14 Partial Derivatives

Context

- The Chain Rule
- The Chain Rule: General Version
- Implicit Differentiation

14.5 The Chain Rule

The Chain Rule (1 of 1)

We know that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

The Chain Rule: Case 1

The Chain Rule: Case 1 (1 of 2)

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function.

The first version (Theorem 1) deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t.

This means that z is indirectly a function of t, $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t. We assume that f is differentiable.

The Chain Rule: Case 1 (2 of 2)

We know that this is the case when f_x and f_y are continuous.

1 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Since we often write $\frac{\partial z}{\partial x}$ in place of $\frac{\partial f}{\partial x}$, we can rewrite the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

多选题 1分

设置

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\dfrac{dz}{dt}$ when $t = 0$.

A 0

B 3

C -3

D 6

提交

Example 1

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$.

Solution:

The Chain Rule gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

It's not necessary to substitute the expressions for x and y in terms of t.

Example 1 – Solution

We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$.

Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2\cos 0) + (0+0)(-\sin 0) = 6$$

The Chain Rule: Case 2

The Chain Rule: Case 2 (1 of 4)

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t: $x = g(s, t)$, $y = h(s, t)$.

Then z is indirectly a function of s and t and we wish to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

We know that in computing $\frac{\partial z}{\partial t}$ we hold s fixed and compute the ordinary derivative of z with respect to t.

Therefore we can apply Theorem 1 to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The Chain Rule: Case 2 (2 of 4)

A similar argument holds for $\frac{\partial z}{\partial s}$ and so we have proved the following version of the Chain Rule.

2 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Case 2 of the Chain Rule contains three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the **dependent** variable.

Example 3

If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Solution:

Applying Case 2 of the Chain Rule, we get

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(e^x \sin y\right)\left(t^2\right) + \left(e^x \cos y\right)\left(2st\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(e^x \sin y\right)\left(2st\right) + \left(e^x \cos y\right)\left(s^2\right)$$

Example 3 – Solution

If we wish, we can now express $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ solely in terms of *s* and *t* by substituting $x = st^2, y = s^2t$, to get

$$\frac{\partial z}{\partial s} = t^2 e^{st^2} \sin\left(s^2 t\right) + 2ste^{st^2} \cos\left(s^2 t\right)$$

$$\frac{\partial z}{\partial t} = 2ste^{st^2} \sin\left(s^2 t\right) + s^2 e^{st^2} \cos\left(s^2 t\right)$$

The Chain Rule: Case 2 (3 of 4)

Notice that Theorem 2 has one term for each intermediate variable and each of these terms resembles the one-dimensional Chain Rule in Equation 1.

To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2.

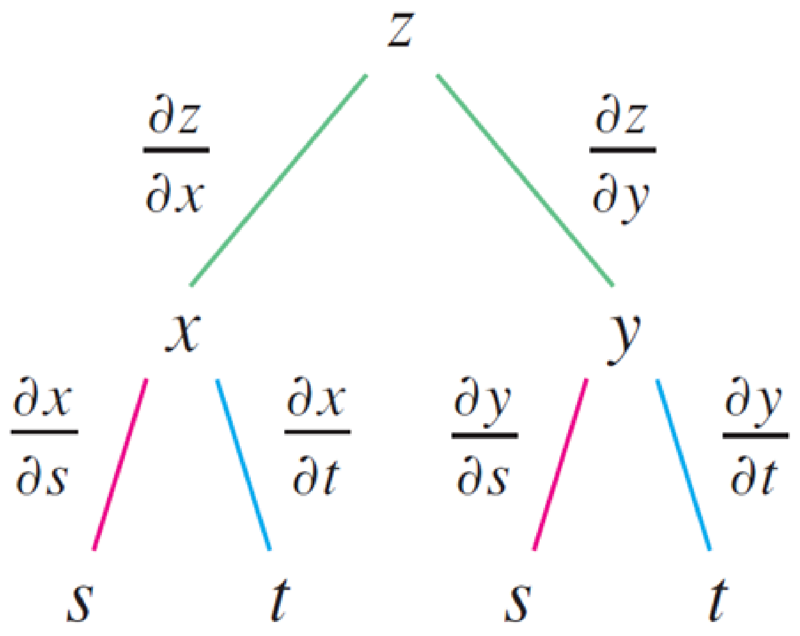

Figure 2

The Chain Rule: Case 2 (4 of 4)

We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y. Then we draw branches from x and y to the independent variables s and t.

On each branch we write the corresponding partial derivative. To find $\frac{\partial z}{\partial s}$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Similarly, we find $\frac{\partial z}{\partial t}$ by using the paths from z to t.

The Chain Rule: General Version

The Chain Rule: General Version (1 of 2)

Now we consider the general situation in which a dependent variable u is a function of n intermediate variables $x_1, \ldots, x_n$, each of which is, in turn, a function of m independent variables $t_1, \ldots, t_m$.

Notice that there are n terms, one for each intermediate variable. The proof is similar to that of Case 1.

The Chain Rule: General Version (2 of 2)

3 The Chain Rule (General Version) Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

Example 4

Write out the Chain Rule for the case where $\omega = f(x, y, z, t)$ and

$x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

Solutions:

We apply Theorem 3 with $n = 4$ and $m = 2$. Figure 3 shows the tree diagram. Although we haven't written the derivatives on the branches, it's understood that if a branch leads from y to u, then the partial derivative for that branch is

$\dfrac{\partial y}{\partial u}$.

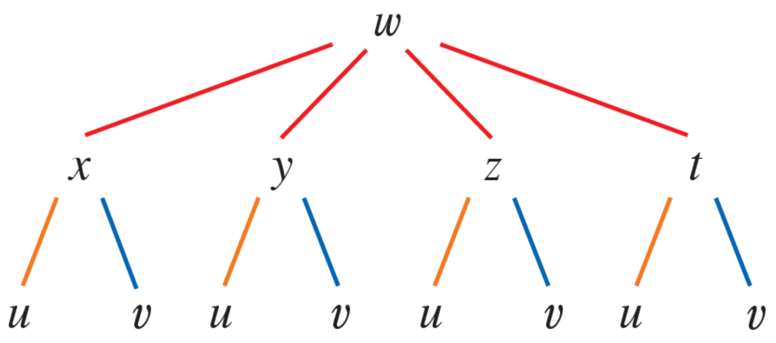

Figure 3

Example 4 – Solution

With the aid of the tree diagram, we can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

多选题 1分

设置

MAQ: Let

$$z = f(x,y) = \ln\left(1 + x^2 y\right),$$
$$x = g(s,t) = se^t, y = h(s,t) = s^2 - t$$

Which of the followings are true at $(s,t) = (1,0)$?

A $\frac{\partial z}{\partial s} = 1$

B $\frac{\partial z}{\partial t} = 0.5$

C Let $w(h) = z(1+h,h)$ and z is function of (s, t), then $w'(0) = \frac{3}{2}$

D For any (s,t), $\frac{\partial z}{\partial t} = \frac{2xy}{1+x^2y} x_t + \frac{x^2}{1+x^2y} y_t$

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Example

Solutions:

Option D is true and we can use this method.

$x(1,0) = 1, y(1,0) = 1,\ f_x = \frac{2xy}{1+x^2y} = 1,\ f_y = \frac{x^2}{1+x^2y} = \frac{1}{2}$

$x_s = e^t\Big|_{(1,0)} = 1, y_s = 2s\Big|_{(1,0)} = 2, x_t = se^t\Big|_{(1,0)} = 1, y_t = -1$

$$\frac{\partial z}{\partial s} = \frac{2xy}{1+x^2y}x_s + \frac{x^2}{1+x^2y}y_s = 2$$ A is false

$$\frac{\partial z}{\partial t} = \frac{2xy}{1+x^2y}x_t + \frac{x^2}{1+x^2y}y_t = 0.5$$ B is true

$w'(0) = \frac{\partial z}{\partial s} + \frac{\partial z}{\partial t} = \frac{5}{2}$ C is false

Implicit Differentiation

Implicit Differentiation (1 of 6)

The Chain Rule can be used to give a more complete description of the process of implicit differentiation.

We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f.

If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x.

Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Implicit Differentiation (2 of 6)

But $\frac{dx}{dx} = 1$, so if $\frac{\partial F}{\partial y} \neq 0$ we solve for $\frac{dy}{dx}$ and obtain

5 $$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x.

Implicit Differentiation (3 of 6)

The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid: it states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 5.

Example 8

Find y' if $x^3 + y^3 = 6xy$.

Solution:

The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

so Equation 5 gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{3x^2 - 6y}{3y^2 - 6y} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Implicit Differentiation (4 of 6)

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.

This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Implicit Differentiation (5 of 6)

But $\frac{\partial}{\partial x}(x) = 1$ and $\frac{\partial}{\partial x}(y) = 0$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\frac{\partial F}{\partial z} \neq 0$, we solve for $\frac{\partial z}{\partial x}$ and obtain the first formula in Equations 6.

The formula for $\frac{\partial z}{\partial y}$ is obtained in a similar manner.

Implicit Differentiation (6 of 6)

6 $$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid: if F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by (6).

多选题 1分

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MAQ: Let $F(x,y,z)=\ln(z)+xy+x^3-1$ and suppose $F(x,y,z)=0$ defines $z=f(x,y)$ near the point $P(1,1,\frac{1}{e})$, which of the followings are true?

A $\left.\frac{\partial z}{\partial x}\right|_P=-\frac{4}{e}$

B $\left.\frac{\partial z}{\partial y}\right|_P=-\frac{2}{e}$

C If instead we view x as $x=g(y,z)$, then $\left.\frac{\partial x}{\partial y}\right|_P=-\frac{1}{4}$

D For any smooth path $t\mapsto(x(t),y(t))$ with $x(0)=y(0)=1$ and $z(t)$ satisfying $F(x,y,z)=0$ we have $\left.\frac{dz}{dt}\right|_{t=0}=-\frac{4x'(0)+y'(0)}{e}$

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Example

Solutions:

$$F_x = y + 3x^2,\ F_y = x,\ F_z = \frac{1}{z} \qquad F_x(P) = 4, F_y(P) = 1, F_z(P) = e$$

$$\left.\frac{\partial z}{\partial x}\right|_P = -\frac{F_x}{F_z} = -\frac{4}{e} \qquad \left.\frac{\partial z}{\partial y}\right|_P = -\frac{F_y}{F_z} = -\frac{1}{e} \qquad \left.\frac{\partial x}{\partial y}\right|_P = -\frac{F_y}{F_x} = -\frac{1}{4}$$

A is true, B is false, C is true

$$F_x x' + F_y y' + F_z z' = 0$$

$$\left.\frac{dz}{dt}\right|_{t=0} = -\frac{F_x x' + F_y y'}{F_z} = -\frac{4x'(0) + y'(0)}{e}$$

D is true

Recap

- The Chain Rule
- The Chain Rule: General Version
- Implicit Differentiation