Leveraged Equity Investing

First draft: November 3, 2022
Current draft: April 4, 2024

Lifecycle Investing

Ayres and Nalebuff's *Lifecycle Investing* advocates leveraging equitites investments while young. The reason is to diversify over time. Investors usually have large portfolios late in life and small portfolios earlier in life. That means the investment returns in the later period affect the portfolio outcome much more than the returns in the earlier period. The portfolio is too concentrated in the later period; it is not diversified over time.

The solution is to move some of the portfolio balance from the later period to the earlier period. That is, borrow money to invest early in life. They show this strategy gives better diversification over time, which can be spent on higher expected returns, lower variance, or both. Diversification is one of the few free lunches that economics offers. Investors should be diversifying over time.

Time Diversification

Diversification is spreading a fixed investment budget across more investments to reduce variance. Investors often diversify over *assets*, by owning mutual funds and ETFs. But investors often ignore diversifying over time. Investors usually have much larger exposure to the later years' returns than the early years, because they have accumulated assets by the end. If the market tanks in final years before retirement, that hurts a portfolio more than the market tanking in the early years.

Let's show an example[1]. Suppose you consider an investment in a risky portfolio over 2 periods. We'll show that you should invest half in the first period and half in the second period, rather than invest all in (only) period one or all in (only) period 2. This is spreading a fixed risk budget over more investments; which is the definition of diversification.

Now for the details. The fraction invested in a risky asset, A, is y, leaving the fraction $1-y$ invested in the risk-free asset. Asset A's risk premium is R, and its standard deviation is σ. The risk premium of the complete portfolio P is $R_P = yR$, its standard deviation is $\sigma_P = y\sigma$, and the Sharpe ratio is $S_P = \frac{R}{\sigma}$. The returns in the two periods are uncorrelated.

The returns on the complete portfolio, for one period, consisting of a risk asset and a risk-free asset are:

$$R_c = yr_p + (1-y)r_f \tag{1}$$
$$E(r_c) = yE(r_p) + (1-y)r_f \tag{2}$$
$$\sigma_c = y\sigma_p \tag{3}$$

Now consider two strategies.

1. Invest the whole portfolio in the risky asset in the first period, then withdraw the entire proceeds, placing them in a risk-free asset in the other period. Because you are invested in the risky asset for only 1 year, the risk premium over the whole investment period is R, the whole-period standard deviation is σ, and the whole-period Sharpe ratio is $S = R/\sigma$.

[1] Example from *Investments*; Bodie, Kane, Marcus, 10th edition, chapter 7.5, *(Risk Pooling, Risk Sharing, and the Risk of Long-Term Investments)*

2. In each period, invest half of the portfolio in the risky asset and half in the risk-free asset. The 2-year risk premium is R, the 2-year variance is $2 \cdot (\frac{1}{2}\sigma)^2 = \frac{1}{2}\sigma^2$, the standard deviation is $\frac{1}{\sqrt{2}}\sigma$, and the Sharpe ratio is $S = \sqrt{2}\frac{R}{\sigma}$. This Sharpe ratio is bigger than strategy 1, so strategy 2 is better because of diversification.

Strategy 2 demonstrates time diversification. The investment budget is split between two periods, rather than one, thus reducing variance. Strategy 1 is "putting all your eggs in one basket" by investing in only one period's returns rather than two. So strategy 2 is better.

Now consider a third strategy:

3. Invest the whole budget in the risky asset for both periods. The 2-year risk premium is $2R$ (assuming continuously compounded rates), the 2-year variance is $2\sigma^2$, the 2-year standard deviation is $\sqrt{2}\sigma$, and the 2-year Sharpe ratio is $S = \sqrt{2}\frac{R}{\sigma}$

This strategy does not diversify; it takes more risky bets than strategy 2 by investing in the risky asset for both periods, rather than one. This strategy has higher risk premium but higher standard deviation. Moving from strategy to 1 to 3 is not diversifying because strategy 3 is using a larger risk budget. To diversify over time, we should spread a fixed risk budget over different periods.

We can show that this equal split over periods in the above example minimizes risk. Let's generalize by saying we have N periods. We invest asset A_i in period i, and for the other $N-1$ periods we invest in the risk-free asset. That is, at the beginning of this investment, we choose an asset allocation over N possible assets. We've transformed the multi-period problem to a one-period problem, and we can apply standard mean-variance optimization.

We assume the return of the risky asset and risk-free asset are both iid, that is, the same in all periods. We these assumptions, the expected return for the portfolio is fixed, regardless of the asset allocation.

$$\begin{aligned} E(A_t) &= \mu + (N-1)r_f \\ var(A_t) &= \sigma^2 \\ cov(A_t, A_s) &= 0 \end{aligned}$$

for all investments t. Remember that these are treated as different investments, even though they are really the same asset just in different time periods.

We can now plug in these numbers to the one-period mean-variance optimization problem to find the minimum-variance portfolio, as done here [2]. The result is to invest equal weights ($1/N$) in each "asset". That is, you should spread your risk budget equally over all the assets. Remember, each "asset" has the same expected return, and covariances are zero, so only variance matter. It makes intuitive sense to spread as widely as possible, i.e. equally, for maximum variance reduction.

What does this show us? This shows that we want a constant asset allocation over time. We want to put an equal fraction of our initial wealth in the risky asset (stocks). This is the goal of the lifecycle strategy. The problem is your initial wealth is trapped in the future as earnings. If you include the present value of your future income in your initial wealth, you'll realize that you need to leverage your current *financial* portfolio to reach the target stock allocation. Leveraging becomes necessary.

Myths about Time Diversification

The phrase "time diversification" is often misunderstood. We will clear up a common confusion, since it appears in the investing discussion. Here are some examples of some common fallacies:

[2] Course notes, eq. 12.6

> Stocks are more stable in the long run. If you hold them for a long time, the good times average out the bad times, and the returns converge to the long-run mean.

This confuses the annualized returns with the final returns. It's true that the average *annualized* returns converges to the mean as the investment horizon increases. But the final returns don't. The variance of final returns increases with the horizon. And the final returns are what matters.

Here's an explanation copied from Cochrane [3]. Suppose we have a two-period investment horizon. The two period gross return is

$$R_{0\to 2} = R_{0\to 1} R_{1\to 2}$$

so

$$\ln R_{0\to 2} = \ln R_{0\to 1} + \ln R_{1\to 2}$$

Let's look at the mean and standard deviation fo the two period return. We assume that mean returns are the same every year and that returns are independent over time.

$$\begin{aligned} E(\ln R_{0\to 2}) &= E(\ln R_{0\to 1}) + E(\ln R_{1\to 2}) = 2E(\ln R) \\ \sigma^2(\ln R_{0\to 2}) &= \sigma^2(\ln R_{0\to 1}) + \sigma^2(\ln R_{1\to 2}) + cov... = 2\sigma^2(\ln R) \end{aligned}$$

Look. The ratio of the mean return to variance of return is independent of horizon (again, if the mean is the same very year and returns are independent over time).

Where did the fallacy come from? Look at the *annualized* returns:

$$\begin{aligned} R^{ann}_{0\to 2} &= (R_{0\to 1} R_{1\to 2})^{1/2} \\ E \ln R^{ann}_{0\to 2} &= \frac{1}{2}\left[E(\ln R_{0\to 1}) + E(\ln R_{1\to 2})\right] = E(\ln R) \\ \sigma^2(\ln R^{ann}_{0\to 2}) &= \sigma^2\left[\frac{1}{2}(\ln R_{0\to 1}) + \sigma^2(\ln R_{1\to 2})\right] = \frac{1}{4}2\sigma^2(\ln R) = \frac{1}{2}\sigma^2(\ln R) \end{aligned}$$

The mean of annualized returns is the same as the horizon gets longer, but the variance of annualized returns goes down as the horizon gets longer.

But who cares if the variance of annualized returns gets smaller? You care about the total return, which is the mean annualized return raised to the power of the horizon. The explosive effect of compounding exactly undoes the stabilizing effects of longer horizon.

How to Borrow

Now that the theory of time diversification is done, we'll discuss practicalities of leveraging equity investments. There are a few ways to borrow money to invest in equities:

1. **Margin Loans**: This approach is the easiest, and most straightforward. Fidelity won't let me borrow from any other broker, and Fidelity has very high margin loan rates (around 9%).

2. **ITM Call Options**: Deep in-the-money call options behave as loans. Since the option is deep in the money, you will almost certainly exercise. And the price of the option is much lower than the strike, so you effectively pay for half of the stock initially (when you buy the option) and pay the rest when you exercise. Since you're delaying payment, that is loan behavior.

   The problem is the implied borrowing rate was pretty high - around 5%.

[3] Taken from https://static1.squarespace.com/static/5e6033a4ea02d801f37e15bb/t/5f6e70c5abd4ac5b167c7626/1601073350521/notes.pdf

3. **Index Futures**: These basically allow you to access index funds now, for the cost of posting collateral. The collateral is called "margin", but it's different from a margin loan. If you keep the leverage low, around 2:1, then the chances of a margin call are quite low. One downside is that part of the margin can earn interest at IBKR. The margin which is in the securities segment can earn interest but the portion in the commodities segment cannot.

4. **Synthetic Futures**: Buy a call and sell a put with the same strike price. The payoffs are similar to that of a futures contract; a sloped line with a slight shift downward. This is similar to the ITM call options approach, except you buy ATM call options, and you sell at ATM put option so that you aren't paying for downside protection. One potential benefit is that all of the margin could be held in the securities segment, and thus earn interest, rather than in the commodities segment.

   I've found rates around 4%, which is the best I've found. For this reason I chose to use futures.

Next, I show how to compute the borrowing rate and other metrics for options and futures.

Call Options

Here is the interest rate implied by the call option price [4]:

$$r = \frac{1}{T}\frac{(C_0 + K - S_0) + D \cdot T + TC}{S_0 - C_0}$$

where

$$\begin{aligned}
r &= implied\ interest\ rate \\
C_0 &= price\ of\ call\ option \\
K &= strike\ price \\
S_0 &= spot\ price\ of\ SPY \\
D &= annual\ divididends\ per\ share\ of\ SPY \\
T &= time\ until\ expriy,\ in\ years \\
TC &= transaction\ cost\ per\ share\ (\$0.59\ for\ Fidelity)
\end{aligned}$$

The first term in parentheses is the extra amount paid $(C_0 + K - S_0)$, the second term is the forgone dividends over the period, and the denominator is the amount borrowed.

As we'll see below, you can get the implied rate for futures can be obtained by setting the price of the call option $C_0 = 0$, and the strike price $K = F_0$, where F_0 is the price of the futures contract. This recognizes that K and F_0 are the prices that you'll have to buy stock at later. Setting $C_0 = 0$ reflects the fact that you don't need to pay any money to enter a futures contract.

Leveraged ETFs

Leveraged ETFs give a leveraged return over one day, but that doesn't equal the same leveraged return over the whole period.

Suppose you have a stock for two days with a final return of

$$1 + r_{1,2} = (1 + r_1)(1 + r_2) = 1 + r_1 + r_2 + r_1 r_2$$

Now suppose you want to achieve a l-x leveraged return, where l is the leverage ratio (e.g. $l = 2$:

[4] Lifecycle Investing, Ayres and Nalebuff

$$1 + lr_{1,2} = 1 + lr_1 + lr_2 + lr_1 r_2$$

but you try to achieve that by leveraging each individual day's returns:

$$(1 + lr_1)(1 + lr_2) = 1 + lr_1 + lr_2 + l^2 r_1 r_2$$

The daily l-x leverage does not achieve the desired total l-times-leveraged return. The last term differs between the daily leveraged return and the total period leveraged return.

Notice that with daily l-x leverage, we don't always want the second period's returns to be higher. This is counter-intuitive; don't we always want any given day's returns to be higher? Surprisingly, no. Let's call f' the derivative of the daily leveraged returns with the second day's return:

$$\begin{aligned} f' = \frac{\partial}{\partial r_2}[(1 + lr_1)(1 + lr_2)] &= (1 + lr_1)l \\ &= l + l^2 r_1 \end{aligned}$$

Notice that if one day is positive and the other is negative (volatility), then this approach will give a lower return than the desired 2x-leveraged total return. This phenomena is called *volatility decay*. Volatility, specifically daily returns that are both positive and negative, hurts leveraged ETFs performance.

$$\begin{cases} f' > 0, & \text{if } r_1 > -\frac{1}{l} \\ f' < 0, & \text{if } r_1 < -\frac{1}{l} \end{cases}$$

So if the first day's returns are below $-1/l$, then getting higher second-day returns r_2 will *lower* total-period returns. That's counter-intuitive.

As a sanity check, plug in the "unlevered" return of $l = 1$. Then $f' > 0$ if $r_1 \geq -1$, which it always will be since $r_1 \geq -1$ by construction. In other words, no matter the first day's return, you want the second day's return to be higher, when unlevered.

This shows that you cannot use daily leverage to create a total-period leveraged result. Mathematically, it just doesn't work. You'd have to find another way.

Resetting Leverage and Volatility Drag

Volatility drag refers to the fact that the larger the volatility, the larger the difference between the arithmetic mean returns and geometric (compound) mean returns. With unlevered (i.e. leverage=1) portfolios, the difference between arithmetic mean return and geometric mean return isn't very important, it's just a mathematical sidenote. But when leverage is introduced, the same math implies that leverage affects the geometric mean return, which is of course the return that you actually get from the investment.

If we start the portfolio as levered but don't reset the leverage in any subsequent periods, then the return *will* be the levered return[5]. You can proove this by seeing that if you lever, l_1 in the first period, then recompute where the leverage has changed to to start the second period, "apply" that leverage in the second period, then you'll see the final returns are exactly the unlevered returns times the initial leverage ratio l_1. Here's a proof:

Suppose your initial exposure e_1 is capital and debt: $e_1 = c_1 + d_1$. Your initial leverage ratio is $l_1 = \frac{e_1}{c_1} = \frac{c_1 + d_1}{c_1}$. After some algebra, we have $c_1 l_1 = c_1 + d_1$ and $d_1 = c_1(l_1 - 1)$, which will both be useful later. After period 1, the amount of debt hasn't changed; we haven't paid off nor taken out any new debt: $d_1 = d_2$. After period 1, your capital has changed to

$$\begin{aligned} c_2 &= c_1 + e_1 r_1 \\ &= c_1 + (c_1 + d_1) r_1 \end{aligned}$$

[5] https://blog.thinknewfound.com/2018/01/levered-etfs-long-run/

and thus leverage amount as changed to:

$$
\begin{aligned}
l_2 &= \frac{c_2 + d_2}{c_2} \\
&= \frac{c_1(1 + l_1 r_1) + d_1}{c_1(1 + l_1 r_1)} \\
&= \frac{c_1(1 + l_1 r_1) + c_1(l_1 - 1)}{c_1(1 + l_1 r_1)} \\
&= \frac{(1 + l_1 r_1) + (l_1 - 1)}{1 + l_1 r_1} \\
&= \frac{(l_1 r_1) + l_1}{1 + l_1 r_1} \\
&= \frac{l_1(1 + r_1)}{1 + l_1 r_1}
\end{aligned}
$$

Note that this leverage ratio is just the result of letting the portfolio grow or shrink in period 1. At the start of period 2, we don't reset the leverage, we just keep it as-is. Now we can compute the two-period returns:

$$
\begin{aligned}
(1 + l_1 r_1)(1 + l_2 r_2) &= (1 + l_1 r_1)\left(1 + \frac{l_1(1 + r_1)}{1 + l_1 r_1} r_2\right) \\
&= (1 + l_1 r_1)\left(\frac{(1 + l_1 r_1) + l_1(1 + r_1)}{1 + l_1 r_1} r_2\right) \\
&= 1 + l_1 r_1 + (l_1 + l_1 r_1) r_2 \\
&= 1 + l_1 r_1 + l_1 r_2 + l_1 r_1 r_2
\end{aligned}
$$

which is exactly the levered return we're aiming for over the whole period. In summary, we levered at the beginning then never reset the leverage, which gave us the desired whole-period levered return.

One final note is that when returns are negative in the first period, the leverage increases just enough to return us to the desired leverage ratio after the second period. And vice-versa if the stocks rise in the first period.

Futures Contracts

A key cost factor of a leveraged strategy is borrowing costs. The borrowing rate must be less than the investment returns, or else you've lost money. This makes computing the implied borrowing rate essential.

The price of futures can be found through arbitrage. You can replicate the payoff of futures by borrowing at the risk-free rate, buying stock, and selling it upon expiration to pay off the loan. Along the way, you're collecting dividends.

You can find the implied borrowing rate in at least two way. First, you can use the futures price and the spot price. Second, you can use two futures contracts of different maturity. The latter method is more applicable when you're rolling over your futures position. This should be done quarterly, since futures with farther explorations are too illiquid.

Now we'll show the two ways to compute the implied interest rate.

Implied Borrowing Rate Using Different Expirations

The interest rate from two contracts of different maturities comes from a CME report [6]. $F(t_1)$ is the price of the futures contract at time t which expires at time t_1. $D_{t \to t_1}$ is the dividends payed out from time t to t_1.

[6] https://www.cmegroup.com/education/files/S-and-P-500-Implied-Financing.pdf

$$Roll = F(t_2) - F(t_1)$$

$$r = \left(\frac{1}{t_2 - t_1}\right) \frac{F(t_2) - F(t_1) + D_{t_1 \to t_2}}{F(t_1) + D_{t \to t_1}}$$

This formula was given by the CME report. To prove that it's right, we'll show that it gives the correct price of a futures contract depending on the spot price. So we'll solve for $F(t_2)$:

$$r(t_2 - t_1)\left[F(t_1) + D_{t \to t_1}\right] = F(t_2) - F(t_1) + D_{t_1 \to t_2}$$

$$\begin{aligned}
F(t_2) &= r(t_2 - t_1)\left[F(t_1) + D_{t \to t_1}\right] + F(t_1) - D_{t_1 \to t_2} \\
&= F(t_1)\left[r(t_2 - t_1) + 1\right] + r(t_2 - t_1)D_{t \to t_1} - D_{t_1 \to t_2} \\
&= \left(S_0\left[1 + r(t_1 - t)\right] - D_{t \to t_1}\right)\left[r(t_2 - t_1) + 1\right] + r(t_2 - t_1)D_{t \to t_1} - D_{t_1 \to t_2} \\
&= S_0\left[1 + r(t_1 - t)\right]\left[r(t_2 - t_1) + 1\right] - D_{t \to t}\left[r(t_2 - t_1) + 1\right] + r(t_2 - t_1)D_{t \to t_1} - D_{t_1 \to t_2} \\
F(t_2) &= S_0\left(1 + r(t_1 - t)\right)\left(1 + r(t_2 - t_1)\right) - D_{t \to t} - D_{t_1 \to t_2}
\end{aligned}$$

This shows the price of the futures expiring at t_2 equals a strategy where you borrow S_0, invest at at the risk-free rate for two periods, and forgo dividends for these two periods. This is a generalization of the usual price based on cost-of-carry and no-arbitrage. It extends from one period to two. The risk-free rate is assumed to be the same in both periods, but the dividends can be unequal.

This shows that the implied borrowing rate given by CME is correct.

Implied Borrowing Rate using the Spot Price

The value of a futures contract is F_0, where S_0 is the spot price, r is the annual risk-free rate, T is the number of years from spot to expiration, and q is the annual dividend yield.

$$F_0 = S_0\left(1 + (r - q)T\right)$$

$$F_0 = S_0\left(1 + r - q\right)^T$$

$$F_0 = S_0 e^{(r-q)T}$$

The first equation treats r as compounded once and uses a Taylor Series expansion first term for simplification.[7] The second does not use this simplification. [8] The third treats r as continuously compounded [9].

Note that dividend yield is usually reported as the 12-month dividend per share divided by price, so it is an annual yield, as desired. [10]

Solving for the implied rate r:

$$r = \frac{1}{T}\left(\frac{F_0}{S_0} - 1\right) + q$$

$$r = \left(\frac{F_0}{S_0}\right)^{1/T} - 1 + q$$

[7] https://www.cmegroup.com/education/files/S-and-P-500-Implied-Financing.pdf

[8] Investments, Bodie et. al. eq. 22.2

[9] Options, Futures, and Other Derivatives chap. 5.9

[10] For example on September 30, 2022, the dividend yield was $q = 0.0182$.

$$r = \frac{1}{T} \log\left(\frac{F_0}{S_0}\right) + q$$

This is the risk-free rate implied by futures prices.

Next, we show the annual implied borrowing rate r, given by LI[11], is the same as the implied risk-free rate above:

$$\begin{aligned} r &= \left(\frac{1}{T}\right) \frac{F_0 - S_0 + qS_0T}{S_0} \\ &= \left(\frac{1}{T}\right) \left(\frac{F_0}{S_0} - 1 + \frac{qS_0T}{S_0}\right) \\ &= \left(\frac{1}{T}\right) \left(\frac{F_0}{S_0} - 1\right) + q \end{aligned}$$

Here's a numerical example from p.162 of LI:

$$\begin{aligned} r &= \left(\frac{1}{T}\right) \frac{F_0 - S_0 + qS_0T}{S_0} \\ &= \left(\frac{365}{144}\right) \frac{975.50 - 979.26 + 0.0194 \cdot 979.26 \frac{144}{365}}{979.26} \\ &= \left(\frac{365}{144}\right) \frac{-3.76 + 7.495}{979.26} \\ &= 0.00967 = 0.967\% \end{aligned}$$

Ayres and Nalebuff get 1.2% because they round.

Returns of Futures Contracts

This strategy relies on borrowing at a lower rate than the return rate of equities. Now I'll show that the growth rate in an index futures price equals the excess return on the portfolio underlying the index over the risk-free rate[12]. I assume that the risk-free interest rate and the dividend yield are constant.

$$\begin{aligned} F_0 &= S_0 e^{(r-q)T} \\ F_1 &= S_1 e^{(r-q)(T-t_1)} \\ \frac{F_1}{F_0} &= \frac{S_1}{S_0} e^{-(r-q)t_1} \end{aligned}$$

Now note the return realized by the stock is

$$\begin{aligned} S_1 &= S_0 e^{(r_I - q)t_1} \\ S_1 &= S_0 e^{(r + (r_I - r) - q)t_1} \end{aligned}$$

where we explicitly write the excess return of the portfolio over the risk-free return $r_I - r$. Plugging this in, we see:

$$\begin{aligned} \frac{F_1}{F_0} &= \frac{S_1}{S_0} e^{-(r-q)t_1} \\ &= \frac{S_0}{S_0} e^{(r + (r_I - r) - q)t_1} e^{-(r-q)t_1} \\ &= e^{(r_I - r)t_1} \end{aligned}$$

This shows the leveraged portfolio's returns is just the portfolio returns minus the borrowing costs. The dividends q cancel out of the equation. Since the borrowing costs are around 0.04 these days [13], the equities returns should be higher to make this strategy work.

[11] https://www.lifecycleinvesting.net/index.html

[12] Options, Futures, and Other Derivatives 9th ed., q. 19

[13] 11/4/2022

Returns of a Portfolio with Futures

As we saw before, a futures contract is equivalent to borrowing money, investing the proceeds in equities, then paying back the loan. That means the returns on a futures contract will depend on the return of equities and the return of borrowing rate.

In the real world, you can't hold a portfolio of *just* futures contracts. The brokerage requires you put down margin to serve as collateral on your position. As the futures contract is marked to market daily, you want to avoid your margin dropping below the *maintenance margin* amount set by the brokers, so you hold more margin than is required. This means your "portfolio" with futures requires some additional positions, which affect your return.

Suppose you have an amount W_0 to invest, and you want to use futures. We choose to enter into enough futures contracts to give us an amount FUT of stock market exposure (FUT = futures price $\cdot$ number of contracts). With a futures exposure FUT with W_0 of capital put in, the amount of effective debt you've taken on is given by $FUT = W_0 + D$.

Finally, You must put a certain fraction of your futures position $FUT \cdot M$ to meet the maintenance margin requirement. The remainder of the initial capital W_0 can earn interest, but only at a fraction C of the return of a money market. This is because IBKR pays interest in proportionally lower rates for cash balances below \$100,000.

The return on your futures position is

$$R_F = \frac{FUT \cdot (R_{eq} - R_{rf}) + (W_0 - FUT \cdot M)CR_{mm}}{W_0}$$

The futures contract has return $R_{eq} - R_{rf}$, which we showed above. The amount $FUT \cdot M$ must be held as margin, which earns no interest. On Interactive Brokers, the margin must be in the *Commodities* segment (rather than the *Securities* segment) which earns no interest. The remaining amount of the initial wealth $W_0 - FUT \cdot M$ can sit as cash and earn interest (assuming the cash is paid at rates equal to a money market fund), but at a lower rate C.

Now, assume the return of a money market fund equals the risk-free return ($R_{mm} = R_{rf}$), which is realistic. Also, notice that the futures exposures equals the amount you put in to the strategy plus an amount borrowed ($FUT = W_0 + D$). We rewrite the return as

$$\begin{aligned}
R_F &= \frac{FUT(R_{eq} - R_{mm}) + (W_0 - FUT \cdot M)CR_{mm}}{W_0} && (4) \\
&= \frac{(W_0 + D)(R_{eq} - R_{mm}) + [W_0 - (W_0 + D)M]CR_{mm}}{W_0} \\
&= \frac{W_0 R_{eq} + DR_{eq} - W_0 R_{mm} - DR_{mm} + W_0 CR_{mm} - (W_0 + D)CMR_{mm}}{W_0} && (5) \\
&= R_{eq} + \frac{D}{W_0}(R_{eq} - R_{mm}) - \frac{W_0 - W_0 CR_{mm} + (W_0 + D)CMR_{mm}}{W_0} \\
R_F &= \underbrace{R_{eq}}_{\text{unlevered}} + \underbrace{\frac{D}{W_0}(R_{eq} - R_{mm})}_{\text{from leverage}} - \underbrace{\left[1 + \frac{[(W_0 + D)M - W_0]C}{W_0}\right] R_{mm}}_{\text{doesn't earn interest}} && (6)
\end{aligned}$$

This shows the return on a futures position is the return on the underlying (equities) plus return from leverage. Leverage increases returns when the return on equities is higher than the rate you're paying on your "debt". Also, leveraging more (increasing D or putting down less initial capital W_0 given a fixed futures notional) increases returns. The increase in value is unchanged, but you put in less capital to achieve those gains (or losses), thus raising the returns.

The last term is a drag from cash that serves as margin, which must sit in the *commodities* segment and thus doesn't earn interest. Putting in less capital to the position (lower W_0) for a given exposure (FUT) will increase the return due to leverage, but also increases the returns drag from margin.

We can also rewrite the equation to show that the futures position is equivalent to a portfolio of equities and debt, by starting with equation (5) above:

$$\begin{aligned} R_F &= \frac{W_0 R_{eq} + D R_{eq} - W_0 R_{mm} - D R_{mm} + W_0 C R_{mm} - (W_0 + D) C M R_{mm}}{W_0} \\ &= \frac{(W_0 + D) R_{eq} - [(W_0 + D) - W_0 C + (W_0 + D) C M] R_{mm}}{W_0} \end{aligned}$$

If we assume that all margin can be invested ($M = 0$) at the full money market rate ($C = 1$), then the third term drops out:

$$R_F = \frac{(W_0 + D) R_{eq} - D R_{mm}}{W_0}$$

Now we can see that, at its core, a futures position is equivalent to a portfolio of equities the size of the initial capital plus the borrowed amount, plus a debt which is paid at a certain interest rate.

Leverage of a Futures Portfolio

The leverage is the market exposure divided by the amount you paid to enter the position:

$$\text{leverage} = \frac{\text{stock exposure}}{\text{initial amt.}} = \frac{FUT}{W_0} = \frac{W_0 + D}{W_0} = 1 + \frac{D}{W_0}$$

The futures position's sensitivity to market changes is:

$$\beta = \frac{dR_F}{dR_{eq}} = \frac{d}{dR_{eq}} \frac{FUT \cdot R_{eq}}{W_0} = \frac{FUT}{W_0} = \frac{W_0 + D}{W_0}$$

Back Tests

These back test are variations of the back tests in *Lifecycle Investing (LI)*. These back tests use 96 cohorts of workers, starting investing at age 23 and stopping contributions at age 66. I tweak the back test to assumes an investor starts investing \$300k at age 30, investing even chunks monthly through that year, with no contributions after that, for simplicity. The first of the 96 cohorts starts their career in 1871 and the last cohort starts working in 1966.

Borrowing rate data

These simulations use borrowing rates on margin loans, as used by the authors of LI. In general, margin loans interest rates correlate with the Federal Funds rate. Interactive Brokers computes the margin loan rate as the Federal Funds rate [14], plus a second term [15][16]. For example, the interest rate was $3.83\% + 2.50\% = 6.33\%$ on 11/8/2022.

The borrowing rates using call options or futures would be different than margin loan rates, but I'm assuming they are close enough. This is reasonable for a few reasons. First, the implied interest rate on futures and options are pushed towards the risk-free interest rate via arbitrage (though call options also have some embedding downside protection too). Second, the implied interest rate is often computed during the "roll", and seems like it sometimes is less than the risk-free rate. This is surprising but a CME report says this happens [17]. Also, getting historical futures and options price data is tough, making it hard to use them in backtests. So using the margin loan rates seems reasonable. However, this is an assumption I'm least confident about.

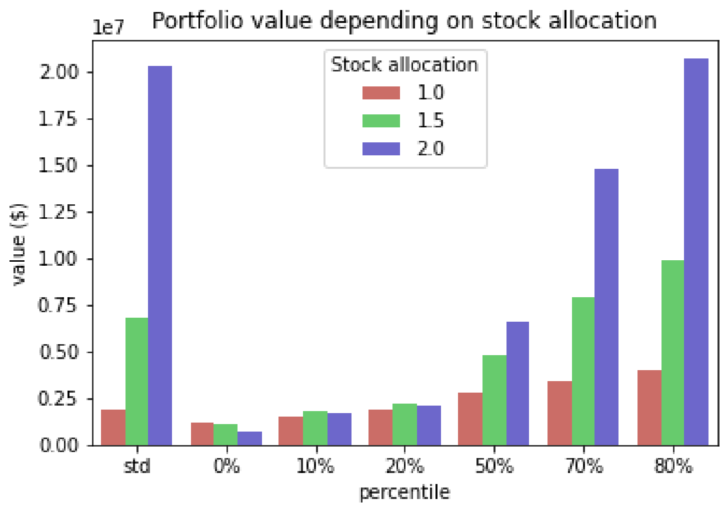
[14] https://www.newyorkfed.org/markets/reference-rates/effr
[15] https://www.interactivebrokers.com/en/trading/margin-rates.php
[16] https://www.interactivebrokers.com/en/trading/margin-benchmarks.php
[17] https://www.cmegroup.com/trading/equity-index/a-cost-comparison-of-futures-and-etfs.html

Stock returns data

LI uses stock returns data from a vendor. Their data goes only until June 2009. So it's missing about 13 years, as of 11/8/2022, of a long bull market with low interest rates. Since these market conditions favor the leveraged strategy, omitting 2010-2022 will give more conservative results. So I'm not worried about removing those years.

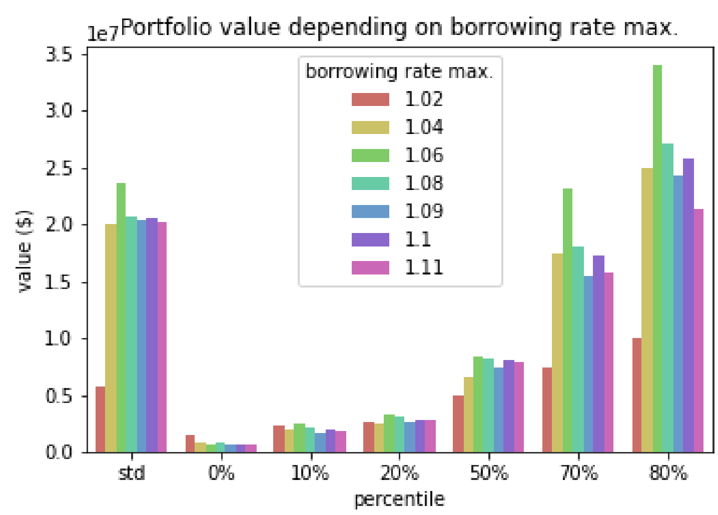
Results

Of the 96 simulations, I show percentiles of the final portfolio value, along with standard deviation. First, I show that leveraging increases final portfolio values for almost all percentiles:



We can see that leverage increases both the expected returns (well, percentiles of returns) and the variance, as expected. Though the variance (standard deviation) is higher, it's mostly to the upside.

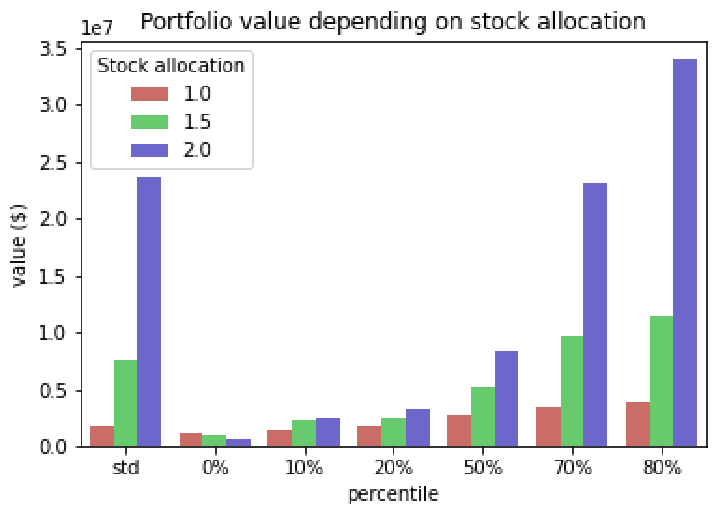
Next, we choose the 2:1 leveraged strategy (stock allocation = 2). Now I make a change. If the interest rate is above a certain value, then I don't leverage for than period:



If we avoid leveraging when the interest rate is high, then it's more likely that the stock return will be higher than the borrowing rate. However, if we avoid leveraging too often, then we will miss too many periods when the stock returns beat the high interest rate. This effect happens more for the high-percentile runs, when stock returns are better. When stock returns are lower, then this cutoff matters less.

To find the optimal borrowing rate threshold, we look at the empirical results above. The optimal value looks like it's around 1.06. That means that if you can borrow for 6% or less, you should use leverage, and don't otherwise.

After including the 6% maximum borrowing rate, here is the expected final portfolio percentiles. Notice that they are much higher than when a cutoff borrowing rate is not applied:



Stress Test

Next, we perform a stress test. We'll add a constant 4% to the annual borrowing rate. This represents a much higher interest rate environment than in the historical data.

When the borrowing rate cutoff is 1, then we're back to the no leverage case. We see that leverage helps for roughy the top 50% of final portfolios. For the bottom half, borrowing hurts. This makes sense because when stocks do well, they overcome even artificially high interest rates.

Again, choosing the interest rate cutoff of 6% seems reasonable here. When stock returns are bad, it hurts only a little. When stock returns are good, levering helps a lot. There is more upside than downside. Also, this stress test scenario is quite pessimistic. It's good to know about, but isn't bad enough to reject the leveraging strategy.

Lingering Questions

1. How long could this strategy under-perform the non-leveraged strategy?
2. What is the role and behavior of maintaining your leveraged position as the market rises or fall?
3. Is it necessary to turn down the equities allocation as Ayres and Nalebuff do? Why not stay leveraged?
4. Why do we want a constant fraction of our lifetime earnings allocated to equities? Why follow Merton's theoretical result?
5. The borrowing cost, implied by futures and index prices, fluctuates by the second. The rate could jump from 2% to 8% within 10 seconds. Is it important to buy only when the rate is good?

Appendix

How much would the market have to fall before triggering a margin call?

This formula shows the fraction that the futures price would have to fall from its current value to trigger a margin call:

$$fall = \frac{EL}{N \cdot 5 \cdot F_0}$$

where

$$\begin{aligned} EL &= excess\ liquidity\ (in\ Interactive\ Brokers) \\ EL &= cash - MM \\ MM &= maintenance\ margin\ (value\ below\ which\ would\ trigger\ a\ margin\ call) \\ F_0 &= current\ price\ of\ the\ futures\ contact \\ N &= number\ of\ MES\ contracts\ in\ your\ position \end{aligned}$$

The numerator is the amount the cash account would have to drop to put the collateral below the minimum allowed (maintenance margin), and trigger a margin call. The 5 in the denominator represents the fact that one futures contract is the obligation to buy 5 times the level of the S&P 500. We used the S&P 500 level, S_0, because the futures price moves rough in line with the S&P 500.

Example

$$fall = \frac{EL}{12 \cdot 5 \cdot F_0} = \frac{40640}{12 \cdot 5 \cdot 4138} = \frac{40640}{248280} = 0.1637$$

So the futures contract, and thus the S&P 500, would have to fall 16.37% to trigger a margin call.