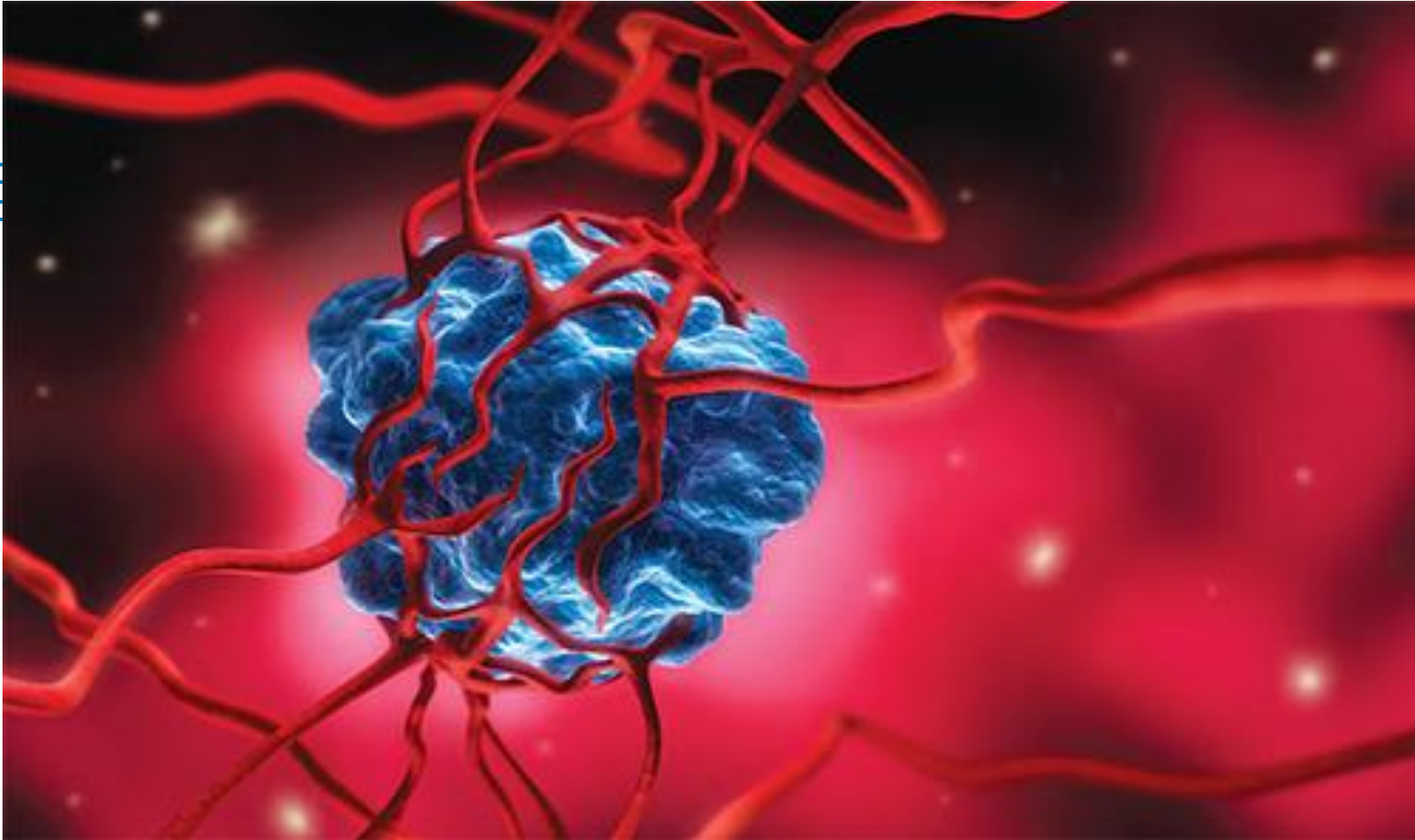


15 Multiple Integrals



A decorative graphic consisting of four horizontal blue lines of varying lengths, stacked vertically, positioned to the left of the section number.

15.7

Triple Integrals in Cylindrical Coordinates

Context

- Cylindrical Coordinates
- Triple Integrals in Cylindrical Coordinates

Triple Integrals in Cylindrical Coordinates (1 of 2)

In plane geometry the polar coordinate system is used to give a convenient description of certain curves and regions.

Figure 1 enables us to recall the connection between polar and Cartesian coordinates.

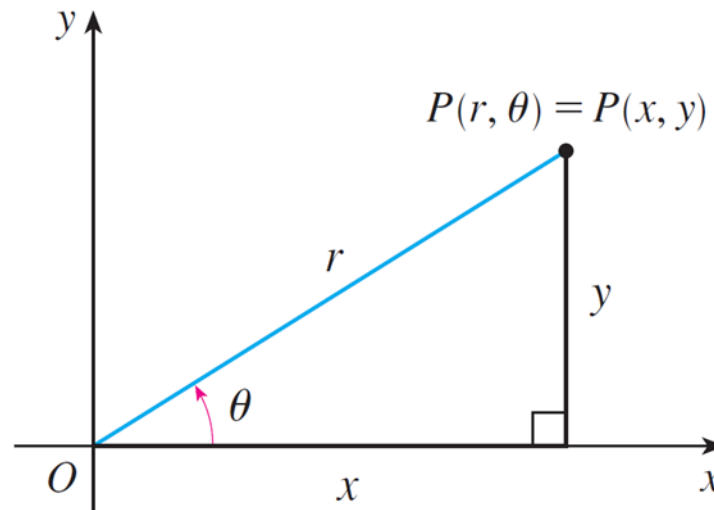


Figure 1

Triple Integrals in Cylindrical Coordinates (2 of 2)

If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ) , then, from the figure,

$$x = r \cos \theta \qquad y = r \sin \theta$$

$$r^2 = x^2 + y^2$$

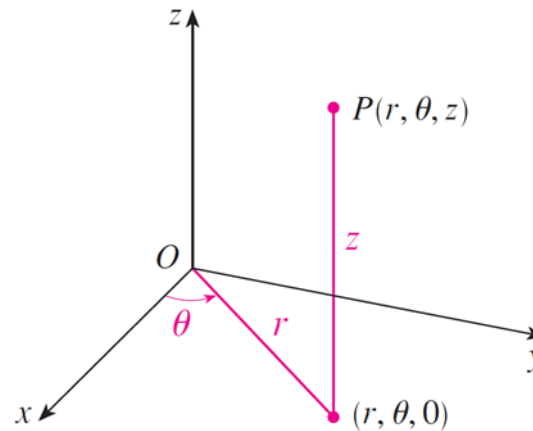
In three dimensions there is a coordinate system, called *cylindrical coordinates*, that is similar to polar coordinates and gives convenient descriptions of some commonly occurring surfaces and solids. As we will see, some triple integrals are much easier to evaluate in cylindrical coordinates.



Cylindrical Coordinates

Cylindrical Coordinates (1 of 2)

In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r, θ, z) , where r and θ are polar coordinates of the projection of P onto the xy -plane and z is the directed distance from the xy -plane to P . (See Figure 2.)



The cylindrical coordinates of a point

Figure 2

Cylindrical Coordinates (2 of 2)

To convert from cylindrical to rectangular coordinates, we use the equations

$$1 \quad x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

$$2 \quad r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad z = z$$

Example 1

- (a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.
- (b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Solution:

- (a) The point with cylindrical coordinates $(2, 2\pi/3, 1)$ is plotted in Figure 3.

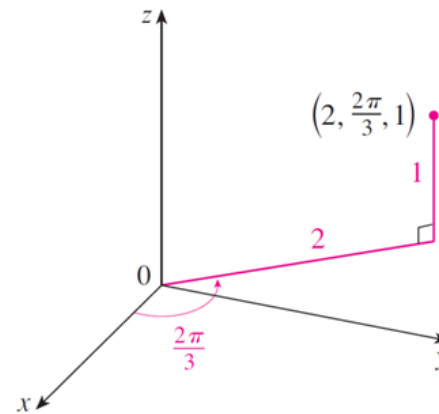


Figure 3

Example 1 – Solution (1 of 2)

From Equations 1, its rectangular coordinates are

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

So the point is $(-1, \sqrt{3}, 1)$ in rectangular coordinates.

Example 1 – Solution (2 of 2)

(b) From Equations 2 and noting that θ is in quadrant IV of the xy -plane, we have

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan \theta = \frac{-3}{3} = -1 \quad \text{so} \quad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore one set of cylindrical coordinates is $(3\sqrt{2}, 7\pi/4, -7)$. Another is $(3\sqrt{2}, -\pi/4, -7)$.

As with polar coordinates, there are infinitely many choices.



Triple Integrals in Cylindrical Coordinates

Triple Integrals in Cylindrical Coordinates (1 of 5)

Suppose that E is a type 1 region whose projection D onto the xy -plane is conveniently described in polar coordinates (see Figure 8).

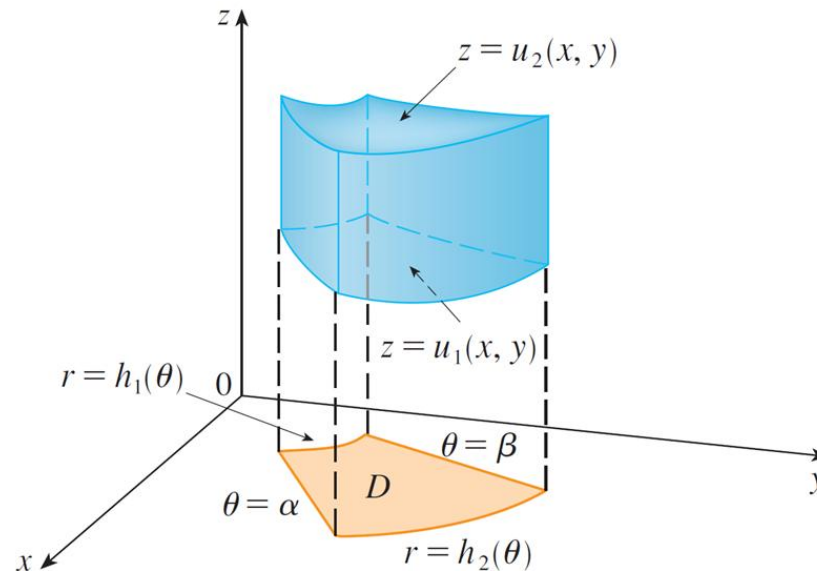


Figure 8

Triple Integrals in Cylindrical Coordinates (2 of 5)

In particular, suppose that f is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

We know

$$\mathbf{3} \quad \iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right] dA$$

Triple Integrals in Cylindrical Coordinates (3 of 5)

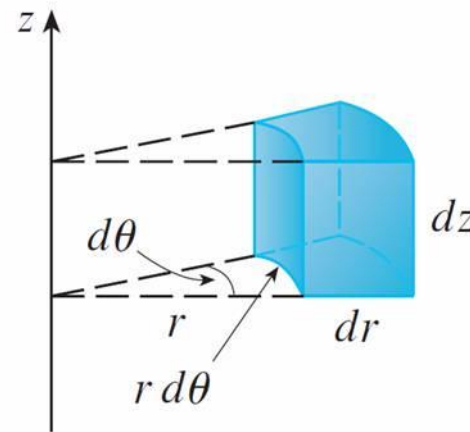
But to evaluate double integrals in polar coordinates, we have the formula

$$4 \quad \iiint_E f(x, y, z) \, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta$$

Formula 4 is the **formula for triple integration in cylindrical coordinates**.

Triple Integrals in Cylindrical Coordinates (4 of 5)

It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r \cos \theta$, $y = r \sin \theta$, leaving z as it is, using the appropriate limits of integration for z , r , and θ , and replacing dV by $r \, dz \, dr \, d\theta$. (Figure 9 shows how to remember this.)



Volume element in cylindrical coordinates: $dV = r \, dz \, dr \, d\theta$

Figure 9

Triple Integrals in Cylindrical Coordinates (5 of 5)

It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates, and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

MCQ: E is bounded below by $z = x^2 + y^2$, above by $z = 4$, between the vertical planes $y = 0$ and $y = x$, with $x \geq 0$. Compute

$$I = \iiint_E z \, dV$$

- ☒ A $\frac{8\pi}{3}$
- ☐ B $\frac{7\pi}{3}$
- ☐ C 2π
- ☐ D $\frac{5\pi}{3}$

提交

Example

Solution:

The bounds are:

$$0 \leq \theta \leq \frac{\pi}{4}, 0 \leq r \leq 2, r^2 \leq z \leq 4.$$

Compute:

$$\iiint_E z \, dV = \int_0^{\pi/4} \int_0^2 \int_{r^2}^4 z r \, dz \, dr \, d\theta$$

Inner integral

$$\int_{r^2}^4 z \, dz = 8 - \frac{r^4}{2}.$$

So

$$\int_0^2 \left(8r - \frac{1}{2} r^5 \right) dr = \frac{32}{3}.$$

Finally

$$\int_0^{\pi/4} \frac{32}{3} d\theta = \frac{8\pi}{3}$$

Example 4

A solid E lies within the cylinder $x^2 + y^2 = 1$, to the right of the xz -plane, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$.

(See Figure 12.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E .

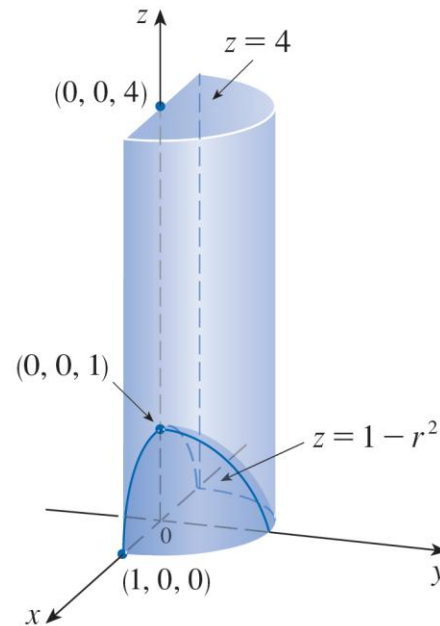


Figure 12

Example 4 – Solution (1 of 2)

In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write

$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq \pi, 0 \leq r \leq 1, 1 - r^2 \leq z \leq 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z -axis, the density function is

$$\rho(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant.

Example 4 – Solution (2 of 2)

Therefore, the mass of E is

$$\begin{aligned} m &= \iiint_E K\sqrt{x^2 + y^2} \, dV = \int_0^\pi \int_0^1 \int_{1-r^2}^4 (Kr) \, r \, dz \, dr \, d\theta \\ &= \int_0^\pi \int_0^1 Kr^2 [4 - (1 - r^2)] \, dr \, d\theta \\ &= K \int_0^\pi d\theta \int_0^1 (3r^2 + r^4) \, dr \\ &= \pi k \left[r^3 + \frac{r^5}{5} \right]_0^1 = \frac{6\pi k}{5} \end{aligned}$$

Recap

- Cylindrical Coordinates
- Triple Integrals in Cylindrical Coordinates