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Algebraic Geometry II
Exercises 4

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Exercise 1. Let $\varphi\colon A \to B$ be morphism of commutative rings and $f\colon Y \to X$ the induced morphism of affine schemes. Let J be an ideal of B.

(i) Show that $V(\varphi^{-1}J)$ is the closure of $f(V(J))$.

(ii) Let I be an ideal of A. Let $T = \operatorname{Spec}(B/J)$ and $Z = \operatorname{Spec}(A/I)$. We assume that $f(Z) = T$ (viewing Z, resp. T, as a closed subset of Y, resp. X). We assume that B/J is reduced. Show that there is a unique morphism of schemes $T \to Z$ fitting into the commutative square of schemes

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

Exercise 2. (i) We say that a scheme is *noetherian* if it admits a finite open covering by spectra of noetherian rings. Show that a commutative ring A is noetherian if and only if the scheme $\operatorname{Spec} A$ is noetherian.

(ii) We say that a scheme is *reduced* if it admits an open covering by spectra of reduced rings. Show that a commutative ring A is reduced if and only if the scheme $\operatorname{Spec} A$ is reduced.

(iii) Show that a scheme X is reduced if and only if the ring $\mathcal{O}_{X,x}$ is reduced for every $x \in X$.