All rings are commutative and unital.

Exercise 1. Let A be a ring and $f \in A$. Consider the localised ring $A_f = A[x]/(xf-1)$. Show that $A_f = 0$ if and only if f is nilpotent in A.

Exercise 2. Let $\varphi\colon A \to B$ be a ring morphism such that each element of $\ker\varphi$ is nilpotent. Show that $\operatorname{Spec}\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is a homeomorphism under any of the following assumptions:

(i) The morphism φ is surjective.

(ii) Let p be a prime number such that $p \cdot 1 = 0$ in A. For each $b \in B$ we may find an integer $n \geq 0$ such that $b^{p^n} \in \operatorname{im}\varphi$.

Exercise 3. Let k be an algebraically closed field. Show that every quasi-projective variety over k is covered by open affine varieties. [Hint : use the covering of $\mathbb{P}^n_k$ by $n+1$ copies of $\mathbb{A}^n_k$.]

Exercise 4. Let n be an integer ≥ 2 and k an algebraically closed field.

(i) Show that the morphism $\mathcal{O}(\mathbb{A}^n_k) \to \mathcal{O}(\mathbb{A}^n_k - 0)$ is bijective. [Hint : use Exercise 1 of the previous sheet, and the opens $U_{X_i} = \mathbb{A}^n_k - Z(X_i)$.]

(ii) Deduce that the variety $\mathbb{A}^n_k - 0$ is not affine.

(iii) Let $f \in \mathcal{O}(\mathbb{A}^n_k - 0)$. Assume that f is $k^\times$-invariant, in other words that for every $\lambda \in k^\times$ and $(x_1, \ldots, x_n) \in k^n - \{0\}$ we have $f(\lambda x_1, \ldots, \lambda x_n) = f(x_1, \ldots, x_n)$. Show that f is constant.

(iv) Deduce that $\mathcal{O}(\mathbb{P}^{n-1}_k) = k$.