GALOIS COHOMOLOGY
EXERCISES 10 (TWISTED FORMS)

The solutions will be discussed during the online session on Jan 26th.

The letter k denotes a field.

Exercise 1. Let Γ be a profinite group, and $f\colon A \to B$ be a morphism of Γ-groups. Composing 1-cocyles with f yields a map $f_*\colon H^1(\Gamma, A) \to H^1(\Gamma, B)$. Describe this map in terms of torsors.

Exercise 2. (i) Let V be a k-vector space of finite dimension n, and $f\colon V \times V \to k$ be a k-bilinear form. We assume that $f(x,x) = 0$ for all $x \in V$ (i.e. f is alternated) and that the k-linear map $V \to \mathrm{Hom}_k(V,k)$ sending x to the map $y \mapsto f(x,y)$ is bijective (i.e. f is nondegenerate). Show that n is even, and that V admits a k-basis $e_1, \dots, e_n$ such that $f(e_{2r+1}, e_{2r+2}) = 1$ and $f(e_{2r+2}, e_{2r+1}) = -1$ for all $0 \leq r < n/2$, and $f(e_i, e_j) = 0$ for all other values of i, j.

(ii) Let $r \in \mathbb{N} - 0$ and consider the matrix (where blank entries are zero)

$$J = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix} \in M_{2r}(k).$$

Show that letting, for every separable field extension L/k,

$$\mathrm{Sp}_{2r}(L) = \{M \in M_{2r}(L) | M^t J M = J\},$$

where M^t denotes the transpose of M, defines a k-group Sp_{2r} such that $H^1(k, \mathrm{Sp}_{2r}) = \{*\}$.

Exercise 3. For every separable extension L/k denote by $L[X]$ the polynomial algebra in one variable over L, and set

$$G(L) = \mathrm{Aut}_{L-\mathrm{alg}}(L[X]).$$

Extension of scalars yields a map $G(L) \to G(L')$ for every morphism $L \to L'$ of separable extensions of k.

(i) Show that G defines a k-group.

(ii) Show that every element of $G(L)$ is of the form $X \mapsto aX + b$, where $a \in L^\times$ and $b \in L$.

(iii) Show that we have an exact sequence of k-groups

$$1 \to \mathbb{G}_a \to G \to \mathbb{G}_m \to 1.$$

(iv) Show that $H^1(k, G) = \{*\}$.

(v) Let A be a k-algebra such that $A_L \simeq L[X]$ as L-algebra, for some separable extension L/k. Show that $A \simeq k[X]$ as k-algebra.

We have thus proved that the k-algebra $k[X]$ admits no nontrivial twisted forms. We now give an example a nontrivial "inseparable twisted form" of $k[X]$, that is a k-algebra B such that $B \not\simeq k[X]$ and $B_K \simeq K[X]$ for some nonseparable extension K/k.

Let us assume that k has positive characteristic p, and that $a \in k$ is such that $a \neq b^p$ for all $b \in k$. We consider the k-algebra $B = k[U, V]/(U^p - aV^p - V)$.

(v) Show that there exists an algebraic field extension K/k such that $B_K \simeq K[X]$ as K-algebra.

(vi) Show that B is not isomorphic to $k[X]$ as k-algebra.

(vii) For every prime p, give an example of a field k of characteristic p, together with an element $a \in k$ such that $a \neq b^p$ for all $b \in k$.