Harvard CS121 and CSCI E-207

Lecture 2: Mathematical Preliminaries

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Reading: Sipser, Chapter 0

Sets

- **Sets** are defined by their members

  $A = B$ means that for every x, $x \in A$ iff $x \in B$

  Example: $\mathcal{N} = \{0, 1, 2, \ldots\}$

- **Cardinality**

  Sets can be **finite** (e.g. $\{1, 3, 5\}$) or **infinite** (e.g. $\mathcal{N}$).

  Q: Is $\{\mathcal{N}\}$ finite?

  If A is finite ($A = \{a_1, \ldots, a_n\}$ for some $n \in \mathcal{N}$), then its **cardinality** (or **size**) $|A|$ is the number of elements in A.

  The **empty set** $\emptyset$ has cardinality 0.

  Cardinality of infinite sets to be discussed later!

Set Operations

$\cup$	union	$\{a, b\} \cup \{b, c\} = \{a, b, c\}$
$\cap$	intersection	$\{a, b\} \cap \{b, c\} = \{b\}$
$-$	difference	$\{a, b\} - \{b, c\} = \{a\}$

- A and B are **disjoint** iff $A \cap B = \emptyset$

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- The **power set** of $S = P(S) = \{X : X \subseteq S\}$

  e.g. $P(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

  Q: What is $|P(S)|$?

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- The **Cartesian product** of sets A, B

  $A \times B = \{(a, b) : a \in A, b \in B\}$

  triples, . . .

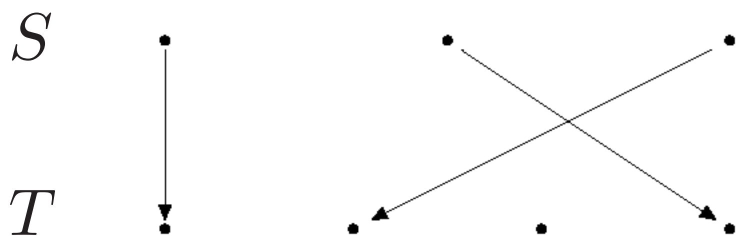
Functions

A **function** $f : S \to T$ maps each element $s \in S$ to (exactly one) element of T, denoted $f(s)$.

For example, $f(n) = n^2$ is a function from $\mathcal{Z} \to \mathcal{N}$

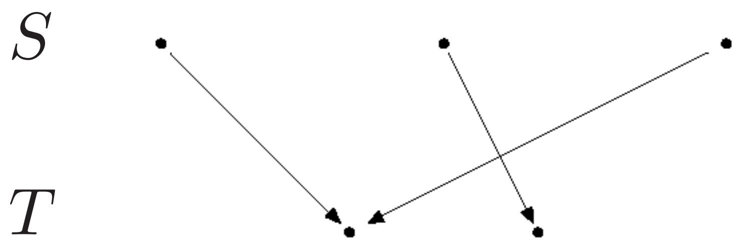
($\mathcal{Z} =$ all integers)

Special varieties of functions



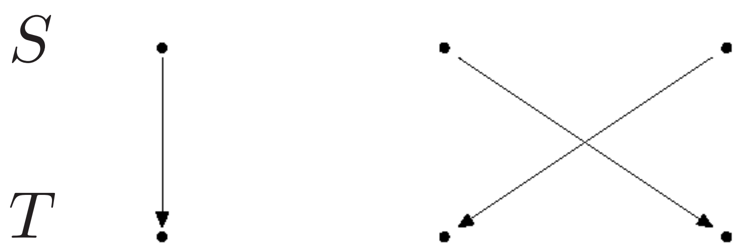
1-1:

$s_1 \neq s_2 \Rightarrow$
$f(s_1) \neq f(s_2)$



Onto:

For every $t \in T$
there is an $s \in S$
such that $f(s) = t$



Bijection:

1-1 and onto
"1-1 Correspondence"

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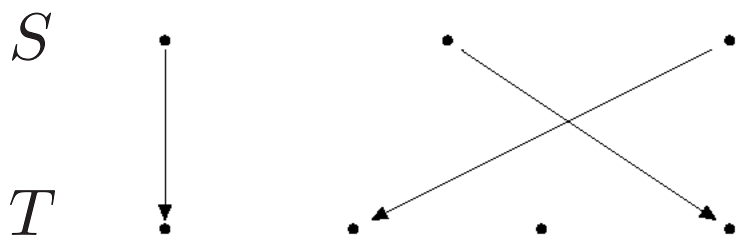

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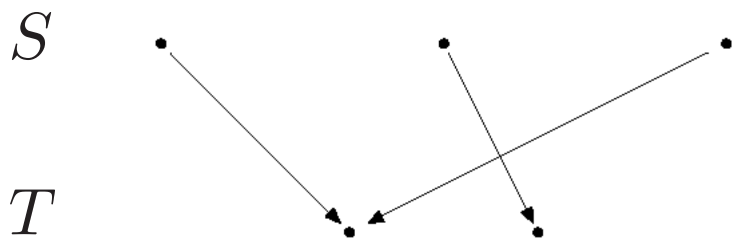

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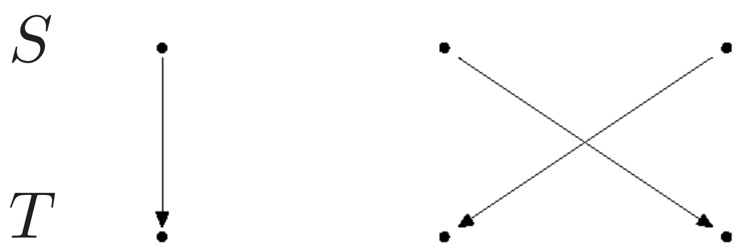


Bijection:
1-1 and onto
"1-1 Correspondence"

- Formal definition of cardinality: S has (finite) cardinality $n \in \mathcal{N}$ iff there is a bijection $f : \{1, \ldots, n\} \to S$.

Relations

- A k-ary **relation** on $S_1, \ldots, S_k$ is a subset of $S_1 \times \ldots \times S_k$

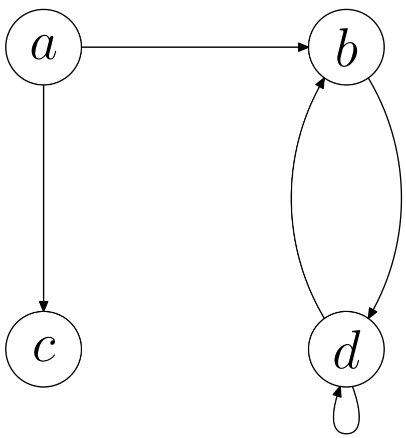
  [A function $f : S \to T$ corresponds to the relation $\{(s, f(s)) : s \in S\} \subseteq S \times T$.]

- A **binary relation** on S is a subset of $S \times S$

- For example, $(\mathrm{GWB}, \mathrm{GHWB}) \in \mathrm{Son}$, where $\mathrm{Son} = \{(x, y) : x \text{ is a son of } y\}$

What is a (directed) graph?

- For finite S, a binary relation can be pictured as a “directed graph”:



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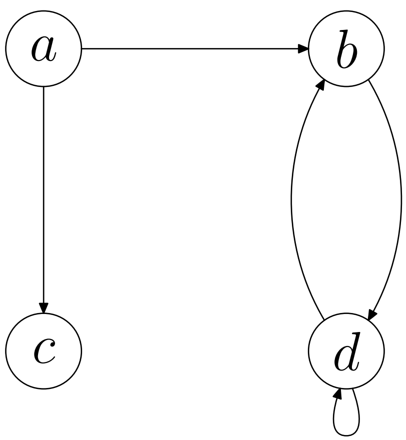

$$= \{(a, b), (a, c), (b, d), (d, b), (d, d)\}$$

- Formally, a **directed graph** G consists of a finite set V of **vertices** (or **nodes**), and a set of edges $E \subseteq V \times V$.

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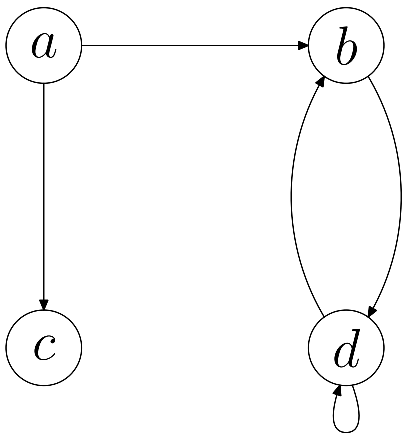

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- Formally, a **directed graph** G consists of a finite set V of **vertices** (or **nodes**), and a set of edges $E \subseteq V \times V$.

- **NB**: Because a relation (or edge set) is a *set* (of ordered pairs), there can be only one arrow from one node to another.

Properties of Binary Relations

A relation $R \subseteq S \times S$ is:

- **reflexive** if $\begin{tikzcd} a \arrow[loop above] \end{tikzcd}$ for each $a \in S$

  i.e., $(a, a) \in R$ for each $a \in S$

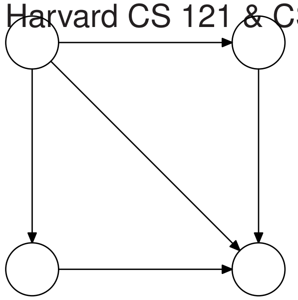
- **symmetric** if $a \longrightarrow b$ whenever $a \longleftarrow b$

  i.e., for any $a, b \in S$, if $(a, b) \in R$ then $(b, a) \in R$

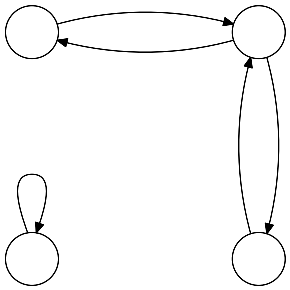
- **transitive** if $a \longrightarrow c$ whenever

  $a \longrightarrow b$ and $b \longrightarrow c$

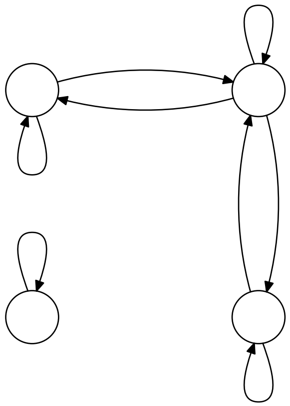
  i.e., if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$



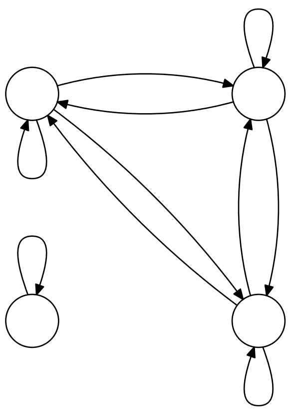
Transitive, not symmetric nor reflexive



Symmetric, not reflexive, nor transitive



Reflexive & symmetric, not transitive



Reflexive, transitive, and symmetric

Equivalence Relations

A relation that satisfies all three properties is called an **equivalence relation**

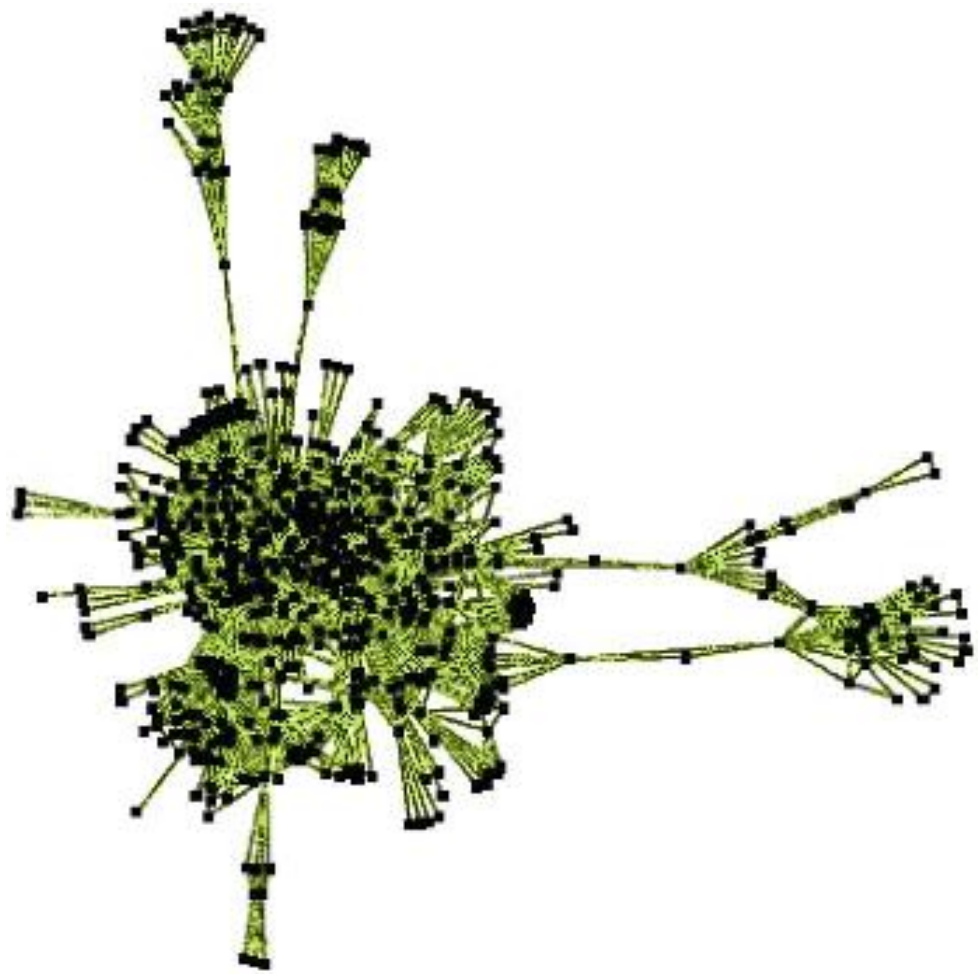
An equivalence relation decomposes S into **equivalence classes**—any two members of the same equivalence class bear the relation to each other.

Which Properties Do These Relations Have?

Domain	**Relation**
People	Lives-In-The-Same-City-As
People	Is-An-Ancestor-Of
People	Is-A-Brother-Of

Undirected graphs

A symmetric relation with no self-loops (that is, for every a, $(a, a) \notin R$) can be depicted as an *undirected graph*



Source: http://www.howweknowus.com/2008/07/23/great-work-lousy-title/

Subject: Re: six degrees to harry lewis

On Friday, January 23, 2004, at 05:09 AM, Mark Elliot Zuckerberg wrote:

Professor,

I've been interested in graph theory and its applications to social networks for a while now, so I did some research (on my own time) that has to do with linking people through articles they appear in from the Crimson.

I thought people would find this interesting, so I've set up a preliminary site that allows people to find the connection (through people and articles) from any person to the most frequently mentioned person in the time frame I looked at. This person is you.

Six degrees, continued

```
From:  Harry Lewis <lewis@deas.harvard.edu>
Date:  January 23, 2004 4:25:44 PM EST
To:  Mark Elliot Zuckerberg <mzuckerb@fas.harvard.edu>
Subject:  Re:  six degrees to harry lewis
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Six degrees, continued

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Sure, what the hell.
Seems harmless.

Strings and Languages

- **Symbol** a, b, ...

- **Alphabet** A finite, nonempty set of symbols

  usually denoted by Σ

- **String** (informal) Finite number of symbols “put together”

  e.g. $abba$, b, bb

  Empty string denoted by ε

- $\Sigma^* =$ set of all strings over alphabet Σ

  e.g. $\{a, b\}^* = \{\varepsilon, a, b, aa, ab, \ldots\}$

More on Strings

- **Length** of a string x is written $|x|$

  $$|abba| = 4$$

  $$|a| = 1$$

  $$|\varepsilon| = 0$$

  The set of strings of length n is denoted Σ^n.

Concatenation

- **Concatenation** of strings

  Written as $x \cdot y$, or just xy

  Just follow the symbols of x by the symbols of y

  $x = abba$, $y = b \Rightarrow xy = abbab$

  $x\varepsilon = \varepsilon x = x$ for any x

- The **reversal** x^R of a string x is x written backwards.

  If $x = x_1 x_2 \cdots x_n$, then $x^R = x_n x_{n-1} \cdots x_1$.

Formal Inductive Definitions

- Like recursive data structures and recursive procedures when programming.

- **Strings** and their **length**:

  ε is a string of length 0.

  If x is a string of length n and $\sigma \in \Sigma$, then $x\sigma$ is a string of length $n+1$.

  (i.e. start with ε and add one symbol at a time, like $\varepsilon aaba$, but we don't write the initial ε unless the string is empty)

Inductive definitions of string operations

- The **concatenation** of x and y, defined by induction on $|y|$.

  $[\,|y| = 0\,]$ $\quad x \cdot \varepsilon = x$

  $[\,|y| = n+1\,]$ write $y = z\sigma$ for some $|z| = n$, $\sigma \in \Sigma$
  define $x \cdot (z\sigma) = (x \cdot z)\sigma$

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  define $x \cdot (z\sigma) = (x \cdot z)\sigma$,

- The **reversal** of x, defined by induction on $|x|$:

  $[\,|x| = 0\,]$ $\quad \varepsilon^R = \varepsilon$

  $[\,|x| = n + 1\,]$ $\quad (y\sigma)^R = \sigma \cdot y^R$,
  for any $|y| = n$, $\sigma \in \Sigma$

A Proof by Induction

Proposition: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \Sigma^*$.

(So it doesn't matter what order we concatenate, so we can just write xyz in the future)

- Proof by "induction" on $|z|$ ...

Proof that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, cont.

Proofs by Induction

To prove $P(n)$ for all $n \in \mathcal{N}$:

1. "Base Case": Prove $P(0)$.
2. "Induction Hypothesis": Assume that $P(k)$ holds for all $k \leq n$ (where n is fixed but arbitrary)
3. "Induction Step": Given induction hypothesis, prove that $P(n+1)$ holds.

If we prove the Base Case and the Induction Step, then we have proved that $P(n)$ holds for $n = 0, 1, 2, \ldots$ (i.e., for all $n \in \mathcal{N}$)

Languages

- A **language** L over alphabet Σ is a set of strings over Σ (i.e. $L \subseteq \Sigma^*$)

  Computational problem: given $x \in \Sigma^*$, is $x \in L$?

  Any YES/NO problems can be cast as a language.

- Examples of simple languages:
  - All words in the *American Heritage Dictionary* $\{a, aah, aardvark, \ldots, zyzzva\}$.
  - $\emptyset$
  - Σ^*
  - Σ
  - $\{x \in \Sigma^* : |x| = 3\} = \{aaa, aab, aba, abb, baa, bab, bba, bbb\}$

More complicated languages

- The set of strings $x \in \{a, b\}^*$ such that x has more a's than b's.
- The set of strings $x \in \{0, 1\}^*$ such that x is the binary representation of a prime number.
- All 'C' programs that do not go into an infinite loop.
- $L_1 \cup L_2$, $L_1 \cap L_2$, $L_1 - L_2$ if L_1 and L_2 are languages.

$\vdots$

The highly abstract and metaphorical term “language”

- A language can be either finite or infinite

- A language need not have any “internal structure”

Be careful to distinguish

ε The empty string (a string)

$\emptyset$ The empty set (a set, possibly a language)

$\{\varepsilon\}$ The set containing one element, which is the empty string (a language)

$\{\emptyset\}$ The set containing one element, which is the empty set (a set of sets, maybe a set of languages)

Operations on Languages

- Set operations $\cup \quad \cap \quad -$

- Concatenation of Languages

$$L_1 L_2 = \{xy : x \in L_1, y \in L_2\}$$

e.g. $\{a, b\}\{a, bb\} = \{aa, ba, abb, bbb\}$

e.g. $\{\varepsilon\}L = L$

e.g. $\emptyset L =$?

Kleene star

$$L^* = \{w_1 \cdots w_n : n \geq 0, w_1, \ldots, w_n \in L\}$$

e.g. $\{aa\}^* = \{\varepsilon, aa, aaaa, \ldots\}$

e.g. $\{ab, ba, aa, bb\}^* =$ all even length strings

e.g. $\Sigma^* =$ Kleene Star of Σ

e.g. $\emptyset^* =$?