Let R be a ring, and $x_1, \cdots, x_n \in R$. We construct the associated *Koszul complex* as follows. Let $e_1, \cdots, e_n$ be the standard basis of the R-module R^n. Let $p \in \mathbb{Z}$. For $p \in \{1, \ldots, n\}$, we let K_p be the free R-module with the basis consisting of the elements $e_{i_1} \wedge \cdots \wedge e_{i_p}$ where $1 \leq i_1 < \cdots < i_p \leq n$. We let $K_0 = R$, and $K_p = 0$ when $p \notin \{0, \ldots, n\}$. We define a R-linear morphism $d \colon K_p \to K_{p-1}$ using the formula

$$d_p(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{r=1}^{p} (-1)^{r-1} x_{i_r} \cdot e_{i_1} \wedge \cdots \wedge e_{i_{r-1}} \wedge e_{i_{r+1}} \wedge \cdots \wedge e_{i_p}.$$

(the vector e_{i_r} is omitted.) When $p = 1$, the above formula must be understood as

$$d_1(e_i) = x_i \in R = K_0.$$

Exercise 1. Show that $d_{p-1} \circ d_p = 0$.

This gives a chain complex $K(x_1, \cdots, x_n) = (K, d)$. Let M be an R-module. We denote by $K(M; x_1, \cdots, x_n)$ the complex $K(x_1, \cdots, x_n) \otimes_R M$. Its p-th homology is denoted $H_p(M; x_1, \cdots, x_n)$.

Exercise 2. (i) Express $H_0(M; x_1, \cdots, x_n)$ and $H_n(M; x_1, \cdots, x_n)$ directly in terms of M and $x_1, \cdots, x_n$.

(ii) Describe the complex $K(M; x_1)$.

Exercise 3. (i) Show that the complexes $K(M; x_1, \cdots, x_n)$ and $K(x_1) \otimes_R \cdots \otimes_R K(x_n) \otimes_R M$ are isomorphic.

(ii) Let L be a chain complex of R-modules and $x \in R$. Show that we have an exact sequence of chain complexes of R-modules

$$0 \to L \to K(x) \otimes_R L \to L[-1] \to 0,$$

(where $L[-1]_n = L_{n-1}$ and $d_n^{L[-1]} = -d_{n-1}^L$) and deduce and exact sequence of R-modules

$$0 \to H_0(H_p(L); x) \to H_p(K(x) \otimes_R L)) \to H_1(H_{p-1}(L); x) \to 0.$$

Exercise 4. Let A be a local (noetherian) ring, M a finitely generated A-module, and $x_1, \cdots, x_n \in \mathfrak{m}$.

(i) Assume that $(x_1, \cdots, x_n)$ is an M-regular sequence. Show that $H_i(M; x_1, \cdots, x_n) = 0$ for $i > 0$.

(ii) Assume that $H_1(M; x_1, \cdots, x_n) = 0$. Show that $(x_1, \cdots, x_n)$ is an M-regular sequence.

Exercise 5. Let A be a local (noetherian) ring, and M a finitely generated A-module. Assume that $(x_1, \cdots, x_n)$ is an M-regular sequence.

(i) Let σ be a permutation of $\{1, \cdots, n\}$. Show that $(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ is an M-regular sequence.

(ii) Let $t_1, \ldots, t_n$ be integers ≥ 1. Show that $(x_1^{t_1}, \ldots, x_n^{t_n})$ is a regular M-sequence.

Exercise 6. (i) Let L be a complex of R-modules and $x \in R$. Show that $x \cdot H_p(K(x) \otimes_R L) = 0$.

(ii) Let $x_1, \cdots, x_n \in R$, and I be the ideal generated by these elements. Let M be an R-module. Show that $I \cdot H_p(M; x_1, \cdots, x_n) = 0$.

Exercise 7. (Depth sensitivity of the Koszul complex) Let A be a local ring, and $\{x_1, \cdots, x_n\}$ a generating set for its maximal ideal. Let M be a nonzero finitely generated A-module. Show that

$$\operatorname{depth} M = n - \max\{i | H_i(M; x_1, \cdots, x_n) \neq 0\}.$$

(use Exercise 6.)

Exercise 8. (A more functorial approach) Let U be a finitely generated free R-module. For an R-module, we denote by $V^\vee = \operatorname{Hom}_R(V, R)$ its dual. We consider the R-module

$$T(U) = \bigoplus_{p \geq 0} U^{\otimes p} = R \oplus U \oplus (U \otimes_R U) \oplus \cdots .$$

The R-module $\Lambda(U)$ is the quotient of $T(U)$ by the submodule generated by elements $x_1 \otimes \cdots \otimes x_p$ which are such that $x_i = x_j$ for some $i \neq j$. It is naturally graded; we denote by $\Lambda^p U$ the image of $U^{\otimes p}$ and by $u_1 \wedge \cdots \wedge u_p$ the image of $u_1 \otimes \cdots \otimes u_p$. An isomorphism $U \simeq R^n$ induces an isomorphism $\Lambda^p U \simeq K_p$.

(i) Show that the natural morphism $\rho_p \colon \Lambda^p(U^\vee) \to (\Lambda^p U)^\vee$ is an isomorphism.

(ii) Let $u \in U$, and $\varphi_u \colon \Lambda^p U \to \Lambda^{p+1} U$ be defined by $\varphi_u(v) = u \wedge v$. Show that if $e_1, \cdots, e_n$ is a basis of the R-module $U^\vee$, and $(x_1, \cdots, x_n)$ are the coordinates of u in the dual basis of U, then the differential d of the Koszul complex may be identified with

$$\Lambda^{p+1}(U^\vee) \xrightarrow{\rho_{p+1}} (\Lambda^{p+1} U)^\vee \xrightarrow{(\varphi_u)^\vee} (\Lambda^p U)^\vee \xrightarrow{\rho_p^{-1}} \Lambda^p(U^\vee).$$

(iii) Reprove without computation that $d \circ d = 0$.