Harvard University
Computer Science 121

Problem Set 0

Due Monday, September 13, 2010 at 1:20 PM.
Submit solutions to cs121+ps0@seas.
See syllabus for collaboration policy.
Questions and comments to cs121@seas.

Note the directions above about submitting problem sets! Problem set 0 counts for 0 points, but you should complete it, and we will grade it and provide a solution set for your edification.

Content covered in Sipser chapter 0. In general, please refer to the Schedule posted on the course web site for information about readings. To prepare your solutions you will need to use LaTeX. Google "download tex" to obtain the software, which is free and available for all platforms. The schedule provides links to some "cheat sheets" for preparing LaTeX documents.

PROBLEM 1

Describe the following sets using formal set notation:
(A) A is the set containing the empty set.
(B) B is the set containing the empty string.
(C) C is the set containing all non-negative, integral powers of 2.
(D) D is the set containing all strings over $\Sigma = \{a, b\}$ whose length is a non-negative, integral power of 2.

PROBLEM 2

Let A be the set $\{x, y, z\}$ and B be the set $\{x, z\}$. Let $\mathcal{P}(S)$ denote the power set of any set S (i.e. the set of subsets of S). You need not justify your answers.

(A) Is $P(B)$ a subset of $P(A)$?
(B) What is $A \cup B$?
(C) What is $A \times B$?
(D) Is $\emptyset \in \mathcal{P}(A)$?
(E) Is $\emptyset \subset \mathcal{P}(A)$?

PROBLEM 3

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers. For each of the following functions, $f : \mathbb{N} \longrightarrow \mathbb{N}$, state whether f is (i) one-to-one, (ii) onto, and/or (iii) bijective. Briefly justify each of your answers.

(A) $f(x) = x^2$
(B) $f(x) = x \bmod 3$
(C) $f(x) = x!$
(D) $f(x) = \begin{cases} x + 1 & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases}$

PROBLEM 4

Consider the binary relation $\sim$ on sets defined by $A \sim B$ if and only if there exists a function $f : A \to B$ such that f is bijective.

(A) Give examples of two finite sets A, B such that $A \sim B$, as well as sets C, D such that $C \not\sim D$.

(B) Prove that $\sim$ is an equivalence relation. (Your proof should work for infinite sets as well as finite ones. Indeed, this relation is how one defines what it means for two infinite sets to have the same cardinality. Later in the course, we will see that not all infinite sets have the same cardinality, i.e. the infinite sets yield multiple equivalence classes under this relation.)

(C) Now consider the relation $\lesssim$ defined by $A \lesssim B$ if there exists a one-to-one function $f : A \to B$. Is $\lesssim$ reflexive? symmetric? transitive? Justify your answers.

PROBLEM 5

Joe the painter has 2010 cans of paint. Show that at least one of the following statements is true about Joe's paint collection:

1. Among the cans, there are at least 42 different colors of paint.
2. Among the cans, there are at least 50 of them with the same color.

(Hint: prove by contradiction)

PROBLEM 6

Define the Fibonacci numbers as follows:

$$F_0 = 0$$

$$F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 2$$

Prove the following statements by induction:

(A) For $n \geq 2$, F_n equals the number of strings of length $n-2$ over alphabet $\Sigma = \{a, b\}$ that do not contain two consecutive a's.

(B)

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$