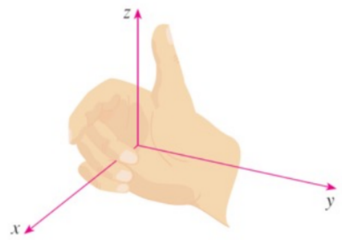
Vectors and the Geometry of Space

1 Three-Dimensional Coordinate Systems

1.1 3D Space

In order to represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate aces** and labeled the x-axis, y-axis, and z-axis.

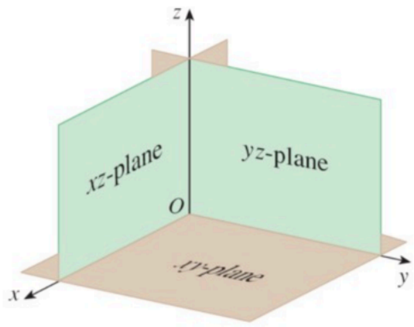
The direction of the z-axis is determined by the **right-hand rule** as illustrated in the figure:



Right-hand rule

If you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

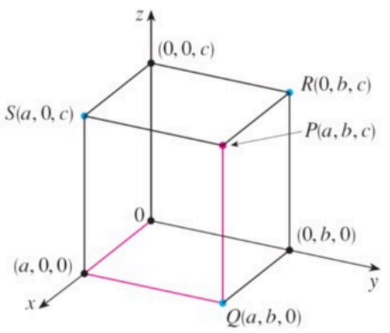
The three coordinate axes determined the three coordinate planes illustrated in the figure.



Coordínate planes

These three coordinate planes divide space into 8 parts, called **octants**(卦限八分区). The **first octant**, in the foreground, is determined by the positive axes.

Thus, to locate the point (a, b, c), we can start at the origin O and move a units along the x-axis, then b units parallel to the y-axis, and then c units parallel to the z-axis. The point $P(a, b, c)$ determins a rectangular box. If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$ called the **projection** of P onto the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P onto the yz-plane and xz-plane, respectively.



The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$.

We have given a one-to-one correspondence between points P in space and an ordered triples (a, b, c) in $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**. Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

1.2 Surfaces and Solids

In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In three-dimensional analytic geometry, an equation in x, y, z represents a surface in $\mathbb{R}^3$

In general, if k is a constant, then $x = k$ represents a plane parallel to the yz-plane, $y = k$ is a plane parallel to the xz-plane, and $z = k$ is a plane parallel tothe xy-plane.

1.3 Distance and Spheres

Distance Formula in Three Dimensions: The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\begin{aligned} |P_1P_2| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(P_1 - P_2)^T (P_1 - P_2)} \end{aligned} \tag{1}$$

Equation of a Sphere: An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \tag{2}$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2 \tag{3}$$

2 Vectors

The term **vector** is used in mathematics to indicate a quantity that has both magnitude and direction.

2.1 Geometric Description of Vectors

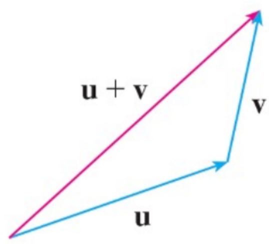
A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. We denote a vector by printing a letter in boldface $\boldsymbol{v}$ or by putting an arrow above the letter $\vec{v}$.

For instance, suppose a particle moves along a line segment from point A to point B. The corresponding **displace ment vector** $\boldsymbol{v}$ has **initial point** A and **terminal point** B and we indicate this by writing $\boldsymbol{v} = \vec{A}B$

We say that $\boldsymbol{u}$ and $\boldsymbol{v}$ are **equivalent** (or **equal**) and we write $\boldsymbol{u} = \boldsymbol{v}$

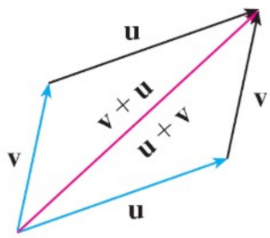
The **zero vector**, denote by **0**, has length 0. It is the only vector with no specific direction.

Definition of Vector Addition: If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors positioned so the initial point of $\boldsymbol{v}$ is at the terminal point of $\boldsymbol{u}$, then the **sum** $\boldsymbol{u} + \boldsymbol{v}$ is the vector from the initial point of $\boldsymbol{u}$ to the bermiinal point of $\boldsymbol{v}$.



The Triangle Law

Parallelogram Law: If we place $\boldsymbol{u}$ and $\boldsymbol{v}$ so they start at the same point, then $\boldsymbol{u} + \boldsymbol{v}$ lies along the diagonal of the parallelogram with $\boldsymbol{u}$ and $\boldsymbol{v}$ as sides.



The Parallelogram Law

Definition of Scalar Multiplication: If c is a scalar and $\boldsymbol{v}$ is a vector, then the **scalar multiple** $c\boldsymbol{v}$ is the vector whose length is $|c|$ times the length of $\boldsymbol{v}$ and whose direction is the same as $\boldsymbol{v}$ if $c > 0$ and is opposite to $\boldsymbol{v}$ if $c < 0$. If $c = 0$ or $\boldsymbol{v} = \mathbf{0}$, then $c\boldsymbol{v} = \mathbf{0}$.

2.2 Components of a Vector

For some purposes it's best to introduce a coordinate system and treat vectors algebraically.

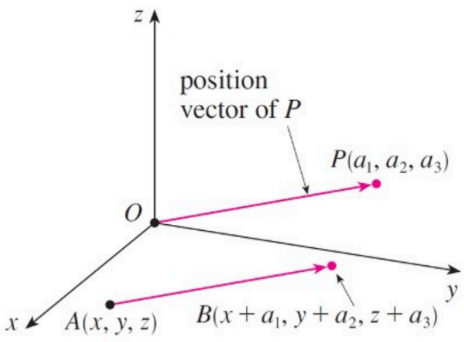
If we place the initial point of a vector $\boldsymbol{a}$ at the origin of a rectangular coordinate system, then the terminal point of a has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether coordinate system is 2 or 3 dimensional.

These coordinates are called the **components** of $\boldsymbol{a}$ and we write

$$\boldsymbol{a} = \langle a_1, a_2 \rangle \quad \text{or} \quad \boldsymbol{a} = \langle a_1, a_2, a_3 \rangle \tag{4}$$

We use the notation $\langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

In three dimensions, the vector $\boldsymbol{a} = \vec{OP} = \langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$.



Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\boldsymbol{a}$ with representation $\vec{AB}$ is

$$\boldsymbol{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \tag{5}$$

The **magnitude** or **length** of the vector $\boldsymbol{v}$ is the length of any of itsrepresentations and is denoted by the symbol $|\boldsymbol{v}|$ or $\|\boldsymbol{v}\|$. By using the distance formula to compute the length of a segment OP, we obtain the following formulas.

The length of the n-dimensional vector $\boldsymbol{a} = \langle a_1, \ldots, a_n \rangle$ is

$$|\boldsymbol{a}| = \sqrt{\sum_{i=1}^{n} a_i^2} \tag{6}$$

We denote by V_2, the set of all two-dimensional vectors and by V_3, the set of allthree-dimensiona vectors. More generally, we will consider the set V_n, of all n-dimensional vectors.

Three vectors in V_3 play a special role. Let

$$\boldsymbol{i} = \langle 1, 0, 0 \rangle \quad \boldsymbol{j} = \langle 0, 1, 0 \rangle \quad \boldsymbol{k} = \langle 0, 0, 1 \rangle \tag{7}$$

These vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are called the standard basis vectors. They have length 1 and point in the directions of the positive x, y, z-axes.

A **unit vector** is a vector whose length is 1. For instance, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are all unit vectors. In general, if $\boldsymbol{a} \neq 0$, then the unit vector that has the same direction as $\boldsymbol{a}$ is

$$\boldsymbol{u} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \tag{8}$$

In order to verify this, we let $c = \frac{1}{|\boldsymbol{a}|}$. Then $\boldsymbol{u} = c\boldsymbol{a}$ and c is a positive scalar, so $\boldsymbol{u}$ has the same direction as $\boldsymbol{a}$.

3 The Dot Product

3.1 Dot Product of Two Vectors

To find the dot product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ we multiply correspondingcomponents and add.

3.1.1 Definition

If $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the number $\boldsymbol{a} \cdot \boldsymbol{b}$ given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{9}$$

3.1.2 Properties

If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors in V_3 and c is a scalar, then

$$\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{a} &= |a|^2 \\
\boldsymbol{a} \cdot \boldsymbol{b} &= \boldsymbol{b} \cdot \boldsymbol{a} \\
\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) &= \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} \\
(c\boldsymbol{a}) \cdot (\boldsymbol{b}) &= c(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot (c\boldsymbol{b}) \\
\boldsymbol{0} \cdot \boldsymbol{a} &= 0
\end{aligned} \tag{10}$$

3.1.3 Theorem

If θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{b} &= |\boldsymbol{a}||\boldsymbol{b}| \cos\theta \\
\cos\theta &= \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}
\end{aligned} \tag{11}$$

3.1.4 Perpendicular or Orthogonal

Two nonzero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \frac{\pi}{2}$. Then the theorem gives

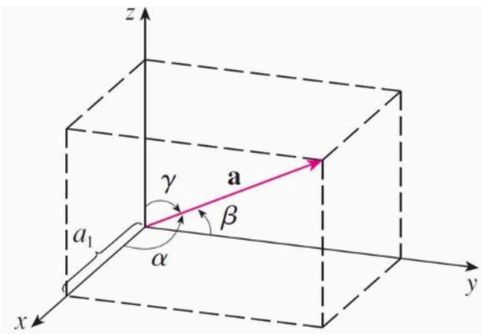
$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\left(\frac{\pi}{2}\right) = 0 \tag{12}$$

and conversely if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, then $\cos\theta = 0$, so $\theta = \frac{\pi}{2}$. The zero vector $\boldsymbol{0}$ is considered to be perpendicular to all vectors. Therefore we have the following method for determining whether two vectors are orthogonal:

$$\boldsymbol{a} \perp \boldsymbol{b} \equiv \boldsymbol{a} \cdot \boldsymbol{b} = 0 \tag{13}$$

3.2 Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector $\boldsymbol{a}$ are the angles α, β, γ (in the interval $[0, \pi]$) that $\boldsymbol{a}$ makes with the positive x, y, z-axes, respectively.

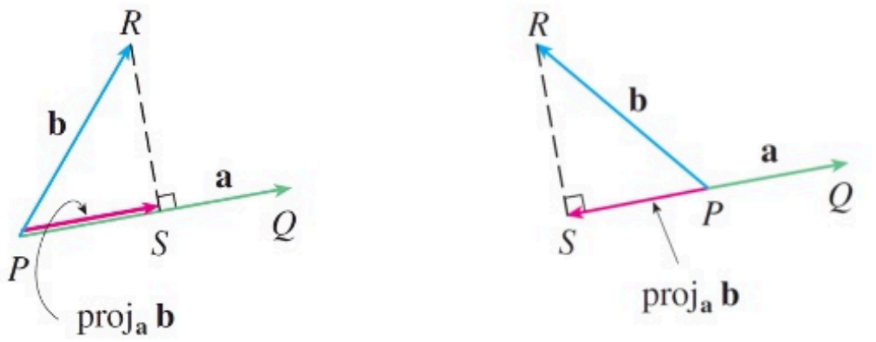


The cosines of these direction angles, $\cos\alpha, \cos\beta, \cos\gamma$ are called the **direction cosines** of the vector $\boldsymbol{a}$.

$$
\begin{aligned}
\cos\alpha &= \frac{\boldsymbol{a}\cdot\boldsymbol{i}}{|\boldsymbol{a}||\boldsymbol{i}|} = \frac{\boldsymbol{a}_1}{|\boldsymbol{a}|} \\
\cos\beta &= \frac{\boldsymbol{a}\cdot\boldsymbol{j}}{|\boldsymbol{a}||\boldsymbol{j}|} = \frac{\boldsymbol{a}_2}{|\boldsymbol{a}|} \\
\cos\gamma &= \frac{\boldsymbol{a}\cdot\boldsymbol{z}}{|\boldsymbol{a}||\boldsymbol{z}|} = \frac{\boldsymbol{a}_3}{|\boldsymbol{a}|} \\
\cos^2\alpha &+ \cos^2\beta + \cos^2\gamma = 1
\end{aligned}
\tag{14}
$$

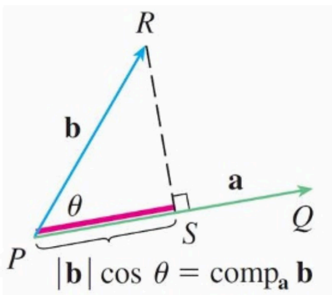
3.3 Projections

If S is the foot of the perpendicular from R to the line containing $\vec{PQ}$, then the vector with representation $\vec{PS}$ is called the **vector projection** of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and is denoted by $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}$



Vector projections

The **scalar projection** of $\boldsymbol{b}$ onto $\boldsymbol{a}$ (also called the **component of $\boldsymbol{b}$ along $\boldsymbol{a}$**) is defined to be the signed magnitude of the vector projection, which is the number $|\boldsymbol{b}|\cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. This is denoted by $\text{comp}_{\boldsymbol{a}}\boldsymbol{b}$. Observe that it's negative if $\frac{\pi}{2} < \theta \le \pi$.



Scalar projection

The equation

$$
\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = |\boldsymbol{a}|(|\boldsymbol{b}|\cos\theta) \tag{15}
$$

Shows that the dot product of $\boldsymbol{a}$ and $\boldsymbol{b}$ can be interpreted as the length of $\boldsymbol{a}$ times the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$. Since

$$
|\boldsymbol{b}|\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}\cdot\boldsymbol{b} \tag{16}
$$

the component of $\boldsymbol{b}$ along $\boldsymbol{a}$ can be computed by taking the dot product of $\boldsymbol{b}$ with the unit vector in the direction of $\boldsymbol{a}$

We summarize these ideas as follows:

- Scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$: $\text{comp}_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}$
- Vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$: $\text{proj}_{\boldsymbol{a}}\boldsymbol{b} = (\text{comp}_{\boldsymbol{a}}\boldsymbol{b})\frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{a}$

4 Cross Product (Vector Product)

4.1 Cross Product of 2 Vectors

Definition of the Cross Product: If $\boldsymbol{a} = \langle a_1, a_2, a_3\rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3\rangle$, then the **cross product** of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector

$$\boldsymbol{a} \times \boldsymbol{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle \tag{17}$$

Notice that the **cross product** $\boldsymbol{a} \times \boldsymbol{b}$ of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector. For this reason it's also called the **vector product**.

A **determinant of order 3** can be defined in terms of second-order determinants

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} a_2 & a_3 \\ b_12 & b_3 \end{vmatrix}\boldsymbol{k} \tag{18}$$

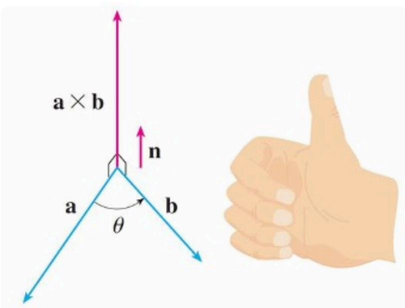
That is the cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

4.2 Properties of the Cross Product

4.2.1 Orthogonality

The vector $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$. This can be easily proved by dot product.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are represented by directed line segments with the same initial point, then the cross product $\boldsymbol{a} \times \boldsymbol{b}$ points in a direction perpendicular to the plane through $\boldsymbol{a}$ and $\boldsymbol{b}$.



The right-hand rule gives the direction of **a** × **b**.

4.2.2 Magnitude

If θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ (so $0 \le \theta \le \pi$), then the length of the cross product $\boldsymbol{a} \times \boldsymbol{b}$ is given by

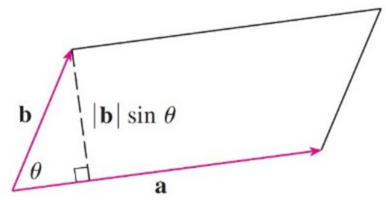
$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta \tag{19}$$

This leads to a corollary: 2 nonzero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if

$$\boldsymbol{a} \times \boldsymbol{b} = \mathbf{0} \tag{20}$$

Since a vector is completely determined by its magnitude and direction, we can now say that for nonparallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a} \times \boldsymbol{b}$ is the vector that is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, whose orientation is determined by the right-hand rule, and whose length is $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$.

4.2.3 Geometric Interpretation



If $\boldsymbol{a}$ and $\boldsymbol{b}$ are represented by directed line segments with the same initial point, then they determined a parallelogram with base $|\boldsymbol{a}|$, altitude $|\boldsymbol{b}|\sin\theta$, and area

$$A = |\boldsymbol{a}|(|\boldsymbol{b}|\sin\theta) = |\boldsymbol{a} \times \boldsymbol{b}| \tag{21}$$

Thus we have the following way of interpreting the magnitude of a cross product. The length of the cross product $\boldsymbol{a} \times \boldsymbol{b}$ is equal to the area of the parallelogram determined by $\boldsymbol{a}$ and $\boldsymbol{b}$.

4.2.4 Others

If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors and c is a scalar, then

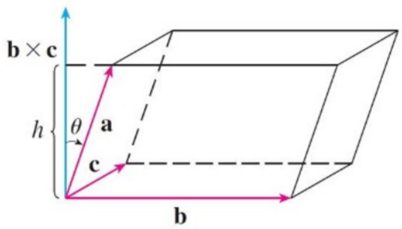
$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= -\boldsymbol{b} \times \boldsymbol{a} \\
(c\boldsymbol{a}) \times \boldsymbol{b} &= c(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \times (c\boldsymbol{b}) \\
\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) &= \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c} \\
(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} &= \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c} \\
\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) &= (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} \\
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}
\end{aligned} \tag{22}$$

4.3 Triple Products

The product $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ is called the **scalar triple product** of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. We can write the scalar triple product as a determinant:

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{23}$$

The geometric significance of the scalar triple product can be seen by considering the parallelpiped determined by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.



If θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b} \times \boldsymbol{c}$, then the height h of the parallelepiped is $h = |\boldsymbol{a}||\cos\theta|$. Therefore, the volume of the parallelepiped is

$$V = Ah = |\boldsymbol{b} \times \mathbf{c}||\boldsymbol{a}||\cos\theta| = |\boldsymbol{a} \cdot (\boldsymbol{b} \times \mathbf{c})| \tag{24}$$

If the volume of the parallelepiped determined by $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

5 Equations of Lines and Planes

5.1 Lines

A line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L, which is conveniently described by a vector $\boldsymbol{v}$ parallel to the line. Let $P(x, y, z)$ be an arbitrary point on L and let $\boldsymbol{r}_0$ and $\boldsymbol{r}$ be the position vectors of P_0 and P. If $\boldsymbol{a}$ is the vector with representation $\vec{P_0P}$, then the Triangle Law for vector addition gives $\boldsymbol{r} = \boldsymbol{r_0} + \boldsymbol{a}$. Since $\boldsymbol{a}$ and $\boldsymbol{v}$ are parallel vectors, there is a scalar t such that $\boldsymbol{a} = t\boldsymbol{v}$. Thus

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v} \tag{25}$$

which is a **vector equation** of L. Each value of the **parameter** t gives the position vector $\boldsymbol{r}$ of a point on L. In other words, as t varies, the line is traced out by the tip of the vector $\boldsymbol{r}$.

Therefore we have the three scalar equations

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct \tag{26}$$

These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\boldsymbol{v} = \langle a, b, c\rangle$. Each value of the parameter t gives a point (x, y, z) on L.

In general, if a vector $\boldsymbol{v} = \langle a, b, c\rangle$ is used to describe the direction of a line L, then the numbers a, b, c are called **direction numbers** of L.

If none of a, b, c is 0, we can solve each of these equations for t:

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{27}$$

These equations are called **symmetric equations** of L.

If one of a, b, c is 0, we can still eliminate t. For instance, if $a = 0$, we could write the equation of L as

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{28}$$

This means that L lies in the vertical plane $x = x_0$.

5.2 Planes

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\boldsymbol{n}$ that is orthogonal to the plane. This orthogonal vector $\boldsymbol{n}$ is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\boldsymbol{r}_0$ and $\boldsymbol{r}$ be the position vectors of P_0 and P. Then the vector $\boldsymbol{r} - \boldsymbol{r}_0$ is represented by $\vec{P_0 P}$. The normal vector $\boldsymbol{n}$ is orthogonal to every vector in the given plane. In particular, $\boldsymbol{n}$ is orthogonal to $\boldsymbol{r} - \boldsymbol{r}_0$ and so we have

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0 \tag{29}$$

which can be written as

$$\boldsymbol{n} \cdot \boldsymbol{r} = \boldsymbol{n} \cdot \boldsymbol{r}_0 \tag{30}$$

This is called a **vector equation of the plane**.

To obtain a scalar for the plane, we write

$$\boldsymbol{n} = \langle a, b, c \rangle, \quad \boldsymbol{r} = \langle x, y, z \rangle, \quad \boldsymbol{r}_0 = \langle x_0, y_0, z_0 \rangle \tag{31}$$

Then the vector equation becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \tag{32}$$

A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\boldsymbol{n} = \langle z, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{33}$$

We can rewrite the equation of a plane as

$$ax + by + cz + d = 0 \tag{34}$$

where $d = -(ax_0 + by_0 + cz_0)$. This is called a **linear equation** in x, y, z. Conversely, it can bes hown that if a, b, c are not all 0, then the linear equation represents a plane with normal vector $\langle a, b, c \rangle$.

Two planes are **parallel** if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.

5.3 Distances

Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$. Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let $\boldsymbol{b}$ be the vector corresponding to $\vec{P_0 P_1}$. Then

$$\boldsymbol{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle \tag{35}$$

The distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\boldsymbol{b}$ onto the normal vector $\boldsymbol{n} = \langle a, b, c \rangle$. Thus

$$
\begin{aligned}
D = |\mathrm{comp}_{\boldsymbol{n}}\boldsymbol{b}| &= \frac{\boldsymbol{n} \cdot \boldsymbol{b}}{|\boldsymbol{n}|} \\
&= \frac{|a(x_1 - x_0) + b(x_1 - x_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)}{\sqrt{a^2 + b^2 + c^2}}
\end{aligned} \tag{36}
$$

Since P_0 lies in the plane, its coordinates satisfy the equation of the plane and so we have

$$
ax_0 + by_0 + cz_0 + d = 0 \tag{37}
$$

Thus the distance D from the point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$
D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{38}
$$

6 Cylinders and Quadric Surfaces

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface.

6.1 Cylinder

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

If one of the variables x, y, z is missing from the equation of a surface, then the surface is a cylinder. When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ represents a cylinder and not a circle. The trace of the cylinder $x^2 + y^2 = 1$ in the xy-plane is the circle with equations $x^2 + y^2 = 1, z = 0$.

6.2 Quadric Surfaces

A **quadric surface** is the graph of a second-degree equation in three variables x, y, z. The most general such equation is

$$
Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0 \tag{39}
$$

where $A, B, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$
Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0 \tag{40}
$$

Quadric surfaces are the counterparts in three dimensions of the conic sections in the plane.

The idea of using traces to draw a surface is employed in three-dimensional graphing software. In most such software, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k.

All surfaces are symmetric with respect to the z-axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.