The topological space underlying a scheme X is denoted by X_{top}.

Exercise 1. Let $f\colon X \to S$ and $g\colon Y \to S$ be two scheme morphisms. Consider the map

$$\varphi\colon (X \times_S Y)_{top} \to X_{top} \times Y_{top}$$

induced by the two projection morphisms $X \times_S Y \to X$ and $X \times_S Y \to Y$.

(i) Let $x \in X_{top}$ and $y \in Y_{top}$ be such that $f(x) = g(y) = s \in S_{top}$. Show that there is a homeomorphism

$$\varphi^{-1}\{(x, y)\} \simeq (\operatorname{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)))_{top}.$$

(ii) What is the image of φ?

Exercise 2. Let S be an affine scheme and $X \to S$ a separated morphism (i.e. the diagonal morphism $(\mathrm{id}_X, \mathrm{id}_X)\colon X \to X \times_S X$ is a closed immersion). Show that the intersection of two affine open subschemes of X is affine.

Exercise 3. Let X be a scheme and $\mathcal{Z}$ a closed subset of X_{top}. The purpose of this exercise it to prove the existence of a reduced scheme Z and a closed immersion $f\colon Z \to X$ inducing a homeomorphism $(Z)_{top} \to \mathcal{Z}$.

(i) Show that Z and f exist when X is affine.

(ii) Assume that Z and f exist. Show that for every morphism $g\colon T \to X$ with T reduced and $g(T_{top}) \subset \mathcal{Z}$, there exists a unique morphism $h\colon T \to Z$ such that $g = f \circ h$. [Hint: Begin with the case T affine.]

(iii) For $n = 1, 2$ let Z_n be a reduced scheme and $f_n\colon Z_n \to X$ a closed immersion inducing a homeomorphism $(Z_n)_{top} \to \mathcal{Z}$. Show that there exists a unique isomorphism $\varphi\colon Z_1 \to Z_2$ such that $f_1 = f_2 \circ \varphi$.

(iv) Conclude.

Exercise 4. Let $A \to B$ be a ring morphism making B a free A-module of rank n. Show that the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is open.