The letter k denotes an algebraically closed field. We denote the polynomial ring $k[X_0, \cdots, X_n]$ by S, and by S_d its homogeneous component of degree d.

Exercise 1. Let $I \subset S$ be a homogeneous ideal. Show that the radical $\sqrt{I}$ is a homogeneous ideal.

Exercise 2. Let $I \subset S$ be a homogeneous ideal. Show that the following conditions are equivalent:

(a) $Z_h(I) = \varnothing \subset \mathbb{P}^n$.

(b) The ideal $\sqrt{I}$ is either equal to S or to $S_+ = \bigoplus_{d>0} S_d$.

(c) There is an integer d such that $S_d \subset I$.

Exercise 3. (i) Show that an algebraic set Y of $\mathbb{P}^n$ is irreducible if and only if its homogeneous ideal $I(Y) \subset S$ is prime.

(ii) Let $f \in S$ be an irreducible homogeneous polynomial. Show that $Z_h(f) \subset \mathbb{P}^n$ is irreducible.

(iii) Show that $\mathbb{P}^n$ is irreducible.

Exercise 4. Let f, g be two elements of $S_1 - \{0\}$. Assume that $Z_h(f) \neq Z_h(g)$. Show that $Z_h(f) \cap Z_h(g) \subset \mathbb{P}^n$ is a linear subspace $\mathbb{P}^{n-2}$ (in other words: find an element of $GL_{n+1}(k)$ such that the induced bijection of $\mathbb{P}^n$ sends $Z_h(f) \cap Z_h(g)$ to $Z_h(X_n, X_{n-1})$).

Exercise 5. (i) Assume that $n = 1$ and $a \in S_d - \{0\}$ with $d \geq 1$. Show that the cardinality of the set $Z_h(a) \subset \mathbb{P}^1$ is between 1 and d.

(ii) Assume that $n = 2$. Let $f \in S_d - \{0\}$ with $d \geq 1$ and $g \in S_1 - \{0\}$. If $Z_h(g) \subset Z_h(f) \subset \mathbb{P}^2$, show that $g \mid f$.

(iii) Assume that $n = 2$. Let $f \in S_d - \{0\}$ with $d \geq 1$ and $g \in S_1 - \{0\}$. If $Z_h(g) \not\subset Z_h(f) \subset \mathbb{P}^2$, show that the cardinality of the set $Z_h(f) \cap Z_h(g)$ is between 1 and d.