Assignment 6

Alex Clemmer
Student number: u0458675

Problem 1:

(a)

		R				
		0	1	2	3	4
	0					$\frac{1}{16}$
	1				$\frac{1}{4}$	
D	2			$\frac{6}{16}$		
	3		$\frac{1}{4}$			
	4	$\frac{1}{16}$				

Not that all the both the marginal distributions and the whole table add to 1, although I didn't draw that into the table.

(b) First, we need to find the $\mathbb{E}[RD]$:

$$\begin{aligned} \mathbb{E}[RD] &= (4 \cdot 0)\left(\frac{1}{16}\right) + (3 \cdot 1)\left(\frac{1}{4}\right) + (2 \cdot 2)\left(\frac{6}{16}\right) + (1 \cdot 3)\left(\frac{1}{4}\right)(0 \cdot 4)\left(\frac{1}{16}\right) \\ &= 3 \end{aligned} \tag{1}$$

We will also need to multiply $\mathbb{E}[R]$ and $\mathbb{E}[D]$ together, so we find them next:

$$\begin{aligned} \mathbb{E}[R] &= (4)\left(\frac{1}{16}\right) + (3)\left(\frac{1}{4}\right) + (2)\left(\frac{6}{16}\right) + (1)\left(\frac{1}{4}\right)(0)\left(\frac{1}{16}\right) \\ &= 2 \end{aligned} \tag{2}$$

$$\begin{aligned} \mathbb{E}[D] &= (4)\left(\frac{1}{16}\right) + (3)\left(\frac{1}{4}\right) + (2)\left(\frac{6}{16}\right) + (1)\left(\frac{1}{4}\right)(0)\left(\frac{1}{16}\right) \\ &= 2 \end{aligned} \tag{3}$$

Now we put them all together:

$$\begin{aligned} \mathrm{Cov}(R, D) &= \mathbb{E}[RD] - \mathbb{E}[R]\mathbb{E}[D] \\ &= 3 - (2 \cdot 2) \\ &= -1 \end{aligned} \tag{4}$$

(c) First we need to find both $\mathrm{Var}(R)$ and $\mathrm{Var}(D)$:

$$\begin{aligned} \mathrm{Var}(R) &= (4^2)\left(\frac{1}{16}\right) + (3^2)\left(\frac{1}{4}\right) + (2^2)\left(\frac{6}{16}\right) + (1^2)\left(\frac{1}{4}\right)(0^2)\left(\frac{1}{16}\right) - \mathbb{E}[R]^2 \\ &= 5 - \mathbb{E}[R]^2 \\ &= 5 - 2 \\ &= 3 \end{aligned} \tag{5}$$

$$\begin{aligned}
\mathrm{Var}(D) &= (4^2)\left(\frac{1}{16}\right) + (3^2)\left(\frac{1}{4}\right) + (2^2)\left(\frac{6}{16}\right) + (1^2)\left(\frac{1}{4}\right)(0^2)\left(\frac{1}{16}\right) - \mathbb{E}[D]^2 \\
&= 5 - \mathbb{E}[D]^2 \\
&= 5 - 2 \\
&= 3
\end{aligned} \tag{6}$$

After that, it's easy to plug them into the equation:

$$\begin{aligned}
\rho(R, D) &= \frac{\mathrm{Cov}(R, D)}{\sqrt{\mathrm{Var}(R)\mathrm{Var}(D)}} \\
&= \frac{-1}{\sqrt{3 \cdot 3}} \\
&= -\frac{1}{3}
\end{aligned} \tag{7}$$

Problem 2:

(a) First we eliminate y from the equation:

$$\begin{aligned}
F_X(x) &= \int_0^1 \frac{2}{3}(x + 2y)\, dx \\
&= \left[\frac{2xy + 2y^2}{3}\right]_0^1 \\
&= \frac{2}{3}(x + 1)
\end{aligned} \tag{8}$$

Now we can integrate the rest of the equation to find $P(\frac{1}{2} \leq X \leq 1)$:

$$\begin{aligned}
P(\tfrac{1}{2} \leq X \leq 1) &= \int_{\frac{1}{2}}^1 F_X(x)\, dx \\
&= \int_{\frac{1}{2}}^1 \frac{2}{3}(x + 1)\, dx \\
&= \left[\frac{x^2}{3} + \frac{2}{3}\right]_{\frac{1}{2}}^1 \\
&= \frac{7}{12}
\end{aligned} \tag{9}$$

(b) First, let's find $\mathbb{E}[XY]$:

$$\begin{aligned}
\mathbb{E}[XY] &= \int_0^1 \int_0^1 x \cdot y \frac{2}{3}(x+2y)\, dy\, dx \\
&= \int_0^1 \left[\frac{x^2y^2}{3} + \frac{4xy^3}{9}\right]_0^1 dx \\
&= \int_0^1 \frac{x^2}{3} + \frac{4x}{9}\, dx \\
&= \left[\frac{x^3}{9} + \frac{2x^2}{9}\right]_0^1 \\
&= \frac{1}{3}
\end{aligned} \tag{10}$$

Next, we find $\mathbb{E}[X]\mathbb{E}[Y]$. We start with the marginal pdfs found earlier:

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^1 x f_X(x)\, dx \\
&= \int_0^1 x \frac{2x+2}{3} dx \\
&= \left[\frac{2x^3}{9} + \frac{x^2}{3}\right]_0^1 \\
&= \frac{5}{9}
\end{aligned} \tag{11}$$

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^1 y f_Y(y)\, dy \\
&= \int_0^1 y \frac{4y+1}{3} dy \\
&= \left[\frac{4y^2}{9} + \frac{y^2}{6}\right]_0^1 \\
&= \frac{11}{18}
\end{aligned} \tag{12}$$

Now we can put it all together:

$$\begin{aligned}
\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] &= \frac{1}{3} - \left(\frac{5}{9}\right)\left(\frac{11}{18}\right) \\
&= \frac{107}{162}
\end{aligned} \tag{13}$$

(c) Variance is still $\mathbb{E}[X^2] - \mathbb{E}[X]^2$. We are going to find the variance of both RVs before anything else. We can build off of what we've determined in the previous problems.

The following is for X:

$$
\begin{aligned}
\mathbb{E}[X^2] &= \int_0^1 x^2 f_X(x)\,dx \\
&= \int_0^1 x^2 \frac{2x+2}{3}\,dx \\
&= \left[\frac{x^4}{6} + \frac{2x^3}{9}\right]_0^1 \\
&= \frac{7}{18}
\end{aligned} \tag{14}
$$

We can use the results from (b) to find the variance:

$$
\begin{aligned}
\mathbb{E}[X^2] - \mathbb{E}[X]^2 &= \frac{7}{18} - \left(\frac{5}{9}\right)^2 \\
&= \frac{13}{162}
\end{aligned} \tag{15}
$$

We use a similar strategy to find the variance for Y:

$$
\begin{aligned}
\mathbb{E}[Y] &= \int_0^1 y^2 f_Y(y)\,dy \\
&= \int_0^1 y^2 \frac{4y+1}{3}\,dy \\
&= \left[\frac{y^4}{3} + \frac{y^3}{9}\right]_0^1 \\
&= \frac{4}{9}
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\mathbb{E}[Y^2] - \mathbb{E}[Y]^2 &= \frac{4}{9} - \left(\frac{11}{18}\right)^2 \\
&= \frac{23}{324}
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\rho(R, D) &= \frac{\mathrm{Cov}(R, D)}{\sqrt{\mathrm{Var}(R)\mathrm{Var}(D)}} \\
&= \frac{\frac{107}{162}}{\sqrt{\frac{23}{324} \cdot \frac{13}{162}}} \\
&= 107\sqrt{\frac{2}{299}}
\end{aligned} \tag{18}
$$