$$I_1 = \int_0^1 \sin(\alpha \ln x) x^{\beta-1} dx \quad I_2 = \int_0^1 \cos(\alpha \ln x) x^{\beta-1} dx$$

Method 1:Euler's formula
Consider

$$I = \int_0^1 e^{i\alpha \ln x} x^{\beta-1} dx = \int_0^1 x^{i\alpha} x^{\beta-1} dx$$

$$= \left[\frac{x^{\beta+i\alpha}}{\beta+i\alpha} \right]_0^1 = \frac{1}{\beta+i\alpha} = \frac{\beta-i\alpha}{\beta^2+\alpha^2}$$

Since

$$I = \int_0^1 e^{ia \ln x} x^{\beta-1} dx = \int_0^1 \cos(\alpha \ln x) x^{\beta-1} dx + i \int_0^1 \sin(\alpha \ln x) x^{\beta-1} dx$$

Compare Real and Imaginary part:

$$I_1 = \int_0^1 \sin(\alpha \ln x) x^{\beta-1} dx = -\frac{\alpha}{\beta^2+\alpha^2} \quad I_2 = \int_0^1 \cos(\alpha \ln x) x^{\beta-1} dx = \frac{\beta}{\beta^2+\alpha^2}$$

Method 2:Laplace Transform
Introduce u-sub for both I_1 and I_2

$$u = -\ln x \qquad -e^{-u} du = dx$$

$$I_1 = -\int_0^\infty e^{-u} \sin(\alpha u) e^{-u(\beta-1)} du \quad I_2 = \int_0^\infty e^{-u} \cos(\alpha u) e^{-u(\beta-1)} du$$

Observe that

$$\mathscr{L}\{\sin \alpha t\} = \int_0^\infty e^{-st} \sin(\alpha t) dt = \frac{\alpha}{s^2+\alpha^2}$$

$$\mathscr{L}\{\cos \alpha t\} = \int_0^\infty e^{-st} \cos(\alpha t) dt = \frac{s}{s^2+\alpha^2}$$

In this case,$s = \beta$
Therefore,

$$I_1 = \int_0^1 \sin(\alpha \ln x) x^{\beta-1} dx = -\frac{\alpha}{\beta^2+\alpha^2} \quad I_2 = \int_0^1 \cos(\alpha \ln x) x^{\beta-1} dx = \frac{\beta}{\beta^2+\alpha^2}$$

$$I = \int_0^1 \frac{x(x+1)\sin(\ln x)}{\ln x} dx$$

Let

$$u = -\ln x \qquad -e^{-u} du = dx$$

$$I = \int_0^\infty \frac{e^{-u}(e^{-u} + e^{-2u})\sin u}{u} du = \int_0^\infty e^{-2u}\frac{\sin u}{u} + e^{-3u}\frac{\sin u}{u} du$$

Observe that

$$\mathscr{L}\left\{\frac{\sin at}{t}\right\} = \int_0^\infty e^{-st}\frac{\sin at}{t} dt = \int_s^\infty \mathscr{L}\{\sin at\} ds$$

$$= \int_s^\infty \frac{a}{a^2+s^2} ds = \left[\frac{a}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

In this case,$a = 1$ and $s = 2/3$
Thus

$$\int_0^\infty e^{-2u}\frac{\sin u}{u} + e^{-3u}\frac{\sin u}{u} du = \frac{\pi}{2} - \tan^{-1}(2) + \frac{\pi}{2} - \tan^{-1}(3) = \frac{\pi}{4}$$

Finally,

$$I = \int_0^1 \frac{x(x+1)\sin(\ln x)}{\ln x} dx = \frac{\pi}{4}$$

Proof:
Set $F(u)$ is the Laplace Transform of $f(t)$

$$\int_s^\infty F(u)du = \int_s^\infty \int_0^\infty e^{-ut} f(t)dtdu = \int_0^\infty f(t)\int_s^\infty e^{-ut} dudt = \int_0^\infty e^{-st}\frac{f(t)}{t} dt$$

Therefore,

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(u)du$$