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Algebraic Geometry II
Exercises 5

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The letter k denotes a field.

Exercise 1. Let $k \subset K$ be a field extension of finite type and of transcendence degree one. Let A be a valuation ring of K containing k. Show that if $A \neq K$, then A is a discrete valuation ring.

Exercise 2. Let $k \subset K$ be a field extension of finite type and of transcendence degree one, and X_K the curve constructed in the lecture (closed points of X_K correspond to valuation rings of K containing k). Show that X_K is proper over k using the valuative criterion.

Exercise 3. We consider the category $\mathcal{B}$ whose objects are finite type integral schemes over k, and morphisms are dominant rational maps. Show that $\mathcal{B}$ is equivalent to the opposite of the category of finite type field extensions of k.

Exercise 4. (i) Let X be an integral scheme, proper over k. Show that the ring extension $k \to \mathcal{O}_X(X)$ is integral (hint: for $f \in \mathcal{O}_X(X)$, what can be the image of the composite morphism $X \xrightarrow{\varphi_f} \mathbb{A}^1_k \to \mathbb{P}^1_k$?).

(ii) Show that k is algebraically closed in $k(T)$.

(iii) Deduce that $\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) = k$.

(iv) (optional) Show that $\mathcal{O}_{\mathbb{P}^n_k}(\mathbb{P}^n_k) = k$.