GALOIS COHOMOLOGY
EXERCISES 7 (PROFINITE GROUPS)

Exercise 1. Let us fix a prime number p.

(i) Let G be a profinite group, and $P \subset G$ a pro-p-Sylow subgroup. Show that:
- for every normal open subgroup U of G containing P, the group G/U has finite order prime to p,
- if $H \subset P$ is a closed subgroup of finite order in P, then $[P : H]$ is a power of p.

(ii) Let k be a field. Show that there exists a separable field extension F/k having the following properties:
- every finite subextension L/k of F/k has degree prime to p,
- the degree of every finite separable extension of F is a power of p.

Exercise 2. Recall that a topological space is called *Hausdorff* if any two distinct points are contained in disjoint opens subsets.

(i) Let Γ be a profinite group. We have seen that Γ is compact. Show that Γ is Hausdorff and that every open subset of Γ containing 1 contains an open normal subgroup.

Let now G be a compact and Hausdorff topological group. We assume that every open subset of G containing 1 contains an open normal subgroup. We are going to show that G is profinite. Let $\mathcal{U}$ be the set of open normal subgroups of G, ordered by setting $U \leq V$ when $V \subset U$.

(ii) Show that the groups G/U for $U \in \mathcal{U}$ form an inverse system, that the group $H = \varprojlim G/U$ is profinite and that the natural morphism $f : G \to H$ is continuous.

(iii) Show that f is injective.

(iv) Show that the image of f is dense (i.e. meets every nonempty open subset of H).

(v) Conclude that $f : G \to H$ is a homeomorphism.

Exercise 3. A topological space all of whose connected subsets are singletons is called *totally disconnected*. We are going to prove that a topological space is profinite if and only it is compact, Hausdorff, and totally disconnected.

(i) Show that a profinite set is Hausdorff and totally disconnected.

Let now X be a compact, Hausdorff, and totally disconnected topological group. Let Ω be the set of open subsets of X. Let $\mathcal{F}$ be the set of finite subsets F of Ω such that $X = \coprod_{U \in \mathcal{F}} U$. We order $\mathcal{F}$ by setting $F \leq F'$ if each element of F' is contained in some element of F. In this case we have a map of finite discrete spaces $F' \to F$.

(ii) Show that the elements $F \in \mathcal{F}$ form an inverse system (indexed by $\mathcal{F}$), and that its inverse limit Y is profinite. Show that there is a natural continuous map $f : X \to Y$.

(iii) Show that f is injective.
(iv) Show that the image of f is dense.
(v) Conclude that $f\colon X \to Y$ is a homeomorphism.