

# Note

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## Abstract

Supplementary note on the proof of the fact that convergence in probability implies convergence in distribution.

**Lemma 1.** *Let  $X, Y$  be random variables, let  $a \in \mathbb{R}$  and  $\epsilon > 0$ . Then*

$$P(Y \leq a) \leq P(X \leq a + \epsilon) + P(|Y - X| > \epsilon)$$

*Proof.*

$$\begin{aligned} P(Y \leq a) &= P(Y \leq a, X \leq a + \epsilon) + P(Y \leq a, X > a + \epsilon) \\ &\leq P(X \leq a + \epsilon) + P(Y - X \leq a - X, a - X < -\epsilon) \\ &\leq P(X \leq a + \epsilon) + P(Y - X \leq -\epsilon) \\ &\leq P(X \leq a + \epsilon) + P(Y - X \leq -\epsilon) + P(Y - X > \epsilon) \\ &= P(X \leq a + \epsilon) + P(|Y - X| > \epsilon) \end{aligned}$$

□

**Theorem 1.**  *$X_n \rightarrow X$  in probability implies  $X_n \rightarrow X$  in distribution.*

*Proof.* Fix  $\epsilon > 0$  and let  $x$  be a continuity point of  $F$ . Then

$$\begin{aligned} F_n(x) &= P(X_n \leq x) \\ &= P(X_n \leq x, X \leq x + \epsilon) + P(X_n \leq x, X > x + \epsilon) \\ &\leq P(X \leq x + \epsilon) + P(|X_n - X| > \epsilon) \\ &= F(x + \epsilon) + P(|X_n - X| > \epsilon). \end{aligned}$$

Also,

$$\begin{aligned} F(x - \epsilon) &= P(x \leq x - \epsilon) \\ &= P(x \leq x - \epsilon, X_n \leq x) + P(x \leq x - \epsilon, X_n > x) \\ &\leq F(x) + P(|X_n - X| > \epsilon). \end{aligned}$$

Hence,

$$F(x - \epsilon) - P(|X_n - X| > \epsilon) \leq F_n(x) \leq F(x + \epsilon) + P(|X_n - X| > \epsilon).$$

Take the limit as  $n \rightarrow \infty$  to conclude that

$$F(x - \epsilon) \leq \liminf_{n \rightarrow \infty} F_n(x) \leq \limsup_{n \rightarrow \infty} F_n(x) \leq F(x + \epsilon).$$

This holds for all  $\epsilon > 0$ . Take the limit as  $\epsilon \rightarrow 0$  and use the fact that  $F$  is continuous at  $x$  and conclude that  $\lim_{n \rightarrow \infty} F_n(x) = F(x)$ . □

## References

- [1] [Proofs of convergence of random variables](#)
- [2] Larry Wasserman, *All of Statistics A Concise Course in Statistical Inference*, Springer, 2004