EXERCISES 6 (INTERSECTION THEORY)

Exercise 1. Let $i\colon D \to X$ be an effective Cartier divisor, $f\colon X \to S$ a flat morphism with a relative dimension. Assume that $f \circ i\colon D \to S$ is flat and has a relative dimension. Show that

$$i^* \circ f^* = (f \circ i)^*\colon \mathrm{CH}(S) \to \mathrm{CH}(D)$$

Exercise 2. Let E be a vector bundle on X, and $E^\vee$ its dual. Show that $c_i(E^\vee) = (-1)^i c_i(E)$.

Exercise 3. Let X be a smooth (or more generally locally factorial) variety. Show that the morphism $\mathrm{Pic}(X) \to \mathrm{CH}^1(X)$ mapping L to $c_1(L)[X]$ is a group isomorphism.

Exercise 4. A vector bundle E of rank r is called *orientable* if its determinant $\det E = \Lambda^r E$ is isomorphic to the trivial line bundle.

(i) Show that $c_1(E) = 0$ if E is orientable.

(ii) Conversely, show that a vector bundle E over a smooth variety X is orientable as soon as $c_1(E)[X] = 0$.

Exercise 5. Let A be a commutative ring. A *characteristic class* φ is the data of a group endomorphism $\varphi(E)$ of $\mathrm{CH}(X) \otimes A$ for every vector bundle $E \to X$, such that for every flat morphism $f\colon Y \to X$,

$$f^* \circ \varphi(E) = \varphi(f^*E) \circ f^*\colon \mathrm{CH}(X) \otimes A \to \mathrm{CH}(Y) \otimes A.$$

(i) Assume that a vector bundle $E \to X$ has a filtration by sub-bundles $E_{n+1} \subset E_n$ such that $L_n = E_n/E_{n+1}$ is a line bundle. Express the i-th Chern class $c_i(E)$ in terms of the classes $c_1(L_n)$.

(ii) Let $F \in A[x_1, \cdots, x_n]$ be a symmetric polynomial. Show that there is a unique characteristic class φ such that whenever E is a vector bundle with a filtration with successive quotients line bundles $L_1, \cdots, L_m$, then

$$\varphi(E) = F(c_1(L_1), \cdots, c_1(L_m)).$$

(iii) Let $P \in A[[t]]$ a power series. Show that there is unique characteristic class π_P such that
- When $0 \to E \to F \to G \to 0$ is an exact sequence of vector bundles over X, then $\pi_P(E) \circ \pi_P(G) = \pi_P(F)$.
- When $L \to X$ is a line bundle, then $\pi_P(L) = P(c_1(L))$.

(iv) Let $P \in A[[t]]$ a power series. Show that there is unique characteristic class γ_P such that
- When $0 \to E \to F \to G \to 0$ is an exact sequence of vector bundles over X, then $\gamma_P(E) + \gamma_P(G) = \gamma_P(F)$.
- When $L \to X$ is a line bundle, then $\gamma_P(L) = P(c_1(L))$.

(v) When $A = \mathbb{Q}$, and

$$P(t) = \sum_{i \geq 0} t^n/n!,$$

we let define the *Chern character* $ch = \gamma_P$. Show that

$$ch(E \otimes F) = ch(E) \circ ch(F).$$