EXERCISES 3 (INTERSECTION THEORY)

Exercise 1. Let A be a domain with fraction field K, and L a finite field extension of K. Show that the integral closure of A in L is of finite type over A under any of the following assumptions:

(i) A is integrally closed (in K), and L/K is separable.
(ii) $A = k[t_1, \cdots, t_n]$ and L/K purely inseparable.
(iii) $A = k[t_1, \cdots, t_n]$.
(iv) A is a k-algebra of finite type.

Exercise 2. Let A be a domain with fraction field K. Show that A is integrally closed (in K) if and only if $A_{\mathfrak{p}}$ is integrally closed (in K) for every $\mathfrak{p} \in \operatorname{Spec} A$.

A variety is called *normal* if the local rings $\mathcal{O}_{X,x}$ are integrally closed (in $K(X)$) for all points x of X.

Exercise 3. Let X be an integral variety, and $L/k(X)$ a finite field extension.

(i) Show that there is a finite morphism of varieties $n \colon X' \to X$ with $k(X') = L$ as $k(X)$-algebras and X' normal, with the following property. For any dominant morphism $f \colon Y \to X$ with Y a normal and integral variety, and any morphism of $k(X)$-algebras $L \subset k(Y)$, there is a unique dominant morphism $g \colon Y \to X'$ such that $f = n \circ g$. The morphism n is the *normalisation of X in L*.
(ii) Let $f \colon Y \to X$ be a finite dominant morphism with Y an integral and normal variety, and $k(Y) = L$ as $k(X)$-algebras. Show that f is the normalisation of X in L.

Exercise 4. Let A be a noetherian local domain, which is integrally closed. Assume that the maximal ideal of A is an associated prime of the module A/aA, for some $a \in A$. Show that A has dimension one.

Exercise 5. Let X be a normal, integral, variety. Let D, D' be two effective Cartier divisors without common irreducible components. Show that $D \cap D' \to D$ is an effective Cartier divisor.