# Worksheet2 Solution

This worksheet is exploratory in nature - reasonable explanations with clear thinking process can earn full 20 points

## Key Concepts for Full Credit

- Question 1 (Elliptic Paraboloid): Vertical translation of parabolas, degenerate cases (paraboloidal cylinders), upward vs downward opening
- Question 2 (Hyperbolic Paraboloid): Intersecting lines at $z = 0$, saddle surface orientation, 90-degree rotation relationship
- Question 3 (Ellipsoid): Equal semi-axes condition, division by zero issue, interior visualization effects
- Question 4 (Double Cone): Conic section theory, degenerate planes, intersecting lines through vertex
- Question 5 (One-Sheet Hyperboloid): Asymptotic behavior toward double cone, special cross-sections, persistent central opening
- Question 6 (Two-Sheet Hyperboloid): Parameter scaling effects, permanent gap between sheets, limiting behavior toward double cone

## 1 Elliptic Paraboloid

First, the general equation for an elliptic paraboloid is:

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

where $a > 0$ and $b > 0$ are constants that control the shape of the surface. Sometimes it's also written in the form $z = Ax^2 + By^2$, where $A = 1/a^2$ and $B = 1/b^2$.

### 1.1 Differences in Parabolic Cross-Sections at $x = c$

When you fix $x = c$, the equation becomes $z = c^2/a^2 + y^2/b^2$, which is indeed a parabola. If you project all these cross-sections onto the same yz-plane, you'll notice an interesting phenomenon: all these parabolas have the same shape (the same openness), but they are shifted differently in the vertical direction.

Specifically, the larger the value of $c$, the higher the vertex of the parabola, because the z-coordinate of the vertex is $c^2/a^2$. In other words, these parabolas are like the same parent parabola shifted upward by different amounts.

### 1.2 The Special Case When $A$ or $B$ is 0

This is a very interesting boundary case. If $A = 0$ (meaning the x-term disappears), the equation becomes $z = By^2$. This is no longer an elliptic paraboloid but a parabolic cylinder—it only curves in the y-direction and is straight in the x-direction. Similarly, when $B = 0$, you get a parabolic cylinder oriented in the other direction.

If both $A$ and $B$ are 0, it's even more extreme, becoming $z = 0$, which is the xy-plane itself. Strictly speaking, these degenerate cases should not be called "elliptic" paraboloids because they lose the typical feature of an elliptic paraboloid—curving in two directions.

The term "elliptic" comes from the fact that when the surface intersects with a plane $z = k$ (for a positive constant $k$), the resulting cross-section is an ellipse ($Ax^2 + By^2 = k$).

### 1.3 The Case When $A$ and $B$ Are Both Negative

This produces a completely different surface! The equation becomes $z = -Ax^2 - By^2$ (where $A$ and $B$ are now negative), and the entire surface is flipped upside down, opening downward instead of upward. Imagine what was once a bowl-shaped surface now looks like an inverted bowl or a mountain peak.

This is still an elliptic paraboloid, just with the opposite orientation. In physical applications, such a downward-opening paraboloid might represent a point of maximum potential energy rather than a minimum.

## 2 Hyperbolic Paraboloid

The hyperbolic paraboloid is a particularly interesting surface, often called a "saddle surface." Let's start with its general equation:

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Or, written more simply, $z = Ax^2 - By^2$. Note the crucial minus sign here, which makes it fundamentally different from an elliptic paraboloid.

### 2.1 Shape of the Cross-Section on the $z = 0$ Plane

This is a point that can be easily misunderstood. When you set $z = 0$, the equation becomes:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

At first glance, this looks like the equation of a hyperbola, but it's not! If you rearrange it, you will find:

$$\frac{x^2}{a^2} = \frac{y^2}{b^2}$$

This means $x/a = \pm y/b$, which indicates that the horizontal cross-section at $z = 0$ is actually two intersecting lines, with the intersection at the origin. The slopes of these two lines are $\pm b/a$.

This result is unique—it is the only cross-section on the hyperbolic paraboloid that is neither a hyperbola nor a parabola. In fact, these two lines are the "asymptotic directions" of the saddle surface, much like the forward-backward and side-to-side directions when you sit on a saddle.

### 2.2 Comparison of the Two Equations: $z = y^2 - x^2$ vs. $z = x^2 - y^2$

These two surfaces are actually reflections of each other, or you could say one is the result of rotating the other by 90 degrees. Let me explain the difference:

For $z = x^2 - y^2$, the curve bends upward along the x-axis (when $y = 0$) and downward along the y-axis (when $x = 0$). Imagine a saddle: when you sit on it, the front and back curve up, while the left and right curve down.

In contrast, $z = y^2 - x^2$ is the opposite: it bends upward along the y-axis and downward along the x-axis. It's like turning the saddle 90 degrees, or a rider sitting sideways.

Mathematically speaking, if you rotate the coordinate system by 90 degrees (by swapping x and y), one equation transforms into the other. They are essentially the same type of surface, just with different orientations. In applications, the choice of which form to use usually depends on which direction you want to be curved upward.

## 3 Ellipsoid

An ellipsoid is a three-dimensional "egg" shape, with its standard equation being:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Here, $a$, $b$, and $c$ control the radii in the three respective directions.

### 3.1 Condition for an Ellipsoid to Become a Sphere

This question is quite intuitive—when all three radii are equal, the ellipsoid becomes a sphere. That is, when $a = b = c = r$, the equation simplifies to the familiar equation of a sphere: $x^2 + y^2 + z^2 = r^2$.

Imagine an ellipsoid as a sphere that has been stretched or compressed. If the "degree of stretching" is exactly the same in all three directions, then it is naturally a standard sphere.

In interactive graphics, if you set all three sliders to the same value, you will see a perfect sphere.

### 3.2 Why the Sliders Cannot Go to 0

What would happen to the equation if any of the parameters (say, $a$) became 0? For example, with $a = 0$, the equation becomes:

$$\frac{x^2}{0} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

This is mathematically meaningless—you cannot divide by zero. From a geometric perspective, if $a = 0$, the ellipsoid is completely "flattened" in the x-direction, degenerating into a two-dimensional ellipse (on the yz-plane). This is no longer a true three-dimensional ellipsoid.

That's why the slider's minimum value is set to 0.1, allowing you to get a very flat but still three-dimensional ellipsoid. This is like a very thin flying saucer; although very flat, it still has thickness.

### 3.3 The Effect of Viewing the Ellipsoid from the Inside

When you set all the sliders to their maximum values and then zoom in until you "enter" the interior of the ellipsoid, you will see a completely different world.

From the inside, the surface of the ellipsoid looks like a dome surrounding you. If you turn on the grid lines, you can clearly see the changes in the curvature of the surface—some areas are more sharply curved, while others are relatively flat. This internal perspective can help you better understand the geometric properties of the ellipsoid.

It's like standing inside a giant eggshell and looking out; you can feel the differences in curvature in different directions. If it were a sphere, every direction would look the same; but for an ellipsoid, you will find that the surface is closer to you in some directions and farther in others.

Try rotating the view from inside; you will find this asymmetry particularly noticeable, which is the essential feature that distinguishes an ellipsoid from a sphere.

## 4 Double Cone

A double cone is a very special quadric surface, with its standard equation being:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Or written as $z^2 = c^2(x^2/a^2 + y^2/b^2)$, it's clear that the absolute value of $z$ increases as the distance of $x, y$ from the origin increases.

## 4.1 Why Neither Vertical nor Horizontal Cross-Sections are Parabolas

This relates to the fundamental properties of conic sections. Let's look at the different cross-sections:

- A horizontal cross-section ($z = k$, where $k$ is a constant), when substituted into the equation, gives $x^2/a^2 + y^2/b^2 = k^2/c^2$, which is the equation of an ellipse. The farther from the origin ($|k|$ is larger), the larger the ellipse.
- A vertical cross-section, for example, along the xz-plane ($y = 0$), gives $x^2/a^2 - z^2/c^2 = 0$, which means $z = \pm(c/a)x$. This represents two lines passing through the origin. Other vertical cross-sections ($y = k$) result in $x^2/a^2 - z^2/c^2 = -k^2/b^2$, which is a standard hyperbola.

So where is the parabola? In fact, to get a parabola, you need to slice the cone with a tilted plane that is parallel to one of the cone's generating lines. Neither vertical nor horizontal cross-sections meet this condition, so you cannot get a parabola. This is why the complete picture of conic sections requires considering slices at various angles.

## 4.2 Degenerate Cases When $A = 0$ or $B = 0$

Suppose the equation is written in the form $z^2 = Ax^2 + By^2$. When $A = 0$, the equation becomes $z^2 = By^2$, which is $z = \pm\sqrt{B} \cdot y$. What does this mean?

This is no longer a cone, but two intersecting planes! Specifically, the equations of these two planes are $z = \sqrt{B} \cdot y$ and $z = -\sqrt{B} \cdot y$, and they intersect along the x-axis. Geometrically, the original cone has been completely "flattened" in the x-direction, leaving only the variations in the y and z directions.

Similarly, when $B = 0$, you get another pair of planes. The reason you don't get a single cone is that a cone needs to expand in two lateral directions. If the coefficient in one direction is zero, the structure collapses.

## 4.3 The Special Nature of the $x = 0$ and $y = 0$ Cross-Sections

When $x = 0$, the equation becomes $y^2/b^2 - z^2/c^2 = 0$, which can be rewritten as $(y/b - z/c)(y/b + z/c) = 0$.

This means either $y/b = z/c$ or $y/b = -z/c$, so the cross-section consists of two intersecting lines that meet at the origin. The slopes of these lines are $\pm c/b$.

This is not a hyperbola, but a "degenerate" form of a hyperbola. You can think of it as the two asymptotes of a hyperbola. In fact, this corresponds precisely to the cross-section of the double cone that passes through the vertex (the origin)—at the vertex, where the upper and lower cones meet, the cross-section is naturally two lines (two generating lines of the cone).

Only when the cross-section does not pass through the vertex (e.g., $x = k$, where $k \neq 0$) do you get a true hyperbola. This phenomenon clearly demonstrates the degenerate behavior of quadric surfaces at special locations.

# 5 Hyperboloid of One Sheet

A hyperboloid of one sheet has a shape like a cooling tower or the waist of an hourglass. Its standard equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Note that the z-term here is negative, but the right side is a positive 1, which is key to distinguishing it from other hyperboloids.

## 5.1 Why the Sliders Cannot Go to 0, and the Limiting Case

Just like with the ellipsoid, if any parameter becomes 0, we encounter the problem of dividing by zero, and the equation becomes mathematically meaningless. More interestingly, if you set all the parameters to be as small as possible (e.g., all 0.1) and then zoom in, you will observe a surprising phenomenon.

The hyperboloid at this point becomes very "skinny," with the "waist" in the middle becoming extremely narrow. If you look closely, it starts to look more and more like a double cone! In fact, this is the limiting behavior of a hyperboloid of one sheet—as the parameters approach 0, the surface tends to degenerate into a double cone. You can imagine the waist of an hourglass getting tighter and tighter until it's about to break, at which point it resembles two cones with their vertices touching.

## 5.2 The Special Nature of the $x = a$ and $x = -a$ Cross-Sections

When $x = 1$ (the problem states $A = 1$), let's see what happens. Substituting into the equation:

$$\frac{1}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

If it happens that $a = 1$ (i.e., $x = a$), then the first term is 1, and the equation simplifies to:

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

This can be rewritten as $y/b = \pm z/c$, which represents two intersecting lines! This is a very special position—slicing right through the "narrowest" part of the hyperboloid. In fact, the two planes $x = \pm a$ cut through the "waist" of the hyperboloid, resulting in lines instead of hyperbolas.

If you choose other values for $x$ (e.g., $x = 2$), you will get standard hyperbolas. The farther you move from $x = \pm a$, the wider the opening of the hyperbolas.

## 5.3 Can the "Hole" in the Middle Close?

On the $z = 0$ plane (which is the "waist"), the cross-section equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is always an ellipse, with semi-axes of length $a$ and $b$. No matter how you adjust the parameters, this ellipse can become very small (when $a$ and $b$ are both small), but it will never shrink to a single point.

In other words, a hyperboloid of one sheet always has a "hole" or a "passageway" through the middle. This is why it is called "of one sheet"—the entire surface is connected, and you can travel from the top to the bottom along the surface without jumping. If the middle were to close, it would become two separate parts, and it would be a "hyperboloid of two sheets."

Looking directly down from the top, even if the parameters are very small, if you zoom in carefully, you can still see that small elliptical opening in the middle. This feature is an essential property of the hyperboloid of one sheet and cannot be eliminated by adjusting the parameters.

# 6 Hyperboloid of Two Sheets

A hyperboloid of two sheets looks like two separate bowls, one opening upward and the other downward. Its standard equation is:

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Note that here the z-term is positive, while the x and y terms are negative, which determines its unique shape.

## 6.1 Why Increasing $a$ and $b$ Flattens the Hyperboloid

This phenomenon might seem a bit counterintuitive at first. Let's understand it from the equation. If we write it in the form $z^2 = c^2(1 + x^2/a^2 + y^2/b^2)$, what happens when you fix the values of $x$ and $y$ and then increase $a$ and $b$ (which means decreasing $A = 1/a^2$ and $B = 1/b^2$)?

The term $x^2/a^2 + y^2/b^2$ in the parentheses will get smaller, so the value of $z$ will also become smaller. In other words, for the same horizontal position $(x, y)$, the height of the surface decreases. This is like pressing down on the two bowls to make them flatter—although they might extend farther horizontally, they become gentler in the vertical direction.

You can think of it this way: $a$ and $b$ control the degree of "stretching" in the horizontal directions. The larger they are, the faster the surface spreads out horizontally, and relatively speaking, the change in the vertical direction becomes more gradual. It's like spreading out a piece of clay; the more you spread it, the thinner it gets, and its slope naturally becomes gentler.

## 6.2 The Issue of the Gap Between the Two Sheets

This is an essential feature of the hyperboloid of two sheets—the two sheets never meet! Let's see why.

From the equation $z^2/c^2 - x^2/a^2 - y^2/b^2 = 1$, we can see that the left side must equal 1. When $x = y = 0$ (i.e., on the z-axis), the equation becomes $z^2/c^2 = 1$, so $z = \pm c$. This means the closest points of the two sheets are at $z = c$ and $z = -c$, respectively, and there is always a distance of $2c$ between them.

Even if you make $c$ very small, the gap will shrink but will never disappear. If you were to let $c \to 0$, the entire equation would degenerate and no longer be a hyperboloid of two sheets. This gap is the reason for the name "of two sheets"—the two parts are always separate.

## 6.3 The Limiting Case When $a = b = c$ and They Are Very Small

When $a = b = c$ are all equal and very small, the equation can be written as $z^2 - x^2 - y^2 = 1/A$ (here $A$ is very small, so $1/A$ is very large).

Now imagine $A$ getting smaller and smaller (approaching 0), so $1/A$ gets larger and larger. If we were to rescale the coordinates, letting $z' = z/\sqrt{1/A}$, $x' = x/\sqrt{1/A}$, and $y' = y/\sqrt{1/A}$, the equation would return to $z'^2 - x'^2 - y'^2 = 1$. But when viewed in the original coordinate system, this surface has been extremely stretched.

More intuitively, as $A \to 0$, the hyperboloid of two sheets increasingly resembles a double cone! The two sheets become sharper and sharper, the gap between them becomes relatively smaller, and the overall shape tends toward two cones with their vertices facing each other.

This can be proven algebraically: when $A$ is very small, the equation has solutions only where $|z|$ is very large, and in these regions, the equation is approximately $z^2 \approx x^2 + y^2$, which is the equation of a cone. So, it is mathematically reasonable to say that the "limit" of the family of hyperboloids of two sheets as the parameter approaches 0 is a double cone.

This relationship is quite beautiful—both the hyperboloid of one sheet and the hyperboloid of two sheets approach a double cone in their limiting cases, as if they are, in a sense, different members of the same family of geometric objects.