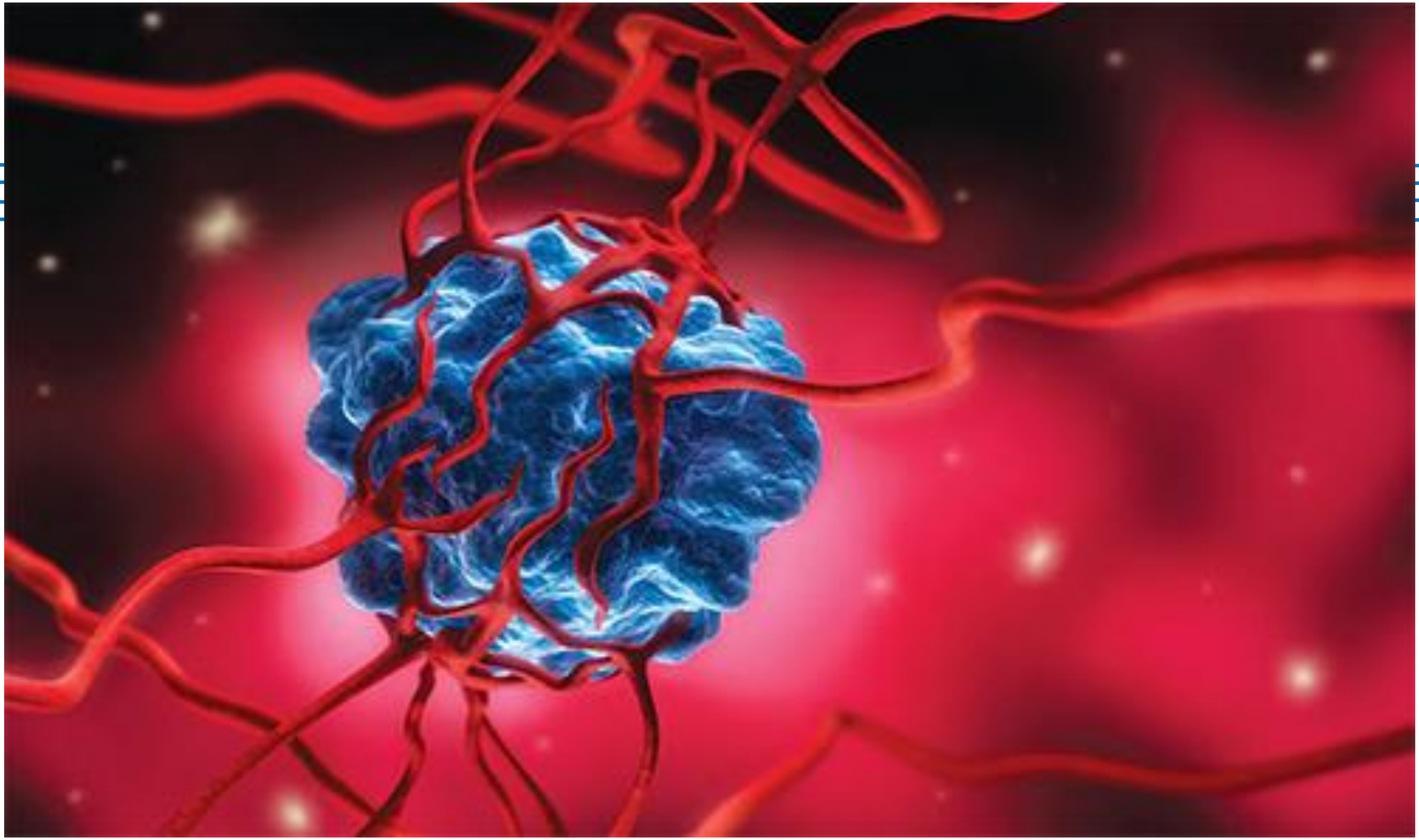
15 Multiple Integrals



15.8 Triple Integrals in Spherical Coordinates

Context

- Spherical Coordinates
- Triple Integrals in Spherical Coordinates

Triple Integrals in Spherical Coordinates (1 of 1)

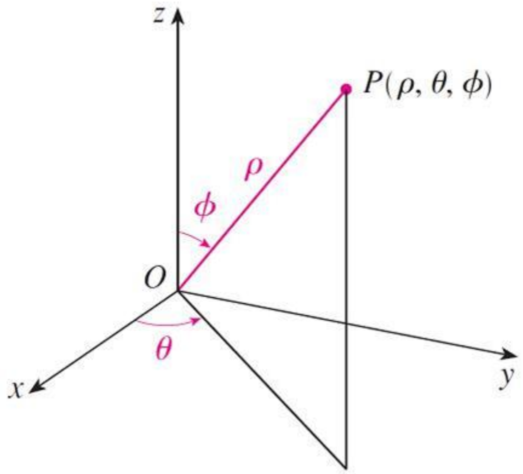
Another useful coordinate system in three dimensions is the *spherical coordinate system*.

It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

Spherical Coordinates

Spherical Coordinates (1 of 6)

The **spherical coordinates** (ρ, θ, ϕ) of a point *P* in space are shown in Figure 1 where $\rho = |OP|$ is the distance from the origin to *P*, *θ* is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive *z*-axis and the line segment *OP*.



The spherical coordinates of a point

Figure 1

Spherical Coordinates (2 of 6)

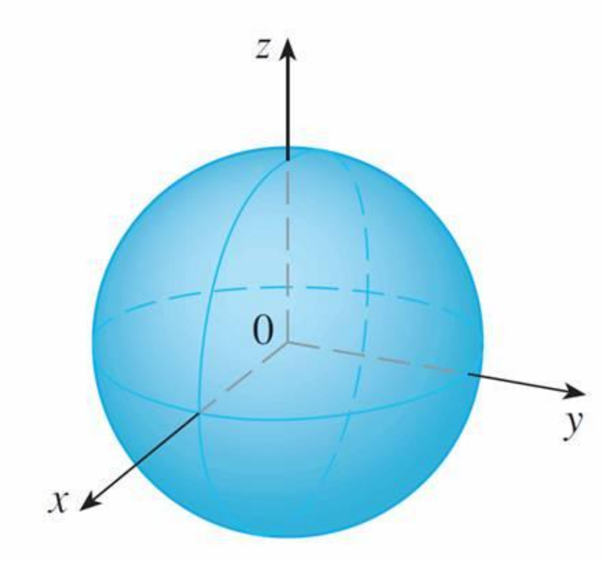
Note that

$$\rho \geq 0 \qquad 0 \leq \phi \leq \pi$$

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

Spherical Coordinates (3 of 6)

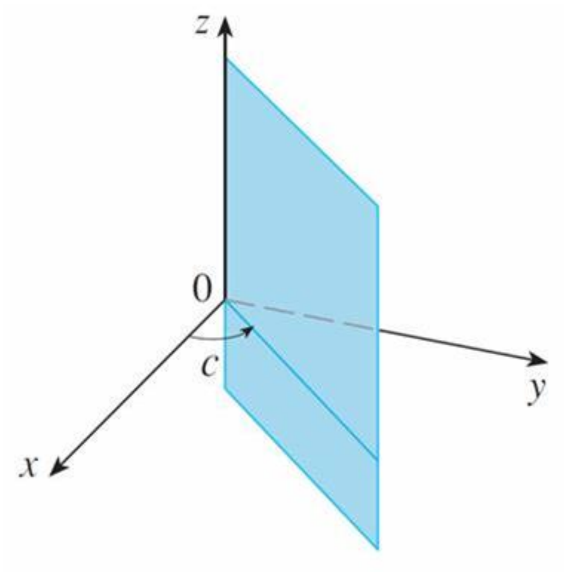
For example, the sphere with center the origin and radius c has the simple equation $\rho = c$ (see Figure 2); this is the reason for the name "spherical" coordinates.



$\rho = c$, a sphere

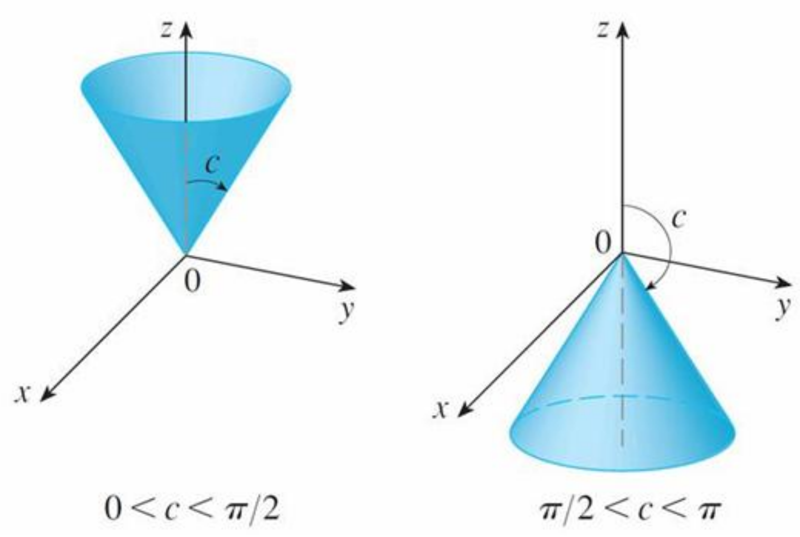
Figure 2

The graph of the equation $\theta = c$ is a vertical half-plane (see Figure 3), and the equation $\phi = c$ represents a half-cone with the *z*-axis as its axis (see Figure 4).



$\theta = c$, a half-plane

Figure 3



$\phi = c$, a half-cone

Figure 4

Spherical Coordinates (5 of 6)

The relationship between rectangular and spherical coordinates can be seen from Figure 5.

From triangles *OPQ* and *OPP′* we have

$$z = \rho\cos\phi \qquad r = \rho\sin\phi$$

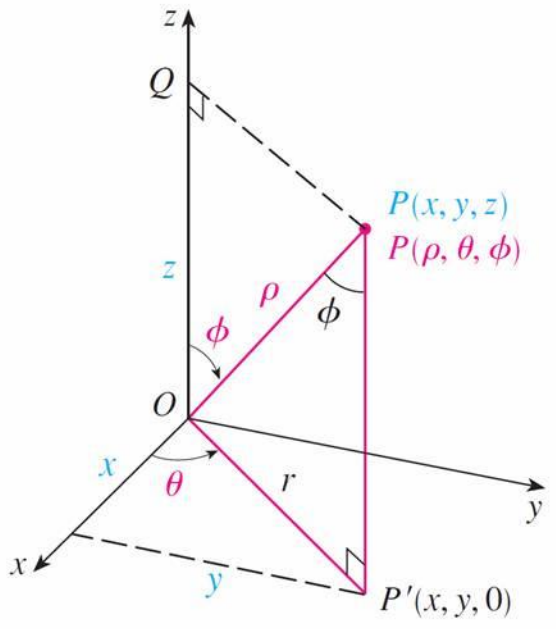


Figure 5

Spherical Coordinates (6 of 6)

But $x = r \cos \theta$ and $y = r \sin \theta$, so to convert from spherical to rectangular coordinates, we use the equations

$$\textbf{1} \qquad x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

Also, the distance formula shows that

$$\textbf{2} \qquad \rho^2 = x^2 + y^2 + z^2$$

We use this equation in converting from rectangular to spherical coordinates.

Example 1

The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Solution:

We plot the point in Figure 6.

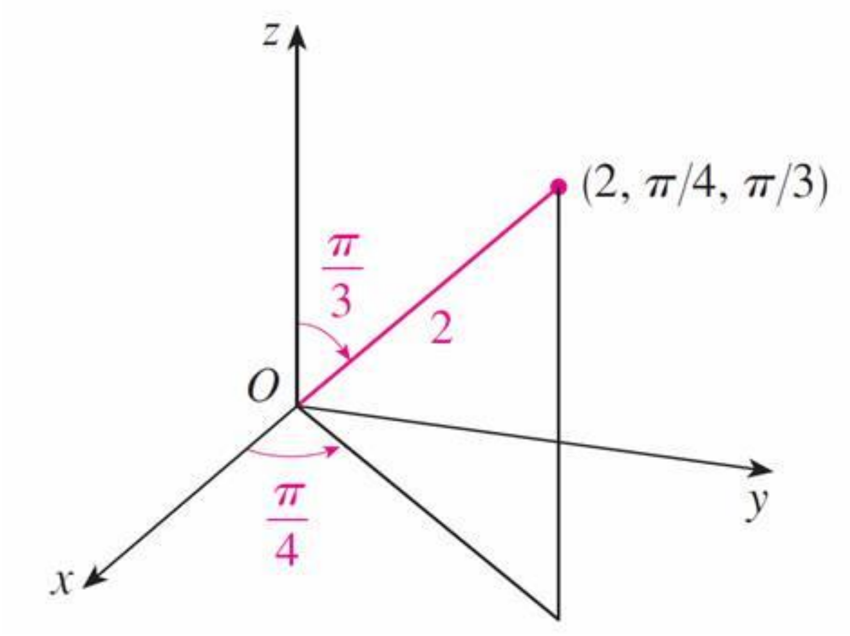


Figure 6

Example 1 – Solution

From Equations 1 we have

$$x = \rho \sin\phi \cos\theta \qquad = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin\phi \sin\theta \qquad = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos\phi \qquad = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

Thus the point $(2,\ \pi/4,\ \pi/3)$ is $\left(\sqrt{3/2},\ \sqrt{3/2},\ 1\right)$ in rectangular coordinates.

Triple Integrals in Spherical Coordinates

Triple Integrals in Spherical Coordinates (1 of 8)

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge**

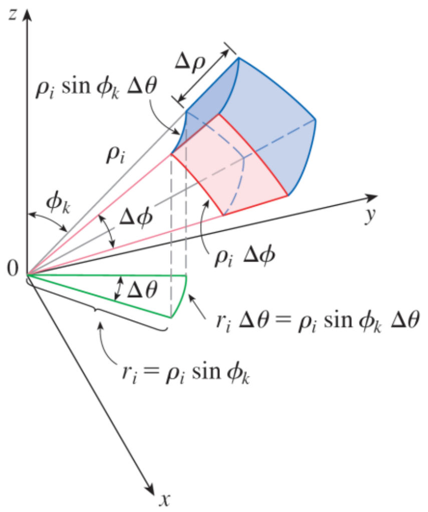
$$E = \{(\rho,\ \theta,\ \phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$$

where $a \ge 0$ and $\beta - \alpha \le 2\pi$, and $d - c \le \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result.

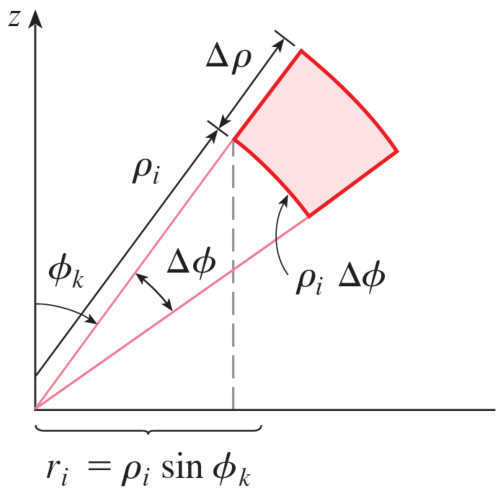
So we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$.

Triple Integrals in Spherical Coordinates (2 of 8)

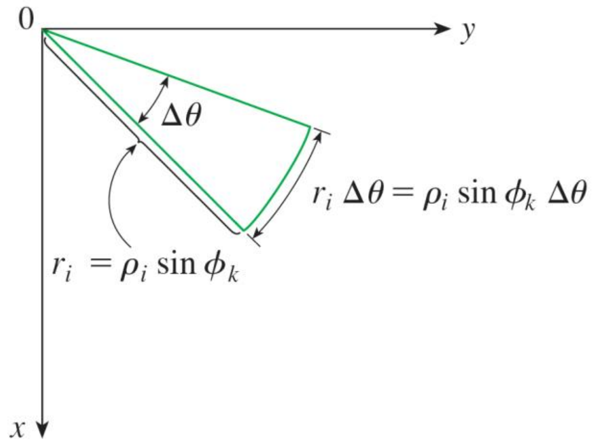
Figure 7 shows that E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i\ \Delta\phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin\phi_k\ \Delta\theta$ (arc of a circle with radius $\rho_i \sin\phi_k$, angle $\Delta\theta$).



(a) A spherical wedge



(b) Side view



(c) Top view

Figure 7

Triple Integrals in Spherical Coordinates (3 of 8)

So an approximation to the volume of E_{ijk} is given by

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i \Delta\phi)(\rho_i \sin\phi_k \Delta\theta) = \rho_i^2 \sin\phi_k \Delta\rho\,\Delta\theta\,\Delta\phi$$

In fact, it can be shown, with the aid of the Mean Value Theorem, that the volume of E_{ijk} is given exactly by

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin\tilde{\phi}_k \,\Delta\rho\,\Delta\theta\,\Delta\phi$$

where $(\rho_i, \theta_j, \phi_k)$ is some point in E_{ijk}.

Triple Integrals in Spherical Coordinates (4 of 8)

Let $\left(x_{ijk}^{*},\ y_{ijk}^{*},\ z_{ijk}^{*}\right)$ be the rectangular coordinates of this point.

Then

$$\iiint_{F} f\left(x,\ y,\ z\right) dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x_{ijk}^{*},\ y_{ijk}^{*},\ z_{ijk}^{*}\right) \Delta V_{ijk}$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(\rho_i \sin\phi_k \cos\theta_j,\ \rho_i \sin\phi_k \sin\theta_j,\ \rho_i \cos\phi_k\right) \rho_i^{2} \sin\phi_k \Delta\rho\, \Delta\theta\, \Delta\phi$$

Triple Integrals in Spherical Coordinates (5 of 8)

But this sum is a Riemann sum for the function

$$F(\rho,\ \theta,\ \phi) = f(\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho \cos\phi)\,\rho^2\ \sin\phi$$

Consequently, we have arrived at the following **formula for triple integration in spherical coordinates**.

3 $$\iiint_E f(x,\ y,\ z)\ dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi\ \cos\ \theta,\ \rho\ \sin\phi\ \sin\theta,\ \rho\ \cos\phi)\,\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho,\ \theta,\ \phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$$

Triple Integrals in Spherical Coordinates (6 of 8)

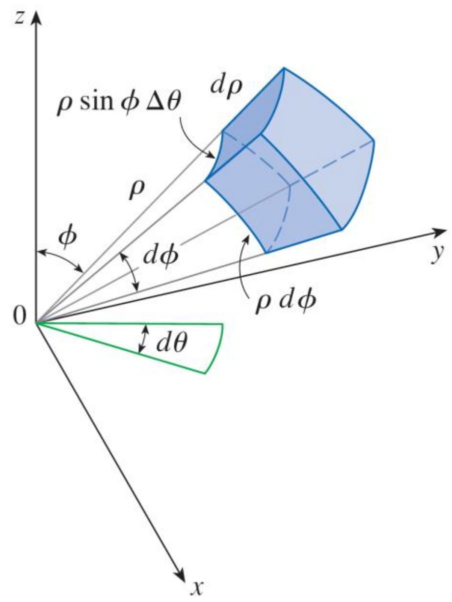
Formula 3 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

using the appropriate limits of integration, and replacing dv by $\rho^2 \sin\phi \; d\rho \; d\theta \; d\phi$.

Triple Integrals in Spherical Coordinates (7 of 8)

This is illustrated in Figure 8.



Volume element in spherical coordinates: $dv = \rho^2 \sin\phi \; d\rho \; d\theta \; d\phi.$

Figure 8

Triple Integrals in Spherical Coordinates (8 of 8)

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho,\ \theta,\ \phi) \mid \alpha \le \theta \le \beta,\ c \le \phi \le d,\ g_1(\theta,\ \phi) \le \rho \le g_2(\theta,\ \phi)\}$$

In this case the formula is the same as in (3) except that the limits of integration for ρ are $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$.

Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

单选题 1分

设置

MCQ: $E = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, x \geq 0, |y| \leq x, x^2 + y^2 \geq z^2\}$
Compute $I = \iiint_E 1 \; dV$

A $\dfrac{4\sqrt{3}\pi}{3}$

B $\dfrac{3\sqrt{3}\pi}{4}$

C $\dfrac{5\sqrt{2}\pi}{4}$

D $\dfrac{4\sqrt{2}\pi}{3}$

提交

Example

Solution:

Sphere:

$$x^2 + y^2 + z^2 \leq 4 \Rightarrow 0 \leq \rho \leq 2$$

Cone condition:

$$x^2 + y^2 \geq z^2 \Rightarrow \rho^2 (\sin\phi)^2 \geq \rho^2\,(\cos\phi)^2 \Rightarrow (\tan\phi)^2 \geq 1 \Rightarrow \phi \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

Wedge in the xy-plane:

$$|\,y\,| \leq x \Rightarrow |\tan\theta\,| \leq 1 \text{ and } \cos\theta \geq 0 \Rightarrow \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

So $x \geq 0$ is effectively redundant.

Evaluate

$$\int_0^2 \rho^2\,d\rho = \frac{8}{3}, \quad \int_{\pi/4}^{3\pi/4} \sin\phi\,d\phi = \sqrt{2}, \int_{-\pi/4}^{\pi/4} d\,\theta = \frac{\pi}{2}$$

Thus

$$\iiint_E 1\ dV = \int_{-\pi/4}^{\pi/4} \int_{\pi/4}^{3\pi/4} \int_0^2 \rho^2 \sin\phi\,d\rho d\phi d\theta = \frac{8}{3} \cdot \sqrt{2} \cdot \frac{\pi}{2} = \frac{4\sqrt{2}\pi}{3}$$

Example 4

Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figure 9.)

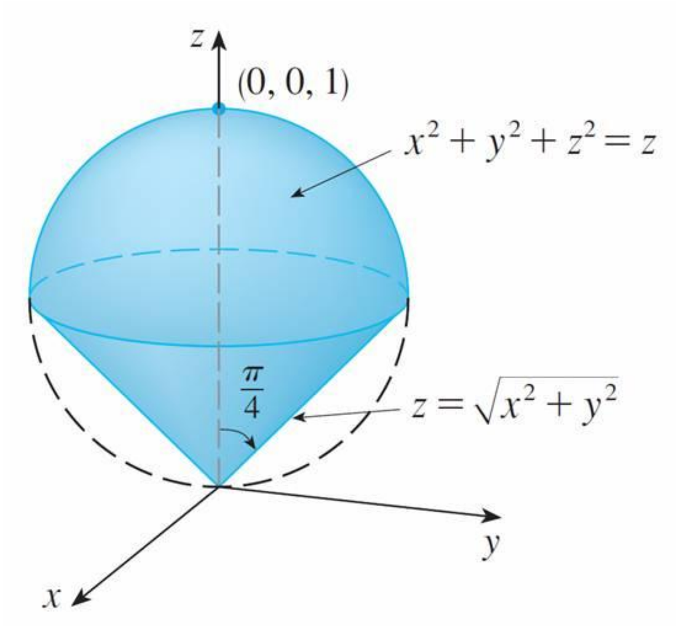


Figure 9

Example 4 – Solution (1 of 4)

Notice that the sphere passes through the origin and has center $\left(0,\ 0,\ \frac{1}{2}\right)$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho\cos\phi \qquad \text{or} \qquad \rho = \cos\phi$$

The equation of the cone can be written as

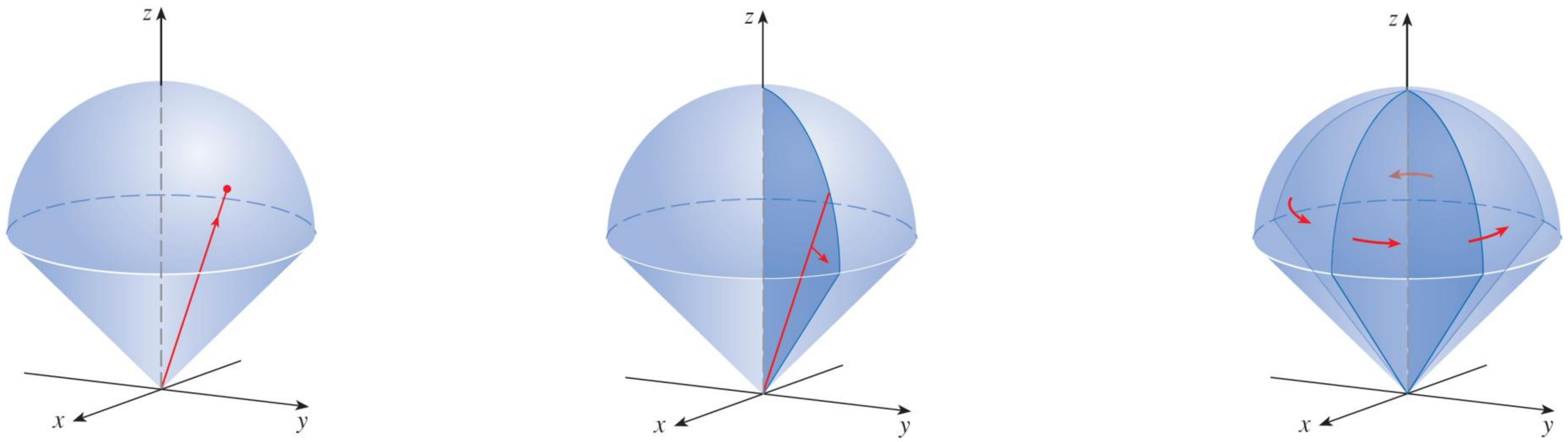
$$\begin{aligned} \rho\cos\phi &= \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} \\ &= \rho\sin\phi \end{aligned}$$

Example 4 – Solution (2 of 4)

This gives $\sin\phi = \cos\phi$, or $\phi = \pi/4$. Therefore the description of the solid *E* in spherical coordinates is

$$\text{E} = \{(\rho,\ \theta,\ \phi) | 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4,\ 0 \leq \rho \leq \cos\phi\}$$

Figure 10 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ.



ρ varies from 0 to $\cos \phi$ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.

Figure 10

Example 4 – Solution (4 of 4)

The volume of E is

$$V(E) = \iiint\limits_E dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\phi} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\phi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \sin\phi \cos^3\phi \; d\phi = \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4} \right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}$$

Recap

- Spherical Coordinates
- Triple Integrals in Spherical Coordinates