All rings a unital and commutative.

Exercise 1. Let A, B be two rings and $S \subset A$ a mutliplicatively closed subset. Show that the natural map $\operatorname{Hom}(S^{-1}A, B) \to \operatorname{Hom}(A, B)$ (Hom refers to ring morphisms) is injective, and that its image is the set of ring morphisms $\varphi\colon A \to B$ such that $\varphi(S) \subset B^{\times}$.

Exercise 2. Let A be a ring.

(i) Show that the set of nilpotent elements in A is the intersection of all prime ideals of A. [Hint: Use Sheet 5, Exercise 1.]

(ii) Let $I \subset A$ be an ideal. Show that the radical $\sqrt{I}$ is the intersection of all prime ideals of A containing I. [Hint: Use (i) for the ring A/I.].

(iii) Show that $I \mapsto V(I)$ induces a bijection between the set of radical ideals of A and the set of closed subsets of $\operatorname{Spec} A$.

Exercise 3. Let A be a ring and $S \subset A$ a multiplicatively closed subset.

(i) Show that (or recall how) an A-module morphism $f\colon M \to N$ induces a $S^{-1}A$-module morphism $S^{-1}f\colon S^{-1}M \to S^{-1}N$.

(ii) If the sequence of A-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact, show that the induced sequence of $S^{-1}A$-modules

$$0 \to S^{-1}M_1 \to S^{-1}M_2 \to S^{-1}M_3 \to 0$$

is exact.

Exercise 4. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Let $\kappa(\mathfrak{p})$ be the fraction field of $A/\mathfrak{p}$. We recall the ring $A_{\mathfrak{p}}$ is local with maximal ideal $\mathfrak{m}_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$. Show that the fields $\kappa(\mathfrak{p})$ and $A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ are isomorphic.