

# Vector Calculus

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## Vector Fields

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### Vector Field in $\mathbb{R}^2$ and $\mathbb{R}^3$

**Definition:** Let  $D$  be a set in  $\mathbb{R}^2$  (a plane region). A **vector field on  $\mathbb{R}^2$**  is a function  $\mathbf{F}$  that assigns to each point  $(x, y)$  in  $D$  a two-dimensional vector  $\mathbf{F}(x, y)$ .

The best way to picture a vector field is to draw the arrow representing the vector  $\mathbf{F}(x, y)$  starting at the point  $(x, y)$ . Since  $\mathbf{F}(x, y)$  is a two-dimensional vector, we can write it in terms of its **component functions**  $P$  and  $Q$  as follows:

$$\begin{aligned}\mathbf{F}(x, y) &= P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y) \rangle \\ \mathbf{F} &= Pi + Qj\end{aligned}$$

Notice that  $P$  and  $Q$  are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

**Definition:** Let  $E$  be a subset of  $\mathbb{R}^3$ . A **vector field on  $\mathbb{R}^3$**  is a function  $\mathbf{F}$  that assigns to each point  $(x, y, z)$  in  $E$  a three-dimensional vector  $\mathbf{F}(x, y, z)$ .

We can express it in terms of its component functions  $P, Q, R$  as

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

As with the vector functions, we can define continuity of vector fields and show that  $\mathbf{F}$  is continuous if and only if its component functions  $P, Q, R$  are continuous. We sometimes identify a point  $(x, y, z)$  with its position vector  $\mathbf{x} = \langle x, y, z \rangle$  and write  $\mathbf{F}(\mathbf{x})$  instead of  $\mathbf{F}(x, y, z)$ . Then  $\mathbf{F}$  becomes a function that assigns a vector  $\mathbf{F}(\mathbf{x})$  to a vector  $\mathbf{x}$ .

## Gradient Fields

If  $f$  is a scalar function of two variables, we know that its gradient  $\nabla f$  (or  $\text{grad } f$ ) is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

Therefore  $\nabla f$  is really a vector field on  $\mathbb{R}^2$  and is called a **gradient vector field**. Likewise, if  $f$  is a scalar function of three variables, its gradient is a vector field on  $\mathbb{R}^3$  given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

A vector field  $f$  is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function  $f$  such that  $\mathbf{F} = \nabla f$ . In this situation  $f$  is called a **potential function** for  $\mathbf{F}$ . Not all vector fields are conservative, but such fields do arise frequently in physics.

## Line Integrals

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In this section we define an integral that is similar to a single integral except that instead of integrating over an interval  $[a, b]$ , we integrate over a curve  $C$ . Such integrals are called line integrals, although "curve integrals" would be better terminology.

They were invented in the early 19th century to solve problems involving fluidflow, forces, electricity, and magnetism.

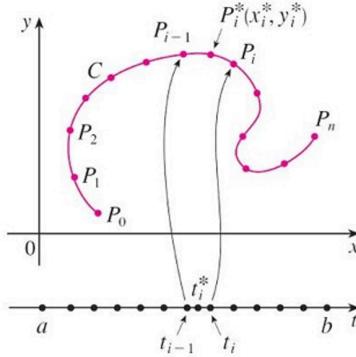
## Line Integrals in the Plane

We start with a plane curve  $C$  given by the parametric equations

$$x = x(t) \quad y = y(t) \quad a \leq t \leq b$$

or, equivalently, by the vector equation  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ , and we assume that  $C$  is a smooth curve.

If we divide the parameter interval  $[a, b]$  into  $n$  subintervals  $[t_{i-1}, t_i]$  of equal width and we let  $x_i = x(t_i)$ , and  $y_i = y(t_i)$ , then the corresponding points  $P_i(x_i, y_i)$  divide  $C$  into  $n$  subarcs with lengths  $\Delta s_1, \Delta s_2, \dots, \Delta s_n$ .



We choose any point  $P_i^*(x_i^*, y_i^*)$  in the  $i$ -th subarc. (This corresponds to a point  $t_j^*$  in  $[t_{j-1}, t_j]$ ). Now if  $f$  is any function of two variables whose domain includes the curve  $C$ , we evaluate  $f$  at the point  $(x_i^*, y_i^*)$ , multiply by the length  $\Delta s_i$  of the subarc, and form the sum

$$\sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

which is similar to a Riemann sum.

**Definition:** If  $f$  is defined on a smooth curve  $C$ , then the **line integral of  $f$  along  $C$**  is

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

if this limit exists.

We have found that the length of  $C$  is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

A similar type of argument can be used to show that if  $f$  is a continuous function, then the limit always exists and the following formula can be used to evaluate the line integral:

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as  $t$  increases from  $a$  to  $b$ .

Suppose now that  $C$  is a **piecewise-smooth curve**; that is,  $C$  is a union of a finite number of smooth curves  $C_1, C_2, \dots, C_n$ , where the initial point of  $C_{i+1}$  is the terminal point of  $C_i$ . Then we define the integral of  $f$  along  $C$  as the sum of the integrals of  $f$  along each of the smooth pieces of  $C$ :

$$\int_C f(x, y) ds = \sum_{i=1}^n \int_{C_i} f(x, y) ds$$

The **mass**  $m$  of the wire:

$$m = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho(x_i^*, y_i^*) \Delta s_i = \int_C \rho(x, y) ds$$

The **center of mass** of the wire with density function  $\rho$  is located at the point  $(\bar{x}, \bar{y})$ , where

$$\bar{x} = \frac{1}{m} \int_C x \rho(x, y) ds$$

$$\bar{y} = \frac{1}{m} \int_C y \rho(x, y) ds$$

## Line Integrals with Respect to $x$ or $y$

The **line integrals of  $f$  along  $C$  with respect to  $x$  and  $y$** :

$$\int_C f(x, y) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta x_i = \int_a^b f(x(t), y(t)) x'(t) dt$$

$$\int_C f(x, y) dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta y_i = \int_a^b f(x(t), y(t)) y'(t) dt$$

In particular, we often need to parametrize a line segment, so it's useful to remember that a vector representation of the line segment that starts at  $\mathbf{r}_0$  and ends at  $\mathbf{r}_1$  is given by

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

In general, a given parametrization  $x = x(t)$ ,  $y = y(t)$ ,  $a \leq t \leq b$ , determines an **orientation** of a curve  $C$ , with the positive direction corresponding to increasing values of the parameter  $t$ . If  $-C$  denotes the curve consisting of the same points as  $C$  but with the opposite orientation, then we have

$$\int_{-C} f(x, y) dx = - \int_C f(x, y) dx$$

$$\int_{-C} f(x, y) dy = - \int_C f(x, y) dy$$

But if we integrate with respect to arc length, the value of the line integral does not change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y) ds = \int_C f(x, y) ds$$

This is because  $\Delta s_i$  is always positive, whereas  $\Delta x_i$  and  $\Delta y_i$  change sign when we reverse the orientation of  $C$ .

## Line Integrals in Space

We evaluate it using a formula similar to the line integrals in plane

$$\begin{aligned}\int_C f(x, y, z) ds &= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\ &= \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt\end{aligned}$$

Similarly, Line integrals along  $C$  with respect to  $x, y, z$  can also be defined.

$$\begin{aligned}\int_C f(x, y, z) dx &= \int_a^b f(x(t), y(t), z(t)) x'(t) dt \\ \int_C f(x, y, z) dy &= \int_a^b f(x(t), y(t), z(t)) y'(t) dt \\ \int_C f(x, y, z) dz &= \int_a^b f(x(t), y(t), z(t)) z'(t) dt\end{aligned}$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

by expressing everything  $(x, y, z, dx, dy, dz)$  in terms of the parameter  $t$ .

## Line Integrals of Vector Fields; Work

We know that the work done by a variable force  $f(x)$  in moving a particle from  $a$  to  $b$  along the  $x$ -axis is  $W = \int_a^b f(x) dx$ . Then we have found that the work done by a constant force  $\mathbf{F}$  in moving an object from a point  $P$  to another point  $Q$  in space is  $W = \mathbf{F} \cdot \mathbf{D}$ , where  $\mathbf{D} = \vec{PQ}$  is the displacement vector.

We can divide the curve  $C$  into subarcs  $P_{i-1}P_i$  with lengths  $\Delta s_i$  by dividing the parameter interval  $[a, b]$  into subintervals of equal width. Thus the work done by the force  $\mathbf{F}$  in moving the particle from  $P_{i-1}$  to  $P_i$  is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)] \Delta s_i$$

and the total work done in moving the particle along  $C$  is approximately

$$\sum_{i=1}^n [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)] \Delta s_i$$

where  $\mathbf{T}(x, y, z)$  is the unit tangent vector at the point  $(x, y, z)$  on  $C$ .

Therefore we define the **work**  $W$  done by the force field  $\mathbf{F}$  as the limit of the Riemann sums, namely

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) ds = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

If the curve  $C$  is given by the vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

then  $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ , so we can rewrite the equation in the form

$$W = \int_a^b \left[ \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

This integral is often abbreviated as  $\int_C \mathbf{F} \cdot d\mathbf{r}$ .

**Definition:** Let  $\mathbf{F}$  be a continuous vector field vector field defined on a smooth curve  $C$  given by a vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ . Then the **line integral of  $F$  along  $C$**  is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$