

GALOIS COHOMOLOGY

EXERCISES 11 (CYCLIC ALGEBRAS)

Let k be a field. A finite-dimensional central simple k -algebra A of degree n is called *cyclic* if there exists a Galois \mathbb{Z}/n -algebra L and $a \in k^\times$ such that A is isomorphic to the cyclic algebra (L, a) . The purpose of these exercises is to prove that every central simple algebra of degree 2 or 3 is cyclic (on the other hand one may construct central simple algebras of degree 4 which are not cyclic).

Exercise 1. We have seen that central simple k -algebra of degree 2 are cyclic (in fact quaternion algebras) when k has characteristic $\neq 2$. In this exercise, we consider the case when the characteristic of k is arbitrary.

- (i) Let D be a finite-dimensional central division k -algebra of degree 2. Show that D contains a Galois $\mathbb{Z}/2$ -algebra as a k -subalgebra, and deduce that D is cyclic.
- (ii) Conclude that every finite-dimensional central simple k -algebra of degree 2 is cyclic.

Exercise 2. Let A be a finite-dimensional central simple k -algebra.

- (i) Show that the map

$$\nu: A \times A \rightarrow k \quad ; \quad (a, b) \mapsto \text{Trd}_A(ab)$$

is a symmetric k -bilinear form.

- (ii) Show that the form ν is nondegenerate, i.e. that the set

$$\{a \in A \mid \text{Trd}_A(ax) = 0 \text{ for all } x \in A\}$$

is reduced to $\{0\}$.

Exercise 3. Let A be a finite-dimensional central simple k -algebra of degree n . Let $x \in A$ and $P = \text{Cprd}_A(x) \in k[X]$ its reduced characteristic polynomial.

- (i) Show that $P(x) = 0 \in A$.
- (ii) Assume that $x \in A^\times$, and let $Q = \text{Cprd}_A(x^{-1}) \in k[X]$. Show that

$$P(X) = (-X)^n \cdot \text{Nrd}_A(x) \cdot Q(X^{-1}) \in k[X] \subset k[X, X^{-1}].$$

Exercise 4. Let D be a finite-dimensional central division k -algebra of degree 3. When $E \subset D$ is a subset, we write

$$E^\perp = \{x \in D \mid \text{Trd}_D(ex) = 0 \text{ for all } e \in E\}.$$

- (i) If $V \subset D$ is a k -subspace, show that $\dim_k V^\perp = 9 - \dim_k V$. (Hint: Use Exercise 2.)

- (ii) Let K be a commutative k -subalgebra of D . Show that $K = k$ or that K/k is a field extension of degree 3.
- (iii) Let $x \in D^\times$ be such that $\text{Trd}_D(x) = \text{Trd}_D(x^{-1}) = 0$. Show that $x^3 = \text{Nrd}_D(x) \in k \subset D$. (Hint: Use Exercise 3.)
- (iv) Let $E \subset D$ be a maximal subfield. Find $z \in D - k$ such that $\text{Trd}_D(z) = \text{Trd}_D(z^{-1}) = 0$. (Hint: Pick a nonzero element $u_1 \in E^\perp$, and find $u_2 \in \{u_1^{-1}\}^\perp \cap E$ such that $u_2 \notin u_1 k$. Set $z = u_1 u_2^{-1}$.)
- (v) Let F be the k -subalgebra of D generated by z . Find $y \in D - F$ such that

$$\text{Trd}_D(yz) = \text{Trd}_D(yz^2) = \text{Trd}_D(z^{-1}y^{-1}) = \text{Trd}_D(z^{-2}y^{-1}) = 0.$$
 (Hint: Pick $v_1 \in F^\perp - F$. Let $V = \{z^{-1}, z^{-2}\}^\perp$, and find a nonzero $v_2 \in (v_1 V) \cap F$. Set $y = v_2^{-1} v_1$.)
- (vi) Let L be the k -subalgebra of D generated by y . Show that zyz^{-1} commutes with y and deduce that $zyz^{-1} \in L$. (Hint: Show that $\text{Nrd}_D(yz^2) \text{Nrd}_D(z^{-1}) = \text{Nrd}_D(yz)$, and expand using (iii).)
- (vii) Show that $y \mapsto zyz^{-1}$ defines a structure of Galois $\mathbb{Z}/3$ -algebra on L .
- (viii) Deduce that D is cyclic.
- (ix) Conclude that every finite-dimensional central simple k -algebra of degree 3 is cyclic (this is a theorem of Wedderburn).