Brauer Groups of fields

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Contents

Note on the literature

The main references that we used in preparing these notes is the book of Gille and Szamuely [**GS17**]. As always, Serre's books [**Ser62, Ser02**] provide excellent accounts. There is also very useful material contained in the Stack's project [**Sta**] (available online). Kersten's book [**Ker07**] (in German, available online) provides a very gentle introduction to the subject.

For the first part (on noncommutative algebra), we additionally used Draxl's [**Dra83**] and Pierce's [**Pie82**], as well as Lam's book [**Lam05**] (which uses the language of quadratic forms) for quaternion algebras. For the second part (on torsors), we used the book of involutions [**KMRT98**, Chapters V and VII].

Part 1

Noncommutative Algebra

CHAPTER 1

Quaternion algebras

This chapter will serve as an introduction to the theory of central simple algebras, by developing some aspects of the general theory in the simplest case of quaternion algebras. The results proved here will not really be used in the sequel, and many of them will be in fact substantially generalised by other means. Rather we would like to show what can be done "by hand", which may help appreciate the more sophisticated methods developed in the sequel.

Quaternions are historically very significant; since their discovery by Hamilton in 1843, they have played an influential role in various branches of mathematics. A particularity of these algebras is their deep relations with quadratic forms, which is not really a systematic feature of central simple algebras. For this reason, we will merely hint at the connections with quadratic form theory.

1. The norm form

All rings will be assumed to be unital and associative (but often noncommutative!). The set of elements of a ring R admitting a two-sided inverse is a group, that we denote by $R^\times$.

We fix a base field k. A k-algebra is a ring A equipped with a structure of k-vector space such that the multiplication map $A \times A \to A$ is k-bilinear. A morphism of k-algebras is a ring morphism which is k-linear. If A is nonzero, the map $k \to A$ given by $\lambda \mapsto \lambda 1$ is injective, and we will view k as a subring of A. Observe that the bilinearity of the multiplication map implies that for any $\lambda \in k$ and $a \in A$

$$\lambda a = (\lambda a)1 = a(\lambda 1) = a\lambda. \tag{1.1.a}$$

In this chapter on quaternion algebras, we will assume that the characteristic of k is not equal to two (i.e. $2 \neq 0$ in k).

Definition 1.1.1. Let $a, b \in k^\times$. We define a k-algebra (a, b) as follows. A basis of (a, b) as k-vector space is given by $1, i, j, ij$. It is easy to verify that (a, b) admits a unique k-algebra structure such that

$$i^2 = a, \quad j^2 = b, \quad ij = -ji. \tag{1.1.b}$$

We will call i, j the *standard generators* of (a, b). An algebra isomorphic to (a, b) for some $a, b \in k^\times$ will be called a *quaternion algebra*.

Lemma 1.1.2. *Let A be a 4-dimensional k-algebra. If $i, j \in A$ satisfy the relations (1.1.b) for some $a, b \in k^\times$, then $A \simeq (a, b)$.*

Proof. It will suffice to prove that the elements $1, i, j, ij$ are linearly independent over k. Since i anticommutes with j, the elements $1, i, j$ must be linearly independent (recall that the characteristic of k differs from 2). Now assume that $ij = u + vi + wj$, with $u, v, w \in k$. Then

$$0 = i(ij + ji) = i(ij) + (ij)i = i(u + vi + wj) + (u + vi + wj)i = 2ui + 2av,$$

hence $u = v = 0$ by linear independence of $1, i$. So $ij = wj$, hence $ij^2 = wj^2$ and thus $bi = bw$, a contradiction with the linear independence of $1, i$. $\square$

The following observations will be used without explicit mention.

LEMMA 1.1.3. *Let $a, b \in k^\times$. Then*
(i) $(a, b) \simeq (b, a)$,
(ii) $(a, b) \simeq (a\alpha^2, b\beta^2)$ *for any* $\alpha, \beta \in k^\times$.

PROOF. (i) : We let i', j' be the standard generators of (b, a), and apply Lemma 1.1.2 with $i = j'$ and $j = i'$.

(ii): We let i'', j'' be the standard generators of $(a\alpha^2, b\beta^2)$, and apply Lemma 1.1.2 with $i = \alpha^{-1}i''$ and $j = \beta^{-1}j''$. $\square$

LEMMA 1.1.4. *For any $b \in k^\times$, the k-algebra $(1, b)$ is isomorphic to the algebra $M_2(k)$ of 2 by 2 matrices with coefficients in k.*

PROOF. The matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, J = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} \in M_2(k)$$

satisfy $I^2 = 1, J^2 = b, IJ = -JI$. Thus the statement follows from Lemma 1.1.2. $\square$

From now on, the letter Q will denote a quaternion algebra over k.

DEFINITION 1.1.5. An element $q \in Q$ such that $q^2 \in k$ and $q \notin k^\times$ will be called a *pure quaternion*.

LEMMA 1.1.6. *Let $a, b \in k^\times$ and $x, y, z, w \in k$. The element $x + yi + zj + wij$ in the quaternion algebra (a, b) is a pure quaternion if and only if $x = 0$.*

PROOF. This follows from the computation

$$(x + yi + zj + wij)^2 = x^2 + ay^2 + bz^2 - abw^2 + 2x(yi + zj + wij). \qquad \square$$

LEMMA 1.1.7. *The subset $Q_0 \subset Q$ of pure quaternions is a k-subspace, and we have $Q = k \oplus Q_0$ as k-vector spaces.*

PROOF. Letting $a, b \in k^\times$ be such that $Q \simeq (a, b)$, this follows from Lemma 1.1.6. $\square$

It follows from Lemma 1.1.7 that every $q \in Q$ may be written uniquely as $q = q_1 + q_2$, where $q_1 \in k$ and q_2 is a pure quaternion. We define the *conjugate of* q as $\overline{q} = q_1 - q_2$. The following properties are easily verified, for any $p, q \in Q$:

(i) $q \mapsto \overline{q}$ is k-linear.
(ii) $\overline{\overline{q}} = q$.
(iii) $q = \overline{q} \iff q \in k$.
(iv) $q = -\overline{q} \iff q \in Q_0$.
(v) $q\overline{q} \in k$.
(vi) $q\overline{q} = \overline{q}q$.
(vii) $\overline{pq} = \overline{q}\,\overline{p}$.

DEFINITION 1.1.8. We define the *(quaternion) norm map* $N\colon Q \to k$ by $q \mapsto q\overline{q} = \overline{q}q$.

Observe that the norm map is multiplicative:

$$N(pq) = N(p)N(q) \quad \text{for all } p, q \in Q.$$

If $a, b \in k^\times$ are such that $Q = (a, b)$ and $q = x + yi + zj + wij$ with $x, y, z, w \in k$, then

$$N(q) = x^2 - ay^2 - bz^2 + abw^2. \tag{1.1.c}$$

LEMMA 1.1.9. *An element $q \in Q$ admits a two-sided inverse if and only if $N(q) \neq 0$.*

PROOF. If $N(q) \neq 0$, then q is a two-sided inverse of $N(q)^{-1}\overline{q}$. Conversely, if $p \in Q$ is such that $pq = 1$, then $N(p)N(q) = 1$, hence $N(q) \neq 0$. □

We will give below a list of criteria for a quaternion algebra to be isomorphic to $M_2(k)$. In order to do so, we first need some definitions.

DEFINITION 1.1.10. A ring (resp. a k-algebra) D is called *division* if it is nonzero and every nonzero element of D admits a two-sided inverse. Such rings are also called skew-fields in the literature.

REMARK 1.1.11. Let A be a finite-dimensional k-algebra and $a \in A$. We claim that a left inverse of a is automatically a two-sided inverse. Indeed, assume that $u \in A$ satisfies $ua = 1$. Then the k-linear morphism $A \to A$ given by $x \mapsto ax$ is injective (as $ax = 0$ implies $x = uax = 0$), hence surjective by dimensional reasons. In particular 1 lies in its image, hence there is $v \in A$ such that $av = 1$. Then $u = u(av) = (ua)v = v$.

A similar argument shows that a right inverse of a is automatically a two-sided inverse.

DEFINITION 1.1.12. Let A be a commutative finite-dimensional k-algebra. The (algebra) norm map $\mathrm{N}_{A/k}\colon A \to k$ is defined by mapping $a \in A$ to the determinant of the k-linear map $A \to A$ given by $x \mapsto ax$.

It follows from the multiplicativity of the determinant that

$$\mathrm{N}_{A/k}(ab) = \mathrm{N}_{A/k}(a)\,\mathrm{N}_{A/k}(b) \quad \text{for all } a, b \in A.$$

When $a \in k$, we consider the field extension

$$k(\sqrt{a}) = \begin{cases} k & \text{if } a \text{ is a square in } k, \\ k[X]/(X^2 - a) & \text{if } a \text{ is not a square in } k. \end{cases}$$

In the second case, let $\alpha \in k(\sqrt{a})$ be such that $\alpha^2 = a$ (such an element is determined only up to sign by the field extension $k(\sqrt{a})/k$). Every element of $k(\sqrt{a})$ is represented as $x + y\alpha$ for uniquely determined $x, y \in k$, and

$$\mathrm{N}_{k(\sqrt{a})/k}(x + y\alpha) = x^2 - ay^2. \tag{1.1.d}$$

PROPOSITION 1.1.13. *Let $a, b \in k^\times$. The following are equivalent.*

(i) $(a, b) \simeq M_2(k)$.
(ii) (a, b) *is not a division ring.*
(iii) The quaternion norm map $(a, b) \to k$ *has a nontrivial zero.*
(iv) We have $b \in \mathrm{N}_{k(\sqrt{a})/k}(k(\sqrt{a}))$.
(v) There are $x, y \in k$ *such that* $ax^2 + by^2 = 1$.
(vi) There are $x, y, z \in k$*, not all zero, such that* $ax^2 + by^2 = z^2$.

Proof. (i) $\Rightarrow$ (ii) : The nonzero matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(k)$$

is not invertible.

(ii) $\Rightarrow$ (iii) : This follows from Lemma 1.1.9.

(iii) $\Rightarrow$ (iv) : We may assume that a is not a square in k, and choose $\alpha \in k(\sqrt{a})$ such that $\alpha^2 = a$. Let $q = x + yi + zj + wij$ be a nontrivial zero of the norm map, where $x, y, z, w \in k$. Then by the formula (1.1.c)

$$0 = x^2 - ay^2 - bz^2 + abw^2,$$

hence $x^2 - ay^2 = b(z^2 - aw^2)$. Assume that $z^2 - aw^2 = 0$. Then $z = w = 0$, because a is not a square. Also $x^2 - ay^2 = 0$, and for the same reason $x = y = 0$. Thus $q = 0$, a contradiction. Therefore $z^2 - aw^2 \neq 0$, and by (1.1.d)

$$b = \frac{x^2 - ay^2}{z^2 - aw^2} = \frac{\mathrm{N}_{k(\sqrt{a})/k}(x + y\alpha)}{\mathrm{N}_{k(\alpha)/k}(z + w\alpha)} = \mathrm{N}_{k(\alpha)/k}\left(\frac{x + y\alpha}{z + w\alpha}\right).$$

(iv) $\Rightarrow$ (v) : Let $\alpha \in k(\sqrt{a})$ be such that $\alpha^2 = a$. If $\alpha \in k$, then we may take $x = \alpha^{-1}$ and $y = 0$. If $\alpha \notin k$, then by (iv) there are $u, v \in k$ such that $b = \mathrm{N}_{k(\sqrt{a})/k}(u + v\alpha)$. Then $b = u^2 - av^2$ by (1.1.d). If $u \neq 0$, we may take $x = vu^{-1}$ and $y = u^{-1}$. Assume that $u = 0$. Then $b = -av^2$, and in particular $v \neq 0$. Let

$$x = \frac{a+1}{2a} \quad \text{and} \quad y = \frac{a-1}{2av}.$$

Then

$$ax^2 + by^2 = ax^2 - av^2y^2 = \frac{a^2 + 2a + 1}{4a} - \frac{a^2 - 2a + 1}{4a} = 1.$$

(v) $\Rightarrow$ (vi) : Take $z = 1$.

(vi) $\Rightarrow$ (i) : By Lemma 1.1.4 (and Lemma 1.1.3 (ii)) we may assume that a is not a square in k, so that $y \neq 0$. Applying Lemma 1.1.14 below with $u = xy^{-1}, v = zy^{-1}$ and $c = b$ yields $(a, b) \simeq (a, b^2)$. Since $(a, b^2) \simeq (1, a)$ (by Lemma 1.1.3), we obtain (i) using Lemma 1.1.4. $\square$

Lemma 1.1.14. *Let $a, b, c \in k^\times$, and assume that $au^2 + c = v^2$ for some $u, v \in k$. Then $(a, b) \simeq (a, bc)$.*

Proof. Denote by i', j' the standard generators of (a, bc). Set

$$i = i', \quad j = c^{-1}(vj' + ui'j') \in (a, bc).$$

The relation $i'j' + j'i' = 0$ implies that $ij + ji = 0$. We have $i^2 = i'^2 = a$, and

$$j^2 = c^{-2}(bcv^2 - abcu^2) = bc^{-1}(v^2 - au^2) = b.$$

It follows from Lemma 1.1.2 that $(a, bc) \simeq (a, b)$. $\square$

Definition 1.1.15. A quaternion algebra satisfying the conditions of Proposition 1.1.13 will be called *split* (observe that this does not depend on the choice of $a, b \in k^\times$).

Example 1.1.16. Assume that k is *quadratically closed*, i.e. that every element of k is a square. Then for every $a, b \in k^\times$, we have $(a, b) \simeq (1, b) \simeq M_2(k)$ by Lemma 1.1.4 (and Lemma 1.1.3 (ii)). Therefore every quaternion k-algebra splits.

EXAMPLE 1.1.17. Assume that the field k is finite, with q elements. As the group $k^\times$ is cyclic of order $q-1$, there are exactly $1+(q-1)/2$ squares in k. Thus the sets $\{ax^2|x \in k\}$ and $\{1-by^2|y \in k\}$ both consist of $1+(q-1)/2$ elements; as subsets of the set k having q elements, they must intersect. It follows from the criterion (v) in Proposition 1.1.13 that (a,b) splits. Therefore *every quaternion algebra over a finite field is split.*

EXAMPLE 1.1.18. Let $k=\mathbb{R}$. The quaternion algebra $(-1,-1)$ is not split, by Proposition 1.1.13 (v). Since $k^\times/k^{\times 2} = \{1,-1\}$, and taking into account Lemma 1.1.4 (as well as Lemma 1.1.3), we see that there are exactly two isomorphism classes of k-algebras, namely $M_2(k)$ and $(-1,-1)$.

Let us record another useful consequence of Lemma 1.1.14.

PROPOSITION 1.1.19. *Let $a,b,c \in k^\times$. If (a,c) is split, then $(a,bc) \simeq (a,b)$.*

PROOF. Since (a,c) is split, by Proposition 1.1.13 (iv) and (1.1.d) there are $u,v \in k$ such that $c = v^2 - au^2$. The statement follows from Lemma 1.1.14. □

PROPOSITION 1.1.20. *Let Q,Q' be quaternion algebras, with respective pure quaternion subspaces Q_0,Q'_0. Then $Q \simeq Q'$ if and only if there is a k-linear map $\varphi\colon Q_0 \to Q'_0$ such that $\varphi(q)^2 = q^2 \in k$ for all $q \in Q_0$.*

PROOF. Let $\psi\colon Q \to Q'$ be an isomorphism of k-algebras. If $q \in Q_0$, then

$$\psi(q)^2 = \psi(q^2) = q^2 \in k, \quad \text{and } \psi(q) \notin \psi(k^\times) = k^\times,$$

so that $\psi(q) \in Q'_0$. So we may take for φ the restriction of ψ.

Conversely, let $\varphi\colon Q_0 \to Q'_0$ be a k-linear map such that $\varphi(q)^2 = q^2 \in k$ for all $q \in Q_0$. We may assume that $Q=(a,b)$ with its standard generators i,j. We have $\varphi(i)^2 = i^2 = a$ and $\varphi(j)^2 = j^2 = b$, and

$$\varphi(i)\varphi(j) + \varphi(j)\varphi(i) = \varphi(i+j)^2 - \varphi(i)^2 - \varphi(j)^2 = (i+j)^2 - i^2 - j^2 = ij + ji = 0.$$

By Lemma 1.1.2 (applied to the elements $\varphi(i),\varphi(j) \in Q'$), we have $Q' \simeq (a,b)$. □

The norm map $N\colon Q \to k$ is in fact a quadratic form. The next corollary is a reformulation of Proposition 1.1.20, assuming some basic quadratic form theory. It can be safely ignored, and will not be used in the sequel.

COROLLARY 1.1.21. *Two quaternion algebras are isomorphic if and only if their norm forms are isometric.*

PROOF. Let Q be a quaternion algebra and $N\colon Q \to k$ its norm form. Note that $N(q) = -q^2$ for all $q \in Q_0$. The subspaces k and Q_0 are orthogonal in Q with respect to the norm form N, and $N|_k = \langle 1 \rangle$. So we have a decomposition $N \simeq \langle 1 \rangle \perp (N|_{Q_0})$. This quadratic form is nondegenerate (e.g. by (1.1.c)), hence a morphism φ as in Proposition 1.1.20 is automatically an isometry. The corollary follows, by Witt's cancellation Theorem (see for instance [**Lam05**, Theorem 4.2]). □

2. Quadratic splitting fields

DEFINITION 1.2.1. The *center* of a ring R is the set of elements $r \in R$ such that $rs = sr$ for all $s \in R$. As observed in (1.1.a), the center of a nonzero k-algebra always contains k. A nonzero k-algebra is called *central* if its center equals k.

LEMMA 1.2.2. *Every quaternion algebra is central.*

PROOF. We may assume that the algebra is equal to (a,b) with $a,b \in k^\times$. Consider an arbitrary element $q = x+yi+zj+wij$ of (a,b), where $x,y,z,w \in k$. Easy computations show that $qi = iq$ if and only if $z = w = 0$, and that $qj = jq$ if and only if $y = w = 0$. □

REMARK 1.2.3. Let $a,b \in k^\times$. We claim that (a,b) contains a subfield isomorphic to $k(\sqrt{a})$. To see this, we may assume that a is not a square in k. Then the morphism of k-algebras $k(\sqrt{a}) = k[X]/(X^2 - a) \to (a,b)$ given by $X \mapsto i$ is injective (because its source is a field, and its target is nonzero).

PROPOSITION 1.2.4. *Let D be a central division k-algebra of dimension 4. Assume that D contains a k-subalgebra isomorphic to $k(\sqrt{a})$ for some $a \in k$ which is not a square in k. Then $D \simeq (a,b)$ for some $b \in k^\times$.*

PROOF. Let $L \subset D$ be a subalgebra isomorphic to $k(\sqrt{a})$, and $\alpha \in L$ such that $\alpha^2 = a$. Since α does not lie in the center of D, there is $x \in D$ such that $x\alpha \neq \alpha x$. Then $\beta = \alpha^{-1}x\alpha - x$ is nonzero. Using the fact that $\alpha^2 = a$ is in the center of D, we see that

$$\beta\alpha = \alpha^{-1}x\alpha^2 - x\alpha = \alpha x - x\alpha = -\alpha\beta.$$

Multiplying with β on the left, resp. right, we obtain $\beta^2\alpha = -\beta\alpha\beta$, resp. $\beta\alpha\beta = -\alpha\beta^2$. It follows that β^2 commutes with α. Since β does not commute with α, we have $\beta \notin L$. Therefore the L-subspace of D generated by $1, \beta$ has dimension 2 over L, hence dimension 4 over k, and thus coincides with D by dimensional reasons. In particular the k-algebra D is generated by α, β. Since β^2 commutes with α and β, it lies in center of D, so that $b = \beta^2 \in k^\times$. It follows from Lemma 1.1.2 (applied with $i = \alpha, j = \beta$) that $D \simeq (a,b)$. □

LEMMA 1.2.5. *Let D be a central division k-algebra of dimension 4 and $d \in D - k$. Then the k-subalgebra of D generated by d is a quadratic field extension of k.*

PROOF. The powers d^i for $i \in \mathbb{N}$ are linearly dependent over k (as D is finite-dimensional), hence there is a nonzero polynomial $P \in k[X]$ such that $P(d) = 0$. Since D contains no nonzero zerodivisors (being division), we may assume that P is irreducible. Then $X \mapsto d$ defines a morphism of k-algebras $k[X]/P \to D$. Since $k[X]/P$ is a field and D is nonzero, this morphism is injective. Its image L is a field, and coincides with the k-subalgebra of D generated by d. Now D is a vector space over L, and $\dim_L D \cdot \dim_k L = \dim_k D = 4$. We cannot have $\dim_k L = 4$, for $D = L$ would then be commutative, and so would not be central over k. The case $\dim_k L = 1$ is also excluded, since by assumption $d \notin k$. So we must have $\dim_k L = 2$. □

COROLLARY 1.2.6. *Every central division k-algebra of dimension 4 is a quaternion algebra.*

PROOF. Since k has characteristic different from 2, every quadratic extension of k has the form $k(\sqrt{a})$ for some $a \in k^\times$. Thus D contains such an extension by Lemma 1.2.5, and the statement follows from Proposition 1.2.4. □

If L/k is a field extension and Q is a quaternion k-algebra, then $Q_L = Q \otimes_k L$ is naturally a quaternion L-algebra. Note that for any $q \in Q$ and $\lambda \in L$ we have

$$\overline{q \otimes \lambda} = \overline{q} \otimes \lambda \quad ; \quad N(q \otimes \lambda) = N(q) \otimes \lambda^2. \tag{1.2.a}$$

DEFINITION 1.2.7. We will say that Q *splits over* L, or that L is a splitting field for Q, if the quaternion L-algebra Q_L is split.

EXAMPLE 1.2.8. Let Q be a quaternion k-algebra which splits over the purely transcendental extension $k(t)$. By Proposition 1.1.13, this means that $ax^2 + by^2 = z^2$ has a nontrivial solution in $k(t)$. Clearing denominators we may assume that $x, y, z \in k[t]$, and that one of x, y, z is not divisible by t. Then $x(0), y(0), z(0)$ is a nontrivial solution in k, hence Q splits. Therefore *every quaternion algebra splitting over $k(t)$ splits over k.*

PROPOSITION 1.2.9. *Let $a \in k^\times$ and Q be a quaternion algebra. Assume that a is not a square in k. Then the following are equivalent:*

(i) $Q \simeq (a, b)$ for some $b \in k^\times$.
(ii) Q splits over $k(\sqrt{a})$.
(iii) The k-algebra Q contains a subalgebra isomorphic to $k(\sqrt{a})$.

PROOF. (i) $\Rightarrow$ (ii) : Since a is a square in $k(\sqrt{a})$, we have $(a, b) \simeq (1, b)$ over $k(\sqrt{a})$, which splits by Lemma 1.1.4.

(ii) $\Rightarrow$ (iii) : If Q is split, then $Q \simeq (1, a) \simeq (a, 1)$ by Lemma 1.1.4, and (iii) was observed in Remark 1.2.3. Thus we assume that Q is division. Let $\alpha \in k(\sqrt{a})$ be such that $\alpha^2 = a$. Then there are $p, q \in Q$ not both zero such that $N(p \otimes 1 + q \otimes \alpha) = 0$ by Proposition 1.1.13. Set $r = p\overline{q} \in Q$. In view of (1.2.a), we have

$$0 = (p \otimes 1 + q \otimes \alpha)(\overline{p} \otimes 1 + \overline{q} \otimes \alpha) = (N(p) + aN(q)) \otimes 1 + (r + \overline{r}) \otimes \alpha.$$

We deduce that $N(p) = -aN(q)$ and that r is a pure quaternion. Now

$$r^2 = -r\overline{r} = -p\overline{q}q\overline{p} = -N(p)N(q) = aN(q)^2.$$

Note that $N(q) \neq 0$, for otherwise $N(p) = -aN(q) = 0$, and thus $q = p = 0$ (by Lemma 1.1.9, as Q is division), contradicting the choice of p, q. The element $s = N(q)^{-1}r \in Q$ satisfies $s^2 = a$. Mapping X to s yields a morphism of k-algebras $k[X]/(X^2 - a) \to Q$, and (iii) follows.

(iii) $\Rightarrow$ (i) : If Q is not division, then $Q \simeq (1, a) \simeq (a, 1)$ by Lemma 1.1.4, so we may take $b = 1$ in this case. If Q is division, the implication has been proved in Proposition 1.2.4. □

3. Biquaternion algebras

LEMMA 1.3.1. *For any $a, b, c \in k^\times$, we have*

$$(a, b) \otimes_k (a, c) \simeq (a, bc) \otimes_k M_2(k).$$

PROOF. Let i, j, resp. i', j', be the standard generators of (a, b), resp. (a, c). Consider the k-subspace A of $(a, b) \otimes_k (a, c)$ generated by

$$1 \otimes 1, \quad i \otimes 1, \quad j \otimes j', \quad ij \otimes j'.$$

Then A is stable under multiplication. So is the k-subspace A' generated by

$$1 \otimes 1, \quad 1 \otimes j', \quad i \otimes i', \quad i \otimes j'i'.$$

There are isomorphisms of k-algebras

$$A \simeq (a, bc) \quad ; \quad A' \simeq (c, a^2) \simeq (c, 1) \simeq M_2(k).$$

Moreover every element of A commutes with every element of A'. Therefore the k-linear map $f\colon A \otimes_k A' \to (a,b) \otimes_k (a,c)$ given by $x \otimes y \mapsto xy = yx$ is a morphism of k-algebras; its image visibly contains the elements

$$i \otimes 1, \quad 1 \otimes i', \quad j \otimes 1, \quad 1 \otimes j'.$$

Since these elements generate the k-algebra $(a,b) \otimes_k (a,c)$, we conclude that f is surjective, hence an isomorphism by dimensional reasons. $\square$

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