Exercise 1. If A, B are two local subrings of a given field, with respective maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$, we say that B *dominates* A if $A \subset B$ and $\mathfrak{m}_A \subset \mathfrak{m}_B$.

Let R be a local domain with fraction field K, and X a scheme. Show that to give a morphism $\operatorname{Spec} R \to X$ is equivalent to giving:

- a pair of points z, y such that y is in the (reduced) closure Z of $\{z\}$,
- and an inclusion $k(z) \subset K$ such that R dominates $\mathcal{O}_{Z,y}$ (as subrings of K).

Exercise 2. Show that a morphism $X \to S$ of schemes is separated if and only if the subset $\Delta_{X/S}(X)$ is closed in $X \times_S X$.

Exercise 3. (i) Let $R \to S$ be an injective ring morphism. Show that every minimal prime of R is in the image of $\operatorname{Spec} S \to \operatorname{Spec} R$.

(ii) Let $f \colon X \to Y$ be a quasi-compact morphism of schemes. Assume that the closure of every point of $f(X)$ is contained in $f(X)$. Show that $f(X)$ is closed in Y.

Exercise 4. Prove the valuative criterion of separatedness. You may use without proof the following result:

Let A be a local domain with fraction field K. Then there is a valuation ring R of K which dominates A.

Exercise 1. Let X be a scheme and n an integer. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n.

(i) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \simeq \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}$.

(ii) Let $f\colon Y \to X$ be a morphism of schemes and $\mathcal{G}$ an $\mathcal{O}_Y$-module. Show that $f_*(\mathcal{G} \otimes_{\mathcal{O}_Y} f^*\mathcal{E}) \simeq (f_*\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{E}$.

Exercise 2. (i) Let A be a noetherian ring, and M an A-module of finite type. Let r be an integer. Assume that the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is free of rank r for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Show that the A-module M is locally free of rank r.

(ii) Let X be a regular curve over a field, and Z a closed subscheme of X satisfying $Z \neq X$. Show that the sheaf of ideals $\mathcal{I}_Z$ is a locally free $\mathcal{O}_X$-module of rank one.

Exercise 3. Let X be a scheme and $\mathcal{L}$ a locally free $\mathcal{O}_X$-module of rank one. Let $s \in H^0(X,\mathcal{L})$, and consider the set X_s consisting of those points $x \in X$ such that s_x generates the $\mathcal{O}_{X,x}$-module $\mathcal{L}_x$.

(i) Show that X_s is an open subset of X.

(ii) Show that the induced open immersion of schemes $X_s \to X$ is an affine morphism.

Exercise 4. Let X be a scheme. Show that the morphisms $X \to \mathbb{P}^n$ are in bijective correspondence with the data of:

- a locally free $\mathcal{O}_X$-module of rank one $\mathcal{L}$,
- sections $s_0,\dots,s_n \in H^0(X,\mathcal{L})$ such that the induced morphism $\mathcal{O}_X^{n+1} \to \mathcal{L}$ is surjective,

where we identify $(\mathcal{L}, s_0,\dots,s_n)$ and $(\mathcal{L}', s'_0,\dots,s'_n)$ if there is an isomorphism of $\mathcal{O}_X$-modules $\mathcal{L} \to \mathcal{L}'$ mapping each s_i to s'_i.

(Hint: To construct $X \to \mathbb{P}^n$, let $i \in \{0,\dots,n\}$. Consider the open subscheme X_{s_i} of X of the previous exercise, use the elements s_j for $j \neq i$ to define a morphism $X_{s_i} \to \mathbb{A}^n = \Omega_i \to \mathbb{P}^n$, and proceed with the glueing. Conversely given a morphism $X \to \mathbb{P}^n$, observe that the $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{O}(1)$ and its $n+1$ canonical sections pull back to an $\mathcal{O}_X$-module $\mathcal{L}$ and $n+1$ sections of $\mathcal{L}$.)

Exercise 1. (i) Let X be a noetherian scheme, and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{F}_\alpha$, for $\alpha \in A$, be a collection of sheaves of $\mathcal{O}_X$-modules and $\bigoplus_{\alpha\in A} \mathcal{F}_\alpha \to \mathcal{F}$ a surjective morphism (i.e. surjective on stalks). Show that there is a finite subset B of A such that the induced morphism $\bigoplus_{\beta\in B} \mathcal{F}_\beta \to \mathcal{F}$ is surjective.

(ii) Let X be an affine scheme, and $\mathcal{G}$ a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Show that the natural morphism $\bigoplus_{\alpha\in A} \mathcal{G}_\alpha \to \mathcal{G}$ is surjective, where $\mathcal{G}_\alpha$ runs over the coherent sheaves of $\mathcal{O}_X$-submodules of $\mathcal{G}$.

(iii) Let X be an affine noetherian scheme and U an open of X. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_U$-modules. Show that $\mathcal{F}$ is the restriction to U of some coherent sheaf of $\mathcal{O}_X$-modules. (Hint: Let $j\colon U \to X$ be the open immersion. Apply (ii) with $\mathcal{G} = j_*\mathcal{F}$. Observe that the morphism $\mathcal{F} \to j^*j_*\mathcal{F}$ is an isomorphism, and use (i).)

Exercise 2. Let S be a graded ring, generated as an S_0-algebra by S_1. Let $d \geq 1$ be an integer, and consider the graded ring R such that $R_n = S_{nd}$ (with ring structure by that of S). Show that there is an isomorphism $\varphi\colon \operatorname{Proj}(S) \to \operatorname{Proj}(R)$ such that $\varphi^*\mathcal{O}_{\operatorname{Proj}(R)}(1) \simeq \mathcal{O}_{\operatorname{Proj}(S)}(d)$.

The letter k denotes a field.

Exercise 1. Let X be an irreducible separated k-scheme and U an open subscheme of X. Assume that U is proper over k. Show that $U = \varnothing$ or $U = X$.

Exercise 2. Give an example of a separated, finite type, closed morphism which is not proper.

Exercise 3. (i) Let A be a local ring. Show that any scheme morphism $\operatorname{Spec} A \to Y$ factors through an affine open subscheme of Y.

(ii) Let X, Y be two k-schemes with Y of finite type. Let $x \in X$ and $\operatorname{Spec} \mathcal{O}_{X,x} \to Y$ be a scheme morphism. Show that there exists an open subscheme U of X containing x, and a morphism $U \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{X,x} \arrow[r] \arrow[d] & Y \\ U \arrow[ur] & \end{tikzcd}$$

[Hint: Reduce to the case when Y and X are affine. The special case of X integral is easier.]

Exercise 4. (i) Let $f, g\colon X \to Y$ be two k-morphisms. Assume that X is reduced and Y is separated over k. Let $h\colon T \to X$ be a k-morphism with dense set-theoretic image. If $g \circ h = f \circ h$, show that $f = g$.

(ii) Let A be a k-algebra. We assume that the ring A is a principal ideal domain, but not a field. Let U be the open complement of a closed point in $X = \operatorname{Spec} A$, and Y a proper k-scheme. Show that any k-morphism $U \to Y$ extends uniquely to a k-morphism $X \to Y$.

(iii) Let U be a non-empty open subscheme of $\mathbb{A}^1_k$ and Y a proper k-scheme. Show that any k-morphism $U \to Y$ extends uniquely to a k-morphism $\mathbb{A}^1_k \to Y$.

(iv) Let U be a non-empty open subscheme of $\mathbb{P}^1_k$ and Y a proper k-scheme. Show that any k-morphism $U \to Y$ extends uniquely to a k-morphism $\mathbb{P}^1_k \to Y$.

Exercise 1. Let L/K be a field extension of degree n. We assume that L/K is separable, which in this exercise means the following: there is an algebraically closed field Ω containing K and n distinct K-algebra morphisms $\sigma_i \colon L \to \Omega$. For an element $x \in L$, we denote by m_x the K-linear endomorphism of L induced by multiplication with x, by $\chi(m_x) \in K[X]$ its characteristic polynomial, and $\operatorname{Tr}(m_x) \in K$ its trace.

(i) Let $x \in L$. Show that the image of $\chi(m_x)$ in $\Omega[X]$ is $\prod_{i=1}^{n}(X - \sigma_i(x))$. (Hint: Reduce to the case when the extension L/K is generated by x.)

(ii) Show that a family $x_1, \ldots, x_n \in L$ is a K-basis if and only the matrix $(\operatorname{Tr}(m_{x_i x_j}))_{i,j}$ is invertible. (Hint: Use that the set of ring morphisms $L \to \Omega$ is linearly independent over Ω.)

Exercise 2. Let k be a field. Consider the polynomial ring $A = k[X_1, \ldots, X_n]$ and its fraction field $K = k(X_1, \ldots, X_n)$. Let L be a finite purely inseparable field extension of K. Show that the integral closure of A in L is an A-module of finite type. (Hint: find a finite purely inseparable extension k' of k such that L is a subfield of $k'(Y_1, \ldots, Y_n)$ where $Y_i^q = X_i$ for an appropriate p-th power q. Reduce to the case $L = k'(Y_1, \ldots, Y_n)$.)

Exercise 3. Let p be prime number and k a field. Consider an integer $n > 1$ prime to p, and let $R = k[X, Y]/(X^p - Y^n)$.

(i) Let K be a field and $a \in K$. We assume that there is no $b \in K$ such that $a = b^p$. Show that the polynomial $X^p - a \in K[X]$ is irreducible. (Hint: Let $Q \in K[X]$ be a non-trivial factor of $X^p - a$, and consider the endomorphism α of the K-vector space $K[X]/Q$ induced by multiplication with X. Compute $\det(\alpha)^p \in K$.)

(ii) Show that the ring R is an integral domain.

(iii) We consider the unique k-algebra morphism $\varphi \colon R \to k[T]$ such that $\varphi(X) = T^n$ and $\varphi(Y) = T^p$. Show that φ is injective (hint: Localise at the set of powers of X), and identifies $k[T]$ with the integral closure of R in its fraction field.

(iv) Show that there is a unique prime $\mathfrak{p} \in \operatorname{Spec} R$ such that $R_{\mathfrak{p}}$ is not a discrete valuation ring. (Hint: What is the normalisation of $\operatorname{Spec} R$?)

The letter k denotes a field. A curve is an integral finite type separated k-scheme of dimension one. A curve is regular if the local ring at each closed point is a discrete valuation ring.

Exercise 1. Let X and Y be two curves, with Y regular. Show that any birational k-morphism $X \to Y$ is an open immersion. (Hint: Use the valuative criterion of separatedness.)

Exercise 2. Let X and Y be two curves, with X regular. Show that any proper dominant k-morphism $X \to Y$ is finite.

Exercise 3. (i) Let $P \in k[X,Y]$ be an irreducible polynomial such that $P(0,0) = 0$. We assume that

$$\frac{\partial P}{\partial X}(0,0) \neq 0 \qquad \text{or} \qquad \frac{\partial P}{\partial Y}(0,0) \neq 0.$$

Show that the localisation of the ring $k[X,Y]/P$ at the maximal ideal generated by (the images modulo P of) X and Y is a discrete valuation ring (use the Taylor expansion).

(ii) Let n be an integer prime to the characteristic of k. Show that the polynomial $X^n + Y^n - 1$ is irreducible.

(iii) Assume that k is algebraically closed and let n be an integer prime to the characteristic of k. Let $Z = V(X^n + Y^n - 1) \subset \mathbb{A}^2_k = \operatorname{Spec} k[X,Y]$. Show that Z is a regular curve.

Exercise 4. (i) Let X be a regular curve and U a non-empty open subscheme of X. Let Y be a proper k-scheme. Show that any k-morphism $U \to Y$ is the restriction of a unique k-morphism $X \to Y$.

(ii) Let X be a regular curve. Assume that there are two open immersions $X \to X_1$ and $X \to X_2$ where X_1 and X_2 are regular curves. We assume that X_1 and X_2 are proper over k. Show that X_1 and X_2 are k-isomorphic.

The letter k denotes a field.

Exercise 1. Let $k \subset K$ be a field extension of finite type and of transcendence degree one. Let A be a valuation ring of K containing k. Show that if $A \neq K$, then A is a discrete valuation ring.

Exercise 2. Let $k \subset K$ be a field extension of finite type and of transcendence degree one, and X_K the curve constructed in the lecture (closed points of X_K correspond to valuation rings of K containing k). Show that X_K is proper over k using the valuative criterion.

Exercise 3. We consider the category $\mathcal{B}$ whose objects are finite type integral schemes over k, and morphisms are dominant rational maps. Show that $\mathcal{B}$ is equivalent to the opposite of the category of finite type field extensions of k.

Exercise 4. (i) Let X be an integral scheme, proper over k. Show that the ring extension $k \to \mathcal{O}_X(X)$ is integral (hint: for $f \in \mathcal{O}_X(X)$, what can be the image of the composite morphism $X \xrightarrow{\varphi_f} \mathbb{A}^1_k \to \mathbb{P}^1_k$?).

(ii) Show that k is algebraically closed in $k(T)$.

(iii) Deduce that $\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) = k$.

(iv) (optional) Show that $\mathcal{O}_{\mathbb{P}^n_k}(\mathbb{P}^n_k) = k$.

The letter k denotes a field. When Z is a closed subscheme of a separated scheme X, we denote by $\widetilde{X}_Z$ the blow-up of Z in X.

Exercise 1. Let X be a reduced separated scheme, and Z a closed subscheme of X. Show that $\widetilde{X}_Z$ is reduced.

Exercise 2. Let $R = k[x,y]/(xy)$, and $E = \operatorname{Spec} R$.

(i) Show that the scheme E is reduced, and has exactly two irreducible components X, Y, both isomorphic to $\mathbb{A}^1_k$. Show that the scheme $X \cap Y$ (defined as $X \times_E Y$) is isomorphic to $\operatorname{Spec} k$.

(ii) Show the blow-up morphism $\widetilde{E}_X \to E$ coincides with the closed immersion $Y \to E$. (Hint: use the functoriality of the blow-up to construct a morphism $Y \to \widetilde{E}_X$; compute the fiber of $\widetilde{E}_X \to E$ over $X \cap Y$.)

(iii) Describe the blow-up of $X \cap Y$ in E.

Exercise 3. Let X be a curve over k, and Z a closed subscheme of X such that $Z \neq X$. Let $b\colon \widetilde{X}_Z \to X$ be the blow-up morphism, and $n\colon \widetilde{X} \to X$ the normalisation morphism (of X in its own function field).

(i) Show that $X - (Z \cup X_{Sing})$ is an open subscheme of $\widetilde{X}$ and $\widetilde{X}_Z$, and deduce the existence of a birational morphism $\widetilde{X} \to \widetilde{X}_Z$ over X.

(ii) Let $z \in Z$. Denote by $\#E$ the cardinality of a set E. Show that

$$1 \leq \#(b^{-1}\{z\}) \leq \#(n^{-1}\{z\}).$$

(iii) Let $X = \operatorname{Spec} k[x,y]/(x^2 - y^3)$ and Z be the closed subscheme of X defined by the ideal (x,y). Show that the morphism $\widetilde{X}_Z \to X$ induces a bijection on the underlying sets.

Exercise 4. Consider the graded k-algebra $S_* = k[T_0, \dots, T_n]$ where T_i has degree 1.

(i) Show that the set $\operatorname{Spec} S_* - \operatorname{Spec} S_0$ consists of those primes $\mathfrak{p} \subset S_*$ such that $S_+ \not\subset \mathfrak{p}$.

(ii) For a prime $\mathfrak{p} \subset S_*$, let us denote by $\mathfrak{p}^h$ the ideal of S_* generated by the homogeneous elements of $\mathfrak{p}$. Observe that $\mathfrak{p}^h \subset \mathfrak{p}$ (in particular $S_+ \not\subset \mathfrak{p}^h$ if $S_+ \not\subset \mathfrak{p}$). Show that $\mathfrak{p}^h$ is a homogeneous prime ideal of S_*. Show that if I is a homogeneous ideal of S_* such that $\mathfrak{p}^h \subset I \subset \mathfrak{p}$, then $I = \mathfrak{p}^h$.

(iii) Show that the set map underlying the canonical morphism $\mathbb{A}^{n+1}_k - 0 \to \mathbb{P}^n_k$ is defined by $\mathfrak{p} \mapsto \mathfrak{p}^h$.

The letter k denotes a field. When n is an integer, we denote by $\mathbb{A}^n = \operatorname{Spec} k[X_1, \dots, X_n]$ the affine space, and write $\times$ instead of $\times_{\operatorname{Spec} k}$ for the fiber product of k-schemes.

Exercise 1. (i) Let R be a commutative ring and $f \in R$. Assume that f is a nonzerodivisor, i.e. for all $a \in R - \{0\}$ we have $fa \neq 0$. Show that the open subscheme $D(f)$ is dense in $\operatorname{Spec} R$.

(ii) Let $X = \operatorname{Spec} A$ be an affine scheme, and $Z = \operatorname{Spec} A/I$ a closed subscheme. Let $\pi \colon \widetilde{X}_Z \to X$ be the blow-up morphism. Show that the open subscheme $X - Z \simeq \widetilde{X}_Z - \pi^{-1}(Z)$ is dense in $\widetilde{X}_Z$. (Hint : $\widetilde{X}_Z$ is covered by the open affine subschemes $D_h(f_i) = \operatorname{Spec} A_i$, where (f_i) is a generating set of the ideal I; use the first question for the ring A_i.)

Exercise 2. We consider the $\mathbb{G}_m$-action on $\mathbb{A}^n \times \mathbb{A}^n = \operatorname{Spec} k[T_1, \dots, T_n, X_1, \dots, X_n]$ given by letting T_i be of degree 1 and X_i of degree 0 (canonical action on the first factor and trivial action on the second factor). Let Y be the $\mathbb{G}_m$-invariant open subscheme $(\mathbb{A}^n - 0) \times \mathbb{A}^n$.

(i) Show that the $\mathbb{G}_m$-action on Y is locally free.

(ii) Let Z be the closed subscheme of Y defined by the equations $T_i X_j = T_j X_i$ for $1 \leq i, j \leq n$. Show that Z is $\mathbb{G}_m$-equivariantly isomorphic to $(\mathbb{A}^n - 0) \times \mathbb{A}^1$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1 = \operatorname{Spec} k[X]$ by letting X be of degree -1 (and canonically on the first factor). Deduce that the blow-up of $\mathbb{A}^n$ at the closed subscheme 0 may be identified with the quotient scheme $Z/\mathbb{G}_m$.

Exercise 3. Let Z and Y be two closed subschemes of X.

(i) Show that the natural morphism $\widetilde{Y}_{Y \cap Z} \to \widetilde{X}_Z$ is a closed embedding.

(ii) Let $\pi \colon \widetilde{X}_Z \to X$ be the blow-up morphism. Show that, as sets,

$$\pi^{-1}(Y) = \widetilde{Y}_{Y \cap Z} \cup \pi^{-1}(Y \cap Z),$$

and that $\pi^{-1}(Y)$ is the disjoint union of $Y - Y \cap Z$ and $\pi^{-1}(Y \cap Z)$.

(iii) For $i = 1, \dots, n$ we let L_i be the closed subscheme of $\mathbb{A}^n = k[X_1, \dots, X_n]$ given by the ideal generated by the elements X_j for $j \neq i$. Describe the inverse image of $L_1 \cup \cdots \cup L_n$ in the blow-up of 0 in $\mathbb{A}^n$.

Exercise 4. Let X be the blow-up of 0 in $\mathbb{A}^n$.

(i) Show that X is covered by n open subschemes, each isomorphic to $\mathbb{A}^n$.

(ii) Assume that $n > 1$. Let x be a closed point of X mapping to 0 in $\mathbb{A}^n$, and Y the blow-up of $\{x\}$ (with reduced structure) in X. Show that the natural open immersion $\mathbb{A}^n - 0 \to Y$ does not extend to a morphism $X \to Y$.

All schemes are assumed to be noetherian.

Exercise 1. Let A be a discrete valuation ring and K its fraction field.

(i) Show that the datum of a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is equivalent to the data of an A-module M and a K-vector space V together with a morphism of A-modules $M \to V$.

(ii) Show that the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the corresponding morphism $M \otimes_A K \to V$ is an isomorphism.

Exercise 2. (i) Let $f\colon X \to Y$ be a morphism and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Show that $f_*\mathcal{F}$ is naturally a sheaf of $\mathcal{O}_Y$-modules.

(ii) Let $\varphi\colon A \to B$ be a ring morphism, and $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ the induced scheme morphism. Let M be a B-module. We denote by M_φ the set M viewed as an A-module using φ and the B-module structure on M. Show that

$$f_*\widetilde{M} = \widetilde{M_\varphi}.$$

(iii) Let $f\colon X \to Y$ be a finite morphism (of schemes) and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules. Show that the sheaf of $\mathcal{O}_Y$-modules $f_*\mathcal{F}$ is coherent. (Hint: Reduce to the case when Y is affine, and use the previous question.)

(iv) Give an example of a morphism $f\colon X \to Y$ and a coherent sheaf of $\mathcal{O}_X$-modules such that the sheaf of $\mathcal{O}_Y$-modules $f_*\mathcal{F}$ is not coherent.

Exercise 3. Let X be a scheme and $\mathcal{J}$ be a sheaf of $\mathcal{O}_X$-ideals. We consider the subset $V(\mathcal{J}) \subset X$ consisting of those points x such that $\mathcal{J}_x \neq \mathcal{O}_{X,x}$, and denote by $j\colon V(\mathcal{J}) \to X$ the inclusion (of sets).

(i) Show that $V(\mathcal{J})$ is closed in X.

(ii) Show that the natural morphism $\mathcal{O}_X/\mathcal{J} \to j_*j^{-1}(\mathcal{O}_X/\mathcal{J})$ is an isomorphism.

(iii) Assume that $\mathcal{J} = \mathcal{I}_Z$ for a closed subscheme Z of X. Show that $Z = V(\mathcal{J})$ as subsets of X, and that $j^{-1}(\mathcal{O}_X/\mathcal{J}) \simeq \mathcal{O}_Z$.

(iv) Assume that $\mathcal{J}$ is quasi-coherent. Show that the ringed space $(V(\mathcal{J}), j^{-1}(\mathcal{O}_X/\mathcal{J}))$ is isomorphic to a closed subscheme of X.

(v) Deduce a correspondence between closed subschemes of X and quasi-coherent sheaves of $\mathcal{O}_X$-ideals.

Exercise 1. Let X be a scheme, and consider a short exact sequence of quasi-coherent sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0.$$

Assume that $\mathcal{M}''$ is locally free of finite type. Show that for any scheme morphism $f\colon Y \to X$ the sequence

$$0 \to f^*\mathcal{M}' \to f^*\mathcal{M} \to f^*\mathcal{M}'' \to 0$$

is exact. Deduce that $\mathcal{F} \mapsto f^*\mathcal{F}$ induces a group morphism $K_0(X) \to K_0(Y)$.

Exercise 2. Let k be a field, and $f\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ a dominant morphism. Recall that f is finite. Show that the $\mathcal{O}_{\mathbb{P}^1_k}$-module $f_*\mathcal{O}_{\mathbb{P}^1_k}$ is locally free of finite type. (Hint: Use the covering of $\mathbb{P}^1_k$ by two copies of $\mathbb{A}^1_k$.)

Exercise 3. Let A be a (not necessarily noetherian) commutative unital ring, and M a locally free A-module of finite type.

(i) Show that M is of finite presentation, i.e. there is an exact sequence

$$A^r \to A^s \to M \to 0$$

for some $r, s \in \mathbb{N}$. (Hint: Find a surjection $A^s \to M$, and let R be its kernel. Choose $f_1, \dots, f_n \in A$ such that the A_{f_i}-modules M_{f_i} are free. Show that the A_{f_i}-module R_{f_i} is finitely generated, and deduce that R is finitely generated.)

(ii) Let N be an A-module. Show that the natural morphism $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, N_f)$ induces an isomorphism $(\operatorname{Hom}_A(M, N))_f \simeq \operatorname{Hom}_A(M, N_f)$. (Hint: reduce to the case when M is free and finitely generated using the first question.)

(iii) Show that there exists an A-module F such that $M \oplus F \simeq A^s$. (Hint: Show using (ii) that the morphism $\operatorname{Hom}_A(M, A^s) \to \operatorname{Hom}_A(M, M)$, induced by the morphism $A^s \to M$ of (i), is surjective.)

We now let P, Q be two A-modules such that $P \oplus Q \simeq A^t$ with $t \in \mathbb{N}$. Let $\mathfrak{p} \in \operatorname{Spec}(A)$.

(iv) Show that the A-module P (and also Q) is projective.

(v) Show that the $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$ (and also $Q_{\mathfrak{p}}$) is free (Hint: Use Nakayama's Lemma). Deduce the existence of a morphism of A-modules $\theta\colon A^m \to P$ for some $m \in \mathbb{N}$ such that $\theta_{\mathfrak{p}}$ is an isomorphism.

(vi) Deduce that we may find $f \in A - \mathfrak{p}$ such that θ_f is bijective. (Hint: Show first that we may find $g \in A - \mathfrak{p}$ such that θ_g is surjective. Observe that the A_g-module P_g is projective.)

(vii) Conclude that an A-module is locally free of finite type if and only if it is a direct summand of a free A-module of finite type.