16 Vector Calculus

16.1 Vector Fields

Context

- Vector Field in $\mathbb{R}^2$and $\mathbb{R}^3$
- Gradient Fields

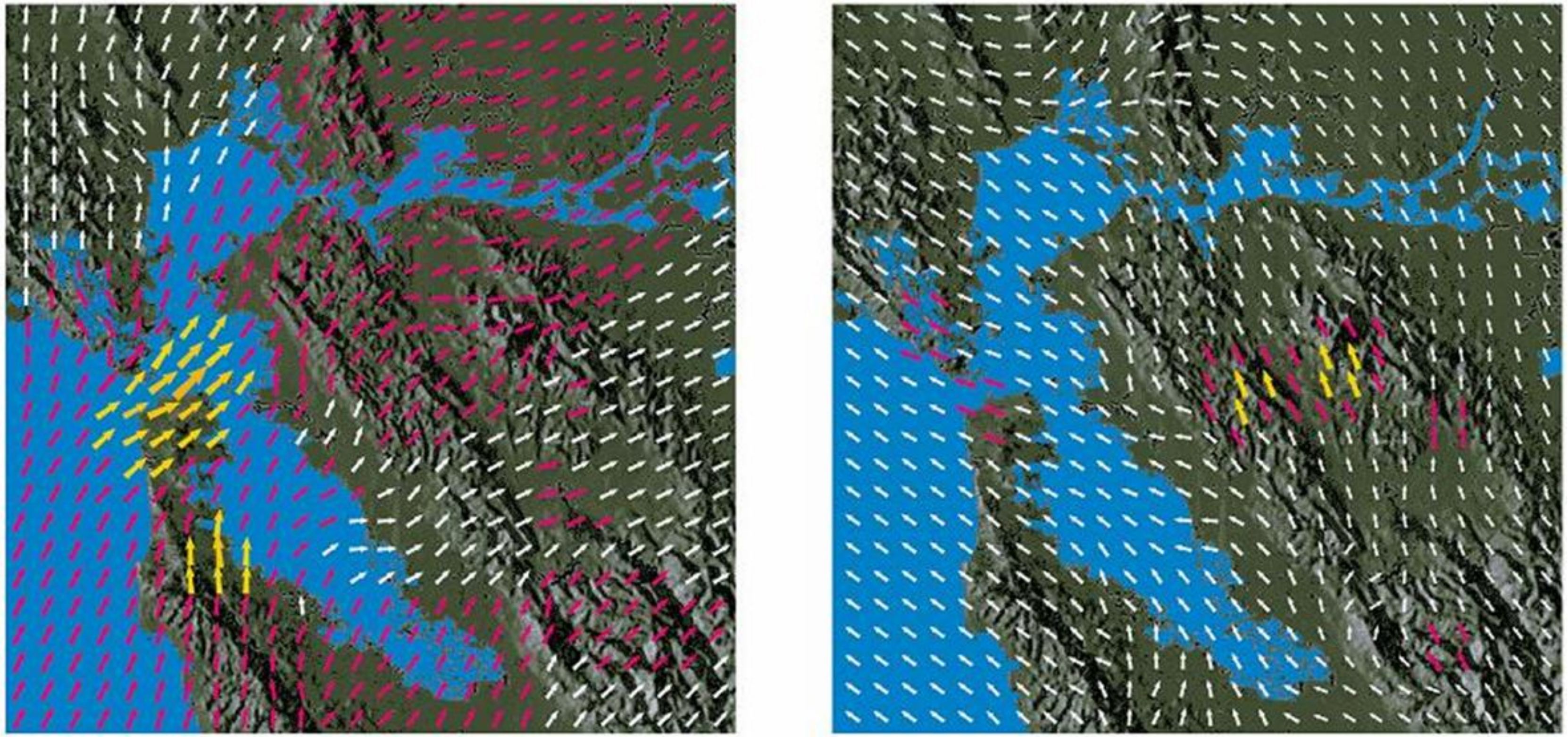
Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (1 of 8)

The vectors in Figure 1 are air velocity vectors that indicate the wind speed and direction at points 10 m above the surface elevation in the San Francisco Bay area.

We see at a glance from the largest arrows in part (a) that the greatest wind speeds at that time occurred as the winds entered the bay across the Golden Gate Bridge. Part (b) shows the very different wind pattern 12 hours earlier.

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (2 of 8)



(a) 6:00 PM, March 1, 2010

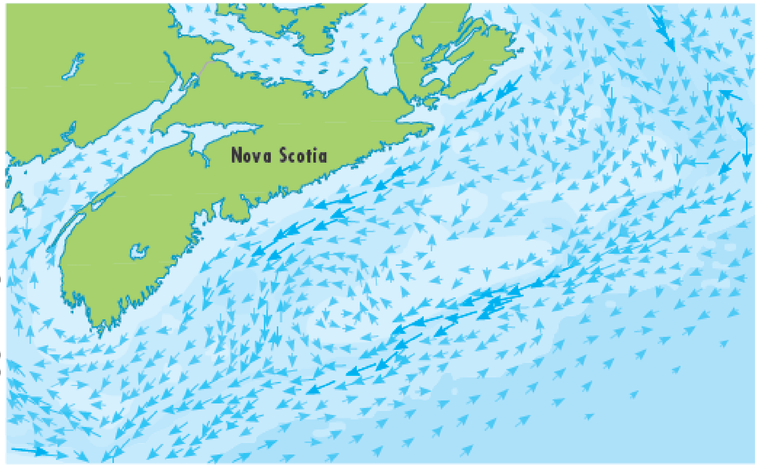
(b) 6:00 AM, March 1, 2010

Velocity vector fields showing San Francisco Bay wind patterns on a particular spring day

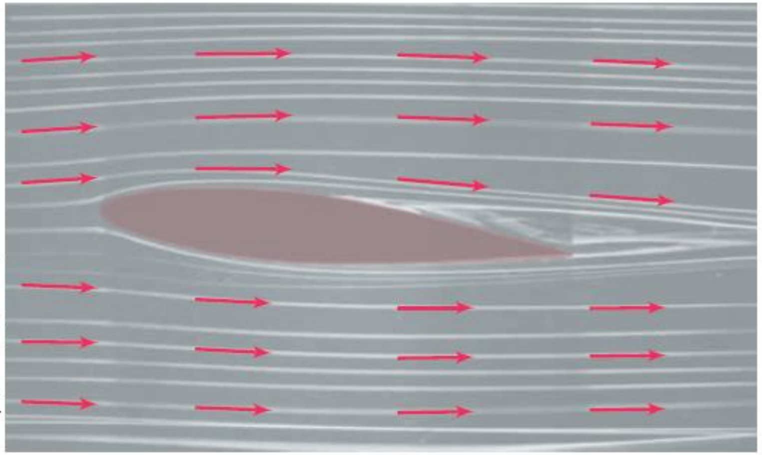
Figure 1

Associated with every point in the air we can imagine a wind velocity vector. This is an example of a *velocity vector field.*

Other examples of velocity vector fields are illustrated in Figure 2: ocean currents and flow past an airfoil.



(a) Ocean currents off the coast of Nova Scotia



(b) Airflow past an inclined airfoil

Velocity vector fields

Figure 2

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (4 of 8)

Another type of vector field, called a *force field*, associates a force vector with each point in a region. An example is the gravitational force field.

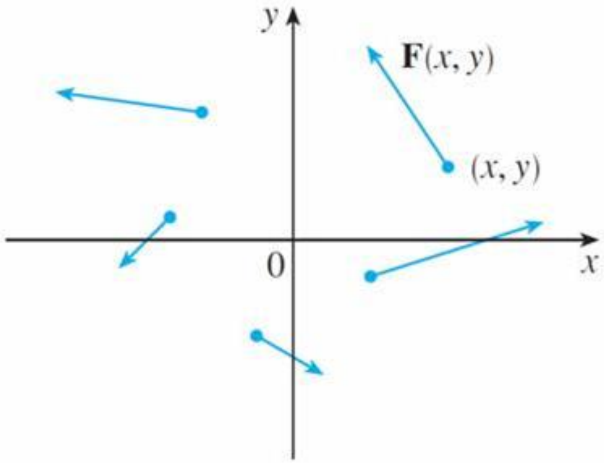
In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in V_2 (or V_3).

1 Definition Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function **F** that assigns to each point (x, y) in D a two-dimensional vector **F**(x, y).

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (5 of 8)

The best way to picture a vector field is to draw the arrow representing the vector **F**(x, y) starting at the point (x, y).

Of course, it's impossible to do this for all points (x, y), but we can gain a reasonable impression of **F** by doing it for a few representative points in D as in Figure 3.



Vector field on $\mathbb{R}^2$

Figure 3

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (6 of 8)

Since **F**(*x*, *y*) is a two-dimensional vector, we can write it in terms of its **component functions** *P* and *Q* as follows:

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y) \rangle$$

or, for short,

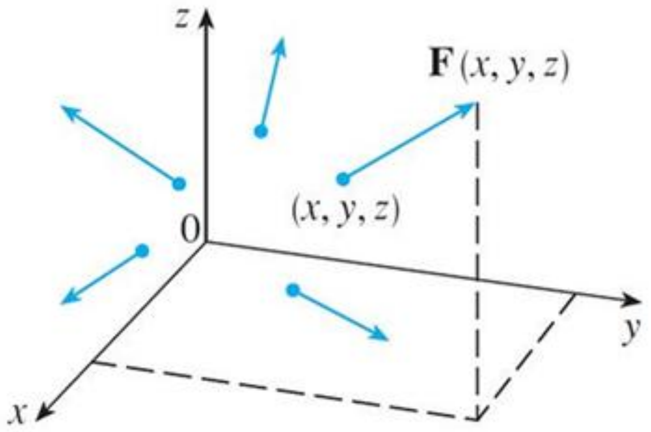
$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$

Notice that *P* and *Q* are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

2 Definition Let *E* be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function **F** that assigns to each point (*x*, *y*, *z*) in E a three-dimensional vector **F**(*x*, *y*, *z*).

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (7 of 8)

A vector field **F** on $\mathbb{R}^3$ is pictured in Figure 4.



Vector field on $\mathbb{R}^3$

Figure 4

We can express it in terms of its component functions *P*, *Q*, and *R* as

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (8 of 8)

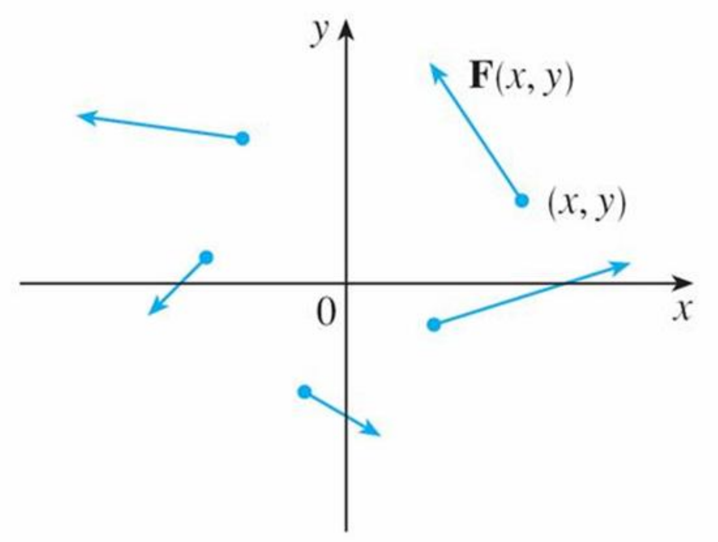
As with the vector functions, we can define continuity of vector fields and show that **F** is continuous if and only if its component functions *P*, *Q*, and *R* are continuous.

We sometimes identify a point (x, y, z) with its position vector $\mathbf{x} = \langle x, y, z \rangle$ and write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$.

Then **F** becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector $\mathbf{x}$.

Example 1

A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$ as in Figure 3.

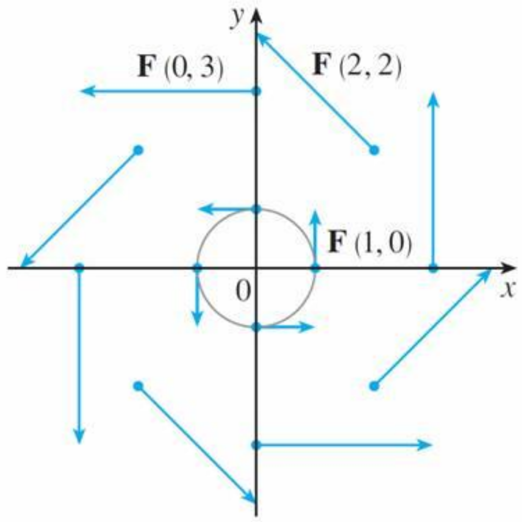


Vector field on?$\mathbb{R}^2$

Figure 3

Example 1 – Solution (1 of 5)

Since **F**(1, 0) = **j**, we draw the vector $\mathbf{j} = \langle 0, 1 \rangle$ starting at the point (1, 0) in Figure 5.



F (x, y) = −y **i** + x **j**

Figure 5

Since **F**(0, 1) = −**i**, we draw the vector $\langle -1, 0 \rangle$ with starting point (0, 1).

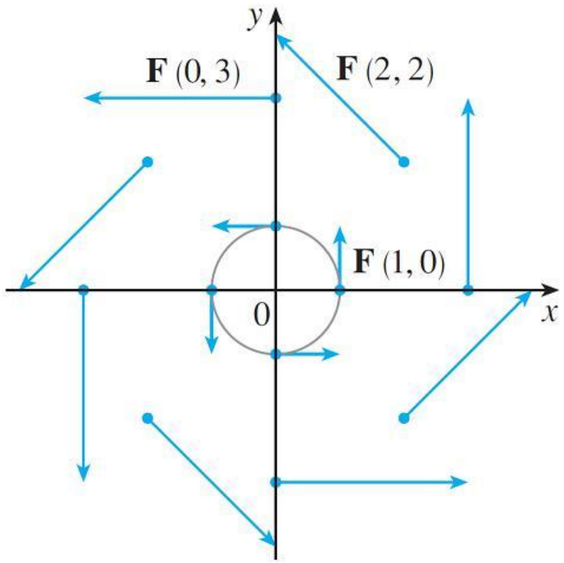
Example 1 – Solution (2 of 5)

Continuing in this way, we calculate several other representative values of $\mathbf{F}(x, y)$ in the table and draw the corresponding vectors to represent the vector field in Figure 5.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1 \rangle$	$(-1, 0)$	$\langle 0, -1 \rangle$
$(2, 2)$	$\langle -2, 2 \rangle$	$(-2, -2)$	$\langle 2, -2 \rangle$
$(3, 0)$	$\langle 0, 3 \rangle$	$(-3, 0)$	$\langle 0, -3 \rangle$
$(0, 1)$	$\langle -1, 0 \rangle$	$(0, -1)$	$\langle 1, 0 \rangle$
$(-2, 2)$	$\langle -2, -2 \rangle$	$(2, -2)$	$\langle 2, 2 \rangle$
$(0, 3)$	$\langle -3, 0 \rangle$	$(0, -3)$	$\langle 3, 0 \rangle$

Example 1 – Solution (3 of 5)

It appears from Figure 5 that each arrow is tangent to a circle with center the origin.



$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

Figure 5

Example 1 – Solution (4 of 5)

To confirm this, we take the dot product of the position vector **x** = x **i** + y **j** with the vector **F**(**x**) = **F**(x, y):

$$\begin{aligned} \mathbf{x} \cdot \mathbf{F}(\mathbf{x}) &= (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j}) \\ &= -xy + yx \\ &= 0 \end{aligned}$$

This shows that **F**(x, y) is perpendicular to the position vector $\langle x, y \rangle$ and is therefore tangent to a circle with center the origin and radius $|\mathbf{x}| = \sqrt{x^2 + y^2}$.

Example 1 – Solution (5 of 5)

Notice also that

$$\begin{aligned} |\mathbf{F}(x, y)| &= \sqrt{(-y)^2 + x^2} \\ &= \sqrt{x^2 + y^2} \\ &= |\mathbf{x}| \end{aligned}$$

so the magnitude of the vector $\mathbf{F}(x, y)$ is equal to the radius of the circle.

Example 4 (1 of 5)

Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses *m* and *M* is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where *r* is the distance between the objects and *G* is the gravitational constant. (This is an example of an inverse square law.)

Let's assume that the object with mass *M* is located at the origin in $\mathbb{R}^3$.

(For instance, *M* could be the mass of the earth and the origin would be at its center.)

Example 4 (2 of 5)

Let the position vector of the object with mass *m* be $\mathbf{x} = \langle x, y, z \rangle$. Then $r = |\mathbf{x}|$, so $r^2 = |x|^2$.

The gravitational force exerted on this second object acts toward the origin, and the unit vector in this direction is

$$-\frac{\mathbf{x}}{|\mathbf{x}|}$$

Therefore the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z \rangle$ is

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x} \qquad (3)$$

Example 4 (3 of 5)

[Physicists often use the notation **r** instead of **x** for the position vector, so you may see Formula 3 written in the form $\mathbf{F} = -\left(\frac{mMG}{r^3}\right)\mathbf{r}$.]

The function given by Equation 3 is an example of a vector field, called the **gravitational field**, because it associates a vector [the force $\mathbf{F}(\mathbf{x})$] with every point **x** in space.

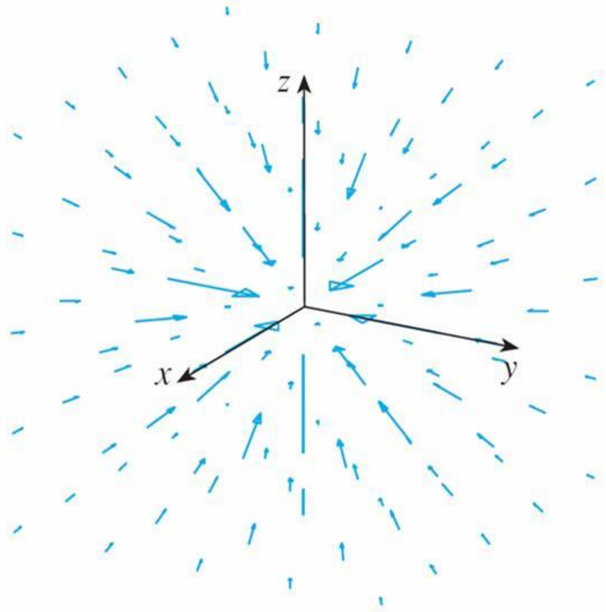
Example 4 (4 of 5)

Formula 3 is a compact way of writing the gravitational field, but we can also write it in terms of its component functions by using the facts that **x** = x **i** + y **j** + z **k** and $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$:

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}\mathbf{i} + \frac{-mMGy}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}\mathbf{j} + \frac{-mMGz}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}\mathbf{k}$$

Example 4 (5 of 5)

The gravitational field **F** is pictured in Figure 14.



Gravitational force field

Figure 14

多选题 1分

设置

MAQ: Given the vector field $F(x, y) = -y\mathbf{i} + x\mathbf{j}$.
Which statements are true? (Select all that apply.)

A At $(2,0)$, $F(2,0)$ points in the $+\mathbf{j}$ direction

B For any $(x, y) \neq (0,0)$, $\mathbf{x} \cdot F(\mathbf{x}) = 0$, so $F(x, y)$ is tangent to a circle centered at the origin.

C For any (x, y), $|F(x, y)| = \sqrt{x^2 + y^2}$

D The vector field is radial outward (points along the position vector).

E At $(0, -3)$, $F(0, -3)$ points in the $-\mathbf{i}$ direction.

提交

Solution

- $F(2,0) = -(0)\mathbf{i} + 2\mathbf{j} = 2\mathbf{j}$, so it points in $+\mathbf{j}$.
- Let $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$. Then $\mathbf{x} \cdot F(\mathbf{x}) = x(-y) + y(x) = 0$,

so F is perpendicular to the position vector, hence tangent to circles centered at the origin.

- $|F(x,y)| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2}$.
- At $(0,-3)$, $F(0,-3) = 3\mathbf{i}$, which is $+\mathbf{i}$, not $-\mathbf{i}$.

- Since F is perpendicular to $\mathbf{x}$, it is not radial outward.

Gradient Fields

Gradient Fields (1 of 3)

If f is a scalar function of two variables, we know that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

Therefore ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

Example 6

Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of *f*. How are they related?

Solution:

The gradient vector field is given by

$$
\begin{aligned}
\nabla f(x, y) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \\
&= 2xy\,\mathbf{i} + \left(x^2 - 3y^2\right)\mathbf{j}
\end{aligned}
$$

Example 6 – Solution (1 of 2)

Figure 15 shows a contour map of *f* with the gradient vector field.

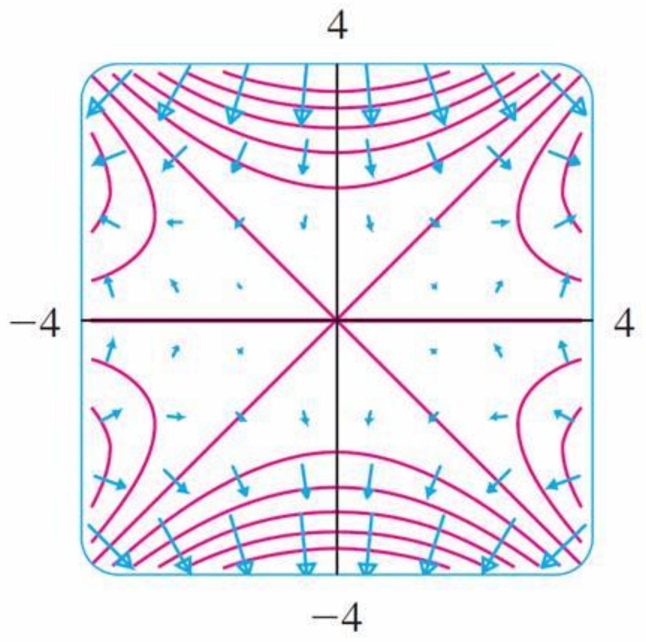


Figure 15

Notice that the gradient vectors are perpendicular to the level curves.

Example 6 – Solution (2 of 2)

Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart.

That's because the length of the gradient vector is the value of the directional derivative of *f* and closely spaced level curves indicate a steep graph.

Gradient Fields (2 of 3)

A vector field **F** is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function *f* such that $\mathbf{F} = \nabla f$.

In this situation *f* is called a **potential function** for **F**.

Not all vector fields are conservative, but such fields do arise frequently in physics.

Gradient Fields (3 of 3)

For example, the gravitational field **F** in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}
\nabla f(x, y, z) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\
&= \frac{-mMGx}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}\mathbf{i} + \frac{-mMGy}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}\mathbf{j} + \frac{-mMGz}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}\mathbf{k} \\
&= \mathbf{F}(x, y, z)
\end{aligned}$$

单选题 1分

设置

MCQ: Given the vector field
$F(x, y) = (2xy)\mathbf{i} + (x^2)\mathbf{j}$.
Which scalar function f is a potential function such that $F = \nabla f$ (so F is a conservative vector field)?

A $f(x, y) = x^2 y$

B $f(x, y) = x^2 y + y$

C $f(x, y) = x^2 y + y^2$

D No such function f exists.

提交

Recap

- Vector Field in $\mathbb{R}^2$and $\mathbb{R}^3$
- Gradient Fields