

14 Partial Derivatives





14.4

Tangent Planes and Linear Approximations

Context

- Tangent Planes
- Linear Approximations
- Differentials
- Functions of Three or More Variables



Tangent Planes

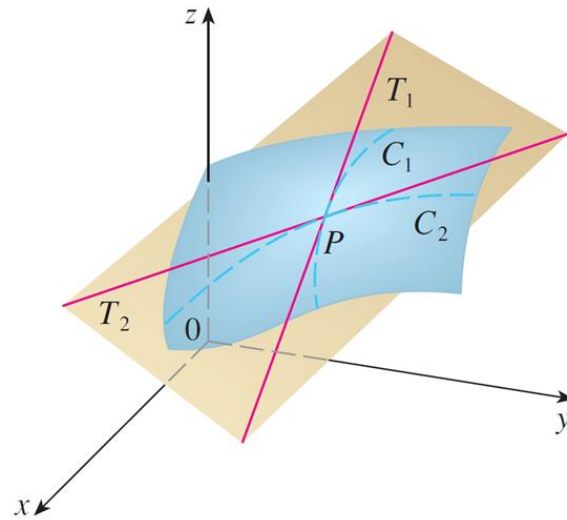
Tangent Planes (1 of 8)

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S .

Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S . Then the point P lies on both C_1 and C_2 . Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P .

Tangent Planes (2 of 8)

Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2 . (See Figure 1.)



The tangent plane contains the tangent lines T_1 and T_2 .

Figure 1

Tangent Planes (3 of 8)

If C is any other curve that lies on the surface S and passes through P , then its tangent line at P also lies in the tangent plane.

Therefore you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P . The tangent plane at P is the plane that most closely approximates the surface S near the point P .

We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Tangent Planes (4 of 8)

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$\mathbf{1} \quad z - z_0 = a(x - x_0) + b(y - y_0)$$

If Equation 1 represents the tangent plane at P , then its intersection with the plane $y = y_0$ must be the tangent line T_1 . Setting $y = y_0$ in Equation 1 gives

$$z - z_0 = a(x - x_0) \quad \text{where } y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a .

Tangent Planes (5 of 8)

But we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$. Therefore $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in Equation 1, we get $z - z_0 = b(y - y_0)$, which must represent the tangent line T_2 , so $b = f_y(x_0, y_0)$.

2 Equation of a Tangent Plane Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution:

Let $f(x, y) = 2x^2 + y^2$.

Then

$$f_x(x, y) = 4x \quad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \quad f_y(1, 1) = 2$$

Then (2) gives the equation of the tangent plane at $(1, 1, 3)$ as

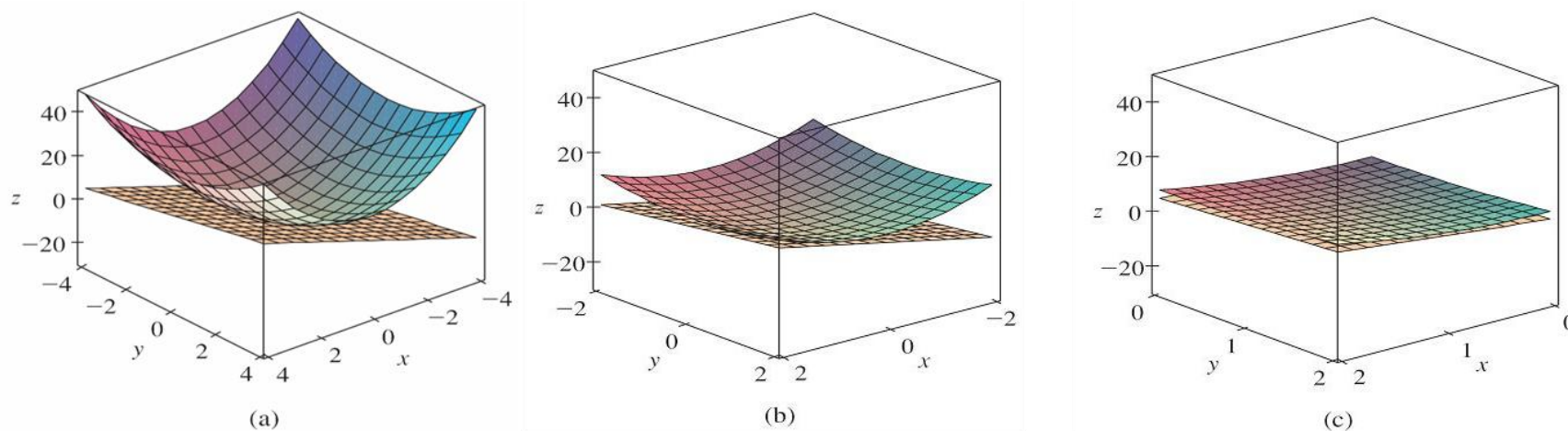
$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

Tangent Planes (6 of 8)

Figure 2(a) shows the elliptic paraboloid and its tangent plane at $(1, 1, 3)$ that we found in Example 1. In parts (b) and (c) we zoom in toward the point $(1, 1, 3)$.

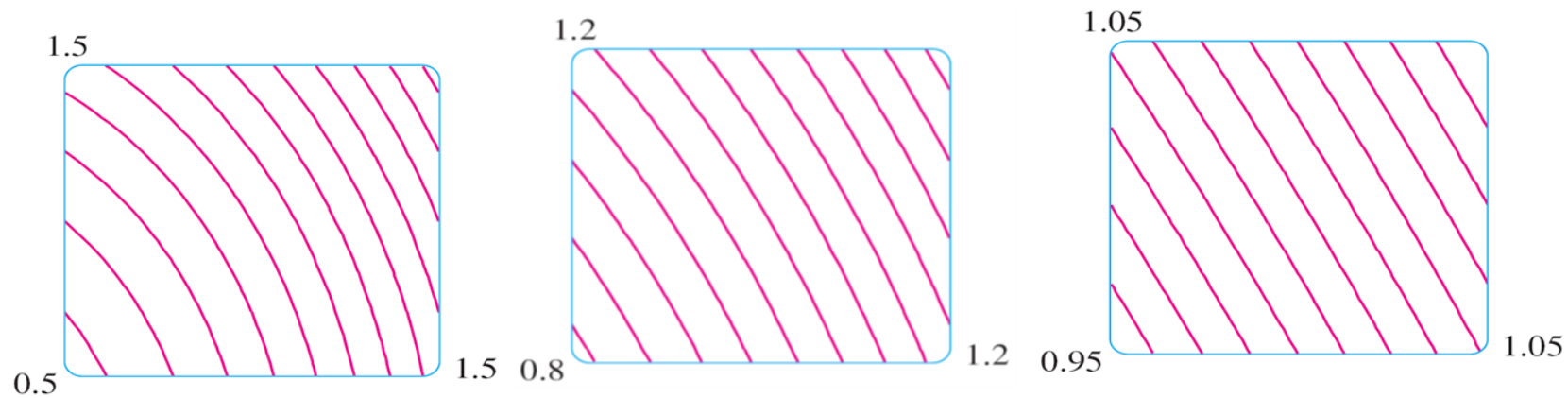


The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward $(1, 1, 3)$.

Figure 2

Tangent Planes (7 of 8)

Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane. In Figure 3 we corroborate this impression by zooming in toward the point $(1, 1)$ on a contour map of the function $f(x, y) = 2x^2 + y^2$.



Zooming in toward $(1, 1)$ on a contour map of $f(x, y) = 2x^2 + y^2$

Figure 3

Tangent Planes (8 of 8)

Notice that the more we zoom in, the more the level curves look like equally spaced parallel lines, which is characteristic of a plane.

MCQ: If $z = f(x, y)$ is differentiable at (a, b) , then the equation of the tangent plane at $(a, b, f(a, b))$ is

- ☐ A $z = f(a, b)$
- ☒ B $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$
- ☐ C $z = f_x(a, b)x + f_y(a, b)y$
- ☐ D $z = z_0$

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Linear Approximations

Linear Approximations (1 of 9)

In Example 1 we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point $(1, 1, 3)$ is $z = 4x + 2y - 3$.

Therefore, the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near $(1, 1)$. The function L is called the *linearization* of f at $(1, 1)$ and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the *linear approximation* or *tangent plane approximation* of f at $(1, 1)$.

Linear Approximations (2 of 9)

For instance, at the point $(1.1, 0.95)$ the linear approximation gives

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

which is quite close to the true value of $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225$.

But if we take a point farther away from $(1, 1)$, such as $(2, 3)$, we no longer get a good approximation.

In fact, $L(2, 3) = 11$ whereas $f(2, 3) = 17$.

Linear Approximations (3 of 9)

In general, we know from (2) that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

$$\mathbf{3} \quad L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) .

Linear Approximations (4 of 9)

The approximation

$$4 \quad f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

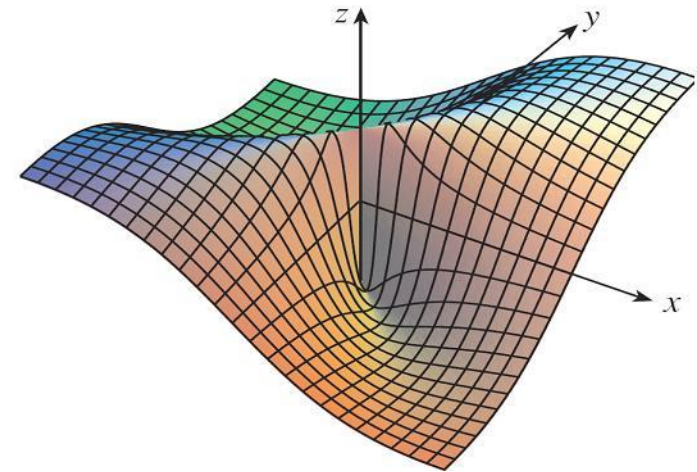
is called the **linear approximation** or the **tangent plane approximation** of f at (a, b) .

Linear Approximations (5 of 9)

We have defined tangent planes for surfaces $z = f(x, y)$, where f has continuous first partial derivatives. What happens if f_x and f_y are not continuous? Figure 4 pictures such a function; its equation is

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

You can verify that its partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous.



$$f(x, y) = \frac{xy}{x^2 + y^2} \text{ if } (x, y) \neq (0, 0), \\ f(0, 0) = 0$$

Figure 4

Linear Approximations (6 of 9)

The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$.

So a function of two variables can behave badly even though both of its partial derivatives exist. To rule out such behavior, we formulate the idea of a differentiable function of two variables.

We know that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

Linear Approximations (7 of 9)

If f is differentiable at a , then

$$\mathbf{5} \quad \Delta y = f'(a) \Delta x + \varepsilon \Delta x \quad \text{where } \varepsilon \rightarrow 0 \text{ as } \Delta x \rightarrow 0$$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

$$\mathbf{6} \quad \Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Thus the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

Linear Approximations (8 of 9)

By analogy with (5) we define the differentiability of a function of two variables as follows.

7 Definition If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 are functions of Δx and Δy such that ε_1 and $\varepsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$.

Definition 7 says that a differentiable function is one for which the linear approximation (4) is a good approximation when (x, y) is near (a, b) . In other words, the tangent plane approximates the graph of f well near the point of tangency.

Linear Approximations (9 of 9)

It's sometimes hard to use Definition 7 directly to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b) , then f is differentiable at (a, b) .

Example 2

Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Solution:

The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2e^{xy}$$

$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

Both f_x and f_y are continuous functions, so f is differentiable by Theorem 8. The linearization is

$$\begin{aligned} L(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \\ &= 1 + 1(x - 1) + 1 \cdot y = x + y \end{aligned}$$

Example 2 – Solution

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

so

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of

$$\begin{aligned} f(1.1 - 0.1) &= 1.1e^{-0.11} \\ &\approx 0.98542. \end{aligned}$$

MAQ: Which of the following gives a sufficient condition for f to be differentiable at (a, b) ?

- ☒ A f_x, f_y exists in a neighborhood of (a, b) and are continuous at (a, b)
- ☐ B f_x, f_y exists at (a, b)
- ☐ C f has directional derivatives at (a, b)
- ☒ D Continuity of f_x, f_y at (a, b) implies differentiability

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Differentials

Differentials (1 of 5)

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable; that is, dx can be given the value of any real number.

The differential of y is then defined as

$$9 \quad dy = f'(x) dx$$

Differentials (2 of 5)

Figure 6 shows the relationship between the increment Δy and the differential dy : Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

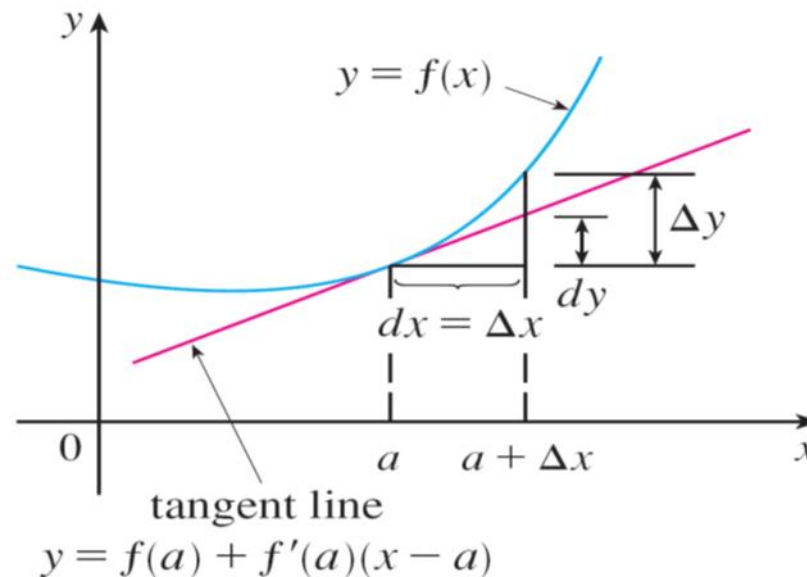


Figure 6

Differentials (3 of 5)

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz , also called the **total differential**, is defined by

$$10 \quad dz = f_x(x, y) dx + f_y(x, y) dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Sometimes the notation df is used in place of dz .

Differentials (4 of 5)

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 10, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So, in the notation of differentials, the linear approximation (4) can be written as

$$f(x, y) \approx f(a, b) + dz$$

Differentials (5 of 5)

Figure 7 is the three-dimensional counterpart of Figure 6 and shows the geometric interpretation of the differential dz and the increment Δz : dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

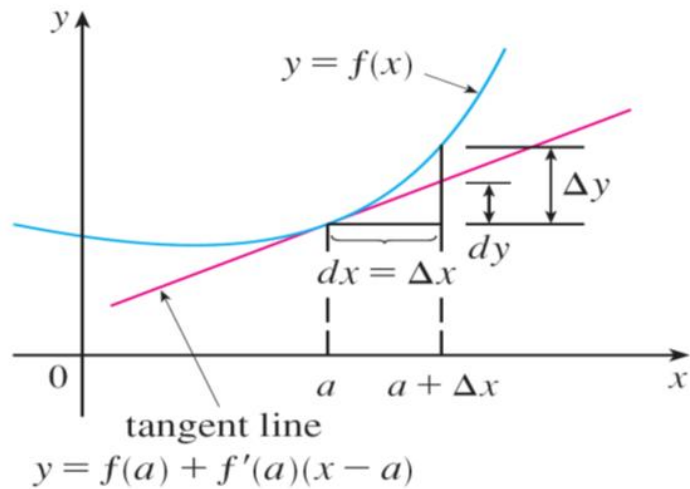


Figure 6

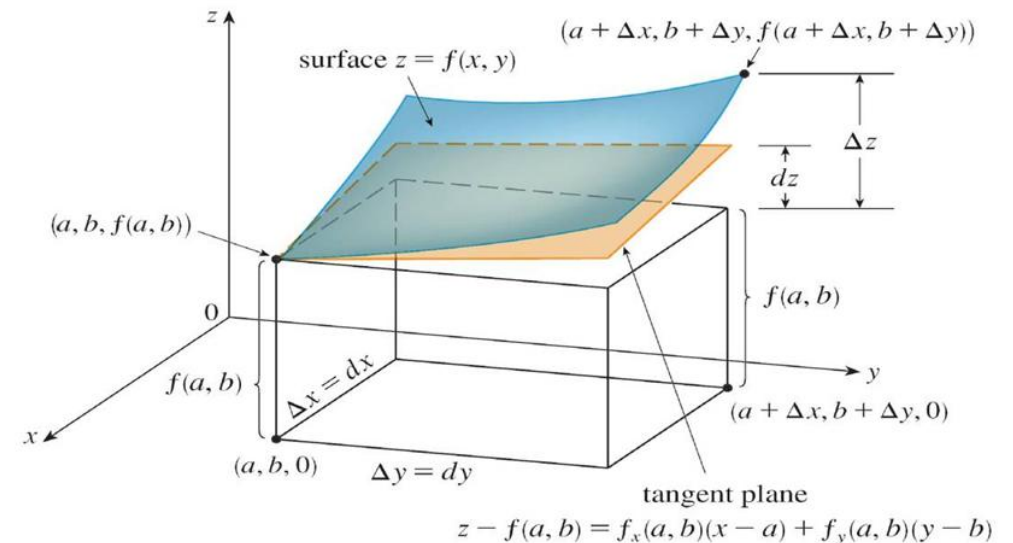


Figure 7

MAQ: Which statements are correct?

- ☐ A $\Delta z = dz$ exactly
- ☒ B For small dx, dy , $\Delta z \approx dz$
- ☒ C Δz is the true change, dz is the linear estimate
- ☐ D Δz and dz are always equal

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Example 4

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz .

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz .

Solution:

(a) Definition 10 gives

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= (2x + 3y) dx + (3x - 2y) dy \end{aligned}$$

Example 4 – Solution

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3) \\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 - 3(2)(3) - 3^2] \\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.



Functions of Three or More Variables

Functions of Three or More Variables (1 of 2)

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one in Definition 7. For such functions the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

Functions of Three or More Variables (2 of 2)

If $w = f(x, y, z)$, then the **increment** of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The **differential** dw is defined in terms of the differentials dx , dy , and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

Example 6

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within ε cm.

- (a) Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.
- (b) What is the estimated maximum error in the calculated volume if the measured dimensions are correct to within 0.2 cm.

Solution:

- (a) If the dimensions of the box are x , y , and z , its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = yz \, dx + xz \, dy + xy \, dz$$

Example 6 – Solution (1 of 2)

We are given that $|\Delta x| \leq \varepsilon$, $|\Delta y| \leq \varepsilon$, and $|\Delta z| \leq \varepsilon$.

To estimate the largest error in the volume, we therefore use $dx = \varepsilon$, $dy = \varepsilon$, and $dz = \varepsilon$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\begin{aligned}\Delta V \approx dV &= (60)(40)\varepsilon + (75)(40)\varepsilon + (75)(60)\varepsilon \\ &= 9900\varepsilon\end{aligned}$$

Thus the maximum error in the calculated volume is about 9900 times larger than the error in each measurement taken.

Example 6 – Solution (2 of 2)

(b) If the largest error in each measurement is $\varepsilon = 0.2$ cm, then

$dV = 9900(0.2) = 1980$, so an error of only 0.2 cm in measuring each dimension could lead to an error of approximately 1980 cm^3 in the calculated volume. (This may seem like a large error, but you can verify that it's only about 1% of the volume of the box.)

MCQ: Let $f(x, y, z) = xy^2z$. At the point $(2, -1, 3)$, suppose the measurement errors satisfy $|dx|, |dy|, |dz| \leq 0.01$, what is the largest possible error $|\Delta f|$

- ☐ A 0.14
- ☐ B 0.15
- ☒ C 0.17
- ☐ D 0.20

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Solution

$$f_x = y^2 z = 3, f_y = 2xyz = -12, f_z = xy^2 = 2$$

$$|df| \leq |f_x||dx| + |f_y||dy| + |f_z||dz| = 0.17$$

MAQ: Let $z = f(x, y)$ be differentiable at (a, b) , and define
 $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

$$dz = f_x dx + f_y dy$$

Decide which of the following statements are correct

- ☒ A The plane $z = L(x, y)$ has slope $f_x(a, b)$ among the plane $y = b$, and slope $f_y(a, b)$ among the plane $x = a$
- ☒ B When moving from (a, b) to $(a + \Delta x, b + \Delta y)$,

$$f(a + \Delta x, b + \Delta y) \approx f(a, b) + dz$$
- ☐ C If $|\Delta x| \leq \epsilon$ and $|\Delta y| \leq \epsilon$, then the maximum possible $|\Delta z|$ is exactly equal to $(|f_x(a, b)| + |f_y(a, b)|)\epsilon$
- ☐ D If f_x, f_y merely exist at (a, b) (even if not continuous), $L(x, y)$ still gives a good local approximation near (a, b)

Solution

A is correct — each directional slope corresponds to the respective partial derivative.

B is correct — differentiability ensures the first-order (linear) approximation $\Delta z \approx dz$

C is incorrect — that expression is only an upper bound, not an exact equality or guaranteed maximum.

D is incorrect — mere existence of partials does not ensure differentiability or a valid linear approximation

Recap

- Tangent Planes
- Linear Approximations
- Differentials
- Functions of Three or More Variables