6.856 — Randomized Algorithms

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Handout #15, Apr. 4th, 2011 — Homework 8, Due 4/18

1. Submit (by email to me) a 1 or 2 paragraph project proposal outlining your planned project and who you intend to do it with.

2. (Based on MR9.9) Consider any linear programming problem. Prove that if the constraints are given positive weights, and r constraints are sampled at random with probability proportional to the weights, then the expected weight of constraints that violate the optimum of the sample is only a $d/(r-d)$ fraction of the total weight. **Hint:** For every constraint of weight w_h, replace it by w_h "virtual copies" of h, and consider sampling uniformly.

3. Consider the problem of finding the smallest (minimum diameter) circle containing some set H of n points in the plane. We will assume that the points are in "general position"—no 3 points are colinear, and no 4 points are on the boundary of a common circle. This assumption can be achieved by small perturbations in the input. For any set of points S in the plane, let $O(S)$ denote the smallest circle containing S.

   (a) Show that for any 3 non-colinear points, there is a unique circle having all 3 of those points on the circle boundary. This circle (center and radius) can be computed in constant time from the points.

   (b) Show that $O(H)$ contains either 2 or 3 of the input points on its boundary. We will call these points the "basis" of the circle (hint, hint) and refer to them as $B(H)$. Deduce a simple $O(n^4)$-time algorithm for solving the problem.

   (c) Show that if a circle C excludes a point of H, then C cannot be the smallest circle containing $B(H)$.

   (d) Show that if p is *not* contained in $O(S)$ for some S then p is on the boundary of $O(S \cup \{p\})$.

   (e) Consider a set R of r points chosen at random from H. Bound the expected number of points of H outside $O(R)$.

   (f) Generalize the previous part to where you have an "active" subset $S \subseteq H$ and compute the number of points outside $O(R \cup S)$.

   (g) Give an $\tilde{O}(n)$ time algorithm for finding $O(H)$.

4. MIT needs to install new routers in their dorms, then wire every student room to a router. Installing a router at a particular site costs f_i, while connecting student room j to site f_i (if a router is built there) costs c_{ij}. We wish to minimize the total construction cost. This is NP-hard.

   (a) Devise an integer linear program for this problem, using indicator variables y_i for building a route at site i and x_{ij} for wiring room j to site i. Make sure to enforce the constraint that you can only wire to where you built a router.

   (b) Consider the fractional relaxation of the ILP. Devise a randomized rounding scheme that will select some sites to install routers and a *constant fraction* of the rooms to wire to those routers (the ones that have a router built within some "reasonable" distance) at a cost proportional to the optimum.

   (c) Apply the above approach to get wire up all the rooms at cost $O(\log n)$ times optimum.

   (d) (Optional). It's very hard but possible to achieve a constant factor approximation by introducing several other new ideas.

5. MR 5.11. In this problem, we will finish establishing the properties of the pessimistic estimator $\hat{P}(a)$ used in the set balancing derandomization by conditional probabilities.

   (a) Show that for a node a at the i-th level of the computation tree, $\hat{P}(a)$ is of the form $N(a)/2^{n-i}$, where $N(a)$ is a sum of binomial coefficients times powers of two.

   (b) Prove that for any node a, we can compute $\hat{P}(a)$ in polynomial time

   (c) Prove that $\min\{\hat{P}(b), \hat{P}(c)\} \leq \hat{P}(a)$ if a has children b and c.

   (d) Give an upper bound on the running time of the deterministic algorithm for either the unit cost or the log-cost RAM.

6. **This problem should be solved without collaboration.** MR 5.12. Show how the method of conditional expectations can be used to deterministically build a 2-dimensional binary space partition of size $O(n \log n)$.