Exercise 1. Let X be a scheme and n an integer. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n.

(i) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \simeq \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}$.

(ii) Let $f : Y \to X$ be a morphism of schemes and $\mathcal{G}$ an $\mathcal{O}_Y$-module. Show that $f_*(\mathcal{G} \otimes_{\mathcal{O}_Y} f^*\mathcal{E}) \simeq (f_*\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{E}$.

Exercise 2. (i) Let A be a noetherian ring, and M an A-module of finite type. Let r be an integer. Assume that the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is free of rank r for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Show that the A-module M is locally free of rank r.

(ii) Let X be a regular curve over a field, and Z a closed subscheme of X satisfying $Z \neq X$. Show that the sheaf of ideals $\mathcal{I}_Z$ is a locally free $\mathcal{O}_X$-module of rank one.

Exercise 3. Let X be a scheme and $\mathcal{L}$ a locally free $\mathcal{O}_X$-module of rank one. Let $s \in H^0(X, \mathcal{L})$, and consider the set X_s consisting of those points $x \in X$ such that s_x generates the $\mathcal{O}_{X,x}$-module $\mathcal{L}_x$.

(i) Show that X_s is an open subset of X.

(ii) Show that the induced open immersion of schemes $X_s \to X$ is an affine morphism.

Exercise 4. Let X be a scheme. Show that the morphisms $X \to \mathbb{P}^n$ are in bijective correspondence with the data of:

- a locally free $\mathcal{O}_X$-module of rank one $\mathcal{L}$,
- sections $s_0, \dots, s_n \in H^0(X, \mathcal{L})$ such that the induced morphism $\mathcal{O}_X^{n+1} \to \mathcal{L}$ is surjective,

where we identify $(\mathcal{L}, s_0, \dots, s_n)$ and $(\mathcal{L}', s'_0, \dots, s'_n)$ if there is an isomorphism of $\mathcal{O}_X$-modules $\mathcal{L} \to \mathcal{L}'$ mapping each s_i to s'_i.

(Hint: To construct $X \to \mathbb{P}^n$, let $i \in \{0, \dots, n\}$. Consider the open subscheme X_{s_i} of X of the previous exercise, use the elements s_j for $j \neq i$ to define a morphism $X_{s_i} \to \mathbb{A}^n = \Omega_i \to \mathbb{P}^n$, and proceed with the glueing. Conversely given a morphism $X \to \mathbb{P}^n$, observe that the $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{O}(1)$ and its $n+1$ canonical sections pull back to an $\mathcal{O}_X$-module $\mathcal{L}$ and $n+1$ sections of $\mathcal{L}$.)