Exercise 1. Let K be a number field, and $I \subset \mathcal{O}_K$ a nonzero ideal such that $\mathrm{N}(I) = \mathrm{card}(\mathcal{O}_K/I)$ is a prime number. Show that the ideal I is prime.

Exercise 2. Let K be a number field and $\mathfrak{p}$ a nonzero prime ideal of $\mathcal{O}_K$. Show that $\mathrm{N}(\mathfrak{p}) = \mathrm{card}(\mathcal{O}_K/\mathfrak{p}) \in \mathbb{N}$ is a power of a prime number.

Exercise 3. Let A be a local noetherian domain. Assume that the maximal ideal $\mathfrak{m}$ of A is principal. We assume that A is not a field.

(i) Show that $\bigcap_{n\in\mathbb{N}} \mathfrak{m}^n = 0$.

(ii) Let K be the fraction field of A, and $\pi \in A$ a generator of $\mathfrak{m}$. Show that every element $x \in K \smallsetminus \{0\}$ is of the form $x = \pi^n u$ for unique elements $u \in A^\times$ and $n \in \mathbb{Z}$.

(iii) Deduce that A is a discrete valuation ring.

Exercise 4. Let A be a discrete valuation ring with fraction field K. Let π be a uniformising parameter of A. Let $\mathfrak{m} = \pi A$ be the maximal ideal of A, and $k = A/\mathfrak{m}$. We denote by $P \mapsto \overline{P}$ the reduction map $A[X] \to k[X]$.

(i) Let $Q \in A[X]$ be such that $\overline{Q} \neq 0$ in $k[X]$. If $U \in K[X]$ is such that $QU \in A[X]$, show that $U \in A[X]$.

We now let $P \in A[X]$ be a monic polynomial such that $\overline{P} \in k[X]$ is irreducible, and consider the ring $B = A[X]/P$.

(ii) Show that the ring B is a domain. (Hint: use (i)).

(iii) Show that the ring B is a discrete valuation ring, with uniformising parameter π. (Hint: Use Exercise 3.)

(iv) Let
$$Q = X^n + a_{n-1}X^{n-1} + \cdots + a_0, \text{with } a_0, \ldots, a_{n-1} \in A.$$
Assume that a_0 is a uniformising parameter of A, and that $a_0 \mid a_i$ for all $i = 1, \ldots, n-1$. Show that $C = A[X]/Q$ is a discrete valuation ring, where the class of X is a uniformising parameter. (Hint: This is not a direct consequence of (iii).)