Exercise 1. Let $A \subset B \subset K$, be three domains. Assume that K is the common fraction field of A and B, and that B is finitely generated A-algebra. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced morphism. Show that there is a non-empty open subscheme U of $\operatorname{Spec} A$ such that $f^{-1}U \to U$ is an isomorphism.

Exercise 2. Let A be a domain with fraction field K, and B a finitely generated A-algebra. We assume that $\dim_K(K \otimes_A B) = n < \infty$. Show that there is $f \in A$ such that $B[1/f]$ is an $A[1/f]$-module of finite type. What does the condition $n = 0$ mean for the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$?

Exercise 3. Let k be a field, and denote by X, Y, T the three coordinates of $\mathbb{A}^3_k$. Let $Z = V(XY - T) \subset \mathbb{A}^3_k$. Consider the composite $f\colon Z \to \mathbb{A}^3_k \to \mathbb{A}^1_k$ where the last morphism is given by T. Compute the (scheme-theoretic) fibre of f over each closed point of $\mathbb{A}^1_k$.

Exercise 4. (*Time permitting*) Let R be a commutative ring, and denote by $\mathbb{P}^n(R)$ the set $\operatorname{Hom}(\operatorname{Spec} R, \mathbb{P}^n_{\mathbb{Z}})$. Given elements $\alpha_0, \dots, \alpha_n \in R$, we denote by $[\alpha_0 : \dots : \alpha_n]$ their class in R^{n+1} modulo the relations $[\alpha_0 : \dots : \alpha_n] = [\lambda \cdot \alpha_0 : \dots : \lambda \cdot \alpha_n]$ for $\lambda \in R^\times$. We consider the set

$$H(R) = \{[\alpha_0 : \dots : \alpha_n] | \sum_{i=0}^{n} \alpha_i R = R\}.$$

(i) Explain how every element of $H(R)$ defines an element of $\mathbb{P}^n(R)$.

(ii) Show that $H(R) = \mathbb{P}^n(R)$ when R is a field.

(iii) Show that $H(R) = \mathbb{P}^n(R)$ when R is a principal ideal domain.

(iv) Let k be a field. Describe the set $\operatorname{Hom}_{\operatorname{Spec} k}(\mathbb{P}^1_k, \mathbb{P}^n_k)$.