Harvard University
Computer Science 121

Problem Set 2

Due Friday, October 1, 2010 at 1:20 PM.
Submit a single PDF (lastname+ps2.pdf) of your solutions to cs121+ps2@seas.harvard.edu
Late problem sets may be turned in until Monday, October 4, 2010 at 1:20 PM with a 20% penalty.
See syllabus for collaboration policy.

Name

Problem set by !!! Your Name Here !!!

with collaborator !!! Collaborators' names here !!!!

PROBLEM 1 (2+2+2+2 points)

Translate the following languages over $\Sigma = \{a, b\}$ from English description to regular expressions, or vice versa.
(A) $L = \{w \in \Sigma^* :$ the second letter of w is the same as the last letter, $|w| \geq 2\}$
(B) $L = \{w \in \Sigma^* :$ every a is followed by an odd number of b's$\}$
(C) $(a(a \cup b)^*b) \cup (b(a \cup b)^*a)$
(D) $(b^*abbb^*) \cup (b^*bbab^*) \cup (b^*babb^*) \cup (bbb^*)$

PROBLEM 2 (4+4 points)

(A) Construct a DFA for $L = \{w \in \{0, 1\}^* :$ the number represented in binary notation by w is equal to 1 mod 3$\}$. That is, when we interpret w as a number written in binary and divide by 3, we get a remainder of 1. Briefly explain why your DFA works.

Note: for this problem, we ignore leading zeros, so that $0111 = 7 \in L$.

(B) Convert your DFA for L to a regular expression, using the GNFA construction described in lecture. Show the full GNFA at each step of the construction.

PROBLEM 3 (12 points)

Let Σ and Δ be alphabets. Consider a function $\varphi : \Sigma \to \Delta^*$. Extend φ to a function from $\Sigma^* \to \Delta^*$ such that:

$$\begin{aligned}\varphi(\varepsilon) &= \varepsilon \\ \varphi(w\sigma) &= \varphi(w)\varphi(\sigma), \text{ for any } w \in \Sigma^*, \sigma \in \Sigma\end{aligned}$$

For example, if $\Sigma = \Delta = \{a, b\}$, $\varphi(a) = ab$, and $\varphi(b) = aab$, then $\varphi(aab) = \varphi(aa)\varphi(b) = \varphi(a)\varphi(a)\varphi(b) = ababaab$. Notice how φ operates on each input symbol individually. Any function $\varphi : \Sigma^* \to \Delta^*$ defined in this way from a function $\varphi : \Sigma \to \Delta^*$ is called a **homomorphism**.

Prove that the set of regular languages is closed under homomorphism. Specifically, prove by induction on the length of R that $\varphi[L(R)]$ is regular for any regular expression R and any homomorphism φ. (In other words, prove that, if L is regular, then $\varphi[L]$ is also regular, for any homomorphism φ.)

PROBLEM 4 (12 points)

Let $\text{SUBSTRINGS}(L, A) = \{w \in L : \text{every substring of } w \text{ is in A}\}$. Prove that if L and A are regular, then so is $\text{SUBSTRINGS}(L, A)$. Hint: can you make use of any closure properties we already know?

PROBLEM 5 (4+4+4+4 points)

(A) Define the $\lesssim$ relation on nonempty sets by $S \lesssim T$ if there is an onto map from T to S. [1] Show that $S \lesssim T$ if and only if there is a one-to-one map from S to T, so this relation is in fact the same as the one that was defined in Problem Set 0.
(B) Show that for every set S, $S \lesssim P(S)$.
(C) Show that for every nonempty set S, $P(S) \not\lesssim S$. (Hint: The diagonalization technique does not require enumerating the set in question.)
(D) Conclude that there are infinitely many different equivalence classes under the relation $\sim$ defined in Problem Set 0, i.e. infinitely many different "infinite cardinalities".

PROBLEM 6 (Challenge! 2+1 points)

(A) Let $L/A = \{x : wx \in A \text{ for some } w \in L\}$. Show that if A is regular and L is *any* language, then L/A is regular.
(B) Suppose $L = \{a^n : n \text{ is greater than 2, is even, and cannot be expressed as the sum of two primes}\}$, and $A = \Sigma^*$. Why doesn't this contradict your proof?

[1] Students of logic may notice that the domain of this relation would be the set of all sets, which cannot be defined without leading to paradoxes. The way around this is to talk about a relation on the 'class' of all sets, but the distinction between the notions of a 'class' and a 'set' is beyond the scope of CS121.