GALOIS COHOMOLOGY
EXERCISES 9 (TORSORS)

Let k be a field.

Exercise 1. Let k_s be a separable closure of k, and $\Gamma = \mathrm{Gal}(k_s/k)$. Let A be an étale k-algebra of dimension n. Consider the associated Γ-set $X = \mathrm{Hom}_{k-\mathrm{alg}}(A, k_s)$. Let $Y \subset X^n$ be the set of those $(x_1, \ldots, x_n)$ such that $x_i \neq x_j$ when $i \neq j$, with the Γ-action given by

$$\gamma(x_1, \ldots, x_n) = (\gamma x_1, \ldots, \gamma x_n) \quad \text{for } \gamma \in \Gamma, \text{ and } x_1, \ldots, x_n \in X.$$

The symmetric group $\mathfrak{S}_n$ acts on Y by

$$\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Denote by Z the quotient of Y be the action of the subgroup $\mathfrak{A}_n$ of even permutations (the kernel of the signature morphism $\mathfrak{S}_n \to \mathbb{Z}/2$).

(i) Show that Z is a Γ-set having two elements.

We denote by Δ the corresponding étale k-algebra of dimension two; it is called the *discriminant algebra* of A.

Assume that k has characteristic $\neq 2$. Let $e_1, \ldots, e_n$ be a k-basis of A, let $f_1, \ldots, f_n$ be the elements of X, and consider the matrix $M = (f_i(e_j)) \in M_n(k_s)$. Set $u = \det M \in k_s$. Let Γ_0 be the subgroup of Γ consisting of those elements acting by even permutations on the set X.

(ii) Let $\gamma \in \Gamma$. Show that $\gamma u = u$ if $\gamma \in \Gamma_0$ and $\gamma u = -u$ otherwise.

Let d be the determinant of the matrix $(\mathrm{Tr}_{A/k}(e_i e_j)) \in M_n(k)$.

(iii) Show that $d = u^2$. (Hint: compute the product $M^t \cdot M$.)

(iv) Conclude that $\Delta = k[X]/(X^2 - d)$.

Exercise 2. Let G be a finite group. Let $H \subset G$ be a subgroup and B a H-algebra over k. Consider the set

$$\mathrm{Ind}_H^G B = \{\text{maps } f \colon G \to B \text{ such that } f(h \cdot g) = h \cdot f(g) \text{ for all } g \in G, h \in H\},$$

viewed as a k-algebra, via pointwise operations on B.

(i) Show that the k-algebra $\mathrm{Ind}_H^G B$ is étale if and only if B is étale

If $f \in \mathrm{Ind}_H^G B$ and $g \in G$, we define an element $g \cdot f \in \mathrm{Ind}_H^G B$ by mapping a $x \in G$ to $f(x \cdot g)$. This gives $\mathrm{Ind}_H^G B$ the structure of a G-algebra.

(ii) Show that the H-algebra B is Galois over k if and only if the G-algebra $\mathrm{Ind}_H^G B$ is Galois over k.

(iii) Let A be a Galois G-algebra over k. Show that there exists a subfield $L \subset A$, which is Galois field extension of k, a subgroup $H \subset G$ isomorphic to $\mathrm{Gal}(L/k)$, and an isomorphism of G-algebras $A \simeq \mathrm{Ind}_H^G L$.

Exercise 3. Let Γ be a profinite group, and $A \to B$ be a morphism of Γ-groups. Describe the map $H^1(\Gamma, A) \to H^1(\Gamma, B)$ in terms of torsors (as opposed to 1-cocyles).