ECE313 Homework 3

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1 Problem 1

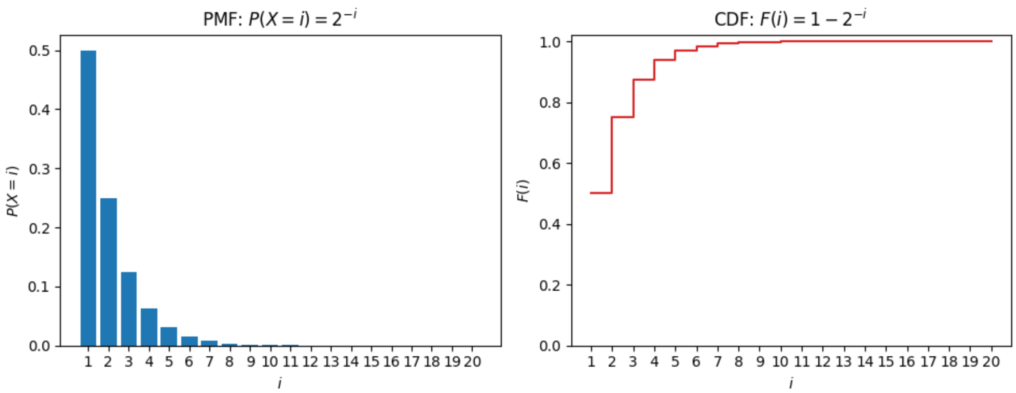
Consider the random variable X with pmf$P(X = i) = 2^{-i}$ for $i \geq 1$.

1. Sketch the pmf (given above) and the distribution function for X.
2. Calculate $P(X \leq 3)$
3. Calculate $P(X > 3)$
4. Calculate $P(X < 1)$
5. Calculate $P(|X - 4| \leq 0.1)$
6. Evaluate the following expression:

$$\sum_{k=5}^{\infty} P(X = k) \tag{50}$$

Answer

1. The plot is as below:



2. $P(X \leq 3) = 1 - 2^{-3} = \frac{7}{8}$
3. $P(X > 3) = 1 - P(X \leq 3) = \frac{1}{8}$
4. $P(X < 1) = 0$, because the pmf's domain is $i > 1$
5. $P(|X - 4| \leq 0.1) = P(3.9 \leq X \leq 4.1) = P(X = 4) = 2^{-4} = \frac{1}{16}$
6. $P(X \leq 4) = 1 - 2^{-4} = \frac{15}{16}$. Therefore, we have

$$\sum_{k=5}^{\infty} P(X = k) = P(X > 4) = 1 - P(X \leq 4) = \frac{1}{16} \tag{51}$$

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2 Problem 2

A communication channel receives independent pulses at the rate of 10 pulses per microsecond. The probability of a transmission error is 0.01 for each pulse. Compute the probabilities of:

1. No errors per microsecond
2. One error per microsecond
3. At least one error per microsecond
4. Exactly two errors per microsecond

Answer:

1. $P(F=0)=(1-0.01)^{10}=0.99^{10}$
2. $P(F=1)=\binom{10}{1}\times(1-0.01)^{9}\times 0.01=0.1\times 0.99^{9}$
3. $P(F\geq 1)=1-P(F=0)=1-0.99^{10}$
4. $P(F=2)=\binom{10}{2}\times(1-0.01)^{8}\times 0.01^{2}=45\times 0.99^{8}\times 10^{-4}$

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3 Problem 3

Solve the following questions.

1. Let $X\sim \mathrm{Bin}(n,p)$. Find $P(X \text{ is odd})$ in terms of n and p.
2. Let $Y\sim \mathrm{Poi}(\lambda)$. Find $P(Y \text{ is odd})$ in terms of λ.
3. Suppose that $n\to\infty$ and $p\to 0$ such that $np=\lambda$. Verify that your answer in part 1 converges to the answer in part 2

Hint: when $n\to\infty$, $p\to 0$, and $np=\lambda$, we have:

$$(1-p)^{m}\approx e^{-np}=e^{-\lambda} \tag{52}$$

Answer:

1. We know the PMF of Binomial Distribution:

$$P(X=k)=\binom{n}{k}p^{k}(1-p)^{n-k},\ k\in\mathbb{N} \tag{53}$$

Consider the two formulas which are derived from binomial theorem:

$$\begin{aligned}(1-p-p)^{n}&=\sum_{k=0}^{n}\binom{n}{k}(-p)^{k}(1-p)^{n-k}=(1-2p)^{n}\\(1-p+p)^{n}&=\sum_{k=0}^{n}\binom{n}{k}p^{k}(1-p)^{n-k}=1\end{aligned} \tag{54}$$

Add them together and we get:

$$\sum_{k=0}^{n}\binom{n}{k}p^{k}(1-p)^{n-k}\left[1+(-1)^{k}\right]=1+(1-2p)^{n} \tag{55}$$

When k is an even number, $1+(-1)^k=2$; when k is an odd number, $1+(-1)^k=0$. Therefore, the left side is equal to $2\times P(X\text{ is even})$.

$$P(X\text{ is odd})=1-P(X\text{ is even})=1-\frac{1+(1-2p)^n}{2}=\frac{1-(1-2p)^n}{2} \tag{56}$$

2. We know the PMF of Poisson distribution:

$$P(Y=k)=\frac{e^{-\lambda}\lambda^k}{k!},\ k\in\mathbb{N} \tag{57}$$

where $\frac{\lambda^k}{k!}$ is the Taylor Expand of e^λ. Therefore, we can consider the following two formulas:

$$e^\lambda=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} \qquad e^{-\lambda}=\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!} \tag{58}$$

The sum of even number and odd number are:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{(2k)!}&=\frac{e^\lambda+e^{-\lambda}}{2}\\ \sum_{k=0}^{\infty}\frac{\lambda^{2k+1}}{(2k+1)!}&=\frac{e^\lambda-e^{-\lambda}}{2}\end{aligned} \tag{59}$$

So:

$$P(Y\text{ is odd})=e^{-\lambda}\cdot\sum_{k=0}^{\infty}\frac{\lambda^{2k+1}}{(2k+1)!}=e^{-\lambda}\cdot\frac{e^\lambda-e^{-\lambda}}{2}=\frac{1-e^{-2\lambda}}{2} \tag{60}$$

3. The question is equivalent to proving that, when $n\to\infty$ and $p\to 0$ wuch that $np=\lambda$, $(1-2p)^n=e^{-2\lambda}$.

$$(1-2p)^n\approx e^{-2np}=e^{-2\lambda} \tag{61}$$

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4 Problem 4

A graduating ECE student goes to career fair booths in the technology sector. His/her likelihood of receiving an off-campus interview invitation from a certain career fair booth depends on how well he/she did in ECE 313. Specifically, an A in 313 results in a probability $p=0.9$ of obtaining an invitation, whereas a C in ECE 313 results in a probability $p=0.2$ of receiving an invitation.

1. What is the probability that an A student receives his/her first interview when visiting the third booth? How about a C student?
2. Each student is allowed to visit 5 booths during this career fair. What is the probability that an A student receives exactly two interviews? How about a C student?
3. Assume that each student visits 5 booths during a typical career fair. Find the probability that an A student in 313 will not get an off-campus interview invitation. Finally, find the probability that a C student in 313 will get at least one invitation.

4. Assume an A student already visited 10 booths without getting an interview, and a C student already visited 5 booths without getting an interview. If they both go visit 5 additional booths, who has a higher probability of landing an interview?

Answer:

1. This is a geometric distribution:

$$\begin{aligned} P(A) &= (1-p_A)^2 \cdot p_A = 0.9 \times 0.01 = 0.009 \\ P(C) &= (1-p_C)^2 \cdot p_C = 0.2 \times 0.64 = 0.128 \end{aligned} \tag{62}$$

2. This is a binomial distribution:

$$\begin{aligned} P(A) &= \binom{5}{2}(1-p_A)^3 \cdot p_A^2 = 0.01 \times 0.81 = 0.0081 \\ P(C) &= \binom{5}{2}(1-p_C)^3 \cdot p_C^2 = 0.2048 \end{aligned} \tag{63}$$

3.

$$\begin{aligned} P(A) &= (1-p_A)^5 = 10^{-5} \\ P(C) &= 1-(1-p_C)^5 = 1-0.8^5 \end{aligned} \tag{64}$$

4. The A student still has a higher probability of landing an interview, because in this "experiment", the former results have no influence to the latter reults.

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5 Problem 5

Assume that the number of jobs arriving to the Blue Waters supercomputer in an interval of t seconds is Poisson distributed with parameter $\lambda = 0.3$. Compute the probabilities of the following events:

1. Exactly 3 jobs will arrive during a 10s interval.
2. More than 10 jobs arrive in a period of 20s.
3. The number of job arrivals in an interval of 10s duration is between 2 and 4 (include 2 and 4).
4. Given that 10 jobs arrive in a period of 30s, what is the conditional probability that 3 jobs arrived in the first 10s?

Hint: Use the Bayes theorem to calculate the conditional probability. Note that the probability of 3 jobs arriving in the first 10s, given that 10 jobs arrived in 30s, equals to the probability of 3 jobs arriving in the first 10s and 7 jobs arriving in the remaining 20s. Also note that the number of arrivals in different time intervals are independent from each other.

Answer:

1. The parameter λ becomes:

$$\lambda_{10} = 0.3 \times 10 = 3 \tag{65}$$

Because $X \sim \text{Poi}(3)$, we will compute $P(X=3)$

$$P(X=3) = \frac{e^{-3} \cdot 3^3}{3!} = 4.5 \cdot e^{-3} \tag{66}$$

2. The parameter λ becomes:

$$\lambda_{20} = 0.3 \times 20 = 6 \tag{67}$$

Because $X \sim \text{Poi}(6)$, we will compute $P(X > 10)$

$$P(X > 10) = 1 - P(X \leq 10) = 1 - \sum_{k=0}^{10} \frac{e^{-6} \cdot 6^k}{k!} \tag{68}$$

3. The parameter λ is still 3, which is the same as question 1

$$\begin{aligned} P(2 \leq X \leq 4) &= P(X=2) + P(X=3) + P(X=4) \\ &= e^{-3} \cdot \left(\frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} \right) \end{aligned} \tag{69}$$

4. Suppose that
   - Event A is: 3 jobs arrived in the first 10s $\rightarrow \lambda_{10} = 3$
   - Event B is: 7 jobs arrived in the latter 20s $\rightarrow \lambda_{20} = 6$
   - Event C is: 10 jobs arrived in the first 30s $\rightarrow \lambda_{30} = 9$

   Because they are independent, so:

$$P(A \cap B) = P(A) \cdot P(B) \tag{70}$$

And we also have:

$$P(C) = \frac{e^{-9} \cdot 9^{10}}{10!} \tag{71}$$

Therefore:

$$\begin{aligned} P(A|C) &= \frac{P(A \cap C)}{P(C)} = \frac{P(A \cap B)}{P(C)} = \frac{P(A) \cdot P(B)}{P(C)} \\ &= \frac{\left(\frac{e^{-3} \cdot 3^3}{3!}\right) \cdot \left(\frac{e^{-6} \cdot 6^7}{7!}\right)}{\frac{e^{-9} \cdot 9^{10}}{10!}} \\ &= \frac{3^3 \cdot 6^7 \cdot 10!}{3! \cdot 7! \cdot 9^{10}} \end{aligned} \tag{72}$$

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6 In-Class Problem

Consider a system with triple modular redundancy (TMR). In such a system there are three components, two of which are required to be in working order for the system to function properly (i.e., $n = 3$ and $m = 2$). This is achieved by feeding the outputs of the three components into a majority voter.

Answer:

1. We assume the property of R to work is p, then the system reliability is:

$$P = \binom{3}{2} p^2 (1-p) + \binom{3}{3} p^3 = 3p^2(1-p) + p^3 \tag{73}$$

2. We can regard one component works or not is a Bernoulli event:

$$X \sim \mathrm{Ber}(n, p) \tag{74}$$