Brauer Groups of fields

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Note on the literature

The main references that we used in preparing these notes is the book of Gille and Szamuely [**GS17**]. As always, Serre's books [**Ser62, Ser02**] provide excellent accounts. There is also very useful material contained in the Stack's project [**Sta**] (available online). Kersten's book [**Ker07**] (in German, available online) provides a very gentle introduction to the subject.

For the first part (on noncommutative algebra), we additionally used Draxl's [**Dra83**] and Pierce's [**Pie82**], as well as Lam's book [**Lam05**] (which uses the language of quadratic forms) for quaternion algebras. For the second part (on torsors), we used the book of involutions [**KMRT98**, Chapters V and VII].

Part 1

Noncommutative Algebra

CHAPTER 1

Quaternion algebras

This chapter will serve as an introduction to the theory of central simple algebras, by developing some aspects of the general theory in the simplest case of quaternion algebras. The results proved here will not really be used in the sequel, and many of them will be in fact substantially generalised by other means. Rather we would like to show what can be done "by hand", which may help appreciate the more sophisticated methods developed in the sequel.

Quaternions are historically very significant; since their discovery by Hamilton in 1843, they have played an influential role in various branches of mathematics. A particularity of these algebras is their deep relations with quadratic forms, which is not really a systematic feature of central simple algebras. For this reason, we will merely hint at the connections with quadratic form theory.

1. The norm form

All rings will be assumed to be unital and associative (but often noncommutative!). The set of elements of a ring R admitting a two-sided inverse is a group, that we denote by $R^\times$.

We fix a base field k. A k-algebra is a ring A equipped with a structure of k-vector space such that the multiplication map $A \times A \to A$ is k-bilinear. A morphism of k-algebras is a ring morphism which is k-linear. If A is nonzero, the map $k \to A$ given by $\lambda \mapsto \lambda 1$ is injective, and we will view k as a subring of A. Observe that the bilinearity of the multiplication map implies that for any $\lambda \in k$ and $a \in A$

$$\lambda a = (\lambda a)1 = a(\lambda 1) = a\lambda. \tag{1.1.a}$$

In this chapter on quaternion algebras, we will assume that the characteristic of k is not equal to two (i.e. $2 \neq 0$ in k).

Definition 1.1.1. Let $a, b \in k^\times$. We define a k-algebra (a, b) as follows. A basis of (a, b) as k-vector space is given by $1, i, j, ij$. It is easy to verify that (a, b) admits a unique k-algebra structure such that

$$i^2 = a, \quad j^2 = b, \quad ij = -ji. \tag{1.1.b}$$

We will call i, j the *standard generators* of (a, b). An algebra isomorphic to (a, b) for some $a, b \in k^\times$ will be called a *quaternion algebra*.

Lemma 1.1.2. *Let A be a 4-dimensional k-algebra. If $i, j \in A$ satisfy the relations (1.1.b) for some $a, b \in k^\times$, then $A \simeq (a, b)$.*

Proof. It will suffice to prove that the elements $1, i, j, ij$ are linearly independent over k. Since i anticommutes with j, the elements $1, i, j$ must be linearly independent (recall that the characteristic of k differs from 2). Now assume that $ij = u + vi + wj$, with $u, v, w \in k$. Then

$$0 = i(ij + ji) = i(ij) + (ij)i = i(u + vi + wj) + (u + vi + wj)i = 2ui + 2av,$$

hence $u = v = 0$ by linear independence of $1, i$. So $ij = wj$, hence $ij^2 = wj^2$ and thus $bi = bw$, a contradiction with the linear independence of $1, i$. □

The following observations will be used without explicit mention.

LEMMA 1.1.3. *Let $a, b \in k^\times$. Then*

(i) $(a, b) \simeq (b, a)$,
(ii) $(a, b) \simeq (a\alpha^2, b\beta^2)$ *for any* $\alpha, \beta \in k^\times$.

PROOF. (i) : We let i', j' be the standard generators of (b, a), and apply Lemma 1.1.2 with $i = j'$ and $j = i'$.

(ii): We let i'', j'' be the standard generators of $(a\alpha^2, b\beta^2)$, and apply Lemma 1.1.2 with $i = \alpha^{-1}i''$ and $j = \beta^{-1}j''$. □

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