GALOIS COHOMOLOGY
EXERCISES 4 (SIMPLE RINGS)

Exercise 1. Prove the following converse of Wedderburn's Theorem: If D is a division ring and $n \geq 1$ an integer, then the ring $M_n(D)$ is artinian simple.

Exercise 2. In Proposition 1.3.5, we proved the following statement : if Q, Q' are quaternion algebras over a field k (of characteristic $\neq 2$), then

$$Q \otimes_k Q' \simeq M_4(k) \Longleftrightarrow Q \simeq Q'.$$

The proof of "$\Longleftarrow$" was easy, while the proof of "$\Longrightarrow$" was comparatively difficult (in particular used Albert's Theorem). Give a new (short) proof of "$\Longrightarrow$", using "$\Longleftarrow$" and the results of §2.1 in the lecture notes.

Exercise 3. Let R be a ring and $n \in \mathbb{N} - 0$. Show that R and $M_n(R)$ have the same center.

Exercise 4. (i) Show that every nonzero ring admits a simple module.

(ii) Let R be a ring, and M a nonzero R-module. Show that there is a submodule N of M and a quotient S of N such that S is simple.

Exercise 5. Let D be a division algebra of positive characteristic (i.e. there is a prime number p such that $pD = 0$.) Show that every finite subgroup of $D^\times$ is cyclic. (Hint: you may use the fact that every subgroup of $k^\times$ is cyclic when k is a finite field).