EXERCISES 2 (INTERSECTION THEORY)

Exercise 1. Let M be a finitely generated A-module (A noetherian), and $x \in A$.

(i) The element x is a nonzerodivisor in M if and only it belongs to no prime of $\mathrm{Ass}(M)$.

(ii) The multiplication by x map $x_M\colon M \to M$ is nilpotent if and only if x belongs to every prime of $\mathrm{Ass}(M)$.

Exercise 2. When $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, we define

$$\mathrm{Ass}(\mathcal{F}) = \{x \in X | \mathfrak{m}_x \in \mathrm{Ass}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)\}.$$

A closed embedding $Z \to X$ is called *locally principal* if there is a covering by open affine subschemes $U_i = \operatorname{Spec} A_i$ and elements $s_i \in A_i$ such that $Z \cap U_i = \operatorname{Spec}(A_i/s_iA_i)$.

(i) If $X = \operatorname{Spec} A$, and $M = H^0(X, \mathcal{F})$, show that $\mathrm{Ass}(M) = \mathrm{Ass}(\mathcal{F})$.

(ii) Show that a closed embedding $D \to X$ is an effective Cartier divisor if and only if:
— $D \to X$ is locally principal,
— and $D \cap \mathrm{Ass}(\mathcal{O}_X) = \emptyset$.

(iii) Let $f\colon Y \to X$ be a morphism, and $Z \to X$ a locally principal closed embedding. Then show that $f^{-1}Z \to Y$ is a locally principal closed embedding.

(iv) Let $f\colon Y \to X$ be a morphism, and $D \to X$ an effective Cartier divisor. Show that $f^{-1}D \to Y$ is an effective Cartier divisor if and only if $f(\mathrm{Ass}(\mathcal{O}_Y)) \cap D = \emptyset$.

(v) Assume that f is flat. Show that $f(\mathrm{Ass}(\mathcal{O}_Y)) \subset \mathrm{Ass}(\mathcal{O}_X)$.

(vi) Explain how we can reprove the lemma concerning pull-backs of effective Cartier divisors.

Exercise 3. (i) Let M be a finitely generated A-module (A noetherian). Show that the following morphism is injective:

$$M \to \bigoplus_{\mathfrak{p} \in \mathrm{Ass}(M)} M_{\mathfrak{p}}.$$

Let X be a variety.

(ii) Show that every generic point of X is in $\mathrm{Ass}(\mathcal{O}_X)$.

(iii) Show that X is reduced if and only if :
— for every generic point $x \in X$, the ring $\mathcal{O}_{X,x}$ is reduced,
— and $\mathrm{Ass}(\mathcal{O}_X)$ is the set of generic points.

Exercise 4. Let $f\colon Y \to X$ be a flat morphism, with X irreducible and Y equidimensional. Show that f has relative dimension $\dim Y - \dim X$.

Exercise 5. Let A be a noetherian local domain of dimension one. Show that A is a discrete valuation ring if and only if it is integrally closed (in its fraction field).