EXERCISES 3 (INTERSECTION THEORY)

Exercise 1. We will view $\mathbb{A}^1 = \operatorname{Spec} k[t]$ as the open complement of ∞ in $\mathbb{P}^1$. This defines an element $t \in k(\mathbb{P}^1)$ such that $\operatorname{div} t = [0] - [\infty] \in \mathcal{Z}(\mathbb{P}^1)$.

(i) Let Z be an integral variety and $f\colon Z \to \mathbb{P}^1$ a morphism whose image is not contained in $\{0, \infty\}$. Denote by f^*t the image of t under the induced morphism $k[t, t^{-1}] \to k(Z)$. Show that

$$\operatorname{div} f^*t = [f^{-1}0] - [f^{-1}\infty] \in \mathcal{Z}(Z).$$

(ii) Let X be an integral variety, and $\varphi \in k(X)^\times$. Show that there is an integral closed subscheme Z of $X \times_k \mathbb{P}^1$ such that $p\colon Z \to X$ is birational, the image of $f\colon Z \to \mathbb{P}^1$ is not contained in $\{0, \infty\}$, and

$$\operatorname{div} \varphi = p_* \circ \operatorname{div} f^*t \in \mathcal{Z}(X).$$

(iii) Let X be a variety. Let $\mathcal{Z}(X; \mathbb{P}^1)$ be the set of integral closed subschemes Z of $X \times_k \mathbb{P}^1$, such that the morphism $f\colon Z \to \mathbb{P}^1$ is dominant. For $\star \in \{0, \infty\}$, show that $f^{-1}\star$ may be identified to a closed subscheme of X, that will be denoted by $Z(\star)$.

(iv) Let X be a variety. Show that the subgroup of rationally trivial classes $\mathcal{R}(X) \subset \mathcal{Z}(X)$ is generated by the elements $[Z(0)] - [Z(\infty)]$, where Z runs over $\mathcal{Z}(X; \mathbb{P}^1)$.

Exercise 2. Let X be an integral variety, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.

(i) Show that we have a correspondence

$$\left\{ \begin{array}{l} \text{Integral closed subschemes } Z \subset \mathbb{P}(\mathcal{L} \oplus 1), \\ \text{with } Z \not\subset \mathbb{P}(1), Z \not\subset \mathbb{P}(\mathcal{L}), \\ \text{and } Z \to X \text{ birational.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{regular meromorphic} \\ \text{sections of } \mathcal{L}. \end{array} \right\}$$

(ii) Let s be a regular meromorphic section of $\mathcal{L}$, and $Z \subset \mathbb{P}(1 \oplus \mathcal{L})$ the corresponding closed subscheme, with morphism $p\colon Z \to X$. Show that

$$\operatorname{div}_{p^*\mathcal{L}}(p^*s) = [Z \cap \mathbb{P}(1)] - [Z \cap \mathbb{P}(\mathcal{L})] \in \mathcal{Z}(Z).$$

(iii) Show that $Z \cap \mathbb{P}(1)$ (resp. $Z \cap \mathbb{P}(\mathcal{L})$) may be viewed as a closed subscheme $Z(1)$ (resp. $Z(\mathcal{L})$) of X, and that we have

$$\operatorname{div}_{\mathcal{L}}(s) = p_*[Z \cap \mathbb{P}(1)] - p_*[Z \cap \mathbb{P}(\mathcal{L})] = [Z(1)] - [Z(\mathcal{L})] \in \mathcal{Z}(X).$$

Exercise 3. Prove directly (that is, without using Chapter 3 of the lecture) Weil's reciprocity law: For any $\varphi \in k(\mathbb{P}^1)^\times$, we have

$$\deg \circ \operatorname{div} \varphi = 0.$$

Exercise 4. Let $i\colon D \to X$ be an effective Cartier divisor, $f\colon X \to S$ a flat morphism with a relative dimension. Assume that $f \circ i\colon D \to S$ is flat and has a relative dimension. Show that

$$i^* \circ f^* = (f \circ i)^*\colon \operatorname{CH}(S) \to \operatorname{CH}(D)$$