Solutions to Quiz 10

1. **(4 pts)** We have
$$x(x-y)(x+y) = x(x^2-y^2) = x^3 - xy^2.$$
Integrate over $0 \le x \le 1$, $0 \le y \le 1$:
$$\int_0^1 \int_0^1 (x^3 - xy^2)\, dy\, dx = \int_0^1 \left(x^3 y - \frac{xy^3}{3} \right)_0^1 dx = \int_0^1 \left(x^3 - \frac{x}{3} \right) dx = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

   Grading (4 pts):

   - 1 pt: Correct algebraic simplification to $x^3 - xy^2$ (or equivalent).
   - 1 pt: Correct setup of iterated integral on $[0,1]^2$.
   - 1 pt: Correct inner integration in y.
   - 1 pt: Correct final value $1/12$.

2. **(6 pts)**

   The region D has vertices $(0,0)$, $(2,1)$, $(0,3)$. The lines through these points are
$$y = \frac{x}{2} \quad \text{and} \quad y = 3 - x,$$
so a convenient description is
$$0 \le x \le 2, \qquad \frac{x}{2} \le y \le 3 - x.$$

   (a) Mass m (3 pts).
$$m = \iint_D (x+y)\, dA = \int_0^2 \int_{x/2}^{3-x} (x+y)\, dy\, dx.$$
Inner integral:
$$\int_{x/2}^{3-x} (x+y)\, dy = \left[xy + \frac{y^2}{2} \right]_{x/2}^{3-x} = \frac{9}{2} - \frac{9}{8}x^2.$$
Thus
$$m = \int_0^2 \left(\frac{9}{2} - \frac{9}{8}x^2 \right) dx = \left[\frac{9}{2}x - \frac{9}{24}x^3 \right]_0^2 = 9 - 3 = 6.$$

   Grading (3 pts):

   - 1 pt: Correct description of D and limits of integration.
   - 1 pt: Correct integrand and inner integral.
   - 1 pt: Correct final mass $m = 6$.

(b) Center of mass $(\bar{x}, \bar{y})$ (3 pts).

We need

$$\bar{x} = \frac{1}{m} \iint_D x(x+y)\,dA, \qquad \bar{y} = \frac{1}{m} \iint_D y(x+y)\,dA.$$

First

$$\iint_D x(x+y)\,dA = \int_0^2 \int_{x/2}^{3-x} (x^2 + xy)\,dy\,dx.$$

Inner integral:

$$\int_{x/2}^{3-x} (x^2 + xy)\,dy = \left[x^2 y + \frac{xy^2}{2}\right]_{x/2}^{3-x} = \frac{9}{2}x - \frac{9}{8}x^3.$$

So

$$\iint_D x(x+y)\,dA = \int_0^2 \left(\frac{9}{2}x - \frac{9}{8}x^3\right) dx = \left[\frac{9}{4}x^2 - \frac{9}{32}x^4\right]_0^2 = 9 - \frac{9}{2} = \frac{9}{2}.$$

Thus

$$\bar{x} = \frac{\frac{9}{2}}{6} = \frac{3}{4}.$$

Next

$$\iint_D y(x+y)\,dA = \int_0^2 \int_{x/2}^{3-x} (xy + y^2)\,dy\,dx.$$

Inner integral:

$$\int_{x/2}^{3-x} (xy + y^2)\,dy = \left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{x/2}^{3-x} = -\frac{9}{2}x + 9.$$

Hence

$$\iint_D y(x+y)\,dA = \int_0^2 (-\tfrac{9}{2}x + 9)\,dx = \left[-\tfrac{9}{4}x^2 + 9x\right]_0^2 = -9 + 18 = 9.$$

So

$$\bar{y} = \frac{9}{6} = \frac{3}{2}.$$

Therefore,

$$m = 6, \qquad (\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{3}{2}\right).$$

Grading (3 pts):

- 1 pt: Correct setup of moments for $\bar{x}$ and/or $\bar{y}$.
- 1 pt: Correct evaluation of at least one moment integral.
- 1 pt: Correct final coordinates $(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{3}{2}\right)$.