Exercise 1. A topological space is called *irreducible* if it cannot be written as a nontrivial reunion of two closed subsets. An *irreducible component* of a topological space is a maximal irreducible subset (unless otherwise stated, subsets of a topological space are endowed with the induced topology). A topological space is called *noetherian* if every decreasing sequence of closed subsets is stationary.

(i) Show that an irreducible component is closed.

(ii) Show that a topological space is the reunion of its irreducible components.

(iii) Show that a noetherian topological space has only a finite number of irreducible components.

Exercise 2. (i) Let $f: Y \to X$ be a continuous map between topological spaces, and $\mathcal{F}$ a sheaf of sets on X. Show that $(f^{-1}\mathcal{F})_y = \mathcal{F}_{f(y)}$ for any $y \in Y$.

(ii) Let now $j: U \to X$ be an open immersion. Give a simple description of the functor j^{-1}. For any sheaf of sets $\mathcal{G}$ on U, we define a sheaf of sets $j_!\mathcal{G}$ on X as the sheafification of the presheaf $\widetilde{j_!}\mathcal{G}$ defined by

$$V \mapsto \widetilde{j_!}\mathcal{G}(V) = \begin{cases} \emptyset & \text{if } V \not\subset U \\ \mathcal{G}(V) & \text{if } V \subset U. \end{cases}$$

Show that j^{-1} is right adjoint to $j_!$.

Exercise 3. Let $f: Y \to X$ be a continuous map between topological spaces, and $\mathcal{F}$ a sheaf of sets on X. Show that the square

$$\begin{array}{ccc} (f^{-1}\mathcal{F})_{et} & \longrightarrow & \mathcal{F}_{et} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

is cartesian in the category of topological spaces (the definition is recalled below).

(We say that a commutative square

$$\begin{array}{ccc} P & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow{b} & X \end{array}$$

in a category is *cartesian*, and that P is the fiber product *fiber product* $A \times_X B$ if for any commutative square

$$\begin{tikzcd} Q \arrow[r, "q_A"] \arrow[d, "q_B"'] & A \arrow[d, "a"] \\ B \arrow[r, "b"] & X \end{tikzcd}$$

there is a unique morphism $f \colon Q \to P$ such that $p_A \circ f = q_A$ and $p_B \circ f = q_B$. If the triple (P, p_A, p_B) exists, it is unique up to a unique isomorphism.)