Solutions

Problem 1.

(a) Critical points. We compute the gradient:

$$f_x(x, y) = 2x - 4y, \qquad f_y(x, y) = 8y - 4x.$$

Critical points satisfy

$$\nabla f(x, y) = (0, 0) \iff \begin{cases} 2x - 4y = 0, \\ 8y - 4x = 0. \end{cases}$$

Both equations reduce to $x = 2y$, so every point on the line $x = 2y$ is a critical point. Hence there are infinitely many critical points:

$$(x, y) = (2t, t), \quad t \in \mathbb{R}.$$

(b) Discriminant. We compute the second partial derivatives:

$$f_{xx} = 2, \quad f_{yy} = 8, \quad f_{xy} = f_{yx} = -4.$$

Thus

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = (2)(8) - (-4)^2 = 16 - 16 = 0.$$

So $D = 0$ at *every* point, in particular at every critical point. The second derivative test is therefore inconclusive here.

(c) Nature of the critical points. We rewrite f by completing the square:

$$f(x, y) = x^2 - 4xy + 4y^2 + 2 = (x - 2y)^2 + 2.$$

Since $(x - 2y)^2 \geq 0$ for all (x, y), we obtain

$$f(x, y) \geq 2 \quad \text{for all } (x, y),$$

with equality if and only if $x - 2y = 0$, i.e. $x = 2y$.

Therefore every point with $x = 2y$ is a point where $f(x, y) = 2$, the global minimum value of f. Thus each critical point on the line $x = 2y$ is both a local and an absolute minimum.

Problem 2.

We have

$$f(x, y) = x, \qquad g(x, y) = y^2 + x^4 - x^3.$$

(a) Lagrange multiplier equations. The gradients are

$$\nabla f(x, y) = (1, 0), \qquad \nabla g(x, y) = (4x^3 - 3x^2, \, 2y).$$

For regular points of the constraint curve ($\nabla g \neq 0$), a constrained extremum should satisfy

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad \text{together with } g(x, y) = 0,$$

that is
$$(1,0) = \lambda(4x^3 - 3x^2,\, 2y) \quad \text{and} \quad y^2 + x^4 - x^3 = 0.$$
From the second component we get
$$0 = 2\lambda y.$$

So either $\lambda = 0$ or $y = 0$.

Case 1: $\lambda = 0$. Then the first component gives $1 = 0$, impossible. Hence $\lambda \neq 0$ and we must have $y = 0$.

With $y = 0$, the constraint becomes
$$0 = y^2 + x^4 - x^3 = x^3(x - 1),$$
so $x = 0$ or $x = 1$.

At $(1,0)$ we have
$$\nabla g(1,0) = (4 - 3,\, 0) = (1,0),$$
and the Lagrange equation
$$(1,0) = \lambda(1,0)$$
is satisfied with $\lambda = 1$. At $(0,0)$ we have $\nabla g(0,0) = (0,0)$, so the Lagrange equation
$$(1,0) = \lambda(0,0)$$
has no solution. Thus the only point on the curve detected by the Lagrange multiplier system is $(1,0)$, which has $x = 1$ and is in fact a maximum for f on the curve, not a minimum.

(b) True minimum and failure of the condition. On the constraint $g(x,y) = 0$ we have
$$y^2 = x^3(1 - x).$$
Since $y^2 \geq 0$, this implies $x^3(1 - x) \geq 0$, so
$$0 \leq x \leq 1.$$

Along the curve, $f(x,y) = x$, so f takes values in the interval $[0,1]$. Thus the minimum value is
$$\min f = 0,$$
attained at $x = 0$, i.e. at the point $(0,0)$.

But at $(0,0)$,
$$\nabla f(0,0) = (1,0), \qquad \nabla g(0,0) = (0,0),$$
and there is no real number λ such that
$$(1,0) = \lambda(0,0).$$

Therefore the Lagrange multiplier condition fails at the true minimum point.

(c) Why Lagrange multipliers fail here. The standard Lagrange multiplier theorem assumes that the constraint curve/surface is smooth at the extremum point, i.e. that $\nabla g(x,y) \neq 0$ there. In this problem, the minimum occurs at $(0,0)$, where
$$\nabla g(0,0) = (0,0),$$
so the constraint has a singular point (a cusp of the piriform) at the minimum. Because the regularity condition fails, the method of Lagrange multipliers does not detect this minimum.