Assignment 6

Alex Clemmer
Student number: u0458675

Problem 1:

(a)

		R				
		0	1	2	3	4
	0					$\frac{188}{3201}$
	1				$4 \cdot \frac{200}{3201}$	
D	2			$6 \cdot \frac{1225}{19206}$		
	3		$4 \cdot \frac{200}{3201}$			
	4	$\frac{188}{3201}$				

Note that all the both the marginal distributions and the whole table add to 1, although I didn't draw that into the table. I figured this out by building this huge tree and putting the values together that had the same *type* of composition (*e.g.*, three of a kind, all are one kind, etc.).

(b) First, we need to find the $\mathbb{E}[RD]$:

$$\begin{aligned} \mathbb{E}[RD] &= (4 \cdot 0)\left(\frac{188}{3201}\right) + (3 \cdot 1)\left(4 \cdot \frac{200}{3201}\right) + (2 \cdot 2)\left(6 \cdot \frac{1225}{19206}\right) + (1 \cdot 3)\left(4 \cdot \frac{200}{3201}\right) + (0 \cdot 4)\left(\frac{188}{3201}\right) \\ &= \frac{100}{33} \end{aligned} \tag{1}$$

We will also need to multiply $\mathbb{E}[R]$ and $\mathbb{E}[D]$ together, so we find them next:

$$\begin{aligned} \mathbb{E}[R] &= 1\left(4 \cdot \frac{200}{3201}\right) + 2\left(6 \cdot \frac{1225}{19206}\right) + \left(4 \cdot \frac{200}{3201}\right) + \left(4 \cdot \frac{188}{3201}\right) \\ &= 2 \end{aligned} \tag{2}$$

The same holds for $\mathbb{E}[D]$, as they are symmetric. Now we put them all together:

$$\begin{aligned} \mathrm{Cov}(R, D) &= \mathbb{E}[RD] - \mathbb{E}[R]\mathbb{E}[D] \\ &= \frac{100}{33} - (2 \cdot 2) \\ &= -\frac{1}{162} \end{aligned} \tag{3}$$

(c) First we need to find both $\mathrm{Var}(R)$ and $\mathrm{Var}(D)$:

$$\begin{aligned} \mathrm{Var}(R) &= 1^2\left(4 \cdot \frac{200}{3201}\right) + 2^2\left(6 \cdot \frac{1225}{19206}\right) + 3^2\left(4 \cdot \frac{200}{3201}\right) + 4^2\left(4 \cdot \frac{188}{3201}\right) \\ &= \frac{164}{33} - 4 \\ &= \frac{32}{33} \end{aligned} \tag{4}$$

$\mathrm{Var}(D)$ will be the same. After that, it's easy to plug them into the equation:

$$
\begin{aligned}
\rho(R, D) &= \frac{\mathrm{Cov}(R, D)}{\sqrt{\mathrm{Var}(R)\mathrm{Var}(D)}} \\
&= \frac{-\dfrac{32}{33}}{\sqrt{\dfrac{32}{33} \cdot \dfrac{32}{33}}} \\
&= -1
\end{aligned} \tag{5}
$$

Problem 2:

(a) First we eliminate y from the equation:

$$
\begin{aligned}
F_X(x) &= \int_0^1 \frac{2}{3}(x + 2y)\, dx \\
&= \left[\frac{2xy + 2y^2}{3} \right]_0^1 \\
&= \frac{2}{3}(x + 1)
\end{aligned} \tag{6}
$$

Now we can integrate the rest of the equation to find $P(\frac{1}{2} \leq X \leq 1)$:

$$
\begin{aligned}
P(\tfrac{1}{2} \leq X \leq 1) &= \int_{\frac{1}{2}}^1 F_X(x)\, dx \\
&= \int_{\frac{1}{2}}^1 \frac{2}{3}(x + 1)\, dx \\
&= \left[\frac{x^2}{3} + \frac{2}{3} \right]_{\frac{1}{2}}^1 \\
&= \frac{7}{12}
\end{aligned} \tag{7}
$$

(b) First, let's find $\mathbb{E}[XY]$:

$$
\begin{aligned}
\mathbb{E}[XY] &= \int_0^1 \int_0^1 x \cdot y \frac{2}{3}(x + 2y)\, dy\, dx \\
&= \int_0^1 \left[\frac{x^2y^2}{3} + \frac{4xy^3}{9} \right]_0^1 dx \\
&= \int_0^1 \frac{x^2}{3} + \frac{4x}{9}\, dx \\
&= \left[\frac{x^3}{9} + \frac{2x^2}{9} \right]_0^1 \\
&= \frac{1}{3}
\end{aligned} \tag{8}
$$

Next, we find $\mathbb{E}[X]\mathbb{E}[Y]$. We start with the marginal pdfs found earlier:

$$\begin{aligned} \mathbb{E}[X] &= \int_0^1 x f_X(x)\,dx \\ &= \int_0^1 x\frac{2x+2}{3}\,dx \\ &= \left[\frac{2x^3}{9}+\frac{x^2}{3}\right]_0^1 \\ &= \frac{5}{9} \end{aligned} \tag{9}$$

$$\begin{aligned} \mathbb{E}[Y] &= \int_0^1 y f_Y(y)\,dy \\ &= \int_0^1 y\frac{4y+1}{3}\,dy \\ &= \left[\frac{4y^2}{9}+\frac{y^2}{6}\right]_0^1 \\ &= \frac{11}{18} \end{aligned} \tag{10}$$

Now we can put it all together:

$$\begin{aligned} \mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y] &= \frac{1}{3}-\left(\frac{5}{9}\right)\left(\frac{11}{18}\right) \\ &= \frac{-1}{162} \end{aligned} \tag{11}$$

(c) Variance is still $\mathbb{E}[X^2]-\mathbb{E}[X]^2$. We are going to find the variance of both RVs before anything else. We can build off of what we've determined in the previous problems.

The following is for X:

$$\begin{aligned} \mathbb{E}[X^2] &= \int_0^1 x^2 f_X(x)\,dx \\ &= \int_0^1 x^2\frac{2x+2}{3}\,dx \\ &= \left[\frac{x^4}{6}+\frac{2x^3}{9}\right]_0^1 \\ &= \frac{7}{18} \end{aligned} \tag{12}$$

We can use the results from (b) to find the variance:

$$\begin{aligned} \mathbb{E}[X^2]-\mathbb{E}[X]^2 &= \frac{7}{18}-\left(\frac{5}{9}\right)^2 \\ &= \frac{13}{162} \end{aligned} \tag{13}$$

We use a similar strategy to find the variance for Y:

$$
\begin{aligned}
\mathbb{E}[Y] &= \int_0^1 y^2 f_Y(y)\, dy \\
&= \int_0^1 y^2 \frac{4y+1}{3} dy \\
&= \left[\frac{y^4}{3} + \frac{y^3}{9}\right]_0^1 \\
&= \frac{4}{9}
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\mathbb{E}[Y^2] - \mathbb{E}[Y]^2 &= \frac{4}{9} - \left(\frac{11}{18}\right)^2 \\
&= \frac{23}{324}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
\rho(R, D) &= \frac{\mathrm{Cov}(R, D)}{\sqrt{\mathrm{Var}(R)\mathrm{Var}(D)}} \\
&= \frac{\frac{-1}{162}}{\sqrt{\frac{23}{324} \cdot \frac{13}{162}}} \\
&= -\sqrt{\frac{2}{299}}
\end{aligned} \tag{16}
$$

Problem 3:

(a) $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ will look pretty similar. They are both binomials, and it is a matter of simply substituting in one variable for another. Here we have a binomial distribution: $p(x_i) = r$, while $1 - p = g + 1 - (r + g)$, as the number of blues is the number of marbles that are neither green nor red. This makes calculating $\mathbb{E}[X]$ *much* easier:

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_n x_n p(x_n) \\
&= np(x_n) \\
&= nr
\end{aligned} \tag{17}
$$

Similarly, $\mathbb{E}[Y] = ng$.

(b) Only slightly trickier, and also made a lot easier by it being binomial:

$$\mathrm{Var}(X) = np(1-p) \tag{18}$$

p in the case of X is r. $(1-p)$ is not that much trickier. We know r and g, so $(1-p) = (1-(r+g))$. So:

$$\mathrm{Var}(X) = nr(1-(r+g)) \tag{19}$$

And:

$$\mathrm{Var}(Y) = ng(1-(r+g)) \tag{20}$$

(c) These variables are not covariant; there is not dependence. Therefore:

$$
\begin{aligned}
\mathrm{Var}(X+Y) &= \mathrm{Var}(X) + \mathrm{Var}(X) \\
&= nr(1-(r+g)) + ng(1-(r+g))
\end{aligned} \tag{21}
$$

(d) As said before, the covariance is 0 when the variables are independent.