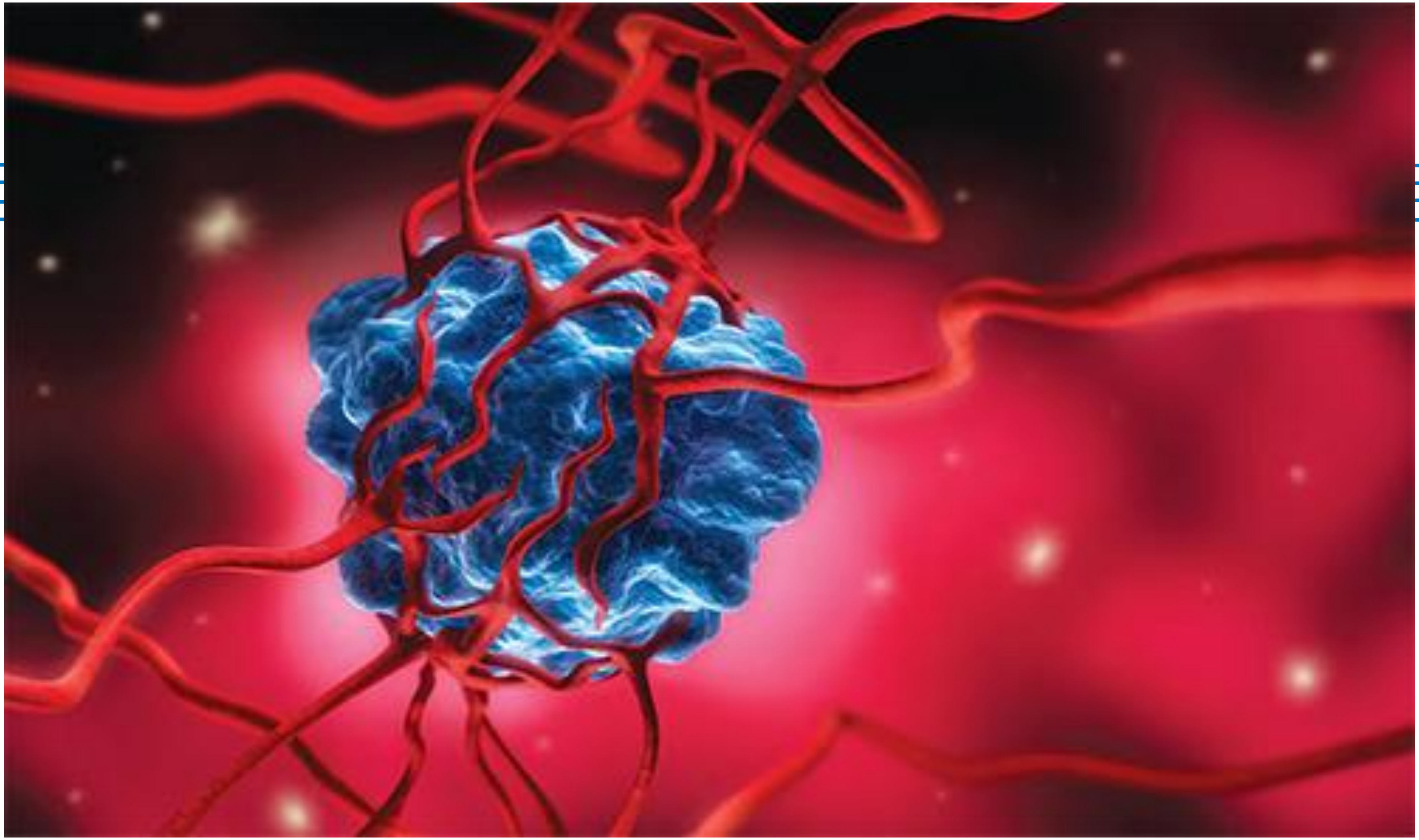
# 15 Multiple Integrals



# 15.9 Change of Variables in Multiple Integrals

# Context

- Change of Variables in Double Integrals
- Change of Variables in Triple Integrals

# Change of Variables in Multiple Integrals (1 of 1)

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. By reversing the roles of *x* and *u*, we can write

$$\mathbf{1} \qquad \int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$. Another way of writing Formula 1 is as follows:

$$\mathbf{2} \qquad \int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

# Change of Variables in Double Integrals

# Change of Variables in Multiple Integrals (1 of 18)

We have already seen one example of this: conversion to polar coordinates. The new variables $r$ and $\theta$ are related to the old variables $x$ and $y$ by the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

and the change of variables formula can be written as

$$\iint_R f(x, y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where $S$ is the region in the $r\theta$-plane that corresponds to the region $R$ in the $xy$-plane.

# Change of Variables in Multiple Integrals (2 of 18)

More generally, we consider a change of variables that is given by a **transformation** $T$ from the $uv$-plane to the $xy$-plane:

$$T(u, v) = (x, y)$$

where $x$ and $y$ are related to $u$ and $v$ by the equations

$$x = g(u, v) \qquad y = h(u, v) \tag{3}$$

or, as we sometimes write,

$$x = x(u, v) \qquad y = y(u, v)$$

# Change of Variables in Multiple Integrals (3 of 18)

We usually assume that $T$ is a **$C^1$ transformation**, which means that $g$ and $h$ have continuous first-order partial derivatives.

A transformation $T$ is really just a function whose domain and range are both subsets of $\mathbb{R}^2$

If $T(u_1, v_1) = (x_1, y_1)$, then the point $(x_1, y_1)$ is called the **image** of the point $(u_1, v_1)$.

If no two points have the same image, $T$ is called **one-to-one**.

Figure 1 shows the effect of a transformation *T* on a region *S* in the *uv*-plane.

*T* transforms *S* into a region *R* in the *xy*-plane called the **image of *S***, consisting of the images of all points in *S*.

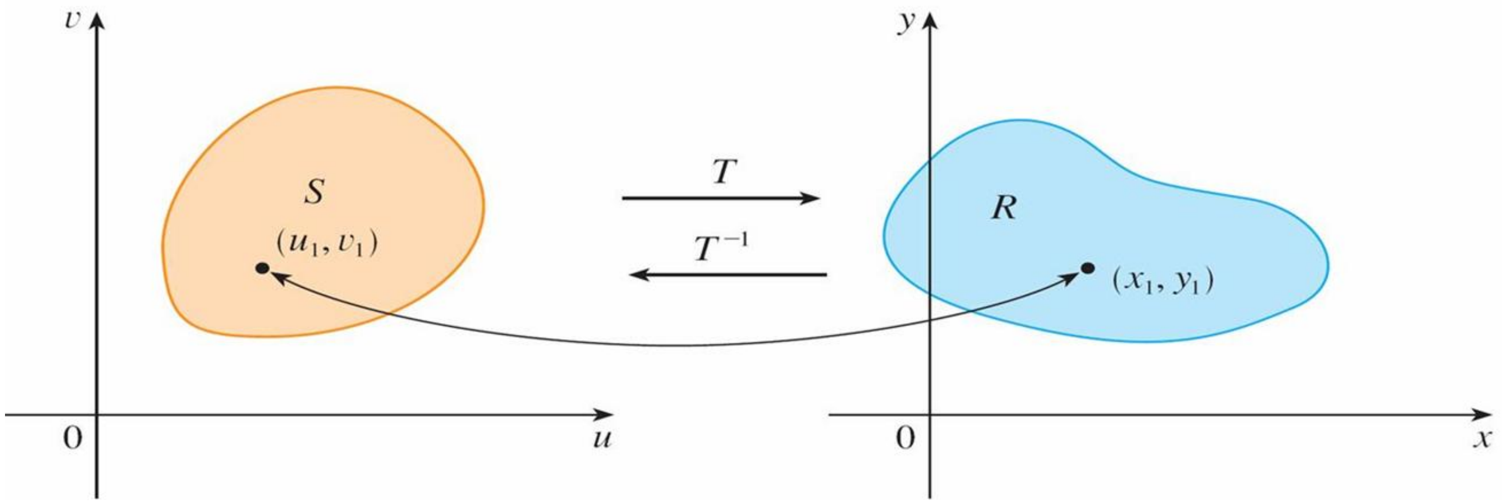



**Figure 1**

# Change of Variables in Multiple Integrals (5 of 18)

If $T$ is a one-to-one transformation, then it has an **inverse transformation** $T^{-1}$ from the $xy$-plane to the $uv$-plane and it may be possible to solve Equations 3 for $u$ and $v$ in terms of $x$ and $y$:

$$u = G(x, y) \qquad v = H(x, y)$$

# Example 1

A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square

$$S = \{(u,\ v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}.$$

Solution:

The transformation maps the boundary of $S$ into the boundary of the image.

So we begin by finding the images of the sides of $S$.

# Example 1 – Solution (1 of 4)

The first side, $S_1$, is given by $v = 0$ ($0 \le u \le 1$). (See Figure 2.)

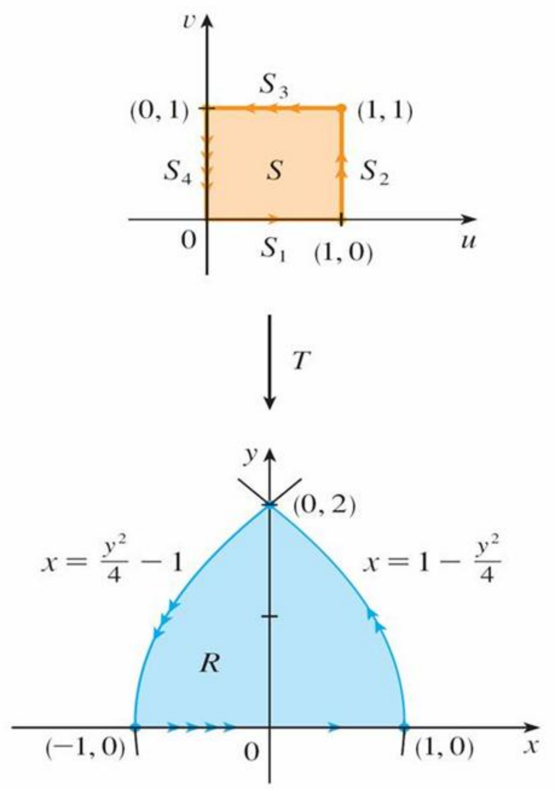



**Figure 2**

# Example 1 – Solution (2 of 4)

From the given equations we have $x = u^2$, $y = 0$, and so $0 \le x \le 1$.
Thus $S_1$ is mapped into the line segment from (0, 0) to (1, 0) in the *xy*-plane.

The second side, $S_2$, is $u = 1$ $(0 \le v \le 1)$ and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2 \qquad y = 2v$$

# Example 1 – Solution (3 of 4)

Eliminating $v$, we obtain

$$\mathbf{4} \qquad x = 1 - \frac{y^2}{4} \qquad 0 \le x \le 1$$

which is part of a parabola.

Similarly, $S_3$ is given by $v = 1$ ($0 \le u \le 1$), whose image is the parabolic arc

$$\mathbf{5} \qquad x = \frac{y^2}{4} - 1 \qquad -1 \le x \le 0$$

# Example 1 – Solution (4 of 4)

Finally, $S_4$ is given by $u = 0$ $(0 \le v \le 1)$ whose image is $x = -v^2$, $y = 0$, that is, $-1 \le x \le 0$. (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.)

The image of $S$ is the region $R$ (shown in Figure 2) bounded by the $x$-axis and the parabolas given by Equations 4 and 5.

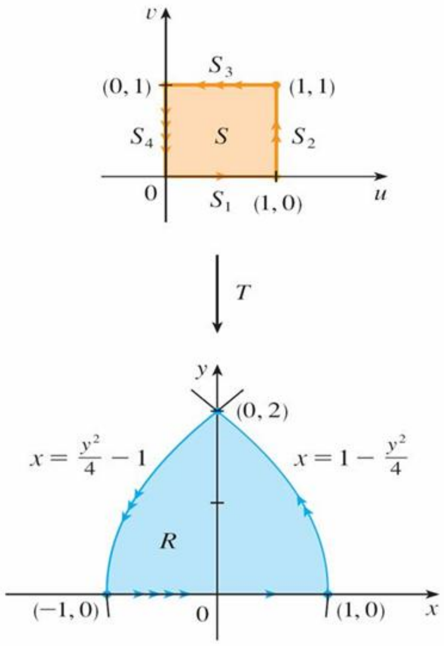



**Figure 2**

# Change of Variables in Multiple Integrals (6 of 18)

Now let's see how a change of variables affects a double integral. We start with a small rectangle $S$ in the $uv$-plane whose lower left corner is the point $(u_0, v_0)$ and whose dimensions are $\Delta u$ and $\Delta v$. (See Figure 3.)

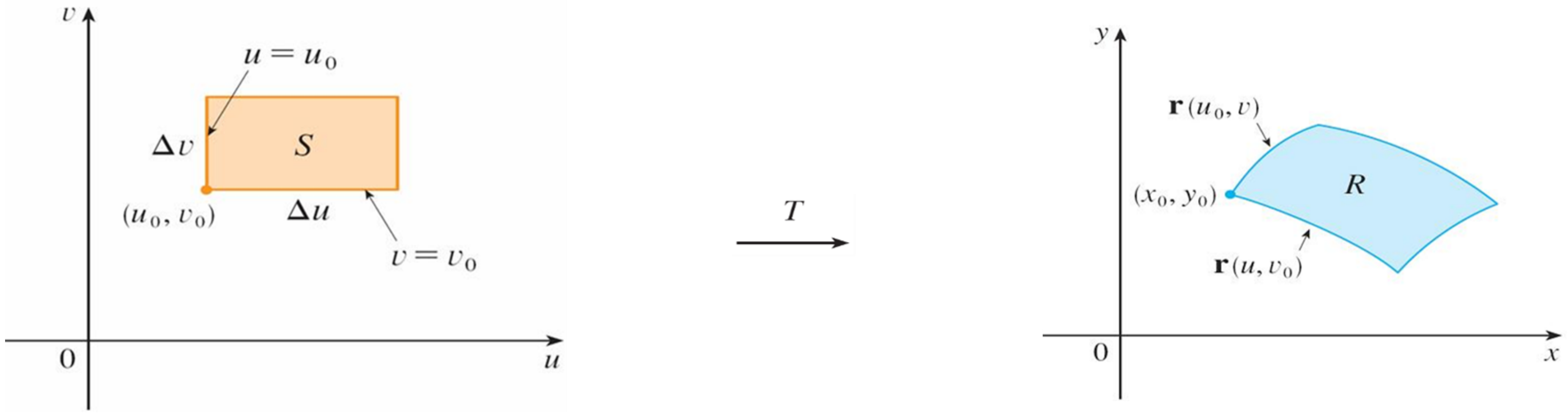



**Figure 3**

# Change of Variables in Multiple Integrals (7 of 18)

The image of $S$ is a region $R$ in the $xy$-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$.

The vector

$$\mathbf{r}(u, v) = g(u, v)\,\mathbf{i} + h(u, v)\,\mathbf{j}$$

is the position vector of the image of the point $(u, v)$.

The equation of the lower side of $S$ is $v = v_0$, whose image curve is given by the vector function $\mathbf{r}(u, v_0)$.

# Change of Variables in Multiple Integrals (8 of 18)

The tangent vector at $(x_0, y_0)$ to this image curve is

$$\begin{aligned}\mathbf{r}_u &= g_u(u_0, v_0)\mathbf{i} + h_u(u_0, v_0)\mathbf{j} \\ &= \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}\end{aligned}$$

Similarly, the tangent vector at $(x_0, y_0)$ to the image curve of the left side of $S$ (namely, $u = u_0$) is

$$\begin{aligned}\mathbf{r}_v &= g_v(u_0, v_0)\mathbf{i} + h_v(u_0, v_0)\mathbf{j} \\ &= \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}\end{aligned}$$

# Change of Variables in Multiple Integrals (9 of 18)

We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u,\ v_0) - \mathbf{r}(u_0,\ v_0) \quad \mathbf{b} = \mathbf{r}(u_0,\ v_0 + \Delta v) - \mathbf{r}(u_0,\ v_0)$$

shown in Figure 4.

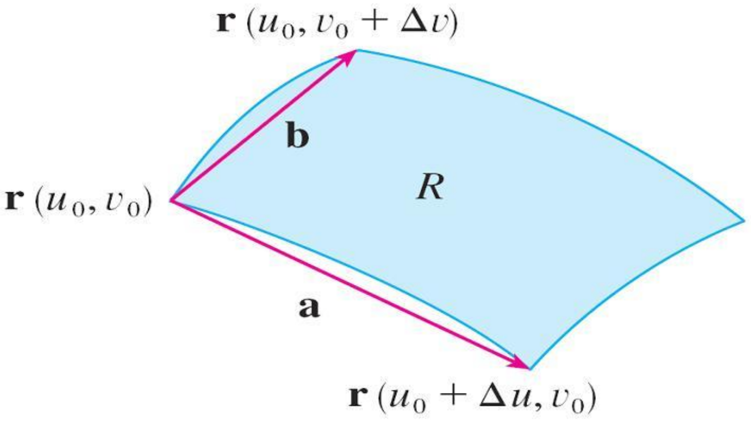



**Figure 4**

# Change of Variables in Multiple Integrals (10 of 18)

But
$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u,\ v_0) - \mathbf{r}(u_0,\ v_0)}{\Delta u}$$

and so
$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u\, \mathbf{r}_u$$

Similarly
$$\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v\, \mathbf{r}_v$$

This means that we can approximate $R$ by a parallelogram determined by the vectors $\Delta u\, \mathbf{r}_u$ and $\Delta v\, \mathbf{r}_v$. (See Figure 5.)

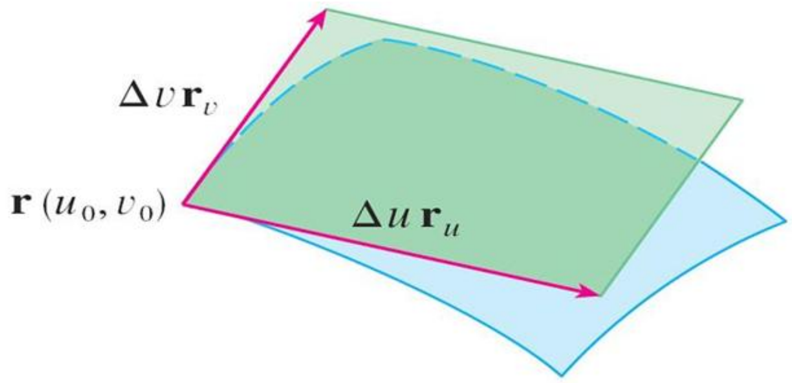



**Figure 5**

Therefore we can approximate the area of *R* by the area of this parallelogram, which is

$$\textbf{6} \qquad |(\Delta u\ \mathbf{r}_u) \times (\Delta v\ \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v$$

Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

# Change of Variables in Multiple Integrals (12 of 18)

The determinant that arises in this calculation is called the *Jacobian* of the transformation and is given a special notation.

**7 Definition** The **Jacobian** of the transformation $T$ given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation we can use Equation 6 to give an approximation to the area $\Delta A$ of $R$:

$$\mathbf{8} \quad \Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u \, \Delta v$$

where the Jacobian is evaluated at $(u_0, v_0)$.

# Change of Variables in Multiple Integrals (13 of 18)

Next we divide a region $S$ in the $uv$-plane into rectangles $S_{ij}$ and call their images in the $xy$-plane $R_{ij}$. (See Figure 6.)

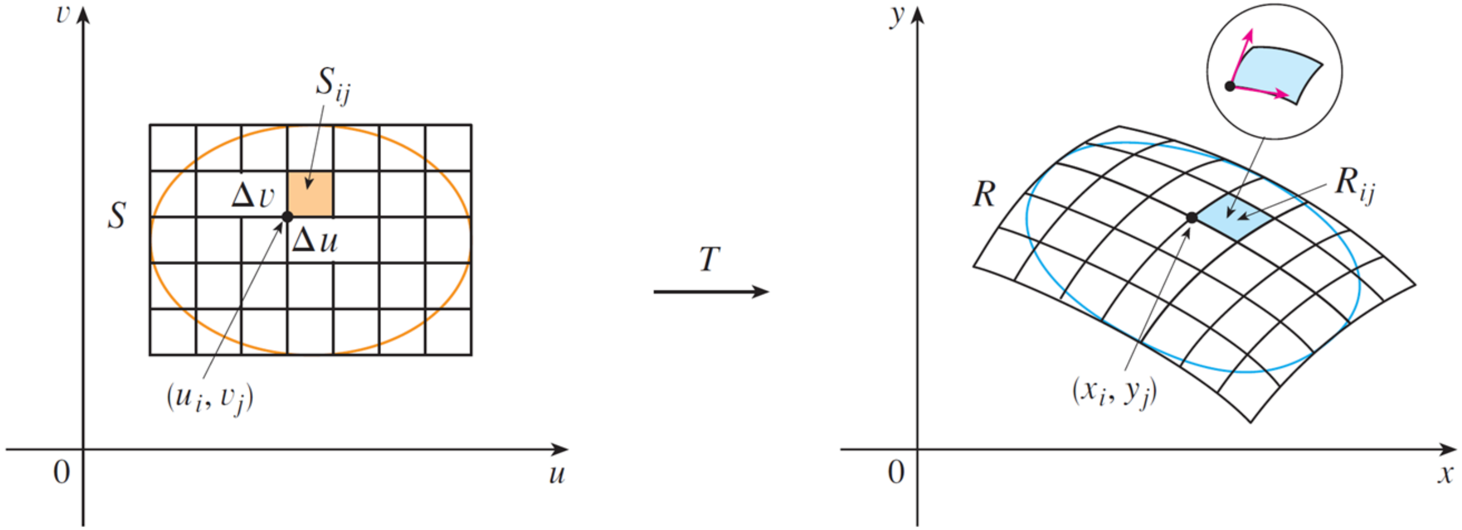



**Figure 6**

# Change of Variables in Multiple Integrals (14 of 18)

Applying the approximation (8) to each $R_{ij}$, we approximate the double integral of $f$ over $R$ as follows:

$$\iint_R f(x,\, y)\,dA \;\approx\; \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,\, y_i)\,\Delta A$$

$$\approx\; \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(g\left(u_i,\, v_j\right),\, h\left(u_i,\, v_i\right)\right)\left|\frac{\partial(x,\, y)}{\partial(u,\, v)}\right|\Delta u\,\Delta v$$

where the Jacobian is evaluated at $(u_i, v_j)$. Notice that this double sum is a Riemann sum for the integral

$$\iint_S f\left(g(u,\, v),\, h(u,\, v)\right)\left|\frac{\partial(x,\, y)}{\partial(u,\, v)}\right| du\, dv$$

# Change of Variables in Multiple Integrals (15 of 18)

The foregoing argument suggests that the following theorem is true.

**9 Change of Variables in a Double Integral** Suppose that $T$ is a $C^1$ transformation whose Jacobian is nonzero and that $T$ maps a region $S$ in the $uv$-plane onto a region $R$ in the $xy$-plane. Suppose that $f$ is continuous on $R$ and that $R$ and $S$ are type I or type II plane regions. Suppose also that $T$ is one-to-one, except perhaps on the boundary of $S$. Then

$$\iint_R f(x, y)\,dA = \iint_S f\left(x(u, v),\, y(u, v)\right)\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

Theorem 9 says that we change from an integral in $x$ and $y$ to an integral in $u$ and $v$ by expressing $x$ and $y$ in terms of $u$ and $v$ and writing

$$dA = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

# Change of Variables in Multiple Integrals (16 of 18)

Notice the similarity between Theorem 9 and the one-dimensional formula in Equation 2.

Instead of the derivative $\frac{dx}{du}$, we have the absolute value of the Jacobian, that is, $\left|\frac{\partial(x, y)}{\partial(u, v)}\right|$.

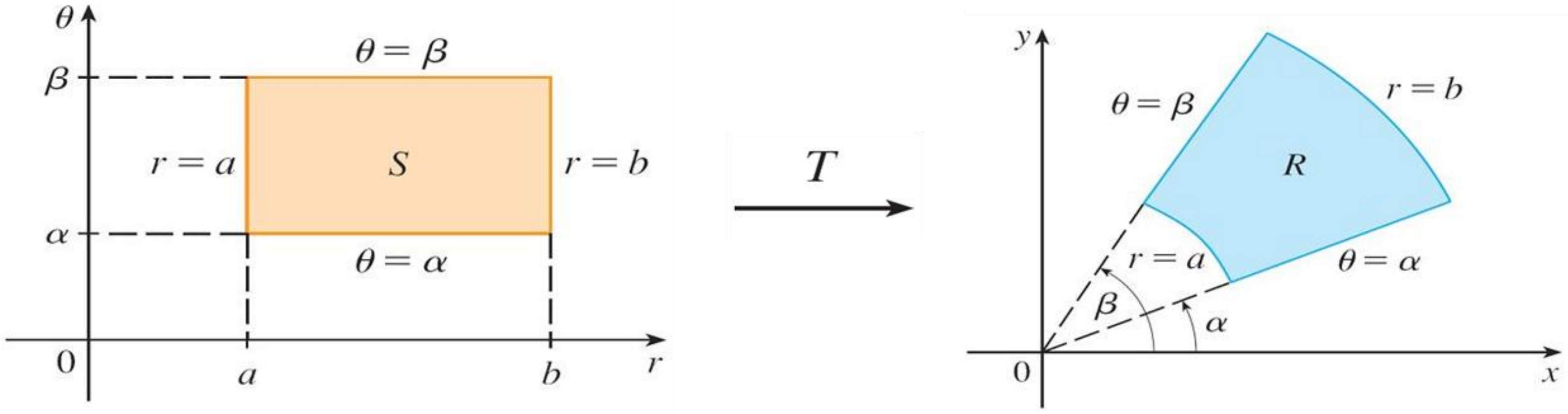
As a first illustration of Theorem 9, we show that the formula for integration in polar coordinates is just a special case.

# Change of Variables in Multiple Integrals (17 of 18)

Here the transformation $T$ from the $r\theta$-plane to the $xy$-plane is given by

$$x = g(r,\ \theta) = r\cos\theta \quad y = h(r,\ \theta) = r\sin\theta$$

and the geometry of the transformation is shown in Figure 7. $T$ maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the $xy$-plane.




The polar coordinate transformation

**Figure 7**

The Jacobian of $T$ is

$$\frac{\partial(x,\ y)}{\partial(r,\ \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$

Thus Theorem 9 gives

$$\begin{aligned} \iint_R f(x,\ y)\,dx\,dy &= \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,\ y)}{\partial(r,\ \theta)}\right| dr\,d\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta,\ r\sin\theta)\,r\,dr\,d\theta \end{aligned}$$

单选题 1分

设置

MCQ: In the $xy$-plane, let $R = \{(x, y) \mid 1 \leq xy \leq 4,\ 1 \leq \frac{x}{y} \leq 9,\ x > 0,\ y > 0\}$. Compute $I = \iint_R \frac{1}{y}\, dA$.

A $2$

B $4$

C $4 \ln 3$

D $8$

提交

# Solution

Using the transformation $T(u,v) = (x,y)$ given by $x = uv,\ y = u/v$, we get $xy = u^2$ and $x/y = v^2$. Hence the region $R = T(S)$ corresponds to the region in the $uv$-plane

$S = \{(u,v) \mid 1 \le u \le 2,\ 1 \le v \le 3\}$.

Compute the Jacobian:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix} = \left|\frac{-2u}{v}\right| = \frac{2u}{v}.$$

Also, $\frac{1}{y} = \frac{v}{u}$.

Therefore the transformed integrand is

$$\frac{1}{y}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{v}{u} \cdot \frac{2u}{v} = 2,$$

So $I = \iint_S 2, du, dv = 2(2-1)(3-1) = 4.$

# Change of Variables in Triple Integrals

# Change of Variables in Triple Integrals (1 of 2)

There is a similar change of variables formula for triple integrals.

Let *T* be a one-to-one transformation that maps a region *S* in *uvw*-space onto a region *R* in *xyz*-space by means of the equations

$$x = g(u,\ v,\ w) \quad y = h(u,\ v,\ w) \quad z = k(u,\ v,\ w)$$

# Change of Variables in Triple Integrals (2 of 2)

The **Jacobian** of *T* is the following 3 × 3 determinant:

**12** $$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Under hypotheses similar to those in Theorem 9, we have the following formula for triple integrals:

**13** $$\iiint_R f(x, y, z)\,dV = \iiint_S f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big)\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\,dv\,dw$$

# Example 4

Use Formula 13 to derive the formula for triple integration in spherical coordinates.

Solution:

Here the change of variables is given by

$$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$$

We compute the Jacobian as follows:

$$\frac{\partial(x,\ y,\ z)}{\partial(\rho,\ \theta,\ \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

# Example 4 – Solution (1 of 2)

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right)$$

$$- \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi$$

$$= -\rho^2\sin\phi$$

Since $0 \le \phi \le \pi$, we have $\sin\phi \ge 0$.

# Example 4 – Solution (2 of 2)

Therefore

$$\left|\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)}\right| = \left|-\rho^2 \sin\phi\right|$$

$$= \rho^2 \sin\phi$$

and Formula 13 gives

$$\iiint_R f(x, y, z)\, dV$$

$$= \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

单选题 1分

设置

MCQ: In $xyz$-space, let the ellipsoid be $E=\left\{(x,y,z)\,\middle|\,x^2+\frac{y^2}{4}+\frac{z^2}{9}\le 1\right\}$. Compute $J=\iiint_E (x^2+y^2+z^2)\,dV$.

A $\frac{112\pi}{15}$

B $\frac{112\pi}{5}$

C $\frac{96\pi}{5}$

D $\frac{224\pi}{15}$

提交

# Solution

The transformation $T(u, v, w) = (x, y, z)$ with $x = u,\ y = 2v,\ z = 3w$ maps the unit ball $B = \{(u, v, w) \mid u^2 + v^2 + w^2 \le 1\}$ onto the ellipsoid $E$.

The Jacobian is the $3 \times 3$ determinant $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$

Moreover, $x^2 + y^2 + z^2 = u^2 + 4v^2 + 9w^2$.

By symmetry on the unit ball, $\iiint_B u^2\, dV = \iiint_B v^2\, dV = \iiint_B w^2\, dV = \frac{4\pi}{15}$.

Thus, $J = \iiint_E (x^2 + y^2 + z^2) dV = \iiint_B (u^2 + 4v^2 + 9w^2) \cdot 6dudvdw = 6(1 + 4 + 9)\frac{4\pi}{15} = \frac{112\pi}{5}$.

# Recap

- Change of Variables in Double Integrals
- Change of Variables in Triple Integrals