Name: _____   Student ID: _____   Major: _____

Question 1 (ca. 11 marks)

Consider the function $f\colon D \to \mathbb{R}$ defined by

$$f(x,y) = \frac{3x}{x^2 + xy + y^2}.$$

Here $D \subseteq \mathbb{R}^2$ is the maximum possible domain for f.

a) Determine D.

b) Which obvious symmetry property does f have? What can you conclude from this about the graph and the contours of f?

c) Determine the limits

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{\substack{(x,y)\to(0,0)\\ x>0,\, y>0}} f(x,y), \qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

(including the possibilities $\pm\infty$), or show that the limit does not exist.

d) Show that f has no critical point (i.e., no point at which ∇f vanishes).

e) Sketch the 1-contour of f as accurately as possible. Your drawing should include the points with a horizontal or vertical tangent.

f) Determine the slope of the graph G_f at $(1,-1)$ in the direction of the origin (NW), and the maximal slope/direction of G_f at $(1,-1)$.

Question 2 (ca. 7 marks)

Consider the differentiable map $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G(x,y) = \left(2xy - 2x,\, x^2 - y^2 + 2y\right).$$

a) For $(x,y) \in \mathbb{R}^2$ compute the Jacobi matrix $\mathbf{J}_G(x,y)$.

b) In which points $(x,y) \in \mathbb{R}^2$ is G conformal?

c) Determine the G-image $S = G(Q)$ of the unit square $Q = [0,1]^2$, and graph Q and the region S on paper (unit length at least $2\,\mathrm{cm}$).

   Hint: It suffices to determine the G-images of the vertices and edges of Q. The edges are segments of the coordinate lines with equations $x = 0$, $x = 1$, $y = 0$, $y = 1$. You should indicate the correspondence between edges and their G-images by using the same color (or line style such as "dashed", "dotted", etc.).

d) Is the figure obtained in c) compatible with the result in b)? Justify your answer!

Question 3 (ca. 4 marks)

The total resistance R produced by two conductors with resistances R_1 and R_2 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

a) Find the linear approximation of $R = R(R_1, R_2)$ near (R_1, R_2).

b) Suppose $R_1 = 10\,\Omega$, $R_2 = 5\,\Omega$ with a possible tolerance of $\pm 0.5\,\Omega$. Using the Mean Value Theorem, state (tight) rigorous upper and lower bounds for the true value of R.

   Note: In b) it is not necessary to determine explicit figures.

Solutions

1 a) $x^2 + xy + y^2 = \left(x + \tfrac{1}{2}y\right)^2 + \tfrac{3}{4}y^2 = 0$ has only the trivial solution $x = y = 0$.
$\implies D = \mathbb{R}^2 \setminus \{(0,0)\}$ $\boxed{\tfrac{1}{2}}$

b) $f(-x,-y) = -f(x,y)$ for $(x,y) \in D$.
$\implies \mathrm{G}_f$ is point-symmetric with respect to the origin in $\mathbb{R}^3$, and the $-k$-contour of f is obtained from the k-contour by reflection at the origin of $\mathbb{R}^2$ (i.e., the map $(x,y) \mapsto (-x,-y)$). $\boxed{1\tfrac{1}{2}}$

c) Since $f(x,0) = 3/x \to +\infty$ for $x \downarrow 0$ and $f(x,0) \to -\infty$ for $x \uparrow 0$, the first limit doesn't exist. $\boxed{1}$

Since $f(x,0) = 3/x \to +\infty$ for $x \downarrow 0$ and $f(y^2,y) = \frac{3y^2}{y^4+y^3+y^2} = \frac{3}{1+y+y^2} \to 3$ for $y \downarrow 0$, the second limit doesn't exist either. $\boxed{1}$

The third limit is zero, since $x^2 + xy + y^2 \geq \tfrac{1}{2}(x^2 + y^2)$ and hence

$$|f(x,y)| \leq \frac{6\,|x|}{x^2 + y^2} \leq \frac{6}{\sqrt{x^2 + y^2}} = \frac{6}{|(x,y)|}.$$ $\boxed{1}$

Alternatively, in polar coordinates we have

$$f(r\cos\phi, r\sin\phi) = \frac{3r\cos\phi}{r^2(\cos^2\phi + \cos\phi\sin\phi + \sin^2\phi)} = \frac{3\cos\phi}{r\left(1 + \tfrac{1}{2}\sin(2\phi)\right)},$$

from which $|f(r\cos\phi, r\sin\phi)| \leq 6/r \to 0$ for $r \to \infty$ follows likewise.
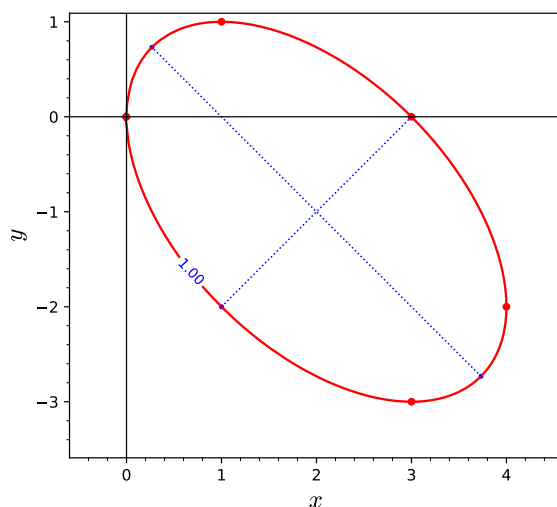
d) We have

$$f_x = \frac{3(x^2 + xy + y^2) - 3x(2x + y)}{(x^2 + xy + y^2)^2} = \frac{3(y^2 - x^2)}{(x^2 + xy + y^2)^2},$$ $\boxed{\tfrac{1}{2}}$

$$f_y = \frac{-3x(x + 2y)}{(x^2 + xy + y^2)^2},$$ $\boxed{\tfrac{1}{2}}$

$$\nabla f = (f_x, f_y).$$

Clearly the only solution of $y^2 - x^2 = 3x(x + 2y) = 0$ is $(x,y) = (0,0)$, but $(0,0) \notin D$. Hence f has no critical point. $\boxed{1}$

e) The 1-contour has the equation $g(x,y) := x^2 + xy + y^2 - 3x = 0$, provided we include the origin $(0,0)$ among its points. From $\nabla g = (2x + y - 3,\, x + 2y)$, $\nabla g(0,0) = (-3,0)$ the tangent at $(0,0)$ is vertical. Other points with vertical tangent must have $f_y = 0$, which implies $x = -2y$ and $4y^2 - 2y^2 + y^2 + 6y = 0$, i.e., $3y^2 + 6y = 0$, giving the point $(4,-2)$. Points with horizontal tangent must have $f_x = 0$, i.e. $x = \pm y$ and $x^2 \pm x^2 + x^2 - 3x = 0$, giving the points $(1,1)$ and $(3,-3)$. For the drawing it is also good to observe that the 1-contour intersects the x-axis in $(3,0)$.

$\boxed{2}$

f) We have

$$\nabla f(1,-1) = (0,3),$$
$$\nabla f(1,-1) \cdot \tfrac{1}{\sqrt{2}}(-1,1) = \tfrac{3}{\sqrt{2}}.$$

$\implies$ The slope of G_f at $(1,-1)$ in the direction of the origin (NW) is $\tfrac{3}{\sqrt{2}}$, $\boxed{1}$
and the maximal slope of G_f at $(1,-1)$, in the direction of the gradient (N), is
$|\nabla f(1,-1)| = 3$. $\boxed{1}$

$$\sum_1 = 11$$

2 a) Writing $G = (u,v)^{\mathsf{T}}$ with $u(x,y) = 2xy - 2x$, $v(x,y) = x^2 - y^2 + 2y$, we have

$$\mathbf{J}_G(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2y - 2 & 2x \\ 2x & -2y + 2 \end{pmatrix}.$$ $\boxed{1}$

b) Using a), we have

$$\mathbf{J}_G(x,y)^{\mathsf{T}}\mathbf{J}_G(x,y) = \begin{pmatrix} (2y-2)^2 + 4x^2 & 0 \\ 0 & (2y-2)^2 + 4x^2 \end{pmatrix}.$$

$\implies G$ is conformal in all points satisfying $(2y-2)^2 + 4x^2 \neq 0$, i.e., in all points of $\mathbb{R}^2$
except $(0,1)$. $\boxed{1}$

c) Writing $P_0 = (0,0)$, $P_1 = (1,0)$, $P_2 = (1,1)$, $P_3 = (0,1)$ for the vertices of Q and
$[P_0, P_1]$, etc., for the edges of Q, we have $G(0,0) = (0,0)$, $G(1,0) = (-2,1)$, $G(1,1) = (0,2)$, and further:

(i) $G(x,0) = (-2x, x^2) = \left(-2x, \tfrac{1}{4}(-2x)^2\right)$, $0 \le x \le 1$
$\implies G\big([P_0, P_1]\big)$ is the arc of the parabola $y = \tfrac{1}{4}x^2$, $-2 \le x \le 0$. $\boxed{\tfrac{1}{2}}$

– 4 –

(ii) $G(1, y) = (2y - 2, 1 - y^2 + 2y) = \big(2(y - 1), 2 - (y - 1)^2\big)$, $0 \le y \le 1$
$\implies G\big([P_1, P_2]\big)$ is the arc of the parabola $y = 2 - \frac{1}{4}x^2$, $-2 \le x \le 0$. $\boxed{\frac{1}{2}}$

(iii) $G(x, 1) = (0, x^2 + 1)$, $0 \le x \le 1$
$\implies G\big([P_2, P_3]\big)$ is the line segment from $(0, 1)$ to $(0, 2)$. $\boxed{\frac{1}{2}}$

(iv) $G(0, y) = (0, -y^2 + 2y) = \big(0, 1 - (y - 1)^2\big)$, $0 \le y \le 1$
$\implies G\big([P_3, P_0]\big)$ is the line segment from $(0, 0)$ to $(0, 1)$. $\boxed{\frac{1}{2}}$
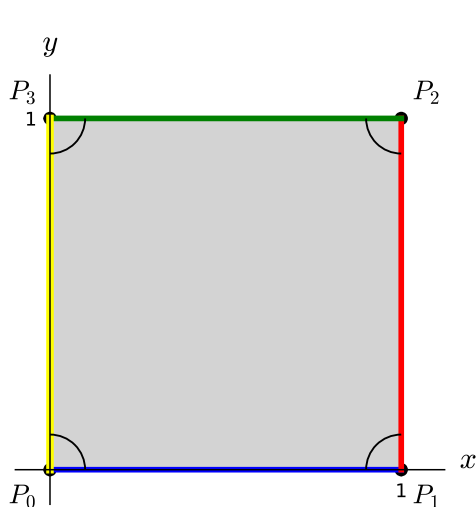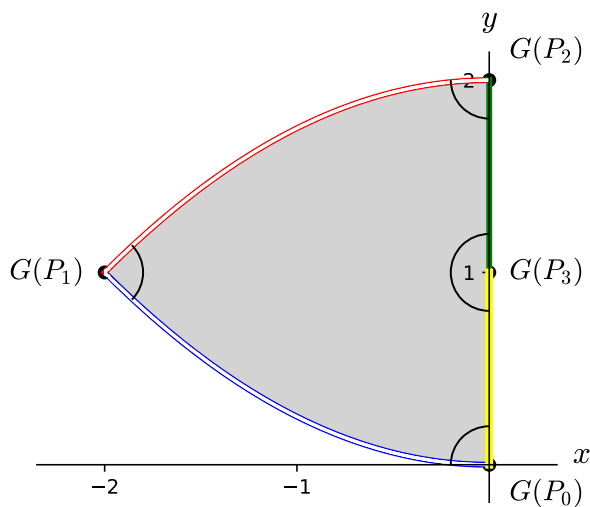
(a) Q (b) $G(Q)$

$\boxed{2}$

d) Since G is conformal in all points except $P_3 = (0, 1)$, the 90°-angles between the edges at P_0, P_1, P_2 are mapped to 90°-angles between the arcs/edges at $G(P_0)$, $G(P_1)$, $G(P_2)$. The 90°-angle between the edges at P_3 is mapped to a 180°-angle between the edges at $G(P_3)$; since G is not conformal in P_3, this doesn't contradict the result in b). $\boxed{1}$

$$\sum_2 = 7$$

3 a) We use Ohm (Ω) as unit of measurement.

$$R = \frac{R_1 R_2}{R_1 + R_2},$$
$$\frac{\partial R}{\partial R_1} = \frac{R_2(R_1 + R_2) - R_1 R_2}{(R_1 + R_2)^2} = \frac{R_2^2}{(R_1 + R_2)^2},$$
$$\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1 + R_2)^2},$$
$$\Delta R = R(R_1 + \Delta R_1, R_2 + \Delta R_2) - R(R_1, R_2)$$
$$\approx \frac{R_2^2}{(R_1 + R_2)^2} \Delta R_1 + \frac{R_1^2}{(R_1 + R_2)^2} \Delta R_2 \qquad \boxed{2}$$

for small ΔR_1, ΔR_2.

b) The Mean Value Theorem gives

$$\Delta R = R - \frac{10 \cdot 5}{10 + 5} = \frac{r_2^2}{(r_1 + r_2)^2} \Delta R_1 + \frac{r_1^2}{(r_1 + r_2)^2} \Delta R_2$$

with $|\Delta R_1| \leq 0.5$, $|\Delta R_2| \leq 0.5$, $9.5 \leq r_1 \leq 10.5$, $4.5 \leq r_2 \leq 5.5$. $\boxed{1}$

Hence we have

$$R \leq \frac{10}{3} + \frac{0.5(5.5^2 + 10.5^2)}{14^2} \approx 3.69\,[\Omega],$$
$$R \geq \frac{10}{3} - \frac{0.5(5.5^2 + 10.5^2)}{14^2} \approx 2.975\,[\Omega].$$ $\boxed{1}$

The explicit figures were not required. A $100\,\%$ rigorous estimate is then $2.97 \leq R \leq 3.70\,[\Omega]$.

$$\sum_{3} = 4$$

$$\sum_{\text{Midterm 2}} = 22 = 20 + 2$$