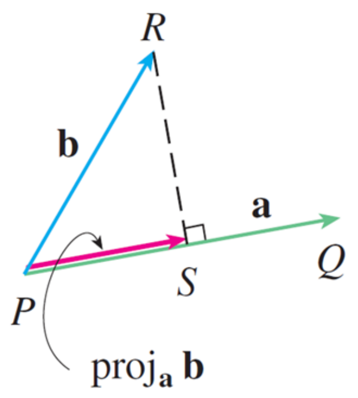
0 Midterm Review

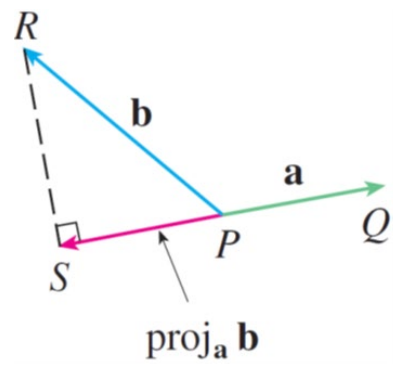
Context

- Projections
- Triple Products
- Lines and Planes
- Arc Length
- Curvature
- Tangent Planes
- Linear Approximations
- Limits of Functions of Two Variables
- Showing That a Limit Does Not Exist
- Partial Derivatives of Functions of Two Variables
- Level Curves and Contour Maps

Projections

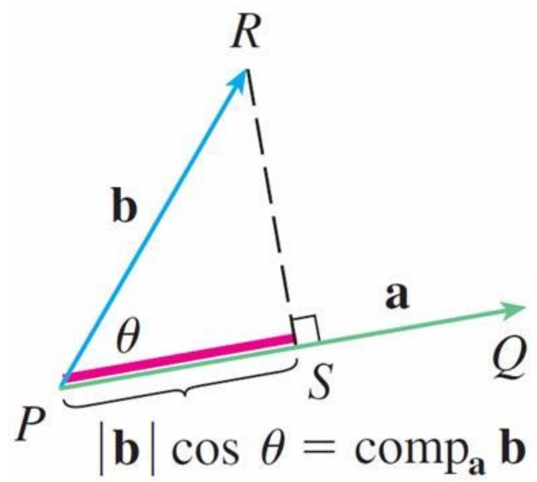
Projections (1 of 3)





Vector projections

Figure 4



Scalar projection

Figure 5

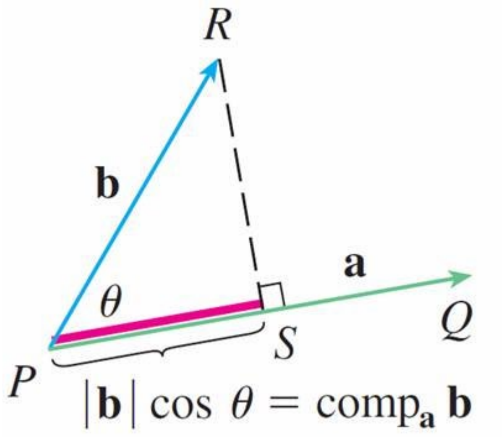
$$\text{comp}_{\mathbf{a}}\boldsymbol{b} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}$$

$$\text{proj}_{\boldsymbol{a}}\boldsymbol{b} = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}\right)\frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{a}$$

it is negative if $\frac{\pi}{2} < \theta \leq \pi$.

Projections (2 of 3)

Fast memory of the formula

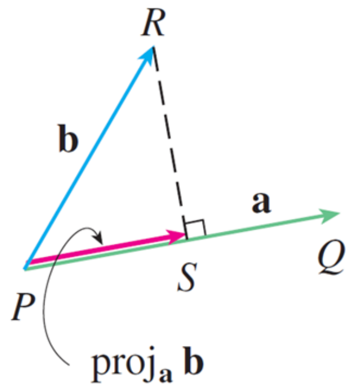


$$\mathrm{\mathbf{comp}}_{\mathbf{a}}\boldsymbol{b} = |\boldsymbol{b}|\cos\theta = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}\cdot\boldsymbol{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**.

Projections (3 of 3)

Fast memory of the formula



$$\mathbf{comp_a}\boldsymbol{b} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \cdot \boldsymbol{b} \longrightarrow \mathbf{proj}_{\boldsymbol{a}}\boldsymbol{b} = \left(\frac{\boldsymbol{a}}{|\boldsymbol{a}|} \cdot \boldsymbol{b}\right)\frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

the $\mathbf{proj}_{\boldsymbol{a}}\boldsymbol{b}$ is the $\mathbf{comp_a}\boldsymbol{b}$ times the unit vector in the direction of **a**.

Calculation steps:

1. Find $\frac{\boldsymbol{a}}{|\boldsymbol{a}|}$

2. Dot product $\frac{\boldsymbol{a}}{|\boldsymbol{a}|} \cdot \boldsymbol{b}$

3. Reuse $\frac{\boldsymbol{a}}{|\boldsymbol{a}|}$ to scale the vector

Triple Products

Triple Products

Definition

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The Other name scalar triple product of the vectors a, b, and c.

The geometric significance

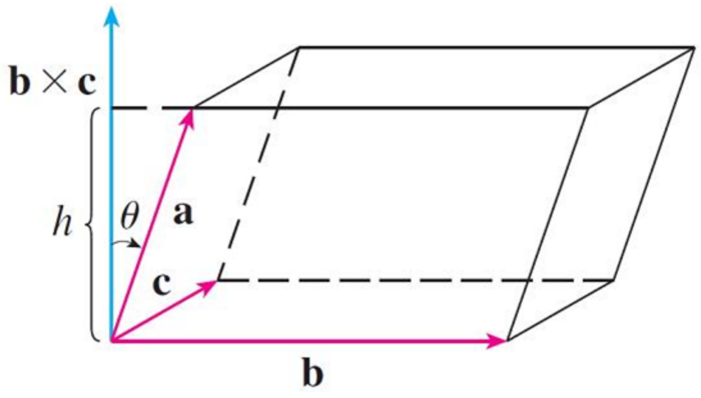


Figure 3

1. The volume of the parallelogram(平行六面体)
2. The method for coplanar(共面)

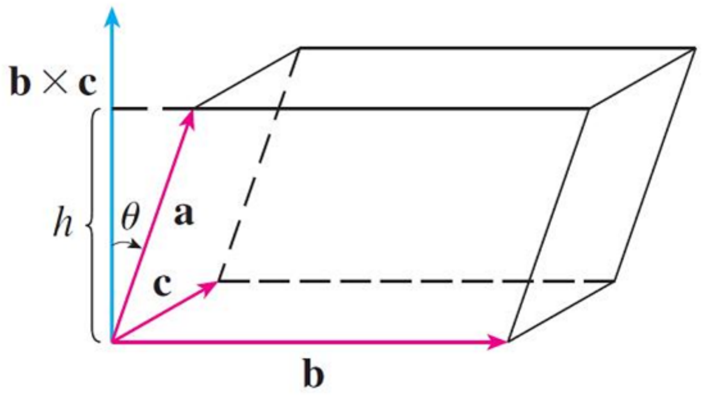
If $V = |\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})| = 0$ then a, b, and c are coplanar

Triple Products

Fast memory of the formula

Key idea: $V = Ah$

1. Use cross product to replace the area of the base $A = |\boldsymbol{b} \times \boldsymbol{c}|$.
2. Use the dot product to replace the height h of the parallelepiped is $h = |\boldsymbol{a}||\cos\theta|$.
3. Remember to add absolute value to avoid negativity cause by dot product



$$V = Ah = |\boldsymbol{b} \times \boldsymbol{c}||\boldsymbol{a}||\cos\theta| = |\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})|$$

Lines and Planes

Lines

the **parametric equations** of the line

$$x = 6 + t \quad y = 5 + 4t \quad z = 1 - 2t$$

In general, if a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of a line *L*, then the numbers *a*, *b*, and *c* are called **direction numbers** of *L*.

The starting point and the direction numbers are not unique

An example of finding another way:

(5, 1, 3) (6, 5, 1) are all on the line,
2i + 8j − 4k = 2*(i + 4j − 2k),

$$x = 5 + 2t \quad y = 1 + 8t \quad z = 3 - 4t$$

Lines

symmetric equations of *L*.

$$3 \qquad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

If none of *a*, *b*, or *c* is 0,

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

If one of *a*, *b*, or *c* is 0,

Lines

4 The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

If we extend the t form [0,1] to R, then the line segment becomes line

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad t \in R$$

Best way if you only know two points on the line

r_0 and r_1 are two points on the line,

Planes

7 A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\mathbf{n} = \langle a, b, c \rangle,\ \mathbf{r} = \langle x, y, z \rangle,\ \text{and } \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle.$$

N is the normal vector, r is any point on the plane, r_0 is a known point on the plane

Distances (1 of 3)

Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let **b** be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

From Figure 12 you can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of **b** onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$.

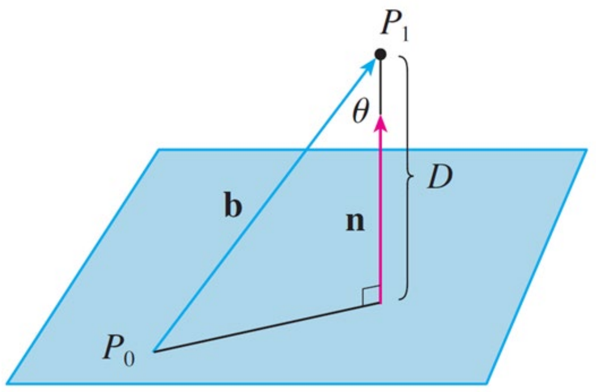


Figure 12

Distances (2 of 3)

Thus

$$D = \left|\text{comp}_{\mathbf{n}}\mathbf{b}\right| = \frac{|\mathbf{n}\cdot\mathbf{b}|}{|\mathbf{n}|}$$

$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

Distances (3 of 3)

Since P_0 lies in the plane, its coordinates satisfy the equation of the plane and so we have

$$ax_0 + by_0 + cz_0 + d = 0.$$

Thus we have the following formula.

9 The distance D from the point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

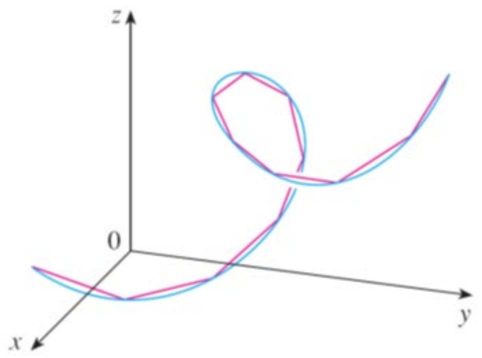
$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Arc Length

Arc Length (3 of 5)

General form

$$\textbf{3} \quad L = \int_a^b |\mathbf{r}'(t)|\, dt$$



for plane curves $\mathbf{r}(t) = f()\mathbf{i} + g(t)\mathbf{j}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

for space curves $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

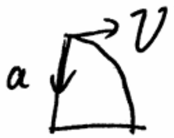
Curvature

Physic about TBN frame

后续

陳楨[illegible]

① Analogy with uniform circular motion,



$v'(t) = \frac{\Delta v}{\Delta t} = a$,

$a \perp v$, $v'(t) \perp v(t)$

more generally, for function $\vec{r}(t)$, $|\vec{r}(t)| = a$ constant

$r'(t) \perp r(t)$,

e.g. 1, Sample 2 Q 5

2, $T(t) \perp N(t)$

Physic about TBN frame

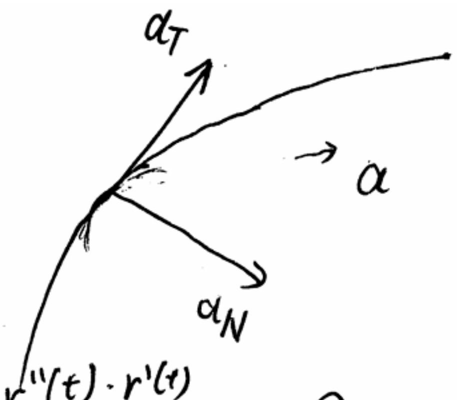
② A way to deduce or remember fomula $\vec{f''}(t) - \frac{\vec{f''}(t) \cdot \vec{f'}(t)}{|\vec{f'}(t)|^2} \vec{f'}(t)$

$r(t) \Leftrightarrow$ path
$r'(t) \Leftrightarrow$ speed
$r''(t) \Leftrightarrow$ acceleration

physic meaning

curve	$r(t)$	path of a particle	s	
1th derivative	$r'(t)$	speed of a particle	speed	v
2nd derivative	$r''(t)$	acceleration of the particle	acceleration a	a

Physic about TBN frame



acceleration has two components

1) Tangential acceleration $\vec{a_T}$

change the magnitude of speed,

~~po~~ direction same as speed, $\vec{a_T} = proj_a v = \frac{r''(t) \cdot r'(t)}{|r'(t)|}$ ... 1

2) Radial acceleration $\vec{a_R}$

change the direction of speed

perpendicular to the speed.

$$|\vec{a_R}| = \sqrt{|\vec{a}|^2 - |\vec{a_T}|^2} = \sqrt{\frac{|r'(t)|^2 \cdot |r''(t)|^2 - |r'(t)|^2 |r''(t)|^2 \cos^2\theta}{|r'(t)|^2}} = \frac{|r'(t)| \cdot |r''(t)| \sin\theta}{|r'(t)|} = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$$

... 2

Physic about TBN frame

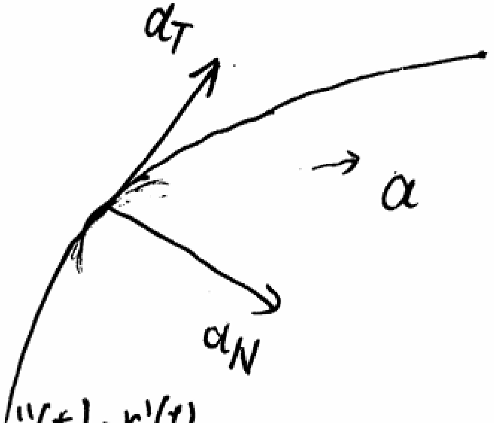
$$\vec{a_N} = \vec{a} - \vec{a_t}$$

$$= \vec{r''(t)} - \mathrm{proj}_{a}\, v$$

$$= \vec{r''(t)} - \mathrm{proj}_{r''(t)}\, r'(t)$$

$$= \vec{r''(t)} - \frac{r''(t) \cdot r'(t)}{|r'(t)|^2} \cdot \vec{r'}(t) \quad \cdots\cdots ③$$

$$N(t) = \frac{\vec{a_N}}{|\vec{a_N}|}$$



Physic about TBN frame

if ρ is radius of curvature circle

(~~oscillating~~ osculating circle) , $\kappa(t) = \frac{1}{\rho}$

$$|\vec{a_r}| = \frac{v^2}{\rho}$$

plug in ③ ~~it is~~

$$\frac{|r'(t) \times r''(t)|}{|r'(t)|} = \frac{|r'(t)|^2}{\rho}$$

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

We recommend using this formula to calculate
Because this avoid absolute value or take
Derivative of the absolute value

TBN的物理意义

陈桢博 陳楨博

Date 10.19

NO.

类比匀速圆周运动

v，长度恒定，方向变化，

$v'(t) = \frac{\Delta v}{\Delta t} = a$，$a = v'(t)$，$v$ 的导数为加速度，$\Delta v \perp v$

$a \perp v$，即 $v'(t) \perp v(t)$

类似地，~~向量~~多维函数 $\vec{r}(t)$，$|\vec{r}(t)| = a$ 恒定，

$r'(t) \perp r(t)$；e.g. 1, Sample 2 Q5

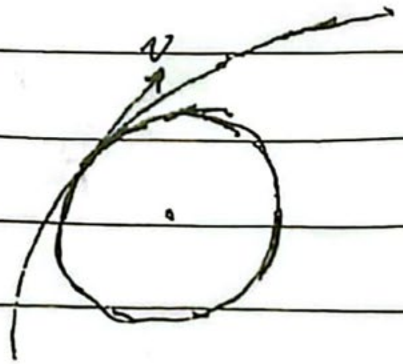
2, $T(t) \perp N(t)$

TBN的物理意义

运用$N(t)$的物理意义，记忆推导公式 ~~N(t)~~：$N(t) = f''(t) - \frac{f'(t)\cdot f''(t)}{|f'(t)|^2} f'(t)$

~~曲线 r(t)，物理意义~~

		物理意义	
曲线	$r(t)$	物体运动轨迹	s
一次导	$r'(t)$	物体运动速度	v
二次导	$r''(t)$	物体运动加速度	a



加速度有两份量，切向加速度改变物体速度大小，径向加速度改变物体运动方向

切向加速度：$r''(t)$在$r(t)$的投影，$a_t = \frac{r''(t)\cdot r'(t)}{|r'(t)|}$

径向加速度：勾股定理，$\sqrt{|r''(t)|^2 - a_t^2} = \frac{|r''(t) \times r'(t)|}{|r'(t)|}$

TBN的物理意义

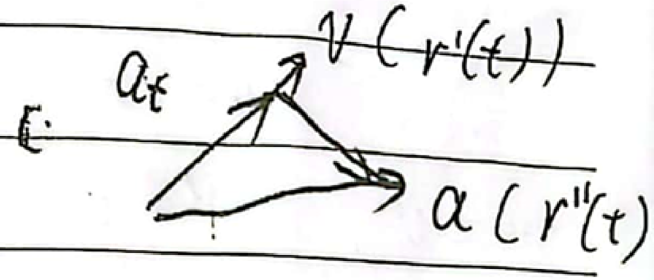
那么，上述公式求得径向加速度的大小，方向如何求

$$\vec{a_r} = \vec{a} - \vec{a_t}$$

$$= \vec{r''}(t) - \text{proj}_a v$$

$$= \vec{r''}(t) - \text{proj}_{r'(t)} r'(t)$$

$$= \vec{r''}(t) - \frac{r''(t)\cdot r'(t)}{|r'(t)|^2}\cdot \vec{r'}(t)$$

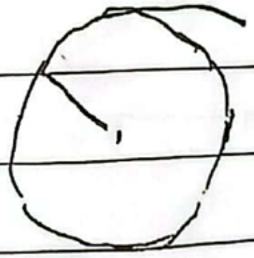


$N(t)$ 即是 $\vec{a_r}$ 方向

$$N(t) = \frac{\vec{a_r}}{|\vec{a_r}|}$$

TBN的物理意义

$K(t)$ is 曲率半径的倒数 $\frac{1}{\rho}$



$a = a_r = \frac{v^2}{\rho}$（数量形式）

即 $\frac{|r'(t)\times r''(t)|}{|r'(t)|} = \frac{|r'(t)|^2}{\rho}$

$$K(t) = \frac{|r'(t)\times r''(t)|}{|r'(t)|^3}$$

定义式简洁，但涉及了向量长度的运算 $\sqrt{(\ )^2+(\ )^2+(\ )^2}$，甚至还要对 $\sqrt{(\ )^2+(\ )^2+(\ )^2}$ 求导，实际运算中并不占优。
当题以三角函数与幂函数(t^3)为主，所以上述公式可以极大地

Curvature (1 of 6)

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I.

A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

If C is a smooth curve defined by the vector function $\mathbf{r}$, recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and indicates the direction of the curve.

Curvature (3 of 6)

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point.

Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the curvature will be independent of the parametrization.) Because the unit tangent vector has constant length, only changes in direction contribute to the rate of change of **T.**

8 Definition The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where **T** is the unit tangent vector.

Curvature (4 of 6)

The curvature is easier to compute if it is expressed in terms of the parameter *t* instead of *s*, so we use the Chain Rule to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \qquad \text{and} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{\dfrac{d\mathbf{T}}{dt}}{\dfrac{ds}{dt}}\right|$$

But $\dfrac{ds}{dt} = |\mathbf{r}'(t)|$ from Equation 7, so

$$\textbf{9} \qquad \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Curvature (5 of 6)

The result of Example 3 shows that small circles have large curvature and large circles have small curvature, in accordance with our intuition.

We can see directly from the definition of curvature that the curvature of a straight line is always 0 because the tangent vector is constant.

Although Formula 9 can be used in all cases to compute the curvature, the formula given by the following theorem is often more convenient to apply.

10 Theorem The curvature of the curve given by the vector function **r** is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Curvature (6 of 6)

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x\,\mathbf{i} + f(x)\,\mathbf{j}$.

Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\mathbf{j}$.

Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\mathbf{k}$.

We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by Theorem 10,

11 $$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{\frac{3}{2}}}$$

Osculating Circle

The **circle of curvature**, or the **osculating circle**, of C at P is the circle in the osculating plane that passes through P with radius $\frac{1}{\kappa}$ and center a distance $\frac{1}{\kappa}$ from P *along* the vector $\mathbf{N}$. The center of the circle is called the **center of curvature** of C at P.

We can think of the circle of curvature as the circle that best describes how C behaves near P— it shares the same tangent, normal, and curvature at P.

Tangent Planes

Tangent Planes

If C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane.

Therefore you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P. The tangent plane at P is the plane that most closely approximates the surface S near the point P.

We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Tangent Planes

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$\textbf{1} \quad z - z_0 = a(x - x_0) + b(y - y_0)$$

If Equation 1 represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1. Setting $y = y_0$ in Equation 1 gives

$$z - z_0 = a(x - x_0) \qquad \text{where } y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a.

Tangent Planes

2 Equation of a Tangent Plane Suppose *f* has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Method for Finding the Tangent Plane to $F(x, y, z) = 0$

Let $F(x, y, z)$ be a differentiable function, and let the surface be defined by the equation $F(x, y, z) = 0$. We want to find the equation of the tangent plane at the point $P_0(x_0, y_0, z_0)$.

$$F_x(x_0, y_0, z_0) \cdot (x - x_0) + F_y(x_0, y_0, z_0) \cdot (y - y_0) + F_z(x_0, y_0, z_0) \cdot (z - z_0) = 0$$

Tangent Planes

Two examples for this formula

$$F_x(x_0, y_0, z_0) \cdot (x - x_0) + F_y(x_0, y_0, z_0) \cdot (y - y_0) + F_z(x_0, y_0, z_0) \cdot (z - z_0) = 0$$

Quiz 6 **MATH 241**

1. (a) Let S be the quadratic surface in $\mathbb{R}^3$ with equation $x^2 + 2yz = 3$. Determine the tangent plane of S in $Q = (0, \frac{3}{2}, 1)$.

67. Show that the sum of the x-, y-, and z-intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

1043 in PDF 1008 in textbook

Reduce the calculation significantly

Linear Approximations

Linear Approximations

The approximation

$$\textbf{4} \quad f(x, y) \approx f(a,\ b) + f_x(a,\ b)(x - a) + f_y(a,\ b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of *f* at (a, b).

W23 *From a previous midterm*

The height and mass ("weight") of a basketball player are known to be $h_0 = 2.00\,\mathrm{m}$ and $m_0 = 100\,\mathrm{kg}$, respectively, with a possible error of 0.5 cm, respectively, 0.5 kg.

a) Find the linear approximation of the *body mass index* $B(m,h) = m/h^2$ near (m_0, h_0).

a) Here $B(m,h) \approx B(m_0, h_0) + \frac{\partial B}{\partial m} \cdot \Delta m + \frac{\partial B}{\partial h} \cdot \Delta h$

$$= 25 + \frac{1}{4}\Delta m - 25\Delta h \quad [\mathrm{kg} \cdot \mathrm{m}^{-2}]$$

Differentiable

Differentiable

The most trival way

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Notice that all elementary function is continuous on its domain

Differentiable

The most rigorous way

7 Definition If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a,b)\ \Delta x + f_y(a,b)\ \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 are functions of Δx and Δy such that ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Definition 7 says that a differentiable function is one for which the linear approximation (4) is a good approximation when (x, y) is near (a, b). In other words, the tangent plane approximates the graph of f well near the point of tangency.

Differentiable

Partial derivative exist and continuous

Differentiable

Original function continuous

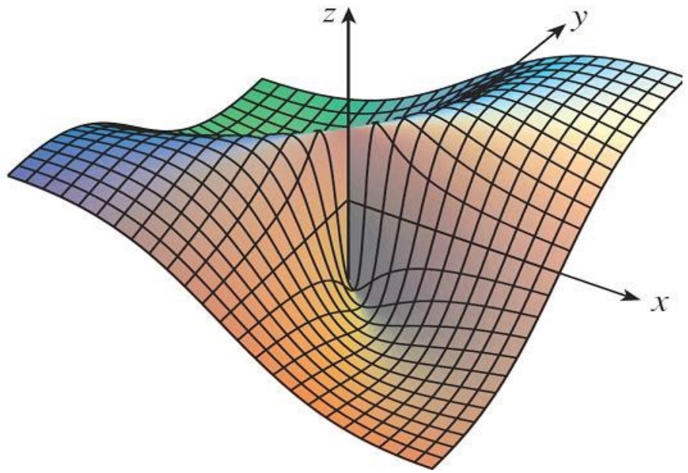
Partial derivative exist

Differentiable

What happens if f_x and f_y are not continuous? Figure 4 pictures such a function; its equation is

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

You can verify that its partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous.



$f(x,y) = \dfrac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$,

$f(0,0) = 0$

Figure 4

Linear Approximations (6 of 9)

The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$.

So a function of two variables can behave badly even though both of its partial derivatives exist. To rule out such behavior, we formulate the idea of a differentiable function of two variables.

We know that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

Linear Approximations (7 of 9)

If *f* is differentiable at *a*, then

$$\mathbf{5} \quad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \quad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

$$\mathbf{6} \quad \Delta z = f(a + \Delta x,\ b + \Delta y) - f(a,\ b)$$

Thus the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

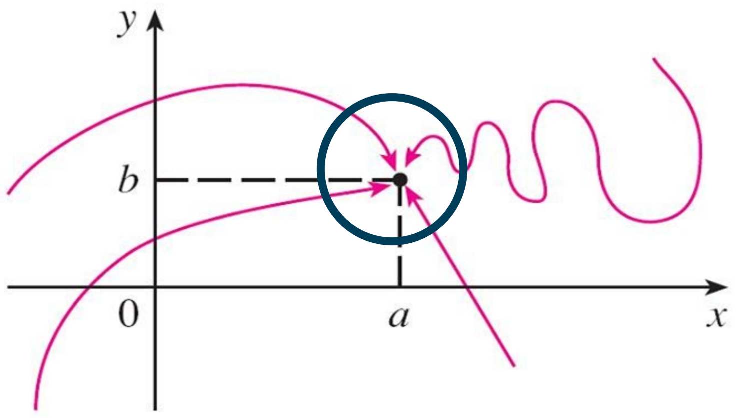
Limits of Functions of Two Variables

Limits of Functions of Two Variables

Limit Existence Condition (Single Equation)

$$\lim_{x \to x_0} f(x) = L \iff \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L$$

Single varible



Multi varible

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x,y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x,y) - L| < \varepsilon$$

Limits of Functions of Two Variables

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L$$

and

$$f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

➕➖✖️➗ Limit Laws for Arithmetic Operations:

Assume $\lim f(x) = A$ and $\lim g(x) = B$ (where $\lim$ can represent $x \to x_0$ or $n \to \infty$, etc.), then:

- **Limit of a Sum/Difference:** $\lim[f(x) \pm g(x)] = \lim f(x) \pm \lim g(x) = A \pm B$
- **Limit of a Product:** $\lim[f(x) \cdot g(x)] = \lim f(x) \cdot \lim g(x) = A \cdot B$
- **Limit of a Constant Multiple:** $\lim[c \cdot f(x)] = c \cdot \lim f(x) = c \cdot A$ (where c is a constant)
- **Limit of a Quotient:** $\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)} = \frac{A}{B}$ (provided that $B \neq 0$)

Limits of Functions of Two Variables

Tool box for solving the limit

1. Path dependent limit

try path y= kx, y = x^a, y = (x-a)^2, can only show the limit does not exist

2. Polar coordinate

(can use to prove the limit exist)

3. Plug in definition

if $(x,y)\in D$ and $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$ then $|f(x,y)-L|<\varepsilon$

We can also use some equation or inequality

4.Change it into single Variable and use Technique from Math 231

Limits of Functions of Two Variables

some equations or inequality

The Cauchy–Schwarz Inequality

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$$

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Description	Formula
Sine Sum	$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
Sine Difference	$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
Cosine Sum	$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
Cosine Difference	$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

Description	Formula
Sine-Cosine	$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$
Cosine-Sine	$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$
Cosine-Cosine	$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$
Sine-Sine	$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

Limits of Functions of Two Variables

use Technique from Math 231

Taylor Series:

$x \to 0$ | $\sin x \sim x, \tan x \sim x, e^x - 1 \sim x, 1 - \cos x \sim \frac{1}{2}x^2$

(L'Hôpital's Rule):

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ **or** If $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

and $g'(x) \neq 0$ near a (except possibly at a),

and the limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists (or is $\pm\infty$),

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Conjugate Method:

$$\lim_{x \to 0} \frac{\sqrt{x+9} - 3}{x}$$

Partial Derivatives of Functions of Two Variables

Partial Derivatives of Functions of Two Variables (9 of 14)

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant.

Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of *f* with respect to *x* at (*a*, *b*)** and denote it by $f_x(a, b)$. Thus

$$\mathbf{1} \qquad f_x(a,b) = g'(a) \quad \text{where} \quad g(x) = f(x,b)$$

By the definition of a derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

and so Equation 1 becomes

2 $$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,\, b) - f(a,\, b)}{h}$$

Partial Derivatives of Functions of Two Variables (11 of 14)

Similarly, the **partial derivative of *f* with respect to *y* at (*a*, *b*)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

3 $$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

With this notation for partial derivatives, we can write the rates of change of the heat index I with respect to the actual temperature T and relative humidity H when $T = 96°F$ and $H = 70\%$ as follows:

$$f_T(96,70) \approx 3.75 \qquad f_H(96,70) \approx 0.9$$

Partial Derivatives of Functions of Two Variables (12 of 14)

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

4 Definition If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Partial Derivatives of Functions of Two Variables (13 of 14)

There are many alternative notations for partial derivatives. For instance, instead of f_x we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the *first* variable) or $\partial f/\partial x$.

But here $\partial f/\partial x$ can't be interpreted as a ratio of differentials.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Level Curves and Contour Maps

Level Curves and Contour Maps (1 of 10)

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour curves*, or *level curves*.

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it is a curve in the xy-plane that shows where the graph of f has height k (above or below the xy-plane). A collection of level curves is called a **contour map**.

Contour maps are most descriptive when the level curves $f(x, y) = k$ are drawn for equally spaced values of k, and we assume that this is the case unless indicated otherwise. You can see from Figure 11 the relation between level curves and horizontal traces.

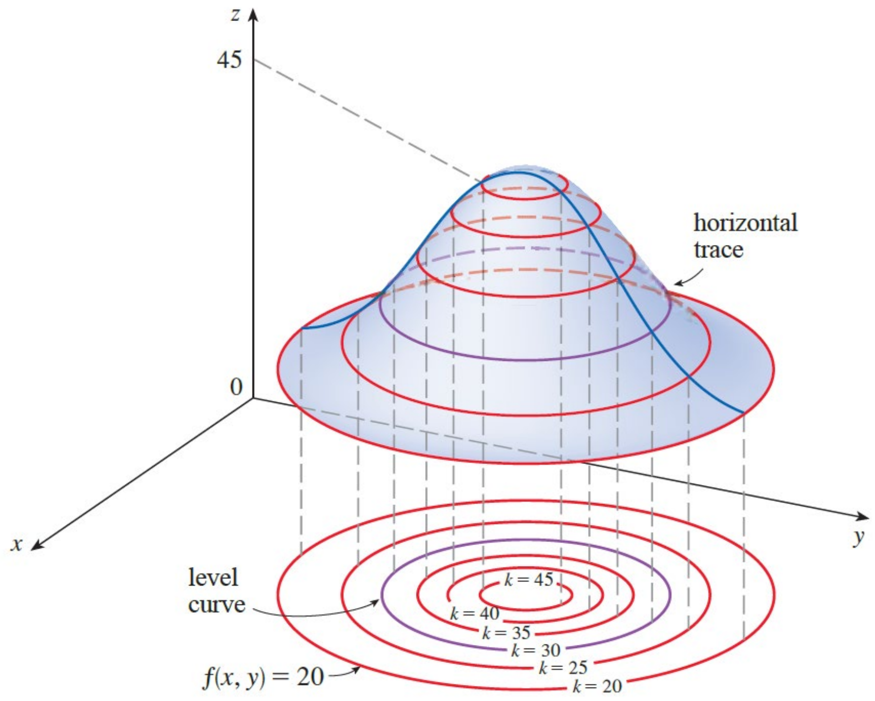


Figure 11

Level Curves and Contour Maps (3 of 10)

The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane. So if you draw a contour map of a function and visualize the level curves being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph.

The surface is steeper where the level curves are close together and somewhat flatter where they are farther apart.

Recap

- Projections
- Triple Products
- Lines and Planes
- Arc Length
- Curvature
- Tangent Planes
- Linear Approximations
- Limits of Functions of Two Variables
- Showing That a Limit Does Not Exist
- Partial Derivatives of Functions of Two Variables
- Level Curves and Contour Maps