8

Computations with random variables

There are many ways to make new random variables from old ones. Of course this is not a goal in itself; usually new variables are created naturally in the process of solving a practical problem. The expectations and variances of such new random variables can be calculated with the change-of-variable formula. However, often one would like to know the *distributions* of the new random variables. We shall show how to determine these distributions, how to compare expectations of random variables and their transformed versions (Jensen's inequality), and how to determine the distributions of maxima and minima of several random variables.

8.1 Transforming discrete random variables

The problem we consider in this section and the next is how the distribution of a random variable X changes if we apply a function g to it, thus obtaining a new random variable Y:

$$Y = g(X).$$

When X is a discrete random variable this is usually not too hard to do: it is just a matter of bookkeeping. We illustrate this with an example. Imagine an airline company that sells tickets for a flight with 150 available seats. It has no idea about how many tickets it will sell. Suppose, to keep the example simple, that the number X of tickets that will be sold can be anything from 1 to 200. Moreover, suppose that each possibility has equal probability to occur, i.e., $\mathrm{P}(X = j) = 1/200$ for $j = 1, 2, \ldots, 200$. The real interest of the airline company is in the random variable Y, which is the number of passengers that have to be refused. What is the distribution of Y? To answer this, note that nobody will be refused when the passengers fit in the plane, hence

$$\mathrm{P}(Y = 0) = \mathrm{P}(X \leq 150) = \frac{150}{200} = \frac{3}{4}.$$

For the other values, $k = 1, 2 \ldots, 50$

$$P(Y = k) = P(X = 150 + k) = \frac{1}{200}.$$

Note that in this example the function g is given by $g(x) = \max\{x - 150, 0\}$.

QUICK EXERCISE 8.1 Let Z be the number of passengers that will be in the plane. Determine the probability distribution of Z. What is the function g in this case?

8.2 Transforming continuous random variables

We now turn to continuous random variables. Since single values occur with probability zero for a continuous random variable, the approach above does not work. The strategy now is to *first* determine the distribution function of the transformed random variable $Y = g(X)$ and then the probability density by differentiating. We shall illustrate this with the following example (actually we saw an example of such a computation in Section 7.3 with the function $g(x) = x^2$).

We consider two methods that traffic police employ to determine whether you deserve a fine for speeding. From experience, the traffic police think that vehicles are driving at speeds ranging from 60 to 90 km/hour at a certain road section where the speed limit is 80 km/hour. They assume that the speed of the cars is uniformly distributed over this interval. The first method is measuring the speed at a fixed spot in the road section. With this method the police will find that about $(90 - 80)/(90 - 60) = 1/3$ of the cars will be fined.

For the second method, cameras are put at the beginning and end of a 1-km road section, and a driver is fined if he spends less than a certain amount of time in the road section. Cars driving at 60 km/hour need one minute, those driving at 90 km/hour only 40 seconds. Let us therefore model the time T an arbitrary car spends in the section by a uniform distribution over $(40, 60)$ seconds. What is the speed V we deduce from this travelling time? Note that for $40 \leq t \leq 60$,

$$P(T \leq t) = \frac{t - 40}{20}.$$

Since there are 3600 seconds in an hour we have that

$$V = g(T) = \frac{3600}{T}.$$

We therefore find for the distribution function $F_V(v) = P(V \leq v)$ of the speed V that

$$F_V(v) = \mathrm{P}\left(\frac{3600}{T} \le v\right) = \mathrm{P}\left(T \ge \frac{3600}{v}\right) = 1 - \frac{(3600/v) - 40}{20} = 3 - \frac{180}{v}$$

for all speeds v between 60 and 90. We can now obtain the probability density f_V of V by differentiating:

$$f_V(v) = \frac{\mathrm{d}}{\mathrm{d}v} F_V(v) = \frac{\mathrm{d}}{\mathrm{d}v}\left(3 - \frac{180}{v}\right) = \frac{180}{v^2}$$

for $60 \le v \le 90$.

It is amusing to note that with the second model the traffic police write fewer speeding tickets because

$$\mathrm{P}(V > 80) = 1 - \mathrm{P}(V \le 80) = 1 - \left(3 - \frac{180}{80}\right) = \frac{1}{4}.$$

(With the first model we found probability 1/3 that a car drove faster than 80 km/hour.) This is related to a famous result in road traffic research, which is succinctly phrased as: "space mean speed < time mean speed" (see [37]). It is also related to Jensen's inequality, which we introduce in Section 8.3.

Similar to the way this is done in the traffic example, one can determine the distribution of $Y = 1/X$ for any X with a continuous distribution. The outcome will be that if X has density f_X, then the density f_Y of Y is given by

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{1}{y^2} f_X\left(\frac{1}{y}\right) \quad \text{for } y < 0 \text{ and } y > 0.$$

One can give $f_Y(0)$ any value; often one puts $f_Y(0) = 0$.

Quick exercise 8.2 Let X have a continuous distribution with probability density $f_X(x) = 1/[\pi(1 + x^2)]$. What is the distribution of $Y = 1/X$?

We turn to a second example. A very common transformation is a change of units, for instance, from Celsius to Fahrenheit. If X is temperature expressed in degrees Celsius, then $Y = \frac{9}{5}X + 32$ is the temperature in degrees Fahrenheit. Let F_X and F_Y be the distribution functions of X and Y. Then we have for any a

$$\begin{aligned} F_Y(a) &= \mathrm{P}(Y \le a) = \mathrm{P}\left(\frac{9}{5}X + 32 \le a\right) \\ &= \mathrm{P}\left(X \le \frac{5}{9}(a - 32)\right) = F_X\left(\frac{5}{9}(a - 32)\right). \end{aligned}$$

By differentiating F_Y (using the chain rule), we obtain the probability density $f_Y(y) = \frac{5}{9} f_X\left(\frac{5}{9}(y - 32)\right)$. We can do this for more general changes of units, and we obtain the following useful rule.

CHANGE-OF-UNITS TRANSFORMATION. Let X be a continuous random variable with distribution function F_X and probability density function f_X. If we change units to $Y = rX + s$ for real numbers $r > 0$ and s, then

$$F_Y(y) = F_X\left(\frac{y-s}{r}\right) \quad \text{and} \quad f_Y(y) = \frac{1}{r} f_X\left(\frac{y-s}{r}\right).$$

As an example, let X be a random variable with an $N(\mu, \sigma^2)$ distribution, and let $Y = rX + s$. Then this rule gives us

$$f_Y(y) = \frac{1}{r} f_X\left(\frac{y-s}{r}\right) = \frac{1}{r\sigma\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}((y-r\mu-s)/r\sigma)^2}$$

for $-\infty < y < \infty$. On the right-hand side we recognize the probability density of a normal distribution with parameters $r\mu + s$ and $r^2\sigma^2$. This illustrates the following rule.

NORMAL RANDOM VARIABLES UNDER CHANGE OF UNITS. Let X be a random variable with an $N(\mu, \sigma^2)$ distribution. For any $r \neq 0$ and any s, the random variable $rX + s$ has an $N(r\mu + s, r^2\sigma^2)$ distribution.

Note that if X has an $N(\mu, \sigma^2)$ distribution, then with $r = 1/\sigma$ and $s = -\mu/\sigma$ we conclude that

$$Z = \frac{1}{\sigma}X + \left(-\frac{\mu}{\sigma}\right) = \frac{X-\mu}{\sigma}$$

has an $N(0, 1)$ distribution. As a consequence

$$F_X(a) = \mathrm{P}(X \le a) = \mathrm{P}(\sigma Z + \mu \le a) = \mathrm{P}\left(Z \le \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right).$$

So any probability for an $N(\mu, \sigma^2)$ distributed random variable X can be expressed in terms of an $N(0, 1)$ distributed random variable Z.

QUICK EXERCISE 8.3 Compute the probabilities $\mathrm{P}(X \le 5)$ and $\mathrm{P}(X \ge 2)$ for X with an $N(4, 25)$ distribution.

8.3 Jensen's inequality

Without actually computing the distribution of $g(X)$ we can often tell how $\mathrm{E}[g(X)]$ relates to $g(\mathrm{E}[X])$. For the change-of-units transformation $g(x) = rx + s$ we know that $\mathrm{E}[g(X)] = g(\mathrm{E}[X])$ (see Section 7.3). It is a common error to equate these two sides for *other* functions g. In fact, equality will *very rarely* occur for nonlinear g.

For example, suppose that a company that produces microelectronic parts has a target production of 240 chips per day, but the yield has only been 40, 60, and 80 chips on three consecutive days. The average production over the three days then is 60 chips, so on average the production should have been 4 times higher to reach the target. However, one can also look at this in the following way: on the three days the production should have been $240/40 = 6$, $240/60 = 4$, and $240/80 = 3$ times higher. On average that is

$$\tfrac{1}{3}(6 + 4 + 3) = \tfrac{13}{3} = 4.3333$$

times higher! What happens here can be explained (take for X the part of the target production that is realized, where you give equal probabilities to the three outcomes 1/6, 1/4, and 1/3) by the fact that if X is a random variable taking positive values, then always

$$\frac{1}{\mathrm{E}[X]} < \mathrm{E}\left[\frac{1}{X}\right],$$

unless $\mathrm{Var}(X) = 0$, which only happens if X is not random at all (cf. Exercise 7.17). This inequality is the case $g(x) = 1/x$ on $(0, \infty)$ of the following result that holds for general convex functions g.

JENSEN'S INEQUALITY. Let g be a convex function, and let X be a random variable. Then

$$g(\mathrm{E}[X]) \le \mathrm{E}[g(X)].$$

Recall from calculus that a twice differentiable function g is *convex* on an interval I if $g''(x) \ge 0$ for all x in I, and *strictly* convex if $g''(x) > 0$ for all x in I. When X takes its values in an interval I (this can, for instance, be $I = (-\infty, \infty)$), and g is strictly convex on I, then *strict* inequality holds: $g(\mathrm{E}[X]) < \mathrm{E}[g(X)]$, unless X is not random.

In Figure 8.1 we illustrate the way in which this result can be obtained for the special case of a random variable X that takes two values, a and b. In the figure, X takes these two values with probability 3/4 and 1/4 respectively. Convexity of g forces any line segment connecting two points on the graph of g to lie above the part of the graph between these two points. So if we choose the line segment from $(a, g(a))$ to $(b, g(b))$, then it follows that the point

$$(\mathrm{E}[X], \mathrm{E}[g(X)]) = \left(\tfrac{3}{4}a + \tfrac{1}{4}b, \tfrac{3}{4}g(a) + \tfrac{1}{4}g(b)\right) = \tfrac{3}{4}(a, g(a)) + \tfrac{1}{4}(b, g(b))$$

on this line lies "above" the point $(\mathrm{E}[X], g(\mathrm{E}[X])$ on the graph of g. Hence $\mathrm{E}[g(X)] \ge g(\mathrm{E}[X])$.

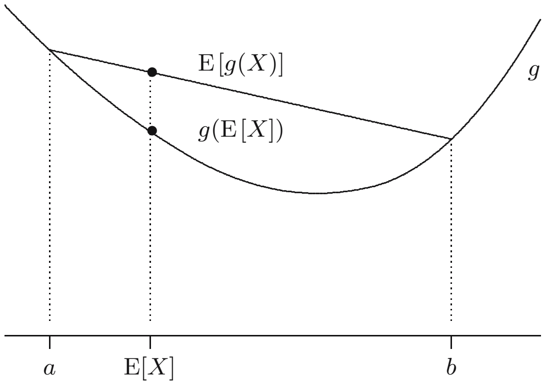

Fig. 8.1. Jensen's inequality.

A simple example is given by $g(x) = x^2$. This function is convex ($g''(x) = 2$ for all x), and hence

$$(\mathrm{E}[X])^2 \leq \mathrm{E}\left[X^2\right].$$

Note that this is exactly the same as saying that $\mathrm{Var}(X) \geq 0$, which we have already seen in Section 7.4.

QUICK EXERCISE 8.4 Let X be a random variable with $\mathrm{Var}(X) > 0$. Which is true: $\mathrm{E}\left[\mathrm{e}^{-X}\right] < \mathrm{e}^{-\mathrm{E}[X]}$ or $\mathrm{E}\left[\mathrm{e}^{-X}\right] > \mathrm{e}^{-\mathrm{E}[X]}$?

8.4 Extremes

In many situations the maximum (or minimum) of a sequence $X_1, X_2, \ldots, X_n$ of random variables is the variable of interest. For instance, let $X_1, X_2, \ldots, X_{365}$ be the water level of a river during the days of a particular year for a particular location. Suppose there will be flooding if the level exceeds a certain height—usually the height of the dykes. The question whether flooding occurs during a year is completely answered by looking at the maximum of $X_1, X_2, \ldots, X_{365}$. If one wants to predict occurrence of flooding in the future, the probability distribution of this maximum is of great interest. Similar models arise, for instance, when one is interested in possible damage from a series of shocks or in the extent of a contamination plume in the subsurface.

We want to find the distribution of the random variable

$$Z = \max\{X_1, X_2, \ldots, X_n\}.$$

We can determine the distribution function of Z by realizing that the maximum of the X_i is smaller than a number a if and only if *all* X_i are smaller than a:

$$F_Z(a) = \mathrm{P}(Z \le a) = \mathrm{P}(\max\{X_1, \dots, X_n\} \le a) = \mathrm{P}(X_1 \le a, \dots, X_n \le a)\,.$$

Now suppose that the events $\{X_i \le a_i\}$ are independent for every choice of the a_i. In this case we call the random variables *independent* (see also Chapter 9, where we study independence of random variables). In particular, the events $\{X_i \le a\}$ are independent for all a. It then follows that

$$F_Z(a) = \mathrm{P}(X_1 \le a, \dots, X_n \le a) = \mathrm{P}(X_1 \le a) \cdots \mathrm{P}(X_n \le a)\,.$$

Hence, if all random variables have the same distribution function F, then the following result holds.

THE DISTRIBUTION OF THE MAXIMUM. Let $X_1, X_2, \dots, X_n$ be n independent random variables with the same distribution function F, and let $Z = \max\{X_1, X_2, \dots, X_n\}$. Then

$$F_Z(a) = (F(a))^n.$$

QUICK EXERCISE 8.5 Let $X_1, X_2, \dots, X_n$ be independent random variables, all with a $U(0,1)$ distribution. Let $Z = \max\{X_1, \dots, X_n\}$. Compute the distribution function and the probability density function of Z.

What can we say about the distribution of the minimum? Let

$$V = \min\{X_1, X_2, \dots, X_n\}.$$

We can now find the distribution function F_V of V by observing that the minimum of the X_i is *larger* than a number a if and only if all X_i are *larger* than a. The trick is to switch to the complement of the event $\{V \le a\}$:

$$\begin{aligned} F_V(a) &= \mathrm{P}(V \le a) = 1 - \mathrm{P}(V > a) = 1 - \mathrm{P}(\min\{X_1, \dots, X_n\} > a) \\ &= 1 - \mathrm{P}(X_1 > a, \dots, X_n > a)\,. \end{aligned}$$

So using independence and switching back again, we obtain

$$\begin{aligned} F_V(a) &= 1 - \mathrm{P}(X_1 > a, \dots, X_n > a) = 1 - \mathrm{P}(X_1 > a) \cdots \mathrm{P}(X_n > a) \\ &= 1 - (1 - \mathrm{P}(X_1 \le a)) \cdots (1 - \mathrm{P}(X_n \le a)). \end{aligned}$$

We have found the following result for the minimum.

THE DISTRIBUTION OF THE MINIMUM. Let $X_1, X_2, \dots, X_n$ be n independent random variables with the same distribution function F, and let $V = \min\{X_1, X_2, \dots, X_n\}$. Then

$$F_V(a) = 1 - (1 - F(a))^n.$$

QUICK EXERCISE 8.6 Let $X_1, X_2, \dots, X_n$ be independent random variables, all with a $U(0,1)$ distribution. Let $V = \min\{X_1, \dots, X_n\}$. Compute the distribution function and the probability density function of V.

8.5 Solutions to the quick exercises

8.1 Clearly Z can take the values $1, \ldots, 150$. The value 150 is special: the plane is full if 150 or more people buy a ticket. Hence $\mathrm{P}(Z = 150) = \mathrm{P}(X \geq 150) = 51/200$. For the other values we have $\mathrm{P}(Z = i) = \mathrm{P}(X = i) = 1/200$, for $i = 1, \ldots, 149$. Clearly, here $g(x) = \min\{150, x\}$.

8.2 The probability density of $Y = 1/X$ is

$$f_Y(y) = \frac{1}{y^2} \frac{1}{\pi(1 + (\frac{1}{y})^2)} = \frac{1}{\pi(1 + y^2)}.$$

We see that $1/X$ has the same distribution as X! (This distribution is called the standard Cauchy distribution, it will be introduced in Chapter 11.)

8.3 First define $Z = (X - 4)/5$, which has an $N(0, 1)$ distribution. Then from Table B.1

$$\mathrm{P}(X \leq 5) = \mathrm{P}\left(Z \leq \frac{5 - 4}{5}\right) = \mathrm{P}(Z \leq 0.20) = 1 - 0.4207 = 0.5793.$$

Similarly, using the symmetry of the normal distribution,

$$\mathrm{P}(X \geq 2) = \mathrm{P}\left(Z \geq \frac{2 - 4}{5}\right) = \mathrm{P}(Z \geq -0.40) = \mathrm{P}(Z \leq 0.40) = 0.6554.$$

8.4 If $g(x) = \mathrm{e}^{-x}$, then $g''(x) = \mathrm{e}^{-x} > 0$; hence g is strictly convex. It follows from Jensen's inequality that

$$\mathrm{e}^{-\mathrm{E}[X]} \leq \mathrm{E}\left[\mathrm{e}^{-X}\right].$$

Moreover, if $\mathrm{Var}(X) > 0$, then the inequality is strict.

8.5 The distribution function of the X_i is given by $F(x) = x$ on $[0, 1]$. Therefore the distribution function F_Z of the maximum Z is equal to $F_Z(a) = (F(a))^n = a^n$. Its probability density function is

$$f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z} F_Z(z) = n z^{n-1} \quad \text{for } 0 \leq z \leq 1.$$

8.6 The distribution function of the X_i is given by $F(x) = x$ on $[0, 1]$. Therefore the distribution function F_V of the minimum V is equal to $F_V(a) = 1 - (1 - a)^n$. Its probability density function is

$$f_V(v) = \frac{\mathrm{d}}{\mathrm{d}v} F_V(v) = n(1 - v)^{n-1} \quad \text{for } 0 \leq v \leq 1.$$

8.6 Exercises

8.1 ⊡ Often one is interested in the distribution of the deviation of a random variable X from its mean $\mu = \mathrm{E}[X]$. Let X take the values $80, 90, 100, 110$, and 120, all with probability 0.2; then $\mathrm{E}[X] = \mu = 100$. Determine the distribution of $Y = |X - \mu|$. That is, specify the values Y can take and give the corresponding probabilities.

8.2 ⊞ Suppose X has a uniform distribution over the points $\{1, 2, 3, 4, 5, 6\}$ and that $g(x) = \sin(\frac{\pi}{2}x)$.

a. Determine the distribution of $Y = g(X) = \sin(\frac{\pi}{2}X)$, that is, specify the values Y can take and give the corresponding probabilities.

b. Let $Z = \cos(\frac{\pi}{2}X)$. Determine the distribution of Z.

c. Determine the distribution of $W = Y^2 + Z^2$. Warning: in this example there is a very special dependency between Y and Z, and in general it is much harder to determine the distribution of a random variable that is a function of *two* other random variables. This is the subject of Chapter 11.

8.3 ⊡ The continuous random variable U is uniformly distributed over $[0, 1]$.

a. Determine the distribution function of $V = 2U + 7$. What kind of distribution does V have?

b. Determine the distribution function of $V = rU + s$ for all real numbers $r > 0$ and s. See Exercise 8.9 for what happens for negative r.

8.4 Transforming exponential distributions.

a. Let X have an $Exp(\frac{1}{2})$ distribution. Determine the distribution function of $\frac{1}{2}X$. What kind of distribution does $\frac{1}{2}X$ have?

b. Let X have an $Exp(\lambda)$ distribution. Determine the distribution function of λX. What kind of distribution does λX have?

8.5 ⊡ Let X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} \frac{3}{4}x(2-x) & \text{for } 0 \le x \le 2 \\ 0 & \text{elsewhere.} \end{cases}$$

a. Determine the distribution function F_X.

b. Let $Y = \sqrt{X}$. Determine the distribution function F_Y.

c. Determine the probability density of Y.

8.6 Let X be a continuous random variable with probability density f_X that takes only positive values and let $Y = 1/X$.

a. Determine $F_Y(y)$ and show that

$$f_Y(y) = \frac{1}{y^2} f_X\left(\frac{1}{y}\right) \quad \text{for } y > 0.$$

b. Let $Z = 1/Y$. Using **a**, determine the probability density f_Z of Z, in terms of f_X.

8.7 Let X have a $Par(\alpha)$ distribution. Determine the distribution function of $\ln X$. What kind of a distribution does $\ln X$ have?

8.8 ⊡ Let X have an $Exp(1)$ distribution, and let α and λ be positive numbers. Determine the distribution function of the random variable

$$W = \frac{X^{1/\alpha}}{\lambda}.$$

The distribution of the random variable W is called the Weibull distribution with parameters α and λ.

8.9 Let X be a continuous random variable. Express the distribution function and probability density of the random variable $Y = -X$ in terms of those of X.

8.10 ⊞ Let X be an $N(3, 4)$ distributed random variable. Use the rule for normal random variables under change of units and Table B.1 to determine the probabilities $\mathrm{P}(X \geq 3)$ and $\mathrm{P}(X \leq 1)$.

8.11 ⊞ Let X be a random variable, and let g be a twice differentiable function with $g''(x) \leq 0$ for all x. Such a function is called a *concave* function. Show that for concave functions always

$$g(\mathrm{E}[X]) \geq \mathrm{E}[g(X)].$$

8.12 ⊞ Let X be a random variable with the following probability mass function:

x	0	1	100	10 000
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

a. Determine the distribution of $Y = \sqrt{X}$.

b. Which is larger $\mathrm{E}\left[\sqrt{X}\right]$ or $\sqrt{\mathrm{E}[X]}$?
Hint: use Exercise 8.11, or start by showing that the function $g(x) = -\sqrt{x}$ is convex.

c. Compute $\sqrt{\mathrm{E}[X]}$ and $\mathrm{E}\left[\sqrt{X}\right]$ to check your answer (and to see that it makes a big difference!).

8.13 Let W have a $U(\pi, 2\pi)$ distribution. What is larger: $\mathrm{E}[\sin(W)]$ or $\sin(\mathrm{E}[W])$? Check your answer by computing these two numbers.

8.14 In this exercise we take a look at Jensen's inequality for the function $g(x) = x^3$ (which is neither convex nor concave on $(-\infty, \infty)$).

a. Can you find a (discrete) random variable X with $\mathrm{Var}(X) > 0$ such that

$$\mathrm{E}\left[X^3\right] = (\mathrm{E}[X])^3?$$

b. Under what kind of conditions on a random variable X will the inequality $\mathrm{E}\left[X^3\right] > (\mathrm{E}[X])^3$ certainly hold?

8.15 Let $X_1, X_2, \ldots, X_n$ be independent random variables, all with a $U(0,1)$ distribution. Let $Z = \max\{X_1, \ldots, X_n\}$ and $V = \min\{X_1, \ldots, X_n\}$.

a. Compute $\mathrm{E}[\max\{X_1, X_2\}]$ and $\mathrm{E}[\min\{X_1, X_2\}]$.

b. Compute $\mathrm{E}[Z]$ and $\mathrm{E}[V]$ for general n.

c. Can you argue directly (using the symmetry of the uniform distribution (see Exercise 6.3) and not the result of the computation in **b**) that $1 - \mathrm{E}[\max\{X_1, \ldots, X_n\}] = \mathrm{E}[\min\{X_1, \ldots, X_n\}]$?

8.16 In this exercise we derive a kind of Jensen inequality for the minimum.

a. Let a and b be real numbers. Show that

$$\min\{a, b\} = \frac{1}{2}(a + b - |a - b|).$$

b. Let X and Y be independent random variables with the same distribution and finite expectation. Deduce from **a** that

$$\mathrm{E}[\min\{X, Y\}] = \mathrm{E}[X] - \frac{1}{2}\mathrm{E}[|X - Y|].$$

c. Show that

$$\mathrm{E}[\min\{X, Y\}] \leq \min\{\mathrm{E}[X], \mathrm{E}[Y]\}.$$

Remark: this is not so interesting, since $\min\{\mathrm{E}[X], \mathrm{E}[Y]\} = \mathrm{E}[X] = \mathrm{E}[Y]$, but we will see in the exercises of Chapter 11 that this inequality is also true for X and Y, which do not have the same distribution.

8.17 Let $X_1, \ldots, X_n$ be n independent random variables with the same distribution function F.

a. Convince yourself that for any numbers $x_1, \ldots, x_n$ it is true that

$$\min\{x_1, \ldots, x_n\} = -\max\{-x_1, \ldots, -x_n\}.$$

b. Let $Z = \max\{X_1, X_2, \ldots, X_n\}$ and $V = \min\{X_1, X_2, \ldots, X_n\}$. Use Exercise 8.9 and the observation in **a** to deduce the formula

$$F_V(a) = 1 - (1 - F(a))^n$$

directly from the formula

$$F_Z(a) = (F(a))^n.$$

8.18 ⊡ Let $X_1, X_2, \ldots, X_n$ be independent random variables, all with an $Exp(\lambda)$ distribution. Let $V = \min\{X_1, \ldots, X_n\}$. Determine the distribution function of V. What kind of distribution is this?

8.19 ⊞ From the "north pole" N of a circle with diameter 1, a point Q on the circle is mapped to a point t on the line by its projection from N, as illustrated in Figure 8.2.

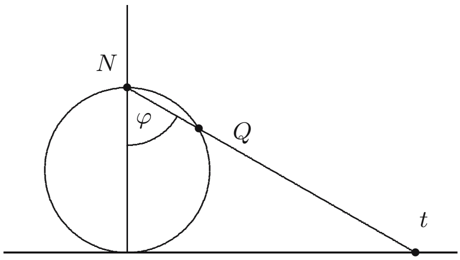

Fig. 8.2. Mapping the circle to the line.

Suppose that the point Q is uniformly chosen on the circle. This is the same as saying that the angle φ is uniformly chosen from the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (can you see this?). Let X be this angle, so that X is uniformly distributed over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This means that $P(X \leq \varphi) = 1/2 + \varphi/\pi$ (cf. Quick exercise 5.3). What will be the distribution of the projection of Q on the line? Let us call this random variable Z. Then it is clear that the event $\{Z \leq t\}$ is equal to the event $\{X \leq \varphi\}$, where t and φ correspond to each other under the projection. This means that $\tan(\varphi) = t$, which is the same as saying that $\arctan(t) = \varphi$.

- **a.** What part of the circle is mapped to the interval $[1, \infty)$?
- **b.** Compute the distribution function of Z using the correspondence between t and φ.
- **c.** Compute the probability density function of Z.

The distribution of Z is called the Cauchy distribution (which will be discussed in Chapter 11).