Exercise 1. Let K be an imaginary quadratic field. Show that the group $(\mathcal{O}_K)^\times$ is finite and cyclic. (A more precise answer is obtained in Exercise 5 below).

Exercise 2. Let K be a real quadratic field. We fix an embedding $K \subset \mathbb{R}$.

(i) Show that $(\mathcal{O}_K)^\times \simeq \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$.

(ii) Deduce that the subset of units in $\mathcal{O}_K$ which are > 0 is a free $\mathbb{Z}$-module of rank 1, which admits a unique generator u such that $u > 1$. This element u is called *the fundamental unit* of K.

Exercise 3. Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field, where $d \in \mathbb{N} \smallsetminus \{0, 1\}$ is square-free. We view K as a subfield of $\mathbb{R}$. In this exercise, we describe a procedure to determine explicitly the fundamental unit of K (see the previous exercise).

(i) Let $x \in (\mathcal{O}_K)^\times$, and write $x = a + b\sqrt{d}$, with $a, b \in \mathbb{Q}$. Show that $a^2 \geq b^2$. *(Hint: the number $a^2 - db^2$ can only take two values...)*

(ii) Let $x \in (\mathcal{O}_K)^\times$, and write $x = a + b\sqrt{d}$, with $a, b \in \mathbb{Q}$. Show that

$$(x > 1) \iff (a > 0 \text{ and } b > 0).$$

(Hint: If $x > 1$, observe that x is the unique maximal element of the set $\{x, x^{-1}, -x, -x^{-1}\}$.)

(iii) Assume that $d = 2, 3 \mod 4$. Show that the fundamental unit of K can be written as $a_1 + b_1\sqrt{d}$ with $a_1, b_1 \in \mathbb{N} \smallsetminus \{0\}$. Let $x = a + b\sqrt{d} \in (\mathcal{O}_K)^\times$, with $a, b \in \mathbb{N} \smallsetminus \{0\}$. Show that $b \geq b_1$, and that $b = b_1$ implies $a = a_1$.

(Hint: consider the sequence $a_n, b_n \in \mathbb{N} \smallsetminus \{0\}$ defined by $(a_1 + b_1\sqrt{d})^n = a_n + b_n\sqrt{d}$.)

(iv) Assume that $d = 2, 3 \mod 4$. Let $b \in \mathbb{N} \smallsetminus \{0\}$ be the smallest integer such that $db^2 - 1$ or $db^2 + 1$ is of the form a^2 with $a \in \mathbb{N} \smallsetminus \{0\}$. Show that $a + b\sqrt{d}$ is the fundamental unit of K.

(v) Assume that $d = 1 \mod 4$. Show that the fundamental unit of K can be written as $\frac{1}{2}(a_1 + b_1\sqrt{d})$ with $a_1, b_1 \in \mathbb{N} \smallsetminus \{0\}$. Let $x = a + b\sqrt{d} \in (\mathcal{O}_K)^\times$, with $a, b \in \mathbb{N} \smallsetminus \{0\}$. Show that $b \geq b_1$. Assume that $b = b_1$ and $a \neq a_1$. Show that $d = 5$, that $a_1 = b_1 = 1$ and $a = 3$.

(Hint: consider the sequence $a_n, b_n \in \mathbb{N} \smallsetminus \{0\}$ defined by $(\frac{1}{2}(a_1 + b_1\sqrt{d}))^n = \frac{1}{2}(a_n + b_n\sqrt{d})$, and analyse the conditions under which $b_2 = b_1$.)

(vi) Assume that $d = 1 \mod 4$ with $d \neq 5$. Let $b \in \mathbb{N} \smallsetminus \{0\}$ be the smallest integer such that $db^2 - 4$ or $db^2 + 4$ is of the form a^2 with $a \in \mathbb{N} \smallsetminus \{0\}$. Show that $\frac{1}{2}(a + b\sqrt{d})$ is the fundamental unit of K.

(vii) Determine the fundamental units of the following quadratic fields:

$$\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{17})$$

Exercise 4 (Pell's equation)**.** (i) Let $d \in \mathbb{N} \smallsetminus \{0, 1\}$ be square-free. Show that the set of solutions $x, y \in \mathbb{N}$ to the equation

$$x^2 - dy^2 = 1,$$

is $\{(x_n, y_n) | n \in \mathbb{N}\}$, where

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n.$$

(Hint: Use the previous exercise.)

(ii) Determine (x_1, y_1) when $d \in \{2, 5, 6, 17\}$.

Exercise 5. Let K be an imaginary quadratic number field. Show that

$$(\mathcal{O}_K)^\times = \begin{cases} \{1, -1, i, -i\} & \text{if } K = \mathbb{Q}(i), \\ \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}, \text{where } \alpha = \frac{1+\sqrt{-3}}{2} & \text{if } K = \mathbb{Q}(\sqrt{-3}) \\ \{1, -1\} & \text{otherwise.} \end{cases}$$