GALOIS COHOMOLOGY
EXERCISES 11 (CYCLIC ALGEBRAS)

Let k be a field. A finite-dimensional central simple k-algebra A of degree n is called *cyclic* if there exists a Galois $\mathbb{Z}/n$-algebra L and $a \in k^\times$ such that A is isomorphic to the cyclic algebra (L, a). The purpose of these exercises is to prove that every central simple algebra of degree 2 or 3 is cyclic (on the other hand one may construct central simple algebras of degree 4 which are not cyclic).

Exercise 1. We have seen that central simple k-algebra of degree 2 are cyclic (in fact quaternion algebras) when k has characteristic $\neq 2$. In this exercise, we consider the case when the characteristic of k is arbitrary.

(i) Let D be a finite-dimensional central simple k-algebra of degree 2. Show that D contains a Galois $\mathbb{Z}/2$-algebra as a k-subalgebra, and deduce that D is cyclic.

(ii) Conclude that every finite-dimensional central simple k-algebra of degree 2 is cyclic.

Exercise 2. Let A be a finite-dimensional central simple k-algebra.

(i) Show that the map
$$\nu\colon A \times A \to k \quad ; \quad (a,b) \mapsto \mathrm{Trd}_A(ab)$$
is a symmetric k-bilinear form.

(ii) Show that the form ν is nondegenerate, i.e. that the set
$$\{a \in A \mid \mathrm{Trd}_A(ax) = 0 \text{ for all } x \in A\}$$
is reduced to $\{0\}$. (Hint: Show that above set is a two-sided ideal of A.)

Exercise 3. Let A be a finite-dimensional central simple k-algebra of degree n. Let $x \in A$ and $P = \mathrm{Cprd}_A(x) \in k[X]$ its reduced characteristic polynomial.

(i) Show that $P(x) = 0 \in A$.

(ii) Assume that $x \in A^\times$, and let $Q = \mathrm{Cprd}_A(x^{-1}) \in k[X]$. Show that
$$P(X) = (-X)^n \cdot \mathrm{Nrd}_A(x) \cdot Q(X^{-1}) \in k[X] \subset k[X, X^{-1}].$$

Exercise 4. Let D be a finite-dimensional central simple division k-algebra of degree 3. When $E \subset D$ is a subset, we write
$$E^\perp = \{x \in D \mid \mathrm{Trd}_D(ex) = 0 \text{ for all } e \in E\}.$$

(i) If $V \subset D$ is a k-subspace, show that $\dim_k V^\perp = 9 - \dim_k V$. (Hint: Use Exercise 2.)

(ii) Let K be a commutative k-subalgebra of D. Show that $K = k$ or that K/k is a field extension of degree 3.

(iii) Let $x \in D^\times$ be such that $\mathrm{Trd}_D(x) = \mathrm{Trd}_D(x^{-1}) = 0$. Show that $x^3 = \mathrm{Nrd}_D(x) \in k \subset D$. (Hint: Use Exercise 3.)

(iv) Let $E \subset D$ be a maximal subfield. Find $z \in D - k$ such that $\mathrm{Trd}_D(z) = \mathrm{Trd}_D(z^{-1}) = 0$. (Hint: Pick a nonzero element $u_1 \in E^\perp$, and find $u_2 \in \{u_1^{-1}\}^\perp \cap E$ such that $u_2 \notin u_1 k$. Set $z = u_1 u_2^{-1}$.)

(v) Let F be the k-subalgebra of D generated by z. Find $y \in D - F$ such that

$$\mathrm{Trd}_D(yz) = \mathrm{Trd}_D(yz^2) = \mathrm{Trd}_D(z^{-1}y^{-1}) = \mathrm{Trd}_D(z^{-2}y^{-1}) = 0.$$

(Hint: Pick $v_1 \in F^\perp - F$. Let $V = \{z^{-1}, z^{-2}\}^\perp$, and find a nonzero $v_2 \in (v_1 V) \cap F$. Set $y = v_2^{-1} v_1$.)

(vi) Let L be the k-subalgebra of D generated by y. Show that zyz^{-1} commutes with y and deduce that $zyz^{-1} \in L$. (Hint: Show that $\mathrm{Nrd}_D(yz^2)\,\mathrm{Nrd}_D(z^{-1}) = \mathrm{Nrd}_D(yz)$, and expand using (iii).)

(vii) Show that $y \mapsto zyz^{-1}$ defines a structure of Galois $\mathbb{Z}/3$-algebra on L.

(viii) Deduce that $D \simeq (L, \mathrm{Nrd}_D(z))$.

(ix) Conclude that every finite-dimensional central simple k-algebra of degree 3 is cyclic (this is a theorem of Wedderburn).