Vector Calculus

1 Vector Fields

1.1 Vector Field in $\mathbb{R}^2$ and $\mathbb{R}^3$

Definition: Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\boldsymbol{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\boldsymbol{F}(x, y)$.

The best way to picture a vector field is to draw the arrow representing the vector $\boldsymbol{F}(x, y)$ starting at the point (x, y). Since $\boldsymbol{F}(x, y)$ is a two-dimentional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\begin{gathered} \boldsymbol{F}(x, y) = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j} = \langle P(x, y), Q(x, y)\rangle \\ \boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} \end{gathered} \tag{136}$$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

Definition: Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\boldsymbol{F}$ that assigns to each point (x, y, z) in E a three-dimentional vector $\boldsymbol{F}(x, y, z)$.

We can express it in terms of its component functions P, Q, R as

$$\boldsymbol{F}(x, y, z) = P(x, y, z)\boldsymbol{i} + Q(x, y, z)\boldsymbol{j} + R(x, y, z)\boldsymbol{k} \tag{137}$$

As with the vector functions, we can define continuity of vector fields and show that $\boldsymbol{F}$ is continuous if and only if its component functions P, Q, R are continuous. We sometimes identify a point (x, y, z) with its position vector $\boldsymbol{x} = \langle x, y, z\rangle$ and write $\boldsymbol{F}(\boldsymbol{x})$ instead of $\boldsymbol{F}(x, y, z)$. Then $\boldsymbol{F}$ becomes a function that assigns a vector $\boldsymbol{F}(\boldsymbol{x})$ to a vector $\boldsymbol{x}$

1.2 Gradient Fields

If f is a scalar function of two variables, we know that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\boldsymbol{i} + f_y(x, y)\boldsymbol{j} \tag{138}$$

Therefore ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\boldsymbol{i} + f_y(x, y, z)\boldsymbol{j} + f_z(x, y, z)\boldsymbol{k} \tag{139}$$

A vector filed $\boldsymbol{f}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\boldsymbol{F} = \nabla f$. In this situation f is called a **potential function** for $\boldsymbol{F}$. Not all vector fields are conservative, but such fields called a **potential function** for F. Not all vector fields are conservative, but such fields do arise frequently in physics.

2 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C. Such integrals are called line integrals, although “curve integrals" would be better terminology.
They were invented in the early 19th century to solve problems involving fluidflow, forces, electricity, and magnetism.

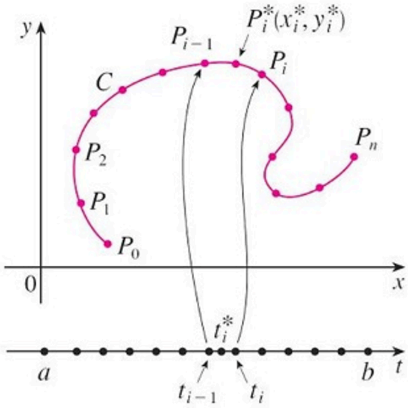
2.1 Line Integrals in the Plane

We start with a plane curve C given by the parametric equations

$$x = x(t) \quad y = y(t) \quad a \leq t \leq b \tag{140}$$

or, equivalently, by the vector equation $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$, and we assume taht C is a smooth curve.

If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$, and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subjarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$.



We choose any point $P_i^*(x_i^*, y_i^*)$ in the i-th subjarc. (This corresponds to a point t_j^* in $[t_{j-1}, t_j]$). Now if f fis any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \tag{141}$$

which is similar to a Riemann sum.

Definition: If f is defined on a smooth curve C, then the **line integral of f along C** is

$$\int_C f(x, y) ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \tag{142}$$

if this limit exists.

We have found that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \tag{143}$$

A similar type of argument can be used to show that if f is a continuous function, then the limit always exists and the following formula can be used to evaluate the line integral:

$$\int_c f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \tag{144}$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)ds = \sum_{i=1}^n \int_{C_i} f(x,y)ds \tag{145}$$

The **mass** m of the wire:

$$m = \lim_{n\to\infty} \sum_{i=1}^n \rho(x_i^*, y_i^*)\Delta s_i = \int_C \rho(x,y)ds \tag{146}$$

The **center of mass** of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where

$$\begin{aligned} \bar{x} &= \frac{1}{m}\int_C x\rho(x,y)ds \\ \bar{y} &= \frac{1}{m}\int_C y\rho(x,y)ds \end{aligned} \tag{147}$$

2.2 Line Integrals with Respect to x or y

The **line integrals of f along C with respect to x and y**:

$$\begin{aligned} \int_C f(x,y)dx &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*, y_i^*)\Delta x_i = \int_a^b f(x(t),y(t))x'(t)dt \\ \int_C f(x,y)dy &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*, y_i^*)\Delta y_i == \int_a^b f(x(t),y(t))y'(t)dt \end{aligned} \tag{148}$$

$$\int_C f(x,y)dy = \lim_{n\to\infty}\sum_{i=1} f(x_i^*,y_i^*)\Delta y_i = \int_a f(x(t),y(t))y'(t)dt$$

In particular, we often need to parametrize a line segment, so it's useful to remember that a vector representation of the line segment that starts at $\boldsymbol{r}_0$ and ends at $\boldsymbol{r}_1$ is given by

$$\boldsymbol{r}(t) = (1-t)\boldsymbol{r}_0 + t\boldsymbol{r}_1 \quad 0 \le t \le 1 \tag{149}$$

In general, a given parametrization $x = x(t), y = y(t), a \le t \le b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t. If -C denotes the curve consisting of the same points as C but with the opposite orientation, then we have

$$\begin{aligned} \int_{-C} f(x,y)dx &= -\int_C f(x,y)dx \\ \int_{-C} f(x,y)dy &= -\int_C f(x,y)dy \end{aligned} \tag{150}$$

But if we integrate with respect to arc length, the value of the line integral does not change when we reverse the orientation of the curve:

$$\int_{-C} f(x,y)ds = \int_C f(x,y)ds \tag{151}$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

2.3 Line Integrals in Space

We evaluate it using a formula similar to the line integrals in plane

$$\begin{aligned} \int_C f(x,y,z)ds &= \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt \\ &= \int_a^b f(\boldsymbol{r}(t))|\boldsymbol{r}'(t)|dt \end{aligned} \tag{152}$$

Similarly, Line integrals along C with respect to x, y, z can also be defined.

$$\begin{aligned} \int_C f(x,y,z)dx &= \int_a^b f(x(t),y(t),z(t))x'(t)dt \\ \int_C f(x,y,z)dy &= \int_a^b f(x(t),y(t),z(t))y'(t)dt \\ \int_C f(x,y,z)dz &= \int_a^b f(x(t),y(t),z(t))z'(t)dt \end{aligned} \tag{153}$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz \tag{154}$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

2.4 Line Integrals of Vector Fields; Work

We know that the work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is $W = \int_a^b f(x)dx$. Then we have found that the work done by a constant force $\boldsymbol{F}$ in moving an object from a point P to another point Q in space is $W = \boldsymbol{F} \cdot \boldsymbol{D}$, where $\boldsymbol{D} = \vec{PQ}$ is the displacement vector.

We can divide the curve C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width. Thus the work done by the force $\boldsymbol{F}$ in moving the particle from P_{i-1} to P_i is approximately

$$\boldsymbol{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \boldsymbol{T}(t_i^*)] = [\boldsymbol{F}(x_i^*, y_i^*, z_i^*) \cdot \boldsymbol{T}(t_i^*)]\Delta s_i \tag{155}$$

and the total work done in moving the particle along C is approximately

$$\sum_{i=1}^{n} [\boldsymbol{F}(x_i^*, y_i^*, z_i^*) \cdot \boldsymbol{T}(x_i^*, y_i^*, z_i^*)]\Delta s_i \tag{156}$$

where $\boldsymbol{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C.

Therefore we define the **work** W done by the force field $\boldsymbol{F}$ as the limit of the Riemann sums, namely

$$W = \int_C \boldsymbol{F}(x,y,z) \cdot \boldsymbol{T}(x,y,z)ds = \int_C \boldsymbol{F} \cdot \boldsymbol{T} ds \tag{157}$$

If the curve C is given by the vector equation

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k} \tag{158}$$

then $\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$, so we can rewrite the equation in the form

$$W = \int_a^b \left[\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}\right] |\boldsymbol{r}'(t)|dt = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt \tag{159}$$

This integral is often abbreviated as $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Definition: Let $\boldsymbol{F}$ be a continuous vector field vector field defined on a smooth curve C given by a vector function $\boldsymbol{r}(t)$, $a \le t \le b$. Then the **line integral of F along C** is

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt = \int_C \boldsymbol{F} \cdot \boldsymbol{T} ds \tag{160}$$