The letter k denotes a field. When Z is a closed subscheme of a separated scheme X, we denote by $\widetilde{X}_Z$ the blow-up of Z in X.

Exercise 1. Let X be a reduced separated scheme, and Z a closed subscheme of X. Show that $\widetilde{X}_Z$ is reduced.

Exercise 2. Let $R = k[x,y]/(xy)$, and $E = \operatorname{Spec} R$.

(i) Show that the scheme E is reduced, and has exactly two irreducible components X, Y, both isomorphic to $\mathbb{A}^1_k$. Show that the scheme $X \cap Y$ (defined as $X \times_E Y$) is isomorphic to $\operatorname{Spec} k$.

(ii) Show the blow-up morphism $\widetilde{E}_X \to E$ coincides with the closed immersion $Y \to E$. (Hint: use the functoriality of the blow-up to construct a morphism $Y \to \widetilde{E}_X$; compute the fiber of $\widetilde{E}_X \to E$ over $X \cap Y$.)

(iii) Describe the blow-up of $X \cap Y$ in E.

Exercise 3. Let X be a curve over k, and Z a closed subscheme of X such that $Z \neq X$. Let $b \colon \widetilde{X}_Z \to X$ be the blow-up morphism, and $n \colon \widetilde{X} \to X$ the normalisation morphism (of X in its own function field).

(i) Show that $X - (Z \cup X_{Sing})$ is an open subscheme of $\widetilde{X}$ and $\widetilde{X}_Z$, and deduce the existence of a birational morphism $\widetilde{X} \to \widetilde{X}_Z$ over X.

(ii) Let $z \in Z$. Denote by $\#E$ the cardinality of a set E. Show that
$$1 \leq \#(b^{-1}\{z\}) \leq \#(n^{-1}\{z\}).$$

(iii) Let $X = \operatorname{Spec} k[x,y]/(x^2 - y^3)$ and Z be the closed subscheme of X defined by the ideal (x,y). Show that the morphism $\widetilde{X}_Z \to X$ induces a bijection on the underlying sets.

Exercise 4. Consider the graded k-algebra $S_* = k[T_0, \ldots, T_n]$ where T_i has degree 1.

(i) Show that the set $\operatorname{Spec} S_* - \operatorname{Spec} S_0$ consists of those primes $\mathfrak{p} \subset S_*$ such that $S_+ \not\subset \mathfrak{p}$.

(ii) For a prime $\mathfrak{p} \subset S_*$, let us denote by $\mathfrak{p}^h$ the ideal of S_* generated by the homogeneous elements of $\mathfrak{p}$. Observe that $\mathfrak{p}^h \subset \mathfrak{p}$ (in particular $S_+ \not\subset \mathfrak{p}^h$ if $S_+ \not\subset \mathfrak{p}$). Show that $\mathfrak{p}^h$ is a homogeneous prime ideal of S_*. Show that if I is a homogeneous ideal of S_* such that $\mathfrak{p}^h \subset I \subset \mathfrak{p}$, then $I = \mathfrak{p}^h$.

(iii) Show that the set map underlying the canonical morphism $\mathbb{A}^{n+1}_k - 0 \to \mathbb{P}^n_k$ is defined by $\mathfrak{p} \mapsto \mathfrak{p}^h$.