GALOIS COHOMOLOGY
EXERCISES 3 (QUATERNIONS)

Let k be a field of characteristic $\neq 2$.

Exercise 1. Show that every quaternion algebra can be realised as a subalgebra of $M_4(k)$.

Exercise 2. Let $k = \mathbb{Q}$, and consider that quaternion algebra $Q = (-1, -1)$ over k. Let $K = k(\xi)$ where $\xi \in \mathbb{C}$ is a primitive 5-th root of 1.

(i) Show that Q_K splits. (Hint : compute $-(\xi^3 + \xi^2)^2 - (\xi - \xi^4)^2$.)
(ii) Determine the subfields of K.
(iii) Deduce that K contains no quadratic extension splitting Q.

Exercise 3. Show that every element of a quaternion k-algebra satisfies a quadratic equation over k.

Exercise 4. Let D be a division k-algebra. We assume that for every $d \in D$, there is a nonzero polynomial $P \in k[X]$ of degree ≤ 2 such that $P(d) = 0$. We are going to prove that one of the following must happen:

— $D = k$,
— D is a quadratic field extension of k,
— D is a quaternion k-algebra.

Let us assume that $D \neq k$.

(i) Show that there is $i \in D - k$ and $a \in k$ such that $i^2 = a$.
(ii) Let K be the k-subalgebra of D generated by i. Show that K is a field and that $[K : k] = 2$.
(iii) Let $\varphi \colon D \to D$ be the map $d \mapsto i^{-1}di$. Show that $\varphi^2 = \mathrm{id}$, and that $D = D_+ \oplus D_-$ as K-vector spaces, where $D_+ = \ker(\varphi - \mathrm{id})$, $D_- = \ker(\varphi + \mathrm{id})$.
(iv) Show that D_+ is a K-subalgebra of D.
(v) Let $\alpha \in D_+$ and F the K-subalgebra of D_+ generated by α. Show that F is a field.
(vi) Show that $\alpha \in K$. (Hint: use the minimal polynomials of α and $\alpha + i$ to construct a linear equation over K having α as a solution.)
(vii) Deduce that $D_+ = K$.
(viii) Let now $\beta, \beta' \in D_-$. Show that $\beta\beta' \in D_+$, and deduce that $\dim_K D_- \in \{0, 1\}$.
(ix) Assume that $\dim_K D_- = 1$, and let j be a nonzero element of D_-. Let $A \in k[X]$ be a nonzero polynomial of degree ≤ 2 such that $A(j) = 0$. Show that $A(-j) = 0$, and deduce that $j^2 \in k$.
(x) Conclude.