GALOIS COHOMOLOGY
EXERCISES 5 (SEMISIMPLE RINGS)

Exercise 1. Let k be a field. Let D be a finite-dimensional central division k-algebra and L/k a finite field extension such that $\operatorname{ind}(D) = [L : k]$. Show that L is a splitting field for D if and only if L can be embedded in D. (Hint: Look at the proof of Proposition 2.5.2 in the notes.)

Exercise 2. Let R be a ring and M an R-module. We are going to prove that the following conditions are equivalent:

(a) The module M is generated by its simple submodules.
(b) The module M is a direct sum of simple R-modules.
(c) Every submodule of M is a direct summand.

The R-module M will be called *semisimple* if it satisfies the above conditions.

(i) Let $S_i \to M$ for $i \in I$ be a collection of morphisms of R-modules, where each S_i is a simple module. When $K \subset I$, let us write $S_K = \bigoplus_{i \in K} S_i$, and denote by N_K the kernel of $S_K \to M$. Using Zorn's lemma, show that there is a maximal subset $K \subset I$ such that $N_K = 0$.
(ii) In the situation of (i), show that $S_I \to M$ and $S_K \to M$ have the same image.
(iii) Prove that (a) $\implies$ (b).
(iv) Prove that (b) $\implies$ (c). (Hint: use (i) and (ii) for an appropriate collection of morphisms $S_i \to Q$.)

For the rest of the exercise, we assume that (c) holds, and prove (a). So we let M' be the submodule of M generated by the simple submodules of M, and choose a submodule M'' such that $M' \oplus M'' = M$. We assume that $M'' \neq 0$ and come to a contradiction. By a previous exercise, we know that there are submodules $P \subset N \subset M''$ such that N/P is simple.

(v) Show that N/P is isomorphic to a submodule of N. (Hint: Introduce a submodule Q such that $P \oplus Q = M$.)
(vi) Conclude that (c) $\implies$ (a).

Exercise 3. A ring is called *semisimple* if it is semisimple as a module over itself (see the previous exercise). Prove the following assertions:

(i) Every semisimple ring is a finite direct sum of simple modules.
(ii) Every semisimple ring is artinian.
(iii) Every artinian simple ring is semisimple.
(iv) Every semisimple ring is isomorphic to a product $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, where $D_1, \ldots, D_r$ are division algebras and $n_1, \ldots, n_r$ are integers. (Hint: Proceed as in the proof of Wedderburn's Theorem.)
(v) The product of two semisimple rings is semisimple.
(vi) A ring is semisimple if and only if it is a finite product of artinian simple rings.