Fabien Morel
Olivier Haution

Algebraic Geometry I
Exercises 1

02.11.2016

All rings are commutative and unital. The letter k denotes an algebraically closed field. If f is an element of a ring A, we denote by $(f) = \{af | a \in A\}$ the ideal generated by f.

Exercise 1. Show that an ideal I of a ring A is maximal if and only if the quotient ring A/I is a field.

Exercise 2. (a) Let $f_1, \cdots, f_r$ be irreducible elements in $R = k[X_1, \cdots, X_n]$. Assume that $(f_i) \neq (f_j)$ for $i \neq j$. Show that

$$(f_1 \cdots f_r) = (f_1) \cap \cdots \cap (f_r).$$

(b) Is it still true when R is an arbitrary ring? (Give a proof or a counterexample.)

Exercise 3. Let $f_1, \cdots, f_r \in k[X_1, \cdots, X_n]$ and $\varphi \colon \mathbb{A}^n_k \to \mathbb{A}^r_k$ the map defined by

$$(x_1, \cdots, x_n) \mapsto (f_1(x_1, \cdots, x_n), \cdots, f_r(x_1, \cdots, x_n)).$$

(a) If Z is an irreducible closed subset of $\mathbb{A}^r_k$, is the subset $\varphi^{-1}(Z) \subset \mathbb{A}^n_k$ closed? irreducible?

(b) Let Y be an irreducible subset of $\mathbb{A}^n_k$. Is the subset $\varphi(Y) \subset \mathbb{A}^r_k$ irreducible? Is its closure $\overline{\varphi(X)} \subset \mathbb{A}^r_k$ irreducible?

(c) Find n, r and $f_1, \cdots, f_r$, and a closed subset $Y \subset \mathbb{A}^n_k$ such that $\varphi(Y) \subset \mathbb{A}^r_k$ is not closed.

Exercise 4. (a) Let A be a ring and I an ideal of A. Show that I is radical if and only if A/I is reduced.

(b) Show that every reduced finitely generated k-algebra is the coordinate ring $A(Y)$ of some algebraic set $Y \subset \mathbb{A}^n_k$ (for some n).

(c) Let $Y \subset \mathbb{A}^n_k$ be an algebraic set. For a closed subset Z of Y, let $I_Y(Z)$ be the kernel of the natural morphism of coordinate rings $A(Y) \to A(Z)$. Show that $Y \mapsto I_Y(Z)$ induces a bijection between the closed subsets of Y and the radical ideals of $A(Y)$.

Exercise 1. For an isomorphism $\varphi\colon \mathbb{A}^1 - 0 \to \mathbb{A}^1 - 0$, we denote by $\mathbb{A}^1 \sqcup_\varphi \mathbb{A}^1$ the glueing of $\mathbb{A}^1$ with itself along φ. Let $\chi\colon \mathbb{A}^1 - 0 \to \mathbb{A}^1 - 0$ be the morphism corresponding to the ring morphism $\mathbb{Z}[x, x^{-1}] \to \mathbb{Z}[x, x^{-1}]$ mapping x to x^{-1}. Show that the schemes $\mathbb{A}^1 \sqcup_{\mathrm{id}} \mathbb{A}^1$ and $\mathbb{A}^1 \sqcup_\chi \mathbb{A}^1$ are not isomorphic. [Hint: Look at the set of morphisms into $\mathbb{A}^1$.]

Exercise 2. Let $f\colon X \to Y$ be a scheme morphism and y a point of Y. Consider the natural morphism $\operatorname{Spec} \kappa(y) \to Y$ and the fibre $X_y = X \times_Y \operatorname{Spec} \kappa(y)$. Show that the projection $X_y \to X$ induces a homeomorphism between X_y and $f^{-1}\{y\}$.

Exercise 3. Let k be a field. Let $X = \operatorname{Spec} k[X, Y, Z]/(XY - Z)$ and $\mathbb{A}^1_k = \operatorname{Spec} k[T]$. Consider the morphism $X \to \mathbb{A}^1_k$ corresponding to the k-algebra morphism $k[T] \to k[X, Y, Z]/(XY - Z)$ mapping T to Z. Describe the fibre over each point of $\mathbb{A}^1_k$.

Exercise 4. (i) Let A be a ring, and $f_1, \ldots, f_n \in A$ elements generating the unit ideal. Assume that each ring A_{f_i} is noetherian. Show that the ring A is noetherian.

(ii) Show that an open subscheme of a locally noetherian scheme is locally noetherian.

(iii) Let X be a locally noetherian scheme, and $U = \operatorname{Spec} A$ an affine open subscheme of X. Using (i) and (ii), show that the ring A is noetherian.

The topological space underlying a scheme X is denoted by X_{top}.

Exercise 1. Let $f\colon X \to S$ and $g\colon Y \to S$ be two scheme morphisms. Consider the map

$$\varphi\colon (X \times_S Y)_{top} \to X_{top} \times Y_{top}$$

induced by the two projection morphisms $X \times_S Y \to X$ and $X \times_S Y \to Y$.

(i) Let $x \in X_{top}$ and $y \in Y_{top}$ be such that $f(x) = g(y) = s \in S_{top}$. Show that there is a homeomorphism

$$\varphi^{-1}\{(x, y)\} \simeq (\mathrm{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)))_{top}.$$

(ii) What is the image of φ?

Exercise 2. Let S be an affine scheme and $X \to S$ a separated morphism (i.e. the diagonal morphism $(\mathrm{id}_X, \mathrm{id}_X)\colon X \to X \times_S X$ is a closed immersion). Show that the intersection of two affine open subschemes of X is affine.

Exercise 3. Let X be a scheme and $\mathcal{Z}$ a closed subset of X_{top}. The purpose of this exercise it to prove the existence of a reduced scheme Z and a closed immersion $f\colon Z \to X$ inducing a homeomorphism $(Z)_{top} \to \mathcal{Z}$.

(i) Show that Z and f exist when X is affine.

(ii) Assume that Z and f exist. Show that for every morphism $g\colon T \to X$ with T reduced and $g(T_{top}) \subset \mathcal{Z}$, there exists a unique morphism $h\colon T \to Z$ such that $g = f \circ h$. [Hint: Begin with the case T affine.]

(iii) For $n = 1, 2$ let Z_n be a reduced scheme and $f_n\colon Z_n \to X$ a closed immersion inducing a homeomorphism $(Z_n)_{top} \to \mathcal{Z}$. Show that there exists a unique isomorphism $\varphi\colon Z_1 \to Z_2$ such that $f_1 = f_2 \circ \varphi$.

(iv) Conclude.

Exercise 4. Let $A \to B$ be a ring morphism making B a free A-module of rank n. Show that the morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is open.

The letter k denotes a field.

Exercise 1. Let Y and Z be two closed subschemes of X. Show that $(Y \times_X Z)_{top} = Y_{top} \cap Z_{top}$ as subspaces of X_{top}.

Exercise 2. Let X a scheme of finite type over $\operatorname{Spec} k$. Assume that X is integral with function field K (the residue field $\kappa(\eta)$ at the generic point η). Show that $\dim X$ coincides with the transcendence degree of K over k.

Exercise 3. Let K/k be a field extension, and A a finitely generated k-algebra. Show that $\dim(A \otimes_k K) = \dim A$.

Exercise 4. Let A, B two finitely generated k-algebras. Show that $\dim(A \otimes_k B) = \dim A + \dim B$.

Exercise 5. Let Z be a closed subscheme of $\mathbb{A}^2_k$. Consider the two projections $p, q \colon \mathbb{A}^2_k = \mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k \to \mathbb{A}^1_k$. If $\dim Z = 1$, show that $p|_Z$ or $q|_Z$ is dominant (i.e. has dense image).

Exercise 6. Let S be a commutative $\mathbb{N}$-graded ring, and $f \in S_0$. Describe the open subscheme $D_h(f) \subset \operatorname{Proj}(S)$, and give an example where it is not affine.

Exercise 7 (Optional)**.** Assume that k is algebraically closed. Let X be an integral scheme of finite type over k. Denote by X_{Var} the set of closed points in X, with its induced topology. Show that $X \mapsto X_{Var}$ induces an equivalence of categories between:

- integral, quasi-projective schemes over $\operatorname{Spec} k$ and morphisms of schemes over $\operatorname{Spec} k$,
- and quasi-projective k-varieties and their morphisms.

The letter k denotes an algebraically closed field.

Exercise 1. Let X be a topological space.

(i) If Y is a subset of X, show that $\dim Y \leq \dim X$.

(ii) Assume that $X = \bigcup_{\alpha \in A} U_\alpha$ with U_α an open subset of X for each $\alpha \in A$. Show that $\dim X = \sup_{\alpha \in A} \dim U_\alpha$.

Exercise 2. Given $f \in k[x,y]$, we denote by $\varphi_f \colon \mathbb{A}^2_k \to \mathbb{A}^1_k$ the map $(u,v) \mapsto f(u,v)$.

(i) For $t \in k = \mathbb{A}^1_k$, recall why $\varphi_f^{-1}\{t\}$ is an algebraic set in $\mathbb{A}^2_k$.

(ii) Find f such that $\varphi_f^{-1}\{t\}$ is irreducible for all $t \in k$.

(iii) Find f such that $\varphi_f^{-1}\{t\}$ is irreducible for all $t \in k - \{0\}$ but $\varphi_f^{-1}\{0\}$ is not irreducible.

Exercise 3. (i) Let R be a principal ideal domain. Show that $\dim R \in \{0,1\}$.

(ii) Let $Y = V(y - x^2) \subset \mathbb{A}^2_k$. Show $A(Y)$ is a polynomial ring in one variable over k.

(iii) Let $Z = V(xy-1) \subset \mathbb{A}^2_k$. Show $A(Z)$ is not a polynomial ring in one variable over k.

(iv) Show that Y and Z are irreducible, and compute their dimensions.

Exercise 4. Consider the map $\varphi \colon \mathbb{A}^1_k \to \mathbb{A}^2_k$ given by $t \mapsto (t^2, t^3)$. Show that $\varphi(\mathbb{A}^1_k)$ is an irreducible closed subset $Z \subset \mathbb{A}^2_k$, and that the induced map $\mathbb{A}^1_k \to Z$ is bijective.

The letter k denotes an algebraically closed field. We denote the polynomial ring $k[X_0, \cdots, X_n]$ by S, and by S_d its homogeneous component of degree d.

Exercise 1. Let $I \subset S$ be a homogeneous ideal. Show that the radical $\sqrt{I}$ is a homogeneous ideal.

Exercise 2. Let $I \subset S$ be a homogeneous ideal. Show that the following conditions are equivalent:

(a) $Z_h(I) = \varnothing \subset \mathbb{P}^n$.

(b) The ideal $\sqrt{I}$ is either equal to S or to $S_+ = \bigoplus_{d>0} S_d$.

(c) There is an integer d such that $S_d \subset I$.

Exercise 3. (i) Show that an algebraic set Y of $\mathbb{P}^n$ is irreducible if and only if its homogeneous ideal $I(Y) \subset S$ is prime.

(ii) Let $f \in S$ be an irreducible homogeneous polynomial. Show that $Z_h(f) \subset \mathbb{P}^n$ is irreducible.

(iii) Show that $\mathbb{P}^n$ is irreducible.

Exercise 4. Let f, g be two elements of $S_1 - \{0\}$. Assume that $Z_h(f) \neq Z_h(g)$. Show that $Z_h(f) \cap Z_h(g) \subset \mathbb{P}^n$ is a linear subspace $\mathbb{P}^{n-2}$ (in other words: find an element of $GL_{n+1}(k)$ such that the induced bijection of $\mathbb{P}^n$ sends $Z_h(f) \cap Z_h(g)$ to $Z_h(X_n, X_{n-1})$).

Exercise 5. (i) Assume that $n = 1$ and $a \in S_d - \{0\}$ with $d \geq 1$. Show that the cardinality of the set $Z_h(a) \subset \mathbb{P}^1$ is between 1 and d.

(ii) Assume that $n = 2$. Let $f \in S_d - \{0\}$ with $d \geq 1$ and $g \in S_1 - \{0\}$. If $Z_h(g) \subset Z_h(f) \subset \mathbb{P}^2$, show that $g \mid f$.

(iii) Assume that $n = 2$. Let $f \in S_d - \{0\}$ with $d \geq 1$ and $g \in S_1 - \{0\}$. If $Z_h(g) \not\subset Z_h(f) \subset \mathbb{P}^2$, show that the cardinality of the set $Z_h(f) \cap Z_h(g)$ is between 1 and d.

Exercise 1. (i) Let X be a variety and U a dense open subset of X. Show that the morphism $\mathcal{O}(X) \to \mathcal{O}(U)$ is injective.

(ii) Let X be a variety and U, V two dense open subsets of X such that $X = U \cup V$. Show that $\mathcal{O}(X) = \mathcal{O}(U) \cap \mathcal{O}(V) \subset \mathcal{O}(U \cap V)$.

(iii) Use the two open subsets of $\mathbb{P}^1$

$$\Omega_i = \mathbb{P}^1 - Z_h(X_i) = \{(x_0 : x_1) | x_i \neq 0\}$$

for $i = 0, 1$ to compute $\mathcal{O}(\mathbb{P}^1)$.

(iv) Is $\mathbb{P}^1$ affine?

Exercise 2. Let $f \in k[X_1, \cdots, X_n]$, and $U = \mathbb{A}^n - Z(f)$. Let $Y = Z(X_{n+1} \cdot f - 1) \subset \mathbb{A}^{n+1}$. Show that the morphism $\mathbb{A}^{n+1} \to \mathbb{A}^n$ given by $(x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_n)$ induces an isomorphism of varieties $Z \to U$.

Exercise 3. We say that a subset E of $\mathbb{A}^n$ is stable under the action of $k^\times$ when for all $\lambda \in k^\times$

$$(x_1, \cdots, x_n) \in E \Longrightarrow (\lambda x_1, \cdots, \lambda x_n) \in E.$$

Let Y be a closed subset of $\mathbb{A}^n$ stable under the action of $k^\times$. Show that each irreducible component of Y is stable under the action of $k^\times$.

Exercise 4. (Time permitting, the solution will be explained on Nov 23th, otherwise on Nov 30th.) Let X be a quasi-projective variety.

(i) Show that the set of morphisms $X \to \mathbb{A}^n$ may be identified with the product of n copies of $\mathcal{O}(X)$.

(ii) Deduce that for any affine variety Y, the set of morphisms of varieties $X \to Y$ may be identified with the set of k-algebra morphisms $\mathcal{O}(Y) \to \mathcal{O}(X)$.

(iii) Give a counterexample with Y non-affine.

All rings are commutative and unital.

Exercise 1. Let A be a ring and $f \in A$. Consider the localised ring $A_f = A[x]/(xf-1)$. Show that $A_f = 0$ if and only if f is nilpotent in A.

Exercise 2. Let $\varphi\colon A \to B$ be a ring morphism such that each element of $\ker\varphi$ is nilpotent. Show that $\operatorname{Spec}\varphi\colon \operatorname{Spec}B \to \operatorname{Spec}A$ is a homeomorphism under any of the following assumptions:

(i) The morphism φ is surjective.

(ii) Let p be a prime number such that $p \cdot 1 = 0$ in A. For each $b \in B$ we may find an integer $n \geq 0$ such that $b^{p^n} \in \operatorname{im}\varphi$.

Exercise 3. Let k be an algebraically closed field. Show that every quasi-projective variety over k is covered by open affine varieties. [Hint : use the covering of $\mathbb{P}^n_k$ by $n+1$ copies of $\mathbb{A}^n_k$.]

Exercise 4. Let n be an integer ≥ 2 and k an algebraically closed field.

(i) Show that the morphism $\mathcal{O}(\mathbb{A}^n_k) \to \mathcal{O}(\mathbb{A}^n_k - 0)$ is bijective. [Hint : use Exercise 1 of the previous sheet, and the opens $U_{X_i} = \mathbb{A}^n_k - Z(X_i)$.]

(ii) Deduce that the variety $\mathbb{A}^n_k - 0$ is not affine.

(iii) Let $f \in \mathcal{O}(\mathbb{A}^n_k - 0)$. Assume that f is $k^\times$-invariant, in other words that for every $\lambda \in k^\times$ and $(x_1, \dots, x_n) \in k^n - \{0\}$ we have $f(\lambda x_1, \dots, \lambda x_n) = f(x_1, \dots, x_n)$. Show that f is constant.

(iv) Deduce that $\mathcal{O}(\mathbb{P}^{n-1}_k) = k$.

When $\mathcal{F}$ is a presheaf on X, we denote by $\mathcal{F}_x$ the stalk at $x \in X$, and by $a(\mathcal{F})$ the sheaf associated with $\mathcal{F}$.

Exercise 1. Let $\pi : Y \to X$ be a local homeomorphism, and Γ_π its sheaf of sections. Show that for any $x \in X$ the natural map $(\Gamma_\pi)_x \to \pi^{-1}\{x\}$ is a bijection.

Exercise 2. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on a topological space X.

(i) Let $x \in X$. Show that the natural map $(\mathcal{F} \times \mathcal{G})_x \to \mathcal{F}_x \times \mathcal{G}_x$ is a bijection.

(ii) Deduce that the natural morphism $a(\mathcal{F} \times \mathcal{G}) \to a(\mathcal{F}) \times a(\mathcal{G})$ is an isomorphism.

Exercise 3. Let E be a set and X a topological space. The value of the *constant sheaf* $\underline{E}$ on an open subset U of X is the set of continuous maps $U \to E$, where E is endowed with the discrete topology. Show that $\underline{E}$ is isomorphic to the sheaf associated with the presheaf taking the value E on every open subset of X.

Exercise 4. Let $\mathcal{F}$ be a presheaf on a topological space X.

(i) Let $x \in X$ and $i : \{x\} \to X$ be the inclusion. Let E be a set, and $\underline{E}$ the constant sheaf on $\{x\}$ associated with E (i.e. $\underline{E}(\{x\}) = E$ and $\underline{E}(\varnothing) = \{*\}$). Show that the set of presheaf morphisms $\mathcal{F} \to i_*\underline{E}$ is in bijection with the set of maps $\mathcal{F}_x \to E$.

(ii) Let $j : U \to X$ be the inclusion of an open subset. Let $\mathcal{G}$ be a presheaf on U. Show that the set of presheaf morphisms $\mathcal{F} \to j_*\mathcal{G}$ and $\mathcal{F}|_U \to \mathcal{G}$ are in bijection.

All rings a unital and commutative.

Exercise 1. Let A, B be two rings and $S \subset A$ a mutliplicatively closed subset. Show that the natural map $\mathrm{Hom}(S^{-1}A, B) \to \mathrm{Hom}(A, B)$ (Hom refers to ring morphisms) is injective, and that its image is the set of ring morphisms $\varphi\colon A \to B$ such that $\varphi(S) \subset B^\times$.

Exercise 2. Let A be a ring.

(i) Show that the set of nilpotent elements in A is the intersection of all prime ideals of A. [Hint: Use Sheet 5, Exercise 1.]

(ii) Let $I \subset A$ be an ideal. Show that the radical $\sqrt{I}$ is the intersection of all prime ideals of A containing I. [Hint: Use (i) for the ring A/I.].

(iii) Show that $I \mapsto V(I)$ induces a bijection between the set of radical ideals of A and the set of closed subsets of $\operatorname{Spec} A$.

Exercise 3. Let A be a ring and $S \subset A$ a multiplicatively closed subset.

(i) Show that (or recall how) an A-module morphism $f\colon M \to N$ induces a $S^{-1}A$-module morphism $S^{-1}f\colon S^{-1}M \to S^{-1}N$.

(ii) If the sequence of A-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact, show that the induced sequence of $S^{-1}A$-modules

$$0 \to S^{-1}M_1 \to S^{-1}M_2 \to S^{-1}M_3 \to 0$$

is exact.

Exercise 4. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Let $\kappa(\mathfrak{p})$ be the fraction field of $A/\mathfrak{p}$. We recall the ring $A_\mathfrak{p}$ is local with maximal ideal $\mathfrak{m}_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$. Show that the fields $\kappa(\mathfrak{p})$ and $A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$ are isomorphic.

Exercise 1. Let A be a commutative unital ring, and $\mathfrak{p}$ a prime ideal of A. Show that the set $\{\mathfrak{p}\}$ is closed in $\operatorname{Spec} A$ if and only if the ideal $\mathfrak{p}$ is maximal in A.

Exercise 2. Let $(X, \mathcal{O}_X)$ be a scheme and U an open subset of X. Show that $(U, \mathcal{O}_X|_U)$ is a scheme.

Exercise 3. A topological space X is called *quasi-compact* if for every family $\{U_i, i \in I\}$ of open subsets of X such that $X = \bigcup_{i \in I} U_i$ we may find a finite subset $J \subset I$ such that $X = \bigcup_{i \in J} U_i$. Show that $\operatorname{Spec} A$ is quasi-compact, when A is commutative unital ring.

Exercise 4. Let k be a field, and $A = k[X_i | i \in \mathbb{N}]$ the polynomial ring in a countable infinite set of variables. Let $I \subset A$ be the ideal generated by the variables X_i for $i \in \mathbb{N}$. Show that the topological space $\operatorname{Spec} A - V(I)$ is not quasi-compact. [Hint: Look at the chain of closed subsets $\cdots \subset V(X_1, \ldots, X_{s+1}) \subset V(X_1, \ldots, X_s) \subset \ldots$]

Exercise 5. Let $(X, \mathcal{O}_X)$ be a scheme and $e \in \mathcal{O}_X(X)$ be such that $e^2 = e$. Let $f = 1 - e$. Show that there exists open subsets X_e and X_f of X such that $X = X_e \cup X_f$ and $X_e \cap X_f = \varnothing$, and such that $e|_{X_e} \in \mathcal{O}_X(X_e)^\times$ and $f|_{X_f} \in \mathcal{O}_X(X_f)^\times$. (For a ring R we denote by $R^\times$ its set of invertible elements.)

Exercise 1. Find a morphism of ringed spaces $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$ which is not a scheme morphism.

Exercise 2. Let X and Y be two schemes. Show that the functor associating to an open subset U of X the set of scheme morphisms $U \to Y$ is a sheaf on the topological space X.

Exercise 3. Let $f \in \mathcal{O}_X(X)$ and $\varphi_f \colon X \to \mathbb{A}^1$ the corresponding morphism. Consider the open subset $X_f = \varphi_f^{-1}(\mathbb{A}^1 - 0)$. Show that $x \in X_f$ if and only $f_x \in (\mathcal{O}_{X,x})^\times$.

Exercise 4. Show that a scheme X is affine if and only if the canonical morphism $X \to \operatorname{Spec}(\mathcal{O}_X(X))$ is an isomorphism.

Exercise 5. Let X be a scheme. Show that every irreducible closed subset of X is the closure of a unique point.

Exercise 6. Let X be a scheme and K a field. Show that a scheme morphism $\operatorname{Spec} K \to X$ corresponds to the data of a point $x \in X$ and a field extension $\kappa(x) \to K$.

Exercise 7. Let $f \colon X \to Y$ be a scheme morphism of finite type. Assume that X and Y are integral with generic points η_X and η_Y, and that $f(\eta_X) = \eta_Y$. Show that the following are equivalent:

(a) The natural field extension $\kappa(\eta_Y) \to \kappa(\eta_X)$ is an isomorphism.

(b) There exists non-empty open subschemes U of X and V of Y such that $f(U) \subset V$ and f induces an isomorphism $U \to V$.

Exercise 8. (i) Let X be a quasi-compact scheme, and $f, a \in \mathcal{O}_X(X)$. Assume that $a|_{X_f} = 0$. Show that there exists an integer $n > 0$ such that $f^n a = 0$.

(ii) Assume that X has a finite cover by open affine subschemes U_i such that $U_i \cap U_j$ is quasi-compact for each pair (i, j). Let $f \in \mathcal{O}_X(X)$ and $b \in \mathcal{O}_{X_f}(X_f)$. Show that for some $n \geq 0$ the section $f^n b \in \mathcal{O}_{X_f}(X_f)$ is the restriction of a section in $\mathcal{O}_X(X)$.

(iii) Assume that X has a finite cover by open affine subschemes U_i such that $U_i \cap U_j$ is quasi-compact for each pair (i, j). Show that the natural morphism $\mathcal{O}_X(X)[1/f] \to \mathcal{O}_{X_f}(X_f)$ is a bijection for any $f \in \mathcal{O}_X(X)$.

(iv) Show that a scheme X is affine if and only if there are elements $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ generating the unit ideal, and such that each X_{f_i} is affine. [Hint: Use (iii) and Exercise 4.]

(v) Deduce that a morphism $f \colon X \to Y$ is affine if and only if for any open affine subscheme V of Y the open subscheme $f^{-1}V$ of X is affine.