Exercise 1. Let X be a scheme. Show that the decompositions $X = X_1 \cup X_2$ with X_i open in X and $X_1 \cap X_2 = \emptyset$, are in bijection with the pairs of elements $(p_1, p_2) \in \Gamma(X, \mathcal{O}_X)^2$ such that $1 = p_1 + p_2$ and $p_1 p_2 = 0$.

Exercise 2. A morphism of schemes $f \colon Y \to X$ is *separated* if the diagonal $(\mathrm{id}_Y, \mathrm{id}_Y) \colon Y \to Y \times_X Y$ is a closed embedding.

(i) Show that a composite of separated morphisms is separated.

(ii) Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be morphisms of schemes. Assume that f is separated and $f \circ g$ is closed embedding. Show that g is a closed embedding.

(iii) Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be morphisms of schemes. Assume that f and $f \circ g$ are separated. Show that g is separated.

(iv) Let $X \to S$ be a separated morphism with S affine. Show that the intersection of any two open affine subschemes of X is affine.

Exercise 3. Let $f \colon Y \to X$ be a morphism of schemes. We say that f is *quasi-finite* if for every point $x \in X$, the fiber $f^{-1}x$ is a finite set. We say that f is *finite* if there is an open affine cover X_i of X such that for each i, the scheme $Y_i = f^{-1}X_i$ is affine and $Y_i \to X_i$ corresponds to a finite ring morphism (i.e. $\Gamma(Y_i, \mathcal{O}_{Y_i})$ is a $\Gamma(X_i, \mathcal{O}_{X_i})$-module of finite type).

(i) Show that every finite morphism is quasi-finite.

(ii) Show that a finite morphism is closed.

(iii) Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be morphisms of schemes. Assume that f is separated and $f \circ g$ is finite. Show that g is a finite.

(iv) Let k be a field of characteristic not two. Consider the morphism $f \colon \mathbb{A}^1_k - \{1\} \to \mathbb{A}^1_k$ given by $x \mapsto x^2$. Show that f is surjective, quasi-finite, but not finite.

Exercise 4. Let X be a separated S-scheme ($X \to S$ is separated). Assume that X is reduced. Let $T \to X$ be a separated morphism with dense image. Show that $T \to X$ is an epimorphism in the category of separated S-schemes.