Fabien Morel
Olivier Haution

Algebraic Geometry II
Exercises 5

20.05.2015

Exercise 1. *(If you have seen closed subschemes)* Let X be a scheme.

(i) Show that there exists a unique reduced closed subscheme X_{red} of X such that any morphism $T \to X$ with T reduced factors through $X_{red} \to X$.

(ii) Show that $X_{red} \to X$ is surjective, and that X_{red} is the smallest closed subscheme of X with this property (in other words, if a closed subscheme Z of X is such that $Z \to X$ is surjective, then $X_{red} \to X$ factors through Z).

Exercise 2. (i) Let $f\colon Y \to X$ be a scheme morphism, x a point of X, and $f^{-1}x = Y \times_X \operatorname{Spec} k(x)$ the scheme-theoretic fibre of f over x. Show that the underlying set of $f^{-1}x$ is the set-theoretic fibre $\{y \in Y | f(y) = x\}$.

(ii) Let $f\colon X \to S$ and $g\colon Y \to S$ be two scheme morphisms. Show that the underlying set of $Y \times_S X$ is

$$\{(x, y, l)|x \in X, y \in Y \text{ with } f(x) = g(y) =: s, \text{ and } l \in \operatorname{Spec}(k(x) \otimes_{k(s)} k(y))\}.$$

Exercise 3. (*More difficult*) Let k be a field, and $f\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ a morphism. Let S be the subset of points $x \in \mathbb{A}^1_k$ such that the scheme-theoretic fibre $f^{-1}x$ is a disjoint union of spectra of separable field extensions of $k(x)$. Show that S is open in $\mathbb{A}^1_k$.