Question One

a) Giving relevant examples explain what you understand by the following terms.

i) Continuous random variable.

ii) Expectation

i) A continous random variable is a variable that can take any real value within a given range. Example. The height of students in a class is continuous. It could be 160.2cm, 160.23cm, 150cm etc.

ii) ~~E~~ The expectation of a random variable is its average value over numerous repetitions of the experiment.

Example: For a fair 6-sided dice,

$$E(x) = \sum x P(x) = \frac{1+2+3+4+5+6}{6} = 3.5.$$

b) Given a fair coin, perform an experiment using the coin, and hence show that the outcome of your experiment is a random variable.

Let $X = \begin{cases} 1 & \text{if heads} \\ 0 & \text{if tails} \end{cases}$

$$P(x=1) = 0.5, \quad P(x=0) = 0.5$$

Expectation

$$E(x) = 1(0.5) + 0(0.5) = 0.5.$$

c) A fair six-sided dice is tossed. You win \$2 if the result is a "1" you win \$1 if the result is a "6" but otherwise you lose \$1. Using this information, define the random variable X, the probability distribution and find the expected value

Outcome	Value of X	P(X)
1	+2	1/6
6	1	1/6
2 - 5	-1	4/6

Expected Value

$$E(x) = 2(1/6) + 1(1/6) + (-1)(4/6)$$

$$2/6 + 1/6 - 4/6 = -1/6$$

On average you lose 1/6 per ~~d~~ play

d) The random variable X has probability density function

$$f(x) = \begin{cases} 3.6x - 2.4x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and median of the distribution. Also find $P(X > 0.5)$ and Var (X)

Mean

$$E(X) = \int_0^1 x(3.6x - 2.4x^2)dx = \int_0^1 (3.6x^2 - 2.4x^3)dx = [1.2x^3 - 0.6x^4]_0^1 =$$

$$1.2 - 0.6 = 0.6$$

Median

Find m.

$$\int_0^m (3.6x - 2.4x^2)dx = 0.5$$

$$[1.8x^2 - 0.8x^3]_0^m = 0.5$$

$$1.8m^2 - 0.8m^3 = 0.5$$

$m = 0.56 \Rightarrow \sim 0.504 \to$ Median ≈ 0.56

$P(X > 0.5)$

$$\int_{0.5}^1 (3.6x - 2.4x^2)dx = [1.8x^2 - 0.8x^3]_{0.5}^1 = (1.8 - 0.8) - (0.45 - 0.1) =$$

$$1.0 - 0.35 = 0.65$$

Variance

$E(X^2) = \cancel{\int_0^1 3.6x}\ \int_0^1 x^2(3.6x - 24x^2)\,dx = \int_0^1 (3.6x^3 - 24x^4)\,dx =$

$[0.9x^4 - 0.48x^5]_0^1 = 0.9 - 0.48 = 0.42.$

$Var(X) = E(X^2) - (E(X))^2 = 0.42 - (0.6)^2 = 0.42 - 0.36 = 0.06$

Question 2

a)

Let $X \sim Bin(n = 110, P = 0.95)$, mean = 104.5, Variance = 5.225

Approximate with normal:

$P(X > 100) \approx P(Y > 100.5),\ Y \sim N(104.5, \sqrt{5.225}) \approx N(104.5, 2.29)$

$Z = \frac{100.5 - 104.5}{2.29} \approx -1.75 \Rightarrow P(X > 100) \approx 1 - P(Z < -1.75) = 1 - 0.0401$

$= 0.9599$

b)

Let λ be Poisson mean

$P(X = 2) = 3P(X = 4)$, Poisson: $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$

$\frac{\lambda^2}{2!} = 3\frac{\lambda^4}{4!} \Rightarrow \frac{\lambda^2}{2} = 3 \cdot \frac{\lambda^4}{24}$

$\Rightarrow 12\lambda^2 = 3\lambda^4 \Rightarrow 4 = \lambda^2 \Rightarrow \lambda = 2.$

Standard deviation $= \sqrt{\lambda} = \sqrt{2} \approx 1.41$

c) Use $\lambda = np = 800 \cdot 0.04 = 32$.

Poisson:

$$P(X=75) = \frac{32^{75} e^{-32}}{75!} \approx 0$$

d) Use Poisson: $\lambda = 400 \cdot 0.005 = 2$.

i) $P(X=1) = \frac{2^1 e^{-2}}{1!} = 2e^{-2} \approx 2(0.1353) = 0.2706$

ii) $P(X \leq 2) = P(0) + P(1) + P(2)$

$$P(0) = e^{-2} = 0.1353, \quad P(1) = 0.2706, \quad P(2) = \frac{4e^{-2}}{2} = 0.2706$$

$$\Rightarrow P(X \leq 2) = 0.1353 + 0.2706 + 0.2706 = 0.6765.$$

e) Find mean μ, variance

size (x)	$P(x)$
20	0.15
30	0.10
40	0.05
50	0.20
60	0.10
70	0.10
80	0.30

$$\mu = \sum xP = (20)(0.15) + 30(0.10) + 40(0.05) + 50(0.20) + 60(0.10) + 70(0.10) + 80(0.30)$$
$$= 56.$$

$$E(X^2) = \sum x^2 P = 400(0.15) + 900(0.10) + 1600(0.05) + 2500(0.20) + 3600(0.10) +$$
$$4900(0.10) + 64(0.30) = 3370.$$

$Var(X) = 3370 - 56^2 = 3370 - 3136 = 234$

$\sigma = \sqrt{234} \approx 15.3$

$[56 - 15.3, 56 + 15.3] = 40.7, 71.3$

Include 50, 60, 70 → 0.20, + 0.10 + 0.10 = 0.40 or 40%.

f)

$p = 0.05$, $X \sim Geom(p)$.

i) $E(X) = 1/p = \frac{1}{0.05} = 20$.

ii) $P(X=3) = (1-p)^2 \cdot p = (0.95)^2 \cdot 0.05 \approx 0.9025 \cdot 0.05 \approx 0.0451$.

iii) $P(X \le 5) = 1 - (1-p)^5 = 1 - (0.95)^5 \approx 0.2262$.

iv) $P(X > 10) = (1-p)^{10} = (0.95)^{10} \approx 0.5987$.

g)

i) $P(X=10) = \binom{9}{2}(0.2)^3(0.8)^7 = 36 \cdot 0.008 \cdot 0.2097 \approx 0.0604$.

ii) $P(X \le 12) = \sum_{x=3}^{12} P(X=x) = 0.74$.

iii) Variance: $Var(X) = \frac{r(1-p)}{p^2} = \frac{3 \cdot 0.8}{0.04} = 60$.