Harvard University
Computer Science 121
SOLUTIONS – Problem Set 0

PROBLEM 1

Describe the following sets using formal set notation:
(A) A is the set containing the empty set.
(B) B is the set containing the empty string.
(C) C is the set containing all non-negative, integral powers of 2.
(D) D is the set containing all strings over $\Sigma = \{a, b\}$ whose length is a non-negative, integral power of 2.

(A) $A = \{\{\}\} = \{\emptyset\}$
(B) $B = \{\varepsilon\}$
(C) $C = \{2^i \mid i \in \mathbb{N}\}$
(D) $D = \{w \in \Sigma^* : \ |w| = 2^i \text{ for some } i \in \mathbb{N}\} = \{w \in \Sigma^* : \ |w| \in C\}$

PROBLEM 2

Let A be the set $\{x, y, z\}$ and B be the set $\{x, z\}$. Let $\mathcal{P}(S)$ denote the power set of any set S (i.e. the set of subsets of S). You need not justify your answers.

(A) Is $P(B)$ a subset of $P(A)$?
(B) What is $A \cup B$?
(C) What is $A \times B$?
(D) Is $\emptyset \in \mathcal{P}(A)$?
(E) Is $\emptyset \subset \mathcal{P}(A)$?

(A) Yes. Since $B \subset A$, any subset of B is also a subset of A.
(B) $\{x, y, z\}$. Since $B \subset A$, this is equal to A.
(C) $\{(x, x), (x, z), (y, x), (y, z), (z, x), (z, z)\}$
(D) Yes. $\emptyset$ is indeed a subset of A and thus an element of its power set.
(E) Yes. $\emptyset$ is a subset of any set - simply choose no elements.

PROBLEM 3

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers. For each of the following functions, $f : \mathbb{N} \longrightarrow \mathbb{N}$, state whether f is (i) one-to-one, (ii) onto, and/or (iii) bijective. Briefly justify each of your answers.

(A) $f(x) = x^2$
(B) $f(x) = x \bmod 3$
(C) $f(x) = x!$
(D) $f(x) = \begin{cases} x + 1 & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases}$

(A) This is one-to-one because the square of each natural number is unique. More formally, if $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{N}$, i.e. $x_1^2 = x_2^2$, then by taking the square root of both sides we get $x_1 = x_2$; that is, no two distinct elements can be mapped to the same target. On the other hand, this is not onto because, for instance, $1^2 = 1$ and $2^2 = 4$ are consecutive squares, so that $2, 3$ are not in the range (since x^2 is monotonic in the domain). Since this is not onto, it is not bijective.

(B) This is not one-to-one because for instance 0 mod 3 and 6 mod 3 are both zero. This is not onto because the range only contains 0, 1 and 2. Since this is neither one-to-one nor onto, this is not a bijective function.

(C) This is not one-to-one because 0! and 1! are both 1. This is not onto because, for instance, 2! is 2 and 3! is 6, and henceforth each larger number in the domain is mapped to a number bigger than 6 in the range, and therefore the range has no 3 - much as in part A. Since this is neither one-to-one nor onto, this is not a bijective function.

(D) This is one-to-one since each odd number maps to a unique even number and each even number maps to a unique odd number. This is onto because 0 maps to 1 and 1 maps to 0 and henceforth each even number maps to an odd number greater than 1 and each odd number maps to an even number greater than 0. Because it's one-to-one and onto, it is bijective.

PROBLEM 4

Consider the binary relation $\sim$ on sets defined by $A \sim B$ if and only if there exists a function $f : A \to B$ such that f is bijective.

(A) Give examples of two finite sets A, B such that $A \sim B$, as well as sets C, D such that $C \not\sim D$.

(B) Prove that $\sim$ is an equivalence relation. (Your proof should work for infinite sets as well as finite ones. Indeed, this relation is how one defines what it means for two infinite sets to have the same cardinality. Later in the course, we will see that not all infinite sets have the same cardinality, i.e. the infinite sets yield multiple equivalence classes under this relation.)

(C) Now consider the relation $\lesssim$ defined by $A \lesssim B$ if there exists a one-to-one function $f : A \to B$. Is $\lesssim$ reflexive? symmetric? transitive? Justify your answers.

(A) Let $A = \{a\}, B = \{1\}$. Function $f : A \to B$, $f(a) = 1$ is bijective. However, if $A = \{a, b\}, B = \{1\}$, the only function $f : A \to B$ has $f(a) = f(b) = 1$. f is not one-to-one, so it is not bijective.

Notice that there are a number of other possible solutions. When A, B are both finite, then $A \sim B$ if and only if A and B have the same number of elements.

(B)

- It is reflexive: Consider the identity function $Id : A \to A$: $Id(x) = x$ for all $x \in A$. Since Id is bijective, we have that $A \sim A$ for all sets A.

- It is symmetric: Given $A \sim B$, we have a bijective function $f : A \to B$ as defined above. Since f is onto, for every $y \in B$, there is some $x \in A$ such that $y = f(x)$. Since f is one-to-one, such x is unique. Hence the inverse function $g : B \to A$, where $g(y) = x$ (where x is the unique element of A such that $y = f(x)$) for all $y \in B$ is a well-defined function. Furthermore, since f is a well-defined function, g must be onto and one-to-one, hence $B \sim A$
- It is transitive: If $A \sim B$ and $B \sim C$ then there exists bijective functions $f : A \to B$ and $g : B \to C$. Let $h : A \to C$ be such that $h(x) = g(f(x))$, for all $x \in A$.
  - h is one-to-one:
    $h(x) = h(y) \Leftrightarrow g(f(x)) = g(f(y))$;
    since g is one-to-one, $g(f(x)) = g(f(y)) \Leftrightarrow f(x) = f(y)$;
    since f is one-to-one, $f(x) = f(y) \Leftrightarrow x = y$;
    so $h(x) = h(y) \Leftrightarrow x = y$.
  - h is onto:
    if $y \in C$, then since g is onto and $y \in C$, there exists some $z \in B$ s.t. $g(z) = y$.
    Since $z \in B$ and f is onto, then there exists some $x \in A$ such that $f(x) = z$.
    So for $x \in A$, $y = g(f(x)) = h(x)$.

  So h is bijective and $A \sim C$.

(C)

- It is reflective: Consider the identity function $Id : A \to A$: $Id(x) = x$ for all $x \in A$. Again, Id is bijective, so that inparticular it is one-to-one and thus $A \lesssim A$ for all sets A.
- It is not symmetric: Consider sets $A = \{a\}$ and $B = \{1, 2\}$. Notice that there is a one-to-one function $f : A \to B$, $f(a) = 1$, so $A \lesssim B$. However, there is only one function $g : B \to A$: with $g(1) = g(2) = a$. Such function is not one-to-one, so $B \not\lesssim A$
- It is transitive: If $A \lesssim B$ and $B \lesssim C$ then there exist one-to-one functions $f : A \to B$ and $g : B \to C$. Let $h : A \to C$ be such that $h(x) = g(f(x))$, for all $x \in A$. By the same argument from part B, h is one-to-one, so $\lesssim$ is a transitive relation.

PROBLEM 5

Joe the painter has 2010 cans of paint. Show that at least one of the following statements is true about Joe's paint collection:

1. Among the cans, there are at least 42 different colors of paint.
2. Among the cans, there are at least 50 of them with the same color.

(Hint: prove by contradiction)

Suppose neither statement is true. If (1) is false, then Joe has at most 41 different colors of paint. If (2) is false, then Joe has at most 49 cans of paint of each color. Thus Joe has at most $41 \times 49 = 2009$ cans of paint. Contradiction! Since we know Joe has 2010 cans of paint, one of the statements must be true.

PROBLEM 6

Define the Fibonacci numbers as follows:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 2$$

Prove the following statements by induction:

(A) For $n \geq 2$, F_n equals the number of strings of length $n-2$ over alphabet $\Sigma = \{a, b\}$ that do not contain two consecutive a's.

(B)

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Notice that in this problem we will always need to have two base cases, since the formula for each Fibonacci number depends on the two previous ones. For the same reason, when setting up your induction hypothesis, you must assume that the conjecture you want to prove is true at least for the two previous cases.

(A) **Base Cases:**
First note that this is true for $n = 2$ and $n = 3$. There is exactly one string of length 0 with no two consecutive a's: ε, and $F_2 = 1$. Similarly, there are exactly two strings of length 1 with no two consecutive a's: a and b. So we have proven the base case.

Induction Hypothesis:
Let

$$S_k = \{x : |x| = k \text{ and } x \text{ has no two consective } a\text{'s}\}.$$

We want to show that for all natural numbers k, $|S_{k-2}| = F_k$, i.e. the number of elements in S_{k-2} equals F_k.

Suppose that the conjecture holds for $n-1$ and n, i.e. $|S_{n-3}| = F_{n-1}$ and $|S_{n-2}| = F_n$.

Induction Step:
We want to show that the conjecture is true for $n+1$. Split the strings of S_{n-1} into two disjoint sets: A_{n-1}, the subset of strings that end in an a, and B_{n-1}, the subset of strings those that end in a b.

First consider the second set B_{n-1}. The first $n-2$ characters of a string that ends in a b must have no consecutive a's, but can end in any character (since the overall string ends in a b it imposes no conditions on the previous letters). Thus $|B_{n-1}| = |S_{n-2}|$, i.e. there are as many desired strings that end in a b in S_{n-1} as there are strings in S_{n-2}, which we know by the inductive hypothesis is F_n.

Now consider the first set A_{n-1}. The second-to-last character of any string in this set must be a b, since there can't be two consecutive a's. However, this means that the first $n-3$ characters of the string form a string with no consecutive a's, with no other constraints. Thus we know that $|A_{n-1}| = |S_{n-3}|$, i.e. there are as many strings in S_{n-1} that end in an a as there are strings in S_{n-3}, which means that there are F_{n-1} of them.

Thus we know that

$$|S_{n-1}| = |A_{n-1}| + |B_{n-1}| = |S_{n-3}| + |S_{n-2}|$$

The Induction Hypothesis thus implies that

$$|S_{n-1}| = F_{n-1} + F_n = F_{n+1}$$

as desired.

(B) **Base Cases:**
Notice that for $n = 0$ the expression will be

$$\frac{1}{\sqrt{5}}(1-1) = 0$$

which is exactly F_0. For $n = 1$ we get

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = \frac{\sqrt{5}}{\sqrt{5}} = F_1.$$

So we know that this holds for the base cases.

Induction Hypothesis:
Now suppose that it holds for $n-1$ and n.

Induction Step:
We will show it holds for $n+1$. Notice that

$$\begin{aligned}
F_{n+1} &= F_{n-1} + F_n \\
&= \left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right) + \left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}+1\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}+1\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}\right)^{2} \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}
\end{aligned}$$

as desired. Note that in the next-to-last step, we observed that

$$\begin{aligned}
\left(\frac{1+\sqrt{5}}{2}\right)^2 &= \left(\frac{6+2\sqrt{5}}{4}\right) \\
&= \left(\frac{1+\sqrt{5}}{2}+1\right)
\end{aligned}$$

and similarly for the $1-\sqrt{5}$ term.