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Calculus III (Math 241)

Q1 Find the limit, if it exists, or show that the limit does not exist.

a) $\lim_{(x,y)\to(1,-1)} \mathrm{e}^{-xy}\cos(x+y)$;

b) $\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4}$;

c) $\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$;

d) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$.

Q2 Using the ε-δ definition of the limit $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ for functions $f: D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$, and $\mathbf{x}_0 \in D'$, show that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{a} \wedge \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b} \quad \text{implies} \quad \mathbf{a} = \mathbf{b}.$$

Solutions

1 a) The function $f(x,y) = \mathrm{e}^{-xy}\cos(x+y)$ is continuous in $\mathbb{R}^2$, so we have

$$\lim_{(x,y)\to(1,-1)} \mathrm{e}^{-xy}\cos(x+y) = f(1,-1) = \mathrm{e}^{-1(-1)}\cos(1-1) = \mathrm{e}.$$

b) Again denoting the function by $f(x,y)$, we have $f(x,0) = 0$ and $f(x,x) = \frac{5}{2}\cos^2 x \to \frac{5}{2}$ for $x \to 0$. This shows that the limits $\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y)$ along the lines $L = \mathbb{R}(1,0)$ and $\mathbb{R}(1,1)$ are 0 and $\frac{5}{2}$, respectively, and hence that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Note: It's slightly easier to use the y-axis $\mathbb{R}(0,1)$ in place of $\mathbb{R}(1,1)$, since $f(0,y) = 5$ is constant and hence trivially $\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in\mathbb{R}(0,1)}} f(x,y) = \lim_{y\to 0} f(0,y) = 5$.

c) This limit reduces to one considered in the lecture, viz.

$$\begin{aligned}\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2} &= \lim_{(x,y)\to(1,0)} \frac{(x-1)y}{(x-1)^2+y^2} \\ &= \lim_{(x',y)\to(0,0)} \frac{x'y}{x'^2+y^2}. && \text{(Subst. } x' = x-1)\end{aligned}$$

As shown in the lecture, the limit does not exist.

d) Here we can estimate as follows

$$\left|\frac{x^3-y^3}{x^2+y^2}\right| \le \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| \le \left|\frac{x^3}{x^2}\right| + \left|\frac{y^3}{y^2}\right| = |x| + |y|$$

Since $\lim_{(x,y)\to(0,0)} \big(|x|+|y|\big) = 0$, it follows that $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$ as well.

An explicit response to a given $\varepsilon > 0$ is $\delta = \varepsilon/2$, since $|(x,y)| < \varepsilon/2$ implies $|x| < \varepsilon/2$ and $|y| < \varepsilon/2$ and hence $\left|\frac{x^3-y^3}{x^2+y^2} - 0\right| \le |x| + |y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

2 Suppose $\mathbf{a} \ne \mathbf{b}$. Then $|\mathbf{a} - \mathbf{b}| > 0$, and for $\varepsilon = \frac{1}{2}|\mathbf{a} - \mathbf{b}|$ there exist responses $\delta_1, \delta_2 > 0$ such that $\mathbf{x} \in D \wedge 0 < |\mathbf{x} - \mathbf{x}_0| < \delta_1$ implies $|f(\mathbf{x}) - \mathbf{a}| < \varepsilon$ and $\mathbf{x} \in D \wedge 0 < |\mathbf{x} - \mathbf{x}_0| < \delta_2$ im-plies $|f(\mathbf{x}) - \mathbf{b}| < \varepsilon$. For $\delta = \min\{\delta_1, \delta_2\}$ we then have that $\mathbf{x} \in D \wedge 0 < |\mathbf{x} - \mathbf{x}_0| < \delta$ implies $|f(\mathbf{x}) - \mathbf{a}| < \varepsilon \wedge |f(\mathbf{x}) - \mathbf{b}| < \varepsilon$. The premise of this implication can be made true, since $\mathbf{x}_0$ is an accumulation point of D. The conclusion, however, is always false:

$$2\varepsilon = |\mathbf{a} - \mathbf{b}| \le |\mathbf{a} - f(\mathbf{x})| + |f(\mathbf{x}) - \mathbf{b}|,$$

and hence at least one of $|f(\mathbf{x}) - \mathbf{a}|$, $|f(\mathbf{x}) - \mathbf{b}|$ must be $\ge \varepsilon$. This contradiction shows that $\mathbf{a} \ne \mathbf{b}$ is false, i.e., we must have $\mathbf{a} = \mathbf{b}$.