Exercise 1. Let X be a scheme, and consider a short exact sequence of quasi-coherent sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0.$$

Assume that $\mathcal{M}''$ is locally free of finite type. Show that for any scheme morphism $f: Y \to X$ the sequence

$$0 \to f^*\mathcal{M}' \to f^*\mathcal{M} \to f^*\mathcal{M}'' \to 0$$

is exact. Deduce that $\mathcal{F} \mapsto f^*\mathcal{F}$ induces a group morphism $K_0(X) \to K_0(Y)$.

Exercise 2. Let k be a field, and $f: \mathbb{P}^1_k \to \mathbb{P}^1_k$ a dominant morphism. Recall that f is finite. Show that the $\mathcal{O}_{\mathbb{P}^1_k}$-module $f_*\mathcal{O}_{\mathbb{P}^1_k}$ is locally free of finite type. (Hint: Use the covering of $\mathbb{P}^1_k$ by two copies of $\mathbb{A}^1_k$.)

Exercise 3. Let A be a (not necessarily noetherian) commutative unital ring, and M a locally free A-module of finite type.

(i) Show that M is of finite presentation, i.e. there is an exact sequence

$$A^r \to A^s \to M \to 0$$

for some $r, s \in \mathbb{N}$. (Hint: Find a surjection $A^s \to M$, and let R be its kernel. Choose $f_1, \ldots, f_n \in A$ such that the A_{f_i}-modules M_{f_i} are free. Show that the A_{f_i}-module R_{f_i} is finitely generated, and deduce that R is finitely generated.)

(ii) Let N be an A-module. Show that the natural morphism $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N_f)$ induces an isomorphism $(\mathrm{Hom}_A(M, N))_f \simeq \mathrm{Hom}_A(M, N_f)$. (Hint: reduce to the case when M is free and finitely generated using the first question.)

(iii) Show that there exists an A-module F such that $M \oplus F \simeq A^s$. (Hint: Show using (ii) that the morphism $\mathrm{Hom}_A(M, A^s) \to \mathrm{Hom}_A(M, M)$, induced by the morphism $A^s \to M$ of (i), is surjective.)

We now let P, Q be two A-modules such that $P \oplus Q \simeq A^t$ with $t \in \mathbb{N}$. Let $\mathfrak{p} \in \mathrm{Spec}(A)$.

(iv) Show that the A-module P (and also Q) is projective.

(v) Show that the $A_\mathfrak{p}$-module $P_\mathfrak{p}$ (and also $Q_\mathfrak{p}$) is free (Hint: Use Nakayama's Lemma). Deduce the existence of a morphism of A-modules $\theta: A^m \to P$ for some $m \in \mathbb{N}$ such that $\theta_\mathfrak{p}$ is an isomorphism.

(vi) Deduce that we may find $f \in A - \mathfrak{p}$ such that θ_f is bijective. (Hint: Show first that we may find $g \in A - \mathfrak{p}$ such that θ_g is surjective. Observe that the A_g-module P_g is projective.)

(vii) Conclude that an A-module is locally free of finite type if and only if it is a direct summand of a free A-module of finite type.