1. The curves $\mathbf{r}_1(t) = \langle t,\ 2t^2,\ t^3 \rangle, \quad \mathbf{r}_2(t) = \langle \sin t,\ \sin 2t,\ 3t \rangle$ intersect at the origin. Find the cosine value of angle of intersection between the curves at the origin.

2. (a) Let $\mathbf{r}(t)$ be a differentiable vector-valued function such that $\mathbf{r}(t) \neq \mathbf{0}$. Show that: $\frac{d}{dt} |\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|} \mathbf{r}(t) \cdot \mathbf{r}'(t)$ (**Hint:** $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$)

   (b) Let $\mathbf{r} : I \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ be a C^2-curve with nonzero curvature, and let $t \in I$. Prove that the derivative $\frac{d}{dt} \frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$ is perpendicular to $\mathbf{r}(t)$

3. Show that the osculating plane at every point on the curve

$$\mathbf{r}(t) = \langle t + 2,\ 1 - t,\ \tfrac{1}{2} t^2 \rangle$$

is the same plane.

1. **(total 3 points)** The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\mathbf{r}_1'(t) = \langle 1, 4t, 3t^2\rangle, \quad \mathbf{r}_1'(0) = \langle 1, 0, 0\rangle$ **(1 point)** is a tangent vector to $\mathbf{r}_1$ at $(0,0,0)$. Similarly, $\mathbf{r}_2'(t) = \langle \cos t, 2\cos 2t, 3\rangle, \quad \mathbf{r}_2'(0) = \langle 1, 2, 3\rangle$ **(1 point)** is a tangent vector to $\mathbf{r}_2$ at $(0,0,0)$. If θ is the angle between these two tangent vectors, then

$$\cos\theta = \frac{\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(0)}{\|\mathbf{r}_1'(0)\|\,\|\mathbf{r}_2'(0)\|} = \frac{\langle 1,0,0\rangle\cdot\langle 1,2,3\rangle}{1\cdot\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}}.\textbf{(1 point)}$$

2. **(total 4 points: 2 point each)**

(a)

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{d}{dt}\left[(\mathbf{r}(t)\cdot\mathbf{r}(t))\right]^{1/2} = \frac{1}{2}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]^{-1/2}\cdot\frac{d}{dt}\left(\mathbf{r}(t)\cdot\mathbf{r}(t)\right)$$

$$= \frac{1}{2|\mathbf{r}(t)|}\cdot 2\left(\mathbf{r}(t)\cdot\mathbf{r}'(t)\right) = \frac{1}{|\mathbf{r}(t)|}\cdot\left(\mathbf{r}(t)\cdot\mathbf{r}'(t)\right)$$

$$\Rightarrow \frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\,\mathbf{r}(t)\cdot\mathbf{r}'(t)$$

(b) According to

$\boxed{4}$ **Theorem** If $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

$\left|\frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}\right| = 1$, so the derivative $\frac{d}{dt}\frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$ is perpendicular to $\mathbf{r}(t)$

or:

$$\frac{d}{dt}\left[\frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}\right] = \frac{|\mathbf{r}(t)|\cdot\mathbf{r}'(t) - \mathbf{r}(t)\cdot\frac{d}{dt}|\mathbf{r}(t)|}{|\mathbf{r}(t)|^2}$$

$$= \frac{|\mathbf{r}(t)|\cdot\mathbf{r}'(t) - \frac{1}{|\mathbf{r}(t)|}\cdot(\mathbf{r}(t)\cdot\mathbf{r}'(t))\cdot\mathbf{r}(t)}{|\mathbf{r}(t)|^2}$$

$$= \frac{1}{|\mathbf{r}(t)|^2}\left[|\mathbf{r}(t)|\cdot\mathbf{r}'(t) - \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{|\mathbf{r}(t)|}\cdot\mathbf{r}(t)\right]$$

$$\mathbf{r}(t)\cdot\frac{d}{dt}\left[\frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}\right]=\frac{1}{|\mathbf{r}(t)|^2}\left[|\mathbf{r}(t)|\cdot(\mathbf{r}(t)\cdot\mathbf{r}'(t))-\frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{|\mathbf{r}(t)|}\cdot(\mathbf{r}(t)\cdot\mathbf{r}(t))\right]$$
$$=\frac{1}{|\mathbf{r}(t)|^2}\left[|\mathbf{r}(t)|\cdot(\mathbf{r}\cdot\mathbf{r}')-\frac{\mathbf{r}\cdot\mathbf{r}'}{|\mathbf{r}(t)|}\cdot|\mathbf{r}(t)|^2\right]$$
$$=\frac{1}{|\mathbf{r}(t)|^2}\left[|\mathbf{r}(t)|\cdot(\mathbf{r}\cdot\mathbf{r}')-|\mathbf{r}(t)|\cdot(\mathbf{r}\cdot\mathbf{r}')\right]=0$$

3. **total 3 points: 1 point for $\mathbf{T}(t)$; 1 point for $\mathbf{B}(t)$; 1 point for explaining**

60. $\mathbf{r}(t)=\langle t+2,1-t,\frac{1}{2}t^2\rangle \quad\Rightarrow\quad \mathbf{r}'(t)=\langle 1,-1,t\rangle,\quad \mathbf{T}(t)=\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\dfrac{1}{\sqrt{2+t^2}}\langle 1,-1,t\rangle,$

$$\mathbf{T}'(t)=-\tfrac{1}{2}(2+t^2)^{-3/2}(2t)\langle 1,-1,t\rangle+(2+t^2)^{-1/2}\langle 0,0,1\rangle$$
$$=-(2+t^2)^{-3/2}\left[t\langle 1,-1,t\rangle-(2+t^2)\langle 0,0,1\rangle\right]=\frac{-1}{(2+t^2)^{3/2}}\langle t,-t,-2\rangle$$

A normal vector for the osculating plane is $\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)$, but $\mathbf{r}'(t)=\langle 1,-1,t\rangle$ is parallel to $\mathbf{T}(t)$ and $\langle t,-t,-2\rangle$ is parallel to $\mathbf{T}'(t)$ and hence parallel to $\mathbf{N}(t)$, so $\langle 1,-1,t\rangle\times\langle t,-t,-2\rangle=\langle t^2+2,t^2+2,0\rangle$ is normal to the osculating plane for any t. All such vectors are parallel to $\langle 1,1,0\rangle$, so at any point $\left(t+2,1-t,\frac{1}{2}t^2\right)$ on the curve, an equation for the osculating plane is $1[x-(t+2)]+1[y-(1-t)]+0\left(z-\frac{1}{2}t^2\right)=0$ or $x+y=3$. Because the osculating plane at every point on the curve is the same, we can conclude that the curve itself lies in that same plane. In fact, we can easily verify that the parametric equations of the curve satisfy $x+y=3$.