The epsilon-delta definition of the limit

We first define what it means for a function to have **"limit 0 at 0"**.

Definition. Let the function E be defined on an open interval about 0, except possibly at 0. We say that $\lim\limits_{h \to 0} E(h) = 0$ if

> for every challenge number $\epsilon > 0$,
>> there is a response number $\delta > 0$ so that
>>> if $0 < |h| < \delta$,
>>>> then $|E(h)| < \epsilon$.

Example: $E(h) = h^2$ has limit 0 at 0:

> Given an arbitrary $\epsilon > 0$,
>> we can choose $\delta = \sqrt{\epsilon}$. Then
>>> if $0 < |h| < \delta$,
>>>> $|E(h)| = |h^2| < \delta^2 = \epsilon$.

Exercise: Prove these facts using epsilon-delta arguments:

- If functions E_1 and E_2 both have limit 0 at 0, so does their sum.

- If functions E_1 and E_2 both have limit 0 at 0, so does their product.

- If function E_1 has limit 0 at 0, and $|E_2(h)| < |E_1(h)|$ for all h, then E_2 has limit 0 at 0.

- If $|g(h)| < M$ all h, and E has limit 0 at 0, then the function gE has limit 0 at 0.

For a more general function F defined on an open interval about a, except possibly at a, we say

$$\lim_{x \to a} F(x) = L$$

if the "error" function $E(h) = F(a + h) - L$ has limit 0 at 0.

For example, suppose $F(x) = x^2$, and we want to show $\lim\limits_{x \to 2} F(x) = 4$.

We let $E(h) = F(2 + h) - 4 = (2 + h)^2 - 4 = 4h + h^2$, and show $E(h)$ has limit 0 at 0.

If $\epsilon = \frac{1}{10}$, we can choose $\delta = \frac{1}{100}$, since then $|E(h)| = |h^2 + 4h| \leq |h^2| + |4h| < \frac{1}{10000} + \frac{4}{100} < \frac{5}{100} < \frac{1}{10}$.

The case of an arbitrary ϵ is harder. Instead of finding δ directly, we could show $E_1(h) = h^2$ and $E_2(h) = 4h$ both have limit 0 at 0, and then use the fact that their sum also has limit 0 at 0.