GALOIS COHOMOLOGY
EXERCISES 8 (ÉTALE ALGEBRAS)

Let k be a field.

Exercise 1. Let A be an étale k-algebra. Recall that $\mathbf{X}(A)$ denotes the set k-algebra morphisms $A \to k_s$, where k_s is a separable closure of k.

(i) Let B be a quotient algebra of A. Show that B is étale and that the map $\mathbf{X}(B) \to \mathbf{X}(A)$ is injective.
(ii) Let B be a subalgebra of A. Show that B is étale and that the map $\mathbf{X}(A) \to \mathbf{X}(B)$ is surjective. (Hint: assuming that the map is not surjective, produce an element of the kernel of $\mathbf{M}(\mathbf{X}(A)) \to \mathbf{M}(\mathbf{X}(B))$.)
(iii) Show that A has only finitely many subalgebras and quotient algebras.
(iv) Assume that k is infinite. Show that there exists a separable polynomial P such that $A \simeq k[X]/P$. (Hint: to show that A is generated by a single element as a k-algebra, observe that no k-vector space is a finite union of proper subspaces.)

Exercise 2. Let A be a finite-dimensional k-algebra. For an element $a \in A$ recall that $\mathrm{Tr}_{A/k}(a) \in k$ as the trace of the k-linear map $A \to A$ given by $x \mapsto ax$.

(i) Show that a k-algebra A is étale if and only if for every nonzero $a \in A$ there exists $b \in A$ such that $\mathrm{Tr}_{A/k}(ab) \neq 0$.
(ii) Show that a finite field extension L/k is separable if and only if the map $\mathrm{Tr}_{L/k} \colon L \to k$ is nonzero.

Exercise 3. Let K/k be a field extension. We have seen that there is at most one group G (up to isomorphism) such that K is a Galois G-algebra (namely K/k must be Galois, and $G = \mathrm{Gal}(K/k)$). We give here an example of an algebra A admitting G-Galois structures for nonisomorphic group G.

Let K be a separable quadratic extension of k, and $A = K \times K$.

(i) Define a $\mathbb{Z}/4$-Galois algebra structure on A.
(ii) Define a $(\mathbb{Z}/2) \times (\mathbb{Z}/2)$-Galois algebra structure on A.