Exercise 1. Let X be a connected scheme, and $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module. Show that the dimension of the $\kappa(x)$-vector space $\mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ does not depend on the point $x \in X$. Give a counterexample in case $\mathcal{E}$ is coherent but not locally free.

Exercise 2. Let $A \to B$ be a ring morphism, and $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ the corresponding scheme morphism.

(i) Let M, N be two A-modules. Show that

$$\widetilde{M \oplus N} = \widetilde{M} \oplus \widetilde{N} \quad \text{and} \quad \widetilde{M \otimes_A N} = \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}.$$

(ii) Let N be a B-module. What is the A-module M such that $f_*\widetilde{N} = \widetilde{M}$?

(iii) Let M be a A-module. What is the B-module N such that $f^*\widetilde{M} = \widetilde{N}$?

Exercise 3. Let $f\colon Y \to X$ be a separated and quasi-compact morphism of schemes, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_Y$-module. Show that the $\mathcal{O}_X$-module $f_*\mathcal{F}$ is quasi-coherent.

Exercise 4. Let $f\colon Y \to X$ be a scheme morphism.

(i) Let $\mathcal{A}, \mathcal{B}$ be two $\mathcal{O}_X$-modules. Show that

$$f^*\mathcal{A} \otimes_{\mathcal{O}_Y} f^*\mathcal{B} \simeq f^*(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B})$$

(ii) Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module, and $\mathcal{F}$ an $\mathcal{O}_Y$-module. Prove the *projection formula*

$$f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}) \simeq \mathcal{E} \otimes_{\mathcal{O}_X} f_*\mathcal{F}.$$

Exercise 5. Let X be a scheme, and $\pi\colon \mathbb{P}^n_X \to X$.

(i) Show that $\pi_*\mathcal{O}_{\mathbb{P}^n_X} = \mathcal{O}_X$.

(ii) Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module. Show that there is a locally free coherent $\mathcal{O}_{\mathbb{P}^n_X}$-module $\mathcal{F}$ such that $\pi_*\mathcal{F} = \mathcal{E}$ (Hint: use the projection formula).