Final Examination

Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) For any $A \in \mathbb{R}$ the surface in $\mathbb{R}^3$ with equation $x^3 + y^3 + z^3 + A\,xyz = 1$ is smooth.

b) Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies $f(x,y) \to 0$ for $|(x,y)| \to \infty$. Then f has a global extremum.

c) Suppose you start at the point $\mathbf{p} = (1,1)$ in the (x,y)-plane and follow the contour of $f(x,y) = xy^2 + x^2y$ through $\mathbf{p}$ in one of the two possible directions. After some time you reach a point that is closer to $(0,0)$ than $\mathbf{p}$.

d) There exists a function $g\colon \mathbb{R}^2 \to \mathbb{R}$ with at least 2023 saddle points.

e) The set of all real numbers whose decimal expansion doesn't contain the digit 0 (i.e., only digits $1,2,3,4,5,6,7,8,9$ are allowed) has Lebesgue measure zero.

f) For any closed path γ in $\mathbb{R}^2$ and any choice of $a,b,c,d \in \mathbb{R}$ we have $\int_\gamma (ax+by)\,\mathrm{d}x + (cx+dy)\,\mathrm{d}y = \frac{c-b}{2}\int_\gamma x\,\mathrm{d}y - y\,\mathrm{d}x$.

Question 2 (ca. 12 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = x^4 + y^4 - 6\,x^2 - 4\,xy - 6\,y^2.$$

a) Which symmetry properties does f have? What can you conclude from this about the graph of f and the location/type of the critical points of f?

b) Determine all critical points of f and their types.

Hint: There are 9 critical points.

c) Does f have a global extremum?

Question 3 (ca. 7 marks)

Using the method of Lagrange multipliers, solve the optimization problem

$$\begin{array}{rl} \text{Maximize} & \zeta = xy + 6yz + 6zx \\ \text{subject to} & x^2 + y^2 + z^2 = 17. \end{array}$$

Note: Required are (i) a proof that the optimization problem has a solution, (ii) the optimal objective value ζ^*, and (iii) all optimal solutions (x^*, y^*, z^*).

Question 4 (ca. 6 marks)

Consider the function $F\colon (0,\infty) \to \mathbb{R}$ defined by

$$F(a) = \int_0^\infty \frac{\mathrm{d}x}{x^2 + a^2}.$$

a) Show that F is differentiable, and that $F'(a)$ can be obtained by differentation under the integral sign.

b) Using a), evaluate $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}, \quad a>0.$
Hint: The integral defining $F(a)$ can be evaluated using the substitution $x = at$.

Question 5 (ca. 7 marks)

a) Find the mass of the solid K in $\mathbb{R}^3$ consisting of all points (x,y,z) satisfying

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad z^2 \leq 4x, \quad x^2+y^2 \leq 16,$$

whose density is given by $\rho(x,y,z) = xyz^3$.

b) Find the area of the surface P in $\mathbb{R}^3$ consisting of all points (x,y,z) satisfying

$$z = x^{3/2} + y^{3/2}, \quad x \geq 0, \quad y \geq 0, \quad x+y \leq 2.$$