GALOIS COHOMOLOGY
EXERCISES 4–5 (SIMPLE RINGS)

Exercise 1. Prove the following converse of Wedderburn's Theorem: If D is a division ring and $n \geq 1$ an integer, then the ring $M_n(D)$ is artinian simple.

Exercise 2. In Proposition 1.3.5, we proved the following statement : if Q, Q' are quaternion algebras over a field k (of characteristic $\neq 2$), then

$$Q \otimes_k Q' \simeq M_4(k) \iff Q \simeq Q'.$$

The proof of "$\Longleftarrow$" was easy, while the proof of "$\Longrightarrow$" was comparatively difficult (in particular used Albert's Theorem). Give a new (short) proof of "$\Longrightarrow$", using "$\Longleftarrow$" and the results of §2.1 in the lecture notes.

Exercise 3. (i) Show that every nonzero ring admits a simple module.

(ii) Let R be a ring, and M a nonzero R-module. Show that there is a submodule N of M and a quotient S of N such that S is simple.

Exercise 4. Let D be a finite-dimensional central division k-algebra, and n an integer. Show that $M_n(k)$ contains a k-subalgebra isomorphic to D if and only if $\dim_k D \mid n$.

Exercise 5. Let R be a ring and M an R-module. We are going to prove that the following conditions are equivalent:

(a) The module M is generated by its simple submodules.
(b) The module M is a direct sum of simple R-modules.
(c) Every submodule of M is a direct summand.

The R-module M will be called *semisimple* if it satisfies the above conditions.

(i) Let $S_i \to M$ for $i \in I$ be a collection of morphisms of R-modules, where each S_i is a simple module. When $K \subset I$, let us write $S_K = \bigoplus_{i \in K} S_i$, and denote by N_K the kernel of $S_K \to M$. Using Zorn's lemma, show that there is a maximal subset $K \subset I$ such that $N_K = 0$.

(ii) In the situation of (i), show that $S_I \to M$ and $S_K \to M$ have the same image.

(iii) Prove that (a) $\Longrightarrow$ (b).

(iv) Prove that (b) $\Longrightarrow$ (c). (Hint: use (i) and (ii) for an appropriate collection of morphisms $S_i \to Q$.)

For the rest of the exercise, we assume that (c) holds, and prove (a). So we let M' be the submodule of M generated by the simple submodules of M, and choose a submodule M'' such that $M' \oplus M'' = M$. We assume that $M'' \neq 0$ and come to a contradiction. By Exercise 3, we know that there are submodules $P \subset N \subset M''$ such that N/P is simple.

(v) Show that N/P is isomorphic to a submodule of N.

(vi) Conclude that (c) $\Longrightarrow$ (a).

Exercise 6. A ring is called *semisimple* if it is semisimple as a module over itself (see the previous exercise). Prove the following assertions:

(i) Every semisimple ring is a finite direct sum of simple modules.
(ii) Every semisimple ring is artinian.
(iii) Every artinian simple ring is semisimple.
(iv) Every semisimple ring is isomorphic to a product $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, where $D_1, \ldots, D_r$ are division rings and $n_1, \ldots, n_r$ are integers.
(v) The product of two semisimple rings is semisimple.
(vi) A ring is semisimple if and only if it is a finite product of artinian simple rings.

Exercise 7. Let D be a division ring of positive characteristic (i.e. there is a prime number p such that $pD = 0$.) Show that every finite subgroup of $D^\times$ is cyclic. (Hint: you may use the fact that every subgroup of $k^\times$ is cyclic when k is a finite field).