EXERCISES 4 (INTERSECTION THEORY)

Exercise 1. Let k be a field, and consider the ring

$$R = \frac{k[x,y,z]}{(z^2 - xy)},$$

and let $X = \operatorname{Spec} R$.

(i) Let D be the closed subscheme of X defined by the ideal (z^2). Show that D is an effective Cartier divisor.

(ii) Let Z be the closed subscheme of X defined by the ideal (z, x). Show that Z is an irreducible component of D.

(iii) Show that Z is not an effective Cartier divisor in X.

Exercise 2. Let X be an integral variety, and $\mathcal{L}$ a locally free coherent $\mathcal{O}_X$-module of rank one.

(i) Show that we have a correspondence

$$\left\{ \begin{array}{l} \text{Integral closed subschemes } Z \subset \mathbb{P}(\mathcal{L} \oplus 1), \\ \text{with } Z \not\subset \mathbb{P}(1), Z \not\subset \mathbb{P}(\mathcal{L}), \\ \text{and } Z \to X \text{ birational.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{regular meromorphic} \\ \text{sections of } \mathcal{L}. \end{array} \right\}$$

(ii) Let s is a regular meromorphic section of $\mathcal{L}$, and $Z \subset \mathbb{P}(1 \oplus \mathcal{L})$ the corresponding closed subscheme, with morphism $p\colon Z \to X$. Show that

$$\operatorname{div}_{p^*\mathcal{L}}(p^*s) = [Z \cap \mathbb{P}(1)] - [Z \cap \mathbb{P}(L)] \in \mathcal{Z}(Z).$$

(iii) Show that $Z \cap \mathbb{P}(1)$ (resp. $Z \cap \mathbb{P}(L)$) may be viewed as a closed subscheme $Z(1)$ (resp. $Z(L)$) of X, and that we have

$$\operatorname{div}_{\mathcal{L}}(s) = p_*[Z \cap \mathbb{P}(1)] - p_*[Z \cap \mathbb{P}(L)] = [Z(1)] - [Z(L)] \in \mathcal{Z}(X).$$

Exercise 3. Let X be a variety, and $u\colon U \to X$ the immersion of an open subscheme. Show that the morphism

$$\mathcal{O}_X \to u_*\mathcal{O}_U$$

is injective if and only if $\operatorname{Ass}(\mathcal{O}_X) \subset U$.

For a ring A, we define $R(A)$ as the set of nonzerodivisors in A. For an open subscheme U of X, we define

$$\mathcal{R}_X(U) = \{a \in \mathcal{O}_X(U) | a_x \in R(\mathcal{O}_{X,x}), \forall x \in U\}.$$

Then

$$U \mapsto \mathcal{K}'_X(U) = \mathcal{R}_X(U)^{-1}\mathcal{O}_X(U)$$

defines a presheaf of $\mathcal{O}_X$-algebras.

Exercise 4. If U is an affine open subscheme of X, show that $\mathcal{R}_X(U) = R(\mathcal{O}_X(U))$.

Exercise 5. For any $x \in X$, show that $\mathcal{K}'_{X,x}$ is the total ring of fractions $R(\mathcal{O}_{X,x})^{-1}\mathcal{O}_{X,x}$.

Let $\mathcal{K}_X$ be the sheaf associated with $\mathcal{K}'_X$.

Exercise 6. Let U be an open subscheme of X containing $\operatorname{Ass}(\mathcal{O}_X)$, and $u\colon U \to X$ the immersion. Show that

$$\mathcal{K}_X \to u_*\mathcal{K}_U,$$

is an isomorphism