Home Assignment - 01 - Spring 2019
Signals & Systems

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1.1 Signal Power

1) given $x(t) = A\cos(\omega t + \phi)$

$$\text{average power} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)^2|\, dt$$

$$P_{avg} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |A\cos(\omega t+\phi)|^2\, dt$$

$$= \lim_{T\to\infty} \frac{A^2}{2T}\int_{-T}^{T} \frac{1+\cos 2(\omega t+\phi)}{2}\, dt$$

$$= \frac{A^2}{4}\left[\lim_{T\to\infty} \frac{1}{T}\left[\int_{-T}^{T}\cos 2(\omega t+\phi)\, dt + \int_{-T}^{T} 1\, dT\right]\right]$$

$$= \frac{A^2}{4}\left[\lim_{T\to\infty} \frac{1}{T}\left[\left(\frac{\sin 2(\omega t+\phi)}{2\omega}\right)_{-T}^{T} + 2T\right]\right]$$

$$= \frac{A^2}{4}\left[\lim_{T\to\infty} \frac{1}{T}\left[\frac{2\sin(2\omega t)\cos\phi}{2\omega}\right] + 2\right]$$

$$= \frac{A^2}{4}\ \text{Watts}$$

2) $x(t) = A \quad 0 \le t \le T$

$= 0 \quad$ otherwise

$$y(t) = \int_0^t x(\tau)\, d\tau = \begin{cases} 0 & t < 0 \\ At & 0 \le t \le T \\ AT & t > T \end{cases}$$

$$\text{average power} = \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} y(\tau)\, d\tau$$

$$= \lim_{t\to\infty} \frac{1}{2t}\left[\int_0^T (A\tau)^2\, d\tau + \int_T^t (AT)^2\, d\tau\right]$$

$$= \lim_{t\to\infty} \frac{1}{2t}\left[A^2\left(\frac{\tau^3}{3}\right]_0^T + A^2T^2(\tau]_T^t\right]$$

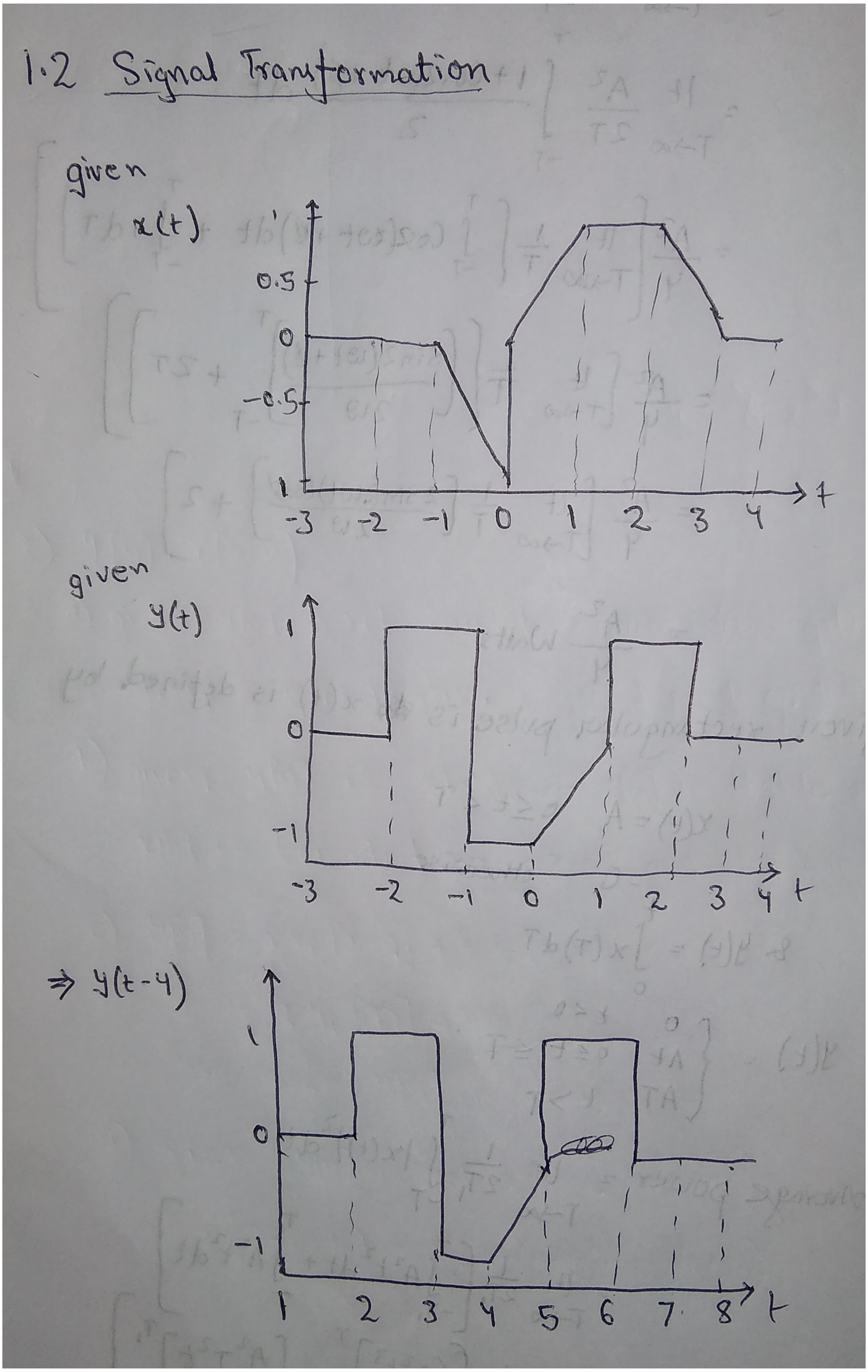
$$= \lim_{t\to\infty} \frac{A^2}{2t}\left[\frac{T^3}{3} + A^2T^2(t-T)\right]$$

$$= \lim_{t\to\infty} \frac{A^2}{2t}\left[T^2 t - \frac{2T^3}{3}\right]$$

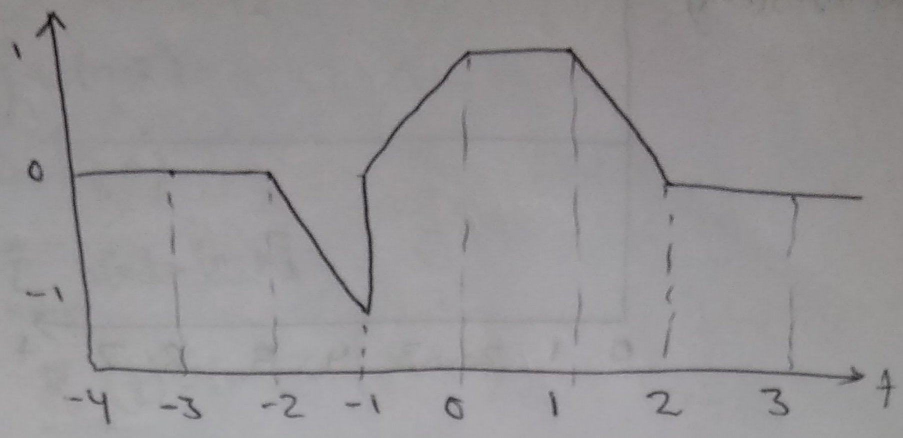
$$= \frac{A^2T^2}{2} - \lim_{t\to\infty}\left(\frac{2A^2T^3}{3t}\right)$$

$$= \frac{A^2T^2}{2}$$

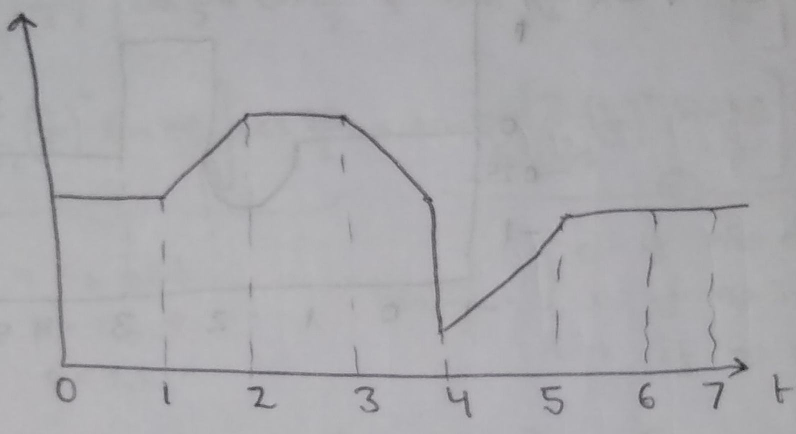
1.2 Signal Transformation



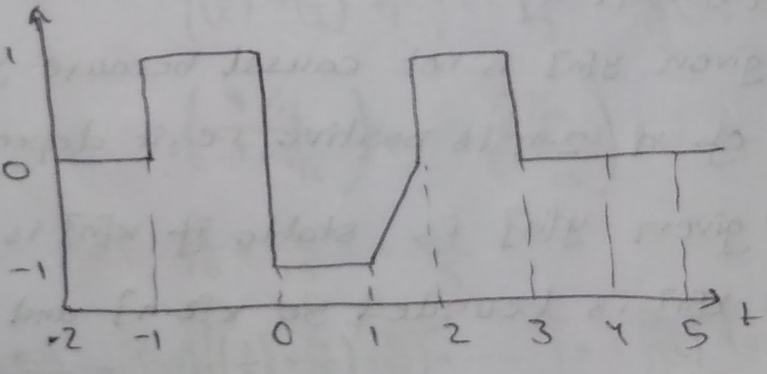
⇒ $x(t+1)$



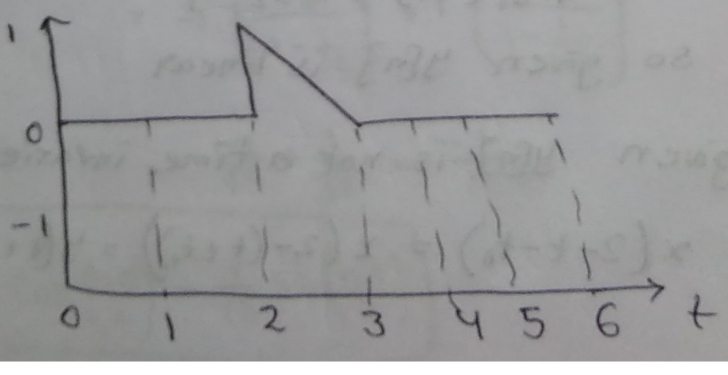
⇒ $x(-t+4)$

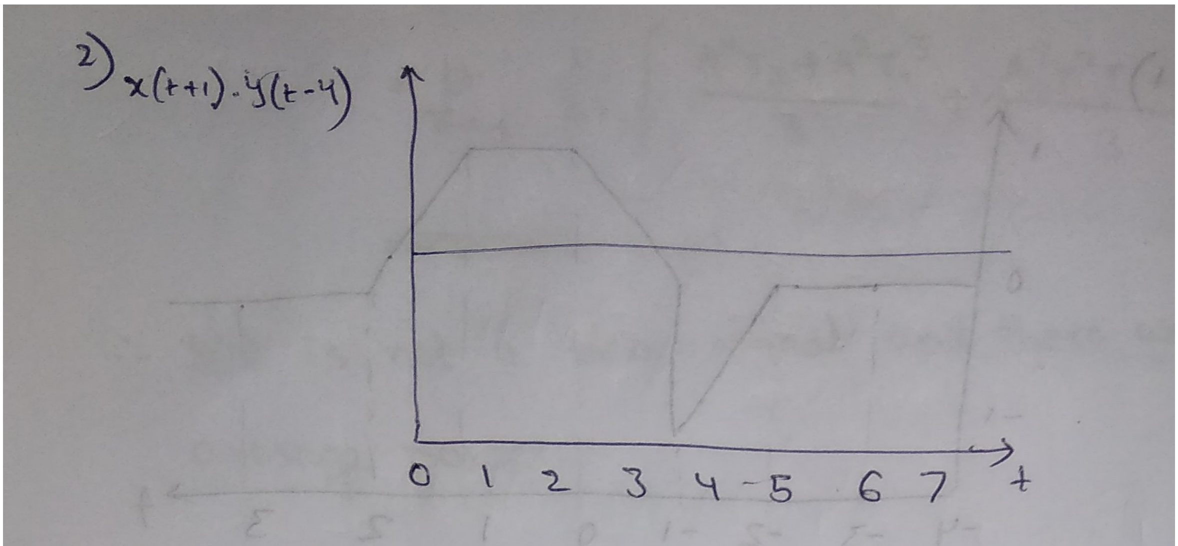


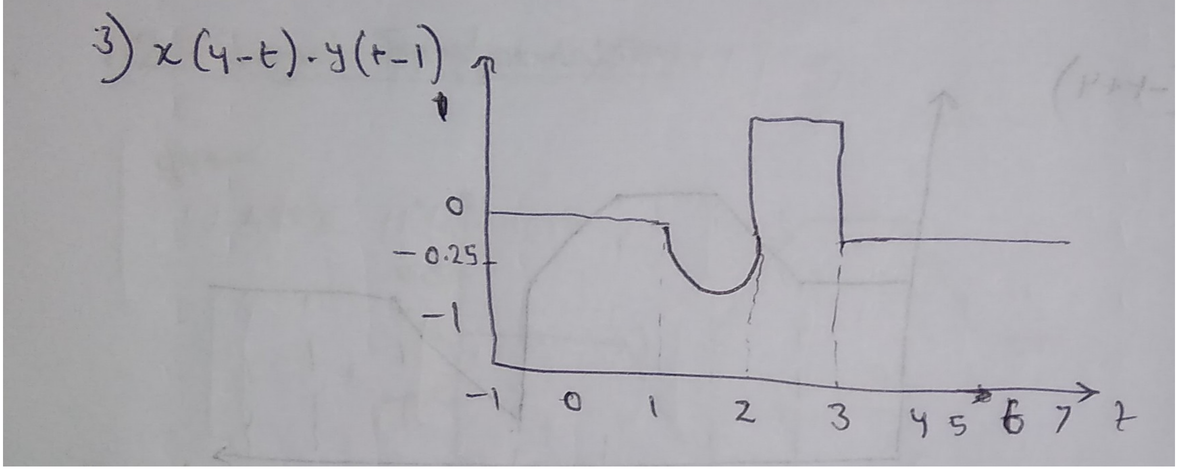
⇒ $y(t-1)$



1) $x(t) \cdot y(t-4)$







2.1 Properties of LTI

2) $y[n] = x[2-n]$

a) given $y[n]$ is not causal because for negative values of n $2-n$ is positive i.e., it depends on future value

b) given $y[n]$ is stable if $x[n]$ is stable since $x[n]$ is bounded so $x[2-n]$ and $y[n]$ are bounded

c) $y[n] = a x_1[2-n] + b x_2[2-n]$

$= a y_1[n] + b y_2[n]$

so given $y[n]$ is linear

d) given $y[n]$ is not a time invarient

$x(2-t-t_0) = x(2-(t+t_0)) = y(t+t_0)$

2.1 Properties of LTI

1) given $y(t) = \frac{d}{dt} x(t)$

a) given $y(t)$ is causal because

$y(t) = \frac{d}{dt} x(t) = \lim_{h \to 0^+} \frac{x(t) - x(t-h)}{h}$ which depends only on past and present values of $x(t)$

b) given $y(t)$ is not stable because

$\frac{d}{dt}(x(t))$ is slope of $x(t)$ it is infinite at some points even $x(t)$ is bounded. so $\frac{d}{dt}(x(t))$ is unbounded and not stable.

c) given $y(t)$ is linear because

$$y(t) = \frac{d}{dt}\left[a\, x_1(t) + b\, x_2(t)\right]$$

$$= a \frac{d}{dt} x_1(t) + b \frac{d}{dt} x_2(t) = a\, y_1(t) + b\, y_2(t)$$

d) given $y(t)$ is timeinvarient because

$$\frac{d}{dt} x(t - t_0) = y(t - t_0)$$

2.2 Linear Convulation

2.2 Linear convulation

1) $h[n] = \left(\frac{1}{2}\right)^n u(n-2)$

$x[n] = \cos\left(\frac{\pi}{2}n\right)$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k-2) \cos\left(\frac{\pi}{2}(n-k)\right)$$

$$= \sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^k \left[\cos\frac{\pi}{2}n \cos\frac{k\pi}{2} + \sin\frac{n\pi}{2} * \sin\frac{k\pi}{2}\right]$$

$$= \underbrace{\left[\sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^k \cos\frac{k\pi}{2}\right]}_{1} \cos\frac{n\pi}{2} + \sin\frac{n\pi}{2} \underbrace{\left[\sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^k \sin\frac{k\pi}{2}\right]}_{2}$$

$$\cos\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n = 2,6,10,\dots \\ -1 & \text{if } n = 4,8,\dots \end{cases}$$

$$\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n = 3,7,11,15 \\ -1 & \text{if } n = 1,5,9,13 \end{cases}$$

$$1 \Rightarrow \sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^k \cos\frac{k\pi}{2} = (-1)\left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^{10} + \dots\right] + 1\left[\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^{12} + \dots\right]$$

$$= -1\left(\frac{1/4}{1-\frac{1}{16}}\right) + 1\left(\frac{1/16}{1-\frac{1}{16}}\right)$$

$$= \frac{-4}{15} + \frac{1}{15} = \frac{-1}{5}$$

$$2 \Rightarrow \sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^k \sin\frac{k\pi}{2} = (-1)\left[\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^{11} + \dots\right] + 1\left[\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^9 + \left(\frac{1}{2}\right)^{13} + \dots\right]$$

$$= -1\left(\frac{1/8}{1-\frac{1}{16}}\right) + 1\left(\frac{1/32}{1-\frac{1}{16}}\right)$$

$$= \frac{-2}{15} + \frac{1}{30} = \frac{-1}{10}$$

$$\Rightarrow \boxed{y[n] = \frac{-1}{5}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{10}\sin\left(\frac{n\pi}{2}\right)}$$

2) given $x[n] = 1 \quad -5 \le n \le 1$

$\quad\quad\quad = 0 \quad$ otherwise

$$h[n] = \begin{cases} 1 & 0 \le n \le 3 \\ 2 & 4 \le n \le 8 \\ 0 & \text{otherwise} \end{cases}$$

$h(n)$ \ $x(n)$		-5	-4	-3	-2	-1	0	1
		1	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1
4	2	2	2	2	2	2	2	2
5	2	2	2	2	2	2	2	2
6	2	2	2	2	2	2	2	2
7	2	2	2	2	2	2	2	2
8	2	2	2	2	2	2	2	2

$y[n] = x[n] * h[n] = \{1, 2, 3, 4, 6, 8, 10, 11, 12, 11, 10, 8, 6, 4, 2\}$

3.1 Fourier Series

$x(t) = 1 - \left|\frac{t}{3}\right| \qquad |t| < 1$

$$a_k = \frac{1}{T_0}\int x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\left[\int_{-1}^{0}\left(1+\frac{t}{3}\right)e^{-jk\omega_0 t}\, dt + \int_{0}^{1}\left(1-\frac{t}{3}\right)e^{-jk\omega_0 t}\, dt\right]$$

$$= \frac{1}{T_0}\left[\int_{-1}^{0} e^{-jk\omega_0 t}\, dt + \frac{1}{3}\int_{-1}^{0} t e^{-jk\omega_0 t}\, dt + \int_{0}^{1} e^{-jk\omega_0 t}\, dt - \frac{1}{3}\int_{0}^{1} t e^{-jk\omega_0 t}\, dt\right]$$

$$= \frac{1}{T_0}\left[\int_{0}^{1}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) dt - \frac{1}{3}\int_{0}^{1} t\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) dt\right]$$

$$= \frac{1}{T_0}\left[2\int_{0}^{1}(\cos k\omega_0 t)\, dt - \frac{2}{3}\int_{0}^{1} t\cos(k\omega_0 t)\, dt\right]$$

$$= \frac{1}{T_0}\left[\frac{2\sin k\omega_0}{k\omega_0} - \frac{2}{3}\left[\frac{\sin k\omega_0}{k\omega_0} - \int_{0}^{1}\frac{\sin k\omega_0 t}{k\omega_0}\, dt\right]\right]$$

$$= \frac{1}{T_0}\left[\frac{2\sin k\omega_0}{k\omega_0} - \frac{2}{3}\frac{\sin k\omega_0}{k\omega_0} - \frac{2\cos(k\omega_0)}{3(k\omega_0)^2} + \frac{2}{3(k\omega_0)^2}\right]$$

3.1 Fourier series

2) $x[n] = \cos\left[\frac{\pi}{5}n\right] \qquad |n| \le 2 \qquad N = 5$

$$a_k = \frac{1}{N}\sum_{N=-\infty}^{\infty}\cos\left[\frac{n\pi}{5}\right] e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{5}\sum_{N=-2}^{2}\cos\left[\frac{n\pi}{5}\right] e^{-jk\frac{2\pi}{5}n}$$

$$= \frac{1}{5}\left[\cos\frac{2\pi}{5} e^{\frac{jk4\pi}{5}} + \cos\frac{\pi}{5} e^{\frac{2jk\pi}{5}} + 1 + \cos\frac{\pi}{5} e^{\frac{-2jk\pi}{5}} + \cos\frac{2\pi}{5} e^{\frac{-jk4\pi}{5}}\right]$$

$$= \frac{1}{5}\left[\cos\frac{2\pi}{5}\left(e^{\frac{jk4\pi}{5}} + e^{-\frac{jk4\pi}{5}}\right) + \cos\frac{\pi}{5}\left(e^{\frac{2jk\pi}{5}} + e^{-\frac{2jk\pi}{5}}\right) + 1\right]$$

$$= \frac{1}{5}\left[\cos\frac{2\pi}{5}\left[2\cos\frac{4\pi}{5}k\right] + 1 + \cos\frac{\pi}{5}\left[2\cos\frac{2k\pi}{5}\right]\right]$$

$$= \frac{1}{5}\left[\cos\left(\frac{2\pi}{5}(2k+1)\right) + \cos\left(\frac{2\pi}{5}(2k-1)\right) + \cos\left(\frac{\pi}{5}(2k+1)\right) + \cos\left(\frac{\pi}{5}(2k-1)\right) + 1\right]$$

3.2 Fourier Transform

3.2 Fourier transform

1) given $x(t) = 2 \quad |t| < 1$

$= 0 \quad$ otherwise

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-1}^{1} 2 e^{-j\omega t} dt$$

$$= 2 \left[\frac{e^{-j\omega}}{-j\omega}\right]_{-1}^{1} = \frac{2(-2j \sin\omega)}{-j\omega}$$

$$\boxed{X(\omega) = \frac{4 \sin\omega}{\omega}}$$

2) given $x(t) = e^{\frac{-|t|}{2}} \quad |t| < 1$

$= 0 \quad$ otherwise

$$X(\omega) = \int_{-1}^{0} e^{t/2} e^{-j\omega t} dt + \int_{0}^{1} e^{-t/2} e^{-j\omega t} dt$$

$$= \int_{-1}^{0} e^{\frac{t}{2}(1-2j\omega)} dt + \int_{0}^{1} e^{-\frac{t}{2}(1+2j\omega)} dt$$

$$= \left[\frac{e^{\frac{t}{2}(1-2j\omega)}}{\frac{1-2j\omega}{2}}\right]_{-1}^{0} + \left[\frac{e^{-\frac{t}{2}(1+2j\omega)}}{\frac{1+2j\omega}{2}}\right]_{0}^{1}$$

$$= 2\left[\frac{1-e^{\frac{-1}{2}(1-2j\omega)}}{1-2j\omega} + \frac{1-e^{-\frac{1}{2}(1+2j\omega)}}{1+2j\omega}\right]$$

3.3 Inverse Transform

3.3) $X(\omega) = \text{Sinc}^2(\omega)$

if $x(t) = \frac{1}{2}\left(1-\frac{|t|}{2}\right) \quad |t|<2$

$$X_1(\omega) = F\{x(t)\} = \frac{1}{2}\int_{-2}^{2}\left(1-\frac{|t|}{2}\right)e^{-j\omega t}dt$$

$$= \frac{1}{2}\left(\int_0^2 1-\frac{t}{2}e^{-j\omega t}dt + \int_{-2}^{0} 1+\frac{t}{2}e^{-j\omega t}dt\right)$$

$$= \frac{1}{2}\left(2\int_0^2\left(1-\frac{t}{2}\right)\cos\omega t\, dt\right)$$

$$= \int_0^2 \cos\omega t\, dt - \int_0^2 \frac{t}{2}\cos\omega t\, dt$$

$$= \left[\frac{\sin\omega t}{\omega}\right]_0^2 - \left(\left[\frac{t\sin\omega t}{2\omega}\right]_0^2 - \int_0^2 \frac{\sin t\omega}{2\omega}dt\right)$$

$$= \frac{\sin 2\omega}{\omega} - \frac{2\sin 2\omega}{2\omega} + \frac{1}{2\omega}\left[-\frac{\cos\omega t}{\omega}\right]_0^2$$

$$= \frac{\sin 2\omega}{\omega} - \frac{\sin 2\omega}{\omega} + \frac{1}{2\omega}\left(\frac{1-\cos 2\omega}{\omega}\right)$$

$$= \frac{1-\cos 2\omega}{2\omega^2}$$

$$= \frac{2\sin^2\omega}{2\omega^2} = \frac{\sin^2\omega}{\omega^2} = \text{Sinc}^2\omega$$

$\because F\{x(t)\} \longrightarrow X_1(\omega)$

$F^{-1}[X_1(\omega)] \longrightarrow x(t)$

$\Rightarrow F^{-1}[X(\omega)] = x(t) = \frac{1}{2}\left(1-\frac{|t|}{2}\right) \quad |t|<2$

2) $x(\omega) = \dfrac{1}{1+j\omega RC}$

if $x(t) = \dfrac{1}{RC} e^{-t/RC} \cdot u(t)$

$$X(\omega) = \frac{1}{RC}\int_{-\infty}^{\infty} u(t)\, e^{-j\omega t} \times e^{-t/RC}\, dt$$

$$= \frac{1}{RC}\int_{0}^{\infty} e^{-t\left[\frac{1}{RC}+j\omega\right]}\, dt$$

$$= \frac{1}{RC}\left[\frac{-e^{-t\left[\frac{1}{RC}+j\omega\right]}}{\frac{1}{RC}+j\omega}\right]_0^{\infty}$$

$$= \frac{1}{RC}\left[0 - \left[\frac{-1}{\frac{1}{RC}+j\omega}\right]\right]$$

$$X(\omega) = \frac{1}{1+jR\omega C}$$

Since $F[x(t)] \rightarrow X_1(\omega)$

$F^{-1}[X_1(\omega)] \rightarrow x(t)$

$$\therefore x(t) = \frac{1}{RC} e^{-t/RC} \cdot u(t)$$

4.1 Discrete Fourier Transform

4.1) Discrete fourier transform

$$w[n] = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right)$$

taking $N = 5$

$$W[k] = \sum_{n=0}^{4}\left[0.42 - 0.5\cos\left(\frac{\pi}{2}n\right) + 0.08\cos(\pi n)\right]$$

$$= 0 + 0.34e^{-jk\frac{2\pi}{5}} + e^{-jk\frac{4\pi}{5}} + 0.34e^{-jk\frac{6\pi}{5}} + 0$$

$$= 0.34\left[e^{-jk\frac{2\pi}{5}} + e^{-jk\frac{6\pi}{5}}\right] + e^{-jk\frac{4\pi}{5}}$$

$$W[0] = 0.34[1+1] + 1 = 1.68$$

$$W[1] = 0.34\left[e^{-j\frac{2\pi}{5}} + e^{-j\frac{6\pi}{5}}\right] + e^{-j\frac{4\pi}{5}}$$

$$W[2] = 0.34\left[e^{-j\frac{4\pi}{5}} + e^{-j\frac{12\pi}{5}}\right] + e^{-j\frac{8\pi}{5}}$$

$$W[3] = 0.34\left[e^{-j\frac{6\pi}{5}} + e^{-j\frac{16\pi}{5}}\right] + e^{-j\frac{12\pi}{5}}$$

$$W[4] = 0.34\left[e^{-j\frac{6\pi}{5}} + e^{-j\frac{24\pi}{5}}\right] + e^{-j\frac{16\pi}{5}}$$

4.2 circular convulation

4.2)

$x_1[n] = \{\underset{\uparrow}{1}, 2, 3, 1\}$ $\quad x_2[n] = \{\underset{\uparrow}{4}, 3, 2, 1\}$

$$X_1(k) = \sum_{n=0}^{N-1} e^{-j\frac{2\pi kn}{N}} x_1[n]$$

$$= \sum_{n=0}^{3} x_1[n] e^{-j\frac{\pi}{2}kn}$$

$$= 1 + 2e^{-j\frac{\pi}{2}k} + 3e^{-j\pi k} + e^{-j\frac{3\pi}{2}k}$$

$$X_1(0) = 1+2+3+1 = 7$$

$$X_1(1) = 1 + 2e^{-j\frac{\pi}{2}} + 3e^{-j\pi} + e^{-j\frac{3\pi}{2}}$$

$$= 1 + 2\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) + 3(\cos\pi - j\sin\pi) + \left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right)$$

$$= 1 - 2j - 2 + j = -2 - j$$

$$X_1(2) = 1 + 2e^{-j\pi} + 3e^{-j2\pi} + e^{-j3\pi}$$

$$= 1 + 2(-1) + 3(1) + (\cos\pi - j\sin\pi)$$

$$= 1 - 2 + 3 - 1$$

$$= 1$$

$$X_1(3) = 1 + 2e^{-j\frac{3\pi}{2}} + 3e^{-3j\pi} + e^{-j\frac{9\pi}{2}}$$

$$= 1 + 2\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) + 3(\cos\pi - j\sin\pi) + \left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$$

$$= 1 + 2j + 3(-1) + (-j)$$

$$= -2 + j \qquad X_1 = \{7, -2-j, 1, -2+j\}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2[n] e^{-j\frac{\pi}{2}kn}$$

$$= 4 + 3e^{-j\frac{\pi}{2}k} + 2e^{-j\frac{\pi}{2}k\times 2} + 2e^{-j\frac{\pi}{2}k\times 3}$$

$$= 4 + 3e^{-j\frac{\pi}{2}k} + 2e^{-j\pi k} + 2e^{-j\frac{3\pi}{2}k}$$

$$X_2(2) = 4 + 3e^{-j\frac{\pi}{2}(2)} + 2e^{-j2\pi} + 2e^{-j\frac{3\pi}{2}(2)}$$

$$= 4 + 3\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) + 2\left(\cos 2\pi - j\sin 2\pi\right) + 2\left(\cos 3\pi - j\sin 3\pi\right)$$

$$= 4 + 3(-1) + 2(1) + 2(-1)$$

$$= 4 - 3 = 1$$

$$X_2(3) = 4 + 3e^{-j\frac{\pi}{2}(3)} + 2e^{-j\pi 3} + 2e^{-j\frac{3\pi}{2}3}$$

$$= 4 + 3\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) + 2\left(\cos 3\pi - j\sin 3\pi\right) + 2\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right)$$

$$= 4 + 3j + 2(-1) + 2(-j)$$

$$= 2 + j$$

$$X_2 = \{11, 2-j, 1, 2+j\}$$

$$X_1 = \{7, -2-j, 1, -2+j\}$$

$$X = X_1 X_2 = \{77, -5, 1, -5\}$$

$$x[n] = \frac{1}{N}\sum_{K=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}nk}$$

$$= \frac{1}{4}\left[77 + (-5)e^{j\frac{\pi}{2}n} + e^{j\frac{\pi}{2}(2n)} - 5e^{j\frac{\pi}{2}(3n)}\right]$$

$$x[0] = \frac{1}{4}[77 - 5 + 1 - 5] = \frac{1}{4}(68) = 17$$

$$x[1] = \frac{1}{4}\left[77 - 5e^{j\frac{\pi}{2}} + e^{j\pi} - 5e^{+j\frac{3\pi}{2}}\right]$$

$$= \frac{1}{4}\left[77 - 5(-j) + (-1) - 5(j)\right] = \frac{1}{4}(76) = 19$$

$$x[2] = \frac{1}{4}\left[77 - 5e^{j\frac{\pi}{2}(2)} + e^{+j\frac{\pi}{2}(2)(2)} - 5e^{j\frac{3\pi}{2}(2)}\right]$$

$$= \frac{1}{4}\left[77 - 5(-1) + 1 - (-5)\right] = \frac{1}{4}[88] = 22$$

$$x[3] = \frac{1}{4}\left[77 - 5e^{+j\frac{\pi}{2}(3)} + e^{+j\frac{\pi}{2}(2)(3)} - 5e^{j\frac{3\pi}{2}(3)}\right]$$

$$= \frac{1}{4}\left[77 - 5(+j) + 1(-1) - 5(-j)\right]$$

$$= \frac{1}{4}\left[77 - 1\right] = \frac{76}{4} = 19$$

$$x[n] = \{17, 19, 22, 19\}$$