ECE313 Homework 2

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1 Problem 1

A factory has two machines M_1 and M_2 producing parts:

- The probability that a part is produced by M_1 is 0.6, and by M_2 is 0.4.
- Among the parts produced by M_1, 2% are defective.
- Among the parts produced by M_2, 5% are defective.

Let event A be "the part is produced by M_1", and event B be "the part is defective".

1. Find the probability that a part is produced by M_1 and is defective
2. Find the probability that a randomly chosen part is defective.

Answer:

1. $P(A) = 0.6$, $P(B|A) = 0.02$. Therefore, the event "a part is produced by M_1 and is defective" is $A \cap B$, and

$$P(AB) = P(A)P(B|A) = 0.6 \times 0.02 = 0.012 \tag{1}$$

2. Let event C be "the part is produced by M_2", then event A and event C are mutually exclusive, so

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|C)P(C) \\ &= 0.6 \times 0.02 + 0.4 \times 0.05 \\ &= 0.032 \end{aligned} \tag{2}$$

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2 Problem 2

In an email system:

- The probability that an email is spam is 0.4, and the probability that it is not spam is 0.6.
- If an email is spam, the probability that it contains the word “discount” is 0.7.
- If an email is not spam, the probability that it contains the word “discount” is 0.2.

Now we observe a new email that contains the word “discount.”

1. Compute the probability that this email is spam given that it contains “discount.”

2. If the system classifies the email by choosing the class with the larger probability, should the email be classified as spam or not spam?

Answer:

1. Let event A be "the email is spam", event B be "the email contains the word 'discount'". Then this question requires me to compute $P(A|B)$.

$$P(A) = 0.4 \quad P(B|A) = 0.7 \quad P(B|A^c) = 0.2 \tag{3}$$

Therefore, we can compute $P(AB)$ and $P(B)$:

$$\begin{aligned} P(AB) &= P(A)P(B|A) = 0.28 \\ P(B) &= P(B|A)P(A) + P(B|A^c)P(A^c) \\ &= 0.28 + 0.12 = 0.4 \end{aligned} \tag{4}$$

Then we can compute $P(A|B)$:

$$P(A|B) = \frac{P(AB)}{P(B)} = 0.7 \tag{5}$$

2. The email should be classified as spam, because $P(A|B) = 0.7 > 0.5$, so the probability that the email is spam must be larger than the probability that the email is not spam.

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3 Problem 3

Suppose you are on a game show. In front of you are three doors:

- Behind one door is a car (the grand prize).
- Behind each of the other two doors is a goat (a consolation prize).

The game proceeds as follows:

1. You choose one of the three doors.
2. The host, who knows what is behind each door, opens one of the other two doors and always opens a door that has a goat.
3. The host then asks: “Would you like to switch your choice to the other remaining unopened door?”

Should you switch? Use Bayes’ Theorem to support your answer.

Answer: I should switch the door.

We can simplify the question by consider a concrete situation, suppose I initially chose door A, and the host open door C, meaning that car is either in door A or in door B.

Let event H_A be "car is behind door A", event H_B be "car is behind door B", event C be "the host open door C", then we need to compute $P(H_A|C)$ and $P(H_B|C)$

According to Bayes' Theorem

$$P(H_i|C) = \frac{P(C|H_i)P(H_i)}{P(C)} \tag{6}$$

We know that $P(H_A) = P(H_B) = P(H_C) = \frac{1}{3}$

- If H_A happens, then the host can randomly choose to open door B or door C, so $P(C|H_A) = \frac{1}{2}$.
- If H_B happens, then the host cannot open door B, but you have chosen A, then he must open C. Therefore, $P(C|H_B) = 1$
- If H_C happens, then the host must not open door C, so $P(C|H_C) = 0$

$$P(C) = \sum_{i=A,B,C} P(C|H_i)P(H_i) = \frac{1}{2} \tag{7}$$

Therefore, we can compute the posterior probability:

$$\begin{aligned} P(H_A|C) &= \frac{1}{3} \\ P(H_B|C) &= \frac{2}{3} \end{aligned} \tag{8}$$

The probability raises to $\frac{2}{3}$ when I choose to switch the door, so I need to switch the door.

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4 Problem 4

In a certain city, there are three traffic lights A, B, and C. The following information is known:

- When A is green, the probability that B turns green is 0.9, and the probability that C turns green is 0.6.
- When A is red, the probability that B turns green is 0.3, and the probability that C turns green is 0.1.
- Given the state of A, the states of B and C are conditionally independent.

1. If it is known that A is green, find the probability that B and C are both green.
2. If the state of A is not given and it is known that both B and C are green, discuss whether the event $\{A = \text{green}\}$ is independent of the event $\{B = \text{green}, C = \text{green}\}$.

Answer

1. Let event A be "A is green", event B be "B is green", event C be "C is green". Then we know that B and C are conditionally independent given A, meaning that $P(BC|A) = P(B|A)P(C|A)$

$$\begin{aligned} P(BC|A) &= P(B|A)P(C|A) \\ &= 0.9 \times 0.6 \\ &= 0.54 \end{aligned} \tag{9}$$

2. This means we need to justify whether $P(A) = P(A|BC)$ is true.

$$P(A|BC) = \frac{P(ABC)}{P(BC)} = \frac{P(BC|A)P(A)}{P(BC)} \tag{10}$$

Therefore, we only need to prove that $P(BC|A) = P(BC)$ because $P(A)$ on both sides cancel out.

$$\begin{gathered} P(BC|A) = 0.54 \\ P(BC) = P(BC|A)P(A) + P(BC|A^c)P(A^c) = 0.54P(A) + 0.03(1 - P(A)) \end{gathered} \tag{11}$$

Solve the equation, we get that $P(A) = 1$, which means that the event A is only independent of the event BC if and only if A is a certain event. Otherwise, they are dependent.

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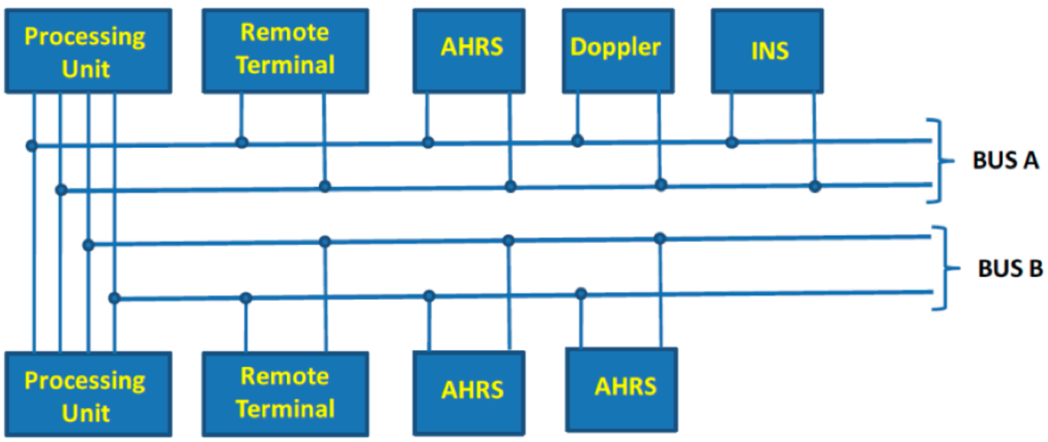
5 Problem 5

The system shown in the figure below is a processing system for a helicopter. The system has dual- redundant processors and dual-redundant remote terminals. Two buses are used in the system, and each bus is also dual-redundant.

The interesting part of the system is the navigation equipment. The aircraft can be completely navigated using the Inertial Navigation System (INS). If the INS fails, the aircraft can be navigated using the combination of the Doppler and the Attitude Heading and Reference System (AHRS). The system contains three AHRS units, of which only one is needed. This is an example of functional redundancy where the data from the AHRS and the Doppler can be used to replace the INS if the INS fails. Because of the other sensors and instrumentation, both buses are required for the system to function properly regardless of which navigation mode is being employed.



1. Identify the components that are in series and those that are in parallel.
2. Draw the reliability block diagram of the system.
3. Calculate the reliability of the system using the component reliabilities given in the table below:

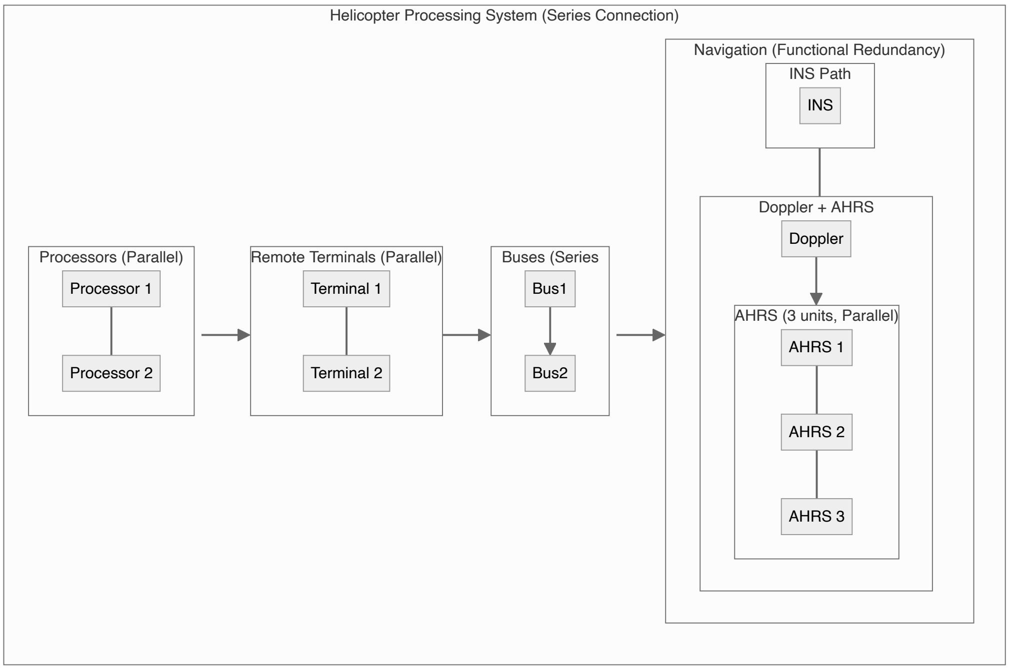
Component	Reliability
Processing Unit, R_{PU}	0.92
Remote Terminal, R_{RT}	0.95
AHRS, R_{AHRS}	0.88
INS, R_{INS}	0.85
Doppler, R_{DOP}	0.90
Bus, R_{BUS}	0.80

Answer:

1.

Components	**Relation**
2 Processing Units	Parallel
2 Remote Terminal	Parallel
Bus1 & Bus2	Series
3 AHRS	Parallel
Doppler & AHRS	Series
INS & (Dopper+AHRS)	Parallel
Whole System	Parallel

2.



3. The reliability of a parallel system and that of a series system are

$$R_{\text{parallel}} = 1 - \prod_i (1 - R_i)$$
$$R_{\text{series}} = \prod_i R_i \tag{12}$$

The reliability of each sub-system is

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\begin{aligned}
R_{PUS} &= 1-(1-0.92)^2 = 0.9936 \\
R_{RTS} &= 1-(1-0.95)^2 = 0.9975 \\
R_{AHRSS} &= 1-(1-0.88)^3 = 0.998272 \\
R_{BUSS} &= 0.8^2 = 0.64 \\
R_{backup} &= R_{DOP} \times R_{AHRSS} = 0.9R_{AHRSS} = 0.8984448 \\
R_{NAV} &= 1-(1-R_{INS})(1-R_{backup}) = 0.98476672
\end{aligned}
\tag{13}
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Therefore, we can calculate the reliability of the whole system:

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R_{\text{system}} = R_{PUS} \times R_{RTS} \times R_{BUSS} \times R_{NAV} \approx 0.625 \tag{14}
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