11

More computations with more random variables

Often one is interested in combining random variables, for instance, in taking the sum. In previous chapters, we have seen that it is fairly easy to describe the expected value and the variance of this new random variable. Often more details are needed, and one also would like to have its probability distribution. In this chapter we consider the probability distributions of the sum, the product, and the quotient of two random variables.

11.1 Sums of discrete random variables

In a solo race across the Pacific Ocean, a ship has one spare radio set for communications. Each of the two radios has probability p of failing each time it is switched on. The skipper uses the radio once every day. Let X be the number of days the radio is switched on until it fails (so if the radio can be used for two days and fails on the third day, X attains the value 3). Similarly, let Y be the number of days the spare radio is switched on until it fails. Note that these random variables are similar to the one discussed in Section 4.4, which modeled the number of cycles until pregnancy. Hence, X and Y are $Geo(p)$ distributed random variables. Suppose that $p = 1/75$ and that the trip will last 100 days. Then at first sight the skipper does not need to worry about radio contact: the number of days the first radio lasts is $X - 1$ days, and similarly the spare radio lasts $Y - 1$ days. Therefore the expected number of days he is able to have radio contact is

$$\mathrm{E}[X - 1 + Y - 1] = \mathrm{E}[X] + \mathrm{E}[Y] - 2 = \frac{1}{p} + \frac{1}{p} - 2 = 148 \text{ days!}$$

The skipper—who has some training in probability theory—still has some concerns about the risk he runs with these two radios. What if the probability $\mathrm{P}(X + Y - 2 \leq 99)$ that his two radios break down before the end of the trip is large?

This example illustrates that it is important to study the probability distribution of the sum $Z = X + Y$ of two discrete random variables. The random variable Z takes on values $a_i + b_j$, where a_i is a possible value of X and b_j of Y. Hence, the probability mass function of Z is given by

$$p_Z(c) = \sum_{(i,j):a_i+b_j=c} \mathrm{P}(X = a_i, Y = b_j)\,,$$

where the sum runs over all possible values a_i of X and b_j of Y such that $a_i + b_j = c$. Because the sum only runs over values a_i that are equal to $c - b_j$, we simplify the summation and write

$$p_Z(c) = \sum_j \mathrm{P}(X = c - b_j, Y = b_j)\,,$$

where the sum runs over all possible values b_j of Y. When X and Y are *independent*, then $\mathrm{P}(X = c - b_j, Y = b_j) = \mathrm{P}(X = c - b_j)\,\mathrm{P}(Y = b_j)$. This leads to the following rule.

Adding two independent discrete random variables. Let X and Y be two independent discrete random variables, with probability mass functions p_X and p_Y. Then the probability mass function p_Z of $Z = X + Y$ satisfies

$$p_Z(c) = \sum_j p_X(c - b_j)p_Y(b_j),$$

where the sum runs over all possible values b_j of Y.

Quick exercise 11.1 Let S be the sum of two independent throws with a die, so $S = X + Y$, where X and Y are independent, and $\mathrm{P}(X = k) = \mathrm{P}(Y = k) = 1/6$, for $k = 1, \ldots, 6$. Use the addition rule to compute $\mathrm{P}(S = 3)$ and $\mathrm{P}(S = 8)$, and compare your answers with Table 9.2.

In the solo race example, X and Y are independent $Geo(p)$ distributed random variables. Let $Z = X + Y$; then by the above rule for $k \geq 2$

$$\mathrm{P}(X + Y = k) = p_Z(k) = \sum_{\ell=1}^{\infty} p_X(k - \ell)p_Y(\ell).$$

Because $p_X(a) = 0$ for $a \leq 0$, all terms in this sum with $\ell \geq k$ vanish, hence

$$\begin{aligned}\mathrm{P}(X + Y = k) &= \sum_{\ell=1}^{k-1} p_X(k - \ell) \cdot p_Y(\ell) = \sum_{\ell=1}^{k-1} (1-p)^{k-\ell-1}p \cdot (1-p)^{\ell-1}p \\ &= \sum_{\ell=1}^{k-1} p^2(1-p)^{k-2} = (k-1)p^2(1-p)^{k-2}.\end{aligned}$$

Note that $X + Y$ does *not* have a geometric distribution.

Remark 11.1 (The expected value of a geometric distribution). The preceding gives us the opportunity to calculate the expected value of the geometric distribution in an easy way. Since the probabilities of Z add up to one:

$$1 = \sum_{k=2}^{\infty} p_Z(k) = \sum_{k=2}^{\infty} (k-1)p^2(1-p)^{k-2} = p \sum_{\ell=1}^{\infty} \ell p(1-p)^{\ell-1};$$

it follows that

$$\mathrm{E}[X] = \sum_{\ell=1}^{\infty} \ell p(1-p)^{\ell-1} = \frac{1}{p}.$$

Returning to the solo race example, it is clear that the skipper does have grounds to worry:

$$\begin{aligned} \mathrm{P}(X+Y-2 \leq 99) = \mathrm{P}(X+Y \leq 101) &= \sum_{k=2}^{101} \mathrm{P}(X+Y=k) \\ &= \sum_{k=2}^{101} (k-1)(\tfrac{1}{75})^2(1-\tfrac{1}{75})^{k-2} = 0.3904. \end{aligned}$$

The sum of two binomial random variables

It is not always necessary to use the addition rule for two independent discrete random variables to find the distribution of their sum. For example, let X and Y be two independent random variables, where X has a $Bin(n,p)$ distribution and Y has a $Bin(m,p)$ distribution. Since a $Bin(n,p)$ distribution models the number of successes in n independent trials with success probability p, heuristically, $X+Y$ represents the number of successes in $n+m$ trials with success probability p and should therefore have a $Bin(n+m,p)$ distribution. A more formal reasoning is the following. Let

$$R_1, R_2, \ldots, R_n, S_1, S_2, \ldots, S_m$$

be independent $Ber(p)$ distributed random variables. Recall that a $Bin(n,p)$ distributed random variable has the same distribution as the sum of n independent $Ber(p)$ distributed random variables (see Section 4.3 or 10.2). Hence X has the same distribution as $R_1 + R_2 + \cdots + R_n$ and Y has the same distribution as $S_1 + S_2 + \cdots + S_m$. This means that $X+Y$ has the same distribution as the sum of $n+m$ independent $Ber(p)$ variables and therefore has a $Bin(n+m,p)$ distribution. This can also be verified analytically by means of the addition rule, using that X and Y are also independent.

QUICK EXERCISE 11.2 For $i = 1,2,3$, let X_i be a $Bin(n_i,p)$ distributed random variable, and suppose that X_1, X_2, and X_3 are independent. Argue that $Z = X_1 + X_2 + X_3$ is a $Bin(n_1+n_2+n_3,p)$ distributed random variable.

11.2 Sums of continuous random variables

Let X and Y be two continuous random variables. What can we say about the probability density function of $Z = X+Y$? We start with an example. Suppose that X and Y are two independent, $U(0,1)$ distributed random variables. One might be tempted to think that Z is also uniformly distributed.

Note that the joint probability density function f of X and Y is equal to the product of the marginal probability functions f_X and f_Y:

$$f(x,y) = f_X(x) f_Y(y) = 1 \quad \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1,$$

and $f(x,y) = 0$ otherwise. Let us compute the distribution function F_Z of Z. It is easy to see that $F_Z(a) = 0$ for $a \le 0$ and $F_Z(a) = 1$ for $a \ge 2$. For a between 0 and 1, let G be that part of the plane below the line $x+y=a$, and let Δ be the triangle with vertices $(0,0)$, $(a,0)$, and $(0,a)$; see Figure 11.1.

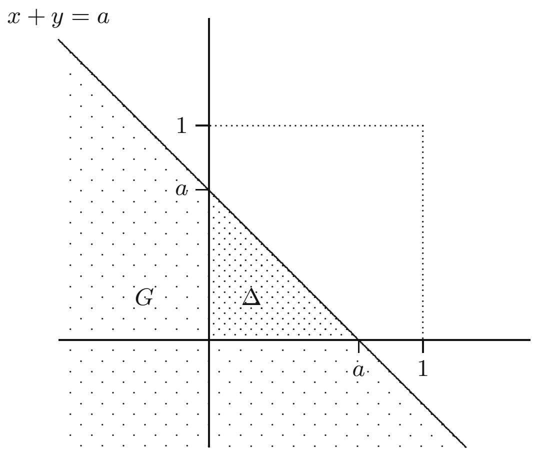

Fig. 11.1. The region G in the plane where $x+y \le a$ (with $0 < a < 1$) intersected with Δ.

Since $f(x,y) = 0$ outside $[0,1] \times [0,1]$, the distribution function of Z is given by

$$\begin{aligned} F_Z(a) &= \mathrm{P}(Z \le a) = \mathrm{P}(X+Y \le a) \\ &= \iint_G f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_\Delta 1\,\mathrm{d}x\,\mathrm{d}y = \text{area of } \Delta = \frac{1}{2}a^2 \end{aligned}$$

for $0 < a < 1$. For the case where $1 \le a < 2$ one can draw a similar figure (see Figure 11.2), from which one can find that

$$F_Z(a) = 1 - \frac{1}{2}(2-a)^2 \quad \text{for } 1 \le a < 2.$$

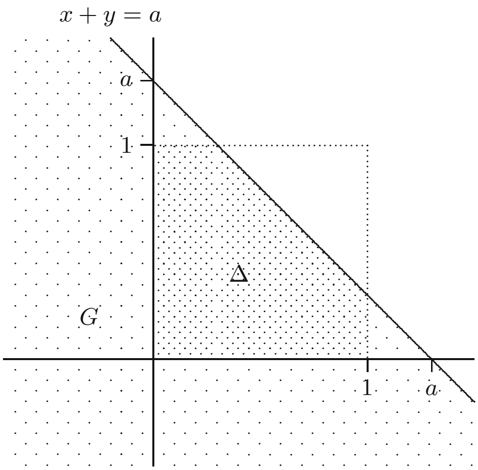

Fig. 11.2. The region G in the plane where $x + y \leq a$ (with $1 \leq a < 2$) intersected with Δ.

We see that Z is *not* uniformly distributed.

In general, the distribution function F_Z of the sum Z of two continuous random variables X and Y is given by

$$F_Z(a) = \mathrm{P}(Z \leq a) = \mathrm{P}(X + Y \leq a) = \iint_{(x,y):x+y \leq a} f(x, y) \,\mathrm{d}x \,\mathrm{d}y.$$

The double integral on the right-hand side can be written as a repeated integral, first over x and then over y. Note that x and y are between minus and plus infinity and that they also have to satisfy $x + y \leq a$ or, equivalently, $x \leq a - y$. This means that the integral over x runs from minus infinity to $y - a$, and the integral over y runs from minus infinity to plus infinity. Hence

$$F_Z(a) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{a-y} f(x, y) \,\mathrm{d}x \right) \mathrm{d}y.$$

In case X and Y are independent, the last double integral can be written as

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{a-y} f_X(x) \,\mathrm{d}x \right) f_Y(y) \,\mathrm{d}y,$$

and we find that

$$F_Z(a) = \int_{-\infty}^{\infty} F_X(a - y) f_Y(y) \,\mathrm{d}y$$

for $-\infty < a < \infty$. Differentiating F_Z we find the following rule.

Adding two independent continuous random variables. Let X and Y be two independent continuous random variables, with probability density functions f_X and f_Y. Then the probability density function f_Z of $Z = X + Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,\mathrm{d}y$$

for $-\infty < z < \infty$.

The single-server queue revisited

In the single-server queue model from Section 6.4, T_1 is the time between the start at time zero and the arrival of the first customer and T_i is the time between the arrival of the $(i-1)$th and ith customer at a well. We are interested in the arrival time of the nth customer at the well. For $n \geq 1$, let Z_n be the arrival time of the nth customer at the well: $Z_n = T_1 + \cdots + T_n$. Since each T_i has an $Exp(0.5)$ distribution, it follows from the linearity-of-expectations rule in Section 10.1 that the expected arrival time of the nth customer is

$$\mathrm{E}[Z_n] = \mathrm{E}[T_1 + \cdots + T_n] = \mathrm{E}[T_1] + \cdots + \mathrm{E}[T_n] = 2n \text{ minutes.}$$

We would like to know whether the pump capacity is sufficient; for instance, when the service times S_i are independent $U(2,5)$ distributed random variables (this is the case when the pump capacity $v = 1$). In that case, *at most* 30 customers can pump water at the well in the first hour. If $\mathrm{P}(Z_{30} \leq 60)$ is large, one might be tempted to increase the capacity of the well.

Recalling that the T_i are independent $Exp(\lambda)$ random variables, it follows from the addition rule that $f_{T_1+T_2}(z) = 0$ if $z < 0$, and for $z \geq 0$ that

$$\begin{aligned} f_{Z_2}(z) = f_{T_1+T_2}(z) &= \int_{-\infty}^{\infty} f_{T_1}(z-y) f_{T_2}(y)\,\mathrm{d}y \\ &= \int_0^z \lambda \mathrm{e}^{-\lambda(z-y)} \cdot \lambda \mathrm{e}^{-\lambda y}\,\mathrm{d}y \\ &= \lambda^2 \mathrm{e}^{-\lambda z} \int_0^z \mathrm{d}y = \lambda^2 z \mathrm{e}^{-\lambda z}. \end{aligned}$$

Viewing $T_1 + T_2 + T_3$ as the sum of T_1 and $T_2 + T_3$, we find, by applying the addition rule again, that $f_{Z_3}(z) = 0$ if $z < 0$, and for $z \geq 0$ that

$$\begin{aligned} f_{Z_3}(z) = f_{T_1+T_2+T_3}(z) &= \int_{-\infty}^{\infty} f_{T_1}(z-y) f_{T_2+T_3}(y)\,\mathrm{d}y \\ &= \int_0^z \lambda \mathrm{e}^{-\lambda(z-y)} \cdot \lambda^2 y \mathrm{e}^{-\lambda y}\,\mathrm{d}y \\ &= \lambda^3 \mathrm{e}^{-\lambda z} \int_0^z y\,\mathrm{d}y = \frac{1}{2}\lambda^3 z^2 \mathrm{e}^{-\lambda z}. \end{aligned}$$

Repeating this procedure, we find that $f_{Z_n}(z) = 0$ if $z < 0$, and

$$f_{Z_n}(z) = \frac{\lambda\,(\lambda z)^{n-1}\,\mathrm{e}^{-\lambda z}}{(n-1)!}$$

for $z \geq 0$. Using integration by parts we find (see Exercise 11.13) that for $n \geq 1$ and $a \geq 0$:

$$\mathrm{P}(Z_n \leq a) = 1 - \mathrm{e}^{-\lambda a} \sum_{i=0}^{n-1} \frac{(\lambda a)^i}{i!}.$$

Since $\lambda = 1/2$, it follows that

$$\mathrm{P}(Z_{30} \leq 60) = 0.524.$$

Even if each customer fills his jerrican in the minimum time of 2 minutes, we see that after an hour with probability 0.524, people will be waiting at the pump!

The random variable Z_n is an example of a *gamma random variable*, defined as follows.

> DEFINITION. A continuous random variable X has a *gamma distribution* with parameters $\alpha > 0$ and $\lambda > 0$ if its probability density function f is given by $f(x) = 0$ for $x < 0$ and
>
> $$f(x) = \frac{\lambda\,(\lambda x)^{\alpha-1}\,\mathrm{e}^{-\lambda x}}{\Gamma(\alpha)} \qquad \text{for } x \geq 0,$$
>
> where the quantity $\Gamma(\alpha)$ is a normalizing constant such that f integrates to 1. We denote this distribution by $Gam(\alpha, \lambda)$.

The quantity $\Gamma(\alpha)$ is for $\alpha > 0$ defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

It satisfies for $\alpha > 0$ and $n = 1, 2, \ldots$

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha) \quad \text{and} \quad \Gamma(n) = (n-1)!$$

(see also Exercise 11.12). It follows from our example that the sum of n independent $Exp(\lambda)$ distributed random variables has a $Gam(n, \lambda)$ distribution, also known as the Erlang-n distribution with parameter λ.

The sum of independent normal random variables

Using the addition rule you can show that the sum of two independent normally distributed random variables is *again* a normally distributed random variable. For instance, if X and Y are independent $N(0,1)$ distributed random variables, one has

$$
\begin{aligned}
f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}(z-y)^2}\right)\left(\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}y^2}\right)\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right)^2 \mathrm{e}^{-\frac{1}{2}(2y^2-2yz+z^2)}\,\mathrm{d}y.
\end{aligned}
$$

To prepare a change of variables, we subtract the term $\frac{1}{2}z^2$ from $2y^2-2yz+z^2$ to complete the square in the exponent:

$$
2y^2 - 2yz + \frac{1}{2}z^2 = \left[\sqrt{2}\left(y-\frac{z}{2}\right)\right]^2.
$$

In this way we find with changing integration variables $t = \sqrt{2}(y - z/2)$:

$$
\begin{aligned}
f_{X+Y}(z) &= \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{4}z^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}(2y^2-2yz+\frac{1}{2}z^2)}\,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{4}z^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\left[\sqrt{2}(y-z/2)\right]^2}\,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{4}z^2}\,\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{4\pi}}\,\mathrm{e}^{-\frac{1}{4}z^2} \int_{-\infty}^{\infty} \phi(t)\,\mathrm{d}t.
\end{aligned}
$$

Since ϕ is the probability density of the standard normal distribution, it integrates to 1, so that

$$
f_{X+Y}(z) = \frac{1}{\sqrt{4\pi}}\,\mathrm{e}^{-\frac{1}{4}z^2},
$$

which is the probability density of the $N(0,2)$ distribution. Thus, $X+Y$ also has a normal distribution. This is more generally true.

> THE SUM OF INDEPENDENT NORMAL RANDOM VARIABLES. If X and Y are independent random variables with a normal distribution, then $X+Y$ also has a normal distribution.

QUICK EXERCISE 11.3 Let X and Y be independent random variables, where X has an $N(3,16)$ distribution, and Y an $N(5,9)$ distribution. Then $X+Y$ is a normally distributed random variable. What are its parameters?

Rather surprisingly, independence of X and Y is not a prerequisite, as can be seen in the following remark.

Remark 11.2 (Sums of dependent normal random variables). We say the pair X, Y is has a bivariate normal distribution if their joint probability density equals

$$\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{1}{(1-\rho^2)}Q(x,y)\right),$$

where

$$Q(x,y) = \left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\}.$$

Here μ_X and μ_Y are the expectations of X and Y, σ_X^2 and σ_Y^2 are their variances, and ρ is the correlation coefficient of X and Y. If X and Y have such a bivariate normal distribution, then X has an $N(\mu_X, \sigma_X^2)$ and Y has an $N(\mu_Y, \sigma_Y^2)$ distribution. Moreover, one can show that $X + Y$ has an $N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y)$ distribution. An example of a bivariate normal probability density is displayed in Figure 9.2. This probability density corresponds to parameters $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1/6$, and $\rho = 0.8$.

11.3 Product and quotient of two random variables

Recall from Chapter 7 the example of the architect who wants maximal variety in the sizes of buildings. The architect wants more variety and therefore replaces the square buildings by rectangular buildings: the buildings should be of width X and depth Y, where X and Y are independent and uniformly distributed between 0 and 10 meters. Since X and Y are independent, the expected area of a building equals $\mathrm{E}[XY] = \mathrm{E}[X]\,\mathrm{E}[Y] = 5 \cdot 5 = 25\,\mathrm{m}^2$. But what can one say about the *distribution* of the area $Z = XY$ of an arbitrary building?

Let us calculate the distribution function of Z. Clearly $F_Z(a) = 0$ if $a < 0$ and $F_Z(a) = 1$ if $a > 100$. For a between 0 and 100 we can compute $F_Z(a)$ with the help of Figure 11.3.

We find

$$\begin{aligned} F_Z(a) &= \mathrm{P}(Z \le a) = \mathrm{P}(XY \le a) \\ &= \frac{\text{area of the shaded region in Figure 11.3}}{\text{area of } [0,10] \times [0,10]} \\ &= \frac{1}{100}\left(\frac{a}{10} \cdot 10 + \int_{a/10}^{10} \frac{a}{x}\,\mathrm{d}x\right) \\ &= \frac{1}{100}\left(a + \left[a \ln x\right]_{a/10}^{10}\right) = \frac{a(1 + 2\ln 10 - \ln a)}{100}. \end{aligned}$$

Hence the probability density function f_Z of Z is given by

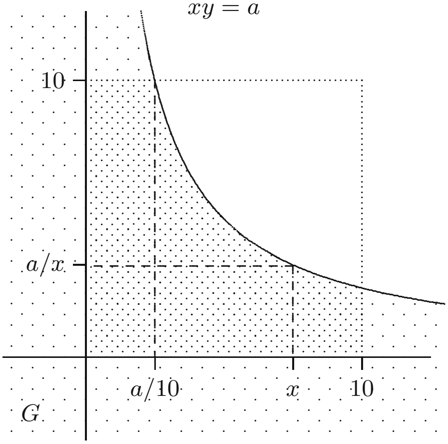

Fig. 11.3. The region G in the plane where $xy \leq a$ intersected with $[0, 10] \times [0, 10]$.

$$f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z} F_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z} \frac{z(1 + 2\ln 10 - \ln z)}{100} = \frac{\ln 100 - \ln z}{100}$$

for $0 < z < 100\,\mathrm{m}^2$.

This computation can be generalized to arbitrary independent continuous random variables, and we obtain the following formula for the probability density function of the product of two random variables.

Product of independent continuous random variables. Let X and Y be two independent continuous random variables with probability densities f_X and f_Y. Then the probability density function f_Z of $Z = XY$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y\left(\frac{z}{x}\right) f_X(x) \frac{1}{|x|}\,\mathrm{d}x$$

for $-\infty < z < \infty$.

For the quotient $Z = X/Y$ of two independent random variables X and Y it is now fairly easy to derive the probability density function. Since the independence of X and Y implies that X and $1/Y$ are independent, the preceding rule yields

$$f_Z(z) = \int_{-\infty}^{\infty} f_{1/Y}\left(\frac{z}{x}\right) f_X(x) \frac{1}{|x|}\,\mathrm{d}x.$$

Recall from Section 8.2 that the probability density function of $1/Y$ is given by

$$f_{1/Y}(y) = \frac{1}{y^2} f_Y\left(\frac{1}{y}\right).$$

Substituting this in the integral, after changing the variable of integration, we find the following rule.

QUOTIENT OF INDEPENDENT CONTINUOUS RANDOM VARIABLES. Let X and Y be two independent continuous random variables with probability densities f_X and f_Y. Then the probability density function f_Z of $Z = X/Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(zx) f_Y(x) |x| \, \mathrm{d}x$$

for $-\infty < z < \infty$.

The quotient of two independent normal random variables

Let X and Y be independent random variables, both having a standard normal distribution. When we compute the quotient Z of X and Y, we find a so-called *standard Cauchy distribution*:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} |x| \left(\frac{1}{\sqrt{2\pi}} \,\mathrm{e}^{-\frac{1}{2}z^2x^2} \right) \left(\frac{1}{\sqrt{2\pi}} \,\mathrm{e}^{-\frac{1}{2}x^2} \right) \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |x| \mathrm{e}^{-\frac{1}{2}(z^2+1)x^2} \,\mathrm{d}x = 2 \cdot \frac{1}{2\pi} \int_0^{\infty} x\mathrm{e}^{-\frac{1}{2}(z^2+1)x^2} \,\mathrm{d}x \\ &= \frac{1}{\pi} \left[\frac{-1}{z^2+1} \mathrm{e}^{-\frac{1}{2}(z^2+1)x^2} \right]_0^{\infty} = \frac{1}{\pi(z^2+1)}. \end{aligned}$$

This is the special case $\alpha = 0$, $\beta = 1$ of the following family of distributions.

DEFINITION. A continuous random variable has a *Cauchy distribution* with parameters α and $\beta > 0$ if its probability density function f is given by

$$f(x) = \frac{\beta}{\pi\left(\beta^2 + (x-\alpha)^2\right)} \qquad \text{for } -\infty < x < \infty.$$

We denote this distribution by $Cau(\alpha, \beta)$.

By integrating, we find that the distribution function F of a Cauchy distribution is given by

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x-\alpha}{\beta} \right).$$

The parameter α is the point of symmetry of the probability density function f. Note that α is *not* the expected value of Z. As a matter of fact, it was shown in Remark 7.1 that the expected value does not exist! The probability density f is shown together with the distribution function F for the case $\alpha = 2$, $\beta = 5$ in Figure 11.4.

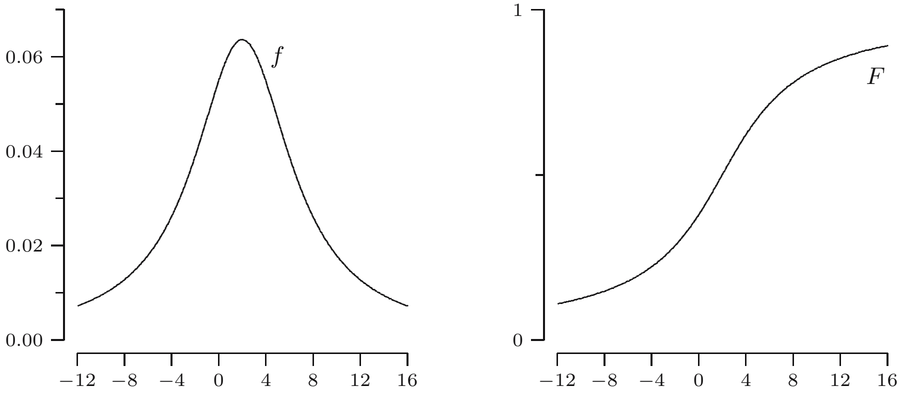

Fig. 11.4. The graphs of f and F of the $Cau(2,5)$ distribution.

Quick exercise 11.4 Argue—without doing *any* calculations—that if Z has a standard Cauchy distribution, $1/Z$ also has a standard Cauchy distribution.

11.4 Solutions to the quick exercises

11.1 Using the addition rule we find

$$\begin{aligned}
\mathrm{P}(S=3) &= \sum_{j=1}^{6} p_X(3-j)p_Y(j) \\
&= p_X(2)p_Y(1) + p_X(1)p_Y(2) + p_X(0)p_Y(3) \\
&\quad + p_X(-1)p_Y(4) + p_X(-2)p_Y(5) + p_X(-3)p_Y(6) \\
&= \frac{1}{36} + \frac{1}{36} + 0 + 0 + 0 + 0 = \frac{1}{18}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{P}(S=8) &= \sum_{j=1}^{6} p_X(8-j)p_Y(j) \\
&= p_X(7)p_Y(1) + p_X(6)p_Y(2) + p_X(5)p_Y(3) \\
&\quad + p_X(4)p_Y(4) + p_X(3)p_Y(5) + p_X(2)p_Y(6) \\
&= 0 + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{5}{36}.
\end{aligned}$$

11.2 We have seen that $X_1 + X_2$ is a $Bin(n_1 + n_2, p)$ distributed random variable. Viewing $X_1 + X_2 + X_3$ as the sum of $X_1 + X_2$ and X_3, it follows that $X_1 + X_2 + X_3$ is a $Bin(n_1 + n_2 + n_3, p)$ distributed random variable.

11.3 The sum rule for two normal random variables tells us that $X + Y$ is a normally distributed random variable. Its parameters are expectation and variance of $X + Y$. Hence by linearity of expectations

$$\mu_{X+Y} = \mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y] = \mu_X + \mu_Y = 3 + 5 = 8,$$

and by the rule for the variance of the sum

$$\sigma^2_{X+Y} = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y) = \sigma^2_X + \sigma^2_Y = 16 + 9 = 25,$$

using that $\mathrm{Cov}(X,Y) = 0$ due to independence of X and Y.

11.4 In the examples we have seen that the quotient X/Y of two independent standard normal random variables has a standard Cauchy distribution. Since $Z = X/Y$, the random variable $1/Z = Y/X$. This is *also* the quotient of two independent standard normal random variables, and it has a standard Cauchy distribution.

11.5 Exercises

11.1 ⊡ Let X and Y be independent random variables with a *discrete uniform* distribution, i.e., with probability mass functions

$$p_X(k) = p_Y(k) = \frac{1}{N}, \quad \text{for } k = 1, \ldots, N.$$

Use the addition rule for discrete random variables on page 152 to determine the probability mass function of $Z = X + Y$ for the following two cases.

a. Suppose $N = 6$, so that X and Y represent two throws with a die. Show that

$$p_Z(k) = \mathrm{P}(X + Y = k) = \begin{cases} \dfrac{k-1}{36} & \text{for } k = 2, \ldots, 6, \\[2ex] \dfrac{13-k}{36} & \text{for } k = 7, \ldots, 12. \end{cases}$$

You may check this with Quick exercise 11.1.

b. Determine the expression for $p_Z(k)$ for general N.

11.2 ⊞ Consider a discrete random variable X taking values $k = 0, 1, 2, \ldots$ with probabilities

$$\mathrm{P}(X = k) = \frac{\mu^k}{k!}\mathrm{e}^{-\mu},$$

where $\mu > 0$. This is the *Poisson* distribution with parameter μ. We will learn more about this distribution in Chapter 12. This exercise illustrates that the sum of independent Poisson variables again has a Poisson distribution.

a. Let X and Y be independent random variables, each having a Poisson distribution with $\mu = 1$. Show that for $k = 0, 1, 2, \ldots$

$$\mathrm{P}(X+Y=k) = \frac{2^k}{k!}\mathrm{e}^{-2},$$

by using $\sum_{\ell=0}^{k} \binom{k}{\ell} = 2^k$.

b. Let X and Y be independent random variables, each having a Poisson distribution with parameters λ and μ. Show that for $k = 0, 1, 2, \ldots$

$$\mathrm{P}(X+Y=k) = \frac{(\lambda+\mu)^k}{k!}\mathrm{e}^{-(\lambda+\mu)},$$

by using $\sum_{\ell=0}^{k} \binom{k}{\ell} p^\ell (1-p)^{k-\ell} = 1$ for $p = \mu/(\lambda+\mu)$.
We conclude that $X+Y$ has a Poisson distribution with parameter $\lambda+\mu$.

11.3 Let X and Y be two independent random variables, where X has a $Ber(p)$ distribution, and Y has a $Ber(q)$ distribution. When $p = q = r$, we know that $X+Y$ has a $Bin(2, r)$ distribution. Suppose that $p = 1/2$ and $q = 1/4$. Determine $\mathrm{P}(X+Y=k)$, for $k = 0, 1, 2$, and conclude that $X+Y$ does not have a binomial distribution.

11.4 ⊞ Let X and Y be two independent random variables, where X has an $N(2, 5)$ distribution and Y has an $N(5, 9)$ distribution. Define $Z = 3X - 2Y + 1$.

a. Compute $\mathrm{E}[Z]$ and $\mathrm{Var}(Z)$.

b. What is the distribution of Z?

c. Compute $\mathrm{P}(Z \leq 6)$.

11.5 ⊡ Let X and Y be two independent, $U(0, 1)$ distributed random variables. Use the rule on addition of independent continuous random variables on page 156 to show that the probability density function of $X+Y$ is given by

$$f_Z(z) = \begin{cases} z & \text{for } 0 \leq z < 1, \\ 2 - z & \text{for } 1 \leq z \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

11.6 ⊡ Let X and Y be independent random variables with probability densities

$$f_X(x) = \frac{1}{4}x\mathrm{e}^{-x/2} \quad \text{and} \quad f_Y(y) = \frac{1}{4}y\mathrm{e}^{-y/2}.$$

Use the rule on addition of independent continuous random variables to determine the probability density of $Z = X+Y$.

11.7 ⊡ The two random variables in Exercise 11.6 are special cases of $Gam(\alpha, \lambda)$ variables, namely with $\alpha = 2$ and $\lambda = 1/2$. More generally, let

$X_1, \ldots, X_n$ be independent $Gam(k, \lambda)$ distributed random variables, where $\lambda > 0$ and k is a positive integer. Argue—without doing any calculations—that $X_1 + \cdots + X_n$ has a $Gam(nk, \lambda)$ distribution.

11.8 We investigate the effect on the Cauchy distribution under a change of units.

a. Let X have a standard Cauchy distribution. What is the distribution of $Y = rX + s$?

b. Let X have a $Cau(\alpha, \beta)$ distribution. What is the distribution of the random variable $(X - \alpha)/\beta$?

11.9 ⊞ Let X and Y be independent random variables with a $Par(\alpha)$ and $Par(\beta)$ distribution.

a. Take $\alpha = 3$ and $\beta = 1$ and determine the probability density of $Z = XY$.

b. Determine the probability density of $Z = XY$ for general α and β.

11.10 Let X and Y be independent random variables with a $Par(\alpha)$ and $Par(\beta)$ distribution.

a. Take $\alpha = \beta = 2$. Show that $Z = X/Y$ has probability density

$$f_Z(z) = \begin{cases} z & \text{for } 0 < z < 1, \\ 1/z^3 & \text{for } 1 \leq z < \infty. \end{cases}$$

b. For general $\alpha, \beta > 0$, show that $Z = X/Y$ has probability density

$$f_Z(z) = \begin{cases} \dfrac{\alpha\beta}{\alpha+\beta} z^{\beta-1} & \text{for } 0 < z < 1, \\ \dfrac{\alpha\beta}{\alpha+\beta} \dfrac{1}{z^{\alpha+1}} & \text{for } 1 \leq z < \infty. \end{cases}$$

11.11 Let X_1, X_2, and X_3 be three independent $Geo(p)$ distributed random variables, and let $Z = X_1 + X_2 + X_3$.

a. Show for $k \geq 3$ that the probability mass function p_Z of Z is given by

$$p_Z(k) = \mathrm{P}(X_1 + X_2 + X_3 = k) = \frac{1}{2}(k-2)(k-1)p^3(1-p)^{k-3}.$$

b. Use the fact that $\sum_{k=3}^{\infty} p_Z(k) = 1$ to show that

$$p^2 \left(\mathrm{E}\left[X_1^2\right] + \mathrm{E}\left[X_1\right] \right) = 2.$$

c. Use $\mathrm{E}\left[X_1\right] = 1/p$ and part **b** to conclude that

$$\mathrm{E}\left[X_1^2\right] = \frac{2-p}{p^2} \quad \text{and} \quad \mathrm{Var}(X_1) = \frac{1-p}{p^2}.$$

11.12 Show that $\Gamma(1) = 1$, and use integration by parts to show that

$$\Gamma(x+1) = x\Gamma(x) \quad \text{for } x > 0.$$

Use this last expression to show for $n = 1, 2, \ldots$ that

$$\Gamma(n) = (n-1)!$$

11.13 Let Z_n have an Erlang-n distribution with parameter λ.

a. Use integration by parts to show that for $a \geq 0$ and $n \geq 2$:

$$\mathrm{P}(Z_n \leq a) = \int_0^a \frac{\lambda^n z^{n-1} \mathrm{e}^{-\lambda z}}{(n-1)!} \, \mathrm{d}z = -\frac{(\lambda a)^{n-1}}{(n-1)!} \mathrm{e}^{-\lambda a} + \mathrm{P}(Z_{n-1} \leq a) .$$

b. Use **a** to show that for $a \geq 0$:

$$\mathrm{P}(Z_n \leq a) = -\sum_{i=1}^{n-1} \frac{(\lambda a)^i}{i!} \mathrm{e}^{-\lambda a} + \mathrm{P}(Z_1 \leq a) .$$

c. Conclude that for $a \geq 0$:

$$\mathrm{P}(Z_n \leq a) = 1 - \mathrm{e}^{-\lambda a} \sum_{i=0}^{n-1} \frac{(\lambda a)^i}{i!} .$$