Solutions (Total: 10 points)

1. [4 points]

Grading idea: (a) correct use of Lagrange multipliers and correct (λ): 3 pts. (b) Correct justification of minimum: 1 pt.

We have

$$\nabla f = (2x, 2y, 2z), \qquad \nabla g = (1, 2, 3).$$

Lagrange's condition gives

$$\nabla f = \lambda \nabla g \quad \Rightarrow \quad 2x = \lambda, \quad 2y = 2\lambda, \quad 2z = 3\lambda.$$

Thus $x = \frac{\lambda}{2}$, $y = \lambda$, $z = \frac{3\lambda}{2}$.

Substitute into the constraint $x + 2y + 3z = 1$:

$$\frac{\lambda}{2} + 2\lambda + 3\left(\frac{3\lambda}{2}\right) = 1 \quad \Rightarrow \quad \frac{1+4+9}{2}\lambda = 1 \quad \Rightarrow \quad \lambda = \frac{1}{7}.$$

Hence

$$(x, y, z) = \left(\frac{1}{14}, \frac{1}{7}, \frac{3}{14}\right).$$

Since $f(x, y, z) = x^2 + y^2 + z^2 \geq 0$ and the constraint is linear, the point on the plane with minimal distance to the origin gives the **minimum** of f. Therefore the critical point found is a global (and local) minimum.

2. [6 points total; 3 points each part]

Suggested breakdown for each function (3 pts): 1 pt: correct critical points; 1 pt: correct second derivatives and D; 1 pt: correct classification.

(a) $f_1(x, y) = x^2 - 3xy + 2y^2$

$$f_x = 2x - 3y, \qquad f_y = -3x + 4y.$$

Setting both equal to zero gives

$$2x - 3y = 0, \quad -3x + 4y = 0.$$

Solving yields $x = y = 0$.

Second partials:

$$f_{xx} = 2, \quad f_{yy} = 4, \quad f_{xy} = -3.$$

Then

$$D = (2)(4) - (-3)^2 = 8 - 9 = -1 < 0.$$

So $(0, 0)$ is a **saddle point**.

(b) $f_2(x, y) = x^3 - 6xy + 3y^2$

$$f_x = 3x^2 - 6y, \quad f_y = -6x + 6y.$$

Setting both to zero:

$$3x^2 - 6y = 0, \quad -6x + 6y = 0.$$

From the second, $y = x$. Substitute into the first:

$$3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow x = 0 \text{ or } 2.$$

Thus critical points: $(0,0)$ and $(2,2)$.

Second partials:

$$f_{xx} = 6x, \quad f_{yy} = 6, \quad f_{xy} = -6.$$

At $(0,0)$:

$$D = (6x)(6) - (-6)^2 = 36x - 36 = -36 < 0,$$

so $(0,0)$ is a **saddle point**.

At $(2,2)$:

$$D = (12)(6) - (-6)^2 = 72 - 36 = 36 > 0, \quad f_{xx} = 12 > 0,$$

so $(2,2)$ is a **local minimum**.

Summary:

$$\begin{cases} f_1: & (0,0) \text{ is a saddle point;} \\ f_2: & (0,0) \text{ is a saddle point, } (2,2) \text{ is a local minimum.} \end{cases}$$