Assignment 5

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Problem 1:

(a) The probability of the function is 1 over the interval. Thus, k will have only one possible value:

$$\begin{aligned}
\int_{-\pi}^{\pi} k(1+\cos x)\,dx &= 1 \\
k\int_{-\pi}^{\pi} 1+\cos x\,dx &= 1 \\
k\left[x+\sin(x)\right]_{-\pi}^{\pi}\,dx &= 1 \\
2k\pi &= 1 \\
k &= \frac{1}{2\pi}
\end{aligned} \tag{1}$$

***(b)** The CDF is given by the antiderivative of the PDF, bounded from $-\infty$ to some point b. In this case:

$$\begin{aligned}
F(x) &= \int_{-\infty}^{b} \frac{1+\cos x}{2\pi}\,dx \\
&= \left[\frac{\sin x + x}{2\pi}\right]_{-\infty}^{b}
\end{aligned} \tag{2}$$

This integral does not actually converge. Given any point b, we're going to get a non-convergent sum. Of course, that doesn't make this useless, as this gives only the probability $P(X \le b)$. We could instead evaluate over an interval, in which case we would conclude that

$$\left[\frac{\sin x + x}{a}\right]_{-\infty}^{b} = \frac{\sin b + b - \sin a - a}{2\pi} \tag{3}$$

***(c)** This is easily calculated given the CDF:

$$\begin{aligned}
P(0 \le X \le \tfrac{\pi}{2}) &= \int_{0}^{\frac{\pi}{2}} \frac{1+\cos x}{2\pi}\,dx \\
&= \left[\frac{\sin x + x}{2\pi}\right]_{0}^{\frac{\pi}{2}} \\
&= \frac{\pi+2}{4\pi}
\end{aligned} \tag{4}$$

***(d)** To find the expected value, we can take the antiderivative of $x * f(x)$, where $f(x)$ is the PDF, and then split the problem into two domains: $(-\infty, 0]$ and $[0, \infty)$. Doing so gives us the following:

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x \left(\frac{1 + \cos x}{2\pi} \right) dx \\
&= \left[\frac{2\cos x + x(2\sin x + x)}{4\pi} \right]_{-\infty}^{\infty} \\
&= \left[\frac{2\cos x + x(2\sin x + x)}{4\pi} \right]_{-\infty}^{0} + \left[\frac{2\cos x + x(2\sin x + x)}{4\pi} \right]_{0}^{\infty} \\
&= -\infty + \infty
\end{aligned} \tag{5}$$

Thus, because each half gives us infinity, the expected value *does not exist.*

(e) This

Problem 2:

(a) This is a pretty straightforward transformation. Given $p_{X,Y}(a, b) = f(a)g(b)$:

$$\begin{aligned}
p_X(a) &= \int_s^t p_{X,Y}(a, b)\, db \\
&= f(a) \left[G(b) \right]_s^t
\end{aligned} \tag{6}$$

$$\begin{aligned}
p_Y(b) &= \int_q^r p_{X,Y}(a, b)\, da \\
&= g(b) \left[F(a) \right]_q^r
\end{aligned} \tag{7}$$

(b) X and Y will of course be independent. If $P(A \cap B)$ for any two sets of discrete quantities is also $P(A)P(B)$, then they are independent. This question just told us that $p_{X,Y}(a, b) = f(a)g(b)$; since $p_{X,Y}(a, b)$ truly is just the intersection of the two, they are clearly independent.

Problem 3:

(a) If we double-integrate $f(x)$ over its entire interval, it should total 1. This is pretty handy for finding k:

$$
\begin{aligned}
1 &= \int_0^1 \int_0^1 k(x^2y + xy + 2y)\,dx\,dy \\
&= k\int_0^1 \int_0^1 (x^2y + xy + 2y)\,dx\,dy \\
&= k\int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y}{2} + 2xy\right]_{x=0}^{x=1} dy \\
&= k\int_0^1 \frac{17y}{6}\,dy \\
&= k\left[\frac{17}{6}y^2\right]_{y=0}^{y=1} \\
&= k\frac{17}{12} \\
\frac{12}{17} &= k
\end{aligned} \tag{8}
$$

(b) Finding the marginal PDF manifests from exactly the same concepts as above:

$$
\begin{aligned}
f_X(x) &= \int_0^1 \frac{12}{17}(x^2y + xy + 2y)\,dy \\
&= \frac{12}{17}\left[\frac{y^2(x^2 + x + 2)}{2}\right]_{y=0}^{y=1} \\
&= \frac{12}{17}\left(\frac{x^2 + x + 2}{2}\right) \\
&= \frac{6(x^2 + x + 2)}{17}
\end{aligned} \tag{9}
$$

(c) And the same principles will hold for $f_Y(y)$ also:

$$
\begin{aligned}
f_Y(y) &= \int_0^1 \frac{12}{17}(x^2y + xy + 2y)\,dx \\
&= \frac{12}{17}\left[\frac{x^3y}{3} + \frac{x^3y}{2} + 2xy\right]_{x=0}^{x=1} \\
&= \frac{12}{17}\left(\frac{17y}{6}\right) \\
&= 2y
\end{aligned} \tag{10}
$$

(d) The conditional PDF $f(x|y)$ is also pretty straightforward to derive:

$$
\begin{aligned}
f(x|Y=y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{\frac{12}{17}(x^2y+xy+2y)}{2y} \\
&= \frac{6(x^2+x+2)}{17}
\end{aligned} \tag{11}
$$

(e) This one is only slightly trickier than the last:

$$
\begin{aligned}
P(X \le \tfrac{1}{2}|Y=\tfrac{1}{2}) &= \int_0^{\frac{1}{2}} \frac{\frac{12}{17}(x^2y+xy+2y)}{2y}dx \\
&= \int_0^{\frac{1}{2}} \frac{\frac{12}{17}(x^2(\frac{1}{2})+x(\frac{1}{2})+2(\frac{1}{2}))}{2(\frac{1}{2})}dx \\
&= \int_0^{\frac{1}{2}} \frac{6(x^2+x+2)}{17}dx \\
&= \left[\frac{x(2x^2+3x+12)}{17}\right]_0^{\frac{1}{2}} \\
&= \frac{7}{17}
\end{aligned} \tag{12}
$$