1. (a) Find an equational representation $a_1x_1 + a_2x_2 + a_3x_3 = b$ of the plane with parametric representation:

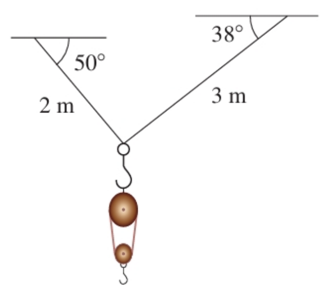
$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}, c_1, c_2 \in \mathbb{R}$$

   What is the geometric meaning of the vector $\mathbf{a} = (a_1, a_2, a_3)$

   (b) Find a parametric representation of the plane $x_1 + x_2 + x_3 = 1$

2. Consider the points P such that the distance from P to $A(-1, 5, 3)$ is twice the distance from P to $B(6, 2, -2)$. Show that the set of all such opints is a sphere, and find its center and radius.

3. A block-and-tackle pulley hoist is suspended in a warehouse by ropes of lengths 2 m and 3 m. The hoist weighs 350 N. The ropes, fastened at different heights, make angles of 50° and 38° with the horizontal. Find the tension in each rope and the magnitude of each tension (cos and sin in results are acceptable).



4. Show that if θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$

1. (a) The desired equation has the form $a_1x_1+a_2x_2+a_3x_3 = b$ with $a_1, a_2, a_3, b \in \mathbb{R}$ and not all a_i equal to zero. Substituting the coordinates x_i gives

$$a_1(1+3c_2) + a_2(2c_1+3c_2) + a_3(-2-2c_1+c_2) = b \quad \text{for all} \quad c_1, c_2 \in \mathbb{R}.$$

This is equivalent to

$$a_1 - 2a_3 + (2a_2 - 2a_3)c_1 + (3a_1 + 3a_2 + a_3)c_2 = b \quad \text{for all} \quad c_1, c_2 \in \mathbb{R},$$

and in turn to

$$a_1 - 2a_3 = b \quad \wedge \quad 2a_2 - 2a_3 = 0 \quad \wedge \quad 3a_1 + 3a_2 + a_3 = 0.$$

Setting $a_3 = 1$ gives $a_2 = 1$, $a_1 = -\frac{4}{3}$, $b = -\frac{10}{3}$. Hence an equation for the plane is

$$-4x_1 + 3x_2 + x_3 = -10,$$

obtained by scaling the original equation by 3 (so that the coefficients become integers).

The geometric meaning of $\mathbf{a} = (a_1, a_2, a_3)$ is taht $\mathbf{a}$ must be orthogonal to any direction vector of H (i.e. a so-called *normal vector* of H)

(b) First we can get a point on the plane $(1, 0, 0)$

Then we can get 2 non-collinear vector that is orthogonal to the normal vector $(1, 1, 1)$. We can choose $(-1, 1, 0)$ and $(-1, 0, 1)$ and obtain

$$H = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mathbb{R}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \mathbb{R}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

2. Let $P = (x, y, z)$. Then $2|PB| = |PA| \iff 4|PB|^2 = |PA|^2 \iff$
$4\left[(x-6)^2 + (y-2)^2 + (z+2)^2\right] = (x+1)^2 + (y-5)^2 + (z-3)^2 \iff$
$4(x^2-12x+36) - x^2 - 2x + 4(y^2-4y+4) - y^2 + 10y + 4(z^2+4z+4) - z^2 + 6z = 35 \iff$
$3x^2 - 50x + 3y^2 - 6y + 3z^2 + 22z = 35 - 144 - 16 - 16 \iff$

$x^2 - \frac{50}{3}x + y^2 - 2y + z^2 + \frac{22}{3}z = -\frac{141}{3}$

By completing the square three times we get

$$\left(x - \frac{25}{3}\right)^2 + (y-1)^2 + \left(z + \frac{11}{3}\right)^2 = -\frac{423 + 625 + 9 + 121}{9} = \frac{332}{9},$$

which is an equation of a sphere with center $\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$ and radius $\sqrt{\frac{332}{3}}$.

3. Call the two tension vectors T_2 and T_3, corresponding to the ropes of length 2 m and 3 m. In terms of vertical and horizontal components, $T_2 = -|T_2|\cos 50°\mathbf{i} + |T_2|\sin 50°\mathbf{j}$ (1) and $T_3 = |T_3|\cos 38°\mathbf{i} + |T_3|\sin 38°\mathbf{j}$ (2).

   The resultant of these forces, $T_2 + T_3$, counterbalances the weight of the hoist (which is $-350\mathbf{j}$), so $T_2 + T_3 = 350\mathbf{j}$, which gives $(-|T_2|\cos 50° + |T_3|\cos 38°)\mathbf{i} + (|T_2|\sin 50° + |T_3|\sin 38°)\mathbf{j} = 350\mathbf{j}$. Equating components, we have $-|T_2|\cos 50° + |T_3|\cos 38° = 0 \Rightarrow |T_2| = |T_3|\frac{\cos 38°}{\cos 50°}$ and $|T_2|\sin 50° + |T_3|\sin 38° = 350$.

   Substituting the first equation into the second gives $|T_3|\frac{\cos 38°}{\cos 50°}\sin 50° + |T_3|\sin 38° = 350 \Rightarrow |T_3|(\cos 38°\tan 50° + \sin 38°) = 350$, so $|T_3| = \frac{350}{\cos 38°\tan 50° + \sin 38°} \approx 225.11\,\mathrm{N}$ and $|T_2| = |T_3|\frac{\cos 38°}{\cos 50°} \approx 275.97\,\mathrm{N}$.

   Finally, from (1) and (2), the tension vectors are $T_2 \approx -177.39\mathbf{i} + 211.41\mathbf{j}$ and $T_3 \approx 177.39\mathbf{i} + 138.59\mathbf{j}$.

4. PROOF:

   Let $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$

   By applying the Law of Cosines to triangle OAB, we get

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$$
$$\Rightarrow |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

   Since $|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}$

   We get

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}$$

   Thus

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$