit 1 point.

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9. (7 pts) Determine if the series convergent (give a complete proof indicating which convergence criteria you are using)

$$\sum_{n=2}^{\infty} \frac{\sin n}{n\,(\ln n)^2}$$

Solution:
We use absolute convergence criterion as follows:

$$\left|\frac{\sin n}{n(\ln n)^2}\right| \leq \frac{1}{n(\ln n)^2}.$$

But the series $\sum_n \frac{1}{n(\ln n)^2}$ converges by integral test as $\int_2^\infty \frac{dx}{x\ln^2 x}$ converges.
Grading criteria: Correct use of inequality 3 points. Using integral test 2 more points.

10. (10 pts) Using Maclaurin series find the sum of the series

(a) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{n!}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$

Solution

(a)

We can write this as $$\sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^n}{n!} = e^{-x^4}.$$

(b) Since

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ for all } x$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},$$

11. (5 pts) Find Taylor polynomial $T_2(x)$ for the function $f(x)$ centered at $a = 1$

$$f(x) = \cot(x).$$

Solution

$$f'(x) = -\frac{1}{\sin^2 x}, f''(x) = 2\frac{\cos(x)}{\sin^3(x)}$$

$$T_2(x) = \cot(1) - \frac{1}{\sin^2 1}(x-1) + \frac{1}{2}2\frac{\cos(1)}{\sin^3(1)}(x-1)^2 = \cot(1) - \frac{1}{\sin^2 1}(x-1) + \frac{\cos(1)}{\sin^3(1)}(x-1)^2.$$

Grading criteria: Correct answer is 5 points. Small mistake like missing factor of 1/2 , reduce by 1 point.

12. (6+6+6 pts) Consider the curve in polar coordinates $r = 1 + \cos\theta$

(a) Sketch the curve. Find the points on the curve where the tangent line is horizontal or vertical.

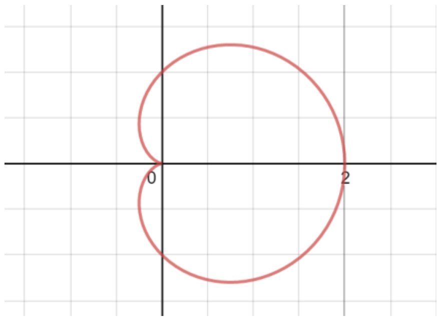
(b) Find the area enclosed by the curve.

(c) Find the length of this curve.

Solution:

(a)

Comment about grading: In general, do not reduce points if they do not find explicitly correct angles, for example if someone forgot $\cos 2\pi/3 = -1/2$. (assuming the qualitative description is correct along with the figure) Also, if integrals in (b),(c) are set up correctly but there is computational error, reduce only by 1/2 point.



To find vertical/horizontal tangents:

$$x' = \frac{d}{d\theta}(\cos\theta(1+\cos\theta)) = -\sin\theta - 2\cos\theta\sin\theta = 0.$$

$$y' = \frac{d}{d\theta}(\sin\theta(1+\cos\theta)) = \cos\theta + 2\cos^2\theta - 1 = 0.$$

Then, $x' = 0$ if $\theta = 0, \pi, 2\pi/3, 4\pi/3$ and $y' = 0$ if $\theta = \pi, \pm\pi/3$.

Next, using that at $\theta = \pm\pi/3$, $x' \neq 0$ and $y' = 0$, we conclude that tangents there are horizontal. Similarly at $\theta = 0, 2\pi/3, 4\pi/3$, the tangent is vertical.

To determine the tangent at $\theta = \pi$, we need to use L'Hospital rule since both x', y' vanish there.

$$\frac{y'}{x'} = \frac{\cos\theta + 2\cos^2\theta - 1}{-\sin\theta - 2\cos\theta\sin\theta} \to (L'H) \to \frac{-\sin\theta - 4\cos\theta\sin\theta}{-\cos\theta + 2\sin^2\theta - 2\cos^2\theta} \to 0$$

as $\theta \to \pi$.

(b)

$$S = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{1}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)d\theta = 3\pi.$$

(c)

$$L = \int_0^{2\pi}\sqrt{r^2 + (r')^2}d\theta = \int_0^{2\pi}\sqrt{(1+\cos\theta)^2 + (\sin\theta)^2}d\theta = \int_0^{2\pi}\sqrt{r^2 + (r')^2}d\theta = \int_0^{2\pi}\sqrt{2 + 2\cos\theta}d\theta = 8.$$

13. (7 pts) Find $f^{(1000)}(0)$ of following function:

$$f(x) = \ln(1 + x + x^2 + x^3)$$

Hint: find $\ln(1 + x)$ Taylor series first.

Solution:

$f(x) = \ln(1 + x + x^2 + x^3) = \ln(1 + x) + \ln(1 + x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{n}$

After 1000 times derivative, only the term with x^{1000} is constant

$$\frac{f^{(1000)}(0)}{1000!} = \frac{(-1)^{999}}{1000} + \frac{(-1)^{499}}{500}$$

$$f^{(1000)}(0) = -\frac{3}{1000} \times 1000! = -3 \times 999!$$

14. (5+5 pts)

$$y = \int_1^x \sqrt{\sqrt{t} - 1}dt, \quad 1 \leq x \leq 16$$

(a) Find the length of the curve

(b) Find the area of the surface obtained by rotating the curve about the y-axis.

Solution.

$y = \int_1^x \sqrt{\sqrt{t} - 1}\, dt \quad \Rightarrow \quad dy/dx = \sqrt{\sqrt{x} - 1} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + \left(\sqrt{x} - 1\right) = \sqrt{x}.$

Thus, $L = \int_1^{16} \sqrt{\sqrt{x}}\, dx = \int_1^{16} x^{1/4}\, dx = \frac{4}{5}\left[x^{5/4}\right]_1^{16} = \frac{4}{5}(32 - 1) = \frac{124}{5}$.

$S = \int_1^{16} 2\pi x\, ds = 2\pi \int_1^{16} x \cdot x^{1/4}\, dx = 2\pi \int_1^{16} x^{5/4}\, dx = 2\pi \cdot \frac{4}{9}\left[x^{9/4}\right]_1^{16} = \frac{8\pi}{9}(512 - 1) = \frac{4088}{9}\pi$

15. (Extra credits: 3 pts) Find all the solutions of the equation

$$1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots = 0$$

(Hint: consider the cases $x \geq 0$ and $x < 0$ separately.)

Solution.

Let $f(x)$ denote the left-hand side of the equation $1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots = 0$. If $x \geq 0$, then $f(x) \geq 1$ and there are no solutions of the equation. Note that $f(-x^2) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \cos x$. The solutions of $\cos x = 0$ for $x < 0$ are given by $x = \frac{\pi}{2} - \pi k$, where k is a positive integer. Thus, the solutions of $f(x) = 0$ are $x = -\left(\frac{\pi}{2} - \pi k\right)^2$, where k is a positive integer.