16 Vector Calculus

16.2 Line Integrals

Context

- Line Integrals in the Plane
- Line Integrals with Respect to x or y
- Line Integrals in Space
- Line Integrals of Vector Fields: Work

Line Integrals (1 of 1)

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval [*a*, *b*], we integrate over a curve *C*.

Such integrals are called *line integrals,* although "curve integrals" would be better terminology.

They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

Line Integrals in the Plane

We start with a plane curve C given by the parametric equations

$$\textbf{1} \quad x = x(t) \quad y = y(t) \quad a \le t \le b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and we assume that C is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \ne 0$.]

If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$, and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. (See Figure 1.)
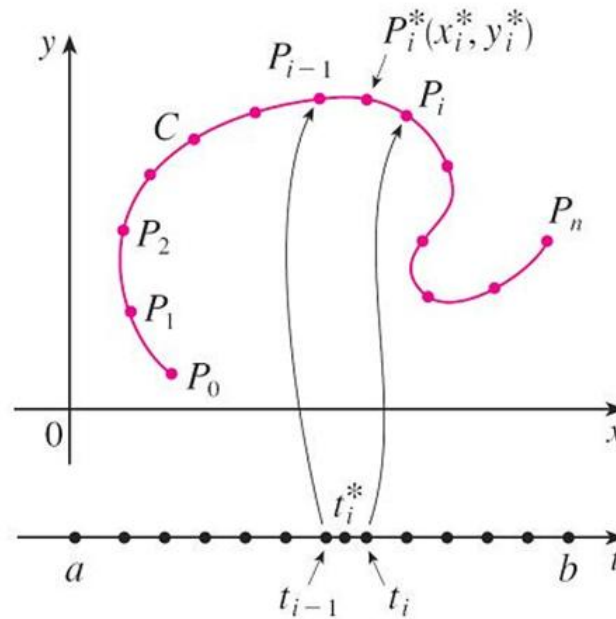

Figure 1

We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.)

Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^{n} f\left(x_i^*, y_i^*\right) \Delta s_i$$

which is similar to a Riemann sum.

Then we take the limit of these sums and make the following definition by analogy with a single integral.

2 Definition If *f* is defined on a smooth curve *C* given by Equations 1, then the **line integral of *f* along *C*** is

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i^*, y_i^*\right) \Delta s_i$$

if this limit exists.

We have found that the length of *C* is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

A similar type of argument can be used to show that if *f* is a continuous function, then the limit in Definition 2 always exists and the following formula can be used to evaluate the line integral:

$$3 \qquad \int_{c} f(x, y)\,ds = \int_{a}^{b} f\big(x(t),\, y(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as *t* increases from *a* to *b*.

If $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = |r'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So the way to remember Formula 3 is to express everything in terms of the parameter t: Use the parametric equations to express x and y in terms of t and write ds as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note In the special case where C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as follows: $x = x$, $y = 0$, $a \leq x \leq b$.

Formula 3 then becomes

$$\int_C f(x, y)\, ds = \int_a^b f(x, 0)\, dx$$

and so the line integral reduces to an ordinary single integral in this case.

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area.

In fact, if $f(x, y) \geq 0$, $\int_C f(x, y)\, ds$ represents the area of one side of the "fence" or "curtain" in Figure 2, whose base is C and whose height above the point (x, y) is $f(x, y)$.
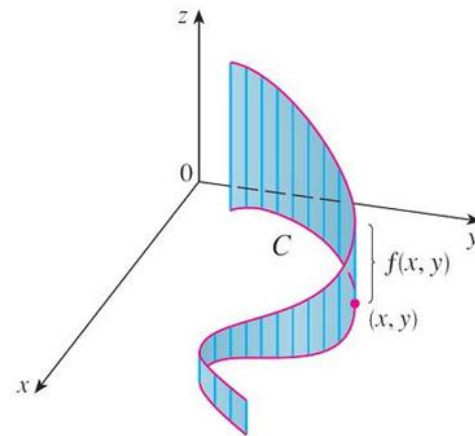

Figure 2

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Example 1

Evaluate $\int_C \left(2 + x^2 y\right) ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

Solution:

In order to use Formula 3, we first need parametric equations to represent C.

We know that the unit circle can be parametrized by means of the equations

$$x = \cos t \quad y = \sin t$$

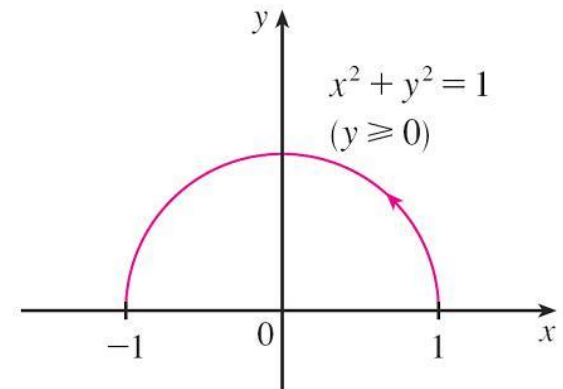and the upper half of the circle is described by the parameter interval $0 \le t \le \pi$. (See Figure 3.)
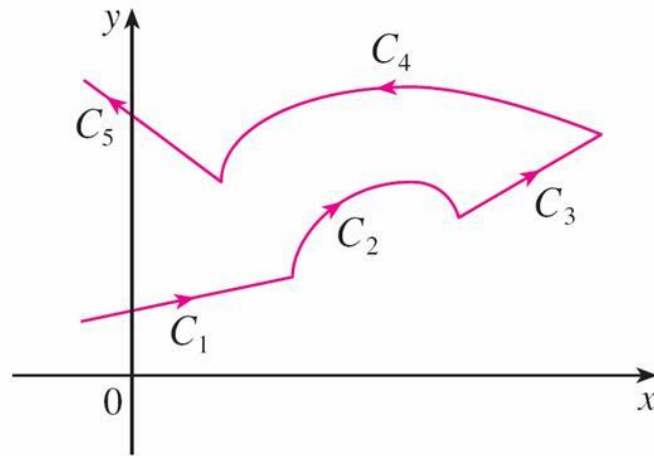

Figure 3

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Example 1 – Solution

Therefore Formula 3 gives

$$\int_C \left(2 + x^2 y\right) ds = \int_0^\pi \left(2 + \cos^2 t \sin t\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^\pi \left(2 + \cos^2 t \sin t\right)\sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^\pi \left(2 + \cos^2 t \sin t\right) dt$$

$$= \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi$$

$$= 2\pi + \frac{2}{3}$$

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves C_1, C_2, ..., C_n, where, as illustrated in Figure 4, the initial point of C_{i+1} is the terminal point of C_i.

A piecewise-smooth curve

Figure 4

Then we define the integral of *f* along *C* as the sum of the integrals of *f* along each of the smooth pieces of *C*:

$$\int_C f(x, y)\,ds = \int_{C_1} f(x, y)\,ds + \int_{C_2} f(x, y)\,ds + \cdots + \int_{C_n} f(x, y)\,ds$$

Any physical interpretation of a line integral $\int_C f(x, y)\, ds$ depends on the physical interpretation of the function f.

Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C.

Then the mass of the part of the wire from P_{i-1} to P_i in Figure 1 is approximately $\rho(x_i^*, y_i^*)\Delta s_i$ and so the total mass of the wire is approximately
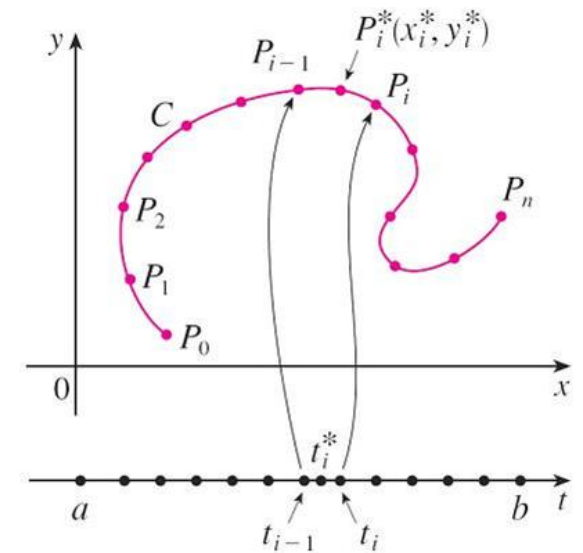
$$\sum \rho\left(x_i^*, y_i^*\right)\Delta s_i.$$

Figure 1

By taking more and more points on the curve, we obtain the **mass** m of the wire as the limiting value of these approximations:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho\left(x_i^*, y_i^*\right) \Delta s_i = \int_C \rho(x, y)\, ds$$

[For example, if $f(x, y) = 2 + x^2 y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.]

The **center of mass** of the wire with density function ρ is located at the point $\left(\bar{x}, \bar{y}\right)$, where

4 $$\bar{x} = \frac{1}{m}\int_C x\rho\left(x, y\right)ds \qquad \bar{y} = \frac{1}{m}\int_C y\rho\left(x, y\right)ds$$

MCQ: Let C be the line segment from (0,0) to (1,1). Compute $\int_C y\, ds$

A $\dfrac{1}{2}$

B $\dfrac{\sqrt{2}}{2}$

C $\sqrt{2}$

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Solution

Use the parametrization $x = t, \; y = t, \; 0 \le t \le 1$. Then

$$ds = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt = \sqrt{1^2 + 1^2}, dt = \sqrt{2}dt$$

So

$$\int_C y, ds = \int_0^1 t\sqrt{2}, dt = \sqrt{2} \cdot \frac{1}{2} = \frac{\sqrt{2}}{2}$$

Line Integrals with Respect to *x* or *y*

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 2.

They *are* called the **line integrals of *f* along *C* with respect to *x* and *y*:**

5 $$\int_C f(x, y)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta x_i$$

6 $$\int_C f(x, y)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta y_i$$

When we want to distinguish the original line integral $\int_C f(x, y)\,ds$ from those in Equations 5 and 6, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to *x* and *y* can also be evaluated by expressing everything in terms of *t*: $x = x(t)$, $y = y(t)$, $dx = x'(t)dt$, $dy = y'(t)dt$.

7 $\displaystyle\int_C f(x, y)\,dx \;=\; \int_a^b f(x(t), y(t))\,x'(t)\,dt$

$\displaystyle\int_C f(x, y)\,dy \;=\; \int_a^b f(x(t), y(t))\,y'(t)\,dt$

It frequently happens that line integrals with respect to *x* and *y* occur together.

When this happens, it's customary to abbreviate by writing

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

When we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given.

In particular, we often need to parametrize a line segment, so it's useful to remember that a vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

$$\mathbf{8} \qquad \mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\,\mathbf{r}_1 \qquad 0 \le t \le 1$$

Example 4

Evaluate $\int_C y^2\,dx + x\,dy$ for two different paths C.

(a) $C = C_1$ is the line segment from (−5, −3) to (0, 2).

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from (−5, −3) to (0, 2).
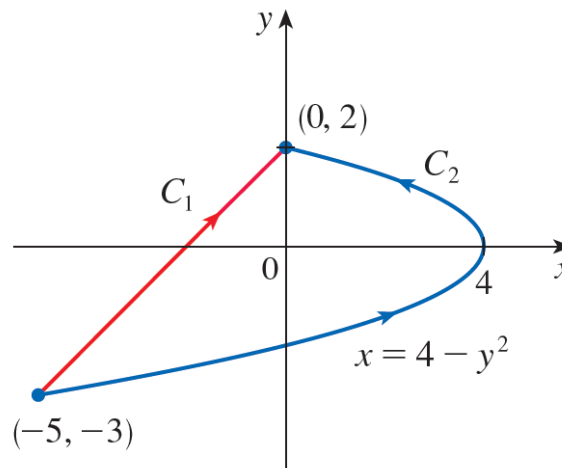
(See Figure 7.)

Figure 7

Example 4 – Solution

(a) A parametric representation for the line segment is

$$x = 5t - 5 \qquad y = 5t - 3 \qquad 0 \le t \le 1$$

(Use Equation 8 with $\mathbf{r}_0 = \langle -5, -3 \rangle$ and $\mathbf{r}_1 = \langle 0, 2 \rangle$.)

Then $dx = 5 \, dt$, $dy = 5 \, dt$, and Formulas 7 give

$$\int_{C_1} y^2 \, dx + x \, dy = \int_0^1 (5t - 3)2(5dt) + (5t - 3)(5dt)$$

$$= 5 \int_0^1 \left(25t^2 - 25t + 4 \right) dt$$

$$= 5 \left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t \right]_0^1 = -\frac{5}{6}$$

Example 4 – Solution (1 of 1)

(b) Since the parabola is given as a function of y, let's take y as the parameter and write C_2 as

$$x = 4 - y^2 \qquad y = y \qquad -3 \le y \le 2$$

Then $dx = -2y\,dy$ and by Formulas 7 we have

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} y^2(-2y)\,dy + (4 - y^2)\,dy$$

$$= \int_{-3}^{2} \left(-2y^3 - y^2 + 4\right)dy$$

$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right]_{-3}^{2} = 40\frac{5}{6}$$

In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, determines an **orientation** of a curve *C*, with the positive direction corresponding to increasing values of the parameter *t*. (See Figure 8, where the initial point *A* corresponds to the parameter value *a* and the terminal point *B* corresponds to *t = b*.)
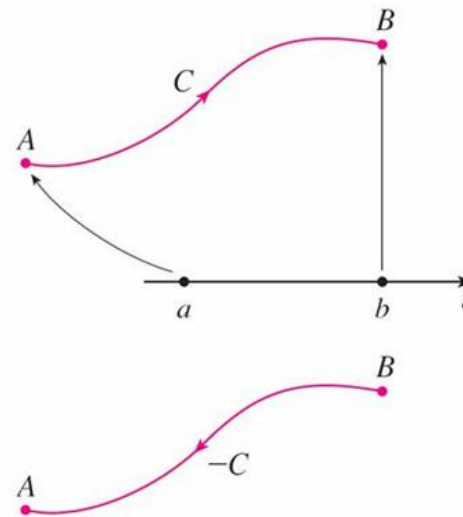

Figure 8

If $-C$ denotes the curve consisting of the same points as *C* but with the opposite orientation (from initial point *B* to terminal point *A* in Figure 8), then we have

$$\int_{-C} f(x, y)\, dx = -\int_{C} f(x, y)\, dx \qquad \int_{-C} f(x, y)\, dy = -\int_{C} f(x, y)\, dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y)\, ds = \int_{C} f(x, y)\, ds$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of *C*.

MAQ: Which statements are true?

A For any f, $\int_C f\, ds$ is independent of the orientation of C

B For any f, reversing the orientation of C changes $\int_C f\, dx$ to its negative.

C If $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$, then reversing the orientation of C does not change $\int_C P dx + Q dy + R dz$

D $\int_C x dy + y dx$ depends only on the endpoints of C (for piecewise smooth curves)

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Solution

- A: $\int_C f\, ds$ uses $ds = |\boldsymbol{r}'(t)|\, dt \geq 0$; reversing orientation leaves the value unchanged.

- B: Reversing orientation gives $dx \rightarrow -dx$, so $\int_C f\, dx$ changes sign (same idea for $\int_C f\, dy$).

- C: Reversing orientation gives $(dx, dy, dz) \rightarrow (-dx, -dy, -dz)$, so $\int_C P\, dx + Q\, dy + R\, dz$ changes sign; it is not invariant.

- D: Since $d(xy) = x\, dy + y\, dx$, $\int_C x\, dy + y\, dx = \int_C d(xy) = (xy)|_{\text{end}} - (xy)|_{\text{start}}$, which depends only on endpoints.

Line Integrals in Space

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \leq t \leq b$$

or by a vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$.

If f is a function of three variables that is continuous on some region containing C, then we define the **line integral of f along C** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z)\,ds = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i^*, y_i^*, z_i^*\right) \Delta s_i$$

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Line Integrals in Space (2 of 3)

We evaluate it using a formula similar to Formula 3:

9 $\displaystyle \int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt$$

For the special case $f(x, y, z) = 1$, we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\,dt = L$$

where L is the length of the curve C.

Line integrals along C with respect to x, y, and z can also be defined. For example,

$$\int_C f(x, y, z)\, dz \;=\; \lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i^*, y_i^*, z_i^*\right) \Delta z_i$$

$$=\; \int_a^b f\left(x(t), y(t), z(t)\right) z'(t)\, dt$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

10 $\qquad \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

Example 5

Evaluate $\int_C y \sin z\, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$. (See Figure 9.)
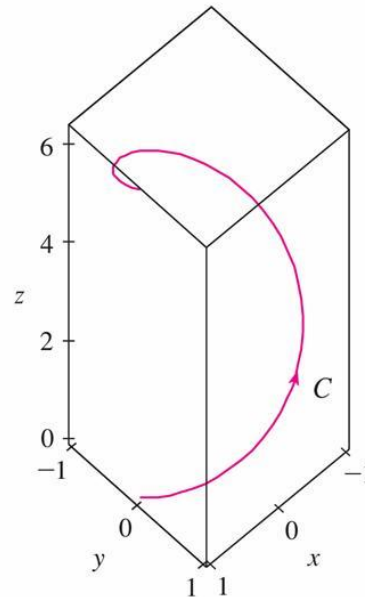

Figure 9

Example 5 – Solution

Formula 9 gives

$$\int_C y \sin z \, ds = \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

$$= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1} \, dt$$

$$= \sqrt{2} \int_0^{2\pi} \frac{1}{2} (1 - \cos 2t) \, dt$$

$$= \frac{\sqrt{2}}{2} \left[t - \frac{1}{2} \sin 2t \right]_0^{2\pi}$$

$$= \sqrt{2}\pi$$

MCQ: Let C be the circular helix $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$. Compute $\int_C z \, ds$

A π^2

B $\sqrt{2}\pi^2$

C $2\sqrt{2}\pi^2$

D $4\sqrt{2}\pi^2$

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Solution

With ᵈᵇ$r(t) = \langle \cos t, \quad \sin t, t \rangle$,

$$r'(t) = \langle -\sin t, \quad \cos t, 1 \rangle, \quad |r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

So $ds = \sqrt{2}dt$ and $z = t$. Hence $\int_C z\, ds = \int_0^{2\pi} t\sqrt{2}\, dt = \sqrt{2} \cdot \frac{(2\pi)^2}{2} = 2\sqrt{2}\pi^2$.

Line Integrals of Vector Fields; Work

We know that the work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is $W = \int_a^b f(x)\,dx$.

Then we have found that the work done by a constant force $\mathbf{F}$ in moving an object from a point P to another point Q in space is $W = \mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D} = \overrightarrow{PQ}$ is the displacement vector.

Now suppose that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$. (A force field on $\mathbb{R}^2$ could be regarded as a special case where $R = 0$ and P and Q depend only on x and y.)

We wish to compute the work done by this force in moving a particle along a smooth curve C.

We divide C into subarcs $P_{i-1} P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width. (See Figure 1 for the two-dimensional case or Figure 12 for the three-dimensional case.)
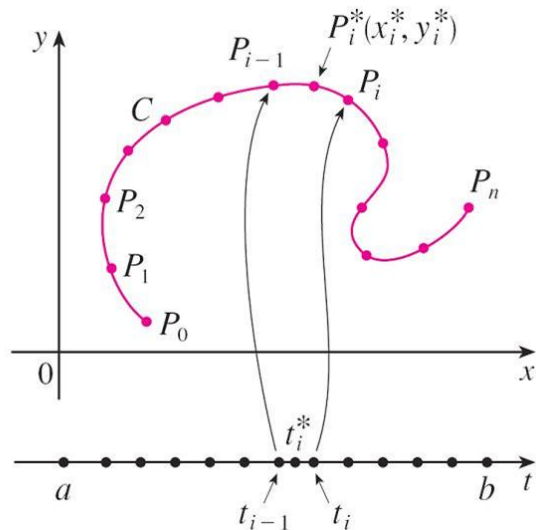

Figure 1
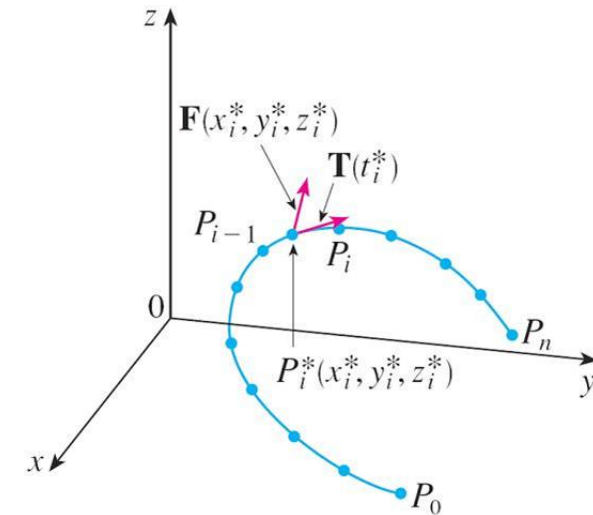
Figure 12

Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the ith subarc corresponding to the parameter value t_i^*.

If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}\left(t_i^*\right)$, the unit tangent vector at P_i^*.

Thus the work done by the force **F** in moving the particle from P_{i-1} to P_i is approximately

$$\mathbf{F}\left(x_i^*, y_i^*, z_i^*\right) \cdot \left[\Delta s_i \mathbf{T}\left(t_i^*\right)\right] = \left[\mathbf{F}\left(x_i^*, y_i^*, z_i^*\right) \cdot \mathbf{T}\left(t_i^*\right)\right]\Delta s_i$$

and the total work done in moving the particle along C is approximately

11 $$\sum_{i=1}^{n}\left[\mathbf{F}\left(x_i^*, y_i^*, z_i^*\right) \cdot \mathbf{T}\left(x_i^*, y_i^*, z_i^*\right)\right]\Delta s_i$$

where **T**(x, y, z) is the unit tangent vector at the point (x, y, z) on C.

Intuitively, we see that these approximations ought to become better as *n* becomes larger.

Therefore we define the **work** *W* done by the force field **F** as the limit of the Riemann sums in (11), namely,

12 $$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\, ds = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

Equation 12 says that *work is the line integral with respect to arc length of the tangential component of the force.*

If the curve *C* is given by the vector equation

$$\mathbf{r}(t) = x(t)\ \mathbf{i} + y(t)\ \mathbf{j} + z(t)\ \mathbf{k},$$

then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, so using Equation 9 we can rewrite Equation 12 in the form

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

This integral is often abbreviated as $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ and occurs in other areas of physics as well.

Therefore we make the following definition for the line integral of *any* continuous vector field.

13 Definition Let **F** be a continuous vector field defined on a smooth curve C given by a vector function **r**(t), $a \leq t \leq b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

When using Definition 13, remember that **F**(**r**(t)) is just an abbreviation for the vector field **F**($x(t)$, $y(t)$, z(t)), so we evaluate **F**(**r**(t)) simply by putting $x = x(t)$, $y = y(t)$, and $z = z(t)$ in the expression for **F**(x, y, z).

Notice also that we can formally write $d\mathbf{r} = \mathbf{r}'(t)\, dt$.

Example 7

Find the work done by the force field $\mathbf{F}(x,\, y) = x^2\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \dfrac{\pi}{2}$.

Solution:

Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\,\mathbf{i} - \cos t \sin t\,\mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Example 7 – Solution

Therefore the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{\frac{\pi}{2}} \left(-\cos^2 t \sin t - \cos^2 t \sin t\right) dt$$

$$= \int_0^{\frac{\pi}{2}} \left(-2\cos^2 t \sin t\right) dt$$

$$= 2\frac{\cos^3 t}{3}\Bigg]_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{3}$$

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field **F** on $\mathbb{R}^3$ is given in component form by the equation **F** = P **i** + Q **j** + R **k**.

We use Definition 13 to compute its line integral along C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b \big(P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}\big) \cdot \big(x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}\big)\, dt$$

$$\int_a^b \left[\begin{array}{l} P\big(x(t), y(t), z(t)\big)x'(t) + Q\big(x(t), y(t), z(t)\big)y'(t) \\ + R\big(x(t), y(t), z(t)\big)z'(t) \end{array} \right] dt$$

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But this last integral is precisely the line integral in (10). Therefore we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz \quad \text{where} \quad \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

For example, the integral $\int_C y\,dx + z\,dy + x\,dz$ could be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$

MAQ: Let $\boldsymbol{F} = \langle y, \; -x \rangle$. Let C be the quarter-circle on the unit circle from (1,0) to (0,1), oriented counterclockwise. For the work $W = \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, which statements are true?

A   Using $x = \cos t$, $y = \sin t$, $0 \le t \le \frac{\pi}{2}$, we get $W = -\frac{\pi}{2}$

B   If the orientation of C is reversed, then $W = \frac{\pi}{2}$

C $W = \int_C P dx + Q dy$ with $P = y$ and $Q = -x$

D   The value of W equals the area of the sector $\frac{\pi}{4}$

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Solution

Parameterize C by $\boldsymbol{r}(t) = \langle \cos t, \ \sin t \rangle$, $0 \le t \le \frac{\pi}{2}$. Then
$$\boldsymbol{r}'(t) = \langle -\sin t, \ \cos t \rangle, \quad \boldsymbol{F}\big(\mathbf{r}(\mathbf{t})\big) = \langle \sin t, \ -\cos t \rangle$$

So $\boldsymbol{F}\big(\mathbf{r}(\mathbf{t})\big) \cdot \boldsymbol{r}'(t) = \sin t \, (-\sin t) + (-\cos t)(\cos t) = -(\sin^2 t + \cos^2 t) = -1$

and therefore $W = \int_0^{\pi/2}(-1), dt = -\frac{\pi}{2}$.

Reversing orientation changes $d\boldsymbol{r} \to -d\boldsymbol{r}$, so the integral changes sign, giving $W = \frac{\pi}{2}$. Also, since $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j}$, $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C P dx + Q dy$.

The work is not generally equal to a geometric area, so D is false.

Recap

- Line Integrals in the Plane

- Line Integrals with Respect to x or y

- Line Integrals in Space

- Line Integrals of Vector Fields: Work