Exercise 1. Let A, B be rings. Show that every ideal of the ring $A \times B$ is of the form $I \times J$, where $I \subset A$ and $J \subset B$ are ideals.

Exercise 2. Let k be a field. A k-algebra is called *diagonalisable* if it is isomorphic to k^n, for some integer $n \in \mathbb{N}$.

(i) Show that a finite-dimensional k-algebra A is diagonalisable if and only if the k-vector space of linear forms $\mathrm{Hom}_k(A, k)$ is generated by morphisms of k-algebras.

(ii) Deduce that every k-subalgebra of a diagonalisable k-algebra is diagonalisable.

(iii) Show that every diagonalisable k-algebra is generated by idempotent elements as a k-vector space. (Recall that an element x in a ring R is called idempotent if $x^2 = x$.)

(iv) Let $(e_1, \dots, e_n)$ be the canonical k-basis of k^n. For $I \subset \{1, \dots, n\}$, set

$$e_I = \sum_{i \in I} e_i.$$

Show that every idempotent of k^n is of the form e_I for some $I \subset \{1, \dots, n\}$.

(v) Deduce that a diagonalisable k-algebra admits only finitely many k-subalgebras.

Exercise 3. Let A be a k-algebra. We assume that there exists a field extension ℓ/k such that the ℓ-algebra $A \otimes_k \ell$ is diagonalisable. Show that the k-algebra A is étale. (N.B.: the converse was established in the lectures).

Exercise 4. Let k be a field, and A an étale k-algebra. (Hint for the questions below: Use the two previous exercises.)

(i) Let $B \subset A$ be a k-subalgebra. Show that B is an étale k-algebra.

(ii) Let C be a quotient k-algebra of A (i.e. $C = A/I$ for some ideal I of A). Show that the k-algebra C is étale.

(iii) Show that the k-algebra A admits only finitely many subalgebras and quotient algebras.

(iv) Assume that k is infinite. Show that there exists a separable polynomial $P \in k[X]$ such that $A \simeq k[X]/P$. (Hint: to show that A is generated by a single element as a k-algebra, recall that no k-vector space is a finite union of proper subspaces.)

Exercise 5. Let L/K be a field extension of finite degree. We are going to prove that the following conditions are equivalent:

(a) The K-algebra L is generated by a single element,

(b) There exist only finitely many subextensions of L/K.

We proceed as follows:

(i) Show that (b) implies (a). (Hint: Treat the cases k finite and infinite using different arguments.)

(ii) Assume that $L = K(\alpha)$ for some $\alpha \in L$. Let E/K be a subextension of L/K, and let

$$P = X^d + a_{d-1}X^{d-1} + \cdots + a_0 \in E[X]$$

be the minimal polynomial of α over E. Show that $E = K(a_0, \ldots, a_{d-1})$.

(iii) Show that in (ii) the image of P in $L[X]$ can take only finitely many values, as E/K varies (the element α being fixed).

(iv) Deduce that (a) implies (b).