EXERCISES 5 (INTERSECTION THEORY)

Let X be a variety, and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The *sheaf of meromorphic sections* of $\mathcal{F}$ is the sheaf of $\mathcal{K}_X$-modules $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$.

Exercise 1. Assume that X is integral with generic point η. Show that $\mathcal{K}_X(\mathcal{F})$ is the constant sheaf with value $\mathcal{F}_\eta$.

Exercise 2. A global section of $\mathcal{K}_X(\mathcal{F})$ is *regular* if the induced morphism $\mathcal{K}_X \to \mathcal{K}_X(\mathcal{F})$ is injective. Assume that X has no embedded points (i.e. $\mathrm{Ass}(X)$ is the set of generic points of X), and that $\mathcal{F}$ is locally free of rank one. Show that $\mathcal{K}_X(\mathcal{F})$ admits a regular global section.

Exercise 3. Let $\mathcal{L}$ be a locally-free $\mathcal{O}_X$-module of rank one, and $s \in H^0(X, \mathcal{L})$. We consider the closed subscheme $Z(s)$ defined by the ideal image of the morphism $\mathcal{L}^\vee \to \mathcal{O}_X$ induced by s. We say that s is a *regular* section of $\mathcal{L}$ if the induced morphism $\mathcal{O}_X \to \mathcal{L}$ is injective.

(i) Show that s is regular if and only if $Z(s)$ is an effective Cartier divisor.

(ii) Show that the associations $D \mapsto (\mathcal{O}(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ induce a correspondence

$$\left\{ \begin{array}{l} \text{pairs } (\mathcal{L}, s) \text{ with} \\ \mathcal{L} \text{ a line bundle on } X, \\ s \text{ a regular section of } \mathcal{L}. \end{array} \right\} \longleftrightarrow \{\text{effective Cartier divisors on } X\}.$$

Exercise 4. Let X be an integral scheme, and $\mathcal{L}$ a locally-free $\mathcal{O}_X$-module of rank one, and s a regular meromorphic section of $\mathcal{L}$. For every open subscheme U of X, we may view $\mathcal{L}(U)$ as an $\mathcal{O}_X(U)$-submodule of $\mathcal{L}_\eta$ (where η is the generic point of X), and we define

$$\mathcal{D}_s(U) = \{\alpha \in \mathcal{O}_X(U) | \alpha s \in \mathcal{L}(U)\}.$$

(i) Show that $\mathcal{D}_s$ is a coherent ideal of $\mathcal{O}_X$, called the *sheaf of denominators of* s, and that we have an injective morphism of $\mathcal{O}_X$-modules $\mathcal{D}_s \to \mathcal{L}$.

(ii) Let $f \colon Y \to X$ be dominant morphism, with Y integral. Show that

$$\mathcal{D}_s \cdot \mathcal{O}_Y \subset \mathcal{D}_{f^*s}.$$

(iii) Let $b \colon B \to X$ be the blow-up of the closed subscheme whose ideal is $\mathcal{D}(s)$ in X. Show that
  (a) B is integral,
  (b) b is birational,
  (c) there are effective Cartier divisors D and E on B such that

$$b^*\mathcal{L} = \mathcal{O}(D) \otimes \mathcal{O}(E)^\vee \text{ and } b^*s = 1_D \otimes 1_E^\vee.$$

(iv) Show that $b \colon B \to X$ is universal among the dominant morphisms $f \colon Y \to X$ with Y integral and $\mathcal{D}_{f^*s}$ locally free of rank one.