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Algebraic Geometry I
Exercises 1

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All rings are commutative and unital. The letter k denotes an algebraically closed field. If f is an element of a ring A, we denote by $(f) = \{af | a \in A\}$ the ideal generated by f.

Exercise 1. Show that an ideal I of a ring A is maximal if and only if the quotient ring A/I is a field.

Exercise 2. (a) Let $f_1, \cdots, f_r$ be irreducible elements in $R = k[X_1, \cdots, X_n]$. Assume that $(f_i) \neq (f_j)$ for $i \neq j$. Show that

$$(f_1 \cdots f_r) = (f_1) \cap \cdots \cap (f_r).$$

(b) Is it still true when R is an arbitrary ring? (Give a proof or a counterexample.)

Exercise 3. Let $f_1, \cdots, f_r \in k[X_1, \cdots, X_n]$ and $\varphi\colon \mathbb{A}^n_k \to \mathbb{A}^r_k$ the map defined by

$$(x_1, \cdots, x_n) \mapsto (f_1(x_1, \cdots, x_n), \cdots, f_r(x_1, \cdots, x_n)).$$

(a) If Z is an irreducible closed subset of $\mathbb{A}^r_k$, is the subset $\varphi^{-1}(Z) \subset \mathbb{A}^n_k$ closed? irreducible?

(b) Let Y be an irreducible subset of $\mathbb{A}^n_k$. Is the subset $\varphi(Y) \subset \mathbb{A}^r_k$ irreducible? Is its closure $\overline{\varphi(X)} \subset \mathbb{A}^r_k$ irreducible?

(c) Find n, r and $f_1, \cdots, f_r$, and a closed subset $Y \subset \mathbb{A}^n_k$ such that $\varphi(Y) \subset \mathbb{A}^r_k$ is not closed.

Exercise 4. (a) Let A be a ring and I an ideal of A. Show that I is radical if and only if A/I is reduced.

(b) Show that every reduced finitely generated k-algebra is the coordinate ring $A(Y)$ of some algebraic set $Y \subset \mathbb{A}^n_k$ (for some n).

(c) Let $Y \subset \mathbb{A}^n_k$ be an algebraic set. For a closed subset Z of Y, let $I_Y(Z)$ be the kernel of the natural morphism of coordinate rings $A(Y) \to A(Z)$. Show that $Y \mapsto I_Y(Z)$ induces a bijection between the closed subsets of Y and the radical ideals of $A(Y)$.