ECE313 Homework 5

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1   Problem 1

Let the random variable X have the following cumulative distribution function (CDF):

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1}$$

1. Find the median of X.

2. Find the first quartiel ($q_{0.25}$) and the third quartile ($q_{0.75}$) of X.

Answer:

1. The median m of X should satisfy the property that $F(m) = 0.5$. Therefore, we can solve the equation

$$x^2 = 0.5 \Rightarrow x = \frac{\sqrt{2}}{2} \tag{1}$$

   So the median is $\frac{\sqrt{2}}{2}$.

2. The first quartile and the third quartile satisfy the following equations

$$F(q_{0.25}) = 0.25 \Rightarrow x^2 = 0.25 \Rightarrow x = 0.5$$

$$F(q_{0.75}) = 0.75 \Rightarrow x^2 = 0.75 \Rightarrow x = \frac{\sqrt{3}}{2} \tag{2}$$

   So the first quartile is 0.5, the first quartile is $\frac{\sqrt{3}}{2}$.

2   Problem 2

Assume that the number of buses arriving at a bus stop in an interval of t seconds, denoted by N, follows a Poisson distribution with parameter $\lambda = 0.3t$. Compute the probabilities of the following events:

1. Exactly 3 buses arrive during a 10-second interval.

2. At most 10 buses arrive during a 20-second interval.

3. The number of arrivals during a 10-second interval is between 2 and 4 (inclusive).

Answer:

1. In this condition, $\lambda = 0.3 \times 10 = 3$. Therefore

$$P(N = 3) = \frac{3^3 e^{-3}}{3!} = \frac{9e^{-3}}{2} \tag{3}$$

2. In this condition, $\lambda = 0.3 \times 20 = 6$. Therefore

$$P(N \le 10) = \sum_{k=0}^{10} \frac{6^k e^{-6}}{k!} \tag{4}$$

3. In this condition, $\lambda = 0.3 \times 10 = 3$. Therefore

$$P(2 \le N \le 4) = \sum_{k=2}^{4} \frac{3^k e^{-3}}{k!} = e^{-3} \left(\frac{9}{2} + \frac{27}{6} + \frac{81}{24} \right) = \frac{99e^{-3}}{8} \tag{5}$$

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3 Problem 3

Let $\{N_t, t \ge 0\}$ be a Poisson process with rate $\lambda > 0$. Answer the following questions; your answers may include λ.

1. Compute the conditional probability

$$P(N_6 - N_4 = 4 \mid N_5 - N_4 = 1) \tag{6}$$

2. Compute the conditional probability

$$P(N_7 - N_2 = 0 \mid N_4 - N_3 = 0) \tag{7}$$

3. The interval $(1, 4]$ is divided into three equal parts: $(1, 2], (2, 3]$, and $(3, 4]$. Given that $N_4 - N_1 = 6$, find the probability that there are $(2, 1, 3)$ arrivals in these three subintervals, respectively.

Answer:

1. Since we already know that $N_5 - N_4 = 1$, then we just need to compute the probability of $N_6 - N_5 = 3$.

$$P(N_6 - N_4 = 4 \mid N_5 - N_4 = 1) = P(N_6 - N_5 = 3) \tag{8}$$

Due to the property of independent increment, we have

$$N_6 - N_5 \sim \text{Poi}(\lambda) \tag{9}$$

Therefore, we can calculate the probability

$$P(N_6 - N_5 = 3) = \frac{\lambda^3 e^{-\lambda}}{3!} \tag{10}$$

2. We can expand the condition probability expression

$$P(N_7 - N_2 = 0 \mid N_4 - N_3 = 0) = \frac{P(N_7 - N_2 = 0 \wedge N_4 - N_3 = 0)}{P(N_4 - N_3 = 0)} = \frac{P(N_7 - N_2 = 0)}{P(N_4 - N_3 = 0)} \tag{11}$$

- We have $N_7 - N_2 \sim \text{Poi}(5\lambda) \Rightarrow P = e^{-5\lambda}$

- We have $N_4 - N_3 \sim \text{Poi}(\lambda) \Rightarrow P = e^{-\lambda}$

Therefore,

$$\frac{e^{-5\lambda}}{e^{-\lambda}} = e^{-4\lambda} \tag{12}$$

3. Under the condition of a given total number of events in a Poisson process, the number of events in each sub-interval follows a multinomial distribution, and the probability is proportional to the length of the interval. The lengths of the three sub-interals are all 1, and the total length is 3, so the probability of each sub-interval is $\frac{1}{3}$

Therefore, the conditional probability is

$$P = \frac{6!}{2!1!3!} \left(\frac{1}{3}\right)^2 \left(\frac{1}{3}\right)^1 \left(\frac{1}{3}\right)^3 = \frac{20}{243} \tag{13}$$

4 Problem 4

The lifetime of the memory chips produced by a factory is exponentially distributed with parameter $\lambda = 0.2(\text{years}^{-1})$. Suppose John bought a computer with a memory chip produced by this factory and after five years it is still working. What is the conditional probability it will still work for at least three more years?

Answer:

The lifetime $T \sim \text{Exp}(\lambda = 0.2)$, given that $T > 5$, we need to compute $P(T > 8 \mid T > 5)$

The exponential distribution is memoryless. Therefore

$$P(T > 8 \mid T > 5) = P(T > 3) = e^{-3\lambda} = e^{-0.6} \tag{14}$$

5 Problem 5

1. Find the PDF of the minimum of two independent exponential random variables with parameter λ.

   Hint: Work with $1 - F_X(x)$, where $F_X(x)$ is the CDF of the minimum. Use the independence property.

2. You have a digital device that requires two batteries to operate. To be on the safe side, you buy three types of batteries (marked as 1, 2, 3), each of which has a lifetime that is exponentially distributed with parameter λ, and operates/fails independently of all the other batteries. Initially, you install two batteries, say 1 and 2. When one of these two batteries fails, you replace it with battery 3. What is the expected total time until your device stops working?

3. In the scenario of part 2, what is the probability that battery 1 is the last battery that still works?

Answer

1. Suppose that $X_1, X_2 \sim \text{Exp}(\lambda)$, and let $Y = \min(X_1, X_2)$

   Then the CDF of Y is

$$F_Y(y) = P(Y \leq y) = 1 - P(X_1 > y, X_2 > y) = 1 - e^{-\lambda y}e^{-\lambda y} = 1 - e^{-2\lambda y} \quad (15)$$

   The PDF is the derivative of the CDF:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = 2\lambda e^{-2\lambda y}, \ y \geq 0 \tag{16}$$

2. The device stop working if any one working batteries fails.

   - Initially, use batteries 1 and 2. Let their lifetimes be $X_1, X_2 \sim \text{Exp}(\lambda)$
   - The first failure time $T_1 = \min(X_1, X_2) \sim \text{Exp}(2\lambda)$, the expectation is $\frac{1}{2\lambda}$
   - Suppose that battery 1 fails first (due to the symmetry, the probability is 0.5), then replace it with battery 3, whose lifetime $X_3 \sim \text{Exp}(\lambda)$.
   - At this time, the device continues working, until battery 2 or battery 3 fails, meaning working for more $\min(X_2 - T_1, X_3)$.

   Given that the exponential distribution is memoryless, $X_2 - T_1 \mid X_2 > T_2 \sim \text{Exp}(\lambda)$, which is i.i.d. with X_3. Therefore, the second stage working time $T_2 = \min(\text{Exp}(\lambda), \text{Exp}(\lambda)) \sim \text{Exp}(2\lambda)$, whose expectation is $\frac{1}{2\lambda}$.

   The total expected working time is

$$\mathbb{E}[T] = \mathbb{E}[T_1] + \mathbb{E}[T_2] = \frac{1}{2\lambda} + \frac{1}{2\lambda} = \frac{1}{\lambda} \tag{17}$$

3. We can seperate the process into 2 stages

   - First stage: $\min(X_1, X_2) \sim \text{Exp}(2\lambda)$, the probability that battery still works is $P(X_1 > X_2) = \frac{1}{2}$
   - Given that bettery 1 is "alive", the second stage we compare $X_1' \sim \text{Exp}(\lambda)$ and $X_3 \sim \text{Exp}(\lambda)$, the probability that battery still works is $P(X_1' > X_3) = \frac{1}{2}$.

   So the total probability is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.