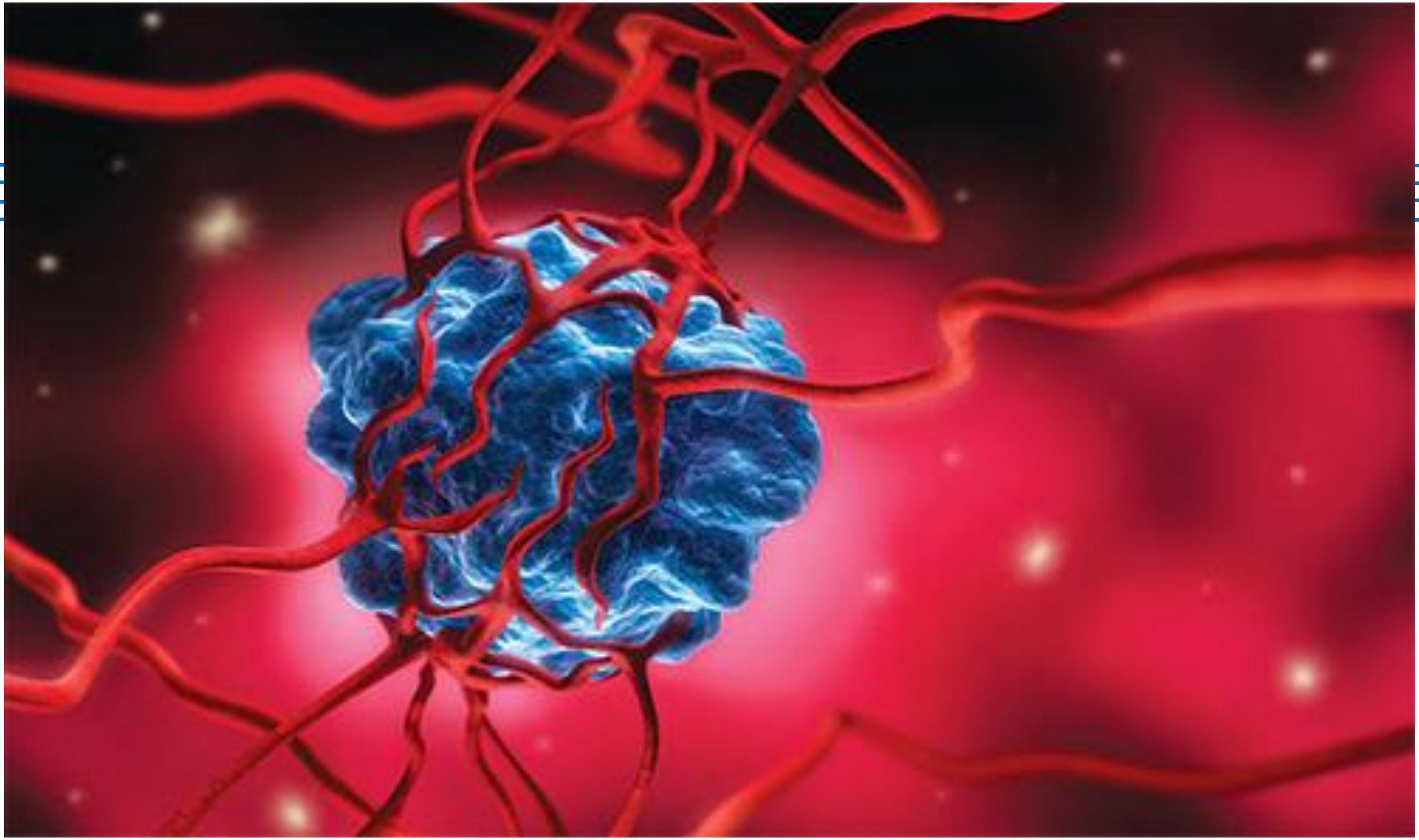
15 Multiple Integrals



15.5 Surface Area

Context

- Surface Area

Surface Area (1 of 8)

In this section we apply double integrals to the problem of computing the area of a surface. Here we compute the area of a surface with equation $z = f(x, y)$, the graph of a function of two variables.

Let S be a surface with equation $z = f(x, y)$, where f has continuous partial derivatives. For simplicity in deriving the surface area formula, we assume that $f(x, y) \geq 0$ and the domain D of f is a rectangle. We divide D into small rectangles R_{ij} with area $\Delta A = \Delta x\, \Delta y$.

If (x_i, y_j) is the corner of R_{ij} closest to the origin, let $P_{ij}\,(x_i, y_j, f(x_i, y_j))$ be the point on S directly above it (see Figure 1).

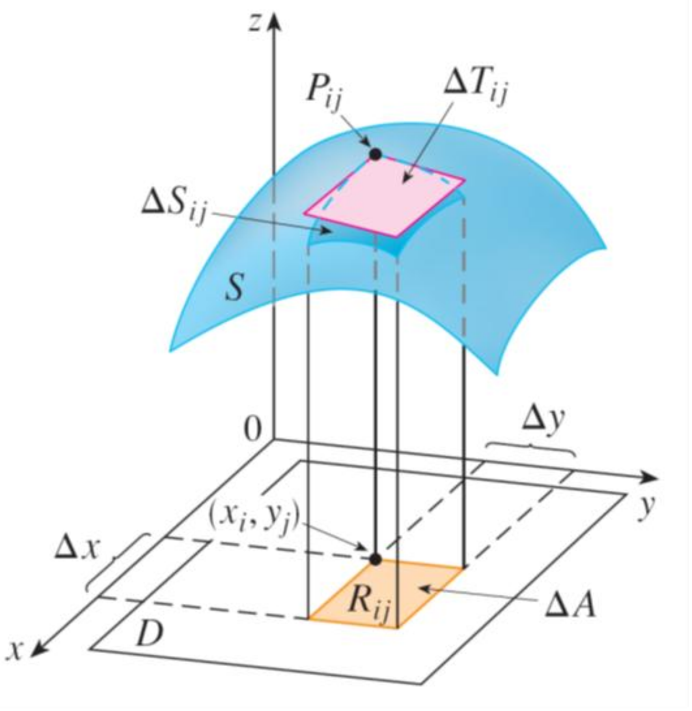


Figure 1

Surface Area (3 of 8)

The tangent plane to S at P_{ij} is an approximation to S near P_{ij}. So the area ΔT_{ij} of the part of this tangent plane (a parallelogram) that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of the part of S that lies directly above R_{ij}.

Thus the sum $\sum\sum \Delta T_{ij}$ is an approximation to the total area of S, and this approximation appears to improve as the number of rectangles increases.

Therefore we define the **surface area** of S to be

$$\mathbf{1} \qquad A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

Surface Area (4 of 8)

To find a formula that is more convenient than Equation 1 for computational purposes, we let **a** and **b** be the vectors that start at P_{ij} and lie along the sides of the parallelogram with area ΔT_{ij}. (See Figure 2.)

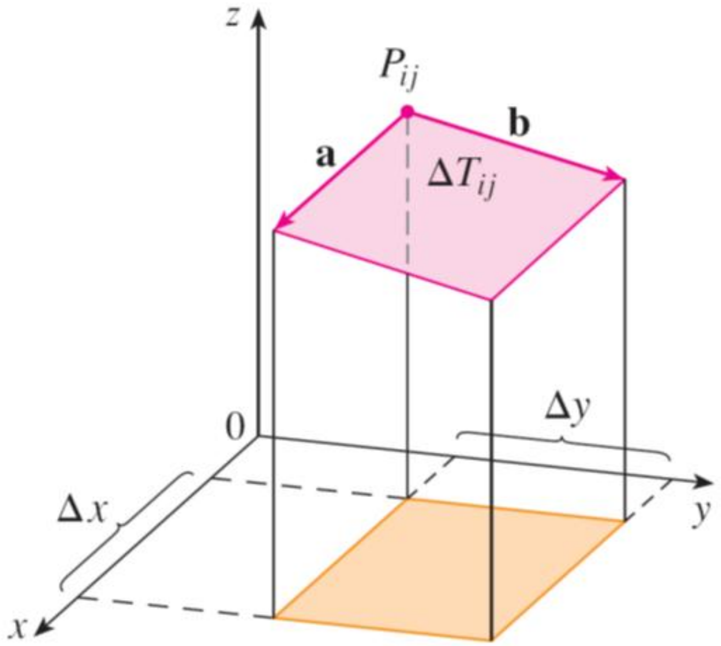


Figure 2

Surface Area (5 of 8)

Then $\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$. We know that $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the directions of **a** and **b**.

Therefore

$$\mathbf{a} = \Delta x\mathbf{i} + f_x(x_i, y_j)\,\Delta x\mathbf{k}$$

$$\mathbf{b} = \Delta y\mathbf{j} + f_y(x_i, y_j)\,\Delta y\mathbf{k}$$

and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\,\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\,\Delta y \end{vmatrix}$$

Surface Area (6 of 8)

$$= -f_x(x_i, y_j)\,\Delta x\,\Delta y\mathbf{i} - f_y(x_i, y_j)\,\Delta x\,\Delta y\mathbf{j} + \Delta x\,\Delta y\mathbf{k}$$

$$= [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A$$

Thus

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}| = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\,\Delta A$$

From Definition 1 we then have

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

Surface Area (7 of 8)

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x_i,\ y_j)]^2 + [f_y(x_i,\ y_j)]^2 + 1}\ \Delta A$$

and by the definition of a double integral we get the following formula.

2 The area of the surface with equation $z = f(x, y), (x, y) \in D,$ where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x,\ y)]^2 + [f_y(x,\ y)]^2 + 1}\ dA$$

Surface Area (8 of 8)

If we use the alternative notation for partial derivatives, we can rewrite Formula 2 as follows:

3 $$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Notice the similarity between the surface area formula in Equation 3 and the arc length formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 1

Find the surface area of the part of the surface $z = x^2 + 2y + 2$ that lies above the triangular region T in the xy-plane with vertices (0, 0), (1, 0), and (1, 1).

Solution:

The region T is shown in Figure 3 and is described by

$$T = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x\}$$

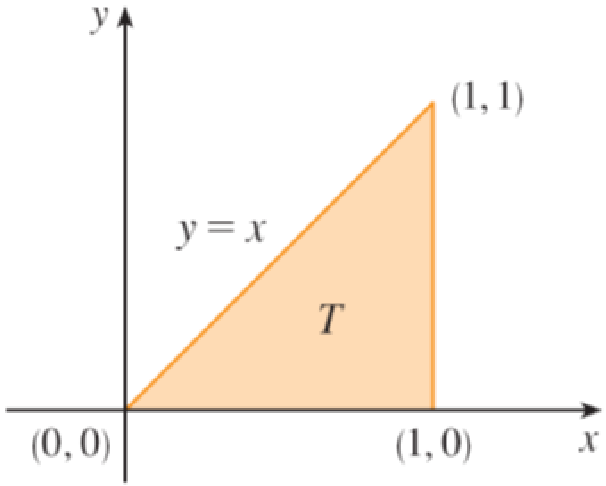


Figure 3

Example 1 – Solution (1 of 2)

Using Formula 2 with $f(x, y) = x^2 + 2y + 2$, we get

$$\begin{aligned}
A &= \iint_T \sqrt{(2x)^2 + (2)^2 + 1}\, dA \\
&= \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy\, dx \\
&= \int_0^1 x\sqrt{4x^2 + 5}\, dx \\
&= \tfrac{1}{8} \cdot \tfrac{2}{3}(4x^2 + 5)^{3/2} \Big]_0^1 \\
&= \tfrac{1}{12}(27 - 5\sqrt{5})
\end{aligned}$$

Example 1 – Solution (2 of 2)

Figure 4 shows the portion of the surface whose area we have just computed.

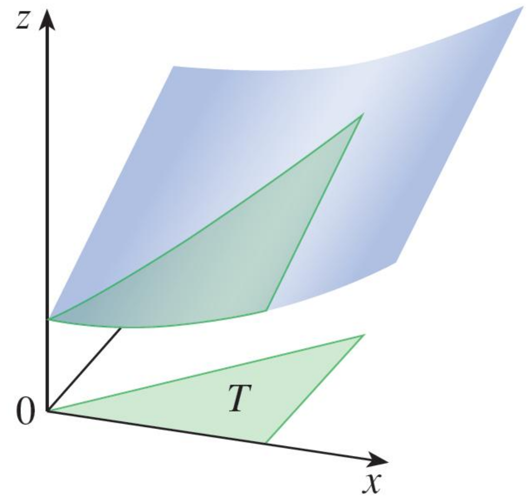


Figure 4

单选题 1分

设置

MCQ: Let $z = x^2 + y^2$ and let S be the part of this surface lying above the disk $D: x^2 + y^2 \leq 1$ in the xy-plane.

A π

B $\frac{\pi}{2}(1+\sqrt{5})$

C $\frac{\pi}{3}(5\sqrt{5}-1)$

D $\frac{\pi}{6}(5\sqrt{5}-1)$

提交

Solution

We have $f(x,y)=x^2+y^2,\quad f_x=\frac{\partial z}{\partial x}=2x,\quad f_y=\frac{\partial z}{\partial y}=2y$

Then $1+f_x^2+f_y^2=1+(2x)^2+(2y)^2=1+4x^2+4y^2$

Use polar coordinates:

$$x=r\cos\theta\,,\quad y=r\sin\theta\,,\quad x^2+y^2=r^2,\quad dA=rdrd\theta$$

The disk D is $0\ \le r\ \le 1,\quad 0\ \le \theta \le 2\pi$

So $\mathrm{Area}(S)=\iint_D \sqrt{1+4x^2+4y^2}\,dA=\int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\,rdrd\theta$

First compute the inner integral: $\int_0^1 r\sqrt{1+4r^2}\,dr=-\frac{1}{12}+\frac{5\sqrt{5}}{12}$

Then integrate with respect to θ:

$$\mathrm{Area}(S)=\int_0^{2\pi}\left(-\frac{1}{12}+\frac{5\sqrt{5}}{12}\right)d\theta=2\pi\left(-\frac{1}{12}+\frac{5\sqrt{5}}{12}\right)=\frac{\pi}{6}\left(5\sqrt{5}-1\right)$$

Therefore, the correct answer is: $\frac{\pi}{6}\left(5\sqrt{5}-1\right)$

Recap

- Surface Area