Solutions

1. (a) C_1: **Positive**. Vectors attached to points on C_1 form an acute angle with the tangent direction of C_1 at that point. Therefore the dot products $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ are all positive, and hence

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

is positive as well.

(b) C_2: **Negative**. Half of the vectors attached to points on C_2 form an acute angle with the tangent direction of C_2 at that point and half an obtuse angle, but those forming an obtuse angle are larger in length (and the angle is more obtuse). Hence the integral over the second quartercircle of C_2 (in the given orientation) is negative and larger in absolute value than the integral over the first quartercircle.

2. (a) We have

$$F(x) = \int_0^1 f(x,t)\, dt \quad \text{with} \quad f : \mathbb{R} \times [0,1] \to \mathbb{R},\ (x,t) \mapsto \frac{e^{xt}}{1+t}.$$

The partial derivative

$$\frac{\partial}{\partial x} f(x,t) = \frac{t\, e^{xt}}{1+t}$$

is continuous on $\mathbb{R} \times [0,1]$ as a function of two variables, and the domain of integration $[0,1]$ is compact. Therefore F is differentiable.

(b) Differentiating under the integral sign gives

$$F'(x) = \int_0^1 \frac{\partial}{\partial x} f(x,t)\, dt = \int_0^1 \frac{t\, e^{xt}}{1+t}\, dt.$$

Hence

$$F(x) + F'(x) = \int_0^1 \left(\frac{e^{xt}}{1+t} + \frac{t\, e^{xt}}{1+t} \right) dt = \int_0^1 \frac{(1+t)e^{xt}}{1+t}\, dt = \int_0^1 e^{xt}\, dt.$$

For $x \neq 0$,

$$\int_0^1 e^{xt}\, dt = \left[\frac{e^{xt}}{x} \right]_0^1 = \frac{e^x - 1}{x}.$$

Thus F solves

$$y' + y = \frac{e^x - 1}{x} \quad \text{on } \mathbb{R} \setminus \{0\}.$$