GALOIS COHOMOLOGY
EXERCISES 1 (TENSOR PRODUCT)

Let k be a field and U, V be k-vector spaces. Let us denote by F the k-vector space having as basis the set $U \times V$; it consists of k-linear combinations of elements of type (u, v) with $u \in U$ and $v \in V$. Consider the subspace R of F generated by the following elements

$$(u + \lambda u', v + \mu v') - (u, v) - \lambda(u', v) - \mu(u, v') - \lambda\mu(u', v'),$$

where $u, u' \in U$ and $v, v' \in V$ and $\lambda, \mu \in k$. The quotient k-vector space F/R will be denoted by $U \otimes_k V$, and the image of (u, v) by $u \otimes v \in U \otimes_k V$.

Exercise 1. Let W be another k-vector space and $\varphi\colon U \times V \to W$ a map. Assume that φ is k-bilinear, i.e. that for all $u \in U$ the map $V \to W$ given by $v \mapsto \varphi(u, v)$ is k-linear and for all $v \in V$ the map $U \to W$ given by $u \mapsto \varphi(u, v)$ is k-linear. Show that there is a unique k-linear map $U \otimes_k V \to W$ sending $u \otimes v$ to $\varphi(u, v)$ for all $u \in U$ and $v \in V$.

Exercise 2.
- (i) Assume that the elements e_α for $\alpha \in A$ form a basis of U, and that f_β for $\beta \in B$ form a basis of V. Show that the elements $e_\alpha \otimes f_\beta$ for $(\alpha, \beta) \in A \times B$ form a basis of $U \otimes_k V$. (Hint: for linear independence, use the dual basis to (e_α) and (f_β) to define linear forms $U \otimes_k V \to k$.)
- (ii) If $U \neq 0$ and $V \neq 0$, show that $U \otimes_k V \neq 0$.
- (iii) Assume that $\dim_k U = m < \infty$ and that $\dim_k V = n < \infty$. What is the dimension of $U \otimes_k V$?

Exercise 3. Let $f\colon U \to U'$ and $g\colon V \to V'$ be k-linear maps.
- (i) Show that there is a unique k-linear map

  $$f \otimes g\colon U \otimes_k V \to U' \otimes_k V'$$

  such that

  $$(f \otimes g)(u \otimes v) = f(u) \otimes g(v) \quad \text{for all } u \in U \text{ and } v \in V.$$
- (ii) Assume that f and g are surjective. Show that $f \otimes g$ is surjective.
- (iii) Assume f and g are injective. Show that $f \otimes g$ is injective.

Exercise 4. Assume that A, B are k-algebras. Show that $A \otimes_k B$ is naturally a k-algebra.

Exercise 5. When $f\colon U \to V$ and $g\colon V \to W$ are k-linear maps, we say that the sequence

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

is exact if f is injective, g is surjective, and $\ker g = \operatorname{im} f$. If F is a k-vector space, show that the induced sequence

$$0 \to U \otimes_k F \xrightarrow{f\otimes \mathrm{id}} V \otimes_k F \xrightarrow{g\otimes \mathrm{id}} W \otimes F \to 0$$

is exact.

Exercise 6. If $U' \subset U', V' \subset V$ are subspaces, by Exercise 3 (iii) we may view $U' \otimes_k V'$ as a subspace of $U \otimes_k V$.

(i) Assume that U_α for $\alpha \in A$ are subspaces of U such that $U = \bigoplus_{\alpha\in A} U_\alpha$. Show that
$$U \otimes_k V = \bigoplus_{\alpha\in A} (U_\alpha \otimes V).$$

(ii) Let U', U'' be subspaces of U. Show that, in $U \otimes_k V$,
$$(U' \otimes_k V) \cap (U'' \otimes_k V) = (U' \cap U'') \otimes_k V.$$

(iii) If $U' \subset U$ and $V' \subset V$ are subspaces, show that, in $U \otimes_k V$,
$$(U' \otimes_k V) \cap (U \otimes_k V') = U' \otimes_k V'.$$