Exercise 1. Find a morphism of ringed spaces $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$ which is not a scheme morphism.

Exercise 2. Let X and Y be two schemes. Show that the functor associating to an open subset U of X the set of scheme morphisms $U \to Y$ is a sheaf on the topological space X.

Exercise 3. Let $f \in \mathcal{O}_X(X)$ and $\varphi_f \colon X \to \mathbb{A}^1$ the corresponding morphism. Consider the open subset $X_f = \varphi_f^{-1}(\mathbb{A}^1 - 0)$. Show that $x \in X_f$ if and only $f_x \in (\mathcal{O}_{X,x})^\times$.

Exercise 4. Show that a scheme X is affine if and only if the canonical morphism $X \to \operatorname{Spec}(\mathcal{O}_X(X))$ is an isomorphism.

Exercise 5. Let X be a scheme. Show that every irreducible closed subset of X is the closure of a unique point.

Exercise 6. Let X be a scheme and K a field. Show that a scheme morphism $\operatorname{Spec} K \to X$ corresponds to the data of a point $x \in X$ and a field extension $\kappa(x) \to K$.

Exercise 7. Let $f \colon X \to Y$ be a scheme morphism of finite type. Assume that X and Y are integral with generic points η_X and η_Y, and that $f(\eta_X) = \eta_Y$. Show that the following are equivalent:

(a) The natural field extension $\kappa(\eta_Y) \to \kappa(\eta_X)$ is an isomorphism.

(b) There exists non-empty open subschemes U of X and V of Y such that $f(U) \subset V$ and f induces an isomorphism $U \to V$.

Exercise 8. (i) Let X be a quasi-compact scheme, and $f, a \in \mathcal{O}_X(X)$. Assume that $a|_{X_f} = 0$. Show that there exists an integer $n > 0$ such that $f^n a = 0$.

(ii) Assume that X has a finite cover by open affine subschemes U_i such that $U_i \cap U_j$ is quasi-compact for each pair (i, j). Let $f \in \mathcal{O}_X(X)$ and $b \in \mathcal{O}_{X_f}(X_f)$. Show that for some $n \geq 0$ the section $f^n b \in \mathcal{O}_{X_f}(X_f)$ is the restriction of a section in $\mathcal{O}_X(X)$.

(iii) Assume that X has a finite cover by open affine subschemes U_i such that $U_i \cap U_j$ is quasi-compact for each pair (i, j). Show that the natural morphism $\mathcal{O}_X(X)[1/f] \to \mathcal{O}_{X_f}(X_f)$ is a bijection for any $f \in \mathcal{O}_X(X)$.

(iv) Show that a scheme X is affine if and only if there are elements $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ generating the unit ideal, and such that each X_{f_i} is affine. [Hint: Use (iii) and Exercise 4.]

(v) Deduce that a morphism $f \colon X \to Y$ is affine if and only if for any open affine subscheme V of Y the open subscheme $f^{-1}V$ of X is affine.