Harvard University
Computer Science 121 — Professor Harry Lewis

Final Examination — Saturday, December 12, 2009

The points total 180 (the exam is 180 minutes long). $\Sigma = \{a, b\}$ throughout. **You may use any result proven in lecture, Sipser, or the homework, provided that you state it clearly.**

PROBLEM 1 (10 points)

Suppose you have two algorithms, one running in time $1000 \cdot n^2$ and the other running in time $10 \cdot 10^n$. How big would n have to be to make it worthwhile to use the first algorithm rather than the second?

PROBLEM 2 (4+4+4 points)

State whether each part is true for all languages A and B. Justify or give a counterexample.
(A) $(A - B) \cup B = A$
(B) A^* is infinite.
(C) $(A^*)^* = A^*$

PROBLEM 3 (16+16 points)

(A) Let $L_1 = (a \cup ab \cup ba \cup bab)^*$, and let L_2 be the set of all strings in which every b is adjacent to at least one a. Show that $L_1 = L_2$. (Hint for $L_2 \subseteq L_1$: Induction on the length of a string, and cases based on the first symbol.)
(B) Draw a DFA that accepts this language.

PROBLEM 4 (4+4+4+4 points)

State whether each part is true for every **infinite** language A. Justify or give a counterexample.
(A) A^* is countable.
(B) A has countably many subsets.
(C) A has countably many regular subsets.
(D) A has countably many non-recursive subsets.

PROBLEM 5 (5+5+5 points)

(A) Define $\mathcal{NP}$-complete.
(B) Draw a diagram illustrating the relations between $\mathcal{P}, \mathcal{NP}$, co-$\mathcal{NP}$, and the $\mathcal{NP}$-complete, recursive, r.e., and co-r.e. sets on the hypothesis that $\mathcal{P} \neq \mathcal{NP}$.
(C) Is it possible for an $\mathcal{NP}$-complete set to be polynomial-time reducible to a proper subset of itself? Explain.

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PROBLEM 6 (10+10+10 points)

(A) Show that $\{\langle G\rangle : G$ is a context-free grammar such that $L(G) \neq \emptyset\}$ is recursive.
(B) Show that $\{\langle G\rangle : G$ is a general grammar such that $L(G) \neq \emptyset\}$ is r.e. but not recursive.
(C) Show that $\{\langle M\rangle : M$ is a Turing machine such that $L(M)$ is regular$\}$ is not recursive.

PROBLEM 7 (10+10 points)

(A) Show that $\{x\$y\$z : x, y, z \in \Sigma^*$ and $y = y^{\mathrm{R}}\}$ is context-free but not regular ($ is a new symbol).
(B) Show that $\{xyz : x, y, z \in \Sigma^*$ and $y = y^{\mathrm{R}}\}$ is regular.

PROBLEM 8 (10+10 points)

(A) Show that there is a polynomial-time algorithm $\mathcal{A}_k$ for determining whether a graph has a vertex cover of size k, if k is fixed (5, for example).
(B) Why doesn't (A) yield a polynomial-time algorithm for Vertex Cover, as follows? Given $\langle G, k\rangle$, run $\mathcal{A}_k$ on G.

PROBLEM 9 (5+5+15 points)

For any languages A and B, let $A/B = \{x : xy \in A$ for some $y \in B\}$.
(A) What is Σ^*/Σ^*?
Suppose that A is context-free and B is regular.
(B) Show that $A - B$ is context-free but $B - A$ need not be.
(C) Show that A/B is context-free.

THE END