ECE 313: Problem Set 13: Problems

Due: Sunday, Dec 22 at 11:59:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.9.2, 4.9.3, 4.10.1, 4.10.2, and 4.11

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: You must upload handwritten homework to BB. No typeset homework will be accepted. No late homework will be accepted. Please write on the top right corner of the first page:
NAME AS IT APPEARS ON BB
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings.

1. **[MMSE Estimation]**
   Consider the joint pdf below describing the dependence between random variables X and Y:

$$f_{X,Y} = \begin{cases} 2e^{-(u+v)}, & 0 < u < v < \infty; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

   We wish to design various types of minimum mean squared error (MMSE) estimators of Y.

   (a) Determine the MMSE optimal constant estimator c^* of Y and the resulting MSE.

(b) Determine the MMSE optimal unconstrained estimator $g^*(X)$ and the resulting MSE. Compare the MSE values obtained in parts (a) and (b).

(c) Determine the MMSE linear estimator $L^*(X)$ and the resulting MSE.

2. **[Linear MMSE Estimation]**
Suppose Y is estimated by a linear estimator, $L(X_1, X_2) = a + bX_1 + cX_2$, such that X_1 and X_2 have mean zero and are uncorrelated with each other.

   (a) Determine a, b and c to minimize the MSE, $E[(Y - (a + bX_1 + cX_2))^2]$. Express your answer in terms of $E[Y]$, the variances of X_1 and X_2, and the covariances $\text{Cov}(Y, X_1)$ and $\text{Cov}(Y, X_2)$.

   (b) Express the MSE for the estimator found in part (a) in terms of the variances of X_1, X_2, and Y and the covariances $\text{Cov}(Y, X_1)$ and $\text{Cov}(Y, X_2)$.

3. **[Random Autograder]**
You take an exam with 100 Short Answer Questions (SAQs), with each problem worth 2 points for a total of 200 points. The exam is graded using an autograding program. Unfortunately (or fortunately) the autograder is faulty and it assigns a score that is a random (real) number that is uniformly distributed in the interval $[1, 2]$ for each problem, regardless of your answer. The scores assigned to the different problems are independent.

   (a) Based on the LLN, what is the total score that you will likely receive on the exam?

(b) Now use the CLT to estimate the probability that the total score will be greater than 155.

4. **[Chebychev inequality vs. CLT for confidence intervals]**
Recall from Section 2.9 that if X has the binomial distribution with parameters n and p, the Chebychev inequality implies that if $\widehat{p} = \frac{X}{n}$, then p is contained in the (random) confidence interval $\left[\widehat{p} - \frac{a}{2\sqrt{n}}, \widehat{p} + \frac{a}{2\sqrt{n}}\right]$ with probability at least $1 - \frac{1}{a^2}$.
A less conservative, commonly used approach is based on the fact that by the CLT,

$$P\left\{ \frac{|X - np|}{\sqrt{np(1-p)}} \geq c \right\} \approx 2Q(c).$$

In this problem you will explore how you might use this approximation for setting confidence intervals.

(a) Calculate the value of n sufficiently large that, by the Chebychev inequality, the random interval $[\widehat{p} - 0.05, \widehat{p} + 0.05]$ contains the true value p with probability at least 99%.

(b) In this part you will calculate the value of n sufficiently large that, according to the CLT, the approximate probability the same random interval as in part (a) contains the true value p is at least 99%.
Hint: Start with the following step:

$$P\left\{|\widehat{p} - p| \geq 0.05\right\} = P\left\{ \frac{|X - np|}{\sqrt{np(1-p)}} \geq \frac{0.05\sqrt{n}}{\sqrt{p(1-p)}} \right\} \leq P\left\{ \frac{|X - np|}{\sqrt{np(1-p)}} \geq 2(0.05)\sqrt{n} \right\},$$

since $\hat{p} = \frac{X}{n}$ and $\sqrt{p(1-p)} \leq \frac{1}{2}$ (see Section 2.9 for details).

5. **[Jointly Gaussian Random Variables]**
Let X and Y be jointly Gaussian random variables with $\mu_X = 0$, $\mu_Y = -1$, $\sigma_X^2 = 1$, and $\rho_{X,Y} = -\frac{1}{2}$.

(a) In order for $X + Y$ and $X - Y$ to be independent, what must σ_Y^2 be?
Hint: One may use the fact that X and Y are jointly Gaussian, which implies the equivalence between independence and uncorrelatedness.

(b) Find $P(2X + Y > 0)$.

(c) Find $P(2X + Y > 0 | X - Y = 0)$.