Note

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January 28, 2023

Abstract

Supplementary note on the proof of the fact that convergence in probability implies convergence in distribution.

Lemma 1. *Let X, Y be random variables, let $a \in \mathbb{R}$ and $\epsilon > 0$. Then*

$$P(Y \leq a) \leq P(X \leq a + \epsilon) + P(|Y - X| > \epsilon)$$

Proof.

$$\begin{aligned} P(Y \leq a) &= P(Y \leq a, X \leq a + \epsilon) + P(Y \leq a, X > a + \epsilon) \\ &\leq P(X \leq a + \epsilon) + P(Y - X \leq a - X, a - X < -\epsilon) \\ &\leq P(X \leq a + \epsilon) + P(Y - X \leq -\epsilon) \\ &\leq P(X \leq a + \epsilon) + P(Y - X \leq -\epsilon) + P(Y - X > \epsilon) \\ &= P(X \leq a + \epsilon) + P(|Y - X| > \epsilon) \end{aligned}$$

□

Theorem 2. *$X_n \longrightarrow X$ in probability implies $X_n \longrightarrow X$ in distribution.*

Proof. Fix $\epsilon > 0$ and let x be a continuity point of F. Then

$$\begin{aligned} F_n(x) &= P(X_n \leq x) \\ &= P(X_n \leq x, X \leq x + \epsilon) + P(X_n \leq x, X > x + \epsilon) \\ &\leq P(X \leq x + \epsilon) + P(|X_n - X| > \epsilon) \\ &= F(x + \epsilon) + P(|X_n - X| > \epsilon). \end{aligned}$$

Also,

$$\begin{aligned} F(x - \epsilon) &= P(x \leq x - \epsilon) \\ &= P(x \leq x - \epsilon, X_n \leq x) + P(x \leq x - \epsilon, X_n > x) \\ &\leq F_x(x) + P(|X_n - X| > \epsilon). \end{aligned}$$

Hence,

$$F(x - \epsilon) - P(|X_n - X| > \epsilon) \leq F_n(x) \leq F(x + \epsilon) + P(|X_n - X| > \epsilon).$$

Take the limit as $n \to \infty$ to conclude that

$$F(x - \epsilon) \leq \liminf_{n \to \infty} F_n(x) \leq \limsup_{n \to \infty} F_n(x) \leq F(x + \epsilon).$$

This holds for all $\epsilon > 0$. Take the limit as $\epsilon \to 0$ and use the fact that F is continuous at x and conclude that $\lim_{n \to \infty} F_n(x) = F(x)$. □

References

[1] Proofs of convergence of random variables

[2] Larry Wasserman, *All of Statistics A Concise Course in Statistical Inference*, Springer, 2004