The letter R denotes a (commutative unital) noetherian ring.

Exercise 1. Let M be a nonzero finitely generated R-module. Prove directly (using Zorn's Lemma) that $\mathrm{Supp}(M)$ possesses a minimal element.

Exercise 2. Let M be a finitely generated R-module, and let $x \in R$. Show that the following are equivalent:

(i) Multiplication by x is a nilpotent endomorphism of M.

(ii) The element x belongs to every prime of $\mathrm{Ass}(M)$.

Exercise 3. Let M be a finitely generated R-module, and $M_i \subset M_{i+1}$ a chain of submodules such that $M_i/M_{i+1} \simeq R/\mathfrak{p}_i$ with $\mathfrak{p}_i$ a prime of R. Let $\mathfrak{p}$ be a minimal element of $\mathrm{Supp}(M)$. Show that the number of indices i such that $\mathfrak{p}_i = \mathfrak{p}$ does not depend on the choice of the chain, and express this number purely in terms of M.

Exercise 4. Let M be an R-module.

(i) Show that $\mathfrak{p} \in \mathrm{Supp}(M)$ if and only if there is a submodule $N \subset M$ such that $\mathfrak{p} \in \mathrm{Ass}(M/N)$. (Hint: take N of the form $\mathfrak{p}m$ for a well-chosen $m \in M$).

(ii) Assume that M is finitely generated, and let $\mathfrak{p} \in \mathrm{Supp}(M)$. Show that there is a chain of submodules $0 = M_0 \subsetneq \cdots \subsetneq M_n = M$ such that $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \mathrm{Spec}(R)$ and moreover $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots \mathfrak{p}_n\}$.