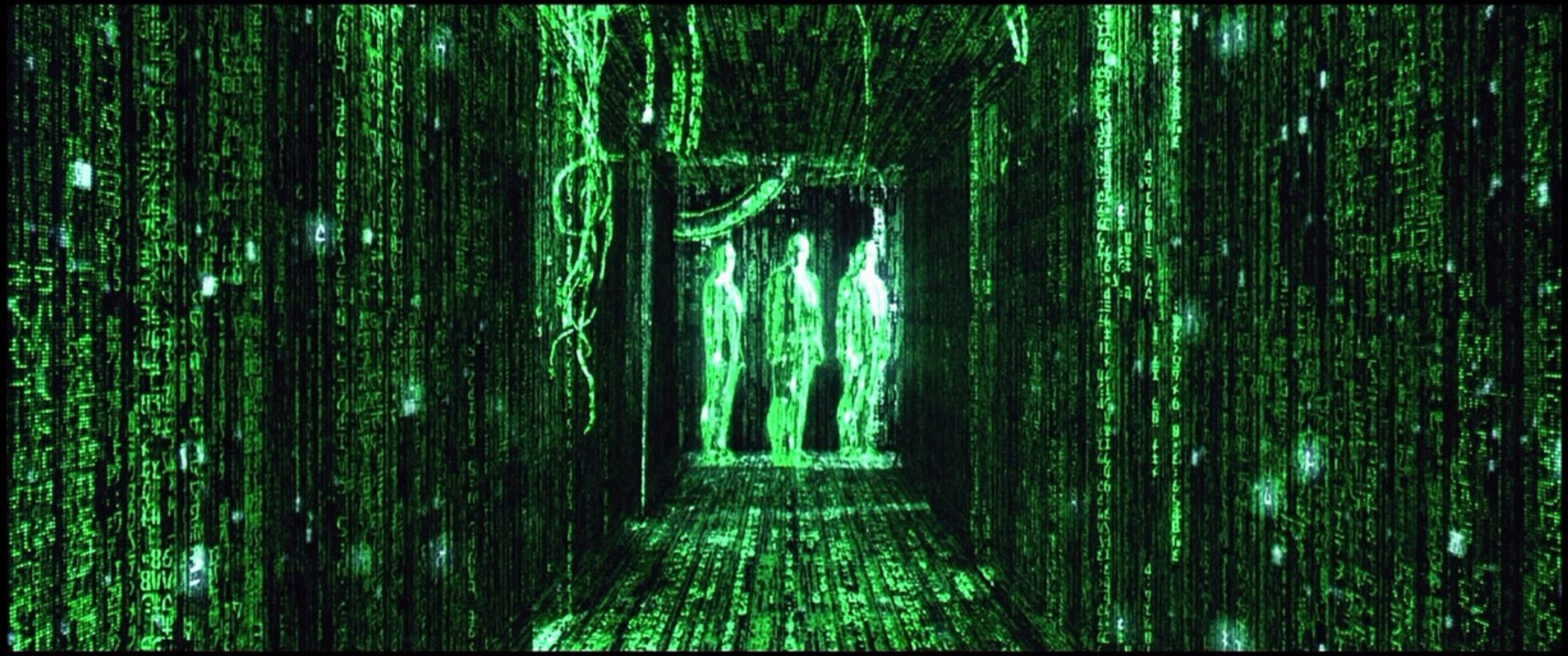
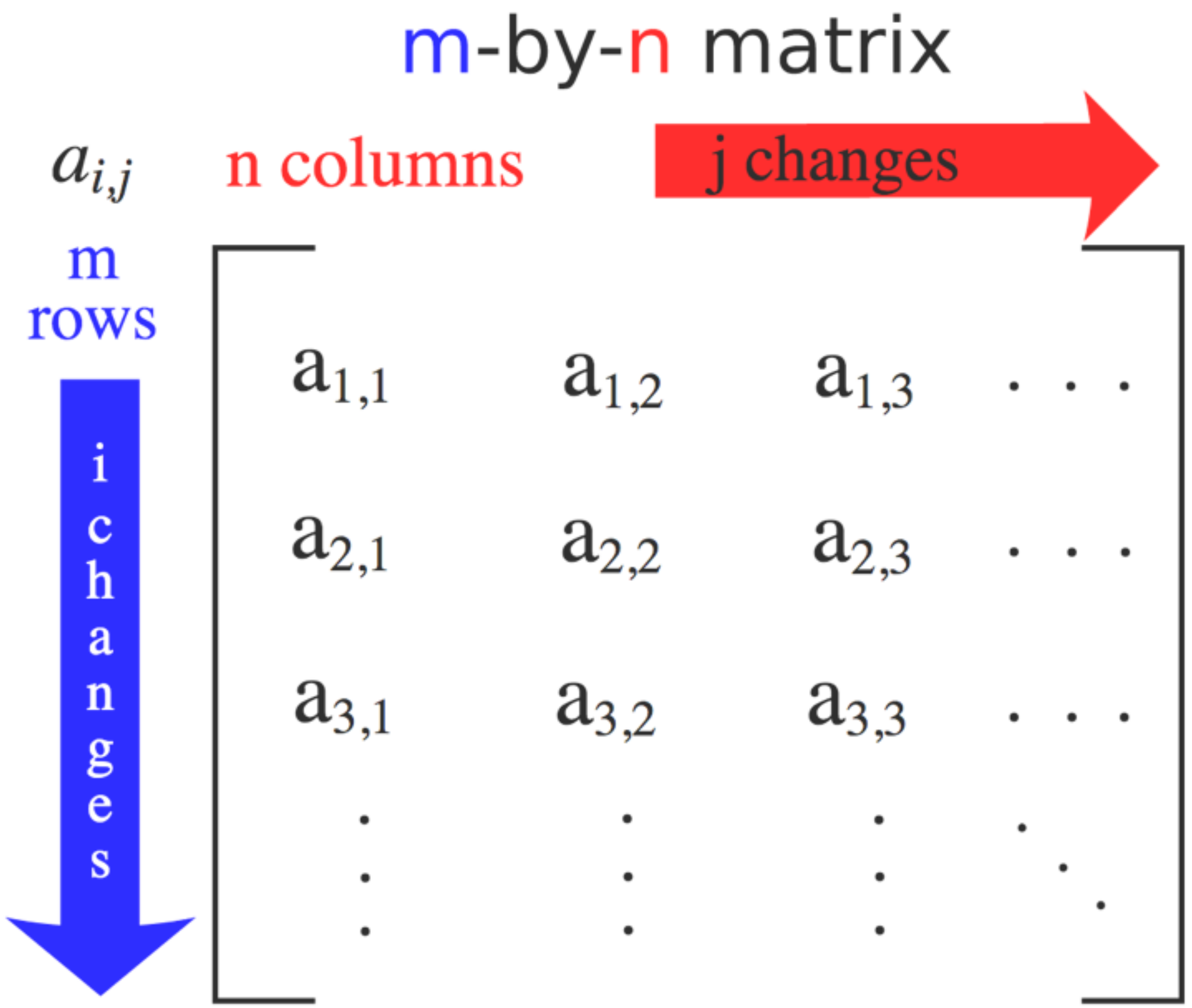
Matrix transformations.

Part 1



Matrix **math**.

A **matrix.**


m-by-n matrix
$a_{i,j}$
n columns
j changes
m rows
i changes
$a_{1,1}$ $a_{1,2}$ $a_{1,3}$ $\cdots$
$a_{2,1}$ $a_{2,2}$ $a_{2,3}$ $\cdots$
$a_{3,1}$ $a_{3,2}$ $a_{3,3}$ $\cdots$
$\vdots$ $\vdots$ $\vdots$ $\ddots$

A **2x3** matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \end{bmatrix}$$

A **3x3** matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \\ 3 & 4 & 2 \end{bmatrix}$$

Matrix **operations**.

Matrix **addition**.

To **add** two matrices, **add** their **corresponding entries**.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} + \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} = \begin{bmatrix} A+J & B+K & C+L \\ D+M & E+N & F+O \\ G+P & H+Q & I+R \end{bmatrix}$$

Matrix **subtraction**.

To **subtract** two matrices, **subtract** their **corresponding entries**.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} - \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} = \begin{bmatrix} A-J & B-K & C-L \\ D-M & E-N & F-O \\ G-P & H-Q & I-R \end{bmatrix}$$

Matrix addition and subtraction can only happen with **matrices that are the same size**!

Transpose of a matrix.

Transpose of a matrix is a matrix whose **columns are the rows** of the original matrix (and its **rows are the columns**).

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \qquad \begin{bmatrix} A & D \\ B & E \\ C & F \end{bmatrix}$$

$$M \qquad\qquad M^T$$

Matrix/**scalar multiplication**.

Multiply each entry of the matrix **by the scalar**.

$$S \times \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = \begin{bmatrix} SxA & SxB & SxC \\ SxD & SxE & SxF \\ SxG & SxH & SxI \end{bmatrix}$$

Matrix/**matrix multiplication**.

You can only multiply **two matrices** if the **number of columns of the first matrix** equals the **number of rows of the second**.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} = \quad ?$$

It results in a matrix that is **number of rows of first matrix** by **number of columns of second matrix**.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

For **each row**, find **dot product with each column**.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}\begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

$$\begin{bmatrix} AxJ + BxM + CxP & \\ & \end{bmatrix}$$

For **each row**, find **dot product with each column**.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

$$\begin{bmatrix} A \times J + B \times M + C \times P & A \times K + B \times N + C \times Q \\ & \end{bmatrix}$$

For **each row**, find **dot product with each column.**

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

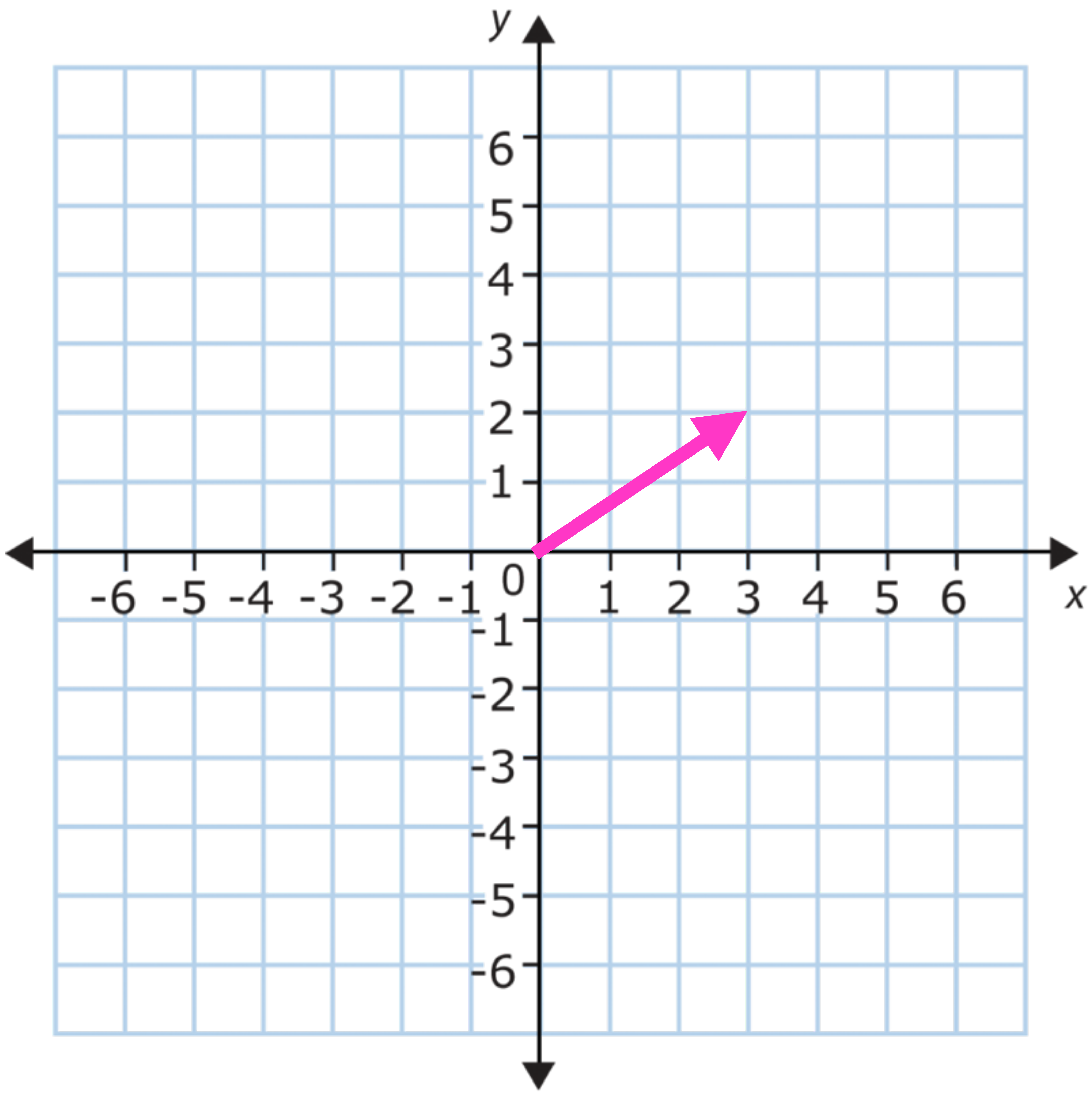
$$\begin{bmatrix} A \times J + B \times M + C \times P & A \times K + B \times N + C \times Q \\ D \times J + E \times M + F \times P & \end{bmatrix}$$

For **each row**, find **dot product with each column**.

$$\begin{bmatrix} A & B & C \\ \boxed{D \quad E \quad F} \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

$$\begin{bmatrix} A \text{x} J + B \text{x} M + C \text{x} P & A \text{x} K + B \text{x} N + C \text{x} Q \\ D \text{x} J + E \text{x} M + F \text{x} P & D \text{x} K + E \text{x} N + F \text{x} Q \end{bmatrix}$$

Vectors.



y
x
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-6 -5 -4 -3 -2 -1
1 2 3 4 5 6

A **2 dimensional** vector can be represented as a **2x1 matrix**.

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

A **3 dimensional** vector can be represented as a **3x1 matrix**.

$$\begin{bmatrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{3} \end{bmatrix}$$

Matrix vector multiplication.

Multiplying a **matrix and a vector** is basically just **multiplying two matrices**.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

Row by row, multiply **each column value** with the **each row of the vector** and **add them together**.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ \\ \\ \end{bmatrix}$$

Row by row, multiply **each column value** with the **each row of the vector** and **add them together**.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ \\ \end{bmatrix}$$

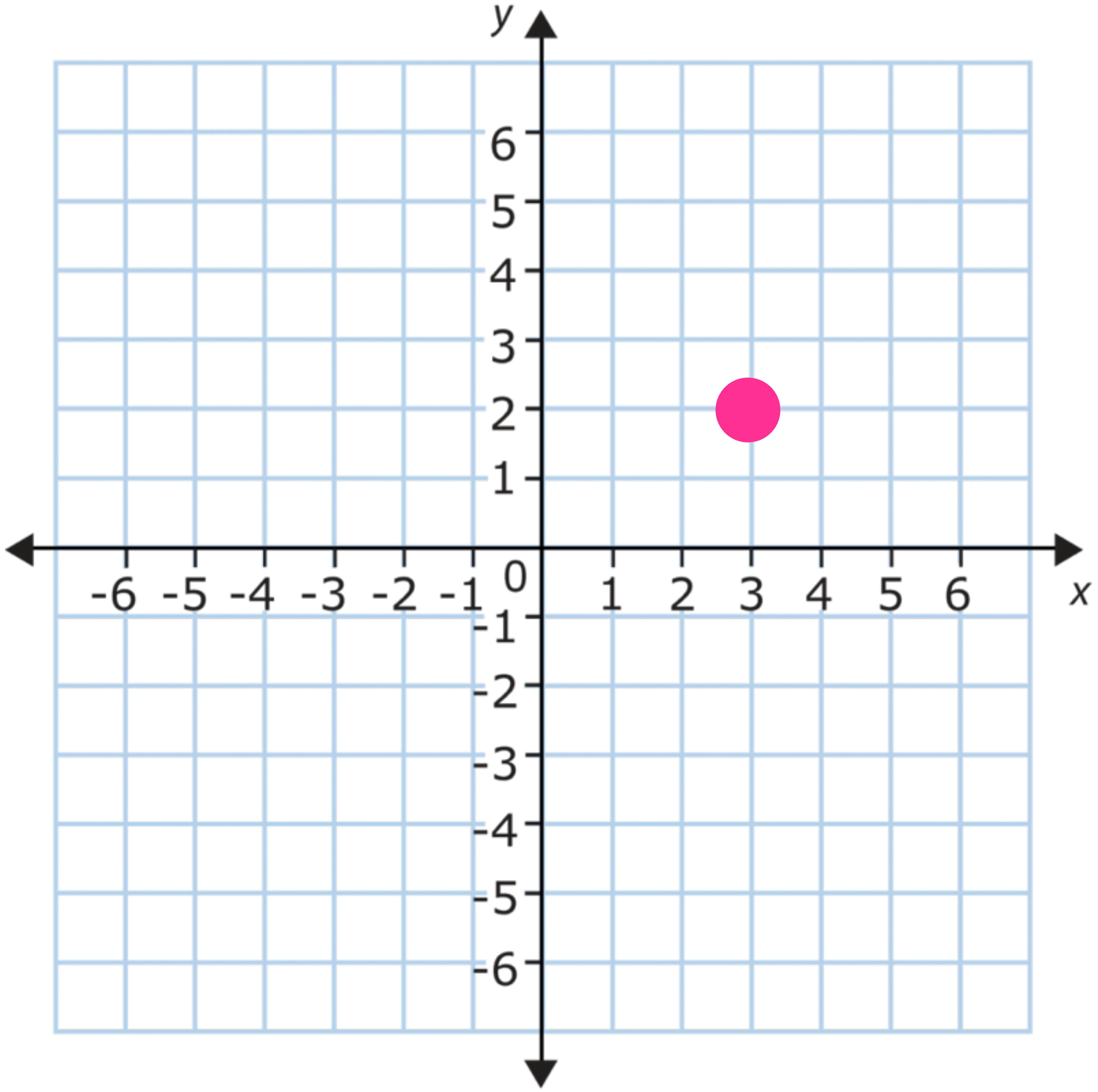
Row by row, multiply **each column value** with the **each row of the vector** and **add them together**.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ GX + HY + IZ \end{bmatrix}$$

Why are we doing all this?

Transformation matrices.

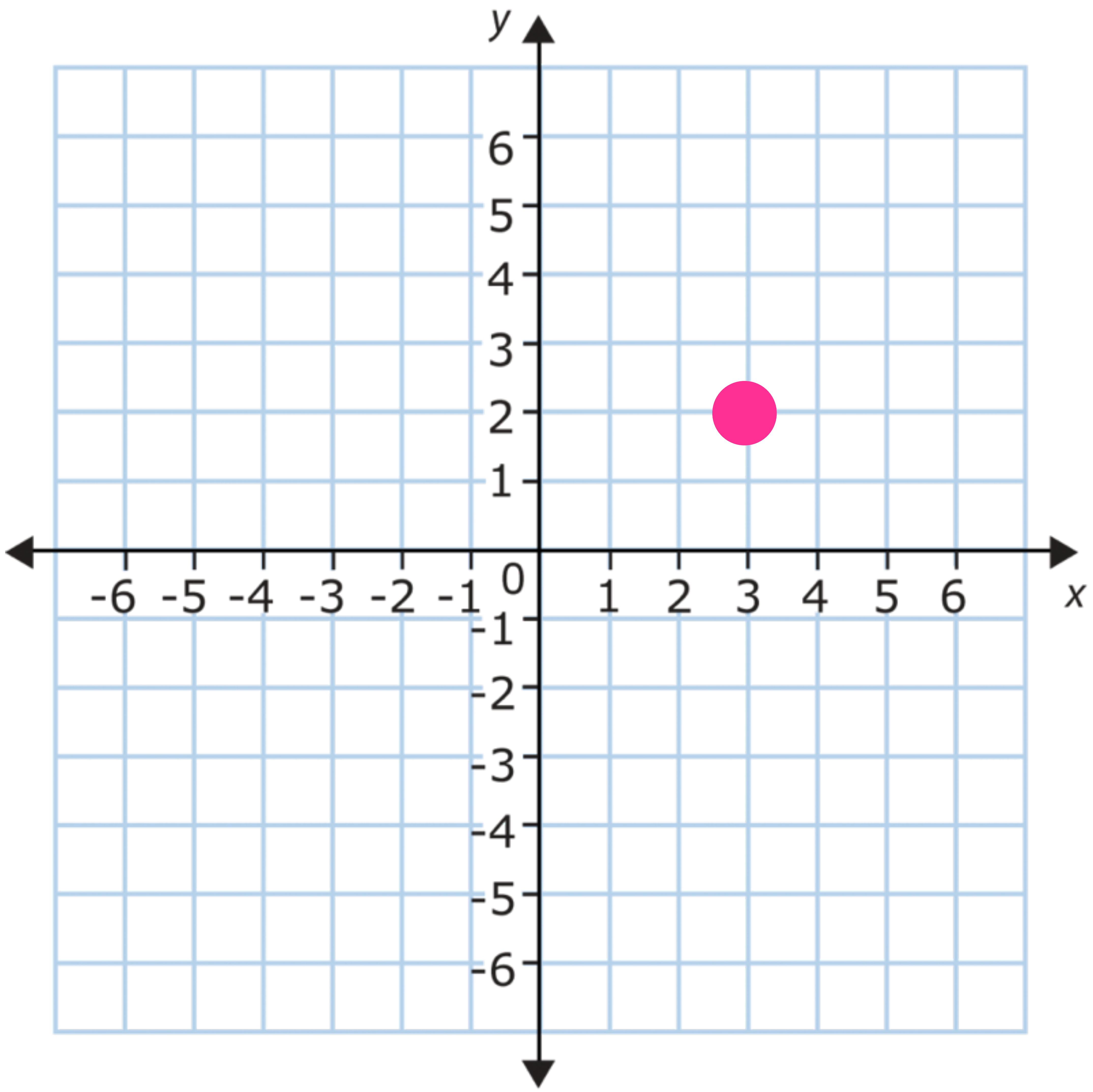
Affine transformations stored as **matrices**.



$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

A **transformation matrix** is a **matrix that we can multiply with a vector** to **transform the vector.**

Example: **scale**



$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Scale

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Scale

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} AX + BY \\ CX + DY \end{bmatrix}$$

$$\begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} sxX + 0Y \\ 0X + syY \end{bmatrix}$$

Example: **translate**?

Homogenous coordinates.

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Translate

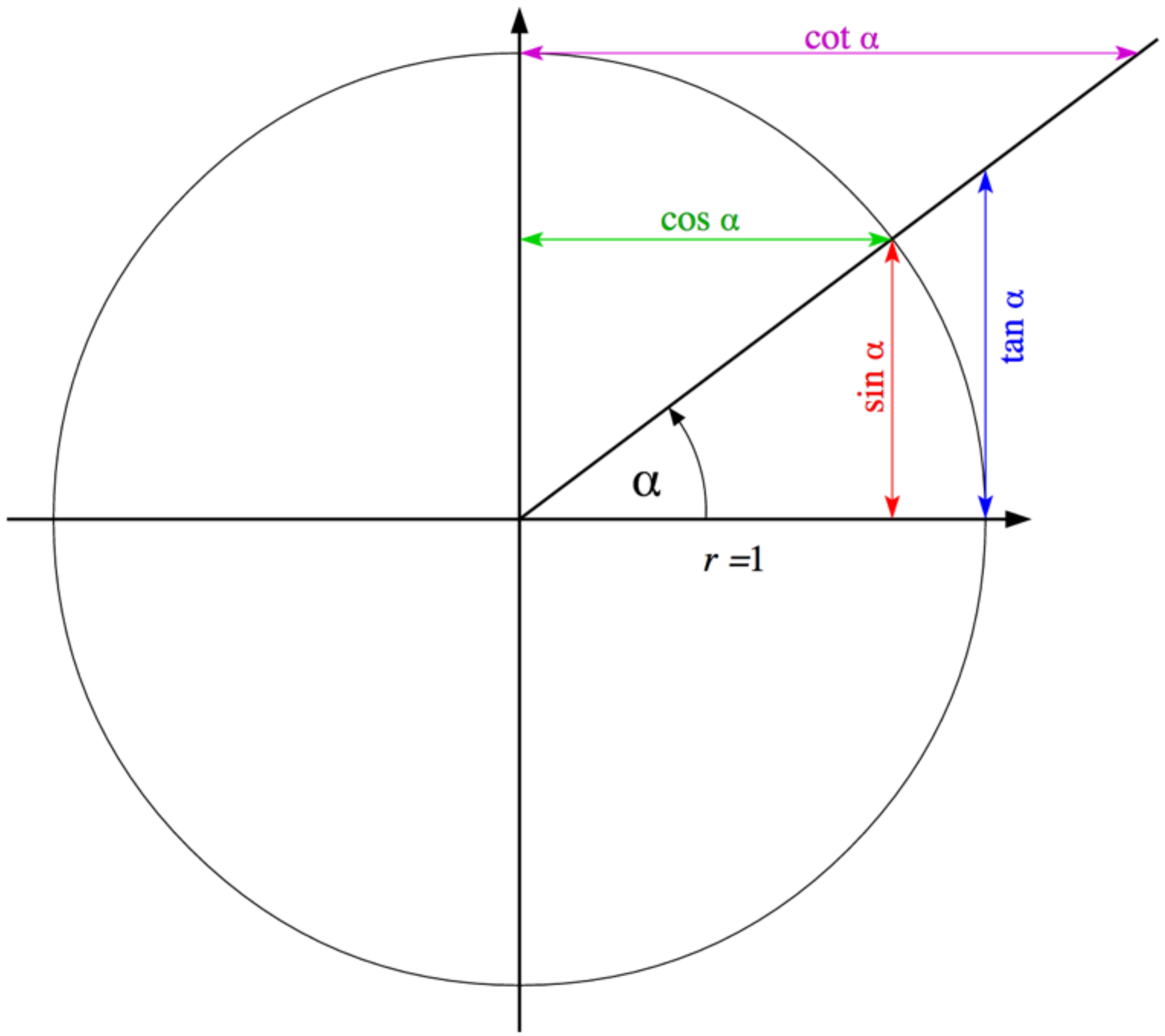
Translate

$$\begin{bmatrix} 1 & 0 & Tx \\ 0 & 1 & Ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 1X + 0Y + 1Tx \\ 0X + 1Y + 1Ty \\ 0X + 0Y + 1x1 \end{bmatrix}$$

Rotation

Rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta X + -\sin\theta Y + 1\mathrm{x}0 \\ \sin\theta X + \cos\theta Y + 1\mathrm{x}0 \\ 0X + 0Y + 1\mathrm{x}1 \end{bmatrix}$$



$$\begin{bmatrix} \cos\theta X + -\sin\theta Y + 1x0 \\ \sin\theta X + \cos\theta Y + 1x0 \\ 0X + 0Y + 1x1 \end{bmatrix}$$

Identity

Identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 1X + 0Y + 1x0 \\ 0X + 1Y + 1x0 \\ 0X + 0Y + 1x1 \end{bmatrix}$$

Multiplying affine transformation **matrices**.

You can only multiply **two matrices** if the **number of columns of the first matrix** equals the **number of rows of the second**.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} = \quad ?$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = ✗$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = ✓$$

```
matrix.identity();
matrix.Translate(5.0, 4.0, 0.0);
matrix.Scale(2.0, 4.0, 1.0);

// draw vertex at 3,2
```

```
matrix.identity();
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

```
matrix.identity();  matrix.Translate(5.0, 4.0, 0.0);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix}$$

```
matrix.identity();     matrix.Translate(5.0, 4.0, 0.0);   matrix.Scale(2.0, 4.0, 1.0);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 5 \\ 0 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ 1 \end{bmatrix}$$

```
matrix.identity();
matrix.Scale(2.0, 4.0, 1.0);
matrix.Translate(5.0, 4.0, 0.0);

// draw vertex at 3,2
```

```
matrix.identity();
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

```
matrix.identity();    matrix.Scale(2.0, 4.0, 1.0);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 1 \end{bmatrix}$$

`matrix.identity();` `matrix.Scale(2.0f, 4.0f, 1.0f);` `matrix.Translate(5.0f, 4.0f, 0.0f);`

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 10 \\ 0 & 4 & 16 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 24 \\ 1 \end{bmatrix}$$

Moving into **3D**

3D identity matrix and **3d position** in **homogenous coordinates**.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

All transformations in 3D

X-Rotation in 3D

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Z-Rotation in 3D

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scale in 3D

$$\begin{bmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Y-Rotation in 3D

$$\begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation in 3D

$$\begin{bmatrix} 1 & 0 & 0 & Tx \\ 0 & 1 & 0 & Ty \\ 0 & 0 & 1 & Tz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection matrices are the same.

matrix.**setOrthoProjection**(l, r, b, t, n, f);

$$\begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{(r+l)}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{(t+b)}{t-b} \\ 0 & 0 & -\frac{2}{f-n} & -\frac{(f+n)}{f-n} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$