1. (a) The desired equation has the form $a_1x_1+a_2x_2+a_3x_3=b$ with $a_1,a_2,a_3,b\in\mathbb{R}$ and not all a_i equal to zero. Substituting the coordinates x_i gives

$$a_1(1+3c_2)+a_2(2c_1+3c_2)+a_3(-2-2c_1+c_2)=b \quad \text{for all} \quad c_1,c_2\in\mathbb{R}.$$

This is equivalent to

$$a_1-2a_3+(2a_2-2a_3)c_1+(3a_1+3a_2+a_3)c_2=b \quad \text{for all} \quad c_1,c_2\in\mathbb{R},$$

and in turn to

$$a_1-2a_3=b \quad \wedge \quad 2a_2-2a_3=0 \quad \wedge \quad 3a_1+3a_2+a_3=0.$$

Setting $a_3=1$ gives $a_2=1$, $a_1=-\frac{4}{3}$, $b=-\frac{10}{3}$. Hence an equation for the plane is

$$-4x_1+3x_2+x_3=-10,$$

obtained by scaling the original equation by 3 (so that the coefficients become integers).

The geometric meaning of $\mathbf{a}=(a_1,a_2,a_3)$ is taht $\mathbf{a}$ must be orthogonal to any direction vector of H (i.e. a so-called *normal vector* of H)

(b) First we can get a point on the plane $(1,0,0)$

Then we can get 2 non-collinear vector that is orthogonal to the normal vector $(1,1,1)$. We can choose $(-1,1,0)$ and $(-1,0,1)$ and obtain

$$H=\begin{bmatrix}1\\0\\0\end{bmatrix}+\mathbb{R}\begin{bmatrix}-1\\1\\0\end{bmatrix}+\mathbb{R}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

2. Let $P=(x,y,z)$. Then $2|PB|=|PA| \iff 4|PB|^2=|PA|^2 \iff$
$4\left[(x-6)^2+(y-2)^2+(z+2)^2\right]=(x+1)^2+(y-5)^2+(z-3)^2 \iff$
$4(x^2-12x+36)-x^2-2x+4(y^2-4y+4)-y^2+10y+4(z^2+4z+4)-z^2+6z=35 \iff$
$3x^2-50x+3y^2-6y+3z^2+22z=35-144-16-16 \iff$

$x^2 - \frac{50}{3}x + y^2 - 2y + z^2 + \frac{22}{3}z = -\frac{141}{3}$

By completing the square three times we get

$$\left(x - \frac{25}{3}\right)^2 + (y-1)^2 + \left(z + \frac{11}{3}\right)^2 = -\frac{423 + 625 + 9 + 121}{9} = \frac{332}{9},$$

which is an equation of a sphere with center $\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$ and radius $\sqrt{\frac{332}{3}}$.

3. Call the two tension vectors T_2 and T_3, corresponding to the ropes of length 2 m and 3 m. In terms of vertical and horizontal components, $T_2 = -|T_2|\cos 50°\mathbf{i} + |T_2|\sin 50°\mathbf{j}$ (1) and $T_3 = |T_3|\cos 38°\mathbf{i} + |T_3|\sin 38°\mathbf{j}$ (2).

   The resultant of these forces, $T_2 + T_3$, counterbalances the weight of the hoist (which is $-350\mathbf{j}$), so $T_2 + T_3 = 350\mathbf{j}$, which gives $(-|T_2|\cos 50° + |T_3|\cos 38°)\mathbf{i} + (|T_2|\sin 50° + |T_3|\sin 38°)\mathbf{j} = 350\mathbf{j}$. Equating components, we have $-|T_2|\cos 50° + |T_3|\cos 38° = 0 \Rightarrow |T_2| = |T_3|\frac{\cos 38°}{\cos 50°}$ and $|T_2|\sin 50° + |T_3|\sin 38° = 350$.

   Substituting the first equation into the second gives $|T_3|\frac{\cos 38°}{\cos 50°}\sin 50° + |T_3|\sin 38° = 350 \Rightarrow |T_3|(\cos 38° \tan 50° + \sin 38°) = 350$, so $|T_3| = \frac{350}{\cos 38° \tan 50° + \sin 38°} \approx 225.11\,\mathrm{N}$ and $|T_2| = |T_3|\frac{\cos 38°}{\cos 50°} \approx 275.97\,\mathrm{N}$.

   Finally, from (1) and (2), the tension vectors are $T_2 \approx -177.39\mathbf{i} + 211.41\mathbf{j}$ and $T_3 \approx 177.39\mathbf{i} + 138.59\mathbf{j}$.

4. PROOF:

   Let $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$

   By applying the Law of Cosines to triangle OAB, we get

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$$
$$\Rightarrow |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

   Since $|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}$

   We get

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}$$

   Thus

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$