EXERCISES 2 (INTERSECTION THEORY)

Exercise 1. When $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, we define
$$\operatorname{Ass}(\mathcal{F}) = \{x \in X | \mathfrak{m}_x \in \operatorname{Ass}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)\}.$$
(Here $\mathfrak{m}_x$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,x}$.)

A closed embedding $Z \to X$ is called *locally principal* if there is a covering by open affine subschemes $U_i = \operatorname{Spec} A_i$ and elements $s_i \in A_i$ such that $Z \cap U_i = \operatorname{Spec}(A_i/s_iA_i)$.

(i) If $X = \operatorname{Spec} A$, and $M = H^0(X, \mathcal{F})$, show that $\operatorname{Ass}(M) = \operatorname{Ass}(\mathcal{F})$.

(ii) Show that a closed embedding $D \to X$ is an effective Cartier divisor if and only if:
— $D \to X$ is locally principal,
— and $D \cap \operatorname{Ass}(\mathcal{O}_X) = \varnothing$.

(iii) Let $f \colon Y \to X$ be a morphism, and $Z \to X$ a locally principal closed embedding. Then show that $f^{-1}Z \to Y$ is a locally principal closed embedding.

(iv) Let $f \colon Y \to X$ be a morphism, and $D \to X$ an effective Cartier divisor. Show that $f^{-1}D \to Y$ is an effective Cartier divisor if and only if $f(\operatorname{Ass}(\mathcal{O}_Y)) \cap D = \varnothing$.

(v) Assume that f is flat. Show that $f(\operatorname{Ass}(\mathcal{O}_Y)) \subset \operatorname{Ass}(\mathcal{O}_X)$.

(vi) Explain how we can reprove the lemma concerning pull-backs of effective Cartier divisors.

Exercise 2. (i) Let M be a finitely generated A-module (A noetherian). Show that the following morphism is injective:
$$M \to \bigoplus_{\mathfrak{p} \in \operatorname{Ass}(M)} M_{\mathfrak{p}}.$$

Let X be a variety.

(ii) Show that every generic point of X is in $\operatorname{Ass}(\mathcal{O}_X)$.

(iii) Show that X is reduced if and only if :
— for every generic point $x \in X$, the ring $\mathcal{O}_{X,x}$ is reduced,
— and $\operatorname{Ass}(\mathcal{O}_X)$ is the set of generic points.

Exercise 3. Let us denote by P the closed point $0 \in \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, that is, the integral closed subscheme defined by the ideal (x,y). Find closed subschemes Z_1, Z_2 of $\mathbb{A}^2_k$ such that
$$[Z_1] = [Z_2] = 3[P] \in \mathcal{Z}(\mathbb{A}^2_k),$$
but $Z_1 \not\simeq Z_2$ as schemes (and thus as closed subschemes of $\mathbb{A}^2_k$).

(more exercises next page)

Exercise 4. (i) Let $f\colon Y \to X$ be a closed immersion. Show that f is an isomorphism if and only if there is an open subscheme U of X containing $\mathrm{Ass}(\mathcal{O}_X)$ such that $Y \cap U \to U$ is an isomorphism.

(ii) Find a closed immersion $Y \to X$ and an open dense subscheme U of X such that $Y \cap U \to U$ is an isomorphism (and thus $[Y] = [X] \in \mathcal{Z}(X)$), but $Y \not\simeq X$.

Exercise 5. Let $R = k[x,y,z]/(zx, zy)$ and $X = \operatorname{Spec} R$. Let D be the closed subscheme of X defined by $(z - x)$.

(i) Show that $D \to X$ is an effective Cartier divisor.
(ii) What is the multiplicity m_i of X at each irreducible component X_i of X?
(iii) Compare $[D]$ and $\sum_i m_i [D \cap X_i]$ in $\mathcal{Z}(X)$.
(iv) Is this compatible with Proposition 1.3.5?

Exercise 6. Prove the snake lemma: A commutative diagram of A-modules

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\varphi'"] & M \arrow[r] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

with exact rows induces a long exact sequence of A-modules

$$0 \to \ker \varphi' \to \ker \varphi \to \ker \varphi'' \to \operatorname{coker} \varphi' \to \operatorname{coker} \varphi \to \operatorname{coker} \varphi'' \to 0.$$

Exercise 7. Prove the going-down theorem: If $Y \to X$ is flat, then every irreducible component of Y dominates an irreducible component of X.

Exercise 8. Let $f\colon Y \to X$ be a flat morphism, with X irreducible and Y equidimensional. Show that f has relative dimension $\dim Y - \dim X$.

Exercise 9. Let $f\colon Y \to X$ be a finite morphism such that the $\mathcal{O}_X$-module $f_*\mathcal{O}_Y$ is locally free of rank $d > 0$. Show that f is flat of relative dimension 0, and that $f_* \circ f^*$ is multiplication with d on $\mathcal{Z}(X)$.