

Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

- a) The surface in \mathbb{R}^3 with equation $x^2y + y^2z + z^2x = 1$ is smooth.
- b) Suppose you start at the point $(1, 1)$ in the (x, y) -plane and follow (and continuously adjust) the direction of steepest ascent of $f(x, y) = \frac{x-y}{x+y}$. After some time (provided you won't get tired) you will cross the x -axis at a point $(x_0, 0)$ with $x_0 > 2$.
- c) If $f: [0, 1] \rightarrow \mathbb{R}$ is continuous and $f(0) = 0$ then $\lim_{n \rightarrow \infty} \int_0^1 f(x^n) dx = 0$.
- d) The equation $x^2 + xy + y^2 = 3$ defines a circle, which is symmetric to the line $y = x$.
- e) There exists a subset D of the upper half plane $\{(x, y) \in \mathbb{R}^2; y > 0\}$ whose set of accumulation points is equal to the real axis (x -axis).
- f) If γ is a closed path in \mathbb{R}^3 satisfying $\int_{\gamma} x dy + y dz + z dx = 0$, we must have $\int_{\gamma} x dz + y dx + z dy = 0$ as well.

Question 2 (ca. 12 marks)

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = x^4 - x^3y - xy + y^2.$$

- a) Which obvious symmetry property does f have? What can you conclude from this about the graph and the contours of f ?
- b) Determine all critical points of f and their types.
Hint: There are 5 critical points.
- c) Does f have a global extremum?
- d) Determine the extrema of f on the unit square $Q = \{(x, y) \in \mathbb{R}^2; 0 \leq x, y \leq 1\}$.

Question 3 (ca. 12 marks)

The sphere $x^2 + y^2 + z^2 = 9$ intersects the surface $xy + yz + zx = 8$ in the first octant $O = \{(x, y, z) \in \mathbb{R}^3; x, y, z \geq 0\}$ in a curve C .

- a) Show that C has no points on the boundary of O .
- b) Show that there exist points on C with minimum, resp., maximum height (z -coordinate).
Hint: $(2, 2, 1) \in C$.
- c) Using the method of Lagrange multipliers on the interior of O , determine all those points.

Note: Don't forget to check for points on C where the Jacobi matrix of the vectorial constraint doesn't have full row rank. In fact there are no such points, but this requires a proof.

- d) At the point $(2, 2, 1)$ the curve C admits locally a parametrization $\gamma(x) = (x, h(x), k(x))$ with functions $h, k: (2 - \epsilon, 2 + \epsilon) \rightarrow \mathbb{R}$. Determine $h'(2)$ and $k'(2)$.

Question 4 (ca. 12 marks)

Consider the transformation

$$T(s, t, u) = (us \cos t, us \sin t, us + ut)$$

from the region $U = \{(s, t, u) \in \mathbb{R}^3; 0 < s < t < 2\pi, 0 < u < 1\}$ to the region $V = T(U) \subset \mathbb{R}^3$, and the “helicoid”

$$S = \{(s \cos t, s \sin t, s + t); 0 < s < t < 2\pi\}$$

bounding V from above.

- Show that $T: U \rightarrow V$ is a diffeomorphism (i.e., T is one-to-one, and both T and T^{-1} are differentiable).
- Determine the volume of V .
- Express the surface area of S as an ordinary 1-dimensional Riemann integral.