

GALOIS COHOMOLOGY

EXERCISES 12

Let k be a field.

Exercise 1. Let $n \geq 1$ be an integer, and $\omega \in k$ a root of unity of order n . Let $a, b \in k^\times$. Consider the Galois \mathbb{Z}/n -algebra $R_a = k[X]/(X^n - a)$, where $i \in \mathbb{Z}/n$ acts by $X \mapsto \omega^i X$. Up to isomorphism R_a depends only on the class of a in $k^\times/k^{\times n}$ (and on n and the choice of ω). Let us denote the cyclic algebra (R_a, b) by $(a, b)_\omega$.

- (i) Show that $(a, b)_\omega \simeq ((b, a)_\omega)^{\text{op}}$.
- (ii) If $a \neq 1$, show that $(1 - a, a)_\omega \simeq M_n(k)$.

We define the extension $K = k(\sqrt[n]{a})$ as the splitting field of the polynomial $X^n - a \in k[X]$.

- (iii) Show that $R_a \simeq K \times \cdots \times K$ as k -algebras.
- (iv) Show that $N_{R_a/k}(R_a^\times) = N_{K/k}(K^\times)$ in k^\times .
- (v) Prove the “reciprocity law”:

$$a \in N_{k(\sqrt[n]{b})/k}(k(\sqrt[n]{b})) \iff b \in N_{k(\sqrt[n]{a})/k}(k(\sqrt[n]{a})).$$

Exercise 2. (Question (ii) uses §10.1.1 in the lectures; please skip it in case we are not there yet, the remaining questions can be treated before if you admit (ii).)

Let \bar{k} be an algebraic closure of k . We first assume that \bar{k}/k is finite of prime order p , where p is unequal to the characteristic of k .

- (i) Show that the extension \bar{k}/k is generated by an element α such that $a = \alpha^p \in k$.
- (ii) (See the remark at the beginning of the exercise.) Show that $\text{Br}(\bar{k}/k) \simeq H^2(k, \mathbb{Z}/p)$ and $k^\times/k^{\times p} \simeq H^1(k, \mathbb{Z}/p)$, and that each of these groups is isomorphic to \mathbb{Z}/p . (Hint: Use the computation of the cohomology of finite cyclic groups.)
- (iii) Deduce that $N_{\bar{k}/k}(\bar{k}^\times) = k^{\times p}$.
- (iv) Show that $N_{\bar{k}/k}(\alpha) = (-1)^{p-1}a$.
- (v) Deduce that $p = 2$, that -1 is not a square in k , and that $\bar{k} \simeq k[X]/(X^2 + 1)$.

We now assume that \bar{k}/k is finite (of possibly nonprime order) and that k has characteristic zero.

- (vi) Assume that -1 is a square in k . Show that $k = \bar{k}$.
- (vii) Assume that -1 is not a square in k . Show that $\bar{k} \simeq k[X]/(X^2 + 1)$.