GALOIS COHOMOLOGY
EXERCISES 7 (PROFINITE GROUPS)

The solutions will be discussed during the online session on Dec 15th.

Exercise 1. Show that a closed subset of a profinite set is profinite.

Exercise 2. Let p be a prime number and Γ a pro-p-group. The purpose of this exercise is to prove that the index of every subgroup of Γ is a power of p, if it is finite.

Let $n \in \mathbb{N}$, and write $n = p^r m$, where m is prime to p and $r \in \mathbb{N}$.

(i) Consider the subset $C_n = \{g^n | g \in \Gamma\}$. Show that C_n is closed in Γ.

Let now $g \in \Gamma$. Let U be an open normal subgroup of Γ.

(ii) Show that $g^{p^s} \in U$ for some $s \geq r$.
(iii) Show that $g^{p^r} \in C_n U$. (Hint: Write $p^r = ap^s + bn$, with $a, b \in \mathbb{Z}$.)
(iv) Deduce that $g^{p^r} \in C_n$.
(v) Let $H \subset \Gamma$ be a normal subgroup of index n. Show that $C_n \subset H$, and deduce that Γ/H is a finite p-group.
(vi) Conclude.

Exercise 3. Let F/k be a Galois extension. Let $H \subset \mathrm{Gal}(F/k)$ be a subgroup, and $\overline{H}$ its closure. Show that $\overline{H}$ is a subgroup, and that $F^H = F^{\overline{H}}$.

Exercise 4. Let us fix a prime number p.

(i) Let G be a profinite group, and $P \subset G$ a pro-p-Sylow subgroup. Show that:
  - for every normal open subgroup U of G containing P, the group G/U has finite order prime to p,
  - if $H \subset P$ is a closed subgroup of finite index in P, then $[P : H]$ is a power of p.

(ii) Let k be a field. Show that there exists a separable field extension F/k having the following properties:
  - every finite subextension L/k of F/k has degree prime to p,
  - the degree of every finite separable extension of F is a power of p.

Exercise 5. Recall that a topological space is called *Hausdorff* if any two distinct points are contained in disjoint opens subsets.

(i) Let Γ be a profinite group. We have seen that Γ is compact. Show that Γ is Hausdorff and that every open subset of Γ containing 1 contains an open normal subgroup.

Let now G be a compact and Hausdorff topological group. We assume that every open subset of G containing 1 contains an open normal subgroup. We are going to show that G is profinite. Let $\mathcal{U}$ be the set of open normal subgroups of G, ordered by setting $U \leq V$ when $V \subset U$.

(ii) Show that the groups G/U for $U \in \mathcal{U}$ form an inverse system, that the group $H = \varprojlim G/U$ is profinite and that the natural morphism $f \colon G \to H$ is continuous.

(iii) Show that f is injective.

(iv) Show that the image of f is dense (i.e. meets every nonempty open subset of H).

(v) Show that f is closed (i.e. $f(Z)$ is closed in H whenever Z is closed in G).

(vi) Conclude that $f \colon G \to H$ is a homeomorphism.