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Algebraic Geometry II
Exercises 9

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Exercise 1. Show that there is a closed embedding $\mathbb{P}^n \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{P}^n \to \mathbb{P}^{(n+1)^2-1}$. Deduce that a composite of projective morphisms is projective.

Exercise 2. (Uses quasi-coherent modules.) Let $f\colon Y \to X$ be a scheme morphism. We assume that Y is noetherian. We construct below the *scheme-theoretic image* Z of the morphism f.

(i) Let $\mathcal{I} = \ker(\mathcal{O}_X \to f_*\mathcal{O}_Y)$. Show that $\mathcal{I}$ is a quasi-coherent ideal of $\mathcal{O}_X$.

(ii) Let $Z = V(\mathcal{I})$. Show that the morphism f factors through Z, and that for any closed subscheme Z' of X such that f factors though Z', we have $Z \subset Z'$ (as closed subschemes of X).

(iii) Show that $X \to Z$ is dominant.

(iv) Show that f is a closed immersion if and only if $Y \to Z$ is an isomorphism.

(v) Let $X \to S$ be a separated morphism of finite type, and assume that the composite $Y \to S$ is proper. Show that $Z \to S$ is proper.

Exercise 3. We give two proofs that any finite morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is projective.

(i) Let A be a commutative ring, and $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0X^0 \in A[X]$ a monic polynomial. Let $\widetilde{P}(T,S) = T^n + a_{n-1}T^{n-1}S^1 + \cdots + a_0T^0S^n$ be the homogeneisation of P. Show that $\operatorname{Spec}(A[X]/P) = \operatorname{Proj}(A[T,S]/\widetilde{P})$.

(ii) Deduce that any finite morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is projective.

(iii) Prove directly that any proper morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is projective (use Exercise 2).

Exercise 4. (Optional)

(i) Let R be a discrete valuation ring with fraction field K, and S a ring such that $R \subset S \subset K$. Show that $R = S$ or $S = K$.

(ii) Let R be a discrete valuation ring with fraction field K, and denote by $i\colon \operatorname{Spec} K \to \operatorname{Spec} R$ the generic point. Consider a commutative diagram of schemes with solid arrows

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "h"] & X \end{tikzcd}$$

If f is proper, show that there is a unique morphism h making the diagram commute.