Calculus III (Math 241)

Consider the curve C in $\mathbb{R}^3$ parametrized by

$$\gamma(t) = \left(t^3 + t^2 + 1, t^2 - t, -t^3 - t - 1\right), \quad t \in \mathbb{R}.$$

a) Is C contained in a plane? Justify your answer!

b) Determine the center and radius of the osculating circle and the TNB frame of C in $(1, 0, -1)$.

3. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \frac{xy^2}{\sqrt{x^2 + y^2}} \quad \text{for } (x, y) \neq \mathbf{0}$$

In this problem, you'll show $\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{h}) = 0$.

(a) For $\epsilon = 1/2$, find some $\delta > 0$ so that when $0 < |\mathbf{h}| < \delta$ we have $|f(\mathbf{h})| < \epsilon$. Hint: As with the example in class, the key is to relate $|x|$ and $|y|$ with $|\mathbf{h}|$.

(b) Repeat with $\epsilon = 1/10$.

(c) Now show that $\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{h}) = 0$. That is, given an arbitrary $\epsilon > 0$, find a $\delta > 0$ so that that when $0 < |\mathbf{h}| < \delta$ we have $|f(\mathbf{h})| < \epsilon$.

(d) Explain why the limit laws that you learned in class on Wednesday aren't enough to compute this particular limit.

The epsilon-delta definition of the limit

We first define what it means for a function to have **"limit 0 at 0"**.

Definition. Let the function E be defined on an open interval about 0, except possibly at 0. We say that $\lim_{h \to 0} E(h) = 0$ if

for every challenge number $\epsilon > 0$,
there is a response number $\delta > 0$ so that
if $0 < |h| < \delta$,
then $|E(h)| < \epsilon$.

Example: $E(h) = h^2$ has limit 0 at 0:

Given an arbitrary $\epsilon > 0$,
we can choose $\delta = \sqrt{\epsilon}$. Then
if $0 < |h| < \delta$,
$|E(h)| = |h^2| < \delta^2 = \epsilon$.

Exercise: Prove these facts using epsilon-delta arguments:

- If functions E_1 and E_2 both have limit 0 at 0, so does their sum.
- If functions E_1 and E_2 both have limit 0 at 0, so does their product.
- If function E_1 has limit 0 at 0, and $|E_2(h)| < |E_1(h)|$ for all h, then E_2 has limit 0 at 0.
- If $|g(h)| < M$ all h, and E has limit 0 at 0, then the function gE has limit 0 at 0.

For a more general function F defined on an open interval about a, except possibly at a, we say

$$\lim_{x \to a} F(x) = L$$

if the "error" function $E(h) = F(a+h) - L$ has limit 0 at 0.

For example, suppose $F(x) = x^2$, and we want to show $\lim_{x \to 2} F(x) = 4$.

We let $E(h) = F(2+h) - 4 = (2+h)^2 - 4 = 4h + h^2$, and show $E(h)$ has limit 0 at 0.

If $\epsilon = \frac{1}{10}$, we can choose $\delta = \frac{1}{100}$, since then $|E(h)| = |h^2 + 4h| \leq |h^2| + |4h| < \frac{1}{10000} + \frac{4}{100} < \frac{5}{100} < \frac{1}{10}$.

The case of an arbitrary ϵ is harder. Instead of finding δ directly, we could show $E_1(h) = h^2$ and $E_2(h) = 4h$ both have limit 0 at 0, and then use the fact that their sum also has limit 0 at 0.