Exercise 1. Let k be a field. Show that $k[X, Y]$ is not a Dedekind domain.

Exercise 2. Let k be a field, and consider the subring $A = k[X^2, X^3]$ of the polynomial ring $k[X]$.

(i) Show that A is a noetherian domain, and that every nonzero prime ideal of A is maximal. (Hint: Use the inclusions $k[X^2] \subset A \subset k[X]$.)

(ii) Let $k(X)$ be the fraction field of $k[X]$. Show that $k(X)$ is the fraction field of A.

(iii) Show that A is not a Dedekind domain.

Exercise 3 (Approximation Lemma)**.** Let A be a Dedekind domain, with fraction field K. For a nonzero prime ideal $\mathfrak{q}$ of A, and a element $y \in K$, we define

$$v_{\mathfrak{q}}(y) = \sup\{n \in \mathbb{Z} | y \in \mathfrak{q}^n\} \in \mathbb{Z} \cup \{\infty\}.$$

(i) For $a, b \in A$ and $\mathfrak{q}$ a nonzero prime ideal of A, show that

$$v_{\mathfrak{q}}(a + b) \geq \min\{v_{\mathfrak{q}}(a), v_{\mathfrak{q}}(b)\} \quad \text{and} \quad v_{\mathfrak{q}}(ab) = v_{\mathfrak{q}}(a) + v_{\mathfrak{q}}(b).$$

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be pairwise distinct nonzero prime ideals of A. Let $x_1, \ldots, x_s \in K$ and $n_1, \ldots, n_s \in \mathbb{N}$. We are going to prove that we may find $x \in K$ such that

$$v_{\mathfrak{p}_i}(x - x_i) \geq n_i \quad \text{for } i \in \{1, \ldots, s\}, \quad \text{and} \quad v_{\mathfrak{q}}(x) \geq 0 \quad \text{for } \mathfrak{q} \notin \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}. \quad (*)$$

(ii) If $s \geq 2$, show that $\mathfrak{p}_1^{n_1} + \mathfrak{p}_2^{n_2} \cdots \mathfrak{p}_s^{n_s} = A$.

(iii) Show that we may find $x \in A$ satisfying $(*)$ when $x_1 \in A$ and $x_2 = \cdots = x_s = 0$.

(iv) Show that we may find $x \in A$ satisfying $(*)$ when $x_1, \ldots, x_s \in A$.

(v) Show that we may find $x \in K$ satisfying $(*)$.

Exercise 4. (Optional) Let A be a Dedekind domain.

(i) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be pairwise distinct nonzero prime ideals of A. Let $n_1, \ldots, n_s \in \mathbb{N}$. Show that we may find an element $x \in A$ such that $v_{\mathfrak{p}_i}(x) = n_i$ for all $i \in \{1, \ldots, s\}$. (Hint: Use the previous exercise.)

(ii) Show that every ideal of A is generated by at most two elements.

(iii) Assume that A has only finitely prime ideals. Reprove (using (i)) that A is a principal ideal domain.