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Algebraic Geometry II
Exercises 4

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The letter k denotes a field. A curve is an integral finite type separated k-scheme of dimension one. A curve is regular if the local ring at each closed point is a discrete valuation ring.

Exercise 1. Let X and Y be two curves, with Y regular. Show that any birational k-morphism $X \to Y$ is an open immersion. (Hint: Use the valuative criterion of separatedness.)

Exercise 2. Let X and Y be two curves, with X regular. Show that any proper dominant k-morphism $X \to Y$ is finite.

Exercise 3. (i) Let $P \in k[X,Y]$ be an irreducible polynomial such that $P(0,0) = 0$. We assume that

$$\frac{\partial P}{\partial X}(0,0) \neq 0 \qquad \text{or} \qquad \frac{\partial P}{\partial Y}(0,0) \neq 0.$$

Show that the localisation of the ring $k[X,Y]/P$ at the maximal ideal generated by (the images modulo P of) X and Y is a discrete valuation ring (use the Taylor expansion).

(ii) Let n be an integer prime to the characteristic of k. Show that the polynomial $X^n + Y^n - 1$ is irreducible.

(iii) Assume that k is algebraically closed and let n be an integer prime to the characteristic of k. Let $Z = V(X^n + Y^n - 1) \subset \mathbb{A}^2_k = \operatorname{Spec} k[X,Y]$. Show that Z is a regular curve.

Exercise 4. (i) Let X be a regular curve and U a non-empty open subscheme of X. Let Y be a proper k-scheme. Show that any k-morphism $U \to Y$ is the restriction of a unique k-morphism $X \to Y$.

(ii) Let X be a regular curve. Assume that there are two open immersions $X \to X_1$ and $X \to X_2$ where X_1 and X_2 are regular curves. We assume that X_1 and X_2 are proper over k. Show that X_1 and X_2 are k-isomorphic.