

GALOIS COHOMOLOGY EXERCISES 9 (TORSORS)

Let k be a field.

Exercise 1. Let k_s be a separable closure of k , and $\Gamma = \text{Gal}(k_s/k)$. Let A be an étale k -algebra of dimension n . Consider the associated Γ -set $X = \text{Hom}_{k\text{-alg}}(A, k_s)$. Let $Y \subset X^n$ be the set of those (x_1, \dots, x_n) such that $x_i \neq x_j$ when $i \neq j$, with the Γ -action given by

$$\gamma(x_1, \dots, x_n) = (\gamma x_1, \dots, \gamma x_n) \quad \text{for } \gamma \in \Gamma, \text{ and } x_1, \dots, x_n \in X.$$

The symmetric group \mathfrak{S}_n acts on Y by

$$\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Denote by Z the quotient of Y by the action of the subgroup \mathfrak{A}_n of even permutations (the kernel of the signature morphism $\mathfrak{S}_n \rightarrow \mathbb{Z}/2$).

(i) Show that Z is a Γ -set having two elements.

We denote by Δ the corresponding étale k -algebra of dimension two; it is called the *discriminant algebra* of A .

Assume that k has characteristic $\neq 2$. Let e_1, \dots, e_n be a k -basis of A , let f_1, \dots, f_n be the elements of X , and consider the matrix $M = (f_i(e_j)) \in M_n(k_s)$. Set $u = \det M \in k_s$. Let Γ_0 be the subgroup of Γ consisting of those elements acting by even permutations on the set X .

(ii) Let $\gamma \in \Gamma$. Show that $\gamma u = u$ if $\gamma \in \Gamma_0$ and $\gamma u = -u$ otherwise.

Let d be the determinant of the matrix $(\text{Tr}_{A/k}(e_i e_j)) \in M_n(k)$.

(iii) Show that $d = u^2$. (Hint: compute the product $M^t \cdot M$.)

(iv) Conclude that $\Delta = k[X]/(X^2 - d)$.

Exercise 2. Let G be a finite group. Let $H \subset G$ be a subgroup and B a H -algebra over k . Consider the set

$$\text{Ind}_H^G B = \{\text{maps } G/H \rightarrow B\},$$

viewed as a k -algebra, via pointwise operations on B .

(i) Show that the k -algebra $\text{Ind}_H^G B$ is étale if and only if B is étale

If $f \in \text{Ind}_H^G B$ and $g \in G$, we define an element $g \cdot f \in \text{Ind}_H^G B$ by mapping a coset uH to $f(ugH)$, where $u \in G$. This gives $\text{Ind}_H^G B$ the structure of a G -algebra.

(ii) Show that the H -algebra B is Galois over k if and only if the G -algebra $\text{Ind}_H^G B$ is Galois over k .

(iii) Let A be a Galois G -algebra over k . Show that there exists a subfield $L \subset A$, which is Galois field extension of k , a subgroup $H \subset G$ isomorphic to $\text{Gal}(L/k)$, and an isomorphism of G -algebras $A \simeq \text{Ind}_H^G L$.

Exercise 3. Let Γ be a profinite group, and $A \rightarrow B$ be a morphism of Γ -group. Describe the map $H^1(\Gamma, A) \rightarrow H^1(\Gamma, B)$ in terms of torsors (as opposed to 1-cocycles).