Final Examination

Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) The function $f(x,y) = \dfrac{\sin(x)\cos(y)}{1+x^2+y^2}$, $(x,y) \in \mathbb{R}^2$ attains a global maximum and a global minimum.

b) The function $g(x,y) = xy(x^2+2y^2-3)$, $(x,y) \in \mathbb{R}^2$ has at least 9 critical points.

c) Suppose you start at the point $(1,0)$ and move 0.5 units in the (x,y)-plane following (and continuously adjusting) the direction of steepest ascent of $h(x,y) = \dfrac{xy}{x^2+y^2}$. Afterwards you are closer to the y-axis than before.

d) If C^2-functions $f(u,v)$ and $g(x,y)$ are related by $g(x,y) = f(ax+by, cx+dy)$ $(a,b,c,d \in \mathbb{R},\ ad-bc \neq 0)$ then $g_{xx}g_{yy} - g_{xy}^2 = (ad-bc)(f_{uu}f_{vv} - f_{uv}^2)$.

e) The line integral of $(\sin y + y\sin x)\,\mathrm{d}x + (x\cos y - \cos x)\,\mathrm{d}y$ along the quarter circle $x^2+y^2=1$, $x \geq 0$, $y \geq 0$ (in the mathematically positive direction) is zero.

f) If P, Q are C^2-functions on $\mathbb{R}^2$ such that both $P\,\mathrm{d}x + Q\,\mathrm{d}y$ and $Q\,\mathrm{d}x - P\,\mathrm{d}y$ are exact, then P and Q solve Laplace's equation $\Delta u = u_{xx} + u_{yy} = 0$.

Question 2 (ca. 9 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = x^4 - 3x^2y + x^2 + 2y^2 - y.$$

a) Which symmetry property does f have? What can you conclude from this about the graph of f and the contours of f?

b) Determine the gradient $\nabla f(x,y)$ and the Hesse matrix $\mathbf{H}_f(x,y)$.

c) Determine all critical points of f and their types.
Hint: There are exactly 3 critical points.

d) Does f have a global extremum?

Question 3 (ca. 9 marks)

Consider the surface S in $\mathbb{R}^3$ with equation

$$xz - y^2 + 2y + 2 = 0.$$

a) Show that S is smooth.

b) Using the method of Lagrange multipliers, determine the point(s) on S that minimize(s) the distance from the origin $(0,0,0)$, and the corresponding distance d. (It need not be proved that at least one such point exists.)

c) Determine an equation for the tangent plane to S in the point $(2,-2,3)$.

d) The surface S is a (central) quadric. Determine its center and type.

Question 4 (ca. 6 marks)

Evaluate the integral

$$\int_0^1 \frac{t^{1010} + t - 2}{\ln t}\,\mathrm{d}t\,.$$

Hint: Consider $F(x) = \displaystyle\int_0^1 \frac{t^x + t - 2}{\ln t}\,\mathrm{d}t$, $x \in (-1, \infty)$. Show first that F is well-defined and can be differentiated under the integral sign.

Question 5 (ca. 6 marks)

a) Let K be the solid in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying

$$x \geq 0, \quad y \geq 0, \quad x + y \leq 2, \quad 0 \leq z \leq 1 + x^2 + 2y.$$

Find the volume of K.

b) Let P be the surface in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying

$$z = x^2 + y^2 \leq 1.$$

Find the area of P.