The letter $k$ denotes a field. When $n$ is an integer, we denote by $\mathbb{A}^n = \operatorname{Spec} k[X_1, \dots, X_n]$ the affine space, and write $\times$ instead of $\times_{\operatorname{Spec} k}$ for the fiber product of $k$-schemes.

**Exercise 1.** (i) Let $R$ be a commutative ring and $f \in R$. Assume that $f$ is a nonzerodivisor, i.e. for all $a \in R - \{0\}$ we have $fa \neq 0$. Show that the open subscheme $D(f)$ is dense in $\operatorname{Spec} R$.

(ii) Let $X = \operatorname{Spec} A$ be an affine scheme, and $Z = \operatorname{Spec} A/I$ a closed subscheme. Let $\pi \colon \widetilde{X}_Z \to X$ be the blow-up morphism. Show that the open subscheme $X - Z \simeq \widetilde{X}_Z - \pi^{-1}(Z)$ is dense in $\widetilde{X}_Z$. (Hint : $\widetilde{X}_Z$ is covered by the open affine subschemes $D_h(f_i) = \operatorname{Spec} A_i$, where $(f_i)$ is a generating set of the ideal $I$; use the first question for the ring $A_i$.)

**Exercise 2.** We consider the $\mathbb{G}_m$-action on $\mathbb{A}^n \times \mathbb{A}^n = \operatorname{Spec} k[T_1, \dots, T_n, X_1, \dots, X_n]$ given by letting $T_i$ be of degree 1 and $X_i$ of degree 0 (canonical action on the first factor and trivial action on the second factor). Let $Y$ be the $\mathbb{G}_m$-invariant open subscheme $(\mathbb{A}^n - 0) \times \mathbb{A}^n$.

(i) Show that the $\mathbb{G}_m$-action on $Y$ is locally free.

(ii) Let $Z$ be the closed subscheme of $Y$ defined by the equations $T_i X_j = T_j X_i$ for $1 \leq i, j \leq n$. Show that $Z$ is $\mathbb{G}_m$-equivariantly isomorphic to $(\mathbb{A}^n - 0) \times \mathbb{A}^1$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1 = \operatorname{Spec} k[X]$ by letting $X$ be of degree $-1$ (and canonically on the first factor). Deduce that the blow-up of $\mathbb{A}^n$ at the closed subscheme 0 may be identified with the quotient scheme $Z/\mathbb{G}_m$.

**Exercise 3.** Let $Z$ and $Y$ be two closed subschemes of $X$.

(i) Show that the natural morphism $\widetilde{Y}_{Y \cap Z} \to \widetilde{X}_Z$ is a closed embedding.

(ii) Let $\pi \colon \widetilde{X}_Z \to X$ be the blow-up morphism. Show that, as sets,

$$\pi^{-1}(Y) = \widetilde{Y}_{Y \cap Z} \cup \pi^{-1}(Y \cap Z),$$

and that $\pi^{-1}(Y)$ is the disjoint union of $Y - Y \cap Z$ and $\pi^{-1}(Y \cap Z)$.

(iii) For $i = 1, \dots, n$ we let $L_i$ be the closed subscheme of $\mathbb{A}^n = k[X_1, \dots, X_n]$ given by the ideal generated by the elements $X_j$ for $j \neq i$. Describe the inverse image of $L_1 \cup \dots \cup L_n$ in the blow-up of 0 in $\mathbb{A}^n$.

**Exercise 4.** Let $X$ be the blow-up of 0 in $\mathbb{A}^n$.

(i) Show that $X$ is covered by $n$ open subschemes, each isomorphic to $\mathbb{A}^n$.

(ii) Assume that $n > 1$. Let $x$ be a closed point of $X$ mapping to 0 in $\mathbb{A}^n$, and $Y$ the blow-up of $\{x\}$ (with reduced structure) in $X$. Show that the natural open immersion $\mathbb{A}^n - 0 \to Y$ does not extend to a morphism $X \to Y$.