12 Vectors and the Geometry of Space

12.4 The Cross Product

Context

- Cross Product of Two Vectors
- Properties of the Cross Product
- Triple Products
- Application

The Cross Product of Two Vectors

Definition of the Cross Product If $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of **a** and **b** is the vector

$$\boldsymbol{a} \times \boldsymbol{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Notice that the **cross product a × b** of two vectors **a** and **b** is a vector (whereas the dot product, product is a scalar). For this reason it is also called the **vector product**.

Note that **a × b** is defined only when **a** and **b** are *three-dimensional* vectors.

The Cross Product of Two Vectors

In order to make the Definition easier to remember, we use the notation of determinants.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

The Cross Product of Two Vectors

A **determinant of order 3** can be defined in terms of second-order determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\begin{aligned} \begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} &= 1 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix} \\ &= 1(0-4) - 2(6+5) + (-1)(12-0) \\ &= -38 \end{aligned}$$

The Cross Product of Two Vectors

If we now rewrite the Definition using second-order determinants and the standard basis vectors **i**, **j**, and **k**, we see that the cross product of the vectors **a** = a_1 **i** + a_2 **j** + a_3 **k** and **b** = b_1 **i** + b_2 **j** + b_3 **k** is

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}$$

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MCQ: $If\ \boldsymbol{a} = \langle 1, 3, 4 \rangle$ and $\boldsymbol{b} = \langle 2,\ 7,\ -5 \rangle$, then $\boldsymbol{a} \times \boldsymbol{b} =$

A -43**i**+13**j**+**k**

B -13**i**+43**j**+**k**

C -23**i**+13**j**+43**k**

D -43**i**+23**j**+13**k**

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Example 1

If $\boldsymbol{a} = \langle 1, 3, 4\rangle$ and $\boldsymbol{b} = \langle 2,\ 7,\ -5\rangle$, then

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \boldsymbol{k} \\
&= (-15 - 28)\boldsymbol{i} - (-5 - 8)\boldsymbol{j} + (7 - 6)\boldsymbol{k} \\
&= -43\boldsymbol{i} + 13\boldsymbol{j} + \boldsymbol{k}
\end{aligned}$$

Properties of the Cross Product

Properties of the Cross Product

We constructed the cross product **a** × **b** so that it would be perpendicular to both **a** and **b**. This is one of the most important properties of a cross product.

8 Theorem The vector **a** × **b** is orthogonal to both **a** and **b**.

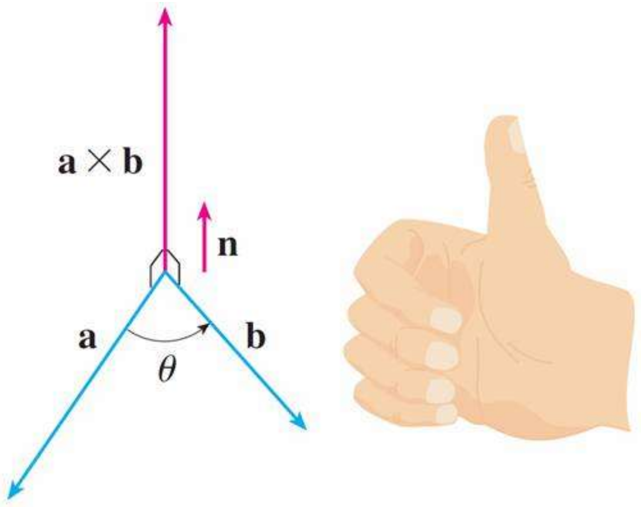
Proof.

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3 \\ &= 0.\end{aligned}$$

Similarly, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0$.

Properties of the Cross Product

If **a** and **b** are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 8 says that the cross product **a × b** points in a direction perpendicular to the plane through **a** and **b**.



The right-hand rule gives the direction of **a** × **b**.

Figure 1

Properties of the Cross Product

It turns out that the direction of **a** × **b** is given by the *right-hand rule*: If the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from to **a** to **b**, then your thumb points in the direction of **a** × **b.**

Now that we know the direction of the vector **a** × **b**, the remaining thing we need to complete its geometric description is its length $|\boldsymbol{a} \times \boldsymbol{b}|$. This is given by following theorem.

9 Theorem If θ is the angle between **a** and **b** (so $0 \leq \theta \leq \pi$), then the length of the cross product **a** × **b** is given by

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \theta$$

Properties of the Cross Product

10 Corollary Two nonzero vectors **a** and **b** are parallel if and only if

$$\boldsymbol{a} \times \boldsymbol{b} = 0$$

Since a vector is completely determined by its magnitude and direction, we can now say that for nonparallel vectors **a** and **b, a × b** is the vector that is perpendicular to both **a** and **b**, whose orientation is determined by the right-hand rule, and whose length is $|\boldsymbol{a}||\boldsymbol{b}| \sin \theta$.

In fact, that is exactly how physicists *define* **a × b**.

Properties of the Cross Product

The geometric interpretation of Theorem 9 can be seen by looking at Figure 2.

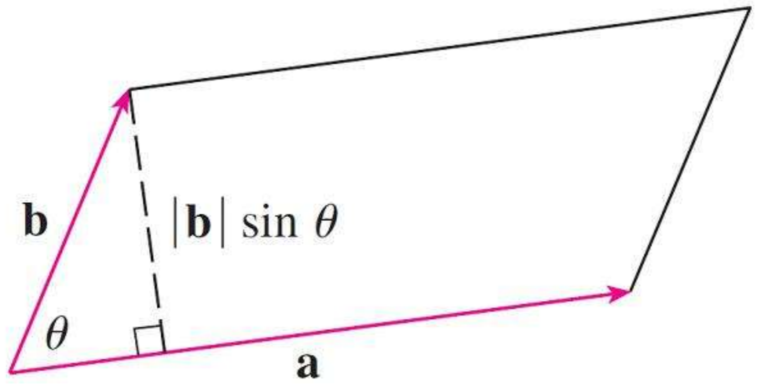


Figure 2

Properties of the Cross Product

If **a** and **b** are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\boldsymbol{a}|$, altitude $|\boldsymbol{b}| \sin \theta$, and area

$$A = |\boldsymbol{a}|(|\boldsymbol{b}| \sin \theta) = |\boldsymbol{a} \times \boldsymbol{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product.

The length of the cross product **a** × **b** is equal to the area of the parallelogram determined by **a** and **b**.

MCQ: Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

A $5\sqrt{82}$

B $5\sqrt{41}$

C $\frac{5}{2}\sqrt{82}$

D $\frac{5}{2}\sqrt{41}$

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Example

Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution:

We computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides *PQ* and *PR* is the length of this cross product:

$$\begin{aligned} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| &= \sqrt{(-40)^2 + (-15)^2 + 15^2} \\ &= 5\sqrt{82} \end{aligned}$$

The area *A* of the triangle *PQR* is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$.

Properties of the Cross Product

If we apply Theorems 8 and 9 to the standard basis vectors **i**, **j**, and **k** using $\theta = \frac{\pi}{2}$, we obtain

$$\begin{array}{ccc} \boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} & \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i} & \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j} \\ \boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k} & \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i} & \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j} \end{array}$$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

Properties of the Cross Product

Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0 \times \mathbf{j} = 0$$

So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra *do* hold for cross products.

Properties of the Cross Product

The following theorem summarizes the properties of vector products.

11 Properties of the Cross Product If **a**, **b**, and **c** are vectors and *c* is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

多选题 1分

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MAQ: Let $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \neq \mathbf{0}$ while $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\mathbf{0}$

Which of the following statements are always true? (Multiple answers may be correct)

A $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{b}\times\boldsymbol{c}=\boldsymbol{c}\times\boldsymbol{a}$

B $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=0$

C $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{b}\times\boldsymbol{c}|=|\boldsymbol{c}\times\boldsymbol{a}|$

D $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$

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Example

Solution:

$$\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c}$$

And similarly, so option A and C will be ture

Because $\boldsymbol{c} = -\boldsymbol{a} - \boldsymbol{b}$

The three vectors are **coplanar**. Thus the scalar triple product

$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \mathbf{0}$ So option B will be ture.

The cross product is not associative in general, option D is false.

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

Triple Products

Triple Products

The product **a** · (**b** × **c**) that occurs in Property 5 is called the **scalar triple product** of the vectors **a**, **b**, and **c**. Notice from Equation 12 that we can write the scalar triple product as a determinant:

$$13 \quad \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Triple Products (2 of 5)

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c**.

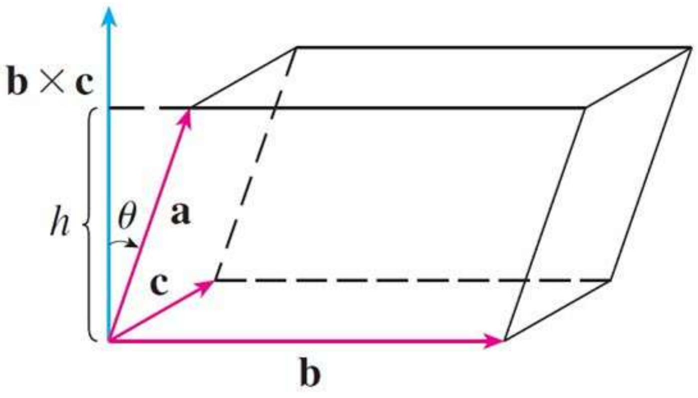

Figure 3

The area of the base parallelogram is $A = |\boldsymbol{b} \times \boldsymbol{c}|$.

Triple Products

If θ is the angle between **a** and **b** × **c**, then the height h of the parallelepiped is $h = |\boldsymbol{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of cos θ in case $\theta > \frac{\pi}{2}$.) Therefore the volume of the parallelepiped is

$$V = Ah = |\boldsymbol{b} \times \boldsymbol{c}||\boldsymbol{a}||\cos\theta| = |\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})|$$

Thus we have proved the following formula.

14 The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})|$$

If we use the formula and discover that the volume of the parallelepiped determined by **a**, **b**, and **c** is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

Example

Use the scalar triple product to show that the vectors $\boldsymbol{a} = \langle 1,\ 4,\ -7\rangle, \boldsymbol{b} = \langle 2, -1, 4\rangle$, and $\boldsymbol{c} = \langle 0, -9, 18\rangle$, are coplanar.

Solution:

We use Equation 13 to compute their scalar triple product:

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1(18) - 4(36) - 7(-18)$$

$$= 0$$

This means that **a**, **b**, and **c** are coplanar.

Application: Torque

Application: Torque (1 of 3)

The idea of a cross product occurs often in physics. In particular, we consider a force **F** acting on a rigid body at a point given by a position vector **r**. (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 4, we produce a turning effect.)

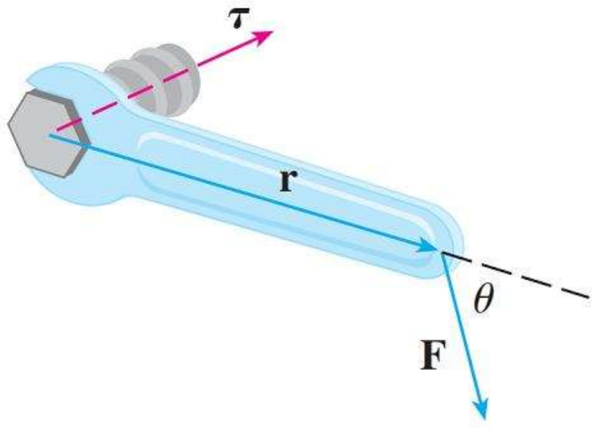

Figure 4

Application: Torque (2 of 3)

The **torque** τ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation.

Application: Torque (3 of 3)

According to Theorem 9, the magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\boldsymbol{r} \times \boldsymbol{F}| = |\boldsymbol{r}||\boldsymbol{F}| \sin \theta$$

where θ is the angle between the position and force vectors. Observe that the only component of **F** that can cause a rotation is the one perpendicular to **r**, that is, $|\boldsymbol{F}| \sin \theta$.

The magnitude of the torque is equal to the area of the parallelogram determined by **r** and **F**.

Example 6

A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5.

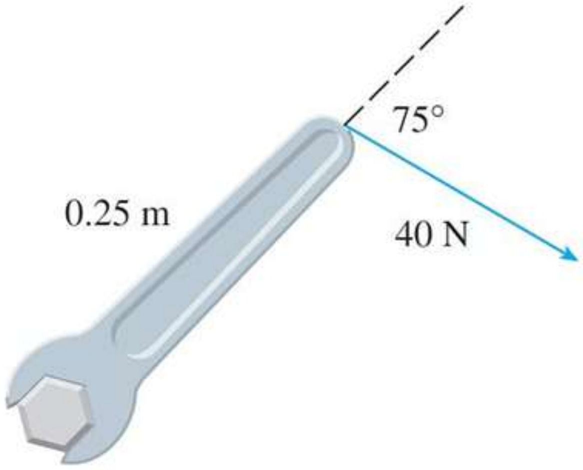

Figure 5

Find the magnitude of the torque about the center of the bolt.

Example 6 – Solution

The magnitude of the torque vector is

$$\begin{aligned} |\boldsymbol{\tau}| = |\boldsymbol{r} \times \boldsymbol{F}| &= |\boldsymbol{r}||\boldsymbol{F}| \sin 75° \\ &= (0.25)(40) \sin 75° \\ &= 10 \sin 75° \approx 9.66 \text{ N} \cdot \text{m} \end{aligned}$$

If the bolt is right-threaded, then the torque vector itself is

$$\boldsymbol{\tau} = |\boldsymbol{\tau}|\boldsymbol{n} \approx 9.66\, \boldsymbol{n}$$

where **n** is a unit vector directed down into the page (by the right-hand rule).

Recap

- Cross Product of Two Vectors
- Properties of the Cross Product
- Triple Products
- Application