Math 241 Calculus III (Prof. Honold) **Midterm Exam 1** 2022/10/19

Name: ____________________ Student ID : ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. The volume of the parallelepiped spanned by the vectors $(1,-1,2)$, $(a,1,-1)$, $(-1,2,2)$ is equal to 1 for

   ☐ $a=-2$ ■ $a=-1$ ☐ $a=0$ ☐ $a=1$ ☐ $a=2$

2. The tangent to $f(t)=(t,t^2,t^4)$ in the point $(1,1,1)$ meets the plane $x+2y-z=3$ in

   ☐ no point. ☐ $(-1,3,-5)$ ■ $(2,3,5)$ ☐ $(-1,-3,-7)$ ☐ $(2,3,-7)$

3. The length of the arc of $\gamma(t)=(t^3-1,6t,3t^2-3)$ between $(0,6,0)$ and $(-2,-6,0)$ is

   ☐ 6 ☐ 8 ☐ 10 ☐ 12 ■ 14

4. For a C^2-curve $\mathbf{r}\colon I\to\mathbb{R}^3\setminus\{\mathbf{0}\}$ with nonzero curvature and $t\in I$, the derivative $\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$ is perpendicular to

   ■ $\mathbf{r}(t)$ ☐ $\mathbf{r}'(t)$ ☐ $\mathbf{r}''(t)$ ☐ $\mathbf{N}(t)$ ☐ $\mathbf{B}(t)$

5. For $\mathbf{A}=\begin{pmatrix}\sqrt{3}/2 & 1/2\\ 1/2 & -\sqrt{3}/2\end{pmatrix}$ the smallest positive integer k such that $\mathbf{A}^k=\mathbf{I}_2$ (the 2×2 identity matrix) is

   ■ 2 ☐ 3 ☐ 6 ☐ 12 ☐ 24

6. The distance between the lines $\mathbb{R}(1,-1,1)$ and $(1,2,-3)+\mathbb{R}(1,1,-1)$ is

   ☐ $1/2$ ■ $1/\sqrt{2}$ ☐ 1 ☐ $\sqrt{2}$ ☐ 2

7. The unit normal vector $\mathbf{N}(1)$ of the curve $f(t)=(t,t^2/2,t^3/3)$ is a positive multiple of

   ☐ $(0,1,-1)$ ☐ $(0,-1,1)$ ☐ $(0,0,1)$ ■ $(-1,0,1)$ ☐ $(1,0,-1)$

8. If $f\colon[0,2\pi]\to\mathbb{R}^3$ satisfies $f(0)=(0,0,0)$, $f'(0)=(0,1,-1)$ and $f''(t)=(1,\cos t,\sin t)$, the point $f(2\pi)$ is equal to

   ☐ $(2\pi^2,0,2\pi)$ ☐ $(\pi^2,0,2\pi)$ ■ $(2\pi^2,2\pi,0)$ ☐ $(0,2\pi,2\pi)$ ☐ $(\pi^2,2\pi,0)$

9. The 2-contour (level-2 set) of $f(x,y)=\dfrac{1}{x^2+y^2-1}$ is

   ☐ empty ☐ a point ☐ a line ■ a circle ☐ a sphere

10. The paths of the curves $f(t)=(t,t^2,t^3)$ and $g_b(t)=\bigl(1+2t,(1-b)t,t\bigr)$ intersect for

    ■ $b=2$ ☐ no $b\in\mathbb{R}$ ☐ $b=1$ ☐ all $b\in\mathbb{R}$ ☐ $b=0$

Time allowed: 45 min CLOSED BOOK ***Good luck!***

Notes

Notes are only provided for Midterm 1-A. Students of Groups B and C, which had the same midterm paper, should locate their questions in Midterm 1-A. Only Q1, Q2, Q10 were different, and the necessary modifications in these cases you can figure out yourself.

1 $\begin{vmatrix} 1 & a & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 2 \end{vmatrix} = 2-1+4a-(-2-2-2a) = 5+6a = -1$ for $a=-1$. The volume is also 1 if $5+6a=1$, i.e. $a=-2/3$, but this answer is not offered.

2 The tangent is $f(1)+\mathbb{R}f'(1) = (1,1,1)+\mathbb{R}(1,2,4) = \{(1+t,1+2t,1+4t); t\in\mathbb{R}\}$. The condition $x+2y-z = 1+t+2(1+2t)-(1+4t) = 2+t = 3$ gives $t=1$ and the point $(1,1,1)+(1,2,4) = (2,3,5)$.

3 The two points are $\gamma(1) = (0,6,0)$, $\gamma(-1) = (-2,-6,0)$. Hence the length of the arc is

$$\int_{-1}^{1} |\gamma'(t)|\,\mathrm{d}t = \int_{-1}^{1} |(3t^2,6,6t)|\,\mathrm{d}t = \int_{-1}^{1} \sqrt{9t^4+36+36t^2}\,\mathrm{d}t = \int_{-1}^{1} 3t^2+6\,\mathrm{d}t = 2\int_{0}^{1} 3t^2+6\,\mathrm{d}t = 2\left[t^3+6t\right]_0^1$$
$$= 2\cdot 7 = 14.$$

4 Since the curve has constant length, it is perpendicular to its derivative (from $(\mathrm{d}/\mathrm{d}t)\,|f|^2 = (\mathrm{d}/\mathrm{d}t)(f\cdot f) = 2f\cdot f'$), and, since the curve is a scalar multiple of $\mathbf{r}(t)$, the same is true of $\mathbf{r}(t)$.

5 Since $\mathbf{A} = S(30^\circ)$ is a reflection matrix, its square muts be the identity matrix. One can also verify this through direct computation, of course.

6 The distance d is the same as the distance from the point $(1,2,-3)$ to the plane $\mathbb{R}(1,-1,1)+\mathbb{R}(1,1,-1)$, which is given by the length of the orthogonal projection of $(1,2,-3)$ onto any normal vector $\mathbf{n}$ of the plane. We can take $\mathbf{n} = (0,1,1)$ and obtain

$$d = \left| \frac{(1,2,-3)\cdot(0,1,1)}{(0,1,1)\cdot(0,1,1)} \begin{pmatrix} 0\\1\\1 \end{pmatrix} \right| = \frac{|(1,2,-3)\cdot(0,1,1)|}{|(0,1,1)|} = \frac{1}{\sqrt{2}}.$$

7 As shown in the lecture, $\mathbf{N}(1)$ can be obtained by subtracting from $f''(1)$ its orthogonal projection onto $f'(1)$ and normalizing to unit length. We have $f'(t) = (1,t,t^2)$, $f''(t) = 0,1,2t)$, and hence

$$f''(1) - \frac{f''(1)\cdot f'(1)}{|f'(1)|^2} f'(1) = \begin{pmatrix} 0\\1\\2 \end{pmatrix} - \frac{(0,1,2)\cdot(1,1,1)}{|(1,1,1)|^2} \begin{pmatrix} 1\\1\\1 \end{pmatrix} = \begin{pmatrix} 0\\1\\2 \end{pmatrix} - \begin{pmatrix} 1\\1\\1 \end{pmatrix} = \begin{pmatrix} -1\\0\\1 \end{pmatrix},$$
$$\mathbf{N}(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1\\0\\1 \end{pmatrix}.$$

Alternatively,

$$\begin{aligned}
\mathbf{T}(t) &= \frac{(1,t,t^2)}{\sqrt{1+t^2+t^4}},\\
\mathbf{T}'(t) &= \frac{(0,1,2t)}{\sqrt{1+t^2+t^4}} - \frac{2t+4t^3}{2(1+t^2+t^4)^{3/2}}(1,t,t^2)\\
&= \frac{(1+t^2+t^4)(0,1,2t)-(t+2t^3)(1,t,t^2)}{(1+t^2+t^4)^{3/2}},\\
\mathbf{T}'(1) &= \frac{3(0,1,2)-3(1,1,1)}{3\sqrt{3}} = \frac{1}{\sqrt{3}}(-1,0,1),
\end{aligned}$$

which gives again $\mathbf{N}(1) = \mathbf{T}'(1)\big/|\mathbf{T}'(1)| = \frac{1}{\sqrt{2}}(-1,0,1)$.

8 Here we obtain

$$\begin{aligned}
f'(t) &= f'(0) + \int_0^t f''(s)\,\mathrm{d}s = (0,1,-1) + \int_0^t (1,\cos s,\sin s)\,\mathrm{d}s = (0,1,-1) + [(s,\sin s,-\cos s)]_0^t \\
&= (0,1,-1) + (t,\sin t,1-\cos t) = (t,1+\sin t,-\cos t), \\
f(t) &= f(0) + \int_0^t f'(s)\,\mathrm{d}s = \int_0^t (s,1+\sin s,-\cos s)\,\mathrm{d}s = \left[(s^2/2,s-\cos s,-\sin s)\right]_0^t \\
&= (t^2/2,t-\cos t+1,-\sin t), \\
f(2\pi) &= (2\pi^2,2\pi,0).
\end{aligned}$$

9 The 2-contour has the equation $\frac{1}{x^2+y^2-1} = 2$, which is equivalent to $x^2+y^2 = 3/2$.

10 The paths intersect if $f(t) = g_b(s)$ is solvable. This gives the system $t = 1+2s$, $t^2 = (1-b)s$, $t^3 = s$. If the system is solvable, $(1+2s)^3 = s$, which has only the solution $s = -1$, since it is equivalent to $8s^3+12s^2+5s+1 = (s+1)(8s^2+4s+1) = 0$. In this case, $t = s = -1$, $b = 2$.