Exercise 1. Show that the polynomial ring $\mathbb{Z}[X]$ is not a principal ideal domain.

Exercise 2. Let A be a nonzero noetherian ring, and M a free A-module of rank n. If m is an integer such that the A-module M is free of rank m, show that $m = n$. (Hint: consider a maximal ideal of A.)

Exercise 3. Let A be a domain, and $P \in A[X]$ a polynomial. Show that $A[X]/P$ is integral over A if and only if the leading coefficient of the polynomial P is a unit in A.

Exercise 4. Let A be a domain having only finitely many elements. Show that A is a field.

Exercise 5. Let A be a domain, with fraction field K. Let L be a field extension of K having finite degree, and B the integral closure of A in L. Show that L is the fraction field of B.

Exercise 6. Let $A \subset R$ be a ring extension. Consider the following conditions

(a) the extension $A \subset R$ is integral,

(b) the A-module R is finitely generated.

Does (a) implies (b)? Does (b) implies (a)? (Justify your answers, either with a proof, reference to the lecture, or counterexample). Same questions when the A-algebra R is additionally assumed to be finitely generated.

Exercise 7. (Time permitting) We let $\sqrt{-5} \in \mathbb{C}$ be one of the roots of the polynomial $X^2 + 5$, and consider the subset

$$R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}\} \subset \mathbb{C}.$$

Show that R is a subring of $\mathbb{C}$, and that R is not a principal ideal domain. (Hint: Assuming that R is a principal ideal domain, consider a prime decomposition of $1 + \sqrt{-5}$.)

Exercise 8. (Time permitting) Let K be a quadratic field.

(i) Let $\sigma \colon K \to K$ the nontrivial morphism of $\mathbb{Q}$-algebras. Express the maps

$$\mathrm{Tr}_{K/\mathbb{Q}} \colon K \to \mathbb{Q} \quad \text{and} \quad \mathrm{N}_{K/\mathbb{Q}} \colon K \to \mathbb{Q}$$

in terms of σ.

(ii) Show that $\mathrm{N}_{K/\mathbb{Q}}(\mathcal{O}_K) \subset \mathbb{Z}$.