EXERCISES 4 (INTERSECTION THEORY)

Exercise 1. Let A be a commutative ring. A *characteristic class* φ is the data of a group endomorphism $\varphi(E)$ of $\mathrm{CH}(X) \otimes A$ for every vector bundle $E \to X$, such that for every flat morphism $f \colon Y \to X$ having a relative dimension,

$$f^* \circ \varphi(E) = \varphi(f^*E) \circ f^* \colon \mathrm{CH}(X) \otimes A \to \mathrm{CH}(Y) \otimes A.$$

(i) Assume that a vector bundle E has a filtration by sub-bundles $E_{n+1} \subset E_n$ such that $L_n = E_n/E_{n+1}$ is a line bundle. Express the i-th Chern class $c_i(E)$ in terms of the classes $c_1(L_n)$.

(ii) Let $F \in A[x_1, \cdots, x_n]$ be a symmetric polynomial. Show that there is a unique characteristic class φ such that whenever E is a vector bundle with a filtration with successive quotients line bundles $L_1, \cdots, L_m$, then

$$\varphi(E) = F(c_1(L_1), \cdots, c_1(L_m)).$$

(iii) Let $P \in A[[t]]$ a power series. Show that there is unique characteristic class π_P such that
- If $0 \to E \to F \to G \to 0$ is an exact sequence of vector bundles, then $\pi_P(E) \circ \pi_P(G) = \pi_P(F)$.
- If $L \to X$ is a line bundle, then $\pi_P(L) = P(c_1(L))$.

(iv) Let $P \in A[[t]]$ a power series. Show that there is unique characteristic class γ_P such that
- If $0 \to E \to F \to G \to 0$ is an exact sequence of vector bundles, then $\gamma_P(E) + \gamma_P(G) = \gamma_P(F)$.
- If $L \to X$ is a line bundle, then $\gamma_P(L) = P(c_1(L))$.

(v) When $A = \mathbb{Q}$, and

$$P(t) = \sum_{n \geq 0} t^n/n!,$$

we define the *Chern character* $\mathrm{ch} = \gamma_P$. Show that

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \circ \mathrm{ch}(F)$$

for any vector bundles E, F.

Exercise 2. Let X be a smooth (or more generally locally factorial) variety. Show that the morphism $\mathrm{Pic}(X) \to \mathrm{CH}^1(X)$ mapping L to $c_1(L)[X]$ is a group isomorphism.

Exercise 3. When E is a vector bundle of rank r, its *determinant* $\det E$ is the line bundle $\Lambda^r E$. We say that E is *orientable* if $\det E$ is the trivial line bundle.

(i) Consider an exact sequence of vector bundles

$$0 \to E \to F \to L \to 0$$

where L is a line bundle. Show that $(\det E) \otimes L \simeq \det F$.

(ii) Show that $c_1(\det E) = c_1(E)$ for any vector bundle E, and deduce that $c_1(E) = 0$ when E is orientable.

(iii) Conversely, show that a vector bundle E over a smooth variety X is orientable as soon as $c_1(E)[X] = 0$.