Assignment 5

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Problem 1:

(a) The probability of the function is 1 over the interval. Thus, k will have only one possible value:

$$\begin{aligned}
\int_{-\pi}^{\pi} k(1+\cos x)\,dx &= 1 \\
k\int_{-\pi}^{\pi} 1+\cos x\,dx &= 1 \\
k\left[x+\sin(x)\right]_{-\pi}^{\pi}\,dx &= 1 \\
2k\pi &= 1 \\
k &= \frac{1}{2\pi}
\end{aligned} \tag{1}$$

(b) The CDF is given by the antiderivative of the PDF, bounded from $-\infty$ to some point b. In this case:

$$\begin{aligned}
F(x) &= \int_{-\infty}^{b} \frac{1+\cos x}{2\pi}\,dx \\
&= \left[\frac{\sin x + x}{2\pi}\right]_{-\infty}^{b}
\end{aligned} \tag{2}$$

This integral does not actually converge. Given any point b, we're going to get a non-convergent sum. Of course, that doesn't make this useless, as this gives only the probability $P(X \leq b)$. We could instead evaluate over an interval, in which case we would conclude that

$$\left[\frac{\sin x + x}{a}\right]_{a}^{b} = \frac{\sin b + b - \sin a - a}{2\pi} \tag{3}$$

(c) This is easily calculated given the CDF:

$$\begin{aligned}
P(0 \leq X \leq \tfrac{\pi}{2}) &= \int_{0}^{\frac{\pi}{2}} \frac{1+\cos x}{2\pi}\,dx \\
&= \left[\frac{\sin x + x}{2\pi}\right]_{0}^{\frac{\pi}{2}} \\
&= \frac{\pi+2}{4\pi}
\end{aligned} \tag{4}$$

(d) To find the expected value, we can take the antiderivative of $x * f(x)$, where $f(x)$ is the PDF, and then constrict the bounds to those delineated. Doing so gives us the following:

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\pi}^{\pi} x\left(\frac{1+\cos x}{2\pi}\right)\,dx \\
&= \left[\frac{2\cos x + x(2\sin x + x)}{4\pi}\right]_{-\pi}^{\pi} \\
&= \frac{\pi^2-2}{4\pi} - \frac{\pi^2-2}{4\pi} = 0
\end{aligned} \tag{5}$$

(e) Variance is given by $\mathbb{E}[X^2] - \mathbb{E}[X]^2$, so we have part of this already solved:

$$\begin{aligned}
\mathbb{E}[X^2] - \mathbb{E}[X]^2 &= \mathbb{E}[X^2] - 0 \\
&= \int_{-\pi}^{\pi} x^2 f(x)\, dx \\
&= \int_{-\pi}^{\pi} x^2 \left(\frac{1+\cos x}{2\pi}\right) dx \\
&= \left[\frac{6x\cos x + 3(x^2-2)\sin x + x^3}{6\pi}\right]_{-\pi}^{\pi} \\
&= \frac{\pi^2 - 6}{3}
\end{aligned} \tag{6}$$

Problem 2:

(a) This is a pretty straightforward transformation. Here we will assume some arbitrary bounds for each equation; further, we will carry the equation through the integration step, assuming that both $g(b)$ and $f(b)$ can be anti-derived. Given $p_{X,Y}(a,b) = f(a)g(b)$:

$$\begin{aligned}
p_X(a) &= \int_s^t p_{X,Y}(a,b)\, db \\
&= f(a)\left[G(b)\right]_s^t \\
&= C \cdot f(a) \text{ for some constant } C
\end{aligned} \tag{7}$$

$$\begin{aligned}
p_Y(b) &= \int_q^r p_{X,Y}(a,b)\, da \\
&= g(b)\left[F(a)\right]_q^r \\
&= C \cdot g(b) \text{ for some constant } C
\end{aligned} \tag{8}$$

(b) X and Y will of course be independent. If $P(A \cap B)$ for any two sets of discrete quantities is also $P(A)P(B)$, then they are independent. This question just told us that $p_{X,Y}(a,b) = f(a)g(b)$; thus they are clearly independent.

Problem 3:

(a) If we double-integrate $f(x)$ over its entire interval, it should total 1. This is pretty handy for finding k:

$$
\begin{aligned}
1 &= \int_0^1 \int_0^1 k(x^2y + xy + 2y)\,dx\,dy \\
&= k\int_0^1 \int_0^1 (x^2y + xy + 2y)\,dx\,dy \\
&= k\int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y}{2} + 2xy\right]_{x=0}^{x=1} dy \\
&= k\int_0^1 \frac{17y}{6}\,dy \\
&= k\left[\frac{17}{6}y^2\right]_{y=0}^{y=1} \\
&= k\frac{17}{12} \\
\frac{12}{17} &= k
\end{aligned} \tag{9}
$$

(b) Finding the marginal PDF manifests from exactly the same concepts as above:

$$
\begin{aligned}
f_X(x) &= \int_0^1 \frac{12}{17}(x^2y + xy + 2y)\,dy \\
&= \frac{12}{17}\left[\frac{y^2(x^2 + x + 2)}{2}\right]_{y=0}^{y=1} \\
&= \frac{12}{17}\left(\frac{x^2 + x + 2}{2}\right) \\
&= \frac{6(x^2 + x + 2)}{17}
\end{aligned} \tag{10}
$$

(c) And the same principles will hold for $f_Y(y)$ also:

$$
\begin{aligned}
f_Y(y) &= \int_0^1 \frac{12}{17}(x^2y + xy + 2y)\,dx \\
&= \frac{12}{17}\left[\frac{x^3y}{3} + \frac{x^3y}{2} + 2xy\right]_{x=0}^{x=1} \\
&= \frac{12}{17}\left(\frac{17y}{6}\right) \\
&= 2y
\end{aligned} \tag{11}
$$

(d) The conditional PDF $f(x|y)$ is also pretty straightforward to derive:

$$
\begin{aligned}
f(x|Y = y) &= \frac{f(x, y)}{f_Y(y)} \\
&= \frac{\frac{12}{17}(x^2y + xy + 2y)}{2y} \\
&= \frac{6(x^2 + x + 2)}{17}
\end{aligned} \tag{12}
$$

(e) This one is only slightly trickier than the last:

$$\begin{aligned}
P(X \le \tfrac{1}{2}|Y=\tfrac{1}{2}) &= \int_0^{\frac{1}{2}} \frac{\frac{12}{17}(x^2y+xy+2y)}{2y}\,dy \\
&= \int_0^{\frac{1}{2}} \frac{\frac{12}{17}(x^2(\frac{1}{2})+x(\frac{1}{2})+2(\frac{1}{2}))}{2(\frac{1}{2})}\,dy \\
&= \int_0^{\frac{1}{2}} \frac{6(x^2+x+2)}{17}\,dy \qquad (13)\\
&= \left[\frac{x(2x^2+3x+12)}{17}\right]_0^{\frac{1}{2}} \\
&= \frac{7}{17}
\end{aligned}$$

(f) This is, in a lot of ways, the same problem as the last one:

$$\begin{aligned}
\mathbb{E}[X|Y=\tfrac{1}{2}] &= \int_a^b xf(x|Y=y)\,dy \\
&= \int_0^1 x\frac{\frac{12}{17}(x^2y+xy+2y)}{2y}\,dy \\
&= \int_0^1 x\frac{6(x^2+x+2)}{17}\,dy \qquad (14)\\
&= \left[\frac{6}{17}\left(\frac{x^4}{4}+\frac{x^3}{3}+x^2\right)\right]_0^1 \\
&= \frac{19}{34}
\end{aligned}$$

Problem 3:

(a)

		a	
b		pr	$r(1-p)$
		$p(1-r)$	$(1-p)(1-r)$

(b)

		outcome $(X=a,Y=b)$			
		11	12	21	22
sum	2	pr			
	3		$(p-1)r$	$p(r-1)$	
	4				$(p-1)(r-1)$

(c) Adding all the probabilities gives us 1: $pr + p(r-1) + r(p-1) + (p-1)(r-1) = 1$. So the way to check this is to add up both the columns and rows. If you check both, you will find that they are consistent and both add to this probability.

		$X = a$	
		1	2
$S = b$	2	pr	
	3	$p(r-1)$	$r(p-1)$
	4		$(p-1)(r-1)$

(d) This is pretty easy. We just take the weighted sum of all the values from the line where $\{S = 3\}$:

$$\mathbb{E}[X|S = 3] = 1p(r-1) + 2r(p-1) \tag{15}$$

(e)