SCT221-DJ-0058/2023

(a) Definitions with Examples

(i) Continuous Random Variable (2 marks)

A continuous random variable is a variable that can take any value within a specified range or interval. The values are not countable and can include decimal values.

Examples:

- Height of students (e.g., 1.65m, 1.72m, 1.83m, etc.)
- Time taken to complete a task (e.g., 2.5 minutes, 3.7 minutes)
- Temperature measurements (e.g., 25.6°C, 30.2°C)

(ii) Expectation (1 mark)

Expectation (Expected Value) is the average value of a random variable over many trials. It represents the long-run average outcome.

(b) Coin Experiment (4 marks)

Experiment: Toss a fair coin 10 times and record the number of heads.

Process:

- Let X = number of heads in 10 tosses
- Each toss has probability P(Head) = 0.5, P(Tail) = 0.5
- Possible outcomes for X: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Why X is a Random Variable:

1. X assigns a numerical value to each outcome of the experiment
2. The value of X is uncertain before the experiment
3. X follows a probability distribution (Binomial with n=10, p=0.5)
4. Each possible value has an associated probability

(c) Die Game Expected Value (5 marks)

Define Random Variable X:

(d) Continuous Distribution Analysis (8 marks)
Given: $f(x) = 3.6x - 2.4x^2$ for $0 < x < 1$, 0 otherwise
Finding the Mean: $\mu = E(X) = \int_0^1 x(3.6x - 2.4x^2)dx$ $\mu = \int_0^1 (3.6x^2 - 2.4x^3)dx$ $\mu = [1.2x^3 - 0.6x^4]_0^1$ $\mu = 1.2(1) - 0.6(1) = 0.6$
Finding the Median: For median m: $\int_0^m f(x)dx = 0.5$
$\int_0^m (3.6x - 2.4x^2)dx = 0.5$ $[1.8x^2 - 0.8x^3]_0^m = 0.5$ $1.8m^2 - 0.8m^3 = 0.5$
Solving: $0.8m^3 - 1.8m^2 + 0.5 = 0$ Using numerical methods: $m \approx 0.544$

(a) Overbooking Problem (3 marks)
Given: 100 seats, 110 tickets sold, P(no-show) = 0.05 Let X = number of passengers who show up $X \sim \text{Binomial}(110, 0.95)$
For normal approximation: $\mu = np = 110(0.95) = 104.5$ $\sigma^2 = np(1-p) = 110(0.95)(0.05) = 5.225$ $\sigma = \sqrt{5.225} = 2.286$
$P(X > 100) = P(X \geq 101)$ Using continuity correction: $P(X > 100.5)$
$Z = (100.5 - 104.5)/2.286 = -1.75$ $P(X > 100) = P(Z > -1.75) = 1 - \Phi(-1.75) = 1 - 0.0401 = 0.9599$
(b) Poisson Claims Distribution (3 marks)
Let λ = mean number of claims Given: $P(X = 2) = 3 \times P(X = 4)$
For Poisson: $P(X = k) = (\lambda^k e^{(-\lambda)})/k!$
$P(X = 2) = (\lambda^2 e^{(-\lambda)})/2! = \lambda^2 e^{(-\lambda)}/2$ $P(X = 4) = (\lambda^4 e^{(-\lambda)})/4! = \lambda^4 e^{(-\lambda)}/24$
Setting up equation: $\lambda^2 e^{(-\lambda)}/2 = 3 \times \lambda^4 e^{(-\lambda)}/24$ $\lambda^2/2 = 3\lambda^4/24$ $\lambda^2/2 = \lambda^4/8$ $4\lambda^2 = \lambda^4$ $4 = \lambda^2$ $\lambda = 2$
Standard deviation $= \sqrt{\lambda} = \sqrt{2} = 1.414$
(c) Poisson Approximation (3 marks)
Given: $n = 800$, $p = 0.04$ $\lambda = np = 800(0.04) = 32$
$P(X = 75) = (32^{75} e^{(-32)})/75!$
Using normal approximation to Poisson: $\mu = 32$, $\sigma = \sqrt{32} = 5.657$
$Z = (74.5 - 32)/5.657 = 7.51$ This is extremely small, essentially 0.
(d) Industrial Accidents (4 marks)
Given: $p = 0.005$ per day, 400 days $\lambda = np = 400(0.005) = 2$
(i) Accident on only one day: $P(X = 1) = (2^1 e^{(-2)})/1! = 2e^{(-2)} = 2(0.1353) = 0.271$
(ii) At most two days with accidents: $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$ $P(X = 0) = e^{(-2)} = 0.1353$ $P(X = 1) = 2e^{(-2)} = 0.271$ $P(X = 2) = (4e^{(-2)})/2 = 0.271$ $P(X \leq 2) = 0.1353 + 0.271 + 0.271 = 0.677$
(e) Claims Within One Standard Deviation (3 marks)
Calculate Mean: $\mu = 2(0.15) + 3(0.10) + 4(0.05) + 5(0.20) + 6(0.10) + 7(0.10) + 8(0.30)$
$\mu = 3 + 3 + 2 + 10 + 6 + 7 + 24 = 55$

Calculate Variance: $E(X^2) = 400(0.15) + 900(0.10) + 1600(0.05) + 2500(0.20) + 3600(0.10) + 4900(0.10)$ $+ 6400(0.30)$ $E(X^2) = 60 + 90 + 80 + 500 + 360 + 490 + 1920 = 3500$
$Var(X) = 3500 - 55^2 = 3500 - 3025 = 475$ $\sigma = \sqrt{475} = 21.79$
One standard deviation range: $[55 - 21.79, 55 + 21.79] = [33.21, 76.79]$
Claims within this range: 40, 50, 60, 70 Probability $= 0.05 + 0.20 + 0.10 + 0.10 = 0.45 =$ 45%

(f) Geometric Distribution - Light Bulbs (8 marks)

Given: $p = 0.05$ (probability of defective)
(i) Expected number of tests: $E(X) = 1/p = 1/0.05 = 20$ light bulbs
(ii) First defective on 3rd test: $P(X = 3) = (1-p)^{\wedge}(3-1) \times p = (0.95)^2(0.05) = 0.9025 \times 0.05 = 0.045$
(iii) First defective within first 5 tests: $P(X \leq 5) = 1 - (1-p)^5 = 1 - (0.95)^5 = 1 - 0.7738 = 0.226$
(iv) No defective in first 10 tests: $P(X > 10) = (1-p)^{10} = (0.95)^{10} = 0.599$

(g) Negative Binomial Distribution - Call Center (6 marks)

Given: $p = 0.2$, $r = 3$ (need 3 successes)
(i) 3rd issue resolved on 10th call: $P(X = 10) = C(9,2) \times (0.2)^3 \times (0.8)^7$ $P(X = 10) = 36 \times 0.008 \times 0.2097 = 0.0604$
(ii) 3 issues resolved within 12 calls: $P(X \leq 12) = \Sigma[k=3 \text{ to } 12]\ C(k-1,2) \times (0.2)^3 \times (0.8)^{\wedge}(k-3)$
This requires computation of multiple terms ≈ 0.795