Problem1

Evaluate the triple integral using only geometric interpretation and symmetry.

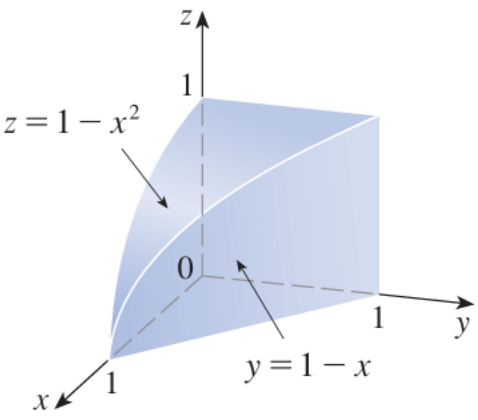
$\iiint_B (z^3 + \sin y + 3)\, dV$, where $B$ is the unit ball $x^2 + y^2 + z^2 \leqslant 1$

Problem2

The figure shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.




Problem3

$\iiint_E z\, dV$, where $E$ is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant

Problem1

**Ex. 42** The solution is similar to Ch.15.3, Ex.66. Using linearity, we can express the integral in terms of those with integrands $(x,y,z) \mapsto z^3$, $(x,y,z) \mapsto \sin y$, and $(x,y,z) \mapsto 1$. Since $z^3$ is odd w.r.t. $z$ and $\sin y$ is odd w.r.t. $y$, the first two integrals are zero. Hence we obtain

$$\iiint_B (z^3 + \sin y + 3)\,\mathrm{dV} = 3\iiint_B 1\,\mathrm{dV} = 3\,\mathrm{vol}(B) = 4\pi$$

Problem2

**Ex. 38** Analytically, the corresponding region is

$$\left\{(x,y,z) \in \mathbb{R}^3; 0 \le x \le 1, 0 \le y \le 1-x, 0 \le z \le 1-x^2\right\}.$$

There are a total of 6 different orders. In what follows we denote the given integral by $\mathbf{I}$.
If we integrate w.r.t. $y$ first, we have

$$\begin{aligned}\mathbf{I} &= \int_{x=0}^{1}\int_{z=0}^{1-x^2}\int_{y=0}^{1-x} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\\ &= \int_{z=0}^{1}\int_{x=0}^{\sqrt{1-z}}\int_{y=0}^{1-x} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z.\end{aligned}$$

If we integrate w.r.t. $z$ first, we have

$$\begin{aligned}\mathbf{I} &= \int_{y=0}^{1}\int_{x=0}^{1-y}\int_{z=0}^{1-x^2} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x^2} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Finally, if we integrate w.r.t. $x$ first, we have for $x$ the two conditions $x \le 1-y$, $x \le \sqrt{1-z}$, which are equivalent to $x \le \min\left\{1-y, \sqrt{1-z}\right\}$. Hence

$$\begin{aligned}\mathbf{I} &= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{\min\left\{1-y,\sqrt{1-z}\right\}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &= \int_{y=0}^{1}\int_{z=0}^{1}\int_{x=0}^{\min\left\{1-y,\sqrt{1-z}\right\}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y\end{aligned}$$

We can make the representation “min-free” by splitting the middle interval into two intervals. The minimum is equal to $1-y$ if $1-y \le \sqrt{1-z}$, and to $\sqrt{1-z}$ otherwise. Solving $1-y \le \sqrt{1-z}$ for $y,z$ gives $y \ge 1-\sqrt{1-z}$ and $z \le 1-(1-y)^2 = 2y-y^2$, respectively, so that

$$\begin{aligned}\mathbf{I} &= \left[\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{\sqrt{1-z}} + \int_0^1\int_{1-\sqrt{1-z}}^{1}\int_0^{1-y}\right] f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &= \left[\int_0^1\int_0^{2y-y^2}\int_0^{1-y} + \int_0^1\int_{2y-y^2}^{1}\int_0^{\sqrt{1-z}}\right] f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y.\end{aligned}$$

Problem3

**61 Ex. 22** The region $E$ is shown in Fig. 3 (with the part of $\partial E$ on the plane $y = 3x$ shaded). Analytically, the region is $E = \{(x,y,z) \in \mathbb{R}^3; 0 \le x \le 1; 3x \le y \le 3; 0 \le z \le \sqrt{9-y^2}\}$.

$$\begin{aligned}
\iiint\limits_E z\,\mathrm{dV} &= \int_{x=0}^{1}\int_{y=3x}^{3}\int_{z=0}^{\sqrt{9-y^2}} z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{x=0}^{1}\int_{y=3x}^{3} \frac{1}{2}(9-y^2)\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\int_{x=0}^{1}\left[9y - \tfrac{1}{3}y^3\right]_{y=3x}^{3}\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 (18 - 27x + 9x^3)\,\mathrm{d}x \\
&= \frac{1}{2}\left(18 - \frac{27}{2} + \frac{9}{4}\right) \\
&= \frac{27}{8}
\end{aligned}$$



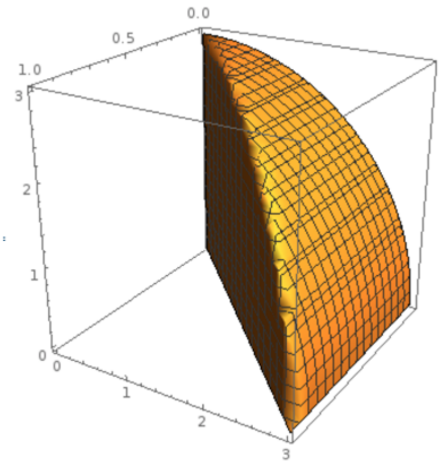



Figure 3: The region of Exercise 22