EXERCISES 1 (INTERSECTION THEORY)

Let A be a noetherian commutative ring with unit, and M a finitely generated A-module.

Exercise 1. The length function is additive.

Exercise 2. The length of any maximal (i.e. saturated) chain of submodules of M is equal to the length of M.

A prime $\mathfrak{p}$ of A is *associated with* M if there is an element $m \in M$ such that $\mathfrak{p} = \operatorname{Ann}(m) = \{x \in A | xm = 0\}$. We write $\operatorname{Ass}(M)$ for the set of associated primes of M.

Exercise 3.
(i) We have $\mathfrak{p} \in \operatorname{Ass}(M)$ if and only if M contains a submodule isomorphic to $A/\mathfrak{p}$.
(ii) Let I be a maximal element of the set $\{\operatorname{Ann}(m) | m \in M - \{0\}\}$. Then I is a prime ideal.
(iii) We have $M = 0$ if and only if $\operatorname{Ass}(M) = \varnothing$.
(iv) Let $\mathfrak{p}$ be a prime of A. Then $\operatorname{Ass}(A/\mathfrak{p}) = \{\mathfrak{p}\}$.

Exercise 4. Consider an exact sequence of finitely generated A-modules

$$0 \to M' \to M \to M'' \to 0.$$

Then $\operatorname{Ass}(M') \subset \operatorname{Ass}(M) \subset \operatorname{Ass}(M') \cup \operatorname{Ass}(M'')$.

Exercise 5. There is a chain of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that $M_i/M_{i-1} \simeq A/\mathfrak{p}_i$ with $\mathfrak{p}_i$ prime, for $i = 1, \cdots, n$. We have

$$\operatorname{Ass}(M) \subset \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}.$$

Exercise 6. Assume that A is local. Then the following are equivalent
(i) $l_A(M) < \infty$.
(ii) There is $n \in \mathbb{N}$ such that $(\mathfrak{m}_A)^n M = 0$.
(iii) We have $\dim M \leq 0$.

Exercise 7. Consider an exact sequence of finitely generated A-modules

$$0 \to M' \to M \to M'' \to 0.$$

Then $\operatorname{Supp}(M) = \operatorname{Supp}(M') \cup \operatorname{Supp}(M'')$.

Exercise 8. Show that the primes $\mathfrak{p}_i$ of Exercise 5 belong to $\operatorname{Supp}(M)$.

Exercise 9. Let $\mathfrak{p} \in \operatorname{Spec} A$. We view $\operatorname{Spec} A_{\mathfrak{p}}$ as a subset of $\operatorname{Spec} A$. Then

$$\operatorname{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = (\operatorname{Spec} A_{\mathfrak{p}}) \cap \operatorname{Ass}(M).$$

Exercise 10. We have $\operatorname{Ass}(M) \subset \operatorname{Supp}(M)$, and these sets have the same minimal elements.

Exercise 11. The set $\operatorname{Ass}(M)$ is finite, and so is the set of minimal primes in $\operatorname{Supp}(M)$.