13 Vector Functions

13.2 Derivatives and Integrals of Vector Functions

Derivatives

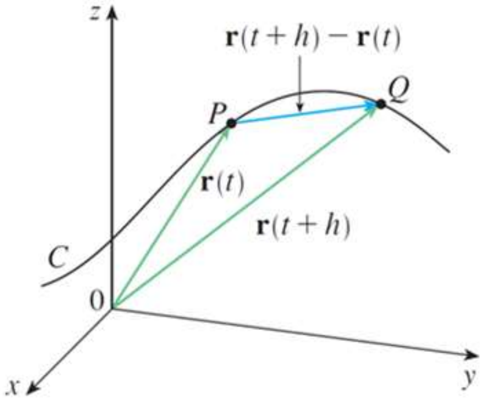
Context

- Derivatives
- Differentiation Rules
- Integrals

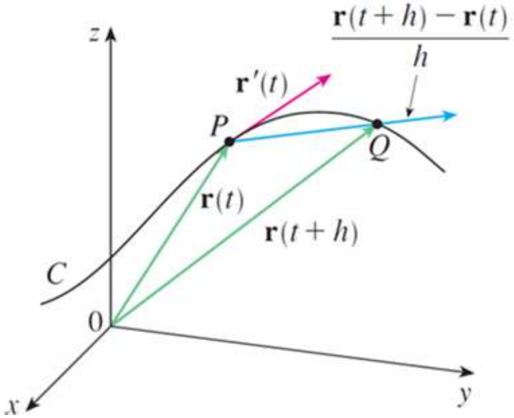
The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in much the same way as for real-valued functions:

$$\mathbf{1} \quad \frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1.



(a) The secant vector $\overrightarrow{PQ}$



(b) The tangent vector $\mathbf{r}'(t)$

Figure 1

Derivatives (2 of 4)

If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t + h)$, then $\overrightarrow{PQ}$ represents the vector $\mathbf{r}(t + h) - \mathbf{r}(t)$, which can therefore be regarded as a secant vector.

If $h > 0$, the scalar multiple $\left(\frac{1}{h}\right)(\mathbf{r}(t+h)-\mathbf{r}(t))$ has the same direction as $\mathbf{r}(t + h) - \mathbf{r}(t)$. As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line.

For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$.

The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$.

Derivatives (3 of 4)

The following theorem gives us a convenient method for computing the derivative of a vector function **r**: just differentiate each component of **r**.

2 Theorem If $\mathbf{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where *f*, *g*, and *h* are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

A unit vector that has the same direction as the tangent vector is called the **unit tangent vector T** and is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

MCQ: Find the unit tangent vector at the point where $t = 0$ for $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin(2t)\,\mathbf{k}$

A $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$

B $\mathbf{i}$

C $\frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$

D $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$

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Example 1

(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

Solution:

(a) According to Theorem 2, we differentiate each component of **r**:

$$\mathbf{r}'(t) = 3t^2\,\mathbf{i} + (1 - t)e^{-t}\,\mathbf{j} + 2\cos 2t\,\boldsymbol{k}$$

Example 1 – Solution

(b) Since $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at the point (1, 0, 0) is

$$\begin{aligned} \mathbf{T}(0) &= \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} \\ &= \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1+4}} \\ &= \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k} \end{aligned}$$

Derivatives (4 of 4)

Just as for real-valued functions, the **second derivative** of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

For instance, the second derivative of the function, $\mathbf{r}(t) = \langle 2\cos t, \sin t, \ t\rangle$, is

$$\mathbf{r}''(t) = \langle -2\cos t, -\sin t, 0\rangle$$

Differentiation Rules

Differentiation Rules (1 of 3)

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

3 Theorem Suppose **u** and **v** are differentiable vector functions, *c* is a scalar, and *f* is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)]=\mathbf{u}'(t)+\mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)]=c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)]=f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)]=\mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)]=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))]=f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Differentiation Rules (2 of 3)

We use Formula 4 to prove the following theorem.

4 Theorem if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

PROOF

Since

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

and c^2 is a constant, Formula 4 of Theorem 3 gives

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

Thus $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Differentiation Rules (3 of 3)

Geometrically, this result says that if a curve lies on a sphere with center the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$. (See Figure 4.)

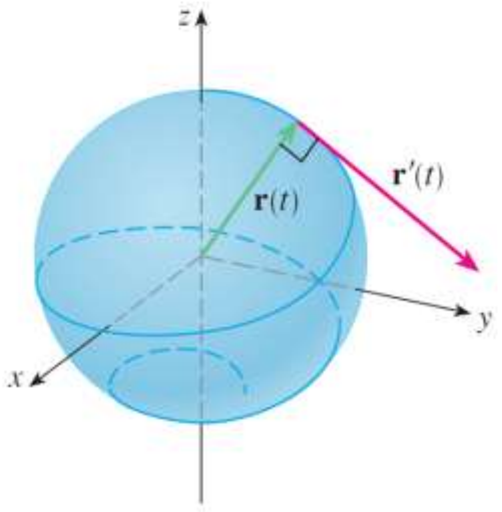

Figure 4

多选题 1分

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MAQ: Let $|\mathbf{r}(t)| \equiv c > 0$ and define $\boldsymbol{L}(r) = \boldsymbol{r}(t) \times \boldsymbol{r}'(t)$

A $\mathbf{r}'(t)$ is orthogonal to both $\mathbf{r}(t)$ and $\mathbf{L}(t)$

B $\mathbf{L}(t)$ is orthogonal to $\mathbf{r}(t)$ for all t

C If $\mathbf{L}(t) \equiv \mathbf{0}$ and $|\mathbf{r}(t)| \equiv c > 0$,then $\mathbf{r}(t)$ is a constant vector

D If $\mathbf{L}(t)$ is a constant vector, then $|\mathbf{r}'(t)|$ must be constant

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Example

Solution:

For option A, from previous PPT $\mathbf{r}'(t)\cdot\mathbf{r}(t)=0$, it's true.

For option B, it's true by properties of the cross product.

For option C, If $\mathbf{L}\equiv 0$, then $\mathbf{r}'$is parallel to $\mathbf{r}$: $\mathbf{r}' = \lambda(t)\mathbf{r}$ .Combining with A,

$$0 = \mathbf{r}\cdot\mathbf{r}' = \lambda(t)|\mathbf{r}|^2 = \lambda(t)c^2 \;\Rightarrow\; \lambda(t)\equiv 0,$$

so $\mathbf{r}'(t)\equiv\mathbf{0}$ and $\mathbf{r}(t)$is a constant vector (a single fixed point on the sphere).

For option D, it's true because $|\mathbf{r}'(\mathrm{t})| = \frac{|\boldsymbol{L}|}{|\boldsymbol{r}(t)|} = \frac{|\boldsymbol{L}|}{c}$

Integrals

Integrals (1 of 3)

The **definite integral** of a continuous vector function **r**(*t*) can be defined in much the same way as for real-valued functions except that the integral is a vector.

But then we can express the integral of **r** in terms of the integrals of its component functions *f*, *g*, and *h* as follows.

$$\begin{aligned}\int_a^b \mathbf{r}(t)\,dt &= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}\left(t_i^*\right)\Delta t \\ &= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f\left(t_i^*\right)\Delta t\right)\mathbf{i} + \left(\sum_{i=1}^{n} g\left(t_i^*\right)\Delta t\right)\mathbf{j} + \left(\sum_{i=1}^{n} h\left(t_i^*\right)\Delta t\right)\mathbf{k}\right]\end{aligned}$$

Integrals (2 of 3)

So

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

Integrals (3 of 3)

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where **R** is an antiderivative of **r**, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

We use the notation $\int \mathbf{r}(t)\,dt$ for indefinite integrals (antiderivatives).

单选题 1分

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MCQ: If $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\mathbf{k}$, find $\int_0^{\frac{\pi}{2}} \mathbf{r}(t)\mathrm{d}t$

A $2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}$

B $\mathbf{i} + 2\mathbf{j} + \frac{\pi^2}{4}\mathbf{k}$

C $2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{2}\mathbf{k}$

D $2\mathbf{i} + \mathbf{j} + \frac{\pi}{4}\mathbf{k}$

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Example 4

If $\mathbf{r}(t)$ = 2 cos t **i** + sin t **j** + 2t **k**, then

$$\int \mathbf{r}(t)\,dt = \left(\int 2\cos t\,dt\right)\mathbf{i} + \left(\int \sin t\,dt\right)\mathbf{j} + \left(\int 2t\,dt\right)\mathbf{k}$$

$$= 2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\mathbf{k} + \mathbf{C}$$

where **C** is a vector constant of integration, and

$$\int_0^{\frac{\pi}{2}} \mathbf{r}(t)\,dt = \left[2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\,\mathbf{k}\right]_0^{\frac{\pi}{2}}$$

$$= 2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}$$

Recap

- Derivatives
- Differentiation Rules
- Integrals