

**Question 1** (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

- The surface in  $\mathbb{R}^3$  with equation  $x^2y + y^2z + z^2x = 1$  is smooth.
- Suppose you start at the point  $(1, 1)$  in the  $(x, y)$ -plane and follow (and continuously adjust) the direction of steepest ascent of  $f(x, y) = \frac{x-y}{x+y}$ . After some time (provided you won't get tired) you will cross the  $x$ -axis at a point  $(x_0, 0)$  with  $x_0 > 2$ .
- If  $f: [0, 1] \rightarrow \mathbb{R}$  is continuous and  $f(0) = 0$  then  $\lim_{n \rightarrow \infty} \int_0^1 f(x^n) dx = 0$ .
- The equation  $x^2 + xy + y^2 = 3$  defines a circle, which is symmetric to the line  $y = x$ .
- There exists a subset  $D$  of the upper half plane  $\{(x, y) \in \mathbb{R}^2; y > 0\}$  whose set of accumulation points is equal to the real axis ( $x$ -axis).
- If  $\gamma$  is a closed path in  $\mathbb{R}^3$  satisfying  $\int_\gamma x dy + y dz + z dx = 0$ , we must have  $\int_\gamma x dz + y dx + z dy = 0$  as well.

**Question 2** (ca. 12 marks)

Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$f(x, y) = x^4 - x^3y - xy + y^2.$$

- Which obvious symmetry property does  $f$  have? What can you conclude from this about the graph and the contours of  $f$ ?
- Determine all critical points of  $f$  and their types.  
*Hint:* There are 5 critical points.
- Does  $f$  have a global extremum?
- Determine the extrema of  $f$  on the unit square  $Q = \{(x, y) \in \mathbb{R}^2; 0 \leq x, y \leq 1\}$ .

**Question 3** (ca. 12 marks)

The sphere  $x^2 + y^2 + z^2 = 9$  intersects the surface  $xy + yz + zx = 8$  in the first octant  $O = \{(x, y, z) \in \mathbb{R}^3; x, y, z \geq 0\}$  in a curve  $C$ .

- Show that  $C$  has no points on the boundary of  $O$ .
- Show that there exist points on  $C$  with minimum, resp., maximum height ( $z$ -coordinate).  
*Hint:*  $(2, 2, 1) \in C$ .
- Using the method of Lagrange multipliers on the interior of  $O$ , determine all those points.

*Note:* Don't forget to check for points on  $C$  where the Jacobi matrix of the vectorial constraint doesn't have full row rank. In fact there are no such points, but this requires a proof.

- d) At the point  $(2, 2, 1)$  the curve  $C$  admits locally a parametrization  $\gamma(x) = (x, h(x), k(x))$  with functions  $h, k: (2 - \epsilon, 2 + \epsilon) \rightarrow \mathbb{R}$ . Determine  $h'(2)$  and  $k'(2)$ .

**Question 4** (ca. 12 marks)

Consider the transformation

$$T(s, t, u) = (us \cos t, us \sin t, us + ut)$$

from the region  $U = \{(s, t, u) \in \mathbb{R}^3; 0 < s < t < 2\pi, 0 < u < 1\}$  to the region  $V = T(U) \subset \mathbb{R}^3$ , and the “helicoid”

$$S = \{(s \cos t, s \sin t, s + t); 0 < s < t < 2\pi\}$$

bounding  $V$  from above.

- a) Show that  $T: U \rightarrow V$  is a diffeomorphism (i.e.,  $T$  is one-to-one, and both  $T$  and  $T^{-1}$  are differentiable).
- b) Determine the volume of  $V$ .
- c) Express the surface area of  $S$  as an ordinary 1-dimensional Riemann integral.