We fix a field k, and a field extension ℓ/k.

Let V be a k-vector space. Consider the ℓ-vector space $\tilde{V}$ on the basis $(e_v, v \in V)$. Let $V \otimes_k \ell$ be the quotient of $\tilde{V}$ by the ℓ-subspace generated by the elements

$$\begin{cases} \lambda e_v - e_{\lambda v} & \text{for } \lambda \in k, v \in V, \\ e_{u+v} - e_u - e_v & \text{for } u, v \in V. \end{cases}$$

For $\mu \in \ell$ and $v \in V$, we denote by $v \otimes \mu \in V \otimes_k \ell$ the image of μe_v.

Exercise 1. Let V be a k-vector space and W an ℓ-vector space. Let $f \colon V \to W$ a k-linear map. Show that there exists a unique ℓ-linear map

$$g \colon V \otimes_k \ell \to W$$

such that $g(v \otimes 1) = f(v)$ for all $v \in V$.

Exercise 2. Let V be a k-vector space. Show that the map $V \to V \otimes_k \ell$ given by $v \mapsto v \otimes 1$ is k-linear and injective. (Hint: injectivity is more subtle point.)

Exercise 3. Let V be a k-vector space, and assume that $e_1, \dots, e_n$ is a k-basis of V. Show that $e_1 \otimes 1, \dots, e_n \otimes 1$ is an ℓ-basis of $V \otimes_k \ell$, and deduce that $\dim_k V = \dim_\ell (V \otimes_k \ell)$.

Exercise 4. Let V, W be a k-vector spaces, and $f \colon V \to W$ a k-linear map.

(i) Show that f induces an ℓ-linear map $g \colon V \otimes_k \ell \to W \otimes_k \ell$.

(ii) If f is surjective, show that g is surjective.

(iii) If f is injective, show that g is injective.

Exercise 5. Let A be a k-algebra.

(i) Show that $A \otimes_k \ell$ is a naturally an ℓ-algebra.

(ii) Let B be an ℓ-algebra, and $f \colon A \to B$ be a morphism of k-algebras. Show that the induced ℓ-linear map $A \otimes_k \ell \to B$ is a morphism of ℓ-algebras.

Exercise 6. (i) Let V, W be k-vector spaces. Show that

$$(V \oplus W) \otimes_k \ell \simeq (V \otimes_k \ell) \oplus (W \otimes_k \ell)$$

as ℓ-vector spaces.

(ii) Let A, B be k-algebras. Show that

$$(A \times B) \otimes_k \ell \simeq (A \otimes_k \ell) \times (B \otimes_k \ell)$$

as ℓ-algebras.

Exercise 7. (i) Show that $(k[X]) \otimes_k \ell \simeq \ell[X]$ as ℓ-algebra.

(ii) Let A be a k-algebra and I an ideal of A. Show that $I \otimes_k \ell$ may be viewed as an ideal of $A \otimes_k \ell$, and that $(A/I) \otimes_k \ell \simeq (A \otimes_k \ell)/(I \otimes_k \ell)$.

(iii) Let $P \in k[X]$, and $A = k[X]/P$. Show that the ℓ-algebra $A \otimes_k \ell$ is naturally isomorphic to $\ell[X]/P$.

Exercise 8. Let A be a k-algebra.

(i) If A is an integral domain, is $A \otimes_k \ell$ an integral domain? Give a proof or a counterexample.

(ii) If A is reduced, is $A \otimes_k \ell$ reduced? Give a proof or a counterexample.