Harvard University
Computer Science 121

Section 3 Handout
Week of 10.4.10

Outline

Part 1 Countability

Part 2 Regular vs. non-regular languages

Part 3 Context Free Grammars

1 Countability

Consider a set S. It is each of the following if...

- **Finite:** if there exists a bijection from $\{1, \ldots, n\}$ to S for some $n \geq 0$
- **Countably infinite:** if there is a bijection $f : \mathbb{N} \to S$
- **Countable:** if it is either finite or countably infinite
- **Uncountable:** it is not countable

Exercise 1.1. *Classify the following as countable or uncountable and provide proof:*

- $\mathbb{N} \times \{0, 1\}$
- $\mathcal{P}(\mathbb{N})$

2 Regular and Nonregular Languages

There are countably many regular expressions over a language, but there are uncountably many languages–so some of these languages *must not* be regular! But how do we find an explicitly non-regular language? We have two techniques: the pumping lemma and the closure properties of regular languages. You can use either of these techniques to prove (by contradiction) that a language is non-regular.

Pumping Lemma for regular languages:

If L is a regular language, then there exists a constant $p > 0$ such that for any string $w \in L$ with $|w| > p$, there exist strings x, y, $z \in \Sigma^*$, such that $w = xyz$, $|xy| \leq p$, $y \neq \epsilon$, and $xy^nz \in L$ for all $n \geq 0$.

Closure Properties:

Recall from lecture (and from last week's section) that regular languages are closed under *union, concatenation, Kleene Star, intersection, difference, complement, reversal.*

Exercise 2.1. *Which of the following are necessarily regular?*

- *A finite language.*
- *A union of finitely many regular languages.*
- $\{x : x \in L_1$ *and* $x \notin L_2\}$ *where* L_1 *and* L_2 *are regular.*
- *A subset of a regular language.*

Exercise 2.2. *Show that* $L = \{a^i b^j : 0 \leq i < j\}$ *is non-regular using the pumping lemma.*

Exercise 2.3. *Let* $L = \{ww | w \in \Sigma^*\}$. *Show that* L *is non-regular using the pumping lemma.*

Exercise 2.4. *Let* $L = \{a^i b^j : i > j \geq 0\}$ *Show that* L *is nonregular using the closure of regular languages under reversal.*

Exercise 2.5. *Show that* $L = \{b^n c^{2^k} : n \geq 1, k \geq 1\}$ *is non-regular.*

3 Context-Free Languages

Context-Free Grammars: A context-free grammar G is a four-tuple, defined as follows: $G = (V, \Sigma, R, S)$, where V (the set of variables) is an alphabet, Σ (the set of terminals) is a set disjoint from V, R is a finite set of rules, with each rule being a variable and a string of variables and terminals, and S (the start symbol) is an element of V.

Exercise 3.1. *Give a context-free grammar for* $L = \{w : w \text{ is an even-length palindrome }\}$

Exercise 3.2. *(a) Give a context-free grammar for* $L = \{w : w \text{ has three more } a\text{'s than } b\text{'s}\}$ *over the alphabet* $\Sigma = \{a, b\}$

(b) Draw a parse tree for the string $baabaaa \in L$.

Exercise 3.3. *Let* $L = \{wy : w, y \in L(a^* \cup b^*) \text{ and } |w| = |y|\}$. *Is* L *regular? Is* L *context-free?*