Exercise 1. (i) Let M be an R-module such that id_M is in the image of the natural morphism

$$\mathrm{Hom}_R(M,R)\otimes_R M \to \mathrm{Hom}_R(M,M).$$

Show that M is projective.

(ii) Let M,N,Q three R-modules. Assume that Q is flat, M is finitely generated, and R is noetherian. Show that the natural morphism

$$\mathrm{Hom}_R(M,N)\otimes_R Q \to \mathrm{Hom}_R(M,N\otimes_R Q)$$

is bijective. (Hint: Introduce a finite presentation of M, that is, an exact sequence $F_1 \to F_0 \to M \to 0$, with F_0, F_1 free and finitely generated R-modules).

(iii) Assume that R is noetherian and let M is a finitely generated flat R-module. Show M is projective.

(iv) Give an example of a flat, non-projective, $\mathbb{Z}$-module.

Exercise 2. Let x be a nonzerodivisor in R. Express $\mathrm{Tor}_1(R/x,M)$ in an elementary way in terms of x and M.

Exercise 3. Let I,J be two ideals in a ring R. Express $\mathrm{Tor}_1^R(R/I,R/J)$ in an elementary way in terms of R,I,J.

Exercise 4. (i) Show that M is flat, resp. projective, if and only if $\mathrm{Tor}_1(N,M)=0$, resp. $\mathrm{Ext}^1(M,N)=0$, for every module N.

(ii) Let $0\to M'\to M\to M''\to 0$ be an exact sequence. Assume that M' and M'' are projective, resp. flat, and show that M is projective, resp. flat.

Exercise 5. Let M,N two R-modules. Assume that R is noetherian and that M is finitely generated. Show that $\mathrm{Tor}_n(M,N)$ and $\mathrm{Ext}^n(M,N)$ are finitely generated.

Exercise 6. Let $R\to S$ be a flat ring morphism, and M,N two R-modules.

(i) Show that

$$\mathrm{Tor}_n^R(M,N)\otimes_R S \simeq \mathrm{Tor}_n^S(M\otimes_R S, N\otimes_R S).$$

(ii) Assume that R is noetherian, and M finitely generated. Show that

$$\mathrm{Ext}^n_R(M,N)\otimes_R S \simeq \mathrm{Ext}^n_S(M\otimes_R S, N\otimes_R S).$$

Exercise 7 (Yoneda description of Ext^1)**.** We fix two modules A and B. Given an exact sequence α of type

$$0\to B\to X\to A\to 0$$

we define $[\alpha]\in \mathrm{Ext}^1(A,B)$ to be the image of id_A under the morphism $\mathrm{Hom}_R(A,A)\to \mathrm{Ext}^1(A,B)$ (which is part of the long exact sequence of Ext-groups associated with the short exact sequence α).

(i) We say that α splits if there is a morphism $A\to X$ such that the composite $A\to X\to A$ is the identity. Show that α splits if and only if $[\alpha]=0$.

We say that two exact sequences $0 \to B \to X \to A \to 0$ and $0 \to B \to X' \to A \to 0$ are *Yoneda equivalent* if there is an isomorphism $X \to X'$ fitting in the commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "="'] & X \arrow[r] \arrow[d] & A \arrow[d, "="] \\ B \arrow[r] & X' \arrow[r] & A \end{tikzcd}$$

(ii) Show that a sequence splits if and if it is Yoneda equivalent to the sequence $0 \to B \to A \oplus B \to A \to 0$.

(iii) We let $E(A, B)$ be the set of exact sequences $0 \to B \to X \to A \to 0$ modulo Yoneda equivalence. Show that $\alpha \mapsto [\alpha]$ induces a map $E(A, B) \to \mathrm{Ext}^1(A, B)$.

We construct a map $\mathrm{Ext}^1(A, B) \to E(A, B)$ as follows. Take an exact sequence $0 \to K \to F \to A \to 0$ with F free. An element $u \in \mathrm{Ext}^1(A, B)$ is represented by a morphism $\varphi_u \colon K \to B$. Let X_u be the cokernel of the morphism $K \to F \oplus B$ given by $k \mapsto (j(k), -\varphi_u(k))$ where j is the injective morphism $K \to F$.

(iv) Show that we have an exact sequence $0 \to B \to X_u \to A \to 0$, and therefore an element of $E(A, B)$.

(v) Show that this gives a map $\mathrm{Ext}^1(A, B) \to E(A, B)$.

(vi) Show that $\mathrm{Ext}^1(A, B)$ and $E(A, B)$ are in bijection.

(vii) Let $\alpha, \beta \in E(A, B)$. Describe the element $\gamma \in E(A, B)$ such that $[\gamma] = [\alpha] + [\beta]$. Describe the functorialities of $E(A, B)$ in A and B.