GALOIS COHOMOLOGY
EXERCISES 2 (QUATERNIONS)

Let k be a field of characteristic $\neq 2$.

Exercise 1. Let $a \in k^\times$. Show that:

(i) $(a, -a)$ splits.
(ii) If $a \neq 1$, then $(a, 1-a)$ splits.
(iii) $(a, a) \simeq (a, -1)$.
(iv) $(a, -1)$ splits if and only if a is a sum of two squares in k.

Exercise 2. (Chain Lemma.) Let $a, b, c, d \in k^\times$ be such that $(a, b) \simeq (c, d)$. We are going to prove that there is $e \in k^\times$ such that

$$(a, b) \simeq (e, b) \simeq (e, d) \simeq (c, d).$$

So we let Q be such that $(a, b) \simeq Q \simeq (c, d)$.

(i) Let i, j, resp. i', j', be the images in Q of the standard generators of (a, b), resp. (c, d). Show that $i, j, i', j' \in Q_0$.
(ii) Let V be the k-subspace of Q_0 generated by j, j'. Show that the morphism $\varphi\colon Q_0 \to \mathrm{Hom}_k(V, k)$ sending $q \in Q_0$ to the map $v \mapsto qv + vq$ is not injective.
(iii) Deduce that there is a nonzero $\varepsilon \in Q_0$ such that $\varepsilon j = -j\varepsilon$ and $\varepsilon j' = -j'\varepsilon$.
(iv) Show that $e = \varepsilon^2 \in k$, and conclude.

Exercise 3. Let L/k be a field extension of odd degree and Q a quaternion k-algebra. Show that Q splits if and only if $Q \otimes_k L$ splits over L. (Hint : use the splitting criterion involving the norm of quadratic field extensions, and the properties of field norms.)