

GALOIS COHOMOLOGY EXERCISES 3 (QUATERNIONS)

Let k be a field of characteristic $\neq 2$.

Exercise 1. Show that every quaternion algebra can be realised as a subalgebra of $M_4(k)$.

Exercise 2. Let $k = \mathbb{Q}$, and consider that quaternion algebra $Q = (-1, -1)$ over k . Let $K = k(\xi)$ where $\xi \in \mathbb{C}$ is a primitive 5-th root of 1.

- (i) Show that Q_K splits. (Hint : compute $-(\xi^3 + \xi^2)^2 - (\xi - \xi^4)^2$.)
- (ii) Determine the subfields of K .
- (iii) Deduce that K contains no quadratic extension splitting Q .

Exercise 3. Show that every element of a quaternion k -algebra satisfies a quadratic equation over k .

Exercise 4. Let D be a division k -algebra. We assume that for every $d \in D$, there is a nonzero polynomial $P \in k[X]$ of degree ≤ 2 such that $P(d) = 0$. We are going to prove that one of the following must happen:

- $D = k$,
- D is a quadratic field extension of k ,
- D is a quaternion k -algebra.

Let us assume that $D \neq k$.

- (i) Show that there is $i \in D - k$ and $a \in k$ such that $i^2 = a$.
- (ii) Let K be the k -subalgebra of D generated by i . Show that K is a field.
- (iii) Let $\varphi: D \rightarrow D$ be the map $d \mapsto i^{-1}di$. Show that $\varphi^2 = \text{id}$, and that $D = D_+ \oplus D_-$ as K -vector spaces, where $D_+ = \ker(\varphi - \text{id})$, $D_- = \ker(\varphi + \text{id})$.
- (iv) Show that D_+ is a K -subalgebra of D .
- (v) Let $\alpha \in D_+$ and F the K -subalgebra of D_+ generated by α . Show that F is a field such that $[F : K] \leq 2$.
- (vi) Show that $\alpha \in K$. (Hint: use the minimal polynomials of α and $\alpha + i$ to construct a linear equation over K having α as a solution.)
- (vii) Deduce that $D_+ = K$.
- (viii) Let now $\beta, \beta' \in D_-$. Show that $\beta\beta' \in D_+$, and deduce that $\dim_K D_- \in \{0, 1\}$.
- (ix) Assume that $\dim_K D_- = 1$, and let j be a nonzero element of D_- . Let $A \in k[X]$ be a nonzero polynomial of degree ≤ 2 such that $A(j) = 0$. Show that $A(-j) = 0$, and deduce that $j^2 \in k$.
- (x) Conclude.