GALOIS COHOMOLOGY
EXERCISES 9 (TORSORS)

The solutions will be discussed during the online session on Jan 12th.

We fix a base field k.

Exercise 1. Let k_s be a separable closure of k, and $\Gamma = \mathrm{Gal}(k_s/k)$. Let A be an étale k-algebra of dimension n. Consider the associated discrete Γ-set $X = \mathrm{Hom}_{k-\mathrm{alg}}(A, k_s)$. Let $Y \subset X^n$ be the set of those $(x_1, \dots, x_n)$ such that $x_i \neq x_j$ when $i \neq j$, with the Γ-action given by

$$\gamma(x_1, \dots, x_n) = (\gamma x_1, \dots, \gamma x_n) \quad \text{for } \gamma \in \Gamma, \text{ and } x_1, \dots, x_n \in X.$$

The symmetric group $\mathfrak{S}_n$ acts on Y by

$$\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

where $\sigma \in \mathfrak{S}_n$ and $x_1, \dots, x_n$ are pairwise distinct elements of X. Denote by Z the quotient of Y be the action of the subgroup $\mathfrak{A}_n$ of even permutations (the kernel of the signature morphism $\mathfrak{S}_n \to \mathbb{Z}/2$).

(i) Show that Z is a discrete Γ-set having two elements.

We denote by Δ the corresponding étale k-algebra of dimension two; it is called the *discriminant algebra* of A.

Assume now that k has characteristic $\neq 2$. Let $e_1, \dots, e_n$ be a k-basis of A, let $f_1, \dots, f_n$ be the elements of X, and consider the matrix $M = (f_i(e_j)) \in M_n(k_s)$. Set

$$u = \det M \in k_s.$$

Let Γ_0 be the subgroup of Γ consisting of those elements acting by even permutations on the set X.

(ii) Let $\gamma \in \Gamma$. Show that $\gamma u = u$ if $\gamma \in \Gamma_0$ and $\gamma u = -u$ otherwise.

Let d be the determinant of the matrix $(\mathrm{Tr}_{A/k}(e_i e_j)) \in M_n(k)$.

(iii) Show that $d = u^2$. (Hint: compute the product $M^t \cdot M$.)

(iv) Conclude that $\Delta \simeq k[X]/(X^2 - d)$.

Exercise 2. Let G be a finite group. Let $H \subset G$ be a subgroup and B a H-algebra over k. Consider the set

$$\mathrm{Ind}_H^G B = \{\text{maps } f \colon G \to B \text{ such that } f(h \cdot g) = h \cdot f(g) \text{ for all } g \in G, h \in H\},$$

viewed as a k-algebra, via pointwise operations on B.

(i) Show that the k-algebra $\mathrm{Ind}_H^G B$ is étale if and only if B is étale.

If $f \in \operatorname{Ind}_H^G B$ and $g \in G$, we define an element $g \cdot f \in \operatorname{Ind}_H^G B$ by mapping a $x \in G$ to $f(x \cdot g)$. This gives $\operatorname{Ind}_H^G B$ the structure of a G-algebra.

(ii) Show that the H-algebra B is Galois over k if and only if the G-algebra $\operatorname{Ind}_H^G B$ is Galois over k.

We consider the morphism of H-algebras $u\colon \operatorname{Ind}_H^G B \to B$ given by $f \mapsto f(1)$.

(iii) Let A be an étale G-algebra, and $\varphi\colon A \to B$ be a morphism of H-algebras. Show that there exists a unique morphism of G-algebras $\widetilde{\varphi}\colon A \to \operatorname{Ind}_H^G B$ such that $u \circ \widetilde{\varphi} = \varphi$.

(iv) Let A be a Galois G-algebra over k. Show that there exists a Galois field extension L/k, a subgroup $H \subset G$ isomorphic to $\operatorname{Gal}(L/k)$, and an isomorphism of G-algebras $A \simeq \operatorname{Ind}_H^G L$.