Problem 1

The introduced technique may have been introduced in ECE313, when normalizing the Gauss distribution, To avoid the usage of polar coordinate transformation, we use another famous example.

Prove Dirichlet Integral $I = \int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}$ in following step:

(a) Prove when x > 0, $\frac{1}{x} = \int_0^\infty e^{-xy}\, dy$

(just calculate the integral according to y)

(b) Use the result of (a), rewrite the Dirichlet Integral into iterated integral

$$I = \int_0^\infty \sin(x) \left(\int_0^\infty e^{-xy} dy \right) dx$$

Since $\sin(x)$ does not depend on y, we can move it inside the inner integral:

$$I = \int_0^\infty \int_0^\infty e^{-xy} \sin(x)\, dy\, dx$$

(c) Use Fubini-Tonelli theorem, calculate the iterated integral

$$I = \int_0^\infty \left(\int_0^\infty e^{-xy} \sin(x)\, dx \right) dy$$

$$\int e^{-xy} \sin(x)\, dx = \frac{e^{-xy}}{(-y)^2 + 1^2}(-y \sin(x) - 1 \cos(x))$$

Evaluating from $x = 0$ to $x = \infty$:

Step 2: Evaluate the outer integral (with respect to y) Substitute J back into the expression for I:

$$I = \int_0^\infty \frac{1}{y^2 + 1} dy$$

This is a standard integral resulting in the arctangent function:

$$I = \Big[\arctan(y) \Big]_0^\infty$$

Problem 2

A question from previous final (around 2022)

Question 2 (ca. 12 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^4 - x^3y - xy + y^2.$$

a) Which obvious symmetry property does f have? What can you conclude from this about the graph and the contours of f?

b) Determine all critical points of f and their types.
Hint: There are 5 critical points.

c) Does f have a global extremum?

d) Determine the extrema of f on the unit square $Q = \{(x, y) \in \mathbb{R}^2; 0 \le x, y \le 1\}$.

（Hint for (d), Extrema located in Q must be critical points, besides we need to plug in values and study the situation of 4 planes）

2 a) $f(-x,-y) = f(x,y)$ for $(x,y) \in \mathbb{R}^2$
$\Longrightarrow$ The graph of f is symmetric with respect to the z-axis. $\boxed{1}$
The contours of f are point-symmetric with respect to the origin. $\boxed{1}$

b) Using the shorthands f, f_x, f_y for $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, we compute

$$\begin{aligned} f_x &= 4\,x^3 - 3\,x^2y - y, \\ f_y &= -x^3 - x + 2\,y, \end{aligned} \qquad \boxed{1}$$

$$\begin{aligned} \nabla f(x,y) = (0,0) &\Longrightarrow y = \tfrac{1}{2}\left(x^3 + x\right) \Longrightarrow 4x^3 - \left(3x^2+1\right)\tfrac{1}{2}\left(x^3+x\right) = 0 \\ &\Longrightarrow 3x^5 - 4x^3 + x = 0 \\ &\Longrightarrow x(x^2-1)(3x^2-1) = 0. \end{aligned}$$

$\Longrightarrow$ The critical points of f are

$$\begin{aligned} &\mathbf{p}_1 = (0,0), \quad \mathbf{p}_2 = (1,1), \quad \mathbf{p}_3 = (-1,-1), \\ &\mathbf{p}_4 = \left(\tfrac{1}{3}\sqrt{3}, \tfrac{2}{9}\sqrt{3}\right), \quad \mathbf{p}_5 = \left(-\tfrac{1}{3}\sqrt{3}, -\tfrac{2}{9}\sqrt{3}\right). \end{aligned} \qquad \boxed{2\tfrac{1}{2}}$$

Further we have

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 - 6xy & -3x^2 - 1 \\ -3x^2 - 1 & 2 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_1) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_{2/3}) = \begin{pmatrix} 6 & -4 \\ -4 & 2 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_{4/5}) = \begin{pmatrix} 8/3 & -2 \\ -2 & 2 \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_1)$ has determinant $-1 < 0$, the point $\mathbf{p}_1$ is a saddle point. $\boxed{\tfrac{1}{2}}$
Since $\mathbf{H}_f(\mathbf{p}_{2/3})$ has determinant $12-16 = -4 < 0$, the points $\mathbf{p}_2$, $\mathbf{p}_3$ are saddle points. $\boxed{1}$
Since $\mathbf{H}_f(\mathbf{p}_{4/5})$ is positive definite ($f_{xx}(\mathbf{p}_{4/5}) = 8/3 > 0$, $\det \mathbf{H}_f(\mathbf{p}_{4/5}) = 16/3 - 4 = 4/3 > 0$), the points $\mathbf{p}_4$, $\mathbf{p}_5$ are strict local minima. $\boxed{1}$
The corresponding value is $f(\mathbf{p}_{4/5}) = -1/27$.

c) No. This follows, e.g., from $f(x,0) = x^4 \to +\infty$ for $x \to \pm\infty$, $f(x,2x) = -x^4 + 2x^2 \to -\infty$ for $x \to \pm\infty$. $\boxed{1}$

d) Extrema located in Q° must be critical points, and hence equal to $\mathbf{p}_4$. On the boundary ∂Q we have

$$\begin{aligned} f(x,0) &= x^4, \\ f(0,y) &= y^2, \\ f(x,1) &= x^4 - x^3 - x + 1 = (x^3-1)(x-1), \\ f(1,y) &= 1 - 2y + y^2 = (1-y)^2. \end{aligned}$$

One sees that the values on ∂Q vary between 0 and 1, with 1 attained at $(1,0)$, $(0,1)$, and 0 attained at $(0,0)$, $(1,1)$. Comparing these with $f(\mathbf{p}_4) = -1/27$ shows that f on Q attains its minimum at $\mathbf{p}_4$ and two maxima at $(1,0)$, $(0,1)$. $\boxed{3}$

Remarks: