12 Vectors and the Geometry of Space

12.3 The Dot Product

Context

- Dot Product of Two Vectors
- Direction Angles and Direction Cosines
- Projections
- Application

The Dot Product of Two Vectors

The Dot Product of Two Vectors

To find the dot product of vectors **a** and **b** we multiply corresponding components and add.

1 Definition of the Dot Product

If $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of **a** and **b** is the number a • b given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Although Definition 1 is given for three-dimensional vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

The Dot Product of Two Vectors

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

2 Properties of the Dot Product If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot (\mathbf{b}) = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

MAQ: Which of the following expressions are meaningful?

A $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$

B $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

C $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$

D $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$

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Solution:

$\mathbf{a} \cdot \mathbf{b}$ is a scalar, and the dot product is defined only for vectors, so $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ has no meaning.

$(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ is a scalar multiple of a vector, so it does have meaning.

Both $\mathbf{a}$ and $\mathbf{b}+\mathbf{c}$ are vectors, so the dot product $\mathbf{a} \cdot (\mathbf{b}+\mathbf{c})$ has meaning.

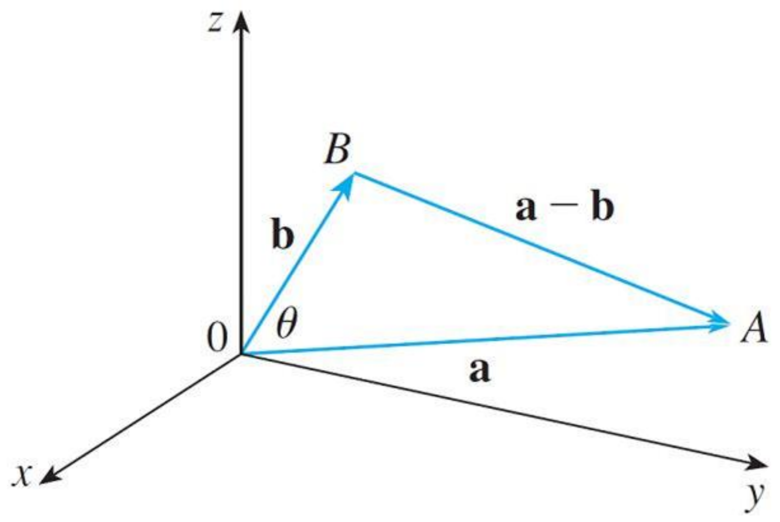
$\mathbf{a} \cdot \mathbf{b}$ is a scalar, but $\mathbf{c}$ is a vector, and so the two quantities cannot be added and $\mathbf{a} \cdot \mathbf{b}+\mathbf{c}$ has no meaning.

The Dot Product of Two Vectors

The formula in the following theorem is used by physicists as the *definition* of the dot product.

3 Theorem If θ is the angle between the vectors **a** and **b**. then

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta$$



The Dot Product of Two Vectors

The formula in Theorem 3 also enables us to find the angle between two vectors.

6 Corollary If θ is the angle between the nonzero vectors **a** and **b,** then

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

单选题 1分

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MCQ： Find the angle between the vectors $\boldsymbol{a} = \langle 2,2,-1\rangle$ and $\boldsymbol{b} = \langle 5,-3,2\rangle$.

A $\cos^{-1}\left(\frac{2}{3\sqrt{38}}\right)$

B $\cos^{-1}\left(\frac{2}{3\sqrt{26}}\right)$

C $\cos^{-1}\left(\frac{3}{4\sqrt{38}}\right)$

D $\cos^{-1}\left(\frac{2}{4\sqrt{38}}\right)$

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Example

Find the angle between the vectors $\boldsymbol{a} = \langle 2,2,-1 \rangle$ and $\boldsymbol{b} = \langle 5,-3,2 \rangle$.

Solution:

Since

$$|\boldsymbol{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \quad \text{and} \quad |\boldsymbol{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\boldsymbol{a} \cdot \boldsymbol{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between **a** and **b** is

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \\ &\approx 1.46 \ \ (\text{or } 84°) \end{aligned}$$

The Dot Product of Two Vectors

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \frac{\pi}{2}$. Then Theorem 3 gives

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\left(\frac{\pi}{2}\right) = 0$$

and conversely if **a** • **b** = 0, then cos θ = 0, so $\theta = \frac{\pi}{2}$. The zero vector **0** is considered to be perpendicular to all vectors.

Therefore we have the following method for determining whether two vectors are orthogonal.

7 Two vectors a and b are orthogonal if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0.$

The Dot Product of Two Vectors

Because cos $\theta > 0$ if $0 \leq \theta < \frac{\pi}{2}$ and cos $< \theta < 0$ if $\frac{\pi}{2} < 0 \leq \pi$,

we see that **a** · **b** is positive for $\theta < \frac{\pi}{2}$ and negative for $\theta > \frac{\pi}{2}$.

We can think of **a** · **b** as measuring the extent to which **a** and **b** point in the same direction.

The dot product **a** · **b** is positive if **a** and **b** point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 2).

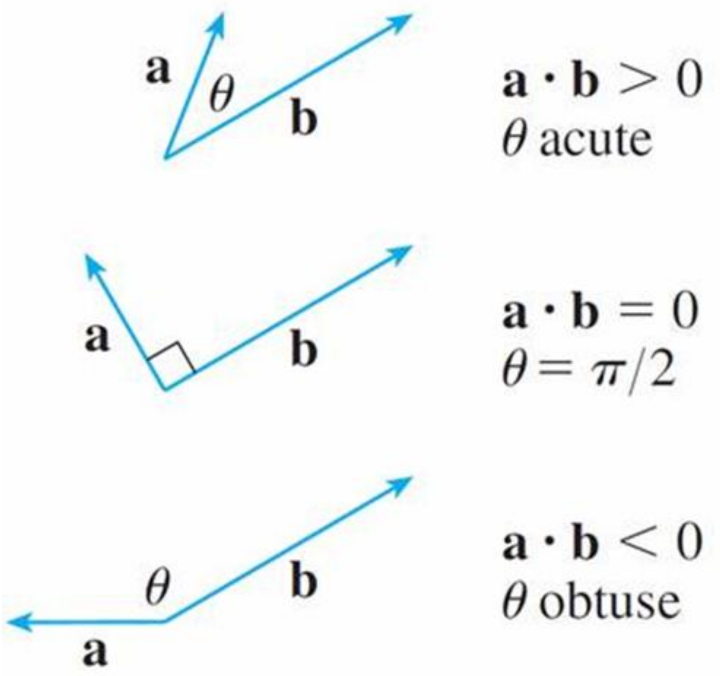
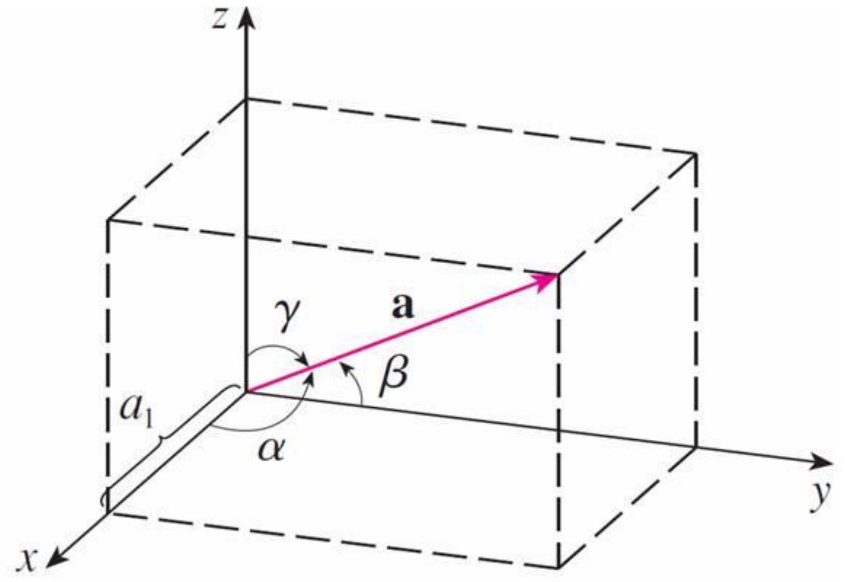


Figure 2

Direction Angles and Direction Cosines

Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector **a** are the angles α, β, and γ (in the interval $[0, \pi]$) that **a** makes with the positive x-, y-, and z-axes, respectively.



Direction Angles and Direction Cosines

The cosines of these direction angles, cos α, cos β, and cos γ, are called the **direction cosines** of the vector **a**. Using Corollary 6 with **b** replaced by **i**, we obtain

8 $$\cos\alpha = \frac{\boldsymbol{a}\cdot\boldsymbol{i}}{|\boldsymbol{a}||\boldsymbol{i}|} = \frac{\boldsymbol{a_1}}{|\boldsymbol{a}|}$$

9 $$\cos\beta = \frac{\boldsymbol{a_2}}{|\boldsymbol{a}|} \qquad \cos\gamma = \frac{\boldsymbol{a_3}}{|\boldsymbol{a}|}$$

10 $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Direction Angles and Direction Cosines

Therefore

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}|\cos\alpha, |\mathbf{a}|\cos\beta, |\mathbf{a}|\cos\gamma \rangle$$

$$= |\mathbf{a}|\langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

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$$\frac{\mathbf{1}}{|\boldsymbol{a}|}\boldsymbol{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

which says that the direction cosines of **a** are the components of the unit vector in the direction of **a**.

Example

Find the direction angles of the vector $\boldsymbol{a} = \langle 1, 2,\ 3 \rangle$.

Solution:

Since $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, Equations 8 and 9 give

$$\cos\alpha = \frac{1}{\sqrt{14}} \quad \cos\beta = \frac{2}{\sqrt{14}} \quad \cos\gamma = \frac{3}{\sqrt{14}}$$

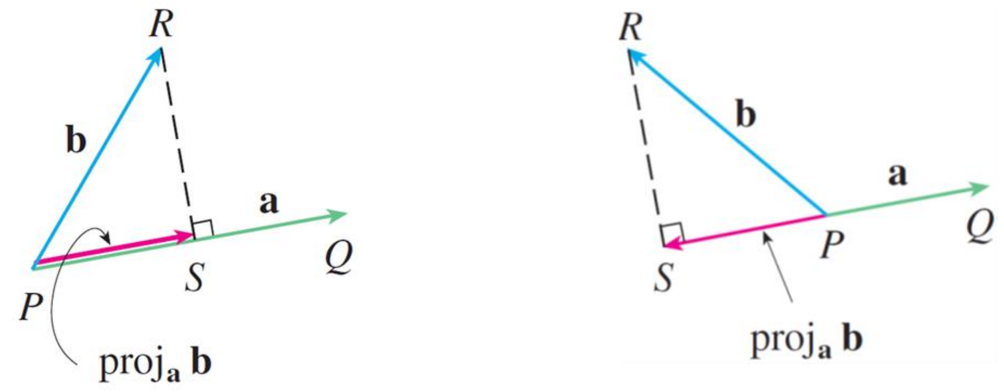
and so

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74° \quad \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58° \quad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37°$$

Projections

Projections (1 of 4)

Figure 4 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point *P*. If *S* is the foot of the perpendicular from *R* to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$. (You can think of it as a shadow of **b**).

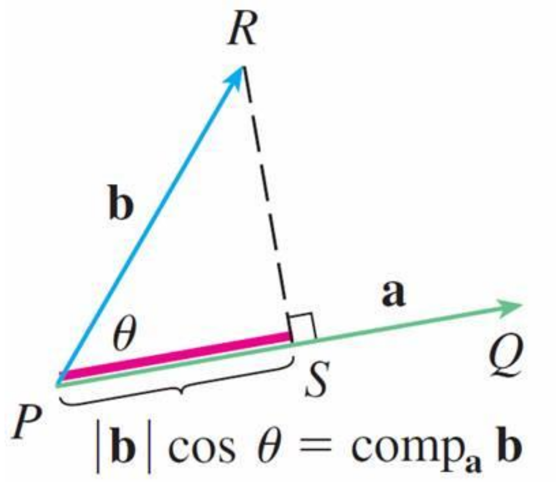


Vector projections

Figure 4

Projections (2 of 4)

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\boldsymbol{b}| \cos\theta$, where θ is the angle between **a** and **b**. (See Figure 5.)



Scalar projection

Figure 5

Projections (3 of 4)

This is denoted by $\text{comp}_{\mathbf{a}}\, \mathbf{b}$. Observe that it is negative if $\frac{\pi}{2} < \theta \leq \pi$.

The equation

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta = |\boldsymbol{a}|(|\boldsymbol{b}| \cos\theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\boldsymbol{b}| \cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \cdot \boldsymbol{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**.

Projections (4 of 4)

We summarize these ideas as follows.

Scalar projection of **b** onto **a**: $\text{comp}_{\mathbf{a}}\boldsymbol{b} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}$

Vector projection of **b** onto **a**: $\text{proj}_{\boldsymbol{a}}\boldsymbol{b} = \left(\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}\right)\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of **a**.

Example

Find the scalar projection and vector projection of $\boldsymbol{b} = \langle 1, 1, 2\rangle$ and $\boldsymbol{a} = \langle -2,\ 3,\ 1\rangle$.

Solution:

Since $|\boldsymbol{a}| = \sqrt{-2^2 + 3^2 + 1^2} = \sqrt{14},$ the scalar projection of **b** onto **a** is

$$\begin{aligned} \mathbf{comp}_{\boldsymbol{a}}\boldsymbol{b} &= \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|} \\ &= \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} \\ &= \frac{3}{\sqrt{14}} \end{aligned}$$

Example

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$
\begin{aligned}
\mathbf{proj_a}\boldsymbol{b} &= \frac{3}{\sqrt{14}}\frac{\boldsymbol{a}}{|\boldsymbol{a}|} \\
&= \frac{3}{14}\boldsymbol{a} \\
&= \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle
\end{aligned}
$$

Application: Work

Application: Work (1 of 2)

The work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 6.

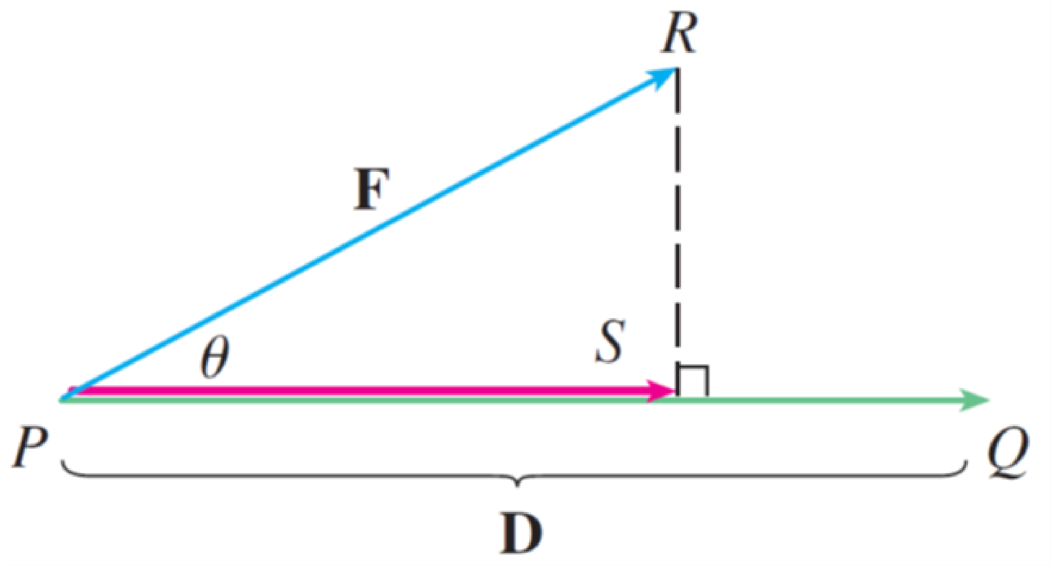


Figure 6

Application: Work (2 of 2)

If the force moves the object from P to Q, then the **displacement vector** is $\boldsymbol{D} = \overrightarrow{PQ}$. The **work** done by this force is defined to be the product of the component of the force along **D** and the distance moved:

$$\boldsymbol{W} = (|\boldsymbol{F}| \cos\theta)|\boldsymbol{D}|$$

But then, from Theorem 3, we have

$$12 \qquad \boldsymbol{W} = |\boldsymbol{F}||\boldsymbol{D}| \cos\theta = \boldsymbol{F} \cdot \boldsymbol{D}$$

Thus the work done by a constant force **F** is the dot product **F** • **D**, where **D** is the displacement vector.

Example

A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

Solution:
If **F** and **D** are the force and displacement vectors, as pictured in Figure 7, then the work done is

$$\begin{aligned} \boldsymbol{W} &= \boldsymbol{F} \cdot \boldsymbol{D} \\ &= |\boldsymbol{F}||\boldsymbol{D}| \cos 35° \\ &= (70)(100) \cos 35° \\ &\approx 5734 \text{ N} \cdot \text{m} \\ &= 5734 \text{ J} \end{aligned}$$

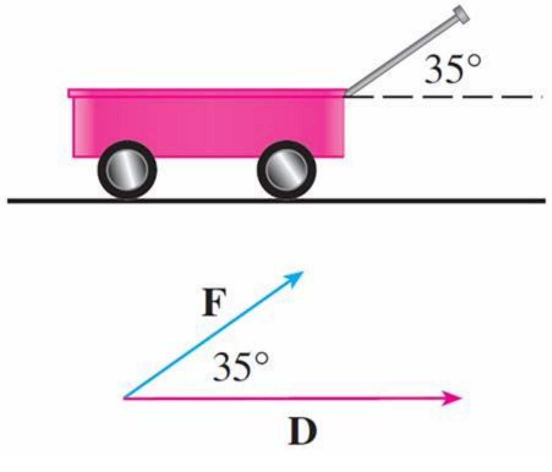


Figure 7