0 Final Review

Context

- Differentiation Rules
- Level Curves and Contour Maps
- Linear Approximations
- The Chain Rule
- Implicit Differentiation
- Directional Derivatives

Before midterm， they are still important but not included in this review

- Triple Products
- Lines and Planes
- Quadric Surfaces
- Arc Length
- Curvature
- The Normal and Binormal Vectors

Context

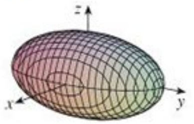
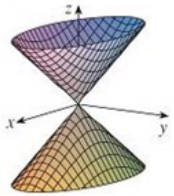
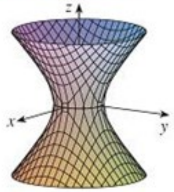
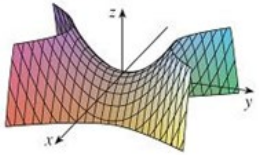
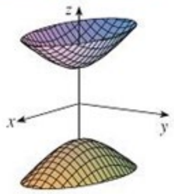
- The Gradient Vector
- Maximizing the Directional Derivative
- Tangent Planes to Level Surfaces
- Local Maximum and Minimum Values
- Lagrange Multipliers: Two Constraints
- Definite Integral
- Iterated Integrals
- Double Integrals in Polar Coordinates
- Triple Integrals in Cylindrical Coordinates
- Triple Integrals in Spherical Coordinates
- Line Integrals

Quadric Surfaces

Quadric Surfaces (3 of 3)

Table 1 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form.

All surfaces are symmetric with respect to the *z*-axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Graphs of Quadric Surfaces

Table 1

Differentiation Rules

Differentiation Rules (1 of 3)

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

3 Theorem Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)]=\mathbf{u}'(t)+\mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)]=c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)]=f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)]=\mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)]=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))]=f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Differentiation Rules (2 of 3)

We use Formula 4 to prove the following theorem.

4 Theorem if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

PROOF

Since

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

and c^2 is a constant, Formula 4 of Theorem 3 gives

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

Thus $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Level Curves and Contour Maps

Level Curves and Contour Maps (1 of 10)

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour curves*, or *level curves*.

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it is a curve in the xy-plane that shows where the graph of f has height k (above or below the xy-plane). A collection of level curves is called a **contour map**.

Many info in Contour Map

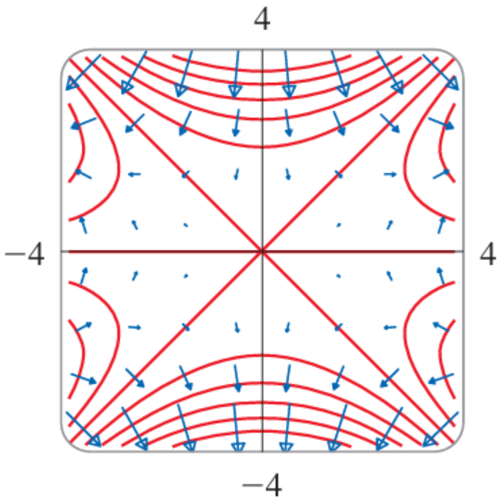


FIGURE 15

EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

SOLUTION The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

Figure 15 shows a contour map of f with the gradient vector field. Notice that the gradient vectors are perpendicular to the level curves, as we would expect from Section 14.6. Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart. That's because the length of the gradient vector is the value of the directional derivative of f and closely spaced level curves indicate a steep graph. ■

Linear Approximations

Linear Approximations (4 of 9)

The approximation

$$\textbf{4} \quad f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of *f* at (*a*, *b*).

The Chain Rule

The Chain Rule: Case 2 (2 of 4)

A similar argument holds for $\frac{\partial z}{\partial s}$ and so we have proved the following version of the Chain Rule.

2 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Case 2 of the Chain Rule contains three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the **dependent** variable.

Question from Quiz

1. $If\ z = e^x \sin y$,where $x = st^3$ and $y = s^3t,\ find\ \frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^3) + (e^x \cos y)\ (3ts^2)$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(3st^2) + (e^x \cos y)(s^3)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(e^{st^3} \sin st^3\right)(t^3) + \left(e^{st^3} \cos s^3t\right)(3st^2)$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(e^{st^3} \sin s^3t\right)(3st^2) + \left(e^{st^3} \cos s^3t\right)(s^3)$$

Remember to replace x(s,t), y(s,t), half of people forget to do so

Implicit Differentiation

Implicit Differentiation (4 of 6)

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.

This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Implicit Differentiation (6 of 6)

6 $$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid: if F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by (6).

My own way of Implicit Differentiation

EXAMPLE 9 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz + 4 = 0$.

$$x^3 + y^3 + z^3 + 6xyz + 4 = 0$$

derivation.

$$3x^2dx + 3y^2dy + 3z^2dz + 6yzdx + 6xzdy + 6xydz = 0$$

$$(3x^2 + 6yz)\,dx + (3y^2 + 6yz)\,dy + (3z^2 + 6xy)\,dz = 0$$

if $dz = 0$, $(x^2 + 2yz)\,\partial x + (y^2 + 2yz)\,\partial y = 0$

$$\frac{\partial x}{\partial y} = -\frac{y^2 + 2yz}{x^2 + 2yz}.$$

if $dy = 0$, $(x^2 + 2yz)\,\partial x + (z^2 + 2xy)\,\partial z = 0$

$$\frac{\partial x}{\partial z} = -\frac{z^2 + 2xy}{x^2 + 2yz}$$

Directional Derivatives

Directional Derivatives (4 of 8)

The slope of the tangent line T to C at the point P is the rate of change of z in the direction of **u**.

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to **u** and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar h. Therefore $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha,\ y_0 + hb) - f(x_0, y_0)}{h}$$

多选题 1分

设置

MAQ: $f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$ Let $\mathbf{u}=(a,b)$ be a unit vector $(a^2+b^2=1)$, which of the followings are true?

A The partial derivatives $f_x(0,0)$ and $f_y(0,0)$ both exist

B The function f is NOT differentiable at $(0,0)$

C One unit vector achieving the maximum directional derivative is $\mathbf{u}=\left(\frac{\sqrt{3}}{3},\frac{\sqrt{6}}{3}\right)$

D The set of all directional derivatives at the origin is

$$\{\mathrm{D}_u f(0,0): |\mathbf{u}|=1\}=[-\frac{2\sqrt{3}}{9},\frac{2\sqrt{3}}{9}]$$

提交

Example

Solution:

Along axes: $f(h,0) = f(0,h) = 0 \Rightarrow f_x(0,0) = f_y(0,0) = 0 \Rightarrow$A true

$\frac{f(ha,hb)}{h} = \frac{(h^2a^2)(hb)}{(h^2a^2+h^2b^2)h} = a^2b$, not always $0 \Rightarrow$ B true

$b = \sin\theta\,, a^2b = \sin\theta - (\sin\theta)^3 \in [-\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9}] \Rightarrow$ D true

It is achieved when $b = \frac{\sqrt{3}}{3},\ a^2 = \frac{2}{3}$, C false

exist of derivative: $\lim_{h\to 0} \frac{\Delta z}{h} = k_0$ (constant)

does not exist, k_0 is function of direction or infinity

The Gradient Vector

The Gradient Vector (2 of 3)

8 Definition If *f* is a function of two variables *x* and *y*, then the **gradient** of *f* is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

The Gradient Vector (3 of 3)

With the notation for the gradient vector, we can rewrite Equation 7 for the directional derivative of a differentiable function as

$$\textbf{9} \quad D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

This expresses the directional derivative in the direction of a unit vector **u** as the scalar projection of the gradient vector onto **u**.

Maximizing the Directional Derivative

Maximizing the Directional Derivatives (1 of 1)

Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point.

These give the rates of change of f in all possible directions.

We can then ask the questions: In which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

15 Theorem Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Tangent Planes to Level Surfaces

Tangent Planes to Level Surfaces (4 of 6)

If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

Using the standard equation of a plane, we can write the equation of this tangent plane as

19 $$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Tangent Planes to Level Surfaces (5 of 6)

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane.

The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, its symmetric equations are

20 $$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Tangent Planes to Level Surfaces (6 of 6)

In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F. Then

$$\begin{aligned} F_x(x_0, y_0, z_0) &= f_x(x_0, y_0) \\ F_y(x_0, y_0, z_0) &= f_y(x_0, y_0) \\ F_z(x_0, y_0, z_0) &= -1 \end{aligned}$$

so Equation 19 becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Local Maximum and Minimum Values

Local Maximum and Minimum Values (3 of 7)

1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b).

[This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).]

The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

Local Maximum and Minimum Values (4 of 7)

Fermat's Theorem states that, for single-variable functions, if f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$. The following theorem states a similar result for functions of two variables.

2 Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Local Maximum and Minimum Values (5 of 7)

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f.

However, as in single-variable calculus, not all critical points give rise to maxima or minima.

Local Maximum and Minimum Values (6 of 7)

The following test, is analogous to the Second Derivative Test for functions of one variable.

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f]. Let

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,\ b) - [f_{xy}(a,\ b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

Local Maximum and Minimum Values (7 of 7)

(c) If $D < 0$, then $f(a, b)$ is a saddle point of f.

Note 1 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b) or (a, b) could be a saddle point of f.

A question from honold

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = x^4 - x^3y - xy + y^2.$$

a) Which obvious symmetry property does f have? What can you conclude from this about the graph and the contours of f?

b) Determine all critical points of f and their types.

Hint: There are 5 critical points.

c) Does f have a global extremum?

2 a) $f(-x,-y) = f(x,y)$ for $(x,y) \in \mathbb{R}^2$

$\Longrightarrow$ The graph of f is symmetric with respect to the z-axis.

The contours of f are point-symmetric with respect to the origin.

b) Using the shorthands f, f_x, f_y for $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, we compute

$$\begin{aligned} f_x &= 4\,x^3 - 3\,x^2y - y, \\ f_y &= -x^3 - x + 2\,y, \end{aligned}$$

$$\begin{aligned} \nabla f(x,y) = (0,0) &\Longrightarrow y = \tfrac{1}{2}\left(x^3+x\right) \Longrightarrow 4x^3 - \left(3x^2+1\right)\tfrac{1}{2}\left(x^3+x\right) = 0 \\ &\Longrightarrow 3x^5 - 4x^3 + x = 0 \\ &\Longrightarrow x(x^2-1)(3x^2-1) = 0. \end{aligned}$$

$\Longrightarrow$ The critical points of f are

$$\begin{aligned} &\mathbf{p}_1 = (0,0), \quad \mathbf{p}_2 = (1,1), \quad \mathbf{p}_3 = (-1,-1), \\ &\mathbf{p}_4 = \left(\tfrac{1}{3}\sqrt{3}, \tfrac{2}{9}\sqrt{3}\right), \quad \mathbf{p}_5 = \left(-\tfrac{1}{3}\sqrt{3}, -\tfrac{2}{9}\sqrt{3}\right). \end{aligned}$$

$\boxed{2\tfrac{1}{2}}$

Further we have

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 - 6xy & -3x^2-1 \\ -3x^2-1 & 2 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_1) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_{2/3}) = \begin{pmatrix} 6 & -4 \\ -4 & 2 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_{4/5}) = \begin{pmatrix} 8/3 & -2 \\ -2 & 2 \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_1)$ has determinant $-1 < 0$, the point $\mathbf{p}_1$ is a saddle point. $\boxed{\tfrac{1}{2}}$

Since $\mathbf{H}_f(\mathbf{p}_{2/3})$ has determinant $12-16 = -4 < 0$, the points $\mathbf{p}_2$, $\mathbf{p}_3$ are saddle points. $\boxed{1}$

Since $\mathbf{H}_f(\mathbf{p}_{4/5})$ is positive definite ($f_{xx}(\mathbf{p}_{4/5}) = 8/3 > 0$, $\det \mathbf{H}_f(\mathbf{p}_{4/5}) = 16/3 - 4 = 4/3 > 0$), the points $\mathbf{p}_4$, $\mathbf{p}_5$ are strict local minima. $\boxed{1}$

The corresponding value is $f(\mathbf{p}_{4/5}) = -1/27$.

The corresponding value is $f(\mathbf{p}_{4/5}) = -1/27$.

c) No. This follows, e.g., from $f(x,0) = x^4 \to +\infty$ for $x \to \pm\infty$, $f(x,2x) = -x^4 + 2x^2 \to -\infty$ for $x \to \pm\infty$. $\boxed{1}$

Maxand Min within a boundary

Example 7

Find the absolute maximum and minimum values of the function $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

Solution:

Since *f* is a polynomial, it is continuous on the closed, bounded rectangle *D*, so Theorem 8 tells us there is both an absolute maximum and an absolute minimum.

According to step 1 in (9), we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

Example 7 – Solution (1 of 5)

So the only critical point is (1, 1). This point is in D and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12.

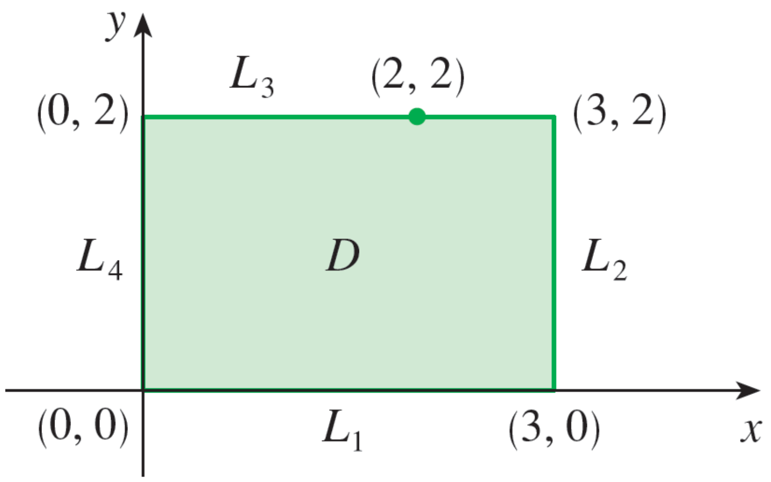


Figure 12

Example 7 – Solution (2 of 5)

On L_1 we have $y = 0$ and

$$f(x,\ 0) = x^2 \qquad 0 \le x \le 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$.

On L_2 we have $x = 3$ and

$$f(3,\ y) = 9 - 4y \qquad 0 \le y \le 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$.

Example 7 – Solution (3 of 5)

On L_3 we have $y = 2$ and

$$f(x,\ 2) = x^2 - 4x + 4 \qquad 0 \le x \le 3$$

Simply by observing that $f(x,\ 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$.

Example 7 – Solution (4 of 5)

Finally, on L_4 we have $x = 0$ and

$$f(0,\ y) = 2y \qquad 0 \le y \le 2$$

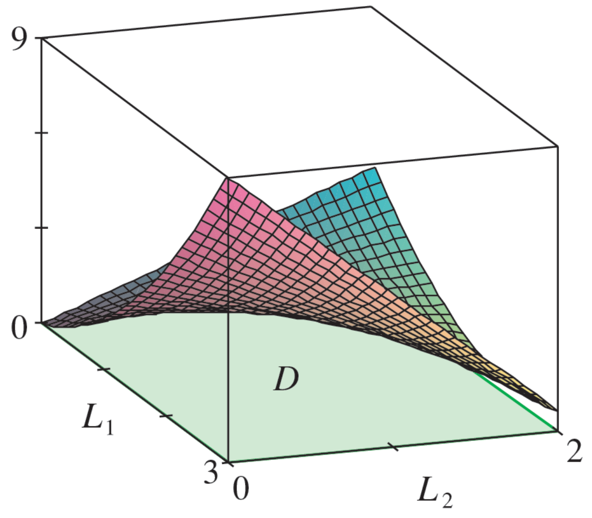
with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$.

Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

Example 7 – Solution (5 of 5)

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$.

Figure 13 shows the graph of f.



$f(x, y) = x^2 - 2xy + 2y$

Figure 13

Lagrange Multipliers: Two Constraints

Lagrange Multipliers: Two Constraints

Constraint Qualification — Extremely Easy to Overlook! This is the most crucial, yet most frequently overlooked condition for beginners.

- **Requirement:** At the extremum point P, the gradient of the constraint function must **not be a zero vector** ($\nabla g(P) \neq \mathbf{0}$).
  - In the case of multiple constraints $g_1, g_2, \ldots$, the gradient vectors of these constraint functions at point P must be **linearly independent**. This is commonly referred to as **LICQ (Linear Independence Constraint Qualification)**.
- **Reason:** If $\nabla g = \mathbf{0}$, it implies that the constraint surface may exhibit a **singularity** (such as a cusp or self-intersection) at that point. In such cases, the geometric concepts of "tangent/normal" break down, causing the Lagrange Multipliers method to fail in locating that extremum.
- **Example:** If the constraint is $x^3 + y^3 = 0$, then at the origin $(0,0)$, we have $\nabla g = (3x^2, 3y^2) = (0,0)$. At this point, the method fails.

多选题 1分

设置

MAQ: Subject to $x+y+z=0$, $2x^2+3y^2+6z^2=6$, find the extrema of $f(x,y,z)=x+2y+3z$. Which statements are correct?

A $\max f=\frac{\sqrt{30}}{3}$

B $\max f=-\frac{\sqrt{30}}{3}$

C A maximizer is $\left(-\frac{\sqrt{30}}{5},\frac{\sqrt{30}}{15},\frac{2\sqrt{30}}{15}\right)$

D A maximizer is $\left(\frac{\sqrt{30}}{5},-2\frac{\sqrt{30}}{15},\frac{\sqrt{30}}{15}\right)$

提交

Solution

We apply the method of Lagrange multipliers with

$$\nabla f = \lambda \nabla(x + y + z) + \mu \nabla(2x^2 + 3y^2 + 6z^2)$$

This gives the system:

$$\begin{cases} 1 = \lambda + 4\mu x \\ 2 = \lambda + 6\mu y \\ 3 = \lambda + 12\mu z \\ x + y + z = 0 \\ 2x^2 + 3y^2 + 6z^2 = 6 \end{cases}$$

From the first 3 equations, we can get $3 \times 1 + 2 \times 2 + 3 = (3 + 2 + 1)\lambda + 12\mu(x + y + z) = 6\lambda \Rightarrow \lambda = \frac{5}{3} \Rightarrow x = -\frac{1}{6\mu}, y = \frac{1}{18\mu}, z = \frac{1}{9\mu}$

Putting them into the last equation, we get $\frac{5}{36}\frac{1}{\mu^2} = 6 \Rightarrow \frac{1}{\mu} = \pm\frac{6\sqrt{30}}{5}$

So critical points: $\left(-\frac{\sqrt{30}}{5}, \frac{\sqrt{30}}{15}, \frac{2\sqrt{30}}{15}\right)$ and $\left(\frac{\sqrt{30}}{5}, -\frac{\sqrt{30}}{15}, -\frac{2\sqrt{30}}{15}\right)$, $f = \pm\frac{\sqrt{30}}{3}$

Definite Integral

Review of the Definite Integral (1 of 2)

First let's recall the basic facts concerning definite integrals of functions of a single variable.

If $f(x)$ is defined for $a \le x \le b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

$$\textbf{1} \qquad \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

$$2 \qquad \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

多选题 1分

设置

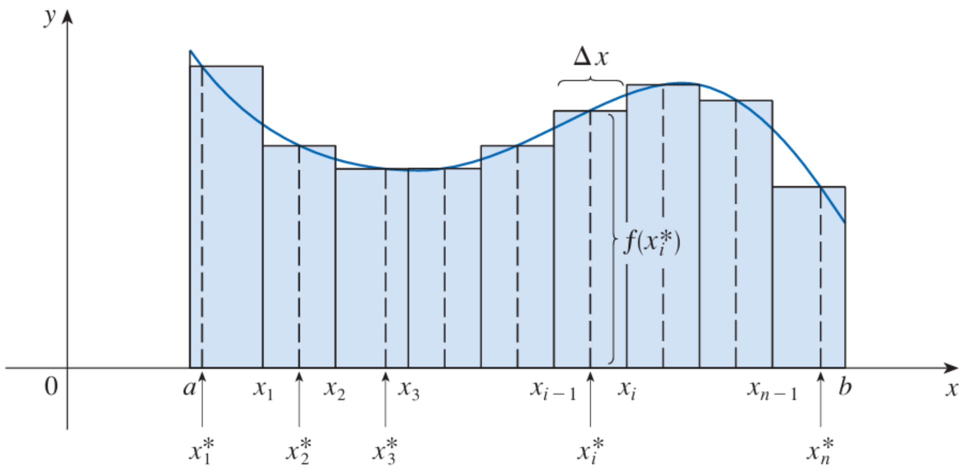
MAQ: Let $f: [0,1] \to \mathbb{R}$ be a continuous function. Consider the following limits (whenever they exist). Which of these limits **must** equal the definite integral $\int_0^1 f(x)\,dx$?

A $\displaystyle \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$

B $\displaystyle \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} f\left(\frac{k^2}{n^2}\right)$

C $\displaystyle \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{2k+1}{2n}\right)$

D $\displaystyle \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} f\left(\frac{2k-1}{n^2}\right)$



提交

Example

Solution:

For option A, here the partition points are $x_k = \frac{k}{n}$ for $k = 0, \dots, n$, so each interval has length $\Delta x = \frac{1}{n}$. The sample point in the k-th interval is the right endpoint $x_k = \frac{k}{n}$.

Thus (A) is exactly the right-endpoint Riemann sum for f on [0,1].

For option B counterexample: take $f(x) \equiv 1$ .Then

$$\int_0^1 f(x)\, dx = 1,$$

but the sum in (B) is as follow. B is false

$$\frac{2}{n}\sum_{k=1}^{n} 1 = \frac{2n}{n} = 2 \ \Rightarrow\ limit = 2.$$

For option C, again use the partition $0, \frac{1}{n}, \dots, \frac{n}{n}$ with width $\Delta x = \frac{1}{n}$.

The sample point in $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ is the midpoint

$$x_k^* = \frac{k + \frac{1}{2}}{n} = \frac{2k+1}{2n}$$

Thus (C) is the midpoint Riemann sum

For option D, Because f is continuous at 0, $f(x_k^*) \to f(0)$ for every fixed k, and uniformly for all such k when n is large. Hence

$$\lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} f(x_k^*) = \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} f(0) = 2f(0).$$

In general $2f(0) \neq \int_0^1 f(x)\, dx$. D is false.

Iterated Integrals

Iterated Integrals (5 of 7)

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

10 Fubini's Theorem If *f* is continuous on the rectangle

$$R = \{(x,\ y) \mid a \le x \le b,\ c \le y \le d\},$$

then

$$\iint_R f(x,\ y)\, dA = \int_a^b \int_c^d f(x,\ y)\, dy\ dx = \int_c^d \int_a^b f(x,\ y)\, dx\ dy$$

More generally, this is true if we assume that *f* is bounded on *R*, *f* is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Iterated Integrals

Prove Dirichlet Integral $I = \int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}$ in following step:

(a) Prove when x > 0, $\frac{1}{x} = \int_0^\infty e^{-xy}\, dy$

(b) Use the result of (a), rewrite the Dirichlet Integral into iterated integral

(c) Use Fubini-Tonelli theorem, calculate the iterated integral

Iterated Integrals

The complex way to do so

Insights:

1. Review Math231 Integration by Parts
2. Reserve place for integration formula

$$= \frac{\pi}{2}$$

Sol 2, $\int_0^\infty \sin(x)e^{-xy}dx = \cancel{\int_0^\infty \frac{1}{-y}\sin(x)\,de^{-xy}}$

$$= \int_0^\infty e^{-xy}\,d(-\cos(x)) \cancel{= -\left(e^{-xy}\cos(x) - \int_0^\infty \cos x(-y)e^{-xy}dx\right)}$$

$$= e^{-xy}(-\cos(x)) - \int_0^\infty -\cos(x)\,de^{-xy}$$

$$= -e^{-xy}\cos(x) + \int_0^\infty \cos(x)(-y)e^{-xy}dx$$

$$= -e^{-xy}\cos(x) - \int_0^\infty ye^{-xy}\,d\sin(x)$$

$$= -e^{-xy}\cos(x) - ye^{-xy}\sin x + y\int_0^\infty \sin(x)\,de^{-xy}$$

$$= -e^{-xy}\cos(x) - ye^{-xy}\sin x - y^2\int_0^\infty \sin(x)e^{-xy}dx$$

$$(1+y^2)\int_0^\infty \sin(x)e^{-xy}dx = -e^{-xy}\cos(x) - ye^{-xy}\sin x$$

$$\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx)).$$

$$\int_0^\infty \sin(x)e^{-xy}dx = -\frac{e^{-xy}}{1+y^2}(\cos(x) + y\sin(x))$$

$$\iint \sin(x)e^{-xy}dxdy = \int_0^\infty \left[-\frac{e^{-xy}}{1+y^2}(\cos(x) + y\sin(x))\right]_0^\infty dy$$

$x\to\infty$, $e^{-xy}\to 0$

$x\to 0$, $e^{-xy}=1$, $\cos(x)=1$ $\sin x=0$

$$\iint \sin e^{-xy}dxdy = \int_0^\infty \frac{1}{1+y^2}dy = \frac{\pi}{2}$$

Iterated Integrals

$$\frac{\pi}{2} = \int_0^\infty\int_0^\infty \sin(x)\, e^{-xy}\, dx\, dy$$

Sol 1. $j\sin(x) = \frac{e^{jx}-e^{-jx}}{2}$

$$\int_0^\infty \frac{1}{j}\,\frac{e^{jx}-e^{-jx}}{2}\, e^{-xy}\, dx dy$$

$$\frac{1}{2j}\int_0^\infty\int_0^\infty e^{(j-y)x} - e^{(-j-y)x}\, dx dy$$

$$= \frac{1}{2j}\int_0^\infty \left(\frac{1}{j-y} e^{(j-y)x} - \frac{1}{-j-y} e^{(-j-y)x} \right)\Big|_0^\infty dy$$

$\cancel{e^{-yx}}$ $\because y > 0,\ x \to \infty$ $\quad e^{-yx} \to 0$

$e^{jx}\cdot e^{-yx} \to 0$

$$= \frac{1}{2j}\int_0^\infty \frac{1}{j-y} + \frac{1}{j+y}\, dy = \int_0^\infty \frac{1}{1+y^2}\, dy$$

$$= \frac{\pi}{\ }$$

corresponding formula:

$e^{j\theta} = \cos\theta + j\sin\theta$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

$j\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2}$

Double Integrals in Polar Coordinates

Double Integrals in Polar Coordinates (6 of 9)

Therefore we have

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*,\ r_i^* \sin\theta_j^*)\,\Delta A_i$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*,\ \theta_j^*)\,\Delta r\ \Delta\theta = \int_{\alpha}^{\beta}\int_{a}^{b} g(r,\ \theta)\,dr\ d\theta$$

$$= \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta,\ r\sin\theta)\,r\ dr\,d\theta$$

2 Change to Polar Coordinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta,\ r\sin\theta)\,r\ dr\,d\theta$$

单选题 1分

设置

MCQ:$D=\{(x,y)|x^2+y^2\leq 4, x\geq 1, y\geq 1\}$ Find area $A(D)$.

A $\dfrac{1+\pi}{3}+\dfrac{\sqrt{3}}{3}$

B $\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{3}$

C $1+\dfrac{\pi}{3}-\sqrt{3}$

D $\dfrac{2+\pi}{3}-\sqrt{3}$

提交

Example

Solution:

$r\cos\theta \geq 1 \rightarrow r \geq \sec\theta \quad r\sin\theta \geq 1 \rightarrow r \geq \csc\theta \quad \text{since } r \leq 2$

we get $\theta \in [\frac{\pi}{6}, \frac{\pi}{3}]$ and with region $\begin{cases} \frac{\pi}{6} \leq \theta \leq \frac{\pi}{4} & \csc\theta \leq r \leq 2 \\ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3} & \sec\theta \leq r \leq 2 \end{cases}$

$$A = \iint_D 1 dA = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{\csc\theta}^{2} r dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_{\sec\theta}^{2} r dr d\theta = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_{\sec\theta}^{2} r dr d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (4 - \sec\theta^2)\, d\theta = [4\theta - \tan\theta]_{\frac{\pi}{3}}^{\frac{\pi}{4}} == 1 + \frac{\pi}{3} - \sqrt{3}$$

Triple Integrals in Cylindrical Coordinates

Triple Integrals in Cylindrical Coordinates (2 of 5)

In particular, suppose that f is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

We know

3 $$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

Triple Integrals in Cylindrical Coordinates (3 of 5)

But to evaluate double integrals in polar coordinates, we have the formula

$$\textbf{4} \qquad \iiint_E f(x, y, z)\, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Formula 4 is the **formula for triple integration in cylindrical coordinates**.

单选题 1分

设置

MCQ: E is bounded below by $z = x^2 + y^2$,above by $z = 4$,between the vertical planes $y = 0$and $y = x$,with $x \geq 0$. Compute

$$I = \iiint_E z \, dV$$

A $\frac{8\pi}{3}$

B $\frac{7\pi}{3}$

C 2π

D $\frac{5\pi}{3}$

提交

Example

Solution:

The bounds are:

$$0 \le \theta \le \frac{\pi}{4}, 0 \le r \le 2, r^2 \le z \le 4.$$

Compute:

$$\iiint_E z\ dV = \int_0^{\pi/4} \int_0^2 \int_{r^2}^4 zrdzdrd\theta$$

Inner integral

$$\int_{r^2}^4 z\ dz = 8 - \frac{r^4}{2}.$$

So

$$\int_0^2 \left(8r - \frac{1}{2}r^5\right) dr = \frac{32}{3}.$$

Finally

$$\int_0^{\pi/4} \frac{32}{3}\ d\theta = \frac{8\pi}{3}$$

Triple Integrals in Spherical Coordinates

Triple Integrals in Spherical Coordinates (1 of 8)

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge**

$$E = \{(\rho,\ \theta,\ \phi) \mid a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$$

where $a \geq 0$ and $\beta - \alpha \leq 2\pi$, and $d - c \leq \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result.

So we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$.

Triple Integrals in Spherical Coordinates (5 of 8)

But this sum is a Riemann sum for the function

$$F(\rho,\ \theta,\ \phi) = f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho^2\ \sin\phi$$

Consequently, we have arrived at the following **formula for triple integration in spherical coordinates**.

3 $$\iiint_E f(x,\ y,\ z)\ dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\ \cos\ \theta,\ \rho\ \sin\phi\ \sin\theta,\ \rho\ \cos\phi)\,\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho,\ \theta,\ \phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$$

单选题 1分

设置

MCQ: $E = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, x \geq 0, |y| \leq x, x^2 + y^2 \geq z^2\}$
Compute $I = \iiint_E 1 \; dV$

A $\frac{4\sqrt{3}\pi}{3}$

B $\frac{3\sqrt{3}\pi}{4}$

C $\frac{5\sqrt{2}\pi}{4}$

D $\frac{4\sqrt{2}\pi}{3}$

提交

Example

Solution:

Sphere:

$$x^2 + y^2 + z^2 \leq 4 \Rightarrow 0 \leq \rho \leq 2$$

Cone condition:

$$x^2 + y^2 \geq z^2 \Rightarrow \rho^2 (\sin\phi)^2 \geq \rho^2 (\cos\phi)^2 \Rightarrow (\tan\phi)^2 \geq 1 \Rightarrow \phi \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

Wedge in the xy-plane:

$$| y | \leq x \Rightarrow | \tan\theta | \leq 1 \text{ and } \cos\theta \geq 0 \Rightarrow \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

So $x \geq 0$ is effectively redundant.

Evaluate

$$\int_0^2 \rho^2 \, d\rho = \frac{8}{3}, \quad \int_{\pi/4}^{3\pi/4} \sin\phi \, d\phi = \sqrt{2}, \int_{-\pi/4}^{\pi/4} d\,\theta = \frac{\pi}{2}$$

Thus

$$\iiint_E 1 \; dV = \int_{-\pi/4}^{\pi/4} \int_{\pi/4}^{3\pi/4} \int_0^2 \rho^2 \sin\phi \, d\rho d\phi d\theta = \frac{8}{3} \cdot \sqrt{2} \cdot \frac{\pi}{2} = \frac{4\sqrt{2}\pi}{3}$$

Line Integrals

Example 4

Evaluate $\int_C y^2 dx + x dy$ for two different paths *C*.

(a) $C = C_1$ is the line segment from (−5, −3) to (0, 2).

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from (−5, −3) to (0, 2).

(See Figure 7.)

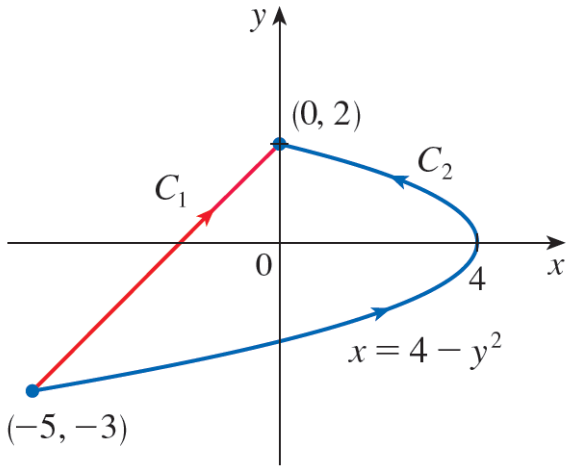


Figure 7

Example 4 – Solution

(a) A parametric representation for the line segment is

$$x = 5t - 5 \qquad y = 5t - 3 \qquad 0 \leq t \leq 1$$

(Use Equation 8 with $\mathbf{r}_0 = \langle -5, -3 \rangle$ and $\mathbf{r}_1 = \langle 0, 2 \rangle$.)

Then $dx = 5\ dt$, $dy = 5\ dt$, and Formulas 7 give

$$\int_{C_1} y^2\,dx + x\,dy = \int_0^1 (5t-3)2(5dt) + (5t-3)(5dt)$$

$$= 5\int_0^1 \left(25t^2 - 25t + 4\right)dt$$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6}$$

Example 4 – Solution (1 of 1)

(b) Since the parabola is given as a function of *y*, let's take *y* as the parameter and write C_2 as

$$x = 4 - y^2 \qquad y = y \qquad -3 \le y \le 2$$

Then $dx = -2y\,dy$ and by Formulas 7 we have

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} y^2(-2y)\,dy + (4 - y^2)\,dy$$

$$= \int_{-3}^{2} \left(-2y^3 - y^2 + 4\right) dy$$

$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right]_{-3}^{2} = 40\frac{5}{6}$$

Line Integrals in Space (2 of 3)

We evaluate it using a formula similar to Formula 3:

$$\textbf{9} \qquad \int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt$$

For the special case $f(x, y, z) = 1$, we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\,dt = L$$

where L is the length of the curve C.

单选题 1分

设置

MCQ: Let C be the circular helix $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$. Compute $\int_C z ds$

A π^2

B $\sqrt{2}\pi^2$

C $2\sqrt{2}\pi^2$

D $4\sqrt{2}\pi^2$

提交

Solution

With $\boldsymbol{r}(t) = \langle \cos t, \quad \sin t,\ t\rangle$,

$$\boldsymbol{r}'(t) = \langle -\sin t\,,\ \cos t\,,\ 1\rangle, \quad |\boldsymbol{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

So $ds = \sqrt{2}dt$ and $z = t$. Hence $\int_C z ds = \int_0^{2\pi} t\sqrt{2}\, dt = \sqrt{2} \cdot \frac{(2\pi)^2}{2} = 2\sqrt{2}\pi^2$.

Recap

- Triple Products
- Lines and Planes
- Quadric Surfaces
- Arc Length
- Curvature
- The Normal and Binormal Vectors
- Differentiation Rules
- Level Curves and Contour Maps
- Linear Approximations
- The Chain Rule
- Implicit Differentiation
- Directional Derivatives

Recap

- The Gradient Vector
- Maximizing the Directional Derivative
- Tangent Planes to Level Surfaces
- Local Maximum and Minimum Values
- Lagrange Multipliers: Two Constraints
- Definite Integral
- Iterated Integrals
- Double Integrals in Polar Coordinates
- Triple Integrals in Cylindrical Coordinates
- Triple Integrals in Spherical Coordinates
- Line Integrals

Before Midterm