EXERCISES 1 (INTERSECTION THEORY)

All rings are noetherian.

Let M be a finitely generated A-module. A prime $\mathfrak{p}$ of A is *associated with* M if there is an element $m \in M$ such that $\mathfrak{p} = \mathrm{Ann}(m) = \{x \in A | xm = 0\}$. We write $\mathrm{Ass}(M)$ for the set of associated primes of M.

Exercise 1.
(i) We have $\mathfrak{p} \in \mathrm{Ass}(M)$ if and only if M contains a submodule isomorphic to $A/\mathfrak{p}$.
(ii) Let I be a maximal element of the set $\{\mathrm{Ann}(m) | m \in M - \{0\}\}$. Then I is a prime ideal.
(iii) We have $M = 0$ if and only if $\mathrm{Ass}(M) = \emptyset$.
(iv) Let $\mathfrak{p}$ be a prime of A. Then $\mathrm{Ass}(A/\mathfrak{p}) = \{\mathfrak{p}\}$.
(v) Consider an exact sequence of finitely generated A-modules
$$0 \to M' \to M \to M'' \to 0.$$
Then $\mathrm{Ass}(M') \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$.

Exercise 2. Recall that
$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}\, A | M_{\mathfrak{p}} \neq 0\}$$
Consider an exact sequence of finitely generated A-modules
$$0 \to M' \to M \to M'' \to 0.$$
Then $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

Exercise 3. Let $\mathfrak{p} \in \mathrm{Spec}\, A$. We view $\mathrm{Spec}\, A_{\mathfrak{p}}$ as a subset of $\mathrm{Spec}\, A$. Then
$$\mathrm{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = (\mathrm{Spec}\, A_{\mathfrak{p}}) \cap \mathrm{Ass}(M).$$

Exercise 4. We have $\mathrm{Ass}(M) \subset \mathrm{Supp}(M)$, and these sets have the same minimal elements.

Exercise 5. Let M be a finitely generated R-module. There is a chain of submodules
$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$
such that $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$ with $\mathfrak{p}_i$ prime, for $i = 1, \cdots, n$. We have
$$\mathrm{Ass}(M) \subset \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\} \subset \mathrm{Supp}(M)$$
and these sets have the same minimal elements.

Exercise 6. The set $\mathrm{Ass}(M)$ is finite, and so is the set of minimal primes in $\mathrm{Supp}(M)$.