Fabien Morel
Olivier Haution

Algebraic Geometry I
Exercises 2

09.11.2016

The letter k denotes an algebraically closed field.

Exercise 1. Let X be a topological space.

(i) If Y is a subset of X, show that $\dim Y \leq \dim X$.

(ii) Assume that $X = \bigcup_{\alpha \in A} U_\alpha$ with U_α an open subset of X for each $\alpha \in A$. Show that $\dim X = \sup_{\alpha \in A} \dim U_\alpha$.

Exercise 2. Given $f \in k[x, y]$, we denote by $\varphi_f \colon \mathbb{A}^2_k \to \mathbb{A}^1_k$ the map $(u, v) \mapsto f(u, v)$.

(i) For $t \in k = \mathbb{A}^1_k$, recall why $\varphi_f^{-1}\{t\}$ is an algebraic set in $\mathbb{A}^2_k$.

(ii) Find f such that $\varphi_f^{-1}\{t\}$ is irreducible for all $t \in k$.

(iii) Find f such that $\varphi_f^{-1}\{t\}$ is irreducible for all $t \in k - \{0\}$ but $\varphi_f^{-1}\{0\}$ is not irreducible.

Exercise 3. (i) Let R be a principal ideal domain. Show that $\dim R \in \{0, 1\}$.

(ii) Let $Y = V(y - x^2) \subset \mathbb{A}^2_k$. Show $A(Y)$ is a polynomial ring in one variable over k.

(iii) Let $Z = V(xy - 1) \subset \mathbb{A}^2_k$. Show $A(Z)$ is not a polynomial ring in one variable over k.

(iv) Show that Y and Z are irreducible, and compute their dimensions.

Exercise 4. Consider the map $\varphi \colon \mathbb{A}^1_k \to \mathbb{A}^2_k$ given by $t \mapsto (t^2, t^3)$. Show that $\varphi(\mathbb{A}^1_k)$ is an irreducible closed subset $Z \subset \mathbb{A}^2_k$, and that the induced map $\mathbb{A}^1_k \to Z$ is bijective.