Solutions

1 a) True. The surface, call it S_A, is the 1-level set of $g_A(x,y,z)=x^3+y^3+z^3+A\,xyz$, which has gradient $\nabla g_A(x,y,z)=(3x^2+A\,yz,3y^2+A\,xz,3z^2+A\,xy)$.

For the proof suppose $(x,y,z)\in S_A$ satisfies $\nabla g(x,y,z)=(0,0,0)$. If $x=0$ then the 2nd and 3rd coordinate of $\nabla g(x,y,z)$ are $3y^2$ resp. $3z^2$, so that $y=z=0$ as well. But $(0,0,0)\notin S_A$, contradiction. Thus $x\neq 0$ and, by symmetry, $y\neq 0$ and $z\neq 0$. Further, $27x^2y^2z^2=(3x^2)(3y^2)(3z^2)=(-Ayz)(-Axz)(-Axy)=-A^3x^2y^2z^2$, which together with $xyz\neq 0$ gives $A=-3$ as the only possible exception. In this case $\nabla g(x,y,z)=(0,0,0)$ reduces to $x^2=yz$, $y^2=xz$, $z^2=xy$. From the 1st equation, $z=x^2/y$. Substituting this into the 2nd equation, $y^2=x^3/y$ and hence $x^3=y^3$, $x=y$. Then, by symmetry, $x=y=z$. But $g_{-3}(x,x,x)=x^3+x^3+x^3-3x^3=0$, and hence $(x,x,x)\notin S_{-3}$. This is the final contradiction. $\boxed{2}$

b) True. If f is the all-zero function, the statement is trivially true. Otherwise we may suppose w.l.o.g. that $f(x_0,y_0)>0$ for some $(x_0,y_0)\in\mathbb{R}^2$. By assumption, there exists $R>0$ such that $f(x,y)<f(x_0,y_0)$ for all points (x,y) with $|(x,y)|>R$. Since f is continuous, f attains a maximum on the closed disk $\mathrm{B}_R(0,0)$, say in (x_1,y_1). Since $(x_0,y_0)\in\mathrm{B}_R(0,0)$, we obtain $f(x,y)<f(x_0,y_0)\leq f(x_1,y_1)$ for all points (x,y) outside $\mathrm{B}_R(0,0)$. Thus the maximum in (x_1,y_1) is global.

Remark: It is not true that such a function f must have global extrema of both kinds, e.g., $f(x,y)=1/(1+x^2+y^2)$ has a global maximum but no global minimum.

c) False. We have

$$\nabla f(x,y)=\begin{pmatrix}y^2+2xy,\\ x^2+2xy\end{pmatrix},$$

$$\begin{vmatrix}y^2+2xy & x\\ x^2+2xy & y\end{vmatrix}=y^3+2xy^2-x^3-2x^2y=(y-x)(x^2+3xy+y^2).$$

Thus moving along the 2-contour from $(1,1)$ means moving in direction NW or SE (since $\nabla f(x,y)$ points to NE in the 1st quadrant and the coordinate axes, which are part of the 0-contour, cannot be reached). At a point on the 2-contour closest to $(0,0)$ (such a point exists by the usual continuity-compactness argument) the gradient $\nabla f(x,y)$ must be orthogonal to (x,y), which is the case only for points on the line $y=x$. But except for the starting point $(1,1)$, no such point can be reached. $\boxed{2}$

d) True. An example is

$$f(x,y)=(x+y-1)(y-x)(y-2x)\cdots(y-2023\,x).$$

For $m\in\{1,2,\dots,2023\}$ the intersection point of the lines $x+y=1$ and $y=mx$, viz. $\left(\frac{1}{m+1},\frac{m}{m+1}\right)$, is a saddle point of f. In order to see this, with m fixed it suffices to consider $g(x,y)=(x+y-1)(y-mx)=y^2-mx^2+(1-m)xy-y+mx$ instead.

$$\begin{aligned}g_x&=-2mx+(1-m)y+m,\\ g_y&=2y+(1-m)x-1,\\ g_{xx}&=-2m,\\ g_{xy}&=1-m=g_{yx},\\ g_{yy}&=2.\end{aligned}$$

One finds that $\nabla g\left(\frac{1}{m+1},\frac{m}{m+1}\right) = (0,0)$ (this also follows from the fact that the 0-contour of g or f is not smooth there), and $\det \mathbf{H}_g(x,y) = -4m-(1-m)^2 < 0$. $\boxed{2}$

e) True. Denote this set by S, and let $S_0 = S\cap[0,1)$. Among the $10^k - 10^{k-1} = 9\cdot 10^{k-1}$ positive integers with exactly k decimal digits, 9^k don't involve the digit 0. Scaling by 10^{-k}, the set of real numbers in $[0,1)$ not involving the digit 0 in the first k digits after the decimal point has Lebesgue measure at most $9^k/(9\cdot 10^{k-1}) = \left(\frac{9}{10}\right)^{k-1}$. Since $\left(\frac{9}{10}\right)^{k-1}\to 0$ for $k\to\infty$, we can conclude that S_0 has Lebesgue measure zero; cf. the corresponding argument for Cantor's Ternary Set. But then S, which is contained in a countable union of translates of S_0, must have Lebesgue measure zero as well. $\boxed{2}$

f) True. Using linearity of the line integral $\int_\gamma \omega$ as a function of ω, the equation can be rewritten as

$$\int_\gamma \left(ax+by+\frac{c-b}{2}\,y\right)dx + \left(cx+dy-\frac{c-b}{2}\,x\right)\mathrm{d}y = 0$$
$$\iff \quad \int \left(ax+\frac{b+c}{2}\,y\right)\mathrm{d}x + \left(\frac{b+c}{2}\,x+dy\right)\mathrm{d}y = 0.$$

Denoting the latter integrand by $\omega = M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y$, we have $M_y = \frac{b+c}{2} = N_x$, i.e., ω is exact in $\mathbb{R}^2$ and hence $\int_\gamma \omega = 0$. $\boxed{2}$

Remarks: No marks were assigned for answers without justification.

a) Most students answered this question correctly. Perhaps the shortest answer, offered by several students, is that $\nabla g(x,y,z) = \mathbf{0}$ implies $x\,g_x + y\,g_y + z\,gz = 3\,(x^3+y^3+z^3+Axyz) = 0$, which contradicts $(x,y,z)\in S_A$. A few students concluded that $\nabla g_A(x,y,z) = \mathbf{0}$ implies $x=y=z=0$, which is wrong (consider $A=-3$).

b) Only a few students found the correct answer, which inevitably must take into account functions who assume only positive (or negative) values.

c) A few students showed directly that the 2-contour of f contains no points (x,y) with $x^2+y^2<2$. A few others used Lagrange multipliers to minimize x^2+y^2 under the constraint $xy^2+x^2y=2$, in which case I have insisted on a detailed argument showing that $(x,y)=(1,1)$ is the only solution. The observation that gradients of f point to NE in the 1st quadrant is definitely not enough to conclude the truth of the statement.

d) A few students argued that from a function with one saddle point, which certainly exists, one can construct a function with infinitely many saddle points by "periodic repetition". This wasn't accepted as answer unless an explicit construction was given. (Additive constructions can only work if the supports of the summands are mutually disjoint.) Most students tried to find explicit examples. In this case, generally I have insisted on a proof that the points offered are indeed saddle points, which requires $\nabla g(x,y) = (0,0) \wedge \det \mathbf{H}_g(x,y) < 0$. A typical example with infinitely many saddle points is $g(x,y) = \sin x + \sin y$ but, since not all critical points of this function are saddle points, a detailed proof identifying the saddle points was required.

e) This (admittedly extremely difficult) question wasn't solved completely by any student. Two students had good ideas towards the proof, for which 1 mark was assigned.

f) Several students observed that the left-hand side of the equation is equal to $\int_\gamma by\,\mathrm{d}x + cx\,\mathrm{d}y$, since $ax\,\mathrm{d}x + dy\,\mathrm{d}y$ is exact. This simplifies the argument. Some students invoked Green's Theorem for the proof, in which case I normally subtracted some marks, because the boundary curves in Green's Theorem are more special than arbitrary closed curves.

$$\sum_1 = 12$$

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2 a) $f(-x,-y) = f(x,y) = f(y,x)$ for $(x,y) \in \mathbb{R}^2$ $\boxed{1}$
$\Longrightarrow$ G_f is symmetric with respect to the z-axis and the plane $x = y$. $\boxed{1}$
Alternatively, $f(y,x)) = f(x,y) = f(-y,-x)$ for $(x,y) \in \mathbb{R}^2$, which says that G_f is symmetric with respect to the two planes $x = \pm y$ (and implies the symmetry with respect to the z-axis).
If (x_0, y_0) is a critical point of f, so are $(-x_0,-y_0)$, (y_0,x_0), and $(-y_0,-x_0)$, and all have the same type. $\boxed{1}$

b) Using the shorthands f, f_x, f_y for $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, we compute

$$\begin{aligned}
f &= x^4 + y^4 - 6\,x^2 - 4\,xy - 6\,y^2 \\
f_x &= 4\,x^3 - 12\,x - 4\,y \\
f_y &= 4\,y^3 - 12\,y - 4\,x, \\
f_x + f_y &= 4(x^3+y^3) - 16\,x - 16\,y = 4(x+y)\left(x^2 - xy + y^2 - 4\right), \\
f_x - f_y &= 4(x^3-y^3) - 8x + 8y = 4(x-y)\left(x^2 + xy + y^2 - 2\right).
\end{aligned}$$

Then $\nabla f(x,y) = (0,0)$ if 2 of the 4 functions $f_x, f_y, f_x+f_y, f_x-f_y$ vanish at (x,y).

Clearly $\mathbf{p}_0 = (0,0)$ is critical. $\boxed{\frac{1}{2}}$
Assuming $(x,y) \neq (0,0)$, we distinguish three mutually exclusive cases:
Case 1: $x = y$ Here $f_x + f_y = 0$ gives $x^2 - x^2 + x^2 = 4$, and hence $x = \pm 2$. This yields the two critical points $\mathbf{p}_1 = (2,2)$, $\mathbf{p}_2 = (-2,-2)$. $\boxed{1}$
Case 2: $x = -y$ Here $f_x - f_y = 0$ gives $x^2 - x^2 + x^2 = 2$, and hence $x = \pm\sqrt{2}2$. This yields the two critical points $\mathbf{p}_3 = \left(\sqrt{2}, -\sqrt{2}\right)$, $\mathbf{p}_4 = \left(-\sqrt{2}, \sqrt{2}\right)$. $\boxed{1}$
Case 3: $x \neq \pm y$ Here we must have $x^2 - xy + y^2 = 4 \wedge x^2 + xy + y^2 = 2$. Adding/subtracting the two equations gives $2\,x^2 + 2\,y^2 = 6$, $-2xy = 2$, i.e., $x^2 + y^2 = 3 \wedge xy = -1$. $\Longrightarrow$ $x^2 + (-1/x)^2 = 3$, i.e., $x^4 - 3\,x^2 + 1 = 0$, $x^2 = \frac{1}{2}\left(3 \pm \sqrt{5}\right)$, $x = \pm\frac{1}{2}\left(1 \pm \sqrt{5}\right)$. This yields the four critical points

$$\mathbf{p}_5 = \left(\tfrac{1+\sqrt{5}}{2}, \tfrac{1-\sqrt{5}}{2}\right), \quad \mathbf{p}_6 = \left(\tfrac{1-\sqrt{5}}{2}, \tfrac{1+\sqrt{5}}{2}\right), \quad \mathbf{p}_7 = \left(\tfrac{-1+\sqrt{5}}{2}, \tfrac{-1-\sqrt{5}}{2}\right), \quad \mathbf{p}_8 = \left(\tfrac{-1-\sqrt{5}}{2}, \tfrac{-1+\sqrt{5}}{2}\right).$$

$\boxed{2}$

When determining the types of the critical points, by a) we need only test $\mathbf{p}_0$ and one from each set $\{\mathbf{p}_1, \mathbf{p}_2\}$, $\{\mathbf{p}_3, \mathbf{p}_4\}$, $\{$

$vekp_5, \mathbf{p}_6, \mathbf{p}_7, \mathbf{p}_8\}$. We have

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12\,x^2 - 12 & -4 \\ -4 & 12\,y^2 - 12 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_0) = \begin{pmatrix} -12 & -4 \\ -4 & -12 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_1) = \begin{pmatrix} 36 & -4 \\ -4 & 36 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_3) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_5) = \begin{pmatrix} 6+6\sqrt{5} & -4 \\ -4 & 6-6\sqrt{5} \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_0)$ is negative definite (determinant > 0, top-left entry < 0), the point $\mathbf{p}_0$ is a strict local maximum. $\boxed{\frac{1}{2}}$

Since $\mathbf{H}_f(\mathbf{p}_1)$, $\mathbf{H}_f(\mathbf{p}_3)$ are positive definite (determinant > 0, top-left entry > 0), the points $\mathbf{p}_1$, $\mathbf{p}_2, \mathbf{p}_3$, $\mathbf{p}_4$ are strict local minima. $\boxed{2}$

Since $\mathbf{H}_f(\mathbf{p}_5)$ is indefinite (determinant < 0), the points $\mathbf{p}_5$, $\mathbf{p}_6, \mathbf{p}_7$, $\mathbf{p}_8$ are saddle points. $\boxed{2}$

c) Yes. The points $\mathbf{p}_5$, $\mathbf{p}_6$, $\mathbf{p}_7$, $\mathbf{p}_8$ are global minima (with value $f(\mathbf{p}_i) = -7$). This follows from $\lim_{|(x,y)|\to\infty} f(x,y) = +\infty$ using an argument analogous to that in the solution to Question 1 b). Indeed, from $x^2 + y^2 \geq 2xy$ we have $6x^2 + 4xy + 6y^2 \leq 8(x^2 + y^2)$ and $x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 \geq \frac{1}{2}(x^2 + y^2)^2$, and hence

$$f(x,y) \geq \frac{1}{2}\,r^4 - 8\,r^2, \quad r = |(x,y)|\,.$$

This clearly implies $\lim_{|(x,y)|\to\infty} f(x,y) = +\infty$. $\boxed{1\frac{1}{2}}$

Since $f(x,0) = x^4 - 6\,x^2$ is unbounded from above, there is no global maximum. $\boxed{\frac{1}{2}}$

Remarks: In a) most students failed to discover one of the two symmetries.

In b) every critical point was worth 0.5 marks and its type another 0.5 marks. Most students obtained 9 marks in this way. It was accepted if you didn't simplify $\sqrt{\frac{3\pm\sqrt{5}}{2}}$ to $\frac{1\pm\sqrt{5}}{2}$, and similarly for other expressions obtained (provided I could verify that they are correct).

In c) a bonus mark was assigned for a rigorous proof that $f(2,2) = f(-2,-2) = -32$ are the global minima of f. The shortest proof of this fact uses the representation $f(x,y) = (x^2-4)^2 + (y^2-4)^2 + 2(x-y)^2 - 32$, from which the assertion follows immediately (well, almost immediately).

$$\sum_2 = 14$$

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3 The continuous function $f(x,y,z) = xy + 6yz + 6zx$ attains a maximum on the sphere $\mathrm{B}_{\sqrt{17}}(0,0,0)$, which is closed and bounded. This shows that the optimization problem has at least one solution. $\boxed{1}$

Setting $g(x,y,z) = x^2 + y^2 + z^2$, the task is to minimize f on $\mathbb{R}^3$ under the constraint $g(x,y,z) = 17$.

$$\nabla f(x,y,z) = (y + 6z, x + 6z, 6x + 6y), \qquad \nabla g(x,y,z) = (2x, 2y, 2z).$$

Since $\nabla g(x,y,z) \neq (0,0,0)$ for all points on the sphere $\mathrm{B}_{\sqrt{17}}(0,0,0)$, the theorem on Langrange multipliers yields that every optimal solution must satisfy $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for some $\lambda \in \mathbb{R}$. This gives the system of equations

$$\begin{aligned} y + 6z &= \lambda x, \\ x + 6z &= \lambda y, \\ 6x + 6y &= \lambda z, \\ x^2 + y^2 + z^2 &= 17. \end{aligned} \qquad \boxed{3}$$

(For simplicity we have replaced λ by $\lambda/2$.)

The solutions (x,y,z,λ) of this system are precisely the unit eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 6 \\ 1 & 0 & 6 \\ 6 & 6 & 0 \end{pmatrix}$$

together with the corresponding eigenvalues.

Since $\chi_{\mathbf{A}}(X) = X^3 - 73\,X - 72 = (X+1)(X^2 - X - 72) = (X+1)(X+8)(X-9)$, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 9$, $\lambda_2 = -1$, $\lambda_3 = -8$. This shows already that the eigenspaces of $\mathbf{A}$ are one-dimensional and that the above system has exactly 6 solutions. Next we compute the corresponding eigenvectors. Unit eigenvectors will be denoted by $\mathbf{u}_i$ and eigenvectors of length $\sqrt{17}$ by $\mathbf{v}_i$.

$\underline{\lambda_1 = 9:}$

$$\mathbf{A} - 9\,\mathbf{I} = \begin{pmatrix} -9 & 1 & 6 \\ 1 & -9 & 6 \\ 6 & 6 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & -9 & 6 \\ 0 & -80 & 60 \\ 0 & 60 & -45 \end{pmatrix} \to \begin{pmatrix} 1 & -9 & 6 \\ 0 & -4 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow \mathbf{u}_1 = \pm\frac{1}{\sqrt{34}}\,(3,3,4)^\mathsf{T}$, $\mathbf{v}_1 = \pm\frac{1}{\sqrt{2}}\,(3,3,4)^\mathsf{T}$; $\boxed{1}$

$\underline{\lambda_2 = -1:}$

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 1 & 1 & 6 \\ 1 & 1 & 6 \\ 6 & 6 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 6 \\ 0 & 0 & -35 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow \mathbf{u}_2 = \pm\frac{1}{\sqrt{2}}\,(1,-1,0)^\mathsf{T}$, $\mathbf{v}_2 = \pm\frac{\sqrt{17}}{\sqrt{2}}\,(1,-1,0)^\mathsf{T}$; $\boxed{1}$

$\underline{\lambda_3 = -8:}$

$$\mathbf{A} + 8\,\mathbf{I} = \begin{pmatrix} 8 & 1 & 6 \\ 1 & 8 & 6 \\ 6 & 6 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 8 & 6 \\ 0 & -63 & -42 \\ 0 & -42 & -28 \end{pmatrix} \to \begin{pmatrix} 1 & 8 & 6 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow \mathbf{u}_3 = \pm\frac{1}{\sqrt{17}}\,(2,2,-3)^\mathsf{T}$, $\mathbf{v}_3 = \pm(2,2,-3)^\mathsf{T}$. $\boxed{1}$

Since $f(3,3,4) = 153 > 0$, $f(1,-1,0) = -1 < 0$, $f(2,2,-3) = -68 < 0$, the maximum is attained at $(x^*,y^*,z^*) = \pm\frac{1}{\sqrt{2}}\,(3,3,4)^\mathsf{T}$, and $\zeta^* = 153/2$. $\boxed{2}$

Remarks: Many students forgot to check the condition $\nabla g(x,y,z) \neq \mathbf{0}$ for the applicability of the Lagrange Multiplier Method. That solving the resulting system of equations amounts to an eigenvalue/eigenvector determination for some 3×3 matrix wasn't recognized by most students, who then solved (in most cases successfully) the system in an adhoc fashion. If no intermediate steps of the computation were shown, 1 mark was subtracted for missing justification.

$$\sum_3 = 9$$

4 a) Writing $f(x,a) = \frac{1}{x^2+a^2}$, we have

$$\int_0^\infty f_a(x,a)\,\mathrm{d}x = \int_0^\infty \frac{-2a}{(x^2+a^2)^2}\,\mathrm{d}x\,. \qquad \boxed{1}$$

Since $x^2 + a^2 \geq 2xa$, we can bound the integrand as follows:

$$|f_a(x,a)| = \frac{2a}{(x^2+a^2)^2} \leq \frac{2a}{2xa(x^2+a^2)} = \frac{1}{x(x^2+a^2)} \leq \frac{1}{x(x^2+\delta^2)} =: \Phi(x), \qquad \boxed{2}$$

provided that $a \geq \delta > 0$. Since $\Phi(x)$ is independent of a and integrable over $[0,\infty)$, this shows that F is differentiable in (δ,∞) and $F'(a)$ can be obtained by differentation under the integral sign. Letting $\delta \downarrow 0$, we then obtain the assertion for the whole domain $(0,\infty)$. $\boxed{1}$

b) From a) we have

$$F'(a) = \int_0^\infty \frac{-2a}{(x^2+a^2)^2}\,\mathrm{d}x, \quad \text{i.e.,} \quad \int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = -\frac{F'(a)}{2a}.$$

On the other hand we have

$$\begin{aligned} F(a) &= \int_0^\infty \frac{\mathrm{d}x}{x^2+a^2} \\ &= a\int_0^\infty \frac{\mathrm{d}t}{a^2t^2+a^2} && \text{(Subst. } x = at,\ \mathrm{d}x = a\,\mathrm{d}t) \\ &= \frac{1}{a}\int_0^\infty \frac{\mathrm{d}t}{t^2+1} = \frac{1}{a}\left[\arctan t\right]_0^\infty = \frac{\pi}{2a}. && \boxed{1} \end{aligned}$$

It follows that

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = -\frac{-\pi/2a^2}{2a} = \frac{\pi}{4a^3}. \qquad \boxed{1}$$

Remarks: No student was able to solve a). A few students obtained some bound $\Phi(x)$ for $|f_a(x,a)|$ independent of a, but unfortunately $\Phi(x)$ in these cases wasn't integrable over $(0,+\infty)$. Some students argued with the continuity of $f_a(x,a)$ as a two-variable function, but this only helps if the domain of integration is a compact intervall.

For b) most students obtained a full score, sometimes by ignoring the hint and evaluating the integral in question directly.

$$\sum_4 = 7$$

5 a) The mass of K is

$$
\begin{aligned}
m &= \int_K xyz^3 \,\mathrm{d}^3(x,y,z) \\
&= \int\limits_{\substack{x^2+y^2\leq 16\\ x,y\geq 0}} \int_{z=0}^{2\sqrt{x}} xyz^3 \,\mathrm{d}z \,\mathrm{d}^2(x,y) \\
&= \int\limits_{\substack{x^2+y^2\leq 16\\ x,y\geq 0}} xy \left[\frac{z^4}{4}\right]_{z=0}^{2\sqrt{x}} \mathrm{d}^2(x,y) \\
&= \int\limits_{\substack{x^2+y^2\leq 16\\ x,y\geq 0}} 4\,x^3y \,\mathrm{d}^2(x,y) \\
&= 4 \int\limits_{\substack{0\leq r\leq 4\\ 0\leq\theta\leq\pi/2}} (r\cos\theta)^3 r\sin\theta\, r \,\mathrm{d}^2(r,\theta) \\
&= 4\int_0^4 r^5\,\mathrm{d}r \int_0^{\pi/2} \cos^3\theta\sin\theta\,\mathrm{d}\theta \\
&= \frac{4^7}{6}\left[-\frac{1}{4}\cos^4\theta\right]_0^{\pi/2} = \frac{4^6}{6} = \frac{2^{11}}{3} = \frac{2048}{3}.
\end{aligned}
$$

$\boxed{3}$

b) Denoting the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(2,0)$, $(0,2)$ by Δ, the surface P is the graph of $f(x,y) = x^{3/2} + y^{3/2}$, $(x,y) \in \Delta$. Using the formula for such surfaces, or going the long way using the parametrization $\gamma(x,y) = \big(x, y, f(x,y)\big)$, we obtain the surface area as

$$
\begin{aligned}
A &= \int_\Delta \sqrt{1+|\nabla f(x,y)|^2}\,\mathrm{d}^2(x,y) = \int_\Delta \sqrt{1+\left|\tfrac{3}{2}\left(\sqrt{x},\sqrt{y}\right)\right|^2}\,\mathrm{d}^2(x,y)\\
&= \int_\Delta \sqrt{1+\tfrac{9}{4}(x+y)}\,\mathrm{d}^2(x,y) && \boxed{1}\\
&= \frac{1}{2}\int_0^2\int_0^{2-x}\sqrt{4+9x+9y}\,\mathrm{d}y\,\mathrm{d}x\\
&= \frac{1}{2}\int_0^2\left[\frac{2}{27}(4+9x+9y)^{3/2}\right]_{y=0}^{2-x}\mathrm{d}x\\
&= \frac{1}{27}\int_0^2 22^{3/2}-(4+9x)^{3/2}\,\mathrm{d}x\\
&= \frac{1}{27}\left(2\cdot 22^{3/2}-\left[\frac{2}{45}(4+9x)^{5/2}\right]_0^2\right)\\
&= \frac{2}{27}\,22^{3/2}-\frac{2}{27\cdot 45}\left(22^{5/2}-4^{5/2}\right)\\
&= \frac{64+46\cdot 22\sqrt{22}}{27\cdot 45}\\
&= \frac{64+1012\sqrt{22}}{1215} && \boxed{3}
\end{aligned}
$$

Remarks: This question was generally answered well. In a) I have subtracted 1 mark for missing justification if no (or virtually no) intermediate steps of the computation were given.

In b) I have insisted on the final simplification of the figure, i.e., if you didn't state the result as $\frac{64+1012\sqrt{22}}{1215}$, 1 mark was subtracted.

$$\sum_5 = 7$$

$$\sum_{\text{Final Exam}} = 12+14+9+7+7 = 49 = 40+9$$