The letter R denotes a (commutative unital) noetherian ring.

Exercise 1. Let M_1 and M_2 be two finitely generated R-modules. Show that

$$\operatorname{Supp}(M_1 \otimes_R M_2) = \operatorname{Supp}(M_1) \cap \operatorname{Supp}(M_2).$$

Exercise 2. Consider the $\mathbb{Z}$-module $N = \bigoplus_{k \in \mathbb{N}} \mathbb{Z}/p^k$. Compute $\operatorname{Supp}_{\mathbb{Z}}(N)$ and $\operatorname{Ann}_{\mathbb{Z}}(N)$.

Exercise 3. Let M be a finitely generated R-module and $\mathfrak{p} \in \operatorname{Spec}(R)$. Show that

$$\mathfrak{p} \in \operatorname{Supp}(M) \Longleftrightarrow \operatorname{Hom}_R(M, R/\mathfrak{p}) \neq 0.$$

Exercise 4. Let k be a field, and $S = k[X_1, X_2, \cdots]/(X_1^2, X_2^2, \cdots)$. Show that S is not noetherian. Compute $\operatorname{Ass}_k(S)$ and $\operatorname{Ass}_S(S)$ (the definition of an associated prime immediately extends to non-noetherian rings).

Exercise 5. Let $x \in R$. For a prime $\mathfrak{p}$ of R, we denote by $x(\mathfrak{p}) \in \kappa(\mathfrak{p}) = R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})$ the image of x. To what (simple) condition on x is each of the following conditions equivalent?

- $x(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Ass}(R)$.
- $x(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in \operatorname{Ass}(R)$.

Exercise 6. (Primary decomposition) Let M be a finitely generated R-module. We are trying to find submodules $Q(\mathfrak{p}) \subset M$ for $\mathfrak{p} \in \operatorname{Ass}(M)$ satisfying

$$\operatorname{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\} \quad \text{and} \quad \bigcap_{\mathfrak{p} \in \operatorname{Ass}(M)} Q(\mathfrak{p}) = 0.$$

(i) Assuming that the $Q(\mathfrak{p})$'s exist, compute $\operatorname{Ass}(Q(\mathfrak{p}))$.

(ii) Show that the $Q(\mathfrak{p})$'s exist.

(iii) If $S \subset R$ is a multiplicatively closed subset, show that we have in $S^{-1}M$

$$\bigcap_{\substack{\mathfrak{p} \in \operatorname{Ass}(M) \\ \mathfrak{p} \cap S = \varnothing}} S^{-1}Q(\mathfrak{p}) = 0.$$

(iv) If $\mathfrak{p} \in \operatorname{Ass}(M)$ is minimal, show that $Q(\mathfrak{p}) = \ker(M \to M_{\mathfrak{p}})$.

Exercise 7. Let M, N be R-modules, with M finitely generated. Show that

$$\operatorname{Ass}(\operatorname{Hom}_R(M, N)) = \operatorname{Supp}(M) \cap \operatorname{Ass}(N).$$

(You may observe that $\operatorname{Hom}_R(M, N)$ is a submodule of $N^n = N \oplus \cdots \oplus N$ for some n.)

Exercise 8. Let $\varphi\colon R \to S$ be a ring morphism, and N an S-module. Show that

$$\operatorname{Ass}_R(N) = \{\varphi^{-1}\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Ass}_S(N)\}.$$

Exercise 9. (*) Let $R \to S$ be a flat ring morphism, and M an R-module. Show that

$$\operatorname{Ass}_S(M \otimes_R S) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \operatorname{Ass}_S(S/\mathfrak{p}S).$$