## GALOIS COHOMOLOGY EXERCISES 1 (TENSOR PRODUCT)

Let k be a field and U, V be k-vector spaces. Let us denote by F the k-vector space having as basis the set  $U \times V$ ; it consists of k-linear combinations of elements of type (u, v) with  $u \in U$  and  $v \in V$ . Consider the subspace R of F generated by the following elements

$$(u + \lambda u', v + \mu v') - (u, v) - \lambda (u', v) - \mu (u, v') - \lambda \mu (u', v'),$$

where  $u, u' \in U$  and  $v, v' \in V$  and  $\lambda, \mu \in k$ . The quotient k-vector space F/R will be denoted by  $U \otimes_k V$ , and the image of (u, v) by  $u \otimes v \in U \otimes_k V$ .

**Exercise 1.** Let W be another k-vector space and  $\varphi \colon U \times V \to W$  a map. Assume that  $\varphi$  is k-bilinear, i.e. that for all  $u \in U$  the map  $V \to W$  given by  $v \mapsto \varphi(u,v)$  is k-linear and for all  $v \in V$  the map  $U \to W$  given by  $u \mapsto \varphi(u,v)$  is k-linear. Show that there is a unique k-linear map  $U \otimes_k V \to W$  sending  $u \otimes v$  to  $\varphi(u,v)$  for all  $u \in U$  and  $v \in V$ .

- **Exercise 2.** (i) Assume that the elements  $e_{\alpha}$  for  $\alpha \in A$  form a basis of U, and that  $f_{\beta}$  for  $\beta \in B$  form a basis of V. Show that the elements  $e_{\alpha} \otimes f_{\beta}$  for  $(\alpha, \beta) \in A \times B$  form a basis of  $U \otimes_k V$ . (Hint: for linear independence, use the dual basis to  $(e_{\alpha})$  and  $(f_{\beta})$  to define linear forms  $U \otimes_k V \to k$ .)
- (ii) If  $U \neq 0$  and  $V \neq 0$ , show that  $U \otimes_k V \neq 0$ .
- (iii) Assume that  $\dim_k U = m < \infty$  and that  $\dim_k V = n < \infty$ . What is the dimension of  $U \otimes_k V$ ?

**Exercise 3.** Let  $f: U \to U'$  and  $g: V \to V'$  be k-linear maps.

(i) Show that there is a unique k-linear map

$$f \otimes g \colon U \otimes_k V \to U' \otimes_k V'$$

such that

$$(f \otimes g)(u \otimes v) = f(u) \otimes g(v)$$
 for all  $u \in U$  and  $v \in V$ .

- (ii) Assume that f and g are surjective. Show that  $f \otimes g$  is surjective.
- (iii) Assume f and g are injective. Show that  $f \otimes g$  is injective.

**Exercise 4.** Assume that A, B are k-algebras. Show that  $A \otimes_k B$  is naturally a k-algebra.

**Exercise 5.** When  $f: U \to V$  and  $g: V \to W$  are k-linear maps, we say that the sequence

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

is exact if f is injective, g is surjective, and  $\ker g = \operatorname{im} f$ . If F is a k-vector space, show that the induced sequence

$$0 \to U \otimes_k F \xrightarrow{f \otimes \mathrm{id}} V \otimes_k F \xrightarrow{g \otimes \mathrm{id}} W \otimes_k F \to 0$$

is exact.

**Exercise 6.** If  $U' \subset U', V' \subset V$  are subspaces, by Exercise 3 (iii) we may view  $U' \otimes_k V'$  as a subspace of  $U \otimes_k V$ .

(i) Assume that  $U_{\alpha}$  for  $\alpha \in A$  are subspaces of U such that  $U = \bigoplus_{\alpha \in A} U_{\alpha}$ . Show that

$$U \otimes_k V = \bigoplus_{\alpha \in A} (U_\alpha \otimes V).$$

(ii) Let U', U'' be subspaces of U. Show that, in  $U \otimes_k V$ ,

$$(U' \otimes_k V) \cap (U'' \otimes_k V) = (U' \cap U'') \otimes_k V.$$

(iii) If  $U' \subset U$  and  $V' \subset V$  are subspaces, show that, in  $U \otimes_k V$ ,

$$(U' \otimes_k V) \cap (U \otimes_k V') = U' \otimes_k V'.$$