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Algebraic Geometry I

Exercises 10

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Exercise 1. For an isomorphism $\varphi\colon \mathbb{A}^1 - 0 \to \mathbb{A}^1 - 0$, we denote by $\mathbb{A}^1 \sqcup_\varphi \mathbb{A}^1$ the glueing of $\mathbb{A}^1$ with itself along φ. Let $\chi\colon \mathbb{A}^1 - 0 \to \mathbb{A}^1 - 0$ be the morphism corresponding to the ring morphism $\mathbb{Z}[x, x^{-1}] \to \mathbb{Z}[x, x^{-1}]$ mapping x to x^{-1}. Show that the schemes $\mathbb{A}^1 \sqcup_{\mathrm{id}} \mathbb{A}^1$ and $\mathbb{A}^1 \sqcup_\chi \mathbb{A}^1$ are not isomorphic. [Hint: Look at the set of morphisms into $\mathbb{A}^1$.]

Exercise 2. Let $f\colon X \to Y$ be a scheme morphism and y a point of Y. Consider the natural morphism $\operatorname{Spec} \kappa(y) \to Y$ and the fibre $X_y = X \times_Y \operatorname{Spec} \kappa(y)$. Show that the projection $X_y \to X$ induces a homeomorphism between X_y and $f^{-1}\{y\}$.

Exercise 3. Let k be a field. Let $X = \operatorname{Spec} k[X, Y, Z]/(XY - Z)$ and $\mathbb{A}^1_k = \operatorname{Spec} k[T]$. Consider the morphism $X \to \mathbb{A}^1_k$ corresponding to the k-algebra morphism $k[T] \to k[X, Y, Z]/(XY - Z)$ mapping T to Z. Describe the fibre over each point of $\mathbb{A}^1_k$.

Exercise 4. (i) Let A be a ring, and $f_1, \ldots, f_n \in A$ elements generating the unit ideal. Assume that each ring A_{f_i} is noetherian. Show that the ring A is noetherian.

(ii) Show that an open subscheme of a locally noetherian scheme is locally noetherian.

(iii) Let X be a locally noetherian scheme, and $U = \operatorname{Spec} A$ an affine open subscheme of X. Using (i) and (ii), show that the ring A is noetherian.