The letter R denotes a (commutative unital) noetherian ring.

Exercise 1. Let $\varphi\colon A \to B$ be a morphism of local noetherian rings making B a finite type A-module. Show that φ is a local morphism.

Exercise 2. Let $\rho\colon R \to S$ be a flat morphism and M a finitely generated R-module. Show that the map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ maps $\operatorname{Ass}_S(S \otimes_R M)$ into $\operatorname{Ass}_R(M)$.

Exercise 3. Assume that $\dim R \geq 2$. Show that $\operatorname{Spec} R$ is infinite.

Exercise 4. (i) Let $\mathfrak{p}$ be a prime of R. Show that the ideal $\mathfrak{p}R[t]$ of $R[t]$ is prime.

(ii) Show that $\dim R[t] \geq 1 + \dim R$

(iii) Show that $\dim R[t_1, \cdots, t_n] = n + \dim R$.

Exercise 5. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and consider the n-th symbolic power

$$\mathfrak{p}^{[n]} = \{u \in R \mid su \in \mathfrak{p}^n \text{ for some } s \in R - \mathfrak{p}\}.$$

(i) Show that $\operatorname{Ass}(R/\mathfrak{p}^n)$ may differ from $\{\mathfrak{p}\}$ by considering the case $R = k[x,y]/(xy)$ with k a field, and $\mathfrak{p} = xR$.

(ii) Show that $\operatorname{Ass}(R/\mathfrak{p}^{[n]}) = \{\mathfrak{p}\}$, and that $\mathfrak{p}^{[n]}$ is minimal among the ideals I containing $\mathfrak{p}^n$ and satisfying $\operatorname{Ass}(R/I) = \{\mathfrak{p}\}$.

Exercise 6. (i) Show that every prime of R has finite height.

(ii) Let M be a possibly non-finitely generated R-module. Assume that $M \neq 0$. Show that $\operatorname{Supp}(M)$ admits at least one minimal element.