GALOIS COHOMOLOGY
EXERCISES 10 (TWISTED FORMS)

The letter k denotes a field.

Exercise 1. (i) Let V be a k-vector space of finite dimension n, and $f\colon V\times V\to k$ be a k-bilinear form. We assume that $f(x,x)=0$ for all $x\in V$ (i.e. f is alternated) and that the k-linear map $V\to \operatorname{Hom}_k(V,k)$ sending x to the map $y\mapsto f(x,y)$ is bijective (i.e. f is nondegenerate). Show that n is even, and that V admits a k-basis $e_1,\dots,e_n$ such that $f(e_{2r+1},e_{2r+2})=1$ and $f(e_{2r+2},e_{2r+1})=-1$ for all $0\leq r<n/2$, and $f(e_i,e_j)=0$ for all other values of i,j.

(ii) When L/k is a separable field extension, consider the matrix (where blank entries are zero)

$$J=\begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}\in M_{2r}(L).$$

Show that letting

$$\operatorname{Sp}_{2r}(L)=\{M\in M_{2r}(L)|M^tJM=J\},$$

where M^t denotes the transpose of M, defines a k-group Sp_{2r} such that $H^1(k,\operatorname{Sp}_{2r})=\{*\}$.

Exercise 2. For every separable extension L/k set

$$G(L)=\operatorname{Aut}_{L-\mathrm{alg}}(L[X]).$$

Extension of scalars yields a map $G(L)\to G(L')$ for every morphism $L\to L'$ of separable extensions of k.

(i) Show that G defines a k-group. (Caution: As $\dim_k k[X]=\infty$, some of the results of the lectures on twisted forms do not apply directly.)

(ii) Show that every element of $G(L)$ is of the form $X\mapsto aX+b$, where $a\in L^\times$ and $b\in L$.

(iii) Show that we have an exact sequence of k-groups

$$1\to\mathbb{G}_a\to G\to\mathbb{G}_m\to 1.$$

(iv) Show that $H^1(k,G)=\{*\}$.

Let A be a k-algebra such that $A_L\simeq L[X]$ as L-algebra, for some separable extension L/k.

(v) For every separable extension L/k, consider the set $I(L)$ of isomorphisms of F-algebras $L[X] \to A_L$. Extension of scalars yields a map $I(L) \to I(L')$ for every morphism $L \to L'$ of separable extensions of k. Show that I defines a G-torsor.

(vi) Conclude that $A \simeq k[X]$ as k-algebra.

We now assume that k has positive characteristic p, and that $a \in k$ is such that $a \neq b^p$ for all $b \in k$. We consider the k-algebra $B = k[U, V]/(U^p - aV^p - V)$.

(vii) Show that there exists an algebraic field extension K/k such that $B_K \simeq K[X]$ as K-algebra.

(viii) Show that B is not isomorphic to $k[X]$ as k-algebra. (Hint: If $\varphi\colon B \to k[X]$ is a morphism of k-algebras, consider the equation satisfied by the polynomials $\varphi(U)$ and $\varphi(V)$ to deduce that $\varphi(B) = k$.)

(ix) Give an example of a field k of characteristic p, together with an element $a \in k$ such that $a \neq b^p$ for all $b \in k$.