Exercise 1. A topological space is called *irreducible* if it cannot be written as a nontrivial reunion of two closed subsets. An *irreducible component* of a topological space is a maximal irreducible subset (unless otherwise stated, subsets of a topological space are endowed with the induced topology). A topological space is called *noetherian* if every decreasing sequence of closed subsets is stationary.

(i) Show that an irreducible component is closed.

(ii) Show that a topological space is the reunion of its irreducible components.

(iii) Show that a noetherian topological space has only a finite number of irreducible components.

Exercise 2. (i) Let $f\colon Y \to X$ be a continuous map between topological spaces, and $\mathcal{F}$ a sheaf of sets on X. Show that $(f^{-1}\mathcal{F})_y = \mathcal{F}_{f(y)}$ for any $y \in Y$.

(ii) Let now $j\colon U \to X$ be an open immersion. Give a simple description of the functor j^{-1}. For any sheaf of sets $\mathcal{G}$ on U, we define a sheaf of sets $j_!\mathcal{G}$ on X as the sheafification of the presheaf $\widetilde{j_!}\mathcal{G}$ defined by

$$V \mapsto \widetilde{j_!}\mathcal{G}(V) = \begin{cases} \emptyset & \text{if } V \not\subset U \\ \mathcal{G}(V) & \text{if } V \subset U. \end{cases}$$

Show that j^{-1} is right adjoint to $j_!$.

Exercise 3. Let $f\colon Y \to X$ be a continuous map between topological spaces, and $\mathcal{F}$ a sheaf of sets on X. Show that the square

$$\begin{tikzcd} (f^{-1}\mathcal{F})_{et} \arrow[r] \arrow[d] & \mathcal{F}_{et} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

is cartesian in the category of topological spaces (the definition is recalled below).

(We say that a commutative square

$$\begin{tikzcd} P \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "a"] \\ B \arrow[r, "b"] & X \end{tikzcd}$$

in a category is *cartesian*, and that P is the fiber product *fiber product* $A \times_X B$ if for any commutative square

$$\begin{tikzcd} Q \arrow[r, "q_A"] \arrow[d, "q_B"'] & A \arrow[d, "a"] \\ B \arrow[r, "b"] & X \end{tikzcd}$$

there is a unique morphism $f\colon Q \to P$ such that $p_A \circ f = q_A$ and $p_B \circ f = q_B$. If the triple (P, p_A, p_B) exists, it is unique up to a unique isomorphism.)

Exercise 1. Let X be a connected scheme, and $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module. Show that the dimension of the $\kappa(x)$-vector space $\mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ does not depend on the point $x \in X$. Give a counterexample in case $\mathcal{E}$ is coherent but not locally free.

Exercise 2. Let $A \to B$ be a ring morphism, and $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ the corresponding scheme morphism.

(i) Let M, N be two A-modules. Show that

$$\widetilde{M \oplus N} = \widetilde{M} \oplus \widetilde{N} \quad \text{and} \quad \widetilde{M \otimes_A N} = \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}.$$

(ii) Let N be a B-module. What is the A-module M such that $f_*\widetilde{N} = \widetilde{M}$?

(iii) Let M be a A-module. What is the B-module N such that $f^*\widetilde{M} = \widetilde{N}$?

Exercise 3. Let $f\colon Y \to X$ be a separated and quasi-compact morphism of schemes, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_Y$-module. Show that the $\mathcal{O}_X$-module $f_*\mathcal{F}$ is quasi-coherent.

Exercise 4. Let $f\colon Y \to X$ be a scheme morphism.

(i) Let $\mathcal{A}, \mathcal{B}$ be two $\mathcal{O}_X$-modules. Show that

$$f^*\mathcal{A} \otimes_{\mathcal{O}_Y} f^*\mathcal{B} \simeq f^*(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B})$$

(ii) Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module, and $\mathcal{F}$ an $\mathcal{O}_Y$-module. Prove the *projection formula*

$$f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}) \simeq \mathcal{E} \otimes_{\mathcal{O}_X} f_*\mathcal{F}.$$

Exercise 5. Let X be a scheme, and $\pi\colon \mathbb{P}^n_X \to X$.

(i) Show that $\pi_*\mathcal{O}_{\mathbb{P}^n_X} = \mathcal{O}_X$.

(ii) Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module. Show that there is a locally free coherent $\mathcal{O}_{\mathbb{P}^n_X}$-module $\mathcal{F}$ such that $\pi_*\mathcal{F} = \mathcal{E}$ (Hint: use the projection formula).

Exercise 1. Let X be a noetherian scheme, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module.

(i) Show that the $\mathcal{O}_X$-module $\mathcal{F}$ is locally free of rank r if and only if the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is free of rank r for every $x \in X$.

(ii) Show that $\mathcal{F}$ is locally free of rank one if and only if there is a coherent $\mathcal{O}_X$-module $\mathcal{G}$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \simeq \mathcal{O}_X$.

(iii) Assume that X is affine. Show that $\mathcal{F}$ is locally free if and only it there is a coherent $\mathcal{O}_X$-module $\mathcal{G}$ such that $\mathcal{F} \oplus \mathcal{G}$ is free.

(iv) Give a counterexample for (iii) when X is not affine.

Exercise 2. Let $f\colon Y \to X$ be a finite morphism of schemes. Assume that $f_*\mathcal{O}_Y$ is locally free $\mathcal{O}_X$-module of rank r. Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_Y$-module of rank e. Show that $f_*\mathcal{E}$ is a locally free coherent $\mathcal{O}_X$-module of rank re.

Exercise 3. Let A be a commutative ring, and $S = [t_0, \ldots, t_n]$. Show that every closed subscheme Y of $\mathbb{P}^n_A$ is given by some homogeneous ideal of S (in other words Y is the closed subscheme $\operatorname{Proj} S/I = V_+(I)$ of $\operatorname{Proj} S$).

Exercise 1. Let A be a local domain, with function field K and residue field κ.

(i) Consider the map $\varphi\colon \operatorname{Spec} K \to \operatorname{Spec} A$ sending the point of $\operatorname{Spec} K$ to the closed point of $\operatorname{Spec} A$. Is φ induced by a morphism of ringed spaces?

(ii) Assume that A is the localisation of $\mathbb{Z}$ at a prime number. Is the morphism φ of (i) induced by a morphism of schemes?

(iii) Assume that $\dim A = 1$. Consider the natural ring morphism $m\colon A \to K \times \kappa$. Is $m^\sharp$ a bijection? a homeomorphism?

Exercise 2. Let X be a scheme, and U an open subset of the underlying topological space. Show that $(U, \mathcal{O}_X|_U)$ is a scheme.

Exercise 3. Let k be a field.

(i) (Hilbert's Nullstellensatz) Let L be a finitely generated k-algebra. If L is a field, show that it is a finite field extension of k. (You may use Noether's normalisation Lemma : *Let A a non-zero finitely generated k-algebra. Then there are $x_1, \dots, x_n \in A$ such that A is integral over $k[x_1, \dots, x_n]$.*)

(ii) Let A be a finitely generated k-algebra. Show that a point of $\mathfrak{p} \in \operatorname{Spec} A$ is closed if and only if its residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is a finite extension of k.

Exercise 4. Let $f\colon A \to B$ be ring morphism such that

- for all $b \in B$ there exists $n \in \mathbb{N}$ such that $b^n \in \operatorname{im} f$.
- $\ker f$ consists of nilpotent elements.

Show that $f^\sharp\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is a homeomorphism.

Exercise 1. (i) Let A be a commutative ring. Describe the set of scheme morphisms $\operatorname{Spec} A \to (\mathbb{A}^n - 0)$ in terms of elements of A.

(ii) Let X be a scheme. Describe the set of scheme morphisms $X \to (\mathbb{A}^n - 0)$ in terms of global sections of $\mathcal{O}_X$.

Exercise 2. Let X be a scheme of finite type over a field k.

(i) Show that a point of X is closed if and only if its residue field is a finite extension of k.

(ii) Show that closed points are dense in X.

(iii) Give an example of a scheme where closed points are not dense.

Exercise 3. Let k be an algebraically closed field.

(i) Let S be an $\mathbb{N}$-graded ring, generated by elements of degree one. Describe the set of closed points of $\operatorname{Proj} S$ in terms of ideals of S.

(ii) Let $(x_0, \ldots, x_n) \in k^{n+1} - 0$. Find a homogeneous prime ideal $\mathfrak{p}$ of $k[T_0, \ldots, T_n]$ such that $V(\mathfrak{p})(k) \subset \mathbb{A}^{n+1}(k)$ is identified with

$$\{(\lambda x_0, \ldots, \lambda x_n) | \lambda \in k\} \subset k^{n+1}.$$

(iii) Let $\mathfrak{p}$ be a closed point of $\mathbb{P}^n_k$. We view $\mathfrak{p}$ as an ideal of $k[T_0, \ldots, T_n]$. Show that we can find $(x_0, \ldots, x_n) \in k^{n+1} - 0$ such that

$$V(\mathfrak{p})(k) = \{(\lambda x_0, \ldots, \lambda x_n) | \lambda \in k\} \subset \mathbb{A}^{n+1}(k).$$

(iv) Deduce a bijection between $\mathbb{P}^n(k)$ and the set of lines in k^{n+1} containing 0.

Exercise 1. Let $\varphi\colon A \to B$ be morphism of commutative rings and $f\colon Y \to X$ the induced morphism of affine schemes. Let J be an ideal of B.

(i) Show that $V(\varphi^{-1}J)$ is the closure of $f(V(J))$.

(ii) Let I be an ideal of A. Let $T = \mathrm{Spec}(B/J)$ and $Z = \mathrm{Spec}(A/I)$. We assume that $f(Z) = T$ (viewing Z, resp. T, as a closed subset of Y, resp. X). We assume that B/J is reduced. Show that there is a unique morphism of schemes $T \to Z$ fitting into the commutative square of schemes

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

Exercise 2. (i) We say that a scheme is *noetherian* if it admits a finite open covering by spectra of noetherian rings. Show that a commutative ring A is noetherian if and only if the scheme $\mathrm{Spec}\, A$ is noetherian.

(ii) We say that a scheme is *reduced* if it admits an open covering by spectra of reduced rings. Show that a commutative ring A is reduced if and only if the scheme $\mathrm{Spec}\, A$ is reduced.

(iii) Show that a scheme X is reduced if and only if the ring $\mathcal{O}_{X,x}$ is reduced for every $x \in X$.

Exercise 1. *(If you have seen closed subschemes)* Let X be a scheme.

(i) Show that there exists a unique reduced closed subscheme X_{red} of X such that any morphism $T \to X$ with T reduced factors through $X_{red} \to X$.

(ii) Show that $X_{red} \to X$ is surjective, and that X_{red} is the smallest closed subscheme of X with this property (in other words, if a closed subscheme Z of X is such that $Z \to X$ is surjective, then $X_{red} \to X$ factors through Z).

Exercise 2. (i) Let $f\colon Y \to X$ be a scheme morphism, x a point of X, and $f^{-1}x = Y \times_X \operatorname{Spec} k(x)$ the scheme-theoretic fibre of f over x. Show that the underlying set of $f^{-1}x$ is the set-theoretic fibre $\{y \in Y | f(y) = x\}$.

(ii) Let $f\colon X \to S$ and $g\colon Y \to S$ be two scheme morphisms. Show that the underlying set of $Y \times_S X$ is

$$\{(x, y, l) | x \in X, y \in Y \text{ with } f(x) = g(y) =: s, \text{ and } l \in \operatorname{Spec}(k(x) \otimes_{k(s)} k(y))\}.$$

Exercise 3. (*More difficult*) Let k be a field, and $f\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ a morphism. Let S be the subset of points $x \in \mathbb{A}^1_k$ such that the scheme-theoretic fibre $f^{-1}x$ is a disjoint union of spectra of separable field extensions of $k(x)$. Show that S is open in $\mathbb{A}^1_k$.

Exercise 1. Let $A \subset B \subset K$, be three domains. Assume that K is the common fraction field of A and B, and that B is finitely generated A-algebra. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced morphism. Show that there is a non-empty open subscheme U of $\operatorname{Spec} A$ such that $f^{-1}U \to U$ is an isomorphism.

Exercise 2. Let A be a domain with fraction field K, and B a finitely generated A-algebra. We assume that $\dim_K(K \otimes_A B) = n < \infty$. Show that there is $f \in A$ such that $B[1/f]$ is an $A[1/f]$-module of finite type. What does the condition $n = 0$ mean for the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$?

Exercise 3. Let k be a field, and denote by X, Y, T the three coordinates of $\mathbb{A}^3_k$. Let $Z = V(XY - T) \subset \mathbb{A}^3_k$. Consider the composite $f\colon Z \to \mathbb{A}^3_k \to \mathbb{A}^1_k$ where the last morphism is given by T. Compute the (scheme-theoretic) fibre of f over each closed point of $\mathbb{A}^1_k$.

Exercise 4. (*Time permitting*) Let R be a commutative ring, and denote by $\mathbb{P}^n(R)$ the set $\operatorname{Hom}(\operatorname{Spec} R, \mathbb{P}^n_{\mathbb{Z}})$. Given elements $\alpha_0, \ldots, \alpha_n \in R$, we denote by $[\alpha_0 : \cdots : \alpha_n]$ their class in R^{n+1} modulo the relations $[\alpha_0 : \cdots : \alpha_n] = [\lambda \cdot \alpha_0 : \cdots : \lambda \cdot \alpha_n]$ for $\lambda \in R^\times$. We consider the set

$$H(R) = \{[\alpha_0 : \cdots : \alpha_n] | \sum_{i=0}^{n} \alpha_i R = R\}.$$

(i) Explain how every element of $H(R)$ defines an element of $\mathbb{P}^n(R)$.

(ii) Show that $H(R) = \mathbb{P}^n(R)$ when R is a field.

(iii) Show that $H(R) = \mathbb{P}^n(R)$ when R is a principal ideal domain.

(iv) Let k be a field. Describe the set $\operatorname{Hom}_{\operatorname{Spec} k}(\mathbb{P}^1_k, \mathbb{P}^n_k)$.

Exercise 1. Let X be a scheme. Show that the decompositions $X = X_1 \cup X_2$ with X_i open in X and $X_1 \cap X_2 = \emptyset$, are in bijection with the pairs of elements $(p_1, p_2) \in \Gamma(X, \mathcal{O}_X)^2$ such that $1 = p_1 + p_2$ and $p_1 p_2 = 0$.

Exercise 2. A morphism of schemes $f \colon Y \to X$ is *separated* if the diagonal $(\mathrm{id}_Y, \mathrm{id}_Y) \colon Y \to Y \times_X Y$ is a closed embedding.

(i) Show that a composite of separated morphisms is separated.

(ii) Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be morphisms of schemes. Assume that f is separated and $f \circ g$ is closed embedding. Show that g is a closed embedding.

(iii) Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be morphisms of schemes. Assume that f and $f \circ g$ are separated. Show that g is separated.

(iv) Let $X \to S$ be a separated morphism with S affine. Show that the intersection of any two open affine subschemes of X is affine.

Exercise 3. Let $f \colon Y \to X$ be a morphism of schemes. We say that f is *quasi-finite* if for every point $x \in X$, the fiber $f^{-1}x$ is a finite set. We say that f is *finite* if there is an open affine cover X_i of X such that for each i, the scheme $Y_i = f^{-1}X_i$ is affine and $Y_i \to X_i$ corresponds to a finite ring morphism (i.e. $\Gamma(Y_i, \mathcal{O}_{Y_i})$ is a $\Gamma(X_i, \mathcal{O}_{X_i})$-module of finite type).

(i) Show that every finite morphism is quasi-finite.

(ii) Show that a finite morphism is closed.

(iii) Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be morphisms of schemes. Assume that f is separated and $f \circ g$ is finite. Show that g is a finite.

(iv) Let k be a field of characteristic not two. Consider the morphism $f \colon \mathbb{A}^1_k - \{1\} \to \mathbb{A}^1_k$ given by $x \mapsto x^2$. Show that f is surjective, quasi-finite, but not finite.

Exercise 4. Let X be a separated S-scheme ($X \to S$ is separated). Assume that X is reduced. Let $T \to X$ be a separated morphism with dense image. Show that $T \to X$ is an epimorphism in the category of separated S-schemes.

Exercise 1. Let k be a field. Show that $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not proper.

Exercise 2. Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be morphisms of schemes. Assume that $f \circ g$ is proper and that f is separated.

(i) Show that g is proper.

(ii) Assume that g is surjective and that f is of finite type. Show that f is proper.

Exercise 3. Let k be a field, and A a commutative k-algebra. We assume that $p\colon \operatorname{Spec} A \to \operatorname{Spec} k$ is proper.

(i) Assume that $\dim_k A = \infty$. Show that p factors through a dominant morphism $\operatorname{Spec} A \to \mathbb{A}^1_k$.

(ii) Deduce that $\dim_k A < \infty$.

(iii) Deduce that a proper affine morphism of schemes is quasi-finite.

Exercise 4. Let S be a graded ring.

(i) Let I be a homogeneous ideal. Show that there is a closed immersion $\operatorname{Proj}(S/I) \to \operatorname{Proj}(S)$.

(ii) Let Z be a reduced closed subscheme of $\operatorname{Proj} S$. Show that there is a homogeneous ideal J of S such that $Z = \operatorname{Proj}(S/J)$ in $\operatorname{Proj}(S)$.

(iii) Find a graded ring S and homogeneous ideals $I \neq I'$ such that $\operatorname{Proj}(S/I) = \operatorname{Proj}(S/I')$ as closed subschemes of $\operatorname{Proj}(S)$.

Exercise 1. Show that there is a closed embedding $\mathbb{P}^n \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{P}^n \to \mathbb{P}^{(n+1)^2-1}$. Deduce that a composite of projective morphisms is projective.

Exercise 2. (Uses quasi-coherent modules.) Let $f \colon Y \to X$ be a scheme morphism. We assume that Y is noetherian. We construct below the *scheme-theoretic image* Z of the morphism f.

(i) Let $\mathcal{I} = \ker(\mathcal{O}_X \to f_*\mathcal{O}_Y)$. Show that $\mathcal{I}$ is a quasi-coherent ideal of $\mathcal{O}_X$.

(ii) Let $Z = V(\mathcal{I})$. Show that the morphism f factors through Z, and that for any closed subscheme Z' of X such that f factors though Z', we have $Z \subset Z'$ (as closed subschemes of X).

(iii) Show that $X \to Z$ is dominant.

(iv) Show that f is a closed immersion if and only if $Y \to Z$ is an isomorphism.

(v) Let $X \to S$ be a separated morphism of finite type, and assume that the composite $Y \to S$ is proper. Show that $Z \to S$ is proper.

Exercise 3. We give two proofs that any finite morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is projective.

(i) Let A be a commutative ring, and $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0X^0 \in A[X]$ a monic polynomial. Let $\widetilde{P}(T,S) = T^n + a_{n-1}T^{n-1}S^1 + \cdots + a_0T^0S^n$ be the homogeneisation of P. Show that $\operatorname{Spec}(A[X]/P) = \operatorname{Proj}(A[T,S]/\widetilde{P})$.

(ii) Deduce that any finite morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is projective.

(iii) Prove directly that any proper morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is projective (use Exercise 2).

Exercise 4. (Optional)

(i) Let R be a discrete valuation ring with fraction field K, and S a ring such that $R \subset S \subset K$. Show that $R = S$ or $S = K$.

(ii) Let R be a discrete valuation ring with fraction field K, and denote by $i \colon \operatorname{Spec} K \to \operatorname{Spec} R$ the generic point. Consider a commutative diagram of schemes with solid arrows

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, "i"] & Y \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "h"] & X \end{tikzcd}$$

If f is proper, show that there is a unique morphism h making the diagram commute.