All schemes are assumed to be noetherian.

Exercise 1. Let A be a discrete valuation ring and K its fraction field.

(i) Show that the datum of a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is equivalent to the data of an A-module M and a K-vector space V together with a morphism of A-modules $M \to V$.

(ii) Show that the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the corresponding morphism $M \otimes_A K \to V$ is an isomorphism.

Exercise 2. (i) Let $f\colon X \to Y$ be a morphism and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Show that $f_*\mathcal{F}$ is naturally a sheaf of $\mathcal{O}_Y$-modules.

(ii) Let $\varphi\colon A \to B$ be a ring morphism, and $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ the induced scheme morphism. Let M be a B-module. We denote by M_φ the set M viewed as an A-module using φ and the B-module structure on M. Show that

$$f_*\widetilde{M} = \widetilde{M_\varphi}.$$

(iii) Let $f\colon X \to Y$ be a finite morphism (of schemes) and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules. Show that the sheaf of $\mathcal{O}_Y$-modules $f_*\mathcal{F}$ is coherent. (Hint: Reduce to the case when Y is affine, and use the previous question.)

(iv) Give an example of a morphism $f\colon X \to Y$ and a coherent sheaf of $\mathcal{O}_X$-modules such that the sheaf of $\mathcal{O}_Y$-modules $f_*\mathcal{F}$ is not coherent.

Exercise 3. Let X be a scheme and $\mathcal{J}$ be a sheaf of $\mathcal{O}_X$-ideals. We consider the subset $V(\mathcal{J}) \subset X$ consisting of those points x such that $\mathcal{J}_x \neq \mathcal{O}_{X,x}$, and denote by $j\colon V(\mathcal{J}) \to X$ the inclusion (of sets).

(i) Show that $V(\mathcal{J})$ is closed in X.

(ii) Show that the natural morphism $\mathcal{O}_X/\mathcal{J} \to j_*j^{-1}(\mathcal{O}_X/\mathcal{J})$ is an isomorphism.

(iii) Assume that $\mathcal{J} = \mathcal{I}_Z$ for a closed subscheme Z of X. Show that $Z = V(\mathcal{J})$ as subsets of X, and that $j^{-1}(\mathcal{O}_X/\mathcal{J}) \simeq \mathcal{O}_Z$.

(iv) Assume that $\mathcal{J}$ is quasi-coherent. Show that the ringed space $(V(\mathcal{J}), j^{-1}(\mathcal{O}_X/\mathcal{J}))$ is isomorphic to a closed subscheme of X.

(v) Deduce a correspondence between closed subschemes of X and quasi-coherent sheaves of $\mathcal{O}_X$-ideals.