Exercise 1. Let A be a domain, and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ prime ideals of A.

(i) Show that the set $S = A \smallsetminus (\mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n)$ is multiplicatively closed.

(ii) Assume that $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for all $i \neq j$. Show that the ring $S^{-1}A$ possesses n maximal ideals.

Exercise 2. Let A be a Dedekind domain. We are going to prove that every ideal of A is generated by at most two elements.

(i) Let $x \in A$ be a nonzero element. Show that x is contained in only finitely many prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of A.

(ii) Let $S = A \smallsetminus (\mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n)$. Show that the ring $S^{-1}A$ is a principal ideal domain. (Hint: use the previous exercise.)

(iii) Show that for any $s \in S$, we have $sA + xA = A$.

(iv) Show that we have a ring isomorphism $A/xA \xrightarrow{\sim} (S^{-1}A)/(xS^{-1}A)$.

(v) Deduce that every ideal of A/xA is principal.

(vi) Conclude that every ideal of A is generated by at most two elements.

Exercise 3. Let A be a Dedekind domain, and $S \subset A$ a multiplicatively closed subset. Show that mapping a nonzero fractional ideal I of A to $S^{-1}I$ induces a surjective group morphism $\mathcal{C}(A) \to \mathcal{C}(S^{-1}A)$ between the ideal class groups.

Exercise 4. Let A be a Dedekind domain, and $f \in A$ a nonzero element. Consider the multiplicatively closed subset $S = \{f^n | n \in \mathbb{N}\}$ in A, and let r be the number of prime ideals of A containing f (recall from Exercise 2 (i) that $r < \infty$).

(i) Let Q be the kernel of the natural morphism $\mathcal{F}(A) \to \mathcal{F}(S^{-1}A)$ (where $\mathcal{F}(A), \mathcal{F}(S^{-1}A)$ denote the respective groups of nonzero fractional ideals). Show that the $\mathbb{Z}$-module Q is free of rank r.

(ii) By considering the morphism

$$(S^{-1}A)^\times \to \mathcal{F}(A), \quad x \mapsto xA$$

show that the $\mathbb{Z}$-module $(S^{-1}A)^\times / A^\times$ is free of rank $\leq r$.

Exercise 5 (Optional)**.** Let B be a noetherian domain, and $A \subset B$ a subring such that B is integral over A. If $\mathfrak{p}$ is a prime ideal of A, show that there exists a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$. (This is called the “going-up” theorem.)