Solutions

1. The region is the half–disk

$$D = \{(x,y) : 0 \le y \le a,\ x^2 + y^2 \le a^2\}.$$

In polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$, with

$$0 \le r \le a, \qquad 0 \le \theta \le \pi.$$

Then

$$2x + y = 2r\cos\theta + r\sin\theta, \qquad dA = r\,dr\,d\theta.$$

Hence

$$\begin{aligned}
\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x+y)\,dx\,dy &= \int_0^\pi \int_0^a (2r\cos\theta + r\sin\theta)\,r\,dr\,d\theta \\
&= \int_0^\pi \int_0^a (2\cos\theta + \sin\theta)\,r^2\,dr\,d\theta \\
&= \int_0^\pi (2\cos\theta + \sin\theta)\,d\theta \int_0^a r^2\,dr.
\end{aligned}$$

Now

$$\int_0^\pi 2\cos\theta\,d\theta = 0, \qquad \int_0^\pi \sin\theta\,d\theta = 2,$$

so the angular integral equals 2, and

$$\int_0^a r^2\,dr = \frac{a^3}{3}.$$

Therefore

$$\boxed{\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x+y)\,dx\,dy = \frac{2}{3}a^3}.$$

2. The region is

$$D = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le \sqrt{2x - x^2}\},$$

which is the upper half of the circle $(x-1)^2 + y^2 \le 1$ (center $(1,0)$, radius 1).

In polar coordinates about the origin, this region is

$$0 \le \theta \le \frac{\pi}{2}, \qquad 0 \le r \le 2\cos\theta$$

(the circle is $r = 2\cos\theta$, and we take the part in the first quadrant). The integrand is $\sqrt{x^2+y^2} = r$, and $dA = r\,dr\,d\theta$, so

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx &= \int_0^{\pi/2} \int_0^{2\cos\theta} r \cdot r\,dr\,d\theta \\
&= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta \\
&= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta \\
&= \int_0^{\pi/2} \frac{8}{3}\cos^3\theta\,d\theta = \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\,d\theta.
\end{aligned}$$

Using $\cos^3\theta = \cos\theta(1-\sin^2\theta)$, or a standard table integral,

$$\int_0^{\pi/2} \cos^3\theta\,d\theta = \frac{2}{3},$$

so

$$\boxed{\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \frac{16}{9}}.$$

3. **(From a previous final)**
$K = \{(x,y,z) : x^2+y^2 \le z \le 1+x\}$.

(a) Closed and bounded.

The set is given by two weak inequalities:

$$x^2+y^2-z \le 0, \qquad z-(1+x) \le 0.$$

Each inequality describes a closed set (inverse image of $(-\infty, 0]$ under a continuous function), so their intersection K is closed.

From

$$x^2+y^2 \le 1+x \quad \Longleftrightarrow \quad \left(x-\frac{1}{2}\right)^2 + y^2 \le \frac{5}{4},$$

we obtain

$$|x| \le \frac{1+\sqrt{5}}{2}, \qquad |y| \le \frac{\sqrt{5}}{2}.$$

Moreover $z \le 1+x$ and $z \ge x^2+y^2 \ge 0$, hence

$$|z| \le 1+|x| \le 1+\frac{1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}.$$

Thus K is bounded. Being closed and bounded in $\mathbb{R}^3$, it is compact.

(b) Volume $\text{vol}(K)$**.**

The projection of K to the xy-plane is the disk

$$D = \{(x, y) : (x - \tfrac{1}{2})^2 + y^2 \leq \tfrac{5}{4}\}.$$

For $(x, y) \in D$, z ranges from $x^2 + y^2$ to $1 + x$, so

$$\text{vol}(K) = \iint_D \big(1 + x - (x^2 + y^2)\big)\, d^2(x, y).$$

Use polar coordinates relative to the center of D:

$$x = \frac{1}{2} + r\cos\phi, \qquad y = r\sin\phi, \qquad 0 \leq r \leq \frac{\sqrt{5}}{2},\ 0 \leq \phi \leq 2\pi,$$

and $d^2(x, y) = r\, dr\, d\phi$. Then

$$\begin{aligned}
\text{vol}(K) &= \int_0^{2\pi} \int_0^{\sqrt{5}/2} \Big(1 + \Big(\frac{1}{2} + r\cos\phi\Big) - \Big(\frac{1}{2} + r\cos\phi\Big)^2 - r^2\sin^2\phi\Big) r\, dr\, d\phi \\
&= \int_0^{2\pi} \int_0^{\sqrt{5}/2} r\Big(\frac{5}{4} - r^2\Big)\, dr\, d\phi \\
&= \int_0^{2\pi} \Big[\frac{5}{8}r^2 - \frac{1}{4}r^4\Big]_0^{\sqrt{5}/2} d\phi = 2\pi\Big(\frac{25}{32} - \frac{25}{64}\Big) = \boxed{\frac{25}{32}\pi}.
\end{aligned}$$

(c) surface area of ∂K**.**

The boundary ∂K is the union

$$S_1 = \{(x, y, z) : z = x^2 + y^2\}, \quad S_2 = \{(x, y, z) : z = 1 + x\}, \quad S_3 = \{(x, y, z) : x^2 + y^2 = z = 1 + x\}.$$

Here S_3 is one-dimensional, so it has surface area 0.

The surface S_1 is the graph of $f(x, y) = x^2 + y^2$ over the disk D. Its area is

$$\begin{aligned}
\text{vol}_2(S_1) &= \iint_D \sqrt{1 + \|\nabla f(x, y)\|^2}\, d^2(x, y) \\
&= \iint_D \sqrt{1 + 4x^2 + 4y^2}\, d^2(x, y) \\
&= \int_0^{2\pi} \int_0^{\sqrt{5}/2} r\sqrt{1 + 4\Big(\frac{1}{2} + r\cos\phi\Big)^2 + 4(r\sin\phi)^2}\, dr\, d\phi \\
&= \int_0^{2\pi} \int_0^{\sqrt{5}/2} r\sqrt{2 + 4r\cos\phi + 4r^2}\, dr\, d\phi.
\end{aligned}$$

Similarly S_2 is the graph of $g(x, y) = 1 + x$ over D. Since $\nabla g = (1, 0)$,

$$\text{vol}_2(S_2) = \iint_D \sqrt{1 + \|\nabla g(x, y)\|^2}\, d^2(x, y) = \iint_D \sqrt{2}\, d^2(x, y) = \sqrt{2}\,\text{vol}_2(D) = \sqrt{2}\cdot\frac{5\pi}{4} = \frac{5\pi}{2\sqrt{2}}.$$

Hence one can write

$$\mathrm{vol}_2(\partial K) = \mathrm{vol}_2(S_1) + \mathrm{vol}_2(S_2) = \frac{5\pi}{2\sqrt{2}} + \int_0^{2\pi} \int_0^{\sqrt{5}/2} r\sqrt{2 + 4r\cos\phi + 4r^2}\, dr\, d\phi.$$

(Explicit evaluation of the last integral is not required here.)