Solutions

1. We compute the line integral

$$\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz,$$

where C consists of two line segments.

First segment: $(0,0,0) \to (1,0,1)$. Let $x = t,\ y = 0,\ z = t$, $0 \le t \le 1$. Then $dx = dz = dt$ and $dy = 0$. Hence

$$\int_{C_1} (y+z)\,dx + (x+z)\,dy + (x+y)\,dz = \int_0^1 2t\,dt = 1.$$

Second segment: $(1,0,1) \to (0,1,2)$. Let $x = t,\ y = 1-t,\ z = 2-t$, $0 \le t \le 1$. Then $dx = dt$ and $dy = dz = -dt$. Hence

$$\int_{C_2} (y+z)\,dx + (x+z)\,dy + (x+y)\,dz = \int_1^0 (-2t)\,dt = 1.$$

Combining C_1 and C_2, we obtain

$$\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz = 1 + 1 = 2.$$

2. We compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt.$$

From $\mathbf{r}(t) = \langle t^3,\ t^2 \rangle$, we have

$$\mathbf{r}'(t) = \langle 3t^2,\ 2t \rangle.$$

Also,

$$\mathbf{F}(\mathbf{r}(t)) = \langle t^3(t^2)^2,\ -(t^3)^2 \rangle = \langle t^7,\ -t^6 \rangle.$$

Thus,

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = t^7(3t^2) + (-t^6)(2t) = 3t^9 - 2t^7.$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3t^9 - 2t^7)\,dt = \left[\frac{3}{10}t^{10} - \frac{1}{4}t^8\right]_0^1 = \frac{1}{20}.$$