Fabien Morel
Olivier Haution

Algebraic Geometry II
Exercises 2

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The letter k denotes a field.

Exercise 1. Let X be an irreducible separated k-scheme and U an open subscheme of X. Assume that U is proper over k. Show that $U = \varnothing$ or $U = X$.

Exercise 2. Give an example of a separated, finite type, closed morphism which is not proper.

Exercise 3. (i) Let A be a local ring. Show that any scheme morphism $\operatorname{Spec} A \to Y$ factors through an affine open subscheme of Y.

(ii) Let X, Y be two k-schemes with Y of finite type. Let $x \in X$ and $\operatorname{Spec} \mathcal{O}_{X,x} \to Y$ be a scheme morphism. Show that there exists an open subscheme U of X containing x, and a morphism $U \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{X,x} \arrow[r] \arrow[d] & Y \\ U \arrow[ur] & \end{tikzcd}$$

[Hint: Reduce to the case when Y and X are affine. The special case of X integral is easier.]

Exercise 4. (i) Let $f, g\colon X \to Y$ be two k-morphisms. Assume that X is reduced and Y is separated over k. Let $h\colon T \to X$ be a k-morphism with dense set-theoretic image. If $g \circ h = f \circ h$, show that $f = g$.

(ii) Let A be a k-algebra. We assume that the ring A is a principal ideal domain, but not a field. Let U be the open complement of a closed point in $X = \operatorname{Spec} A$, and Y a proper k-scheme. Show that any k-morphism $U \to Y$ extends uniquely to a k-morphism $X \to Y$.

(iii) Let U be a non-empty open subscheme of $\mathbb{A}^1_k$ and Y a proper k-scheme. Show that any k-morphism $U \to Y$ extends uniquely to a k-morphism $\mathbb{A}^1_k \to Y$.

(iv) Let U be a non-empty open subscheme of $\mathbb{P}^1_k$ and Y a proper k-scheme. Show that any k-morphism $U \to Y$ extends uniquely to a k-morphism $\mathbb{P}^1_k \to Y$.