Exercise 1. Let X be a noetherian scheme, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module.

(i) Show that the $\mathcal{O}_X$-module $\mathcal{F}$ is locally free of rank r if and only if the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is free of rank r for every $x \in X$.

(ii) Show that $\mathcal{F}$ is locally free of rank one if and only if there is a coherent $\mathcal{O}_X$-module $\mathcal{G}$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \simeq \mathcal{O}_X$.

(iii) Assume that X is affine. Show that $\mathcal{F}$ is locally free if and only it there is a coherent $\mathcal{O}_X$-module $\mathcal{G}$ such that $\mathcal{F} \oplus \mathcal{G}$ is free.

(iv) Give a counterexample for (iii) when X is not affine.

Exercise 2. Let $f \colon Y \to X$ be a finite morphism of schemes. Assume that $f_*\mathcal{O}_Y$ is locally free $\mathcal{O}_X$-module of rank r. Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_Y$-module of rank e. Show that $f_*\mathcal{E}$ is a locally free coherent $\mathcal{O}_X$-module of rank re.

Exercise 3. Let A be a commutative ring, and $S = [t_0, \dots, t_n]$. Show that every closed subscheme Y of $\mathbb{P}^n_A$ is given by some homogeneous ideal of S (in other words Y is the closed subscheme $\operatorname{Proj} S/I = V_+(I)$ of $\operatorname{Proj} S$).