Exercise 1. Let A be a local domain, with function field K and residue field κ.

(i) Consider the map $\varphi\colon \operatorname{Spec} K \to \operatorname{Spec} A$ sending the point of $\operatorname{Spec} K$ to the closed point of $\operatorname{Spec} A$. Is φ induced by a morphism of ringed spaces?

(ii) Assume that A is the localisation of $\mathbb{Z}$ at a prime number. Is the morphism φ of (i) induced by a morphism of schemes?

(iii) Assume that $\dim A = 1$. Consider the natural ring morphism $m\colon A \to K \times \kappa$. Is $m^\sharp$ a bijection? a homeomorphism?

Exercise 2. Let X be a scheme, and U an open subset of the underlying topological space. Show that $(U, \mathcal{O}_X|_U)$ is a scheme.

Exercise 3. Let k be a field.

(i) (Hilbert's Nullstellensatz) Let L be a finitely generated k-algebra. If L is a field, show that it is a finite field extension of k. (You may use Noether's normalisation Lemma : *Let A a non-zero finitely generated k-algebra. Then there are $x_1, \dots, x_n \in A$ such that A is integral over $k[x_1, \dots, x_n]$.*)

(ii) Let A be a finitely generated k-algebra. Show that a point of $\mathfrak{p} \in \operatorname{Spec} A$ is closed if and only if its residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is a finite extension of k.

Exercise 4. Let $f\colon A \to B$ be ring morphism such that

- for all $b \in B$ there exists $n \in \mathbb{N}$ such that $b^n \in \operatorname{im} f$.
- $\ker f$ consists of nilpotent elements.

Show that $f^\sharp\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is a homeomorphism.