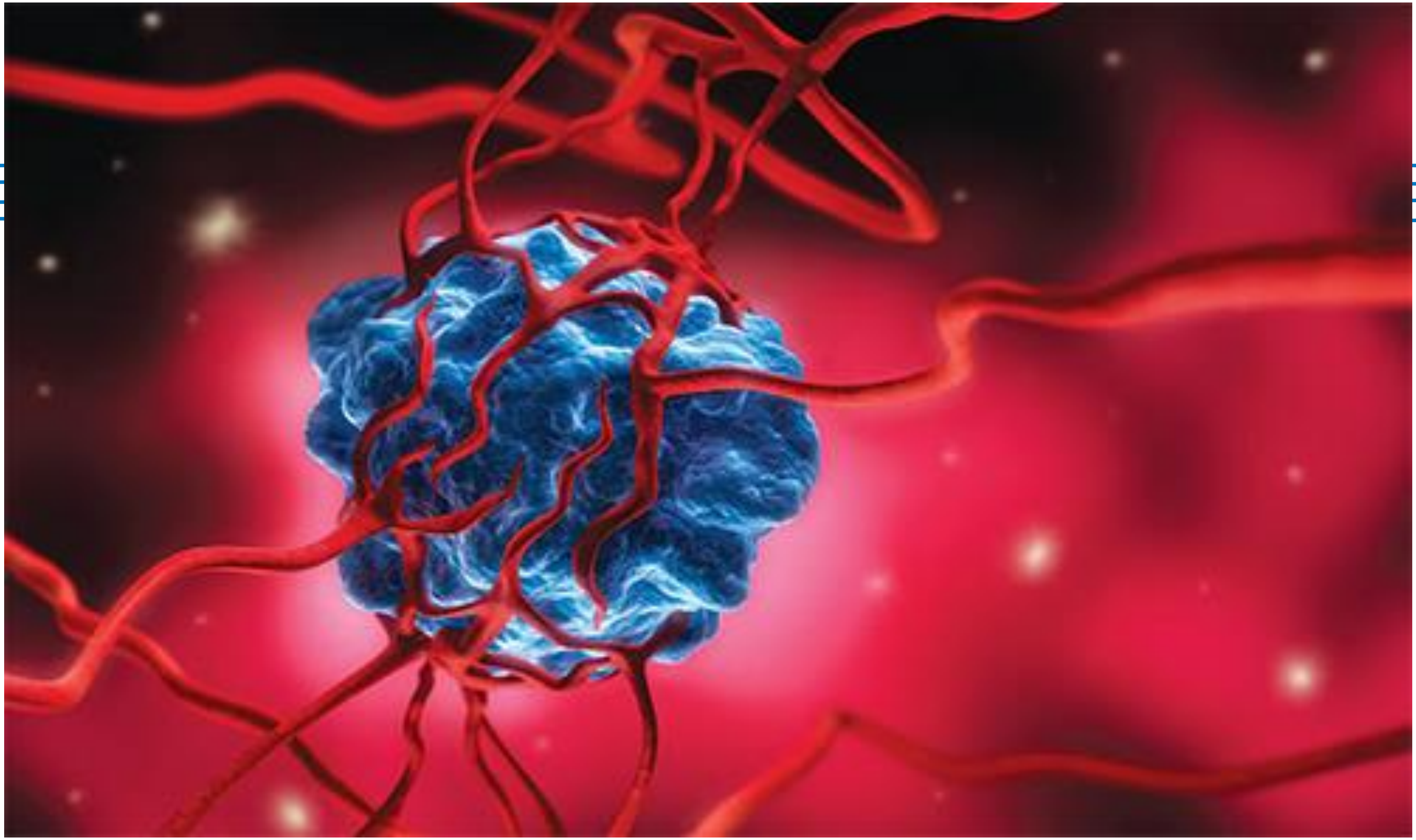
15 Multiple Integrals



15.4 Applications of Double Integrals

Density and Mass

Density and Mass (1 of 5)

We were able to use single integrals to compute moments and the center of mass of a thin plate or lamina with constant density.

But now, equipped with the double integral, we can consider a lamina with variable density.

Suppose the lamina occupies a region D of the xy-plane and its **density** (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D.

Density and Mass (2 of 5)

This means that

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of a small rectangle that contains (x, y) and the limit is taken as the dimensions of the rectangle approach 0. (See Figure 1.)

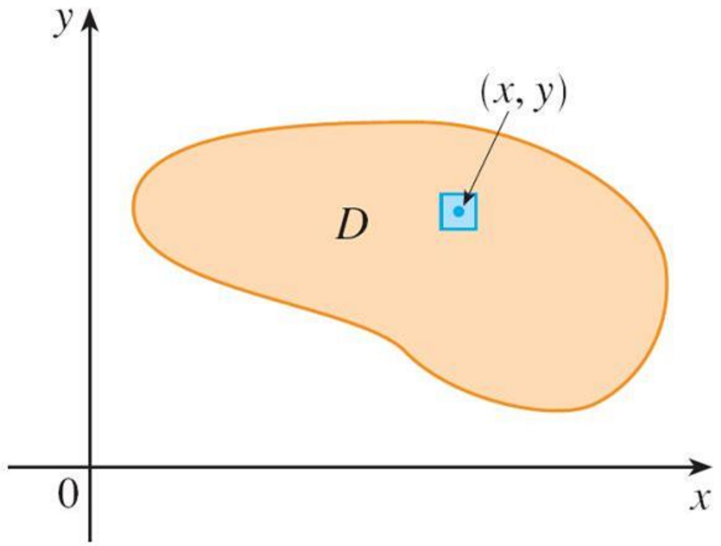


Figure 1

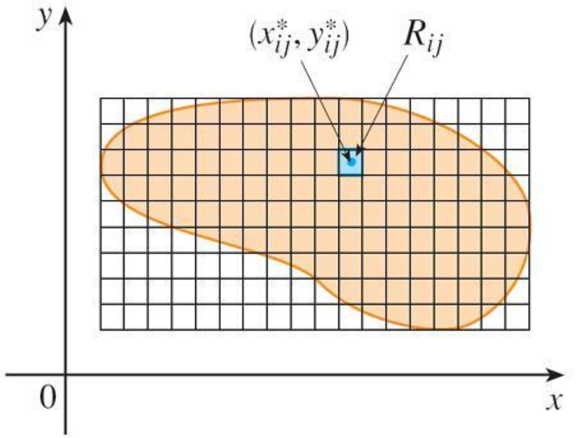
Density and Mass (3 of 5)

To find the total mass m of the lamina we divide a rectangle R containing D into subrectangles R_{ij} of the same size (as in Figure 2) and consider $\rho(x, y)$ to be 0 outside D.

If we choose a point (x_{ij}^*, y_{ij}^*) in R_{ij}, then the mass of the part of the lamina that occupies R_{ij} is approximately $\rho\left(x_{ij}^*, y_{ij}^*\right)\Delta A$, where ΔA is the area of R_{ij}.

If we add all such masses, we get an approximation to the total mass:

$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A$$



The mass of each subrectangle R_{ij} is approximately $\rho\left(x_{ij}^*, y_{ij}^*\right)\Delta A$.

Figure 2

Density and Mass (4 of 5)

If we now increase the number of subrectangles, we obtain the total mass *m* of the lamina as the limiting value of the approximations:

1 $$m = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*,\, y_{ij}^*)\, \Delta A = \iint_D \rho(x,\, y)\, dA$$

Physicists also consider other types of density that can be treated in the same manner.

Density and Mass (5 of 5)

For example, if an electric charge is distributed over a region *D* and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in *D*, then the **total charge** *Q* is given by

$$\mathbf{2} \qquad Q = \iint_D \sigma(x, y)\, dA$$

Example 1

Charge is distributed over the triangular region D in Figure 3 so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.

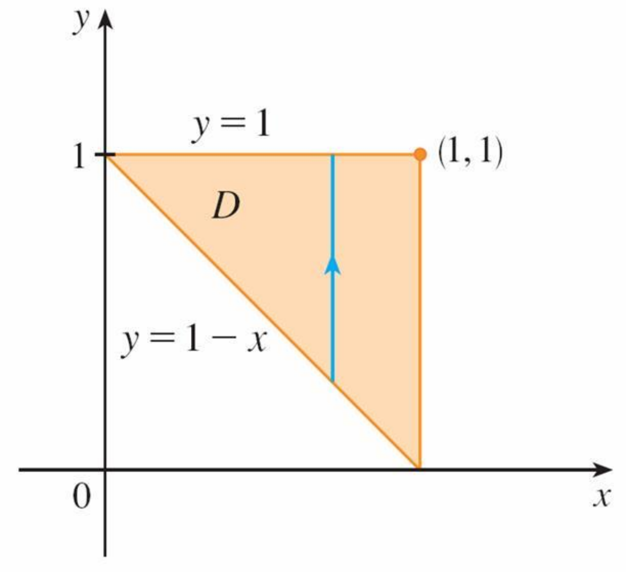


Figure 3

Example 1 – Solution (1 of 2)

From Equation 2 and Figure 3 we have

$$Q = \iint_D \sigma(x, y)\, dA$$

$$= \int_0^1 \int_{1-x}^1 xy \; dy \; dx$$

$$= \int_0^1 \left[x\frac{y^2}{2} \right]_{y=1-x}^{y=1} dx$$

$$= \int_0^1 \frac{x}{2}[1^2 - (1-x)^2]\; dx$$

Example 1 – Solution (2 of 2)

$$= \tfrac{1}{2}\int_0^1 (2x^2 - x^3)\, dx$$

$$= \frac{1}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{5}{24}$$

Thus the total charge is $\frac{5}{24}$ C.

Moments and Centers of Mass

Moments and Centers of Mass (1 of 4)

We have found the center of mass of a lamina with constant density; here we consider a lamina with variable density.

Suppose the lamina occupies a region D and has density function $\rho(x, y)$.

We know that we defined the moment of a particle about an axis as the product of its mass and its directed distance from the axis.

We divide D into small rectangles.

Moments and Centers of Mass (2 of 4)

Then the mass of R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\,\Delta A$, so we can approximate the moment of R_{ij} with respect to the x-axis by

$$\left[\rho(x_{ij}^*, y_{ij}^*)\,\Delta A\right] y_{ij}^*$$

If we now add these quantities and take the limit as the number of subrectangles becomes large, we obtain the **moment** of the entire lamina **about the *x*-axis**:

3

$$M_x = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D y\,\rho(x, y)\,dA$$

Moments and Centers of Mass (3 of 4)

Similarly, the **moment about the *y*-axis** is

4 $$M_y = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x \rho(x, y)\, dA$$

As before, we define the center of mass $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$.

The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass.

Moments and Centers of Mass (4 of 4)

Thus the lamina balances horizontally when supported at its center of mass (see Figure 4).

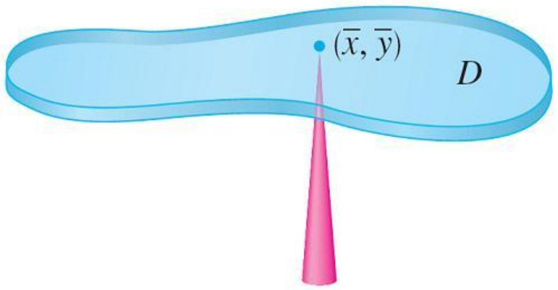


Figure 4

5 The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region *D* and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x, y)\, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x, y)\, dA$$

where the mass *m* is given by

$$m = \iint_D \rho(x, y)\, dA$$

Example 2

Find the mass and center of mass of a triangular lamina with vertices (0, 0), (1, 0), and (0, 2) if the density function is $\rho(x, y) = 1 + 3x + y$.

Solution:

The triangle is shown in Figure 5.

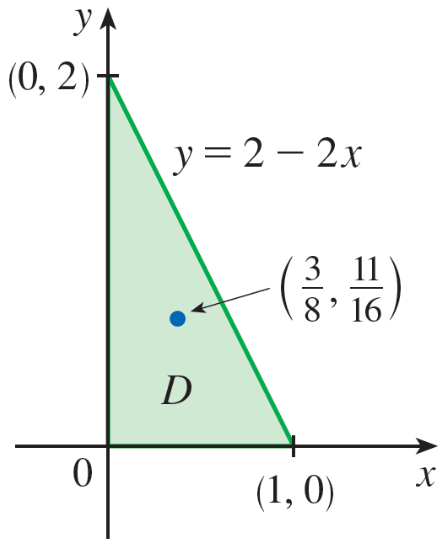


Figure 5

Example 2 – Solution (1 of 2)

(Note that the equation of the upper boundary is $y = 2 - 2x$.) The mass of the lamina is

$$m = \iint_D \rho(x, y)\, dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)\, dy\, dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= 4 \int_0^1 (1 - x^2)\, dx = 4\left[x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3}$$

Example 2 – Solution (2 of 2)

Then the formulas in (5) give

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)\, dA = \tfrac{3}{8}\int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\, dy\, dx$$

$$= \frac{3}{8}\int_0^1 \left[xy + 3x^2y + x\frac{y^2}{2}\right]_{y=0}^{y=2-2x} dx$$

$$= \frac{3}{2}\int_0^1 (x - x^3)\, dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{8}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\, dA = \tfrac{3}{8}\int_0^1 \int_0^{2-2x} (y + 3xy + y^2)\, dy\, dx$$

$$= \frac{3}{8}\int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3}\right]_{y=0}^{y=2-2x} dx = \tfrac{1}{4}\int_0^1 (7 - 9x - 3x^2 + 5x^3)\, dx$$

$$= \frac{1}{4}\left[7x - 9\frac{x^2}{2} - x^3 + 5\frac{x^4}{4}\right]_0^1 = \frac{11}{16}$$

The center of mass is at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$.

Moment of Inertia

Moment of Inertia (1 of 7)

The **moment of inertia** (also called the **second moment**) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis.

We extend this concept to a lamina with density function $\rho(x, y)$ and occupying a region D by proceeding as we did for ordinary moments.

We divide D into small rectangles, approximate the moment of inertia of each subrectangle about the x-axis, and take the limit of the sum as the number of subrectangles becomes large.

Moment of Inertia (2 of 7)

The result is the **moment of inertia** of the lamina **about the *x*-axis:**

$$\mathbf{6} \qquad I_x = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y^2 \rho(x, y)\, dA$$

Similarly, the **moment of inertia about the *y*-axis** is

$$\mathbf{7} \qquad I_y = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x^2 \rho(x, y)\, dA$$

Moment of Inertia (3 of 7)

It is also of interest to consider the **moment of inertia about the origin**, also called the **polar moment of inertia:**

8 $$I_0 = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2 \right] \rho(x_{ij}^*,\, y_{ij}^*)\, \Delta A = \iint_D (x^2 + y^2) \rho(x,\, y)\, dA$$

Note that $I_0 = I_x + I_y$.

Example 4

Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin, and radius a.

Solution:

The boundary of D is the circle $x^2 + y^2 = a^2$ and in polar coordinates D is described by $0 \le \theta \le 2\pi$, $0 \le r \le a$.

By Formula 6,

$$I_x = \iint_D y^2 \rho dA = \rho \int_0^{2\pi} \int_0^a (r \sin\theta)^2 \; r \; dr \; d\theta$$

$$= \rho \int_0^{2\pi} \sin^2 \theta d\theta \int_0^a r^3 dr = \rho \int_0^{2\pi} \frac{1}{2}(1 - \cos 2\theta) d\theta \int_0^a r^3 \; dr$$

Example 4 – Solution

$$= \frac{\rho}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^a = \frac{\pi\rho a^4}{4}$$

Similarly, Formula 7 gives

$$I_y = \iint_D x^2 \rho \, dA = \rho \int_0^{2\pi} \int_0^a (r\cos\theta)^2 \; r \; dr \; d\theta$$

$$= \rho \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta)\, d\theta \int_0^a r^3 \; dr = \frac{\pi\rho a^4}{4}$$

(From the symmetry of the problem, it is expected that $I_x = I_y$.) We could use Formula 8 to compute I_0 directly, or use

$$I_0 = I_x + I_y = \frac{\pi\rho a^4}{4} + \frac{\pi\rho a^4}{4} = \frac{\pi\rho a^4}{2}$$

Moment of Inertia (4 of 7)

In Example 4 notice that the mass of the disk is

$$m = \text{density} \times \text{area} = \rho(\pi a^2)$$

so the moment of inertia of the disk about the origin (like a wheel about its axle) can be written as

$$I_0 = \frac{\pi\rho a^4}{2} = \tfrac{1}{2}(\rho\pi a^2)a^2 = \tfrac{1}{2}ma^2$$

Thus if we increase the mass or the radius of the disk, we thereby increase the moment of inertia.

Moment of Inertia (5 of 7)

In general, the moment of inertia plays much the same role in rotational motion that mass plays in linear motion.

The moment of inertia of a wheel is what makes it difficult to start or stop the rotation of the wheel, just as the mass of a car is what makes it difficult to start or stop the motion of the car.

The **radius of gyration of a lamina about an axis** is the number R such that

$$mR^2 = I \qquad \textbf{9}$$

Moment of Inertia (6 of 7)

Where *m* is the mass of the lamina *I* is the moment of inertia about the given axis. Equation 9 says that if the mass of the lamina were concentrated at a distance *R* from the axis, then the moment of inertia of this "point mass" would be the same as the moment of inertia of the lamina.

Moment of Inertia (7 of 7)

In particular, the radius of gyration $\bar{\bar{y}}$ with respect to the *x*-axis and the radius of gyration $\bar{\bar{x}}$ with respect to the *y*-axis are given by the equations

$$\mathbf{10} \qquad m\bar{\bar{y}}^2 = I_x \qquad m\bar{\bar{x}}^2 = I_y$$

Thus $(\bar{\bar{x}}, \bar{\bar{y}})$ is the point at which the mass of the lamina can be concentrated without changing the moments of inertia with respect to the coordinate axes. (Note the analogy with the center of mass.)

Probability

Probability (1 of 5)

We have considered the *probability density function* f of a continuous random variable X.

This means that $f(x) \geq 0$ for all x, $\int_{-\infty}^{\infty} f(x)\,dx = 1$, and the probability that X lies between a and b is found by integrating f from a to b:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Probability (2 of 5)

Now we consider a pair of continuous random variables *X* and *Y*, such as the lifetimes of two components of a machine or the height and weight of an adult female chosen at random.

The **joint density function** of *X* and *Y* is a function *f* of two variables such that the probability that (*X*, *Y*) lies in a region *D* is

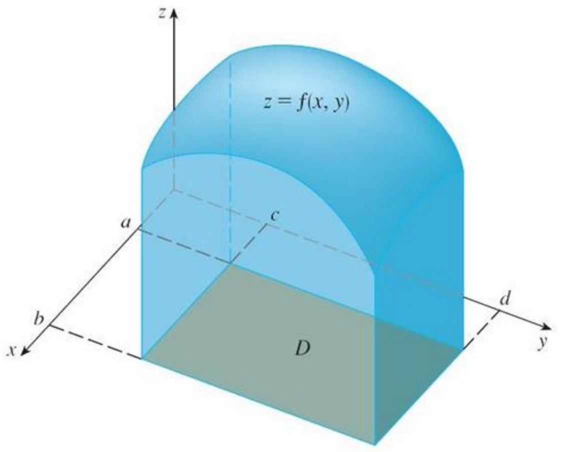
$$P\big((X,\ Y) \in D\big) = \iint_D f(x,\ y)\ dA$$

Probability (3 of 5)

In particular, if the region is a rectangle, then the probability that *X* lies between *a* and *b* and *Y* lies between *c* and *d* is

$$P(a \le X \le b,\ c \le Y \le d) = \int_a^b \int_c^d f(x,\ y)\, dy\, dx$$

(See Figure 7.)



The probability that *X* lies between *a* and *b* and *Y* lies between *c* and *d* is the volume that lies above the rectangle *D* = [*a*, *b*] × [*c*, *d*] and below the graph of the joint density function.

Figure 7

Probability (4 of 5)

Because probabilities aren't negative and are measured on a scale from 0 to 1, the joint density function has the following properties:

$$f(x,\ y) \geq 0 \qquad \iint_{\mathbb{R}^2} f(x,\ y)\ dA = 1$$

The double integral over $\mathbb{R}^2$ is an improper integral defined as the limit of double integrals over expanding circles or squares, and we can write

$$\iint_{\mathbb{R}^2} f(x,\ y)\ dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,\ y)\ dx\ dy = 1$$

Example 6

If the joint density function for X and Y is given by

$$f(x,\ y) = \begin{cases} C(x+2y) & \text{if } 0 \le x \le 10,\ 0 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$

find the value of the constant C. Then find $P(X \le 7,\ Y \ge 2)$.

Example 6 – Solution (1 of 2)

We find the value of C by ensuring that the double integral of f is equal to 1. Because $f(x, y) = 0$ outside the rectangle $[0, 10] \times [0, 10]$, we have

$$\begin{aligned}
\int_{-\infty}^{-\infty}\int_{-\infty}^{\infty} f(x, y)\, dy\, dx &= \int_0^{10}\int_0^{10} C(x + 2y)\, dy\, dx \\
&= C\int_0^{10} \Big[xy + y^2\Big]_{y=0}^{y=10} dx \\
&= C\int_0^{10} (10x + 100)\, dx \\
&= 1500C
\end{aligned}$$

Therefore $1500C = 1$ and so $C = \dfrac{1}{1500}$.

Example 6 – Solution (2 of 2)

Now we can compute the probability that X is at most 7 and Y is at least 2:

$$P(X \leq 7,\ Y \geq 2) = \int_{-\infty}^{7} \int_{2}^{\infty} f(x,\ y)\ dy\ dx$$

$$= \int_{0}^{7} \int_{2}^{10} \tfrac{1}{1500}(x + 2y)\ dy\ dx$$

$$= \frac{1}{1500} \int_{0}^{7} \Big[xy + y^2 \Big]_{y=2}^{y=10} dx$$

$$= \frac{1}{1500} \int_{0}^{7} (8x + 96)\ dx$$

$$= \frac{868}{1500} \approx 0.5787$$

Probability (5 of 5)

Suppose X is a random variable with probability density function $f_1(x)$ and Y is a random variable with density function $f_2(y)$.

Then X and Y are called **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x, y) = f_1(x)f_2(y)$$

We have modeled waiting times by using exponential density functions

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mu^{-1}e^{-\frac{t}{\mu}} & \text{if } t \geq 0 \end{cases}$$

where μ is the mean waiting time.

Expected Values

Expected Values (1 of 3)

We know that if *X* is a random variable with probability density function *f*, then its *mean* is

$$\mu = \int_{-\infty}^{\infty} xf(x)\, dx$$

Now if *X* and *Y* are random variables with joint density function *f*, we define the ***X*-mean** and ***Y*-mean**, also called the **expected values** of *X* and *Y*, to be

11 $$\mu_1 = \iint_{\mathbb{R}^2} xf(x, y)\, dA \qquad \mu_2 = \iint_{\mathbb{R}^2} yf(x, y)\, dA$$

Expected Values (2 of 3)

Notice how closely the expressions for μ_1 and μ_2 in (11) resemble the moments M_x and M_y of a lamina with density function ρ in Equations 3 and 4.

In fact, we can think of probability as being like continuously distributed mass. We calculate probability the way we calculate mass—by integrating a density function.

And because the total "probability mass" is 1, the expressions for $\bar{x}$ and $\bar{y}$ in (5) show that we can think of the expected values of X and Y, μ_1 and μ_2, as the coordinates of the "center of mass" of the probability distribution.

Expected Values (3 of 3)

In the next example we deal with normal distributions. A single random variable is *normally distributed* if its probability density function is of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{(2\sigma^2)}}$$

where μ is the mean and σ is the standard deviation.

Example 8

A factory produces (cylindrically shaped) roller bearings that are sold as having diameter 4.0 cm and length 6.0 cm. In fact, the diameters X are normally distributed with mean 4.0 cm and standard deviation 0.01 cm while the lengths Y are normally distributed with mean 6.0 cm and standard deviation 0.01 cm. Assuming that X and Y are independent, write the joint density function and graph it.

Find the probability that a bearing randomly chosen from the production line has either length or diameter that differs from the mean by more than 0.02 cm.

Example 8 – Solution (1 of 4)

We are given that X and Y are normally distributed with $\mu_1 = 4.0$, $\mu_2 = 6.0$ and $\sigma_1 = \sigma_2 = 0.01$.

So the individual density functions for X and Y are

$$f_1(x) = \frac{1}{0.01\sqrt{2\pi}} e^{\frac{-(x-4)^2}{0.0002}} \qquad f_2(y) = \frac{1}{0.01\sqrt{2\pi}} e^{\frac{-(y-6)^2}{0.0002}}$$

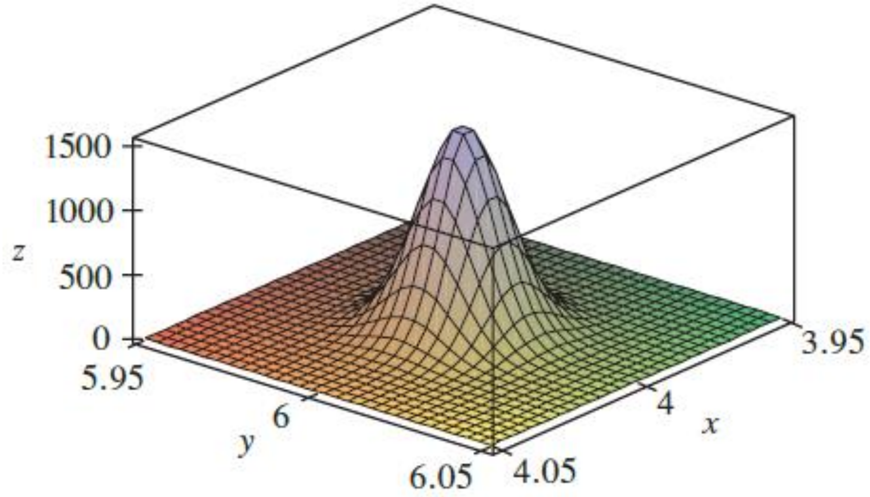
Since X and Y are independent, the joint density function is the product:

$$\begin{aligned} f(x, y) &= f_1(x) f_2(y) \\ &= \frac{1}{0.0002\pi} e^{-\frac{(x-4)^2}{0.0002}} e^{-\frac{(y-6)^2}{0.0002}} \end{aligned}$$

Example 8 – Solution (2 of 4)

$$= \frac{5000}{\pi} e^{-5000[(x-4)^2 + (y-6)^2]}$$

A graph of this function is shown in Figure 9.



Graph of the bivariate normal joint density function in Example 8

Figure 9

Example 8 – Solution (3 of 4)

Let's first calculate the probability that both *X* and *Y* differ from their means by less than 0.02 cm. Using a calculator or computer to estimate the integral, we have

$$P(3.98 < X < 4.02,\ 5.98 < Y < 6.02)$$

$$= \int_{3.98}^{4.02} \int_{5.98}^{6.02} f(x, y)\, dy\, dx$$

$$= \frac{5000}{\pi} \int_{3.98}^{4.02} \int_{5.98}^{6.02} e^{-5000[(x-4)^2 + (y-6)^2]}\, dy\, dx$$

$$\approx 0.91$$

Example 8 – Solution (4 of 4)

Then the probability that either X or Y differs from its mean by more than 0.02 cm is approximately

$$1 - 0.91 = 0.09$$