Exercise 1. Let L/K be a field extension of degree n. We assume that L/K is separable, which in this exercise means the following: there is an algebraically closed field Ω containing K and n distinct K-algebra morphisms $\sigma_i\colon L \to \Omega$. For an element $x \in L$, we denote by m_x the K-linear endomorphism of L induced by multiplication with x, by $\chi(m_x) \in K[X]$ its characteristic polynomial, and $\mathrm{Tr}(m_x) \in K$ its trace.

(i) Let $x \in L$. Show that the image of $\chi(m_x)$ in $\Omega[X]$ is $\prod_{i=1}^n (X - \sigma_i(x))$. (Hint: Reduce to the case when the extension L/K is generated by x.)

(ii) Show that a family $x_1, \ldots, x_n \in L$ is a K-basis if and only the matrix $(\mathrm{Tr}(m_{x_i x_j}))_{i,j}$ is invertible. (Hint: Use that the set of ring morphisms $L \to \Omega$ is linearly independent over Ω.)

Exercise 2. Let k be a field. Consider the polynomial ring $A = k[X_1, \ldots, X_n]$ and its fraction field $K = k(X_1, \ldots, X_n)$. Let L be a finite purely inseparable field extension of K. Show that the integral closure of A in L is an A-module of finite type. (Hint: find a finite purely inseparable extension k' of k such that L is a subfield of $k'(Y_1, \ldots, Y_n)$ where $Y_i^q = X_i$ for an appropriate p-th power q. Reduce to the case $L = k'(Y_1, \ldots, Y_n)$.)

Exercise 3. Let p be prime number and k a field. Consider an integer $n > 1$ prime to p, and let $R = k[X, Y]/(X^p - Y^n)$.

(i) Let K be a field and $a \in K$. We assume that there is no $b \in K$ such that $a = b^p$. Show that the polynomial $X^p - a \in K[X]$ is irreducible. (Hint: Let $Q \in K[X]$ be a non-trivial factor of $X^p - a$, and consider the endomorphism α of the K-vector space $K[X]/Q$ induced by multiplication with X. Compute $\det(\alpha)^p \in K$.)

(ii) Show that the ring R is an integral domain.

(iii) We consider the unique k-algebra morphism $\varphi\colon R \to k[T]$ such that $\varphi(X) = T^n$ and $\varphi(Y) = T^p$. Show that φ is injective (hint: Localise at the set of powers of X), and identifies $k[T]$ with the integral closure of R in its fraction field.

(iv) Show that there is a unique prime $\mathfrak{p} \in \operatorname{Spec} R$ such that $R_{\mathfrak{p}}$ is not a discrete valuation ring. (Hint: What is the normalisation of $\operatorname{Spec} R$?)