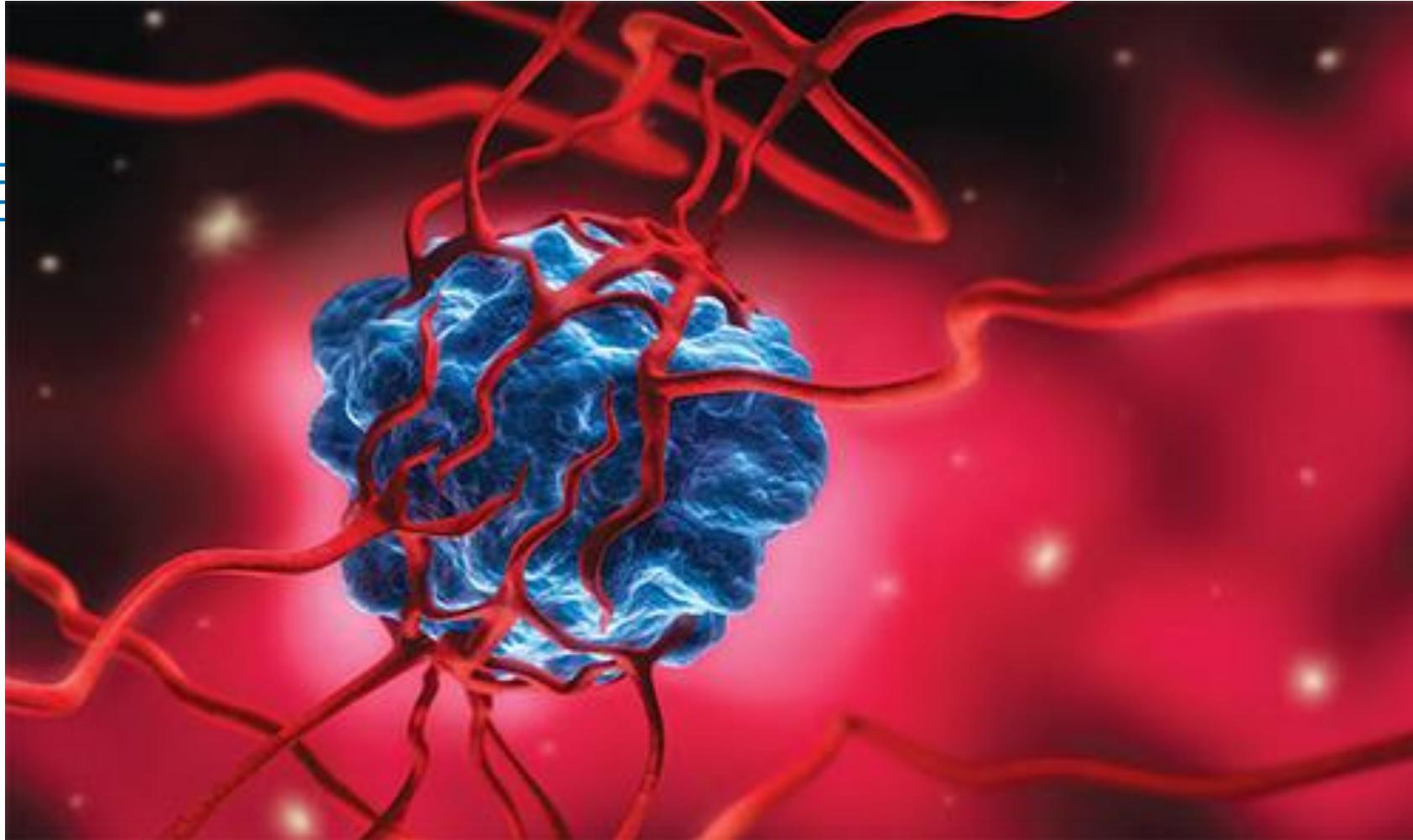


15 Multiple Integrals



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Context

- Review of the Definite Integral
- Volumes and Double Integrals
- The Midpoint Rule
- Iterated Integrals
- Average Value



15.1

Double Integrals over Rectangles

Review of the Definite Integral

Review of the Definite Integral (1 of 2)

First let's recall the basic facts concerning definite integrals of functions of a single variable.

If $f(x)$ is defined for $a \leq x \leq b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

$$1 \quad \sum_{i=1}^n f(x_i^*) \Delta x$$

and take the limit of such sums as $n \rightarrow \infty$ to obtain the definite integral of f from a to b :

$$2 \quad \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Review of the Definite Integral (2 of 2)

In the special case where $f(x) \geq 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x)dx$ represents the area under the curve $y = f(x)$ from a to b .

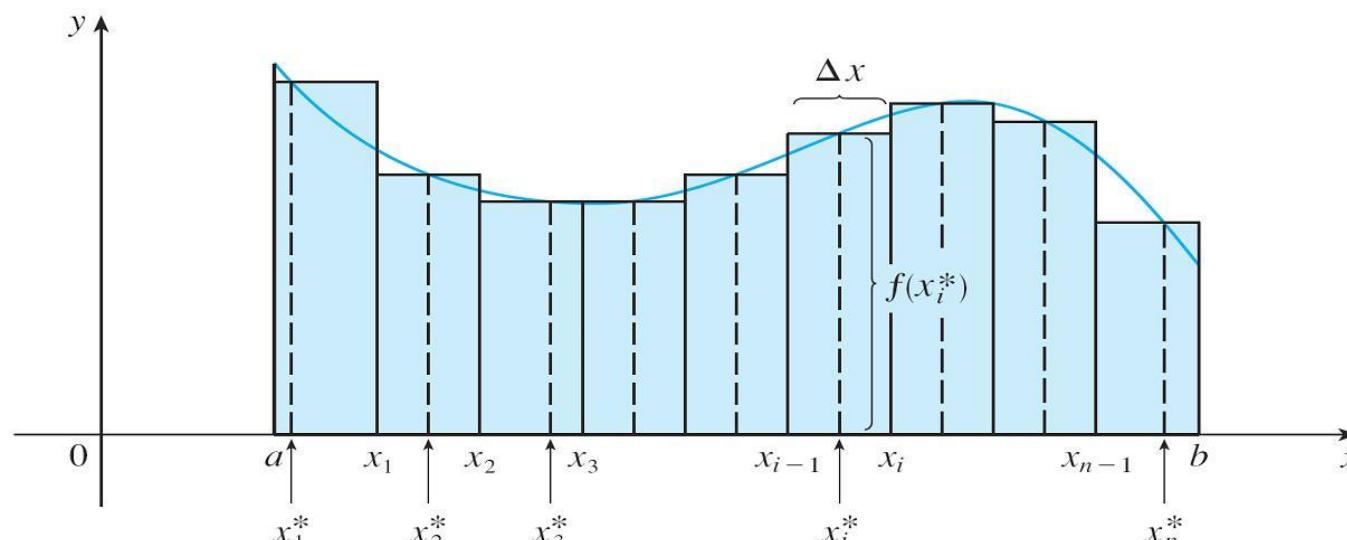


Figure 1

MAQ: Let $f: [0,1] \rightarrow \mathbb{R}$ be a continuous function. Consider the following limits (whenever they exist). Which of these limits **must** equal the definite integral $\int_0^1 f(x) dx$?

A

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right)$$

B

$$\lim_{n \rightarrow \infty} \frac{2}{n} \sum_{k=1}^n f\left(\frac{k^2}{n^2}\right)$$

C

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{2k+1}{2n}\right)$$

D

$$\lim_{n \rightarrow \infty} \frac{2}{n} \sum_{k=1}^n f\left(\frac{2k-1}{n^2}\right)$$

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Example

Solution:

For option A, here the partition points are $x_k = \frac{k}{n}$ for $k = 0, \dots, n$, so each interval has length $\Delta x = \frac{1}{n}$. The sample point in the k -th interval is the right endpoint $x_k = \frac{k}{n}$.

Thus (A) is exactly the right-endpoint Riemann sum for f on $[0,1]$.

For option B counterexample: take $f(x) \equiv 1$. Then

$$\int_0^1 f(x) dx = 1,$$

but the sum in (B) is as follow. B is false

$$\frac{2}{n} \sum_{k=1}^n 1 = \frac{2n}{n} = 2 \Rightarrow \text{limit} = 2.$$

For option C, again use the partition $0, \frac{1}{n}, \dots, \frac{n}{n}$ with width $\Delta x = \frac{1}{n}$.

The sample point in $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ is the midpoint

$$x_k^* = \frac{k + \frac{1}{2}}{n} = \frac{2k + 1}{2n}$$

Thus (C) is the midpoint Riemann sum

For option D, Because f is continuous at 0, $f(x_k^*) \rightarrow f(0)$ for every fixed k , and uniformly for all such k when n is large. Hence

$$\lim_{n \rightarrow \infty} \frac{2}{n} \sum_{k=1}^n f(x_k^*) = \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{k=1}^n f(0) = 2f(0).$$

In general $2f(0) \neq \int_0^1 f(x) dx$. D is false.

Volumes and Double Integrals

Volumes and Double Integrals (1 of 13)

In a similar manner we consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

and we first suppose that $f(x, y) \geq 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f , that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y), (x, y) \in R\}$$

(See Figure 2.)

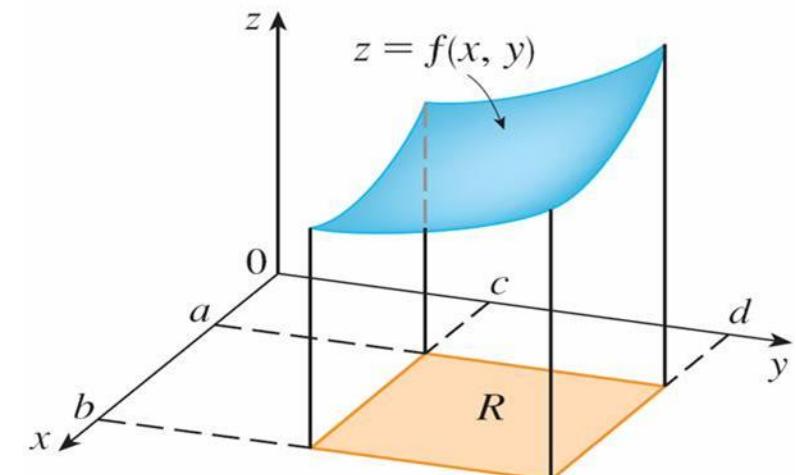


Figure 2

Volumes and Double Integrals (2 of 13)

Our goal is to find the volume of S .

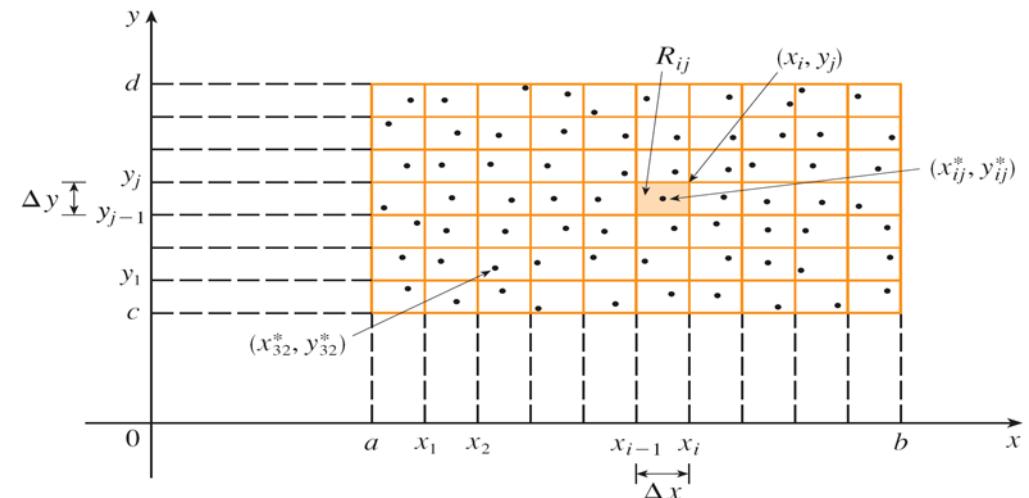
The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$.

Volumes and Double Integrals (3 of 13)

By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, as in Figure 3, we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

each with area $\Delta A = \Delta x \Delta y$.



Dividing R into subrectangles

Figure 3

Volumes and Double Integrals (4 of 13)

If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij} , then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or “column”) with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 4.

The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\Delta A$$

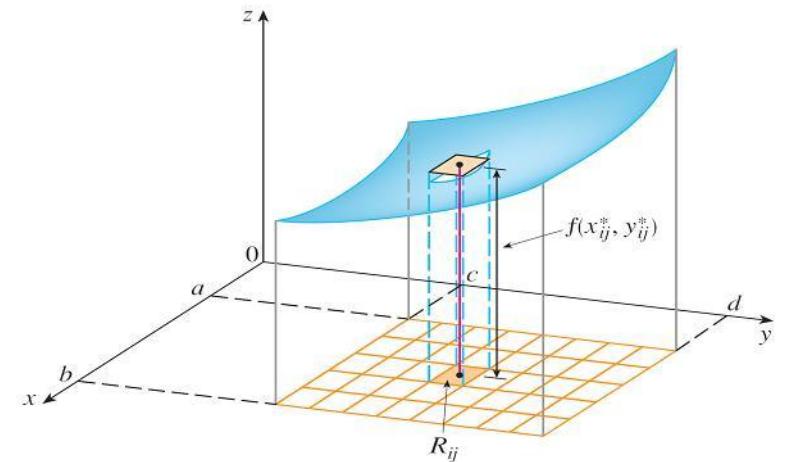


Figure 4

Volumes and Double Integrals (5 of 13)

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S :

$$3 \quad V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

(See Figure 5.) This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results.

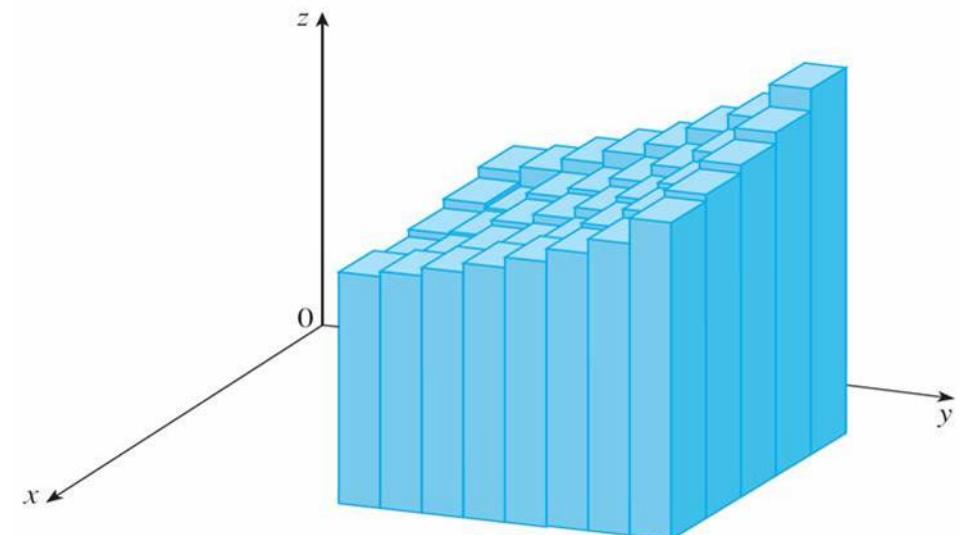


Figure 5

Volumes and Double Integrals (6 of 13)

Our intuition tells us that the approximation given in (3) becomes better as m and n become larger and so we would expect that

$$4 \quad V = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

We use the expression in Equation 4 to define the **volume** of the solid S that lies under the graph of f and above the rectangle R .

Volumes and Double Integrals (7 of 13)

Limits of the type that appear in Equation 4 occur frequently, not just in finding volumes but in a variety of other situations even when f is not a positive function. So we make the following definition.

5 Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x, y) \, dA = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

if this limit exists.

Volumes and Double Integrals (8 of 13)

The precise meaning of the limit in Definition 5 is that for every number $\varepsilon > 0$ there is an integer N such that

$$\left| \iint_R f(x, y) dA - \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A \right| < \varepsilon$$

for all integers m and n greater than N and for any choice of sample points (x_{ij}^*, y_{ij}^*) in R_{ij} .

A function f is called **integrable** if the limit in Definition 5 exists.

Volumes and Double Integrals (9 of 13)

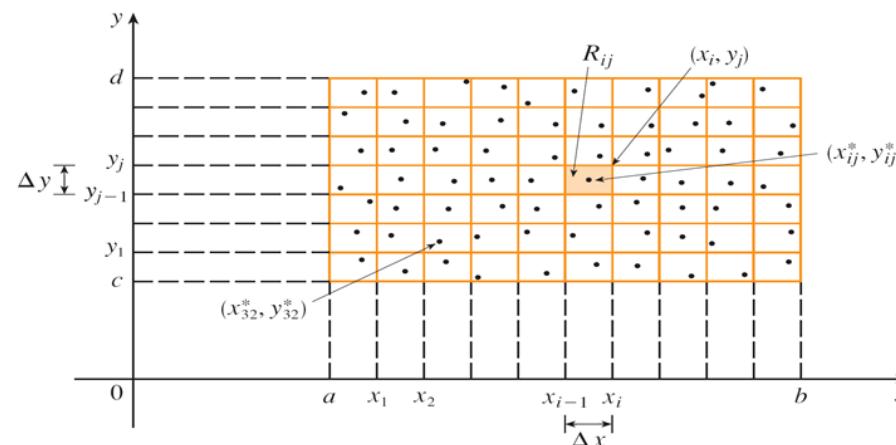
It is shown in courses on advanced calculus that all continuous functions are integrable. In fact, the double integral of f exists provided that f is “not too discontinuous.”

In particular, if f is bounded on R , [that is, there is a constant M such that $|f(x, y)| \leq M$ for all (x, y) in R], and f is continuous there, except on possibly a finite number of smooth curves, then f is integrable over R .

Volumes and Double Integrals (10 of 13)

The sample point (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the subrectangle R_{ij} , but if we choose it to be the upper right-hand corner of R_{ij} [namely (x_i, y_j) , see Figure 3], then the expression for the double integral looks simpler:

$$6 \quad \iint_R f(x, y) dA = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j) \Delta A$$



Dividing R into subrectangles

Figure 3

Volumes and Double Integrals (11 of 13)

By comparing Definitions 4 and 5, we see that a volume can be written as a double integral:

If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y) dA$$

Volumes and Double Integrals (12 of 13)

The sum in Definition 5,

$$\sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. [Notice how similar it is to the Riemann sum in (1) for a function of a single variable.]

Example 1

Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij} . Sketch the solid and the approximating rectangular boxes.

Example 1 – Solution (1 of 3)

The squares are shown in Figure 6.

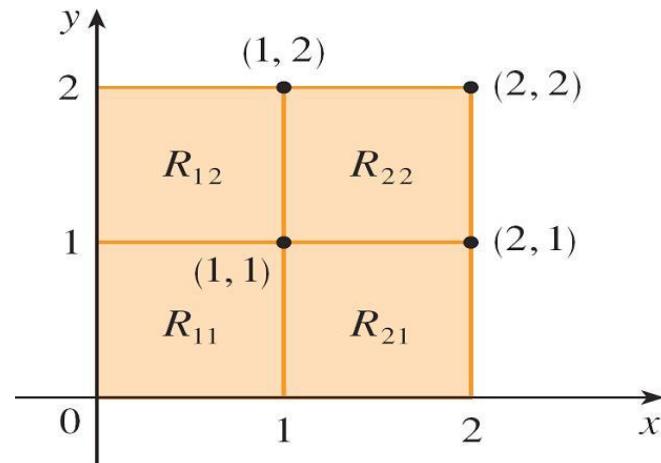


Figure 6

The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$.

Example 1 – Solution (2 of 3)

Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$\begin{aligned} V &\approx \sum_{i=1}^2 \sum_{j=1}^2 f(x_i, y_j) \Delta A \\ &= f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A \\ &= 13(1) + 7(1) + 10(1) + 4(1) \\ &= 34 \end{aligned}$$

Example 1 – Solution (3 of 3)

This is the volume of the approximating rectangular boxes shown in Figure 7.

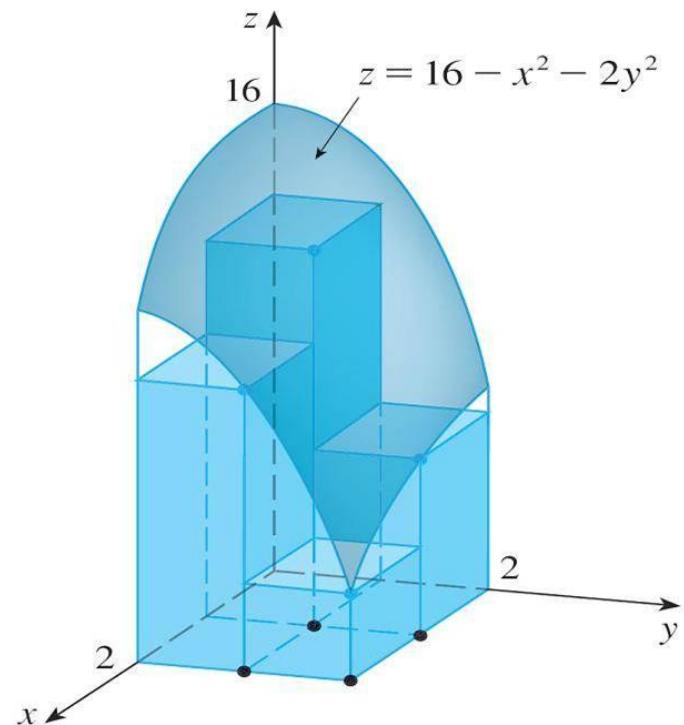


Figure 7

The Midpoint Rule

The Midpoint Rule (1 of 2)

The methods that we used for approximating single integrals (the Midpoint Rule, the Trapezoidal Rule, Simpson's Rule) all have counterparts for double integrals. Here we consider only the Midpoint Rule for double integrals.

This means that we use a double Riemann sum to approximate the double integral, where the sample point (x_{ij}^*, y_{ij}^*) in R_{ij} is chosen to be the center (\bar{x}_i, \bar{y}_j) of R_{ij} . In other words, \bar{x}_i is the midpoint of $[x_{i-1}, x_i]$ and \bar{y}_j is the midpoint of $[y_{j-1}, y_j]$.

The Midpoint Rule (2 of 2)

Midpoint Rule for Double Integrals

$$\iint_R f(x, y) \, dA \approx \sum_{i=1}^m \sum_{j=1}^n f(\bar{x}_i, \bar{y}_j) \Delta A$$

where \bar{x}_i is the midpoint of $[x_{i-1}, x_i]$ and \bar{y}_i is the midpoint of $[y_{i-1}, y_i]$.

Example 3

Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral

$$\iint_R (x - 3y^2) \, dA, \text{ where } R = \{(x, y) | 0 \leq x \leq 2, 1 \leq y \leq 2\}.$$

Solution:

In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = x - 3y^2$ at the centers of the four subrectangles shown in Figure 10.

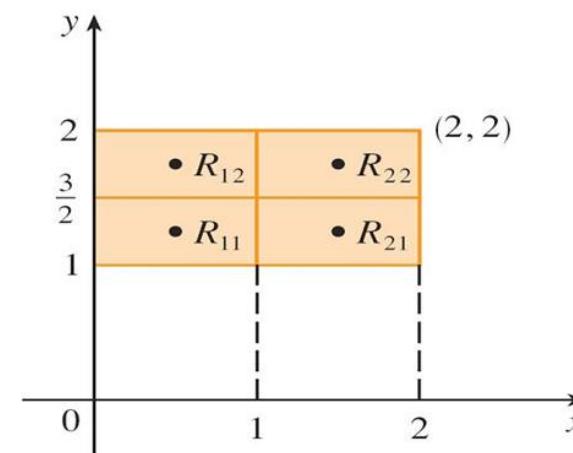


Figure 10

Example 3 – Solution (1 of 2)

So $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{3}{2}$, $\bar{y}_1 = \frac{5}{4}$, and $\bar{y}_2 = \frac{7}{4}$.

The area of each subrectangle is $\Delta A = \frac{1}{2}$.

Thus

$$\begin{aligned}\iint_R (x - 3y^2) dA &\approx \sum_{i=1}^2 \sum_{j=1}^2 f(\bar{x}_i, \bar{y}_j) \Delta A \\&= f(\bar{x}_1, \bar{y}_1) \Delta A + f(\bar{x}_1, \bar{y}_2) \Delta A + f(\bar{x}_2, \bar{y}_1) \Delta A + f(\bar{x}_2, \bar{y}_2) \Delta A \\&= f\left(\frac{1}{2}, \frac{5}{4}\right) \Delta A + f\left(\frac{1}{2}, \frac{7}{4}\right) \Delta A + f\left(\frac{3}{2}, \frac{5}{4}\right) \Delta A + f\left(\frac{3}{2}, \frac{7}{4}\right) \Delta A\end{aligned}$$

Example 3 – Solution (2 of 2)

$$\begin{aligned}&= \left(-\frac{67}{16} \right) \frac{1}{2} + \left(-\frac{139}{16} \right) \frac{1}{2} + \left(-\frac{51}{16} \right) \frac{1}{2} + \left(-\frac{123}{16} \right) \frac{1}{2} \\&= -\frac{95}{8} \\&= -11.875\end{aligned}$$

Thus we have

$$\iint_R (x - 3y^2) \, dA \approx -11.875$$

MAQ: Let $R = [0,2] \times [1,3]$ and let f be a continuous function on R . We approximate the double integral $\iint_R f(x, y) dA$ by the Midpoint Rule with $m = n = 2$. For which of the following functions does this midpoint approximation give the **EXACT** value of the double integral over R ?

 A

$$f(x, y) = 2x + y$$

 B

$$f(x, y) = x^2 + y$$

 C

$$f(x, y) = 3x + y - 5$$

 D

$$f(x, y) = (x + 1)(y - 1)$$

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Example

Solution:

Formally, any function of the form

$$f(x, y) = a + bx + cy + dxy$$

has:

- at most degree 1 in x (no x^2, x^3)
- at most degree 1 in y (no y^2, y^3)

and the 2D midpoint rule on a rectangular grid will integrate it **exactly**.

A,C,D is true.

Iterated Integrals

Iterated Integrals (1 of 7)

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

We use the notation $\int_c^d f(x, y) dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y*. (Notice its similarity to partial differentiation.)

Now $\int_c^d f(x, y) dy$ is a number that depends on the value of x , so it defines a function of x :

$$A(x) = \int_c^d f(x, y) dy$$

Iterated Integrals (2 of 7)

If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

$$7 \quad \int_a^b A(x) dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

The integral on the right side of Equation 7 is called an **iterated integral**. Usually the brackets are omitted. Thus

$$8 \quad \int_a^b \int_c^d f(x, y) dy dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

means that we first integrate with respect to y (holding x fixed) from $y = c$ to $y = d$, and then we integrate the resulting function of x with respect to x from $x = a$ to $x = b$.

Iterated Integrals (3 of 7)

Similarly, the iterated integral

$$9 \quad \int_c^d \int_a^b f(x, y) \, dx \, dy = \int_c^d \left[\int_a^b f(x, y) \, dx \right] dy$$

means that we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$.

Notice that in both Equations 8 and 9 we work *from the inside out*.

Example 4

Evaluate the iterated integrals.

$$(a) \int_0^3 \int_1^2 x^2 y \, dy \, dx \quad (b) \quad \int_1^2 \int_0^3 x^2 y \, dx \, dy$$

Solution:

(a) Regarding x as a constant, we obtain

$$\begin{aligned}\int_1^2 x^2 y \, dy &= \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} \\&= x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) \\&= \frac{3}{2} x^2\end{aligned}$$

Example 4 – Solution (1 of 2)

Thus the function A in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example.

We now integrate this function of x from 0 to 3:

$$\begin{aligned}\int_0^3 \int_1^2 x^2 y \, dy \, dx &= \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx \\&= \int_0^3 \frac{3}{2} x^2 \, dx \\&= \left. \frac{x^3}{2} \right|_0^3 \\&= \frac{27}{2}\end{aligned}$$

Example 4 – Solution (2 of 2)

(b) Here we first integrate with respect to x , regarding y as a constant:

$$\begin{aligned}\int_1^2 \int_0^3 x^2 y \, dx \, dy &= \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy \\&= \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy \\&= \int_1^2 9y \, dy \\&= 9 \left[\frac{y^2}{2} \right]_1^2 = \frac{27}{2}\end{aligned}$$

Iterated Integrals (4 of 7)

Notice that in Example 4 we obtained the same answer whether we integrated with respect to y or x first.

In general, it turns out (see Theorem 10) that the two iterated integrals in Equations 8 and 9 are always equal; that is, the order of integration does not matter. (This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.)

MCQ: $\int_0^\pi \int_0^1 x \cos(xy) dx dy =$

A 0

B 1

C $\frac{1}{\pi}$

D $\frac{2}{\pi}$

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Example

Solution:

Treat x as a constant first, let $u = xy$, so $du = x dy$ and $dy = du/x$. Then

$$\int_0^\pi x \cos(xy) dy = x \int_0^{u=x\pi} \cos(u) \frac{du}{x} = \sin(\pi x)$$

$$\int_0^1 \sin(\pi x) dx = \frac{2}{\pi}$$

Then D is true

Iterated Integrals (5 of 7)

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

10 Fubini's Theorem If f is continuous on the rectangle

$$R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\},$$

then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

More generally, this is true if we assume that f is bounded on R , f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Iterated Integrals (6 of 7)

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form.

To be specific, suppose that $f(x, y) = g(x)h(y)$ and $R = [a, b] \times [c, d]$. Then Fubini's Theorem gives

$$\iint_R f(x, y) \, dA = \int_c^d \int_a^b g(x)h(y) \, dx \, dy = \int_c^d \left[\int_a^b g(x)h(y) \, dx \right] dy$$

Iterated Integrals (7 of 7)

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)h(y) dx \right] dy = \int_c^d \left[h(y) \left(\int_a^b g(x) dx \right) \right] dy = \int_a^b g(x) dx \int_c^d h(y) dy$$

since $\int_a^b g(x) dx$ is a constant.

Therefore, in this case the double integral of f can be written as the product of two single integrals:

11
$$\iint_R g(x)h(y) dA = \int_a^b g(x) dx \int_c^d h(y) dy \quad \text{where } R = [a, b] \times [c, d]$$

MAQ: Let $R = [0,1] \times [0,1]$. Define two functions $f, g: R \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1, & y > x \\ 0, & y \leq x \end{cases} \quad g(x, y) = \begin{cases} 1, & y = x \\ 0, & y \neq x \end{cases}$$

Which of the followings are true?

- A f is integrable over R and $\iint_R f(x, y) dA = \frac{1}{2}$.
- B g is **not** integrable over R , because it is discontinuous at infinitely many points (all points on the line $y = x$)
- C f is **not** integrable over R since the $y = x$ boundary is not included
- D g is integrable over R and $\iint_R g(x, y) dA = 0$.

Example

Solution:

For A, a bounded function which is continuous except on a finite number of smooth curves is integrable over R . Hence f is integrable.

The double integral is simply “height \times area” where height is 1 on the upper triangle and 0 elsewhere, which is 0.5. A is true.

For B, at any point not on the diagonal, there is a small neighborhood where $g \equiv 0$, so g is continuous there. At points on the diagonal, the value of g is 1 but nearby points off the diagonal have value 0, so g is discontinuous exactly on the line $y = x$.

Again, the set of discontinuities is a single smooth curve, which has area 0. The function is bounded ($|g| \leq 1$). By the same theorem, g is integrable over R . It has area 0 and D is true.

Average Value

Average Value (1 of 2)

We know that the average value of a function f of one variable defined on an interval $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$$

In a similar fashion we define the **average value** of a function f of two variables defined on a rectangle R to be

$$f_{\text{avg}} = \frac{1}{A(R)} \iint_R f(x, y) dA$$

where $A(R)$ is the area of R .

Average Value (2 of 2)

If $f(x, y) \geq 0$, the equation

$$A(R) \times f_{\text{avg}} = \iint_R f(x, y) \, dA$$

says that the box with base R and height f_{avg} has the same volume as the solid that lies under the graph of f .

[If $z = f(x, y)$ describes a mountainous region and you chop off the tops of the mountains at height f_{avg} , then you can use them to fill in the valleys so that the region becomes completely flat. See Figure 17.]

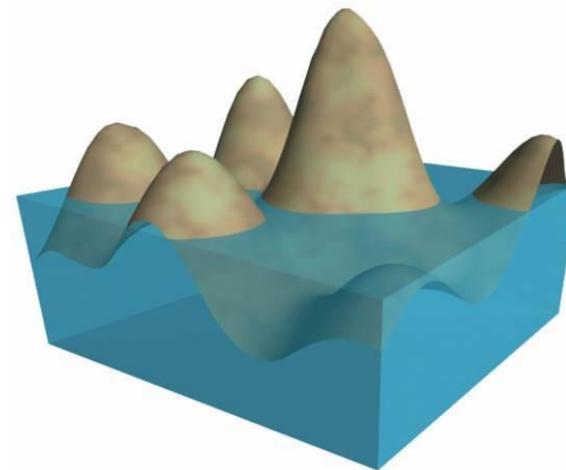


Figure 17

Example 9

The contour map in Figure 18 shows the snowfall, in inches, that fell on the state of Colorado on December 20 and 21, 2006. (The state is in the shape of a rectangle that measures 388 mi west to east and 276 mi south to north.) Use the contour map to estimate the average snowfall for the entire state of Colorado on those days.

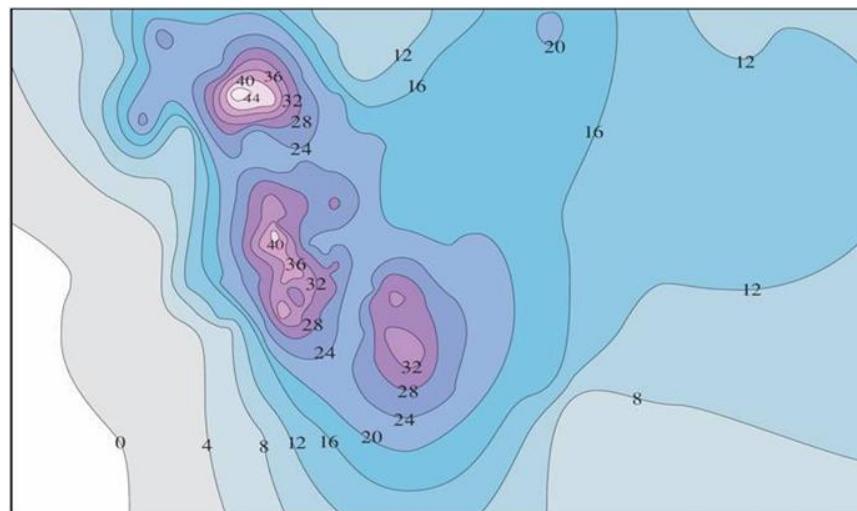


Figure 18

Example 9 – Solution (1 of 4)

Let's place the origin at the southwest corner of the state. Then $0 \leq x \leq 388$, $0 \leq y \leq 276$, and $f(x, y)$ is the snowfall, in inches, at a location x miles to the east and y miles to the north of the origin.

If R is the rectangle that represents Colorado, then the average snowfall for the state on December 20–21 was

$$f_{\text{avg}} = \frac{1}{A(R)} \iint_R f(x, y) \, dA$$

where $A(R) = 388 \cdot 276$.

Example 9 – Solution (2 of 4)

To estimate the value of this double integral, let's use the Midpoint Rule with $m = n = 4$. In other words, we divide R into 16 subrectangles of equal size, as in Figure 19.

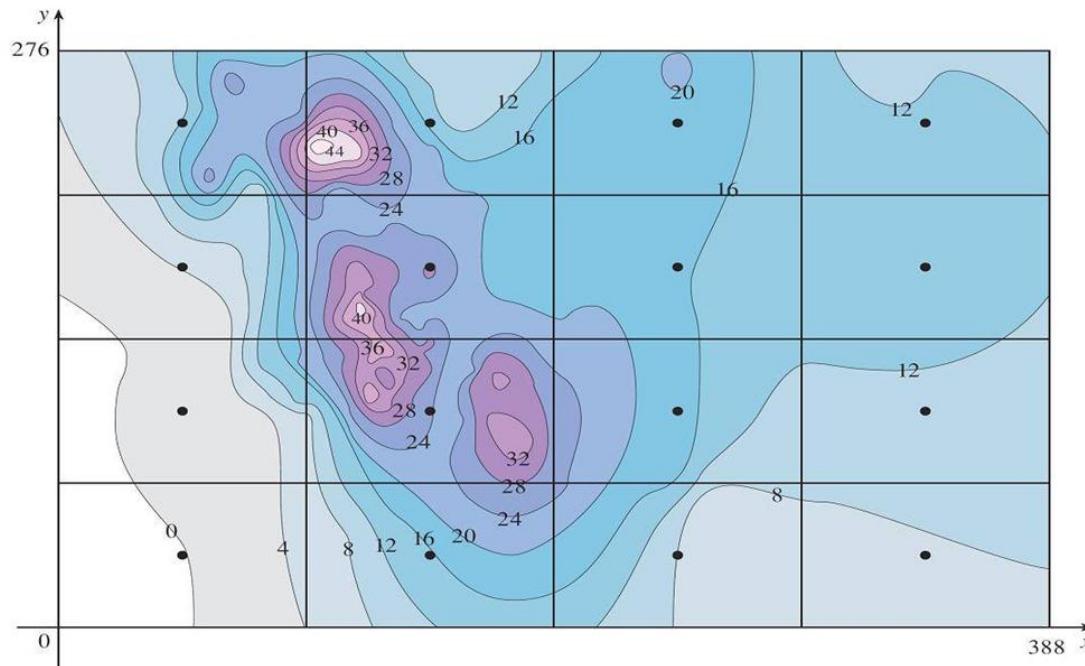


Figure 19

Example 9 – Solution (3 of 4)

The area of each subrectangle is

$$\begin{aligned}\Delta A &= \frac{1}{16}(388)(276) \\ &= 6693 \text{ mi}^2\end{aligned}$$

Using the contour map to estimate the value of f at the center of each subrectangle, we get

$$\begin{aligned}\iint_R f(x, y) dA &\approx \sum_{i=1}^4 \sum_{j=1}^4 f(\bar{x}_i, \bar{y}_j) \Delta A \\ &\approx \Delta A [0 + 15 + 8 + 7 + 2 + 25 + 18.5 + 11 + 4.5 + 28 + 17 + 13.5 + 12 + 15 + 17.5 + 13] \\ &= (6693)(207)\end{aligned}$$

Example 9 – Solution (4 of 4)

Therefore

$$f_{avg} \approx \frac{(6693)(207)}{(388)(276)}$$
$$\approx 12.9$$

On December 20–21, 2006, Colorado received an average of approximately 13 inches of snow.

Recap

- Review of the Definite Integral
- Volumes and Double Integrals
- The Midpoint Rule
- Iterated Integrals
- Average Value