

# GALOIS COHOMOLOGY

## EXERCISES 10 (TWISTED FORMS)

The letter  $k$  denotes a field.

- Exercise 1.** (i) Let  $V$  be a  $k$ -vector space of finite dimension  $n$ , and  $f: V \times V \rightarrow k$  be a  $k$ -bilinear form. We assume that  $f(x, x) = 0$  for all  $x \in V$  (i.e.  $f$  is alternated) and that the  $k$ -linear map  $V \rightarrow \text{Hom}_k(V, k)$  sending  $x$  to the map  $y \mapsto f(x, y)$  is bijective (i.e.  $f$  is nondegenerate). Show that  $n$  is even, and that  $V$  admits a  $k$ -basis  $e_1, \dots, e_n$  such that  $f(e_{2r+1}, e_{2r+2}) = 1$  and  $f(e_{2r+2}, e_{2r+1}) = -1$  for all  $0 \leq r < n/2$ , and  $f(e_i, e_j) = 0$  for all other values of  $i, j$ .
- (ii) When  $L/k$  is a separable field extension, consider the matrix (where blank entries are zero)

$$J = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix} \in M_{2r}(L).$$

Show that letting

$$\text{Sp}_{2r}(L) = \{M \in M_{2r}(L) \mid M^t J M = J\},$$

where  $M^t$  denotes the transpose of  $M$ , defines a  $k$ -group  $\text{Sp}_{2r}$  such that  $H^1(k, \text{Sp}_{2r}) = \{*\}$ .

- Exercise 2.** For every separable extension  $L/k$  set

$$G(L) = \text{Aut}_{L\text{-alg}}(L[X]).$$

Extension of scalars yields a map  $G(L) \rightarrow G(L')$  for every morphism  $L \rightarrow L'$  of separable extensions of  $k$ .

- (i) Show that  $G$  defines a  $k$ -group. (Caution: As  $\dim_k k[X] = \infty$ , some of the results of the lectures on twisted forms do not apply directly.)
- (ii) Show that every element of  $G(L)$  is of the form  $X \mapsto aX + b$ , where  $a \in L^\times$  and  $b \in L$ .
- (iii) Show that we have an exact sequence of  $k$ -groups

$$1 \rightarrow \mathbb{G}_a \rightarrow G \rightarrow \mathbb{G}_m \rightarrow 1.$$

- (iv) Show that  $H^1(k, G) = \{*\}$ .

Let  $A$  be a  $k$ -algebra such that  $A_L \simeq L[X]$  as  $L$ -algebra, for some separable extension  $L/k$ .

(v) For every separable extension  $L/k$ , consider the set  $I(L)$  of isomorphisms of  $L$ -algebras  $L[X] \rightarrow A_L$ . Extension of scalars yields a map  $I(L) \rightarrow I(L')$  for every morphism  $L \rightarrow L'$  of separable extensions of  $k$ . Show that  $I$  defines a  $G$ -torsor.

(vi) Conclude that  $A \simeq k[X]$  as  $k$ -algebra.

We now assume that  $k$  has positive characteristic  $p$ , and that  $a \in k$  is such that  $a \neq b^p$  for all  $b \in k$ . We consider the  $k$ -algebra  $B = k[U, V]/(U^p - aV^p - V)$ .

(vii) Show that there exists an algebraic field extension  $K/k$  such that  $B_K \simeq K[X]$  as  $K$ -algebra.

(viii) Show that  $B$  is not isomorphic to  $k[X]$  as  $k$ -algebra. (Hint: If  $\varphi: B \rightarrow k[X]$  is a morphism of  $k$ -algebras, consider the equation satisfied by the polynomials  $\varphi(U)$  and  $\varphi(V)$  to deduce that  $\varphi(B) = k$ .)

(ix) Give an example of a field  $k$  of characteristic  $p$ , together with an element  $a \in k$  such that  $a \neq b^p$  for all  $b \in k$ .